**MTH3028 Statistical Inference: Theory and Practice**

**Q1.**

Conducted here is a simulation study and comparison of the sampling properties of the Method of Moments (MoM), Bias-Corrected Maximum Likelihood Estimator (BCMLE), and Method of Moments Estimator with Bootstrapping (MoM with Bootstrap) for the distribution. I am simulating multiple datasets using the Monte Carlo method, I will estimate the parameters using each estimator function. Then plot histograms of the estimates for different sample sizes as shown in *Figures 1, 2, and 3*. The bias and mean squared error of the estimators' values are calculated and plotted by sample size *Figure 4*. The mean, median, and standard deviation of the estimator's bias and mean squared error are compared in *Figure 5.* Plotting the mean squared error against the increasing number of samples sizes shows the efficiency of the estimator as shown and compared in *Figure 6.*

Using a true theta value of 0.2, sample sizes of 50, 100, and 200, and 100 bootstrap iterations. Then iterating over 1000 simulations to produce enough estimate values for sufficient variance in each of the estimators, the central tendency (mean or median) and dispersion of estimated values.

The estimated values are the point estimates obtained from each estimator for the true parameter. The consistency of each estimator refers to the variation in change in concentration of estimated values about the true value of theta as the sample sizes increase. Strong consistency exhibits little to no change in this feature. The Method of Moments estimator exhibits strong consistency where the distribution of estimate values centre about the point 0.25 for all sample sizes. The same is true for the other two estimators BCMLE and MoM with Bootstrapping with mean values of 0.25 *Figures 1, 2, 3*.

The bias of the estimated values is a measure of the systematic error in the estimation. A positive bias indicates that the estimator tends to overestimate the true parameter such as MoM, while a negative bias indicates underestimation. A bias close to zero suggests that, on average, the estimator is unbiased. If bias consistently increases or decreases with sample size, it may indicate a trend in the estimator's behaviour. The median bias can assess how sensitive the estimator is to extreme values. The bias decreases for all estimators with increased sample sizes, MoM shows the least bias *Figure 4*.

The mean squared error (MSE) is the average squared difference between the estimated values and the true parameter. It considers both bias and variability. Lower MSE is generally desirable, as it indicates less overall error in the estimation. MSE is decomposable into variance and squared bias components. A decrease in MSE with larger samples indicates improved estimator performance. BCMLE has the lowest MSE while MoM and MoM with bootstrapping share the highest, similar MSE. All estimators decrease in MSE with larger sample sizes. The MoM with bootstrapping estimator changes the most with sample size as it has the largest standard deviation of MSE as shown in *Figure 5.*

Efficiency quantifies how well an estimator uses the data to output precise estimates. Compare the variance or mean squared error (MSE) of different estimators. An estimator with higher efficiency normally has lower variance or MSE. In evaluating efficiency, look how quickly the estimator converges to the true parameter over increasing sample sizes. The BCMLE estimator converges to a lower MSE with a smaller sample and reaches the lowest efficiency overall making it the most efficiency estimator shown in *Figure 6*.

The Cramér-Rao Lower Bound (CRLB) is a theoretical lower bound on the variance of any unbiased estimator. Considering the simplest case, assuming a univariate normal distribution for the data given the estimators follow a roughly normal distribution. For MoM with Bootstrapping, since this involves bootstrapping, obtaining a closed-form expression for the Fisher Information is challenging. In practice, the CRLB for bootstrapped estimators is often difficult to derive analytically. The Cramer-Rao Lower Bound for MoM and BCMLE across the sample sizes (50, 100, 200) come to 0.02, 0.01, 0.005 respectively.

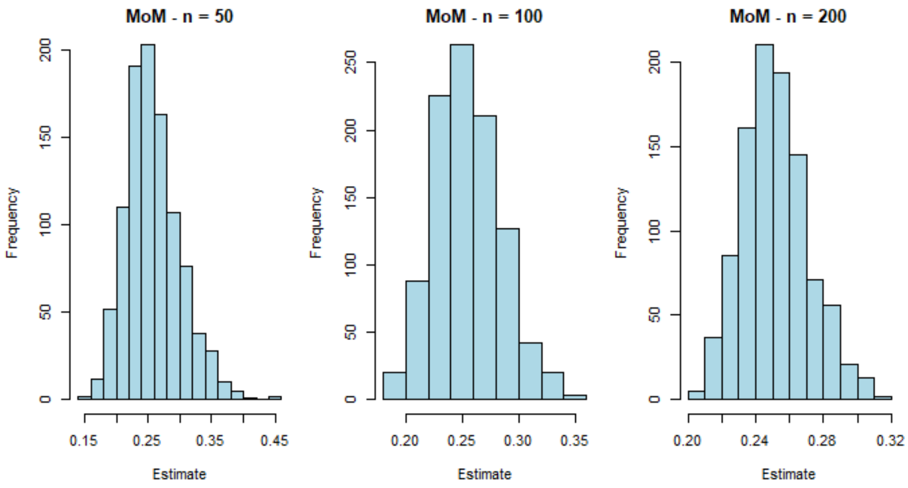


*Figure 1*

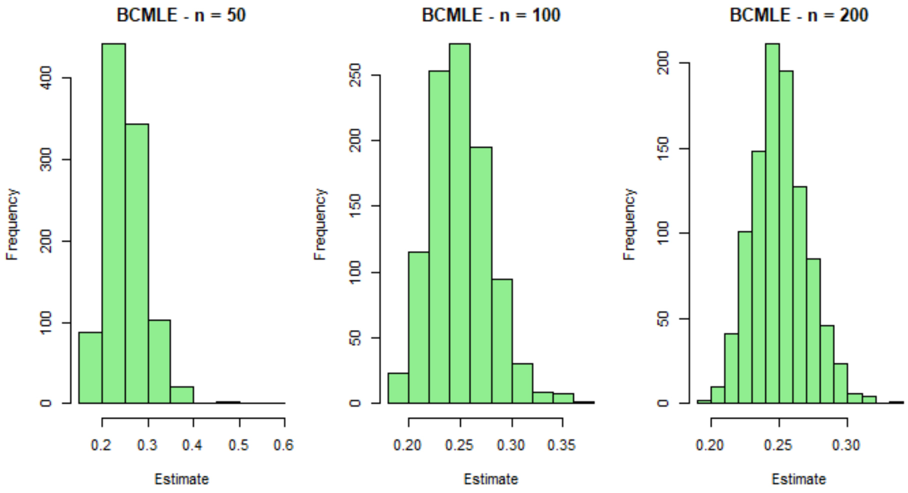


*Figure 2*

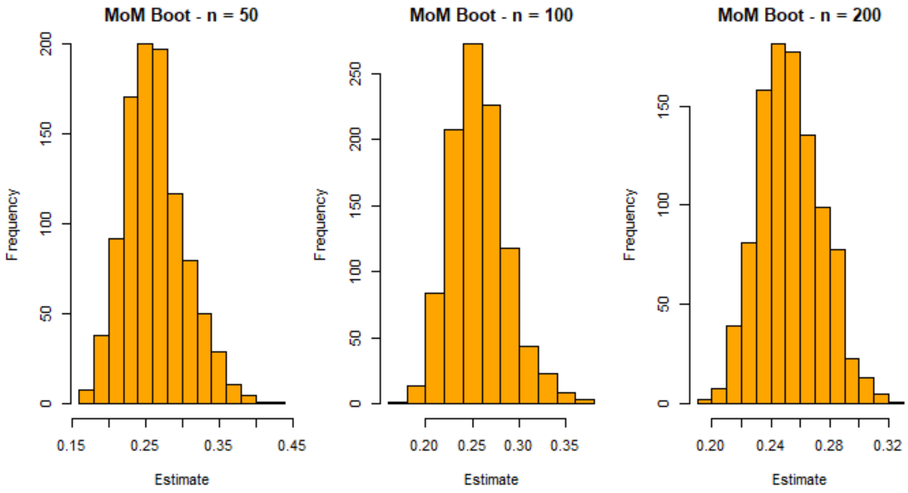


*Figure 3*

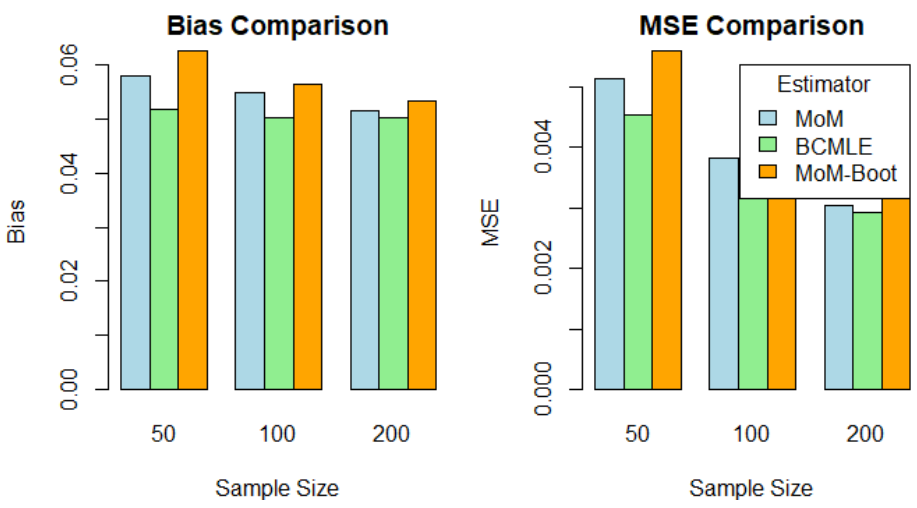


*Figure 4*

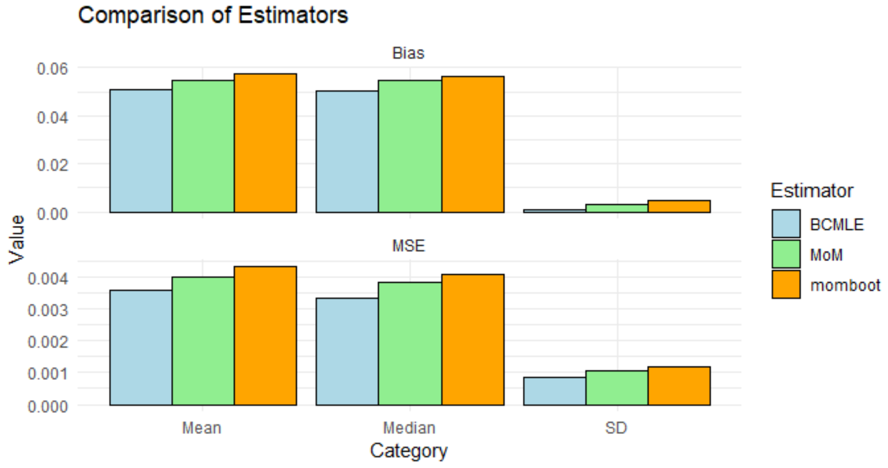


*Figure 5*

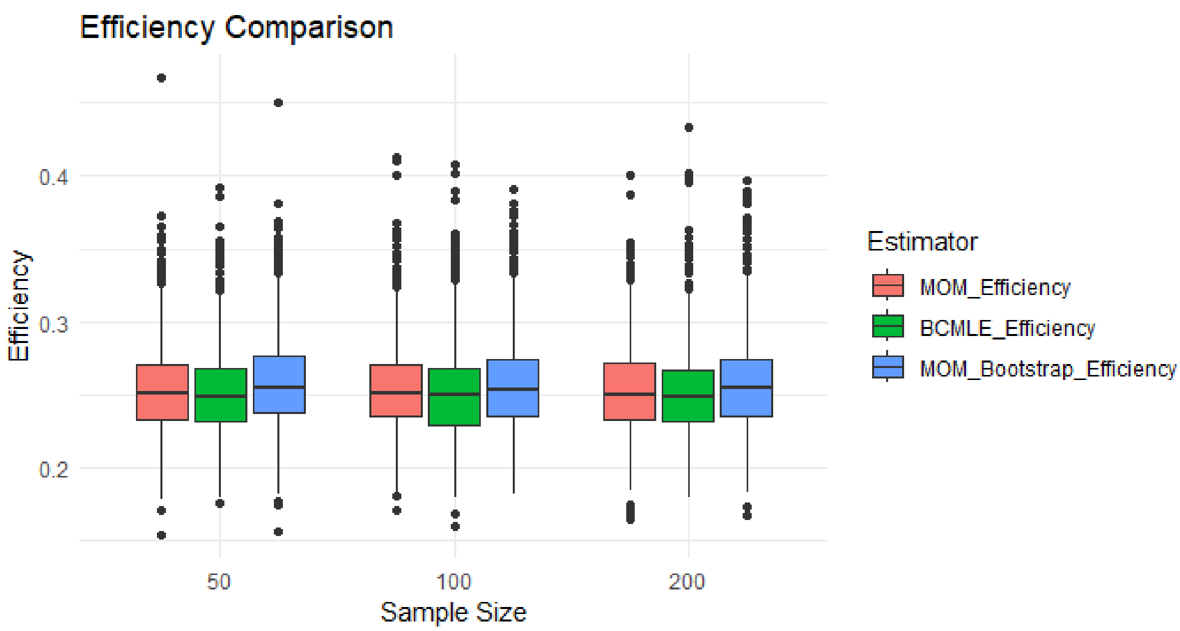


*Figure 6*

**Q2.**

I am comparing the given confidence interval (Variation) with the 90% Wald confidence interval (Wald). The Wald confidence interval and the confidence interval provided differ in their underlying assumptions and construction. The Wald Confidence Interval assumes the asymptotic normality of the estimator, assuming that for large sample sizes, that theta hat is normally distributed, which for our data is true as shown in the previous question.

I am using a true value of theta of 0.2, a sample size of 100, and 1000 simulation iterations for sufficient variability in the data. The simulation is a loop sampling random numbers where the sample mean is calculated. The confidence intervals are computed using both Wald and Variation methods, and the results are stored in a matrix. Several data frames are created to optimize data structures for the visualisation of data. In plotting the data for comparing the I have generated a histogram of confidence intervals for lower and upper bounds *Figure 7, 8*. Also a line plot showing the coverage probability of confidence intervals *Figure 9.* Also, a histogram of confidence intervals widths *Figure 10.* And lastly a line plot of the consistency of confidence intervals widths *Figure 11*.

The distribution of confidence intervals upper and lower bounds can provide insights into the variability and shape of the intervals. For both the upper and lower bounds they display a normal distribution of results *Figures 7, 8*.

The coverage probability of confidence intervals assesses how successfully the confidence intervals cover the true parameter value. The true value should fall within the interval 90% of the time for the 90% confidence interval. Only the Wald confidence interval comes close to a 90% at about 88% coverage probability over 1000 iterations of the simulation. This is a far better result than the ~55% coverage probability of the approximate confidence interval provided *Figure 9*.

The width of the confidence intervals of each confidence intervals summarises estimation precision. By comparing the average width of the confidence intervals for both methods. A narrower interval suggests more precise estimation. The approximate method provided consistency narrower intervals suggesting more precise estimation with a mean around 0.5-0.6, while the Wald method produces larger intervals with a mean in a range of 0.13-0.14 shown in *Figure 10*.

The consistency across simulations of confidence intervals is important for reliable estimation. Both methods display a consistent confidence interval width across the 1000 iterations of the simulation *Figure 11*.

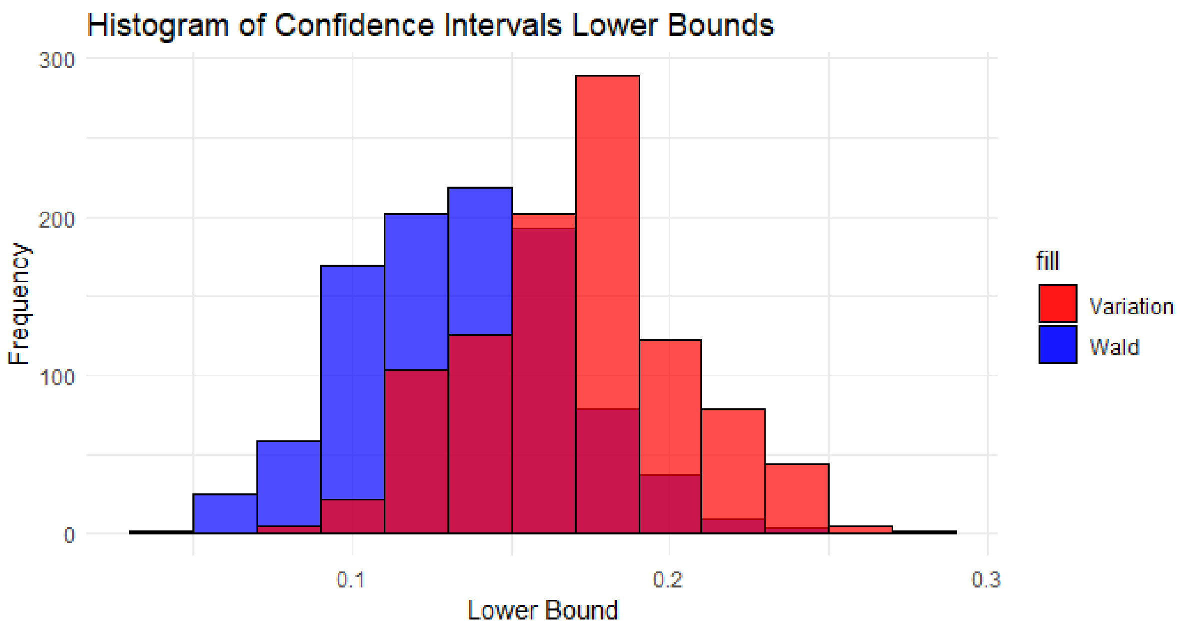


*Figure 7*

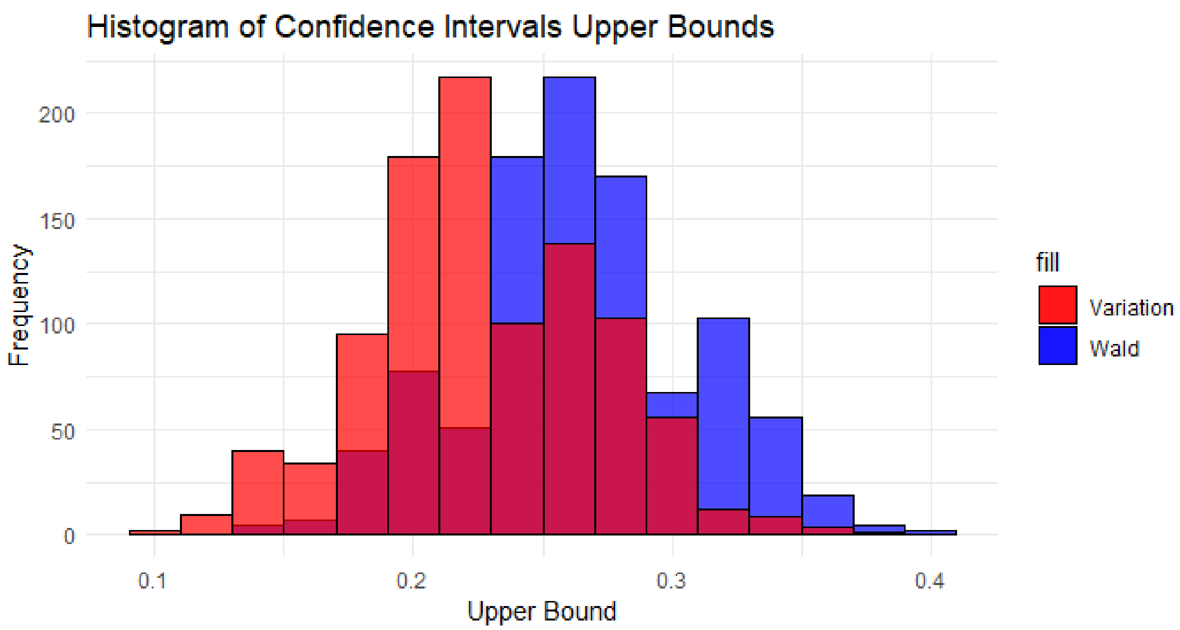


*Figure 8*

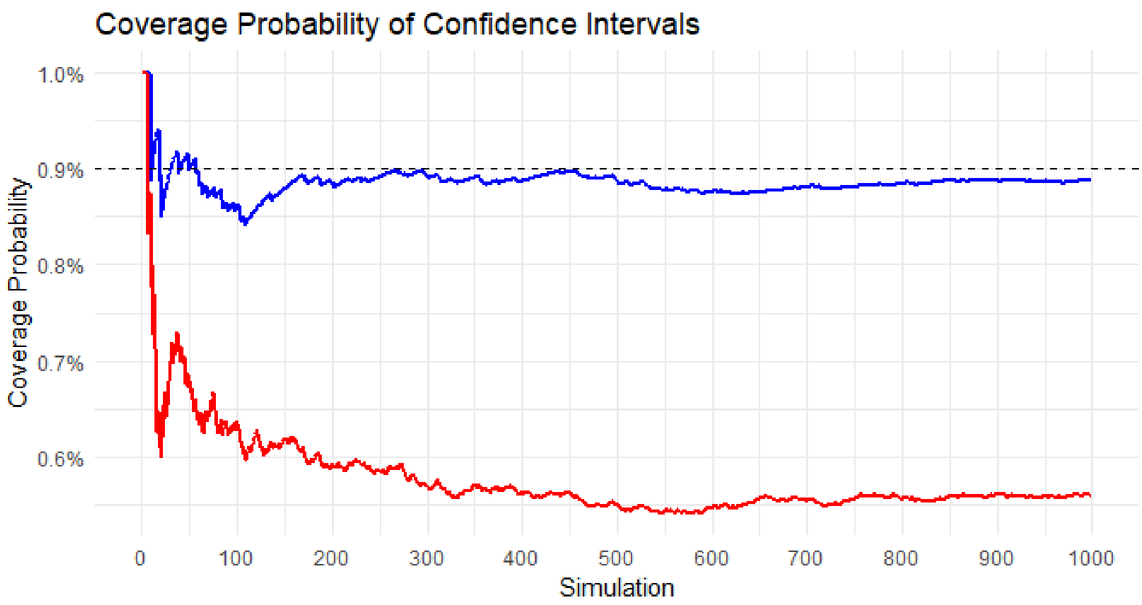


*Figure 9*

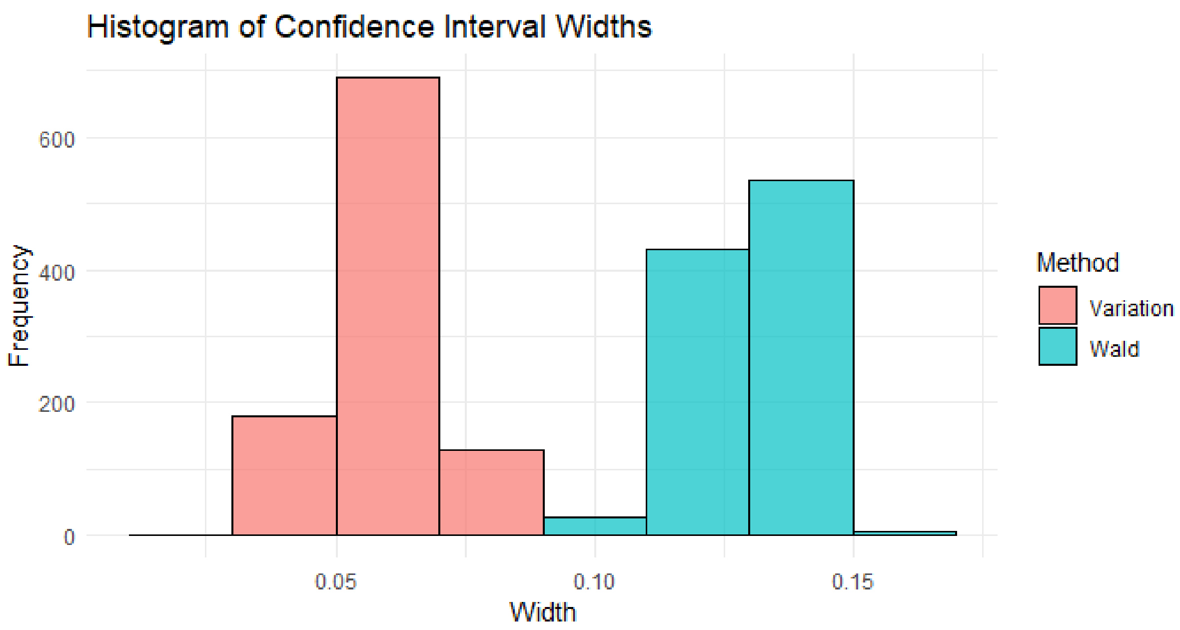


*Figure 10*

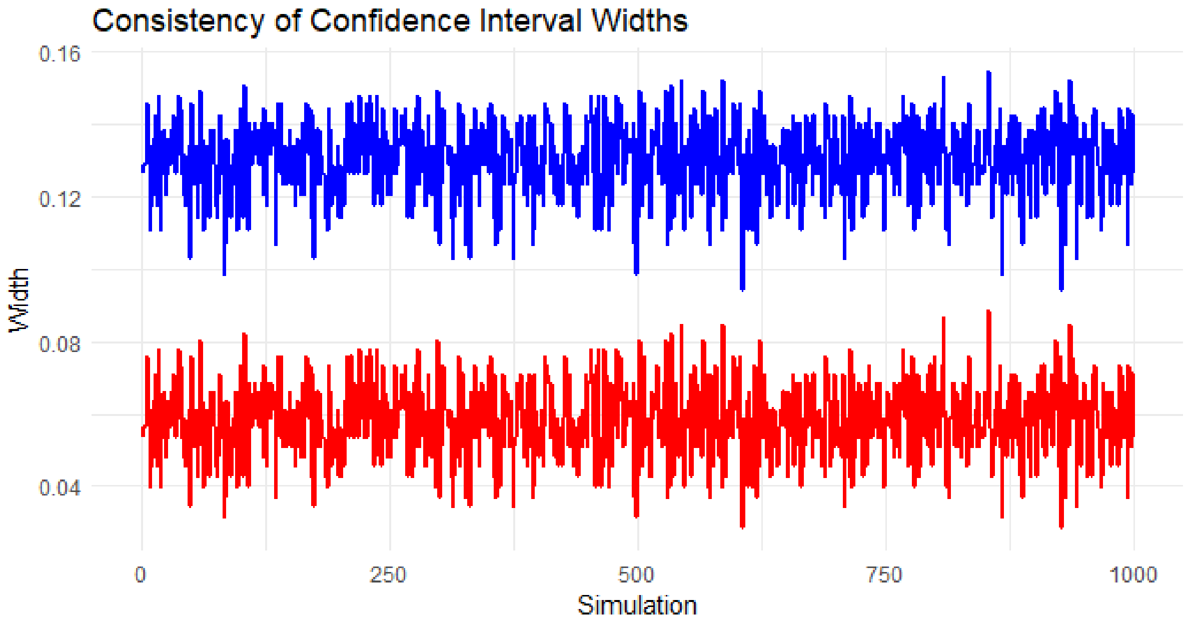
Consistency of Confidence Interval Widths
0.16
0.12
0.08
0.04
Width
0
250
500
750
1000
Simulation

*Figure 11*