Convexity Conditions and Existence Theorems in Nonlinear Elasticity

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0. Introduction

The purpose of this article is to present existence theorems for various equilibrium boundary-value problems of nonlinear elasticity in one, two and three dimensions under realistic hypotheses on the material response. Although some of the results may be extended to cover Cauchy elasticity, we shall restrict our discussion to hyperelastic (Green elastic) materials, that is, to elastic materials possessing a stored-energy function. We ignore thermal effects. For such materials a typical boundary-value problem takes the form of finding a vector field $\boldsymbol{u}_0\colon \Omega \to \mathscr{R}^n$ making the integral

$$I(\boldsymbol{u}, \Omega) = \int_{\Omega} f(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\, d\boldsymbol{x} \tag{0.1}$$

stationary in a suitable class of functions[†]. Here Ω is a non-empty, bounded, open subset of $\mathscr{R}^n$, $n = 1, 2, 3$. The integrand f will usually have the form

$$f(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = \mathscr{W}(\boldsymbol{x}, \nabla \boldsymbol{u}) + \phi(\boldsymbol{x}, \boldsymbol{u}), \tag{0.2}$$

[†] For traction boundary-value problems there will also be a surface integral term.

where $\mathscr{W}$ is the stored-energy function and ϕ is a body force potential. In this introduction we assume that (0.2) holds and that $n=3$.

We attack this problem by using the direct method of the calculus of variations to establish the existence of minimizers for $I(\boldsymbol{u}, \Omega)$ in the class considered. Such a programme has been successfully carried out by ANTMAN [1–8] in an important series of papers on the existence of equilibrium solutions in various problems arising from theories of nonlinear elastic rods and axisymmetric shells (with or without Cosserat structure). In these papers ANTMAN emphasizes the crucial importance of choosing hypotheses on the material response that both ensure the success of the analysis and are reasonable physically. In his work, and in mine, the problem of existence is inextricably linked with that of finding satisfactory constitutive inequalities for nonlinear elasticity (*cf.* TRUESDELL [1]).

As an illustration, consider the effect of imposing the constitutive requirement that $\mathscr{W}$ be convex with respect to $\nabla \boldsymbol{u}$. This mathematically simple hypothesis, when augmented with suitable smoothness and growth assumptions, ensures the existence of minimizers for (0.1), (0.2). Existence theorems under this assumption have been given by several authors (*e.g.*, BEJU [1, 2], ODEN [1]). Unfortunately these results are only of mathematical interest since convexity of $\mathscr{W}$ with respect to $\nabla \boldsymbol{u}$ is unacceptable physically†. Firstly, as was shown by COLEMAN & NOLL [1] (see also TRUESDELL & NOLL [1, p. 163]) such convexity conflicts with the requirement that $\mathscr{W}$ be objective (*cf.* (1.12)). Secondly, consider, for example, a mixed displacement, dead load traction boundary-value problem for such a material. Any equilibrium solution for such a problem must necessarily be an absolute minimizer for $I(\boldsymbol{u}, \Omega)$; in particular, if a strict absolute minimizer exists then it is the only equilibrium solution#. This fact, which is an elementary consequence of the theory of convex functions (*cf.* MOREAU [1], EKELAND & TÉMAM [1]), and for the truth of which $\mathscr{W}$ need not be strictly convex, rules out the nonuniqueness essential for the description of buckling##. Some less restrictive condition on $\mathscr{W}$ is therefore required.

A suitable condition, termed *quasiconvexity*, was introduced by MORREY [1] in a fundamental paper in 1952. $\mathscr{W}$ is said to be *quasiconvex* if

$$\int_D \mathscr{W}(\boldsymbol{x}_0, \boldsymbol{F}_0 + \nabla \boldsymbol{\zeta}(\boldsymbol{x}))\, d\boldsymbol{x} \geqq \int_D \mathscr{W}(\boldsymbol{x}_0, \boldsymbol{F}_0)\, d\boldsymbol{x} = m(D)\, \mathscr{W}(\boldsymbol{x}_0, \boldsymbol{F}_0) \tag{0.3}$$

holds for each fixed $\boldsymbol{x}_0 \in \Omega$, for each constant 3×3 matrix $\boldsymbol{F}_0$, for each bounded open subset $D \subseteq \mathscr{R}^3$, and for all $\boldsymbol{\zeta} \in \mathscr{D}(D)$. Here m denotes Lebesgue measure and $\mathscr{D}(D)$ is the set of all infinitely differentiable functions with compact support contained in D. We regard quasiconvexity as a constitutive restriction on $\mathscr{W}$★. It may be interpreted as follows: For any homogeneous body made from the material found at any point of Ω, and for any displacement boundary-value problem with zero body force for such a body that admits as a possible displacement a

† For hyperelastic rods and shells ANTMAN makes certain convexity hypotheses on the stored-energy functions. Because of the coordinates employed these hypotheses are not subject to the objections made below.

HILL [1] was the first to observe that *strict* convexity with respect to $\nabla \boldsymbol{u}$ implies uniqueness.

The nonuniqueness established by ANTMAN arises from the presence of lower order terms.

★ In the case when $\mathscr{W}$ is not everywhere defined the condition must be slightly modified. See section 3.

homogeneous strain, we require that this homogeneous strain be an absolute minimizer for the total energy. Note that if in the above we admitted for consideration inhomogeneous bodies, or if we considered mixed displacement traction boundary-value problems, then the condition would be unacceptable, as we should expect certain buckled states to have lower total energy than the homogeneous strain. As stated, however, the condition has a certain plausibility.

MORREY showed that if $f(\cdot, \boldsymbol{u}, \cdot)$ is quasiconvex for every $\boldsymbol{u}$, and if certain continuity and growth hypotheses are satisfied, then for various boundary-value problems there exist minimizers for $I(\boldsymbol{u}, \Omega)$. Conversely, if $\boldsymbol{u}$ is a minimizer for $I(\boldsymbol{u}, \Omega)$ among $C^1(\bar{\Omega})$ functions satisfying given Dirichlet boundary conditions, and if $\boldsymbol{x}_0 \in \Omega$, $\boldsymbol{F}_0 = \nabla \boldsymbol{u}(\boldsymbol{x}_0)$, then (0.3) holds. This fact may be used (see Theorem 3.2) to motivate quasiconvexity by showing that it is a necessary condition for the existence of sufficiently regular minimizers for a class of displacement boundary-value problems. The degree of regularity required is, however, fairly severe. Furthermore, if $\mathscr{W}$ is quasiconvex and twice continuously differentiable, then $\mathscr{W}$ satisfies the *Legendre-Hadamard* or *ellipticity* condition†:

$$\frac{\partial^2 \mathscr{W}}{\partial u^i_{,\alpha}\, \partial u^j_{,\beta}} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geqq 0 \quad \text{for all } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{R}^3. \tag{0.4}$$

(It is not known whether the converse holds.) Because we have chosen to impose quasiconvexity as a constitutive restriction, we must therefore regard the Legendre-Hadamard condition also as a constitutive restriction††.

The statement above that quasiconvexity is sufficient for existence must now be qualified. In fact, MORREY's remarkable existence theorem fails to apply directly to nonlinear elasticity. For compressible materials his growth conditions are too stringent; in particular, they prohibit any singular behaviour of $\mathscr{W}$, such as the natural condition

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{F}) \to \infty \quad \text{as} \quad \det \boldsymbol{F} \to 0. \tag{0.5}$$

Moreover, his work gives no indication of how to treat the unilateral constraint $\det \nabla \boldsymbol{u} > 0$ #. Incompressible materials require the constraint $\det \nabla \boldsymbol{u} = 1$, which also poses problems.

† Throughout this article we employ the summation convention for repeated suffices.

†† This contrasts with the views expressed by TRUESDELL & NOLL [1, p. 275], who suggested that the Legendre-Hadamard condition should be regarded not as a constitutive restriction, but as a stability condition. They conjectured that violation of the Legendre-Hadamard condition at a point would lead to wave motion tending to move an elastic body from an unstable to a stable equilibrium configuration and that this process may help explain buckling. While the violation of the Legendre-Hadamard condition at a point may well result in certain kinds of instabilities (*cf.* ERICKSEN [3]), it is by no means necessary for buckling. Indeed, in Section 9 we show that buckling can occur when the Legendre-Hadamard condition holds everywhere. Moreover, other kinds of instabilities, such as necking, may well be compatible with the Legendre-Hadamard condition (*cf.* ANTMAN [6]). Another suggestion of TRUESDELL & NOLL [1, p. 129] concerning internal buckling of a rod would, if true, directly contradict quasiconvexity, but the behaviour described by them as implausible seems typical of buckling.

ANTMAN's material hypotheses for rods and shells are those appropriate under the assumption that (0.4) holds in the three-dimensional theory.

The analogous problem for rods has been studied by ANTMAN [2–5, 7, 8].

To overcome these difficulties we investigate in Section 6 sequential weak continuity properties of functions, defined on Orlicz-Sobolev spaces, having the form

$$\theta:\ \boldsymbol{u} \mapsto \phi(\nabla \boldsymbol{u}(\cdot)), \tag{0.6}$$

where ϕ is a continuous real-valued function defined on the set of all 3×3 matrices. This map is sequentially continuous from $W^{1,\infty}(\Omega)$ with the weak $*$ topology to $L^1(\Omega)$ with the weak topology if and only if θ has the form

$$\theta(\boldsymbol{u}) = A + B_i^\alpha (\nabla \boldsymbol{u})_\alpha^i + C_i^\alpha (\operatorname{adj} \nabla \boldsymbol{u})_\alpha^i + D \det \nabla \boldsymbol{u}, \tag{0.7}$$

where $A, B_i^\alpha, C_i^\alpha, D$ are constants and $\operatorname{adj} \nabla \boldsymbol{u}$ is the transpose of the matrix of cofactors of $\nabla \boldsymbol{u}$. When the domain of θ is a larger Orlicz-Sobolev space, the problem is more delicate. In this case we give various theorems guaranteeing sequential continuity or closure of θ relative to various weak topologies.

We combine these results with standard techniques of the calculus of variations to establish the existence of minimizers for $I(\boldsymbol{u}, \Omega)$ in various classes of functions when $\mathscr{W}$ has the form

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{F}) = g(\boldsymbol{x}, \boldsymbol{F}, \operatorname{adj} \boldsymbol{F}, \det \boldsymbol{F}) \tag{0.8}$$

with $g(\boldsymbol{x}, \cdot, \cdot, \cdot)$ convex for each $\boldsymbol{x}$. We call such functions $\mathscr{W}$ *polyconvex*. Note that $\boldsymbol{F}$, $\operatorname{adj} \boldsymbol{F}$ and $\det \boldsymbol{F}$ govern the deformations of line, surface and volume elements respectively. If $\mathscr{W}$ is polyconvex, then $\mathscr{W}$ is quasiconvex; in fact polyconvexity is equivalent to a sufficient condition for quasiconvexity given by MORREY. However our existence theorems are valid under weaker growth conditions than MORREY's. Moreover, we can handle the pointwise constraints on $\det \nabla \boldsymbol{u}$ mentioned above by using our sequential weak continuity results. Since there are few known examples of quasiconvex functions that are not polyconvex, the restriction to polyconvex functions is not serious. It appears that neither the quasiconvexity nor the polyconvexity condition has been considered previously in the context of elasticity.

A wide variety of realistic models of nonlinear elastic materials satisfy the hypotheses of our existence theorems. In particular, these include the Mooney-Rivlin material and certain stored-energy functions similar to, and for incompressible materials identical to, those of OGDEN [2, 3]. That these stored-energy functions are polyconvex follows from sufficient conditions for the polyconvexity of isotropic functions given in Section 5, where some related results are also discussed.

Our existence theorems apply to displacement, mixed displacement traction, pure traction, mixed displacement pressure, and pure pressure boundary-value problems. Similar methods work for more general classes of mixed boundary conditions. For the most part we consider only polynomial growth hypotheses, which result in a theory based on Sobolev spaces. For stored-energy functions of slower growth an Orlicz-Sobolev space setting is required; for brevity we treat only the displacement boundary-value problem for such functions. An example is given of a stored-energy function requiring this more elaborate theory.

Many of the results may be extended to give new existence theorems for non-linear elliptic systems in higher dimensions; some results in this direction are given in BALL [2].

In Section 9 I apply existence theorems to establish nonuniqueness for the mixed displacement, zero traction boundary-value problem for a Mooney-Rivlin rod under compression. My main result is that nonuniqueness occurs for sufficiently long compressed rods of arbitrary uniform cross-section. Despite the intuitively obvious nature of this result, such nonuniqueness has not previously been established for any mixed boundary-value problem of nonlinear elasticity.

In Sections 3 and 4 the conditions of quasiconvexity, polyconvexity and ellipticity are examined in detail. In particular, necessary and sufficient conditions are given for polyconvexity based upon the work of BUSEMANN, EWALD & SHEPHARD. It would be interesting to have a statical interpretation of polyconvexity. None of these three constitutive restrictions has at present a microscopic or thermodynamic motivation, in contrast, for example, to the situation pertaining to the Navier-Stokes equations, for which the Clausius-Duhem inequality gives conditions closely related to those sufficient for existence.

The reader interested in the constitutive theory of elasticity, but not in the details of existence theorems, can omit Sections 2, 6 and 7 without much loss. With the exception of the existence theorems themselves, the parts of this article which bear most directly on the relevance of the quasiconvexity and polyconvexity conditions to realistic models of elastic materials are Theorems 3.1, 3.2, 3.4, 4.5, 5.1, 5.2 and the discussion in Section 8.

Existence theorems for linear elasticity (*cf.* FICHERA [1]) have a different character from those presented here on account of the geometrical approximation made. In particular we have no need of an analogue of KORN's inequality. The existence theorems of STOPPELLI [1] and VAN BUREN [1] for nonlinear elasticity (*cf.* TRUESDELL & NOLL [1], WANG & TRUESDELL [1]) are based on those of the linear theory. In these theorems the inverse function theorem is used to establish the existence and uniqueness of small solutions to boundary-value problems with small body forces and boundary data. The material response is assumed to be such that existence, uniqueness and regularity theorems hold for the equilibrium equations linearized about the zero data solution. In the case of the pure traction boundary-value problem the degeneracy of the linearized problem forces the authors to assume that the surface and body forces possess no axis of equilibrium. This hypothesis is unnecessary under the assumptions of our existence theorem for the pure traction problem. Note, however, that STOPPELLI and VAN BUREN make assumptions about the material response only for strains close to those of the zero data solution.

This article by no means exhausts the problem of the existence of equilibrium solutions even for hypereleasticity. The most notable shortcoming of this work is that in the two important cases of incompressible materials and compressible materials satisfying (0.5) I have so far been unable to show that the minimizers whose existence has been established are smooth enough even to satisfy a weak form of the equilibrium equations. I hope to discuss this question in later work.

1. Boundary-Value Problems of Nonlinear Hyperelasticity

This section begins with a brief presentation of the basic equations of nonlinear elasticity. We then go on to consider the various boundary-value problems for which we later prove existence theorems. The calculations here are purely formal, in the sense that we assume that the various quantities appearing have sufficient smoothness to justify any operations required (such as integration by parts). The theory of nonlinear elasticity is discussed at length in the books by GREEN & ZERNA [1], TRUESDELL & NOLL [1] and WANG & TRUESDELL [1], and the reader is referred to these texts when clarification is necessary.

We consider a material body $\mathscr{B}$ whose particles are labelled by their positions $\boldsymbol{x}=(x_\alpha)=(x_1,x_2,x_3)$ with respect to a rectangular Cartesian co-ordinate system in a *reference configuration* Ω which is a bounded open subset of $\mathscr{R}^3$. Ω need not be homeomorphic to an open ball. In a given motion the position of the particle $\boldsymbol{x}$ at time t is denoted $\boldsymbol{u}(\boldsymbol{x},t)$†. The *deformation gradient* $\boldsymbol{F}$ is defined by

$$\boldsymbol{F}=\nabla\boldsymbol{u}; \qquad F^i_\alpha=\frac{\partial u^i}{\partial x^\alpha}=u^i{}_{,\alpha}. \tag{1.1}$$

We suppose that $\boldsymbol{u}:\Omega\to\mathscr{R}^3$ is orientation-preserving and locally invertible, so that

$$J=\det\boldsymbol{F}>0. \tag{1.2}$$

Consideration of the stronger requirement that $\boldsymbol{u}$ be globally one-to-one is beyond the scope of this article.

The symmetric, positive-definite *right and left stretch tensors* $\boldsymbol{U},\boldsymbol{V}$ and the *right and left Cauchy-Green tensors* $\boldsymbol{C},\boldsymbol{B}$ are defined by

$$\boldsymbol{C}=\boldsymbol{U}^2=\boldsymbol{F}^T\boldsymbol{F}, \qquad \boldsymbol{B}=\boldsymbol{V}^2=\boldsymbol{F}\boldsymbol{F}^T. \tag{1.3}$$

The following relations hold:

$$\boldsymbol{F}=\boldsymbol{R}\boldsymbol{U}=\boldsymbol{V}\boldsymbol{R}, \qquad \boldsymbol{V}=\boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T, \tag{1.4}$$

where $\boldsymbol{R}$ is the orthogonal *rotation tensor*. The eigenvalues v_1,v_2,v_3 of $\boldsymbol{U}$ and $\boldsymbol{V}$ are positive and are termed the *principal stretches* of the deformation. The *principal invariants* of $\boldsymbol{B}$ and $\boldsymbol{C}$ are given by

$$\begin{aligned} \mathrm{I}_{\boldsymbol{B}}&=\mathrm{I}_{\boldsymbol{C}}=v_1^2+v_2^2+v_3^2,\\ \mathrm{II}_{\boldsymbol{B}}&=\mathrm{II}_{\boldsymbol{C}}=v_2^2v_3^2+v_3^2v_1^2+v_1^2v_2^2,\\ \mathrm{III}_{\boldsymbol{B}}&=\mathrm{III}_{\boldsymbol{C}}=v_1^2v_2^2v_3^2. \end{aligned} \tag{1.5}$$

We suppose that at each particle $\boldsymbol{x}$ the material of the body is *elastic*, so that a constitutive equation of the form

$$\boldsymbol{T}_R(\boldsymbol{x},\mathrm{t})=\hat{\boldsymbol{T}}_R(\boldsymbol{F}(\boldsymbol{x},t),\boldsymbol{x}) \tag{1.6}$$

holds, where $\boldsymbol{T}_R$ denotes the *first Piola-Kirchhoff stress tensor*. $\boldsymbol{T}_R$ is related to the *Cauchy stress tensor* $\boldsymbol{T}$ by

$$\boldsymbol{T}_R=J\boldsymbol{T}(\boldsymbol{F}^{-1})^T. \tag{1.7}$$

† We choose the notation common in partial differential equations rather than that of continuum mechanics where our $\boldsymbol{x}$, $\boldsymbol{u}$ are customarily denoted $\boldsymbol{X}$, $\boldsymbol{x}$ respectively.

The surface traction $\boldsymbol{t}_R$ measured per unit area in the reference configuration, and the actual stress vector $\boldsymbol{t}$ measured per unit area of the deformed configuration, are given by

$$\boldsymbol{t}_R = \boldsymbol{T}_R \boldsymbol{N}; \qquad \boldsymbol{t} = \boldsymbol{T}\boldsymbol{n} \tag{1.8}$$

respectively, where $\boldsymbol{N}$ and $\boldsymbol{n}$ denote the unit outward normals to the boundaries $\partial\Omega$ and $\partial\boldsymbol{u}(\Omega, t)$ respectively.

The pointwise form of the balance laws of linear and angular momentum are given by

$$\operatorname{Div} \boldsymbol{T}_R + \rho_R \boldsymbol{b} = \rho_R \ddot{\boldsymbol{u}}, \tag{1.9}$$

$$\boldsymbol{T} = \boldsymbol{T}^T, \tag{1.10}$$

where

$$(\operatorname{Div} \boldsymbol{T}_R)_i \stackrel{\text{def}}{=} \frac{\partial T^{\alpha}_{Ri}(\boldsymbol{x})}{\partial x^{\alpha}},$$

$\rho_R(\boldsymbol{x})$ is the density in the reference configuration, and $\boldsymbol{b}$ is the body force per unit mass.

Throughout this article we assume that the material is *hyperelastic*; *i.e.*, there exists a real-valued *stored-energy function* $\mathscr{W}(\boldsymbol{x}, \boldsymbol{F})$ such that

$$T_{R_i}^{\alpha} = \frac{\partial \mathscr{W}}{\partial F^i_{\alpha}}. \tag{1.11}$$

$\mathscr{W}$ is *objective* if and only if

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{Q}\boldsymbol{F}) = \mathscr{W}(\boldsymbol{x}, \boldsymbol{F}) \tag{1.12}$$

for all proper orthogonal matrices $\boldsymbol{Q}$. If $\mathscr{W}$ is objective then

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{F}) = \mathscr{W}(\boldsymbol{x}, \boldsymbol{U}), \tag{1.13}$$

and it follows from (1.11) that (1.10) is satisfied identically. $\mathscr{W}$ is *isotropic* if and only if $\mathscr{W}$ is objective and

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{Q}\boldsymbol{F}\boldsymbol{Q}^T) = \mathscr{W}(\boldsymbol{x}, \boldsymbol{F}) \tag{1.14}$$

for all orthogonal matrices $\boldsymbol{Q}$. In this case

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{F}) = \Phi(\boldsymbol{x}, v_1, v_2, v_3), \tag{1.15}$$

where Φ is symmetric in the v_i.

Deformations of *incompressible* materials are restricted by the pointwise constraint

$$J = \det \boldsymbol{F} = 1. \tag{1.16}$$

For incompressible materials the above theory has to be modified by replacing $\boldsymbol{T}$ by the *extra stress*

$$\boldsymbol{T}_E = \boldsymbol{T} + p\mathbf{1}, \tag{1.17}$$

where p is an indeterminate hydrostatic pressure. The stored-energy function $\mathscr{W}$ for an incompressible material need be defined only for $\boldsymbol{F}$ satisfying (1.16).

We shall be concerned only with *equilibrium* configurations of $\mathscr{B}$. If $\mathscr{W}$ is objective we see from (1.9) that $\boldsymbol{u}$ is an equilibrium configuration if and only if

$$A_{ij}^{\alpha\beta} u^j_{,\alpha\beta} + q_i + \rho_R b_i = 0, \tag{1.18}$$

where

$$A_{ij}^{\alpha\beta}(\boldsymbol{x}, \boldsymbol{F}) \stackrel{\text{def}}{=} \frac{\partial^2 \mathscr{W}(\boldsymbol{x}, \boldsymbol{F})}{\partial F^i_\alpha \, \partial F^j_\beta}, \qquad q_i = \frac{\partial^2 \mathscr{W}(\boldsymbol{x}, \boldsymbol{F})}{\partial x^\alpha \, \partial F^i_\alpha}. \tag{1.19}$$

A. The mixed displacement-traction boundary-value problem for a compressible material

In this problem we seek $\boldsymbol{u}$ satisfying (1.18) in Ω and satisfying the boundary conditions

$$\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial\Omega_1, \tag{1.20}$$

$$\boldsymbol{t}_R(\boldsymbol{x}) = \bar{\boldsymbol{t}}_R(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial\Omega_2, \tag{1.21}$$

where $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$, $\partial\Omega_1 \cap \partial\Omega_2 = \phi$, and $\bar{\boldsymbol{u}}\colon \partial\Omega_1 \to \mathscr{R}^3$, $\bar{\boldsymbol{t}}_R\colon \partial\Omega_2 \to \mathscr{R}^3$ are given functions. The boundary condition (1.21) is a condition of *dead loading, i.e.*, the loads acting on $\boldsymbol{u}(\partial\Omega_2)$ have fixed direction and fixed magnitude per unit area of $\partial\Omega_2$. If $\partial\Omega_2 = \phi$ then we have a pure *displacement boundary-value problem*, while if $\partial\Omega_1 = \phi$ we have a *traction boundary-value problem.*

Suppose that the body force $\boldsymbol{b}$ is *conservative*, so that

$$\boldsymbol{b} = -\operatorname{grad} \Psi, \tag{1.22}$$

where $\Psi = \Psi(\boldsymbol{u})$ is a real-valued potential, and where

$$(\operatorname{grad} \Psi)_i \stackrel{\text{def}}{=} \frac{\partial \Psi}{\partial u^i}.$$

Define $f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})$ by

$$f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) = \mathscr{W}(\boldsymbol{x}, \boldsymbol{F}) + \rho_R(\boldsymbol{x})\, \Psi(\boldsymbol{u}). \tag{1.23}$$

Consider the functional

$$J_0(\boldsymbol{u}) \stackrel{\text{def}}{=} \int_\Omega f_1(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x}))\, d\boldsymbol{x} - \int_{\partial\Omega_2} \boldsymbol{u}(\boldsymbol{x}) \cdot \bar{\boldsymbol{t}}_R(\boldsymbol{x})\, dS. \tag{1.24}$$

Let $\partial\Omega_1 \neq \phi$. Then a standard formal calculation shows that $J_0(\boldsymbol{u}_0)$ is stationary with respect to $\boldsymbol{u}$ satisfying (1.20) if and only if the Euler-Lagrange equations (1.18) and the natural boundary conditions (1.21) hold, *i.e.*, if and only if $\boldsymbol{u}_0$ is a solution to the mixed boundary-value problem.

If $\partial\Omega_1 = \phi$ then in general solutions to the boundary value problem will not exist, since a necessary condition for a function $\boldsymbol{u}_0$ to render J_0 stationary subject to (1.21) is that

$$\boldsymbol{a} = \boldsymbol{0} \tag{1.25}$$

where

$$\boldsymbol{a} \stackrel{\text{def}}{=} \frac{1}{m(\Omega)} \Big(\int_\Omega \rho_R \boldsymbol{b}(\boldsymbol{u}_0)\, d\boldsymbol{x} + \int_{\partial\Omega} \bar{\boldsymbol{t}}_R\, dS \Big). \tag{1.26}$$

Condition (1.25) says that the total force on the body due to external loads is zero (*cf.* TRUESDELL & NOLL [1, p. 127]). To describe the effect of this condition we consider two situations corresponding to different types of existence theorems proved in Section 7.

1. $\boldsymbol{b}(\boldsymbol{u})$ is not a constant vector. In this case, under suitable hypotheses on $\boldsymbol{b}$ the set of functions $\boldsymbol{u}_0$ satisfying (1.25) will be nonempty, so that under certain conditions† it is likely that a function $\boldsymbol{u}_0$ such as to render J_0 stationary subject to (1.21) exists. If so, then $\boldsymbol{u}_0$ is a solution to the traction boundary-value problem.

2. $\boldsymbol{b}(\boldsymbol{u})=\boldsymbol{b}_0$, $\boldsymbol{b}_0$ constant. In this case $\boldsymbol{a}$ is independent of $\boldsymbol{u}_0$, so that (1.25) is a condition on the data of the problem. It proves convenient to consider $J_0(\boldsymbol{u})$ as a functional defined on functions $\boldsymbol{u}$ satisfying the constraint

$$\int_{\Omega} \boldsymbol{u}\, dx=\boldsymbol{e}, \tag{1.27}$$

where $\boldsymbol{e}$ is an arbitrary constant vector. The constraint (1.27) removes the indeterminacy resulting from a possible rigid-body translation of $\boldsymbol{u}(\Omega)$. J_0 is stationary at $\boldsymbol{u}=\boldsymbol{u}_0$ subject to (1.27) if and only if (1.21) holds and

$$\operatorname{Div} \boldsymbol{T}_R+\rho_R\, \boldsymbol{b}_0=\boldsymbol{a}. \tag{1.28}$$

To prove this note first that if (1.21) and (1.28) hold, then

$$\delta J_0(\boldsymbol{u}_0)(\boldsymbol{v}) \overset{\text{def}}{=} \frac{d}{d\varepsilon} J_0(\boldsymbol{u}_0+\varepsilon \boldsymbol{v})\Big|_{\varepsilon=0}=-\int_{\Omega} \boldsymbol{a}\cdot \boldsymbol{v}\, dx+\int_{\partial\Omega}(\boldsymbol{t}_R-\bar{\boldsymbol{t}}_R)\, dS, \tag{1.29}$$

which is zero if $\int_{\Omega} \boldsymbol{v}\, dx=\boldsymbol{0}$. The converse statement is a direct application of the multiplier rule for isoperimetric problems with $\boldsymbol{a}$ playing the rôle of the Lagrange multiplier corresponding to the constraint (1.27). A rigorous proof may also be constructed by using a result of SCHWARTZ [1, p. 59].

If $\boldsymbol{a}=\boldsymbol{0}$ then $\boldsymbol{u}_0$ is an equilibrium solution. If $\boldsymbol{a}\neq\boldsymbol{0}$ then

$$\boldsymbol{u}(\boldsymbol{x}, t) \overset{\text{def}}{=} \boldsymbol{u}_0(\boldsymbol{x})+\frac{t^2}{2}\boldsymbol{a} \tag{1.30}$$

is a solution to the dynamic traction boundary-value problem (1.9), (1.21).

Note that any equilibrium solution $\boldsymbol{u}$ must also satisfy the zero moment condition

$$\int_{\Omega} \boldsymbol{u}\wedge \rho_R\, \boldsymbol{b}_0\, dx+\int_{\partial\Omega} \boldsymbol{u}\wedge \bar{\boldsymbol{t}}_R\, dS=\boldsymbol{0}. \tag{1.31}$$

This condition, unlike (1.25), depends explicitly on the unknown function $\boldsymbol{u}$, and so cannot be imposed *a priori*.

B. The mixed displacement pressure boundary-value problem for a compressible material

In this problem we seek $\boldsymbol{u}$ satisfying (1.18) in Ω and satisfying the boundary conditions

$$\boldsymbol{u}(\boldsymbol{x})=\bar{\boldsymbol{u}}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x}\in\partial\Omega_1, \tag{1.32}$$

$$(\boldsymbol{T}\boldsymbol{n})(\boldsymbol{x})=-p_r\, \boldsymbol{n} \qquad \text{for } \boldsymbol{x}\in\partial\Omega_r \quad (r=2, \dots, M), \tag{1.33}$$

† if Ψ takes the form of a powerful potential well, for example.

where $\partial\Omega = \bigcup_{r=1}^{M} \overline{\partial\Omega_r}$, $\partial\Omega_k \cap \partial\Omega_l = \phi$ $(k \neq l)$, $\bar{\boldsymbol{u}}$: $\partial\Omega_1 \to \mathscr{R}^3$ is a given function, and $p_r (r=2, \ldots, M)$ are constant pressures. We assume that for $r \geqq 2$ $\overline{\partial\Omega_r}$ is either a closed surface or is bounded by a closed curve lying in $\overline{\partial\Omega_1}$. Suppose also that there exists a $C^1(\bar{\Omega})$ function p: $\bar{\Omega} \to \mathscr{R}$ taking the value p_r on $\partial\Omega_r$ for each $r=2, \ldots, M$.

Consider the functional

$$J_1(\boldsymbol{u}) = \int_{\Omega} f_2(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x}))\, d\boldsymbol{x}, \tag{1.34}$$

where

$$f_2(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \overset{\text{def}}{=} f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) + p\,J + \tfrac{1}{6} p_{,r} \varepsilon_{ijk} \varepsilon^{pqr} F^i_p F^j_q u^k, \tag{1.35}$$

and where J is defined in (1.16). By the divergence theorem

$$J_1(\boldsymbol{u}) = \int_{\Omega} f_1\, d\boldsymbol{x} + \int_{\partial\Omega} \tfrac{1}{6} p\, \varepsilon_{ijk} \varepsilon^{pqr} u^i_{\ ,p} u^j_{\ ,q} u^k N_r\, dS. \tag{1.36}$$

Suppose now that $\partial\Omega_1 \neq \phi$. Then for $\boldsymbol{v}$ satisfying $\boldsymbol{v}=\boldsymbol{0}$ on $\partial\Omega_1$ we obtain

$$\begin{aligned} \frac{d}{d\varepsilon} J_1(\boldsymbol{u}+\varepsilon\boldsymbol{v})\Big|_{\varepsilon=0} &= -\int_{\Omega} (\operatorname{Div} T_R + \rho_R \boldsymbol{b}) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{t}_R \cdot \boldsymbol{v}\, dS \\ &\quad + \int_{\partial\Omega} \tfrac{1}{2} p\, \varepsilon_{ijk} \varepsilon^{pqr} u^i_{\ ,p} u^j_{\ ,q} v^k N_r\, dS \\ &\quad - \int_{\partial\Omega} \tfrac{1}{3} \varepsilon^{rqp} (p\, \varepsilon_{ijk} u^i_{\ ,p} u^k v^j)_{,q} N_r\, dS \\ &\quad - \int_{\partial\Omega} \tfrac{1}{3} (\varepsilon^{pqr} p_{,q} N_r)(\varepsilon_{ijk} u^i_{\ ,p} u^k v^j)\, dS. \end{aligned} \tag{1.37}$$

The fourth integral in (1.37) is zero by Kelvin's theorem applied to each $\partial\Omega_r$, and the last integral is zero since $\nabla p \wedge \boldsymbol{N} = \boldsymbol{0}$ on $\partial\Omega_r$. Thus

$$\frac{d}{d\varepsilon} J_1(\boldsymbol{u}+\varepsilon\boldsymbol{v})\Big|_{\varepsilon=0} = -\int_{\Omega} (\operatorname{Div} \boldsymbol{T}_R + \rho_R \boldsymbol{b}) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\partial\boldsymbol{u}(\Omega)} (\boldsymbol{t} + p\,\boldsymbol{n}) \cdot \boldsymbol{v}\, dS. \tag{1.38}$$

Thus $J_1(\boldsymbol{u}_0)$ is stationary if and only if $\boldsymbol{u}_0$ is a solution to the mixed displacement pressure boundary-value problem. The calculation above is a slightly simplified version of that of SEWELL [1]; see also BEATTY [1].

If $\partial\Omega_1 = \phi$ then we proceed in a fashion similar to A.

The functionals that we study in later sections include functionals of the form J_0 and J_1. For the purposes of the existence theorems it is not necessary to assume that $\mathscr{W}$ is objective. If, however, this assumption is not made, the resulting minimizers, if smooth, will not necessarily satisfy (1.10).

For incompressible materials the admissible functions are restricted by the additional constraint (1.16).

Finally in this section we discuss briefly the analogous problems of one and two-dimensional hyperelasticity. These problems arise from special deformations of three-dimensional elastic bodies. In two-dimensional *plane strain* we consider deformations having the form

$$\boldsymbol{u} = (u_1(x_1, x_2), u_2(x_1, x_2), \lambda x_3) \qquad \lambda > 0 \text{ constant}. \tag{1.39}$$

For such a deformation

$$F = \begin{pmatrix} u^1_{,1} & u^1_{,2} & 0 \\ u^2_{,1} & u^2_{,2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{1.40}$$

and the incompressibility condition is

$$\lambda(u^1_{,1}\, u^2_{,2} - u^1_{,2}\, u^2_{,1}) = 1. \tag{1.41}$$

In one-dimensional *uniaxial strain* we let

$$u = (u_1(x_1), \mu x_2, \lambda x_3), \tag{1.42}$$

where μ and λ are positive constants. The corresponding deformation gradient is

$$F = \begin{pmatrix} u^1_{,1} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{1.43}$$

We leave to the reader the routine task of formulating the relevant boundary-value problems in these two cases.

2. Technical Preliminaries

For ease of reference we list here various well known spaces used in this article. As general references we cite FRIEDMAN [1], KRASNOSEL'SKII & RUTICKII [1], SCHWARTZ [1] and for functional analytic aspects DUNFORD & SCHWARTZ [1].

Throughout this work Ω denotes a nonempty, bounded, open subset of $\mathscr{R}^m$ with Lebesgue measure dx, where, except in Section 3, m takes the value 1, 2 or 3. $C^\infty(\Omega)$ denotes the space of infinitely-differentiable real-valued functions defined on Ω. $\mathscr{D}(\Omega)$ consists of those elements of $C^\infty(\Omega)$ with compact support contained in Ω. We give $\mathscr{D}(\Omega)$ the strict inductive limit topology of SCHWARTZ. The dual space of $\mathscr{D}(\Omega)$ is denoted $\mathscr{D}'(\Omega)$, and its elements are called *distributions*. Any locally integrable function f defines a distribution T_f through the equation $T_f(\phi) = \int_\Omega f\phi\, dx$, $\phi \in \mathscr{D}(\Omega)$. A sequence $T_r \to T$ in $\mathscr{D}'(\Omega)$ if and only if $T_r(\phi) \to T(\phi)$ for all $\phi \in \mathscr{D}(\Omega)$. (Here and throughout this work we consider only convergence of sequences, rather than nets.) If $T \in \mathscr{D}'(\Omega)$, then we may define $\dfrac{\partial T}{\partial x^\alpha}$ by

$$\frac{\partial T}{\partial x^\alpha}(\phi) = -T\left(\frac{\partial \phi}{\partial x^\alpha}\right).$$

Then $T \mapsto \dfrac{\partial T}{\partial x^\alpha}$ maps $\mathscr{D}'(\Omega)$ into itself and is continuous.

$C(\bar{\Omega})$, $C^1(\bar{\Omega})$ denote respectively the spaces of continuous and continuously differentiable real-valued functions defined on $\bar{\Omega}$ with the usual supremum norms. $\mathscr{R}_+$ denotes the non-negative real numbers, $\bar{\mathscr{R}}$ the extended real line with the usual topology. If E is a subset of $\mathscr{R}^s$ then $\operatorname{Co} E$ denotes the convex hull of E.

The spaces $L^p(\Omega)$, $1 \leqq p \leqq \infty$, of (equivalence classes of) integrable real-valued functions are defined in the standard way. The Sobolev space $W^{1,p}(\Omega)$, $1 \leqq p \leqq \infty$, consists of those functions u belonging to $L^p(\Omega)$ with weak derivatives $\frac{\partial u}{\partial x^\alpha}$ $(1 \leqq \alpha \leqq m)$ belonging to $L^p(\Omega)$. $W^{1,p}(\Omega)$ is a Banach space under the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{\alpha=1}^{m} \left\| \frac{\partial u}{\partial x^\alpha} \right\|_{L^p(\Omega)}. \tag{2.1}$$

The closure of $\mathscr{D}(\Omega)$ in $W^{1,p}(\Omega)$ is denoted $W_0^{1,p}(\Omega)$. $W_{\mathrm{loc}}^{1,1}(\Omega)$ denotes the space of functions u which together with their weak derivatives $\frac{\partial u}{\partial x^\alpha}$ $(1 \leqq \alpha \leqq m)$ are locally integrable. When dealing with the spaces in this paragraph we assume that $p < \infty$ unless otherwise stated.

Weak and weak $*$ convergence of sequences are denoted by $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, respectively. In the case of the Banach space $W^{1,\infty}(\Omega)$ we *define* the weak $*$ topology to be that induced by the natural imbedding of $W^{1,\infty}(\Omega)$ in the product space $(L^\infty(\Omega))^{1+m}$, where each factor has the weak $*$ topology. Thus a sequence $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$ if and only if $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ and $\frac{\partial u_n}{\partial x^\alpha} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial x^\alpha}$ in $L^\infty(\Omega)$ $(1 \leqq \alpha \leqq m)$.

If A is a real-valued, continuous, even, convex function of $t \in \mathscr{R}$ satisfying $A(t) > 0$ for $t > 0$, $\frac{A(t)}{t} \to 0$ as $t \to 0$, $\frac{A(t)}{t} \to \infty$ as $t \to \infty$ then we call A an N-function. If A is an N-function, its conjugate function $\bar{A}$ is defined by $\bar{A}(t) = \sup\{ts - A(s) \colon s \in \mathscr{R}\}$. $\bar{A}$ is also an N-function and satisfies $\bar{\bar{A}} = A$. Furthermore Young's inequality,

$$ts \leqq A(t) + \bar{A}(s), \tag{2.2}$$

holds for all $s, t \in \mathscr{R}$. If A, B are N-functions then we write $A \prec B$ if and only if there exist positive numbers t_0 and k such that

$$A(t) \leqq B(kt) \tag{2.3}$$

for all $t \geqq t_0$. We write $A \sim B$ if and only if $A \prec B$ and $B \prec A$, and $A \ll B$ if and only if

$$\lim_{t \to \infty} \frac{A(t)}{B(\lambda t)} = 0 \tag{2.4}$$

for every $\lambda > 0$. If A is an N-function the Orlicz class $\mathscr{L}_A(\Omega)$ consists of all (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_\Omega A(u(x))\,dx < \infty. \tag{2.5}$$

The Orlicz space $L_A(\Omega)$ is the linear hull of $\mathscr{L}_A(\Omega)$. $L_A(\Omega)$ is a Banach space with respect to the Luxemburg norm

$$\|u\|_{(A)} = \inf\left\{k > 0;\ \int_\Omega A(u/k)\,dx \leqq 1\right\}.$$

If $A \prec B$ then $L_B(\Omega) \subseteq L_A(\Omega)$, while if $A \ll B$ then $L_B(\Omega) \subsetneqq L_A(\Omega)$. The space $E_A(\Omega)$ is defined as the closure of the bounded functions in the $L_A(\Omega)$ norm. We have that $E_A(\Omega) \subseteq \mathscr{L}_A(\Omega) \subseteq L_A(\Omega)$. The dual of $E_A(\Omega)$ can be identified by means of the scalar product $\int_\Omega uv\,dx$ with $L_{\bar{A}}(\Omega)$. The norm on $L_{\bar{A}}(\Omega)$ dual to $\|\ \|_{(A)}$ on $E_A(\Omega)$ is denoted $\| \ \|_{\bar{A}}$ and is equivalent to $\| \ \|_{(A)}$. Hölder's inequality is valid in the form

$$\int_\Omega uv\,dx \leqq \|u\|_{(A)} \|v\|_{\bar{A}} \tag{2.6}$$

for all $u \in L_A(\Omega)$, $v \in L_{\bar{A}}(\Omega)$.

The Orlicz-Sobolev space $W^1 L_A(\Omega)$ $(W^1 E_A(\Omega))$ is defined as the set of functions $u \in L_A(\Omega)$ $(E_A(\Omega))$ such that the weak derivatives $\frac{\partial u}{\partial x^\alpha} \in L_A(\Omega)$ $(E_A(\Omega))$ $(1 \leqq \alpha \leqq m)$. $W^1 L_A(\Omega)$ is a Banach space under the norm

$$\|u\|_{W^1 L_A(\Omega)} = \|u\|_{(A)} + \sum_{\alpha=1}^{m} \left\| \frac{\partial u}{\partial x^\alpha} \right\|_{(A)} \tag{2.7}$$

and similarly for $W^1 E_A(\Omega)$. The closure of $\mathscr{D}(\Omega)$ in $W^1 L_A(\Omega)$ is written $W_0^1 L_A(\Omega)$.

In the special case when $A(t) \sim |t|^p$ $(p > 1)$ we have the equalities

$$L_A(\Omega) = E_A(\Omega) = L^p(\Omega), \qquad W^1 L_A(\Omega) = W^1 E_A(\Omega) = W^{1,p}(\Omega),$$

and $\bar{A}(t) \sim |t|^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Throughout this article we shall be dealing with vector and matrix functions $\boldsymbol{\Psi}$. When we write $\boldsymbol{\Psi} \in X$, where X is any one of the spaces introduced above, we mean that each component of $\boldsymbol{\Psi}$ belongs to X, and we define $\|\boldsymbol{\Psi}\|_X$ to be $\sum_i \|\Psi_i\|_X$ where the sum is over all components of Ψ_i of $\boldsymbol{\Psi}$.

Finally we define some regularity conditions for Ω.

(i) Ω has the *segment property* if there exists a locally finite open covering $\{\theta_i\}$ of $\partial\Omega$ and corresponding vectors $\{y_i\}$ such that $\boldsymbol{x} + t\boldsymbol{y}_i \in \Omega$ for all $\boldsymbol{x} \in \bar{\Omega} \cap \theta_i$ and for all $t \in (0, 1)$.

(ii) Ω satisfies the *cone condition* if there exists a *fixed* cone $k_\Omega \subseteq \mathscr{R}^m$ such that each point $\boldsymbol{x} \in \partial\Omega$ is the vertex of a cone $k_\Omega(\boldsymbol{x})$ that lies in Ω and is congruent to k_Ω.

(iii) Ω satisfies a *strong Lipschitz condition* if each $\boldsymbol{x} \in \partial\Omega$ has a neighbourhood $\mathscr{U}_{\boldsymbol{x}}$ such that in some co-ordinate system, with origin at $\boldsymbol{x}$, $\Omega \cap \mathscr{U}_{\boldsymbol{x}}$ is represented in $\mathscr{U}_{\boldsymbol{x}}$ by $\zeta_m < F(\zeta')$, $\zeta' = (\zeta_1, \ldots, \zeta_{m-1})$ with F a Lipschitz continuous function.

3. Quasiconvexity and the Legendre-Hadamard Condition

We consider integrals of the form

$$I(\boldsymbol{u}, \Omega) = \int_\Omega f(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\,d\boldsymbol{x},$$

where $(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \in \Omega \times \mathscr{R}^n$, $(\nabla \boldsymbol{u}(\boldsymbol{x}))_\alpha^i = \frac{\partial u^i(\boldsymbol{x})}{\partial x^\alpha} = u^i_{,\alpha}(\boldsymbol{x})$ $(1 \leqq i \leqq n, 1 \leqq \alpha \leqq m)$, and where the real-valued function f is defined and continuous on a given relatively open subset S of $\Omega \times \mathscr{R}^n \times M^{n \times m}$. Here $M^{n \times m}$ denotes the space of real $n \times m$ matrices, with the induced norm of $\mathscr{R}^{mn}$. We assume that for each $\boldsymbol{x} \in \Omega$ there exist $\boldsymbol{u} \in \mathscr{R}^n$, $\boldsymbol{F} \in M^{n \times m}$ with $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \in S$.

Definition 3.1 (*cf.* MORREY [1].) Let U be an open subset of $M^{n\times m}$. Let $g\colon U\to\mathscr{R}$ be continuous. Then g is said to be *quasiconvex*† at $\boldsymbol{F}_0\in U$ if and only if

$$\int_D g(\boldsymbol{F}_0+\nabla\zeta(y))\,dy \geqq g(\boldsymbol{F}_0)\,m(D) \tag{3.1}$$

for every bounded open subset $D\subseteq\mathscr{R}^m$ and for every $\zeta\in\mathscr{D}(D)$ which satisfies $\boldsymbol{F}_0+\nabla\zeta(y)\in U$ for all $y\in D$. g is *quasiconvex on* U if it is quasiconvex at each $\boldsymbol{F}_0\in U$.

Note that if g is quasiconvex at $\boldsymbol{F}_0\in U$, and if $\zeta\in W_0^{1,\infty}(D)$ satisfies $\boldsymbol{F}_0+\nabla\zeta(y)\in K$ for almost all $y\in D$ and for some compact subset K of U, then (3.1) holds for ζ. In fact by the definition of $W_0^{1,\infty}(D)$ there exists a sequence of functions $\zeta_r\in\mathscr{D}(D)$ that converges to ζ in $W^{1,\infty}(D)$. Since K is compact there exists an integer N and a compact set K_1 with $K\subset K_1\subset U$ such that $\boldsymbol{F}_0+\nabla\zeta_r(y)\in K_1$ for all $y\in D$ and all $r\geqq N$. If $r\geqq N$ then (3.1) holds for ζ_r. By the compactness of K_1, the continuity of g and the bounded convergence theorem we obtain (3.1) for ζ.

Let $A=\{\boldsymbol{w}\in W_{\mathrm{loc}}^{1,1}(\Omega)\colon (\boldsymbol{x},\boldsymbol{w}(\boldsymbol{x}),\nabla\boldsymbol{w}(\boldsymbol{x}))\in S$ for almost all $\boldsymbol{x}\in\Omega$, and $I(\boldsymbol{w},\Omega)$ exists and is finite$\}$.

Theorem 3.1. *Let $\boldsymbol{u}\in A$ be such that*

$$I(\boldsymbol{u},\Omega)\leqq I(\boldsymbol{w},\Omega)$$

for all $\boldsymbol{w}\in A$ with $\boldsymbol{w}-\boldsymbol{u}\in\mathscr{D}(\Omega)$ and $\|\boldsymbol{w}-\boldsymbol{u}\|_{C(\bar\Omega)}$ sufficiently small. Let $\boldsymbol{x}_0\in\Omega$ and suppose that $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$ have representatives, again denoted by $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$, that are continuous at $\boldsymbol{x}_0$ with $(\boldsymbol{x}_0,\boldsymbol{u}(\boldsymbol{x}_0),\nabla\boldsymbol{u}(\boldsymbol{x}_0))\in S$. Let $U=\{\boldsymbol{F}\colon(\boldsymbol{x}_0,\boldsymbol{u}(\boldsymbol{x}_0),\boldsymbol{F})\in S\}$. Then $f(\boldsymbol{x}_0,\boldsymbol{u}(\boldsymbol{x}_0),\cdot)$ is quasiconvex at $\nabla\boldsymbol{u}(x_0)\in U$.

Proof. Let $\boldsymbol{u}$ satisfy the hypotheses of the theorem, let D be a bounded open subset of $\mathscr{R}^m$ and let $\zeta\in\mathscr{D}(D)$ satisfy $(\boldsymbol{x}_0,\boldsymbol{u}(\boldsymbol{x}_0),\nabla\boldsymbol{u}(\boldsymbol{x}_0)+\nabla\zeta(y))\in S$ for all $y\in D$. For $\varepsilon>0$ define $\boldsymbol{u}_\varepsilon\colon\Omega\to\mathscr{R}^m$ by

$$\begin{aligned}\boldsymbol{u}_\varepsilon(\boldsymbol{x})&=\boldsymbol{u}(\boldsymbol{x})+\varepsilon\zeta\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\varepsilon}\right)\quad\text{if }\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\varepsilon}\in D\\&=\boldsymbol{u}(\boldsymbol{x})\quad\text{otherwise.}\end{aligned}$$

For ε small enough the set $\boldsymbol{x}_0+\varepsilon D$, on which $\boldsymbol{u}$ and $\boldsymbol{u}_\varepsilon$ differ, is contained in Ω, and thus $\boldsymbol{u}_\varepsilon-\boldsymbol{u}\in\mathscr{D}(\Omega)$. Also for $\boldsymbol{x}\in\boldsymbol{x}_0+\varepsilon D$ we have

$$f(\boldsymbol{x},\boldsymbol{u}_\varepsilon(\boldsymbol{x}),\nabla\boldsymbol{u}_\varepsilon(\boldsymbol{x}))=f\left(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x})+\varepsilon\zeta\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\varepsilon}\right),\nabla\boldsymbol{u}(\boldsymbol{x})+\nabla\zeta\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\varepsilon}\right)\right),$$

which by our continuity assumptions and our assumptions about ζ is bounded uniformly above on the set $\boldsymbol{x}_0+\varepsilon D$ for ε small enough. Thus $\boldsymbol{u}_\varepsilon\in A$ and $I(\boldsymbol{u},\Omega)\leqq I(\boldsymbol{u}_\varepsilon,\Omega)$. Making the change of variables $y=\dfrac{\boldsymbol{x}-\boldsymbol{x}_0}{\varepsilon}$ we obtain

$$\begin{aligned}&\int_D f(\boldsymbol{x}_0+\varepsilon y,\boldsymbol{u}(\boldsymbol{x}_0+\varepsilon y)+\varepsilon\zeta(y),\nabla\boldsymbol{u}(\boldsymbol{x}_0+\varepsilon y)+\nabla\zeta(y))\,\varepsilon^m\,dy\\&\qquad\geqq\int_D f(\boldsymbol{x}_0+\varepsilon y,\boldsymbol{u}(\boldsymbol{x}_0+\varepsilon y),\nabla\boldsymbol{u}(\boldsymbol{x}_0+\varepsilon y))\,\varepsilon^m\,dy.\end{aligned}$$

† This is MORREY's original terminology. In his book [2] he calls such functions (for $U=M^{n\times m}$) strongly quasiconvex, retaining the term quasiconvex for functions satisfying the weakened form of the Legendre-Hadamard condition that we shall term rank 1 convexity. The reader is warned that quasiconvexity has other meanings in the literature on convex analysis.

Dividing by ε^m and letting $\varepsilon \to 0$ we obtain (3.1) for $F_0 = \nabla u(x_0)$ by the bounded convergence theorem. □

Corollary 3.1.1. (*Cf.* MORREY [1, p. 43] and MEYERS [1, p. 128].) *Let $U \subseteq M^{n\times m}$ be open, let $g: U \to \mathscr{R}$ be continuous, let $F_0 \in U$ and let* (3.1) *hold for a given bounded open subset $D_0 \subseteq \mathscr{R}^m$ and for every $\zeta \in \mathscr{D}(D_0)$ which satisfies $F_0 + \nabla\zeta(y) \in U$ for all $y \in D_0$. Then* (3.1) *holds for all such D, ζ, i.e., g is quasiconvex at F_0.*

Proof. Apply the theorem to the integrand $g(F)$ with $\Omega = D_0$ and $u^i(x) = (F_0)^i_\alpha x^\alpha$. □

Theorem 3.1 is essentially the same as a result stated by SILVERMAN [1], following earlier work of BUSEMANN & SHEPHARD [1, p. 31].

We next show that for integrands that are independent of x and u the existence of a sufficiently regular minimizer to certain Dirichlet problems implies that the quasiconvexity condition holds.

Theorem 3.2. *Let $U \subseteq M^{n\times m}$ be open and let $g: U \to \mathscr{R}$ be continuous. Suppose that either* (i) *$n=1$, Ω is arbitrary, or* (ii) *Ω is a hypercube, $\Omega = \{x \in \mathscr{R}^m: 0 < x^\alpha < 1, 1 \leqq \alpha \leqq m\}$, say. Let $u_0: \Omega \to \mathscr{R}^n$ be defined by*

$$u_0^i(x) = F^i_\alpha x^\alpha + z^i,$$

where $F \in U$ and $z \in \mathscr{R}^n$ are constants. Let

$$J(u) = \int_\Omega g(\nabla u(x))\, dx$$

and let $A_1 = \{u \in C^1(\bar{\Omega})$: $\nabla u(x) \in U$ for all $x \in \bar{\Omega}$, $J(u)$ exists and is finite, and $u = u_0$ on $\partial\Omega\}$. Suppose there exists $v \in A_1$ such that

$$J(v) \leqq J(u) \quad \text{for all } u \in A_1.$$

Then g is quasiconvex at $F \in U$.

Proof. Let $w = v - u_0$. Then $w \in C^1(\bar{\Omega})$ and $w = 0$ on $\partial\Omega$. In Case (i) there exists $x_0 \in \Omega$ with $\nabla w(x_0) = 0$. Therefore $\nabla v(x_0) = F$ and the result follows from Theorem 3.1. In Case (ii) we have that $\nabla w(x_r) \to 0$ as $r \to \infty$ for any sequence $\{x_r\} \subseteq \Omega$ with $x_r \to 0$ as $r \to \infty$. By Theorem 3.1, g is quasiconvex at $F + \nabla w(x_r) \in U$. Taking the limit $r \to \infty$ in the quasiconvexity condition gives the result. □

Remarks. For $n=1$ or $m=1$ quasiconvexity is equivalent to convexity (see MORREY [1, 2]). The analogue of Theorem 3.2 for $n>1$, Ω arbitrary, is false. As an example, let $m=n=2$, $\Omega = \{(x_1, x_2): x_1^2 + x_2^2 < 1\}$. Define $g: M^{2\times 2} \to \mathscr{R}$ by $g(F) = \rho(r)$, where $r = |F| = \operatorname{tr}(FF^T)^{\frac{1}{2}}$ and where $\rho: \mathscr{R}_+ \to \mathscr{R}$ is zero for $r \geqq 1$ and positive for $0 \leqq r < 1$. We show that for any $u_0 \in C^1(\Omega)$ there exists an absolute minimizer for $J(u) = \int_\Omega g(\nabla u(x))\, dx$ among $C^1(\bar{\Omega})$ functions u satisfying $u = u_0$ on $\partial\Omega$. Let $\zeta \in C^1(\bar{\Omega})$ satisfy $\zeta = 0$ on $\partial\Omega$ and let $|\nabla\zeta(x)| \geqq \varepsilon > 0$ for all $x \in \Omega$; *e.g.*, we may take $\zeta(x) = (h_1(x), h_1(x) + h_2(x))$, where h_1 and h_2 are $C^1(\bar{\Omega})$ functions satisfying $h_1 = h_2 = 0$ on $\partial\Omega$, $\nabla h_i(x) = 0$ if and only if $x = a_i$ $(i = 1, 2)$ with $a_1 \neq a_2$. For large enough $k > 0$ and for all $x \in \bar{\Omega}$ we have

$$|\nabla(u_0 + k\zeta)(x)| \geqq k\varepsilon - \|\nabla u_0\|_{C(\bar{\Omega})} > 1,$$

so that $J(\boldsymbol{u}_0+k\zeta)=0$ and $\boldsymbol{u}_0+k\zeta$ is an absolute minimizer. But g is not quasiconvex, as may be seen by putting $\boldsymbol{u}_0=\boldsymbol{F}\boldsymbol{x}$ in the above argument, where $\boldsymbol{F}\in M^{2\times 2}$ is constant with $|\boldsymbol{F}|<1$. With a little more work one can show that the absolute minimizer $\boldsymbol{u}$ corresponding to $\boldsymbol{u}_0$ may be chosen so that $\det \nabla\boldsymbol{u}(\boldsymbol{x})\geqq c>0$ for all $\boldsymbol{x}\in\Omega$.

Definition 3.2. Let U be an open subset of $M^{n\times m}$. A function $g\colon U\to\mathscr{R}$ is *rank 1 convex on* U if it is convex on all closed line segments in U with end points differing by a matrix of rank 1, *i.e.*,

$$g(\boldsymbol{F}+(1-\lambda)\,\boldsymbol{a}\otimes\boldsymbol{b})\leqq\lambda g(\boldsymbol{F})+(1-\lambda)\,g(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b})$$

for all $\boldsymbol{F}\in U$, $\lambda\in[0,1]$, $\boldsymbol{a}\in\mathscr{R}^m$, $\boldsymbol{b}\in\mathscr{R}^n$, with $\boldsymbol{F}+\mu\,\boldsymbol{a}\otimes\boldsymbol{b}\in U$ for all $\mu\in[0,1]$. Here $(\boldsymbol{a}\otimes\boldsymbol{b})^i_\alpha\overset{\text{def}}{=}a^i\,b_\alpha$ $(1\leqq i\leqq n,\ 1\leqq\alpha\leqq m)$.

Theorem 3.3. *Let U be an open subset of $M^{n\times m}$ and let $g\colon U\to\mathscr{R}$. The following conditions* (i)–(iv) *are equivalent:*

(i) *g is rank 1 convex on U;*

(ii) *for each fixed $\boldsymbol{F}\in M^{n\times m}$, $\boldsymbol{b}\in\mathscr{R}^m$ the function $\boldsymbol{a}\mapsto g(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b})$ is convex on all closed line segments in the set $\{\boldsymbol{a}\colon \boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b}\in U\}$;*

(iii) *for each fixed $\boldsymbol{F}\in M^{n\times m}$, $\boldsymbol{a}\in\mathscr{R}^n$ the function $\boldsymbol{b}\mapsto g(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b})$ is convex on all closed line segments in the set $\{\boldsymbol{b}\colon \boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b}\in U\}$;*

(iv) *the inequality*

$$g(\boldsymbol{H})\leqq\lambda g(\boldsymbol{H}+\boldsymbol{c}\otimes\boldsymbol{d})+(1-\lambda)\,g\left(\boldsymbol{H}-\frac{\lambda}{1-\lambda}\,\boldsymbol{c}\otimes\boldsymbol{d}\right)\tag{3.2}$$

holds for all $\lambda\in[0,1)$ and for all $\boldsymbol{H}\in M^{n\times m}$, $\boldsymbol{c}\in\mathscr{R}^n$, $\boldsymbol{d}\in\mathscr{R}^m$ satisfying $\boldsymbol{H}+\mu\,\boldsymbol{c}\otimes\boldsymbol{d}\in U$ for all $\mu\in\left[\dfrac{\lambda}{\lambda-1},1\right]$.

If $g\in C(U)$, then (i)–(iv) *are equivalent to*

(v) *for each $\boldsymbol{F}\in U$ there exists $A(\boldsymbol{F})\in M^{m\times n}$ such that*

$$g(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b})\geqq g(\boldsymbol{F})+A^\alpha_i(\boldsymbol{F})\,a^i\,b_\alpha\tag{3.3}$$

whenever $\boldsymbol{F}+\lambda\,\boldsymbol{a}\otimes\boldsymbol{b}\in U$ for all $\lambda\in[0,1]$.

If $g\in C^1(U)$, then $A^\alpha_i(\boldsymbol{F})=\dfrac{\partial g(\boldsymbol{F})}{\partial F^i_\alpha}$.

If $g\in C^2(U)$, then (i)–(v) *are equivalent to*

(vi) (*Legendre-Hadamard condition*)

$$\frac{\partial^2 g(\boldsymbol{F})}{\partial F^i_\alpha\,\partial F^j_\beta}\,a^i\,a^j\,b_\alpha\,b_\beta\geqq 0\qquad\textit{for all }\ \boldsymbol{a}\in\mathscr{R}^n,\ \boldsymbol{b}\in\mathscr{R}^m,\ \boldsymbol{F}\in U.$$

Proof. The equivalence of (i), (ii) and (iii) is clear (*cf.* SILVERMAN [1, Thm. 4]). The equivalence of (i) and (iv) is proved by making the change of variables $\boldsymbol{c}=(\lambda-1)\,\boldsymbol{a}$, $\boldsymbol{d}=\boldsymbol{b}$, $\boldsymbol{H}=\boldsymbol{F}+(1-\lambda)\,\boldsymbol{a}\otimes\boldsymbol{b}$. Let $g\in C(U)$. That (v) implies (ii) follows by a well known condition for convexity. To show that (ii) implies (v) one can use the arguments of MORREY [1, p. 47] to establish (3.3) for $\boldsymbol{a}\otimes\boldsymbol{b}$ belonging to some neighbourhood of zero in $M^{n\times m}$, and then deduce (v) from the convexity of the function $\boldsymbol{a}\mapsto g(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b})$. The remaining assertions of the theorem are obvious. $\square$

Condition (iv) was derived by GRAVES [1] and has recently been studied by ERICKSEN [2]. Motivated by (iv) we say that g *is rank 1 convex at* $\boldsymbol{H} \in U$ if the inequality (3.2) holds whenever the right-hand side is defined. In this case it is easy to see that the Legendre-Hadamard condition holds at $\boldsymbol{H}$. The prototype for the following theorem was discovered by HADAMARD [1, 2], the first rigorous proof being that of GRAVES [1]. For other proofs and relevant literature see DUHEM [1], MCSHANE [1], CATTANEO [1], VAN HOVE [1, 2], TRUESDELL & NOLL [1, p. 253] and MORREY [2, p. 10]. The proof here is based on MORREY [1, p. 45] and on Theorem 3.1.

Theorem 3.4. *Let* $\boldsymbol{u}$, $\boldsymbol{x}_0$ *satisfy the hypotheses of Theorem* 3.1. *Let*

$$U_0 = \{\boldsymbol{F} : (\boldsymbol{x}_0, \boldsymbol{u}(\boldsymbol{x}_0), \boldsymbol{F}) \in S\}.$$

Then $f(\boldsymbol{x}_0, \boldsymbol{u}(\boldsymbol{x}_0), \cdot)$ *is rank 1 convex at* $\nabla \boldsymbol{u}(\boldsymbol{x}_0) \in U_0$.

Proof. Define $g: U_0 \to \mathscr{R}$ by $g(\boldsymbol{F}) = f(\boldsymbol{x}_0, \boldsymbol{u}(\boldsymbol{x}_0), \boldsymbol{F})$. Let $H = \nabla \boldsymbol{u}(\boldsymbol{x}_0)$. Let $\lambda \in [0, 1)$, let $\boldsymbol{c}$, $\boldsymbol{d}$ be such that $\boldsymbol{H} + \boldsymbol{c} \otimes \boldsymbol{d}$ and $\boldsymbol{H} - \dfrac{\lambda}{1-\lambda} \boldsymbol{c} \otimes \boldsymbol{d}$ belong to U_0 and assume without loss of generality that $\boldsymbol{d} \neq 0$, $\lambda \neq 0$. Let $\rho > 0$, $\boldsymbol{\mu}_1 = \boldsymbol{d}/|\boldsymbol{d}|$, $h = 1/|\boldsymbol{d}|$ and $k = \dfrac{1-\lambda}{\lambda} h$. Choose vectors $\boldsymbol{\mu}_\beta$ $(1 < \beta \leqq m)$ such that $(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m)$ is an orthonormal set in $\mathscr{R}^m$. Let D denote the rectangular parallelepiped

$$-k \leqq y_1 \leqq h, \qquad |y_\beta| \leqq \rho \qquad (1 < \beta \leqq m),$$

where $y_\beta = \boldsymbol{x} \cdot \boldsymbol{\mu}_\beta$. Let F_1^-, F_1^+, F_β^-, F_β^+ $(1 < \beta \leqq m)$ be the faces $y_1 = -k$, $y_1 = h$, $y_\beta = -\rho$, $y_\beta = 0$, respectively. Let $\pi_\beta^\pm$ $(1 \leqq \beta \leqq m)$ be the pyramid with base $F_\beta^\pm$ and with vertex at the origin. Let $\boldsymbol{\zeta}$ be defined on $\bar{D}$ to be continuous on $\bar{D}$, zero on ∂D, and linear on each π_β^- and π_β^+ with $\boldsymbol{\zeta}(0) = -\boldsymbol{c}$. Then

$$\nabla \boldsymbol{\zeta} = \begin{cases} -k^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_1 & \text{on } \pi_1^- \\ h^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_1 & \text{on } \pi_1^+ \\ -\rho^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_\beta & \text{on } \pi_\beta^- \\ \rho^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_\beta & \text{on } \pi_\beta^+. \end{cases}$$

Provided ρ is large enough Theorem 3.1 and the remark following Definition 3.1 imply that

$$\int_D g(\boldsymbol{H} + \nabla \boldsymbol{\zeta}(\boldsymbol{x}))\, d\boldsymbol{x} \geqq g(\boldsymbol{H})\, m(D) = g(\boldsymbol{H})\, 2^{m-1} \rho^{m-1} (h+k).$$

Hence

$$\frac{1}{2m} \left[\frac{2h}{h+k} g(\boldsymbol{H} + h^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_1) + \frac{2k}{h+k} g(\boldsymbol{H} - k^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_1) + \sum_{\beta=2}^{m} \left(g(\boldsymbol{H} - \rho^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_\beta) + g(\boldsymbol{H} + \rho^{-1} \boldsymbol{c} \otimes \boldsymbol{\mu}_\beta) \right) \right] \geqq g(\boldsymbol{H}).$$

Letting $\rho \to \infty$ we obtain (3.2). $\square$

The proof above shows that, roughly speaking, quasiconvexity implies the Legendre-Hadamard condition. Whether or not the converse holds is an important open question (*cf.* the comments of MORREY [2, p. 122]). The conditions are known to be equivalent only in certain special cases, for example in the quadratic case $f(\boldsymbol{F}) = a_{ij}^{\alpha\beta} F_\alpha^i F_\beta^j$ with $a_{ij}^{\alpha\beta}$ constant and m, n arbitrary (MORREY [1, 2], VAN HOVE [1]), and for certain parametric integrands when $n = m+1$ (MORREY [1, 2]). In particular nothing interesting is known about the case $m = n > 1$, which occurs in nonlinear elasticity.

To discuss this problem, consider a continuous integrand $f(\nabla \boldsymbol{u})$, defined on all of $M^{n\times m}$, and independent of $\boldsymbol{x}$ and $\boldsymbol{u}$. Suppose that f is rank 1 convex on $M^{n\times m}$. It is well known that if D is a bounded open set in $\mathscr{R}^m$ then any function $\zeta \in \mathscr{D}(D)$ can be approximated in $W_0^{1,\infty}(D)$ by piecewise affine functions (EKELAND & TÉMAM [1, p. 286]). Thus a natural method of attack is to follow the lead of the proof of Theorem 3.4 and to seek domains D with a partition into a finite number of disjoint open sets D_k and a set of measure zero, such that the quasiconvexity condition

$$\int_D f(\boldsymbol{F}_0 + \nabla \zeta(\boldsymbol{x}))\, d\boldsymbol{x} \geqq f(\boldsymbol{F}_0)\, m(D) \tag{3.4}$$

holds for any $\boldsymbol{F}_0 \in M^{n\times m}$ and for any $\zeta \in W_0^{1,\infty}(D)$ that is affine on each D_k (*cf.* SILVERMAN [1, Thm. 2]).

For ease of illustration we consider the case $m=2$, n arbitrary; similar comments apply for $m \geqq 3$. First let D be the interior of a triangle in $\mathscr{R}^2$ with vertices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$, and let $\boldsymbol{e}$ be an interior point of D. Let D_1, D_2, D_3 be the interiors of the triangles $\boldsymbol{a}_2 \boldsymbol{e} \boldsymbol{a}_3$, $\boldsymbol{a}_3 \boldsymbol{e} \boldsymbol{a}_1$, $\boldsymbol{a}_1 \boldsymbol{e} \boldsymbol{a}_2$ respectively. Let $\boldsymbol{n}_1$ be the unit outward normal to ∂D on the side $\boldsymbol{a}_2 \boldsymbol{a}_3$, let $l_1 = |\boldsymbol{a}_2 - \boldsymbol{a}_3|$, and let $\boldsymbol{n}_2, \boldsymbol{n}_3, l_2, l_3$ be defined analogously. Let $\zeta \in W_0^{1,\infty}(D)$ be affine on each D_k with $\zeta(\boldsymbol{e}) = \boldsymbol{c}$. Then

$$\nabla \zeta = \frac{l_k}{2m(D_k)}\, \boldsymbol{c} \otimes \boldsymbol{n}_k \quad \text{on} \quad D_k,$$

and (3.4) becomes

$$\sum_{k=1}^{3} \lambda_k f\left(\boldsymbol{F}_0 + \frac{l_k}{2m(D_k)}\, \boldsymbol{c} \otimes \boldsymbol{n}_k\right) \geqq f(\boldsymbol{F}_0),$$

where $\lambda_k = m(D_k)/m(D)$. But this inequality follows from rank 1 convexity of f because

$$\sum_{k=1}^{3} \frac{\lambda_k l_k \boldsymbol{n}_k}{2m(D_k)} = 0.$$

A similar argument shows that (3.4) holds for piecewise affine functions if D is the interior of a convex polygon and the D_k's are triangles formed by joining a single interior point of D to adjacent vertices of the polygon.

A different situation arises if we introduce more interior nodes into the partition of D. For example let D be an equilateral triangle $A_1 A_2 A_3$ of side 1 partitioned into 16 congruent equilateral subtriangles of side $\frac{1}{4}$ (see Fig. 1). Let $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$ be the unit outward normals shown, and let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be the position vectors of the three interior nodes B_1, B_2 and B_3. Let $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ be given and let $\zeta \in W_0^{1,\infty}(D)$ be affine on each subtriangle with $\zeta(\boldsymbol{e}_i) = \frac{8}{\sqrt{3}}\, \boldsymbol{c}_i$. The values of $\nabla \zeta$ in each sub-

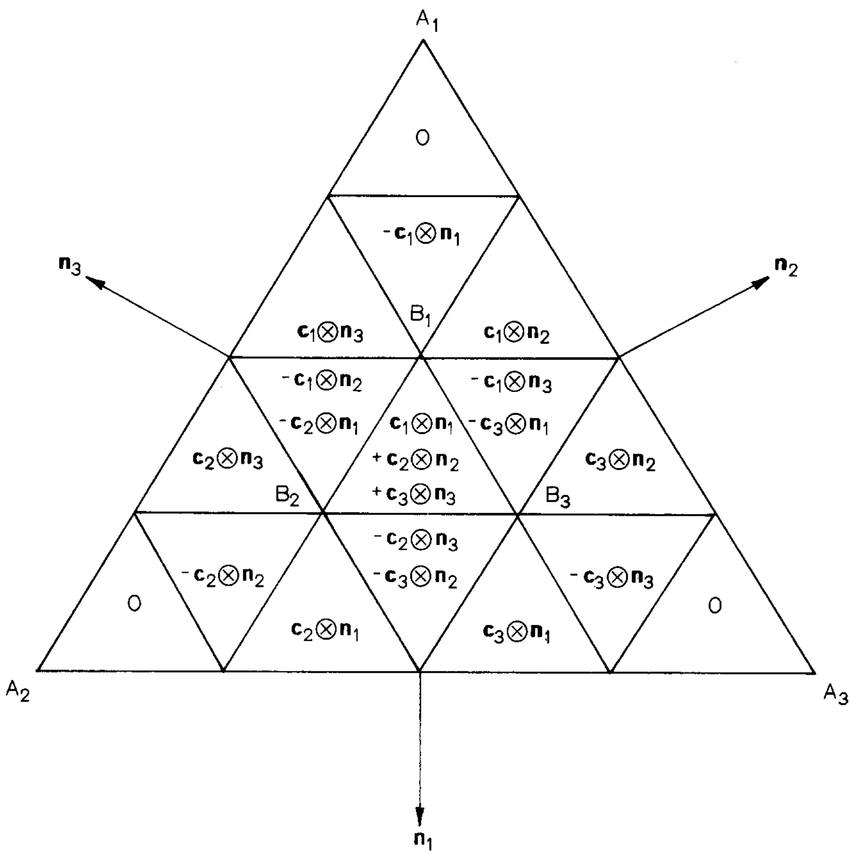


Fig. 1

triangle are shown in Fig. 1. The corresponding quasiconvexity inequality is easy to write down, but does not seem to follow from rank 1 convexity of f. This suggests that the Legendre-Hadamard condition does not imply quasiconvexity. Unfortunately the search for a counter-example is hampered by the fact that, as we have mentioned, any such f cannot be quadratic in $\nabla \boldsymbol{u}$. It is conceivable that if $m=n$ then the Legendre-Hadamard condition implies quasiconvexity for objective functions.

Finally we mention in passing an implication of the configuration in Figure 1 for finite element methods. Let D be as in Figure 1 and suppose that one wishes to solve a boundary-value problem for $\boldsymbol{u}$ subject to the pointwise constraint $\det \nabla \boldsymbol{u}(\boldsymbol{x})=1$ for all $\boldsymbol{x}\in D$ (the two-dimensional analogue of the incompressibility constraint (1.16)) and, for example, the boundary conditions $\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{x}$ for all $\boldsymbol{x}\in\partial D$. Then the only function $\zeta\in W_0^{1,\infty}(D)$ that is affine on each subtriangle with $\det \nabla(\boldsymbol{x}+\zeta(\boldsymbol{x}))=1$ for all $\boldsymbol{x}\in D$ is $\zeta\equiv 0$. Indeed, any such map $\boldsymbol{x}\mapsto\boldsymbol{x}+\zeta(\boldsymbol{x})$ deforms each subtriangle into another triangle with equal area. It follows that, for example, B_1 can be displaced only along the line through B_1 parallel to $A_1 A_3$, and also along the line through B_1 parallel to $A_1 A_2$. Hence B_1 is fixed by the map, and similarly for B_2, B_3. A similar argument applies when the number of congruent subtriangles in the partition of D is increased. Thus either nonlinear interpolation functions or nonconforming elements must be used. A related difficulty for incompressible fluids is discussed by TÉMAM [1].

4. Sufficient Conditions for Quasiconvexity

The quasiconvexity condition is not a pointwise condition on the function f, and is therefore difficult to verify in particular cases. In this section we shall be concerned with more accessible conditions that are sufficient for quasiconvexity. These conditions apply to functions f for which the Legendre-Hadamard condition is not known to be equivalent to quasiconvexity.

Throughout the rest of the article we assume, unless the contrary is stated, that $m=n=1, 2$ *or* 3.

We first study those functions $\phi(\boldsymbol{F})$ that belong to the null-space of the Euler-Lagrange operator; *i.e.*, those functions for which the corresponding Euler-Lagrange equations are identically satisfied. For smooth ϕ the following result is a special case of ERICKSEN [1], EDELEN [1, 2] and RUND [1, 2].

Theorem 4.1. *Let* $\phi\colon M^{n\times n}\to\mathscr{R}$ *be continuous and such that both* ϕ *and* $-\phi$ *are rank 1 convex on* $M^{n\times n}$, *so that*

$$\phi(\boldsymbol{F}+(1-\lambda)\,\boldsymbol{a}\otimes\boldsymbol{b})=\lambda\phi(\boldsymbol{F})+(1-\lambda)\,\phi(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b}) \tag{4.1}$$

for all $\boldsymbol{F}\in M^{n\times n}$, $\boldsymbol{a}, \boldsymbol{b}\in\mathscr{R}^n$, $\lambda\in[0,1]$. *Then* ϕ *has the form*

$$\begin{aligned} &\phi(\boldsymbol{F})=a+b\boldsymbol{F} && \text{if } n=1,\\ &\phi(\boldsymbol{F})=a_1+\beta_i^\alpha F_\alpha^i+\gamma\det\boldsymbol{F} && \text{if } n=2,\\ &\phi(\boldsymbol{F})=A+B_i^\alpha F_\alpha^i+C_i^\alpha(\operatorname{adj}\boldsymbol{F})_\alpha^i+D\det\boldsymbol{F} && \text{if } n=3,\end{aligned} \tag{4.2}$$

where $a, b, a_1, \beta_i^\alpha, \gamma, A, B_i^\alpha, C_i^\alpha, D$ *are arbitrary constants, and where* $\operatorname{adj}\boldsymbol{F}$ *denotes the adjugate matrix of* $\boldsymbol{F}$ (*i.e., the transpose of the matrix of cofactors*).

Proof. We just treat the case $n=3$; the cases $n=1, 2$ are easier. Suppose first that ϕ is C^2. Then by Theorem 3.3 (vi), (4.1) is equivalent to

$$A_{ij}^{\alpha\beta}(\boldsymbol{F})\,a^i a^j b_\alpha b_\beta=0 \qquad \text{for all } \boldsymbol{a}, \boldsymbol{b}\in\mathscr{R}^3 \qquad \text{and for all } \boldsymbol{F}\in M^{3\times 3},$$

where $A_{ij}^{\alpha\beta}(\boldsymbol{F})\overset{\text{def}}{=}\dfrac{\partial^2\phi(\boldsymbol{F})}{\partial F_\alpha^i\,\partial F_\beta^j}$. It follows that $A_{ij}^{\alpha\beta}=A_{ij}^{\alpha\beta}(\boldsymbol{F})$ is alternating; *i.e.*, $A_{ij}^{\alpha\beta}=A_{ji}^{\beta\alpha}=-A_{ji}^{\alpha\beta}$ and $A_{ij}^{\alpha\beta}=0$ if $\alpha=\beta$ or $i=j$. Since $A_{11}^{11}=A_{22}^{22}=A_{33}^{33}=0$ it follows that $\phi(\boldsymbol{F})$ is affine in each of F_1^1, F_2^2 and F_3^3, so that

$$\begin{aligned}\phi(\boldsymbol{F})=\phi_0(\bar{\boldsymbol{F}})\,F_1^1F_2^2F_3^3+\phi_1(\bar{\boldsymbol{F}})\,F_2^2F_3^3+\phi_2(\bar{\boldsymbol{F}})\,F_3^3F_1^1+\phi_3(\bar{\boldsymbol{F}})\,F_1^1F_2^2\\ +\theta_1(\bar{\boldsymbol{F}})\,F_1^1+\theta_2(\bar{\boldsymbol{F}})\,F_2^2+\theta_3(\bar{\boldsymbol{F}})\,F_3^3+\chi(\bar{\boldsymbol{F}}),\end{aligned}$$

where $\bar{\boldsymbol{F}}$ denotes the matrix of off-diagonal elements of $\boldsymbol{F}$, and where the functions ϕ_i, θ_i, χ are C^2. Since $A_{11}^{12}=0$, *etc.*, we obtain the equations

$$\frac{\partial\phi_0}{\partial F_2^1}=\frac{\partial\phi_2}{\partial F_2^1}=\frac{\partial\phi_3}{\partial F_2^1}=\frac{\partial\theta_1}{\partial F_2^1}=0,$$

etc., so that ϕ_0, ϕ_1, ϕ_2, ϕ_3 are constants, $\theta_1=\theta_1(F_3^2, F_2^3)$, $\theta_2=\theta_2(F_1^3, F_3^1)$ and $\theta_3=\theta_3(F_2^1, F_1^2)$. Applying the conditions $A_{12}^{12}=-A_{21}^{12}$, *etc.*, we can reduce ϕ to the form $\phi(\boldsymbol{F})=\psi(\bar{\boldsymbol{F}})+C_i^\alpha(\operatorname{adj}\boldsymbol{F})_\alpha^i+D\det\boldsymbol{F}$, where $\dfrac{\partial^2\psi}{\partial F_\alpha^i\,\partial F_\beta^j}\equiv 0$ for all i, j, α, β. The result follows.

For a general continuous ϕ we use a mollifier argument. Let $\rho\in\mathscr{D}(M^{3\times 3})$ satisfy $\rho\geqq 0$, $\rho(\boldsymbol{F})=0$ if $|\boldsymbol{F}|\geqq 1$, $\int\limits_{M^{3\times 3}}\rho(\boldsymbol{F})\,d\boldsymbol{F}=1$. For $\varepsilon>0$ let $\rho_\varepsilon(\boldsymbol{F})=\varepsilon^{-9}\rho(\boldsymbol{F}/\varepsilon)$. Then $\phi_\varepsilon\stackrel{\text{def}}{=}\rho_\varepsilon*\phi$ clearly satisfies (4.1), is C^∞, and thus

$$\phi_\varepsilon(\boldsymbol{F})=A(\varepsilon)+B_i^\alpha(\varepsilon)\,F_\alpha^i+C_i^\alpha(\varepsilon)(\operatorname{adj}\boldsymbol{F})_\alpha^i+D(\varepsilon)\det\boldsymbol{F}.$$

But since ϕ is continuous, $\phi_\varepsilon\to\phi$ uniformly on bounded subsets of $M^{3\times 3}$ as $\varepsilon\to 0$. For fixed i,α let $F_\alpha^i=t$, $F_\beta^j=0$ if $j\neq i$ or $\beta\neq\alpha$. Then the functions

$$g_\varepsilon(t)=A(\varepsilon)+B_i^\alpha(\varepsilon)\,t$$

converge uniformly on compact subsets of $\mathscr{R}$ to a function $g(t)$, which is easily shown to have the form $g(t)=A+B_i^\alpha t$. Thus $A(\varepsilon)\to A$ and $B_i^\alpha(\varepsilon)\to B_i^\alpha$ as $\varepsilon\to 0$. By choosing $\boldsymbol{F}$ so that $(\operatorname{adj}\boldsymbol{F})_\alpha^i=t$, $(\operatorname{adj}\boldsymbol{F})_\beta^j=0$ if $j\neq i$ or $\beta\neq\alpha$, we obtain similarly that $C_i^\alpha(\varepsilon)\to C_i^\alpha$, $D(\varepsilon)\to D$ as $\varepsilon\to 0$. The result for continuous ϕ follows. □

Corollary 4.1.1. *Let $\phi: M^{n\times n}\to\mathscr{R}$ be continuous. Then both ϕ and $-\phi$ are quasiconvex on $M^{n\times n}$ if and only if ϕ has the form* (4.2).

Proof. If ϕ and $-\phi$ are quasiconvex then by Theorems 3.3 and 3.4 both ϕ and $-\phi$ are rank 1 convex on $M^{n\times n}$, and so by the theorem ϕ has the form (4.2). Conversely any ϕ of the form (4.2) is such that, in the notation of the preceding proof, $A_{ij}^{\alpha\beta}$ is alternating. Then for any $\boldsymbol{F}_0\in M^{n\times n}$, for any bounded open set $D\subseteq\mathscr{R}^n$, and for any $\zeta\in\mathscr{D}(D)$ we have

$$\frac{d}{dt}\int\limits_D\phi(\boldsymbol{F}_0+t\,\nabla\zeta(\boldsymbol{y}))\,d\boldsymbol{y}=-\int\limits_D A_{ij}^{\alpha\beta}(\boldsymbol{F}_0+t\,\nabla\zeta)\,\zeta^j{}_{,\beta\alpha}\,\zeta^i\,d\boldsymbol{y}=0,$$

and the result follows. □

We next recall some results of BUSEMANN, EWALD & SHEPHARD [1] concerning convex functions defined on non-convex sets. Let $s\geqq 1$ and let $M\subseteq\mathscr{R}^s$ be such that the dimension of the convex hull $\operatorname{Co}M$ of M is s (*i.e.*, the linear subspace spanned by M is $\mathscr{R}^s$). *We do not assume that M is convex.* Let $\mathscr{F}$: $M\to\mathscr{R}$. For variable $r\geqq 1$ we denoted by $\Lambda=\{\lambda_1,\ldots,\lambda_r\}$ a variable set of non-negative real numbers λ_i with $\sum\limits_{i=1}^{r}\lambda_i=1$.

Definition 4.1. *$\mathscr{F}$ has a convex lower bound* if and only if there exists a real-valued convex function $C(\boldsymbol{z})$ defined on $\operatorname{Co}M$ such that $\mathscr{F}(\boldsymbol{z})\geqq C(\boldsymbol{z})$ in M. (Without loss of generality C may be assumed to be affine.)

$\mathscr{F}$ is said to be *convex* on M if it is the restriction to M of a real-valued convex function (in the usual sense) defined on $\operatorname{Co}M$; equivalently, $\mathscr{F}$ may be extended to a convex function on $\operatorname{Co}M$.

Theorem 4.2 (BUSEMANN; EWALD & SHEPHARD[1]).

(i) *$\mathscr{F}$ is convex on M if and only if it has a convex lower bound and the inequality*

$$\mathscr{F}(z_\Lambda) \leqq \sum_{i=1}^{r} \lambda_i \mathscr{F}(z_i)$$

holds for all $z_1, \ldots, z_r$ and $z_\Lambda = \sum_{i=1}^{r} \lambda_i z_i$ lying in M. A suitable convex extension to Co *M is given by*

$$g_{\mathscr{F}}(z) = \inf_{z=z_\Lambda} \sum_{i=1}^{r} \lambda_i \mathscr{F}(z_i), \qquad z_i \in M, \ 1 \leqq r < \infty.$$

(ii) *Let* Co *M be open. Then either of the following conditions is necessary and sufficient for $\mathscr{F}$ to be convex on M:*

(a) *$\mathscr{F}$ has a convex lower bound and the inequality*

$$\mathscr{F}(z_\Lambda) \leqq \sum_{i=1}^{s+1} \lambda_i \mathscr{F}(z_i)$$

holds for all $z_1, \ldots, z_{s+1}$ and $z_\Lambda = \sum_{i=1}^{s+1} z_i$ lying in M.

(b) *for each point $z_0 \in M$ there exist numbers $a_i(z_0)$ $(i=1, \ldots, s)$ such that*

$$\mathscr{F}(z) \geqq \mathscr{F}(z_0) + \sum_{i=1}^{s} a_i(z_0)(z^i - z_0^i),$$

for all $z \in M$.

We now define finite-dimensional Euclidean spaces E and E_1 by

$$E = E_1 \times \mathscr{R},$$

where

$$\begin{aligned} &E_1 \text{ is empty} && \text{if } n=1, \\ &E_1 = M^{2\times 2} && \text{if } n=2, \\ &E_1 = M^{3\times 3} \times M^{3\times 3} && \text{if } n=3. \end{aligned}$$

Thus E may be identified with $\mathscr{R}^{s(n)}$, where $s(1)=1$, $s(2)=5$ and $s(3)=19$.

Define the map $T\colon M^{n\times n} \to E$ by

$$\begin{aligned} &T(\boldsymbol{F}) = \boldsymbol{F} && \text{if } n=1, \\ &T(\boldsymbol{F}) = (\boldsymbol{F}, \det \boldsymbol{F}) && \text{if } n=2, \\ &T(\boldsymbol{F}) = (\boldsymbol{F}, \operatorname{adj} \boldsymbol{F}, \det \boldsymbol{F}) && \text{if } n=3. \end{aligned}$$

Let $U \subseteq M^{n\times n}$. By the theorem on the invariance of domain the set $T(U) \subseteq E$ is open if and only if U is open. However, $T(U)$ is not in general convex even if U is convex (except in the case $n=1$). The following result shows, in particular, that in certain important cases when U is open, so is Co $T(U)$.

Theorem 4.3. *Let $K \subseteq \mathscr{R}$ be nonempty and convex, and let $U = \{\boldsymbol{F} \in M^{n\times n} : \det \boldsymbol{F} \in K\}$. Then* Co *$T(U) = E_1 \times K$.*

Proof. We give the proof for $n=3$, that for $n=2$ being similar. For $k \in \mathscr{R}$ define $V_k \subseteq E_1$ by

$$V_k = \{F, \operatorname{adj} F): F \in M^{3\times 3},\ \det F = k\}.$$

It suffices to show that $\operatorname{Co} V_k = E_1$ for all k. Suppose not. Then there is a closed half-space

$$\pi = \{(F, A) \in E_1 : F^i_\alpha G^\alpha_i + A^i_\alpha H^\alpha_i \leqq \mu\},$$

$(G, H) \neq 0$, with $V_k \subseteq \pi$ (ROCKAFELLAR [1, p. 99]). If $R_1, R_2 \in M^{3\times 3}$ are proper orthogonal then

$$F^i_\alpha G^\alpha_i + A^i_\alpha H^\alpha_i = \operatorname{tr}[(R_1 F R_2)(R_2^T G R_1^T) + (R_2^T A R_1^T)(R_1^T H R_2)].$$

Since $\operatorname{adj}(R_1 F R_2) = R_2^T(\operatorname{adj} F) R_1^T$, $\det(R_1 F R_2) = \det F$, we may without loss of generality suppose that H is diagonal. Suppose that $H \neq 0$ and assume without loss of generality that $H^1_1 \neq 0$. Let

$$F = \operatorname{diag}(k N^{-1} \operatorname{sgn} H^1_1, N^{\frac{1}{2}} \operatorname{sgn} H^1_1, N^{\frac{1}{2}}).$$

Then $\operatorname{adj} F = \operatorname{diag}(N \operatorname{sgn} H^1_1, k N^{-\frac{1}{2}} \operatorname{sgn} H^1_1, k N^{-\frac{1}{2}})$ and $\det F = k$. Hence $(F, \operatorname{adj} F) \in V_k$, but for $N > 0$ large enough $(F, \operatorname{adj} F) \notin \pi$. If $H = 0$ then we may assume that $G^1_1 \neq 0$, let $F = (k N \operatorname{sgn} G^1_1, N^{-\frac{1}{2}} \operatorname{sgn} G^1_1, N^{-\frac{1}{2}})$ and proceed similarly. Hence $V_k \nsubseteq \pi$ and this contradiction proves the result. $\square$

For $g: U \to \mathscr{R}$ we may define a function $G: T(U) \to \mathscr{R}$ by $G(T(F)) = g(F)$, *i.e.*,

$$\begin{aligned} g &= G && \text{if } n=1, \\ g(F) &= G(F, \det F) && \text{if } n=2, \\ g(F) &= G(F, \operatorname{adj} F, \det F) && \text{if } n=3. \end{aligned} \tag{4.3}$$

Definition 4.2. A function $g: U \to \mathscr{R}$ is *polyconvex* if and only if $G: T(U) \to R$ defined by (4.3) is convex on $T(U)$.

If $n=1$ and U is convex, then polyconvexity of g is the same as convexity in the usual sense. If $n=2$ or $n=3$, polyconvexity is characterized by the following

Theorem 4.4. *Let U be such that $\operatorname{Co} T(U)$ is open. Then g is polyconvex if and only if one of the following three equivalent conditions holds:*

(i) *If $n=2$:*

there exists a convex function $C(F, \delta)$ on $\operatorname{Co} T(U)$ with

$$g(F) \geqq C(F, \det F) \qquad \textit{for all } F \in U,$$

and the inequality

$$g\left(\sum_{i=1}^{6} \lambda_i F^{(i)}\right) \leqq \sum_{i=1}^{6} \lambda_i g(F^{(i)})$$

holds for all $\lambda_i \geqq 0$ with $\sum_{i=1}^{6} \lambda_i = 1$, and for all $F^{(i)} \in U$ satisfying

$$\sum_{i=1}^{6} \lambda_i \det F^{(i)} = \det\left(\sum_{i=1}^{6} \lambda_i F^{(i)}\right). \tag{4.4}$$

If $n=3$:
there exists a convex function $C(\boldsymbol{F}, \boldsymbol{A}, \delta)$ *on* Co $T(U)$ *with*

$$g(\boldsymbol{F}) \geqq C(\boldsymbol{F}, \operatorname{adj} \boldsymbol{F}, \det \boldsymbol{F}) \quad \text{for all } \boldsymbol{F} \in U,$$

and the inequality

$$g\left(\sum_{i=1}^{20} \lambda_i \boldsymbol{F}^{(i)}\right) \leqq \sum_{i=1}^{20} \lambda_i g(\boldsymbol{F}^{(i)})$$

holds for all $\lambda_i \geqq 0$ *with* $\sum_{i=1}^{20} \lambda_i = 1$, *and for all* $\boldsymbol{F}^{(i)} \in U$ *satisfying*

$$\left.\begin{aligned} &\sum_{i=1}^{20} \lambda_i \operatorname{adj} \boldsymbol{F}^{(i)} = \operatorname{adj}\left(\sum_{i=1}^{20} \lambda_i \boldsymbol{F}^{(i)}\right) \\ \text{and} \\ &\sum_{i=1}^{20} \lambda_i \det \boldsymbol{F}^{(i)} = \det\left(\sum_{i=1}^{20} \lambda_i \boldsymbol{F}^{(i)}\right). \end{aligned}\right\} \tag{4.5}$$

(ii) *If* $n=2$:
for each $\boldsymbol{F} \in U$ *there exist numbers* $a_i^\alpha(\boldsymbol{F})$, $a(\boldsymbol{F})$ *such that*

$$g(\bar{\boldsymbol{F}}) \geqq g(\boldsymbol{F}) + a_i^\alpha(\boldsymbol{F})(\bar{F}_\alpha^i - F_\alpha^i) + a(\boldsymbol{F})(\det \bar{\boldsymbol{F}} - \det \boldsymbol{F}) \tag{4.6}$$

for all $\bar{\boldsymbol{F}} \in U$.

If $n=3$:
for each $\boldsymbol{F} \in U$ *there exist numbers* $a_i^\alpha(\boldsymbol{F})$, $b_i^\alpha(\boldsymbol{F})$, $c(\boldsymbol{F})$ *such that*

$$g(\bar{\boldsymbol{F}}) \geqq g(\boldsymbol{F}) + a_i^\alpha(\boldsymbol{F})(\bar{F}_\alpha^i - F_\alpha^i) + b_i^\alpha(\boldsymbol{F})\big((\operatorname{adj} \bar{\boldsymbol{F}})_\alpha^i - (\operatorname{adj} \boldsymbol{F})_\alpha^i\big) + c(\boldsymbol{F})(\det \bar{\boldsymbol{F}} - \det \boldsymbol{F}) \tag{4.7}$$

for all $\bar{\boldsymbol{F}} \in U$.

(iii) *If* $n=2$:
for each $\boldsymbol{F} \in U$ *there exist numbers* $A_i^\alpha(\boldsymbol{F})$, $a(\boldsymbol{F})$ *such that*

$$g(\boldsymbol{F} + \boldsymbol{\pi}) \geqq g(\boldsymbol{F}) + A_i^\alpha(\boldsymbol{F})\, \pi_\alpha^i + a(\boldsymbol{F}) \det \boldsymbol{\pi} \tag{4.8}$$

for all $\boldsymbol{F} + \boldsymbol{\pi} \in U$.
If $n=3$:
for each $\boldsymbol{F} \in U$ *there exist numbers* $A_i^\alpha(\boldsymbol{F})$, $B_i^\alpha(\boldsymbol{F})$, $c(\boldsymbol{F})$ *such that*

$$g(\boldsymbol{F} + \boldsymbol{\pi}) \geqq g(\boldsymbol{F}) + A_i^\alpha(\boldsymbol{F})\, \pi_\alpha^i + B_i^\alpha(\boldsymbol{F})(\operatorname{adj} \boldsymbol{\pi})_\alpha^i + c(\boldsymbol{F}) \det \boldsymbol{\pi} \tag{4.9}$$

for all $\boldsymbol{F} + \boldsymbol{\pi} \in U$.

Proof. That polyconvexity of g is equivalent to (i) or (ii) follows immediately from Theorem 4.2 (ii) and Theorem 4.3. That (ii) and (iii) are equivalent follows by setting $\bar{\boldsymbol{F}} = \boldsymbol{F} + \boldsymbol{\pi}$ and rewriting the right-hand sides of (4.7) and (4.9). ($c(\boldsymbol{F})$ has the same value in both conditions.) □

Of course, if G is C^1, then the coefficients on the right-hand sides of (4.6) and (4.7) are given by the derivatives of G with respect to its arguments. Condition (iii) is the form given by MORREY [2, p. 123], who proved the following theorem:

Theorem 4.5. *Let U be such that* Co $T(U)$ *is open. If g is polyconvex, then g is quasiconvex on U.*

Proof. We give the proof for $n=3$. Since G is the restriction of a convex function to the set $T(U)$, it is continuous and hence so is g. Let D be a bounded open subset of $\mathscr{R}^3$, let $\boldsymbol{F}_0 \in U$ and let $\zeta \in \mathscr{D}(D)$ satisfy $\boldsymbol{F}_0 + \nabla\zeta(\boldsymbol{y}) \in U$ for all $\boldsymbol{y} \in D$. By Corollary 4.1.1 we have

$$\int_D \zeta^i_{,\alpha}\, d\boldsymbol{y} = \int_D [(\operatorname{adj}(\boldsymbol{F}_0 + \nabla\zeta))^i_\alpha - (\operatorname{adj}\boldsymbol{F}_0)^i_\alpha]\, d\boldsymbol{y} = \int_D [\det(\boldsymbol{F}_0 + \nabla\zeta) - \det \boldsymbol{F}_0]\, d\boldsymbol{y} = 0,$$

and the quasiconvexity of g follows from (4.7). □

The converse to Theorem 4.5 is false if $n=3$. In fact when $U = M^{3\times 3}$ and

$$g(\boldsymbol{F}) = a^{\alpha\beta}_{ij} F^i_\alpha F^j_\beta, \tag{4.10}$$

with $a^{\alpha\beta}_{ij}$ constant, it is easily seen that g is polyconvex if and only if there are constants B^α_i with

$$g(\boldsymbol{F}) - B^\alpha_i (\operatorname{adj}\boldsymbol{F})^i_\alpha \geqq 0 \qquad \text{for all } \boldsymbol{F}. \tag{4.11}$$

As was pointed out by Morrey [1, p. 26], an example of Terpstra [1] shows that if $n=3$ there exist constants $a^{\alpha\beta}_{ij}$ such that (4.11) is violated for any B^α_i, even though

$$a^{\alpha\beta}_{ij} \lambda^i \lambda^j \mu_\alpha \mu_\beta > 0 \qquad \text{for all nonzero } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{R}^3, \tag{4.12}$$

so that g is quasiconvex. Terpstra also showed that for $n=2$ any quasiconvex g of the form (4.10) satisfies (4.11) for suitable B^α_i, and hence is polyconvex. I know of no counterexample to the converse of Theorem 4.5 if $n=2$. Conditions of the type (4.11) were studied by Clebsch [1] and Hadamard [1, 3].

The conditions of quasiconvexity and polyconvexity may be contrasted as follows. Considering for illustration the case in which $n=3$, let λ_i and $\boldsymbol{F}^{(i)}$ be given satisfying (4.5). There is then no reason to suppose that a domain D, and a partition of D into 20 open subsets D_i of volume $\lambda^{(i)}$ and a set of measure zero, can be found such that there is a continuous piecewise affine function on D whose gradient takes the value $\boldsymbol{F}^{(i)}$ on D_i (*cf.* the discussion at the end of Section 3). Only for such D, D_i can the corresponding polyconvexity inequality be deduced from the quasiconvexity condition. I remark that Theorem 4.4 (i) offers a way of proving that a given function is not polyconvex.

By sacrificing the pointwise nature of the condition imposed on g, we may obtain a sufficient condition for quasiconvexity generalizing polyconvexity. For brevity we discuss this new condition just when $n=3$.

We consider continuous functions G: Co $T(U) \to \mathscr{R}$ with $U \subseteq M^{3\times 3}$ arbitrary. Roughly speaking we require that the integral of G, rather than G itself, be convex. To be precise, let $\gamma \geqq 1$, $\mu \geqq 1$, $\nu \geqq 1$ and let $\boldsymbol{u} \in W^{1,\gamma}(\Omega)$ satisfy $\nabla\boldsymbol{u}(\boldsymbol{x}) \in U$ for almost all $\boldsymbol{x} \in \Omega$. Suppose further that adj $\nabla\boldsymbol{u} \in L^\mu(\Omega)^9$, det $\nabla\boldsymbol{u} \in L^\nu(\Omega)$. Thus if $\boldsymbol{\Sigma}(\boldsymbol{u})$ denotes $(\nabla\boldsymbol{u}, \operatorname{adj}\nabla\boldsymbol{u}, \det\nabla\boldsymbol{u})$, then $\boldsymbol{\Sigma}(\boldsymbol{u}) \in \mathscr{B} \overset{\text{def}}{=} L^\gamma(\Omega)^9 \times L^\mu(\Omega)^9 \times L^\nu(\Omega)$. Let

$$R_{\boldsymbol{u}} = \{\boldsymbol{\Sigma}(\boldsymbol{u} + \zeta) \colon \zeta \in \mathscr{D}(\Omega)\}$$

and let $\mathscr{C}_{\boldsymbol{u}}$ denote the closed affine subspace of $\mathscr{B}$ spanned by $R_{\boldsymbol{u}}$. Let

$$\mathscr{K}_{\boldsymbol{u}} = \{\boldsymbol{\sigma} \in \mathscr{C}_{\boldsymbol{u}} \colon \boldsymbol{\sigma}(\boldsymbol{x}) \in \text{Co } T(U) \text{ almost everywhere in } \Omega\}.$$

Define

$$J_{\boldsymbol{u}}(\boldsymbol{\sigma})=\int_{\Omega} G(\boldsymbol{\sigma}(\boldsymbol{x}))\,d\boldsymbol{x}.$$

Suppose that $J_{\boldsymbol{u}}$ exists and is finite or $+\infty$ for all $\sigma\in\mathscr{K}_{\boldsymbol{u}}$. Let G be such that $J_{\boldsymbol{u}}$ is a convex function on the convex set $\mathscr{K}_{\boldsymbol{u}}$.

Definition 4.3. If $g\colon U\to\mathscr{R}$ then g is said to satisfy condition $(P_{\gamma,\mu,\nu})$ at $\boldsymbol{u}$ if and only if there exists $G=G_{\boldsymbol{u}}$: Co $T(U)\to\mathscr{R}$ with the above properties and such that

$$g(\boldsymbol{F})=G(\boldsymbol{F},\operatorname{adj}\boldsymbol{F},\det\boldsymbol{F}) \tag{4.13}$$

for all $\boldsymbol{F}\in U$.

Theorem 4.6. *Let U be such that* Co $T(U)$ *is open. Let γ,μ,ν be arbitrary.*

(i) *Let $g\colon U\to\mathscr{R}$ be polyconvex and bounded below, and let $\boldsymbol{u}$ be as above. Then g satisfies $P_{\gamma,\mu,\nu}$ at $\boldsymbol{u}$.*

(ii) *Let $u^i(\boldsymbol{x})=F^i_{0\alpha}x^\alpha+z^i$ with $\boldsymbol{F}_0\in U$ and $\boldsymbol{z}\in\mathscr{R}^3$ both constant. Let g satisfy $P_{\gamma,\mu,\nu}$ at $\boldsymbol{u}$. Then g is quasiconvex at $\boldsymbol{F}_0\in U$.*

Proof. (i) Since g is polyconvex there exists a convex function G: Co $T(U)\to\mathscr{R}$ satisfying (4.13). Since Co $T(U)$ is open G is continuous, and hence $G(\boldsymbol{\sigma}(\cdot))$ is measurable for each $\boldsymbol{\sigma}\in\mathscr{K}_{\boldsymbol{u}}$. By Theorem 4.2 (i) we may suppose that G is bounded below on Co $T(U)$. Thus $J_{\boldsymbol{u}}$ exists and is finite or $+\infty$ for all $\boldsymbol{\sigma}\in\mathscr{K}_{\boldsymbol{u}}$, and $J_{\boldsymbol{u}}$ is clearly convex.

(ii) Suppose first that G is C^1 on Co $T(U)$. Let $\zeta\in\mathscr{D}(\Omega)$ satisfy $\boldsymbol{F}_0+\nabla\zeta(\boldsymbol{x})\in U$ for all $\boldsymbol{x}\in\Omega$. Let $\boldsymbol{\sigma}=\boldsymbol{\Sigma}(\boldsymbol{u})$, $\bar{\boldsymbol{\sigma}}=\boldsymbol{\Sigma}(\boldsymbol{u}+\zeta)$. Then $\boldsymbol{\sigma},\bar{\boldsymbol{\sigma}}\in\mathscr{K}_{\boldsymbol{u}}$. A standard argument shows that if

$$\Theta(t)=\int_{\Omega} G(\boldsymbol{\sigma}+t(\bar{\boldsymbol{\sigma}}-\boldsymbol{\sigma}))\,d\boldsymbol{x},$$

then $\Theta\in C^1([0,1])$ with the obvious derivative. By the convexity of $J_{\boldsymbol{u}}$ we have that $\Theta(0)\leqq\Theta(1)-\Theta'(0)$. Hence

$$\int_{\Omega} g(\boldsymbol{F}_0)\,d\boldsymbol{x}\leqq\int_{\Omega} g(\boldsymbol{F}_0+\nabla\zeta)\,d\boldsymbol{x}-\int_{\Omega}\left[\frac{\partial G}{\partial F^i_\alpha}(\boldsymbol{\Sigma}(\boldsymbol{u}))\,\zeta^i_{,\alpha}+\frac{\partial G}{\partial(\operatorname{adj}\boldsymbol{F})^i_\alpha}(\boldsymbol{\Sigma}(\boldsymbol{u}))\,(\operatorname{adj}\nabla\zeta)^i_\alpha\right.$$
$$\left.+\frac{\partial G}{\partial(\det\boldsymbol{F})}(\boldsymbol{\Sigma}(\boldsymbol{u}))\det\nabla\zeta\right]d\boldsymbol{x},$$

and so g is quasiconvex at $\boldsymbol{F}_0\in U$.

The result follows for general continuous G by a mollifier argument. $\square$

Condition $P_{\gamma,\mu,\nu}$ does not in general imply polyconvexity. Indeed let $g(\boldsymbol{F})$ be given by (4.10) with $a^{\alpha\beta}_{ij}$ satisfying (4.12) but not (4.11) for any B^α_i, so that g is not polyconvex. Let $\gamma=2$ with μ and ν arbitrary and let $U=M^{3\times 3}$. Define $G\colon E\to\mathscr{R}$ by $G(\boldsymbol{F},\boldsymbol{A},\delta)=g(\boldsymbol{F})$ for all $(\boldsymbol{F},\boldsymbol{A},\delta)\in E$. Let $\boldsymbol{u}$ be as above. For any $\zeta\in W^{1,2}_0(\Omega)$ we have (VAN HOVE [1], MORREY [2])

$$\int_{\Omega} a^{\alpha\beta}_{ij}\,\zeta^i_{,\alpha}\,\zeta^j_{,\beta}\,d\boldsymbol{x}\geqq 0.$$

It follows that $J_{\boldsymbol{u}}$ is convex on $\mathscr{K}_{\boldsymbol{u}}$, and hence g satisfies $P_{\gamma,\mu,\nu}$ at $\boldsymbol{u}$.

There are some grounds, connected with the results of Section 6, for believing that condition $P_{\gamma,\mu,\nu}$ at affine $\boldsymbol{u}$ and the quasiconvexity condition are equivalent, but I have been unable to prove or disprove this.

To complete this section I remark that many of the results may be extended without undue difficulty to arbitrary m and n; the polyconvexity condition is then a requirement of convexity with respect to the basis elements of the null space of the Euler-Lagrange operator. (See BALL [2].)

5. Isotropic Convex and Polyconvex Functions

The purpose of this section is to give a method for producing a wide variety of nontrivial *isotropic* polyconvex functions. These functions will prove valuable in Section 8 when we apply our existence theorems to certain models which have been proposed for rubbers. We begin by discussing isotropic *convex* functions of $n \times n$ matrices for arbitrary $n \geqq 1$. We recall that the *singular values* of an $n \times n$ matrix $\boldsymbol{F}$ are by definition the eigenvalues of the positive semidefinite symmetric matrix $\sqrt{\boldsymbol{F}\boldsymbol{F}^T}$. When $\boldsymbol{F}$ is the deformation gradient these eigenvalues are the principal stretches of the deformation. When examining the results below the reader should bear in mind equation (1.15).

Notation. Vectors in $\mathscr{R}^n$ are denoted by $\boldsymbol{x}=(x_1, \ldots, x_n)$ and the inner product of two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{R}^n$ is written $(\boldsymbol{x}, \boldsymbol{y})$. $\mathscr{R}^n_+$ denotes the positive orthant

$$\{\boldsymbol{x}\colon x_i \geqq 0 \text{ for } 1 \leqq i \leqq n\}$$

of $\mathscr{R}^n$. $\mathscr{P}_n$ denotes the permutation group on n symbols (an element P of $\mathscr{P}_n$ acts on an n vector by permuting its entries).

We shall prove the following theorem:

Theorem 5.1. *Let $n \geqq 1$. Let $\Phi(v_1, \ldots, v_n)$ be a symmetric real-valued function defined on $\mathscr{R}^n_+$. For $\boldsymbol{F} \in M^{n \times n}$ define*

$$W(\boldsymbol{F}) = \Phi(v_1, \ldots, v_n), \tag{5.1}$$

where $v_1, \ldots, v_n$ are the singular values of $\boldsymbol{F}$. Then

(i) *W is convex on $K = \{\boldsymbol{V} \in M^{n \times n}\colon \boldsymbol{V}$ is positive-semidefinite and symmetric$\}$ if and only if Φ is convex.*

(ii) *W is convex on $M^{n \times n}$ if and only if Φ is convex and nondecreasing in each variable v_i.*

Remarks: Part (i), which is probably known to matrix theorists, was stated by HILL [3] for $n=3$. HILL's proof relies on a property of the trace of the product of two symmetric matrices, a recent proof of which has been given by THEOBALD [1]. HILL assumed that W is differentiable *everywhere*, an assumption which rules out several simple and useful functions and which can be surprisingly tedious to verify. Proofs using differentiability obscure the geometric nature of the result.

The harder implication in (ii) is due to THOMPSON & FREEDE [3]. Our method of proof is broadly similar in approach, but rather different in detail. The result extends work of VON NEUMANN [1] on gauge-functions and matrix norms (see especially his Remark 5). For further information and references on singular value inequalities see AMIR-MOEZ [1], AMIR-MOEZ & HORN [1], THOMPSON [1], THOMPSON & FREEDE [1-3].

Lemma 5.1. (VON NEUMANN [1]; see also MIRSKY [1, 2]). *Let $\boldsymbol{A}, \boldsymbol{B} \in M^{n \times n}$ have singular values $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n \geqq 0$ and $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n \geqq 0$. Then*

$$|\operatorname{tr}(\boldsymbol{A}\boldsymbol{B})| \leqq (\boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{5.2}$$

Lemma 5.2. (VON NEUMANN [1]). *Under the hypotheses of Lemma 5.1*

$$\max_{\boldsymbol{Q}, \boldsymbol{R}} \operatorname{tr}(\boldsymbol{A}\boldsymbol{Q}\boldsymbol{B}\boldsymbol{R}) = (\boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{5.3}$$

where the maximum is taken over all pairs $\boldsymbol{Q}, \boldsymbol{R}$ of orthogonal matrices.

Proof. There exist orthogonal matrices $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{R}_1, \boldsymbol{R}_2$ such that $\boldsymbol{A} = \boldsymbol{Q}_1 \operatorname{diag}(\boldsymbol{\alpha}) \boldsymbol{R}_1$, $\boldsymbol{B} = \boldsymbol{Q}_2 \operatorname{diag}(\boldsymbol{\beta}) \boldsymbol{R}_2$. Choose $\boldsymbol{Q} = \boldsymbol{R}_1^T \boldsymbol{Q}_2^T$, $\boldsymbol{R} = \boldsymbol{R}_2^T \boldsymbol{Q}_1^T$. Then

$$\operatorname{tr}(\boldsymbol{A}\boldsymbol{Q}\boldsymbol{B}\boldsymbol{R}) = (\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

But for any orthogonal $\boldsymbol{Q}, \boldsymbol{R}$ the matrices $\boldsymbol{A}\boldsymbol{Q}$ and $\boldsymbol{B}\boldsymbol{R}$ have singular values $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. Hence $\operatorname{tr}(\boldsymbol{A}\boldsymbol{Q}\boldsymbol{B}\boldsymbol{R}) \leqq (\boldsymbol{\alpha}, \boldsymbol{\beta})$ by Lemma 5.1. □

Lemma 5.3. *Let $r_1 \geqq r_2 \geqq \cdots \geqq r_n \geqq 0$. Then $(\boldsymbol{r}, \boldsymbol{v})$ is a convex function of $\boldsymbol{F}$, where $v_1 \geqq v_2 \geqq \cdots \geqq v_n \geqq 0$ are the singular values of $\boldsymbol{F}$.*

Proof. In Lemma 5.2 put $\boldsymbol{A} = \boldsymbol{F}$, $\boldsymbol{B} = \operatorname{diag} \boldsymbol{r}$. Then

$$(\boldsymbol{r}, \boldsymbol{v}) = \max_{\boldsymbol{Q}, \boldsymbol{R}} \operatorname{tr}(\boldsymbol{F}\boldsymbol{Q}\boldsymbol{B}\boldsymbol{R}), \tag{5.4}$$

and each $\operatorname{tr}(\boldsymbol{F}\boldsymbol{Q}\boldsymbol{B}\boldsymbol{R})$ is a convex function of $\boldsymbol{F}$. □

Remark. By letting $\boldsymbol{r} = (1, \ldots, \underset{\substack{k^{\text{th}} \\ \text{place}}}{1}, 0, \ldots, 0)$ in Lemma 5.3 it follows that for $1 \leqq k \leqq n$, $\sum_{i=1}^{k} v_i$ is a convex function of $\boldsymbol{F}$.

Lemma 5.4. *Let $c_1 \geqq c_2 \geqq \cdots \geqq c_n \geqq 0$. Define the sets*

$$L = \{\boldsymbol{y} \in \mathscr{R}_+^n : (\boldsymbol{r}, P\boldsymbol{y}) \leqq (\boldsymbol{r}, \boldsymbol{c}) \quad \textit{for all } r_1 \geqq r_2 \geqq \cdots \geqq r_n \geqq 0 \quad \textit{and all } P \in \mathscr{P}_n\},$$

$$M = \left\{\boldsymbol{y} \in L : \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} c_i\right\},$$

$$L_1 = \operatorname{Co}\{\boldsymbol{0}, P(c_1, c_2, \ldots, c_l, 0, \ldots, 0) : P \in \mathscr{P}_n, 1 \leqq l \leqq n\},$$

$$M_1 = \operatorname{Co}\{P\boldsymbol{c} : P \in \mathscr{P}_n\}.$$

Then $L = L_1$ and $M = M_1$.

Proof. L is a convex set containing $\boldsymbol{0}$ and the points $P(c_1, c_2, \ldots, c_l, 0, \ldots, 0)$. Thus $L_1 \subseteq L$. To show that $L \subseteq L_1$ we prove that any closed half-space in $\mathscr{R}^n$ containing L_1 also contains the closed convex set L. Let such a half-space be $\pi = \{\boldsymbol{y} \in \mathscr{R}^n : (\boldsymbol{y}, \boldsymbol{x}) \leqq \mu\}$, where $\boldsymbol{x} \in \mathscr{R}^n$, $\mu \in \mathscr{R}$ are fixed. Let the coordinates of $\boldsymbol{x}$ in some order be

$$\bar{x}_1 \geqq \bar{x}_2 \geqq \cdots \geqq \bar{x}_k \geqq 0 > \bar{x}_{k+1} \geqq \cdots \geqq x_n,$$

where the symbols to the right of 0 are omitted if $k = n$. Let $\boldsymbol{y} \in L$ and let

$$\bar{y}_1 \geqq \bar{y}_2 \geqq \cdots \geqq \bar{y}_n \geqq 0$$

be the coordinates of $\boldsymbol{y}$ in some order. Then

$$\begin{aligned}(\boldsymbol{y}, \boldsymbol{x}) &\leqq \bar{y}_1 \bar{x}_1 + \cdots + \bar{y}_k \bar{x}_k \\ &\leqq c_1 \bar{x}_1 + \cdots + c_k \bar{x}_k \\ &\leqq \mu,\end{aligned}$$

so that $\boldsymbol{y} \in \pi$. Hence $L_1 = L$.

M is a convex set containing the points $P\boldsymbol{c}$. Thus $M_1 \subseteq M$. Let π contain M_1, let $\boldsymbol{y} \in M$ and let $\bar{y}_1 \geqq \bar{y}_2 \geqq \cdots \geqq \bar{y}_n \geqq 0$ be the coordinates of $\boldsymbol{y}$ in some order. Then

$$\begin{aligned}(\boldsymbol{y}, \boldsymbol{x}) &\leqq \sum_{i=1}^{n} \bar{y}_i \bar{x}_i \\ &= \sum_{i=1}^{n-1} \bar{y}_i \bar{x}_i + \left(\sum_{i=1}^{n} c_i - \sum_{i=1}^{n-1} \bar{y}_i\right) \bar{x}_n \\ &= \sum_{i=1}^{n-1} \bar{y}_i (\bar{x}_i - \bar{x}_n) + \bar{x}_n \sum_{i=1}^{n} c_i.\end{aligned}$$

Choosing $r_i = x_i - x_n (1 \leqq i < n)$, $r_n = 0$ in the definition of L we obtain

$$(\boldsymbol{y}, \boldsymbol{x}) \leqq \sum_{i=1}^{n-1} c_i (\bar{x}_i - \bar{x}_n) + \bar{x}_n \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} c_i x_i \leqq \mu.$$

Thus $\boldsymbol{y} \in \pi$ and $M_1 = M$. $\square$

Proof of Theorem 5.1.

(i) That the convexity of W implies the convexity of Φ is immediate. Thus let Φ be convex and symmetric. Let $\boldsymbol{U}, \boldsymbol{V} \in K$ have singular values (eigenvalues)

$$u_1 \geqq u_2 \geqq \cdots \geqq u_n \geqq 0, \qquad v_1 \geqq v_2 \geqq \cdots \geqq v_n \geqq 0.$$

Let $\lambda \in [0, 1]$ and let $\boldsymbol{A} \stackrel{\text{def}}{=} \lambda \boldsymbol{U} + (1-\lambda) \boldsymbol{V}$ have singular values $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$. Let $\boldsymbol{c} = \lambda \boldsymbol{u} + (1-\lambda) \boldsymbol{v}$. Note that for some orthogonal $\boldsymbol{Q}, \boldsymbol{R}$ we have

$$\sum_{i=1}^{n} a_i = \sum_{i,j=1}^{n} (\lambda Q_{ij}^2 u_j + (1-\lambda) R_{ij}^2 v_j) = \sum_{i=1}^{n} c_i. \tag{5.5}$$

By (5.5) and Lemma 5.3 we have $\boldsymbol{a} \in M$. Hence by Lemma 5.4 $\boldsymbol{a} \in M_1$. Therefore

$$W(\boldsymbol{A}) = \Phi(\boldsymbol{a}) = \Phi\left(\sum_{i=1}^{n} \lambda_i P_i \boldsymbol{c}\right), \tag{5.6}$$

where $P_i \in \mathscr{P}_n$, $\lambda_i \geqq 0$, $\sum_{i=1}^{n} \lambda_i = 1$. Thus

$$W(\boldsymbol{A}) \leqq \sum_{i=1}^{n} \lambda_i \Phi(P_i \boldsymbol{c}) = \sum_{i=1}^{n} \lambda_i \Phi(\boldsymbol{c}) = \Phi(\boldsymbol{c}) \leqq \lambda \Phi(\boldsymbol{u}) + (1-\lambda) \Phi(\boldsymbol{v}),$$

where we have used the symmetry of Φ. Hence.

$$W(\boldsymbol{A}) \leqq \lambda W(\boldsymbol{U}) + (1-\lambda) W(\boldsymbol{V}) \tag{5.7}$$

as required.

(ii) Let W be convex on $M^{n\times n}$. Then clearly Φ is convex. Also for fixed non-negative $v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$

$$g(v) \stackrel{\text{def}}{=} W(\operatorname{diag}(v_1, \ldots, v_{k-1}, v, v_{k+1}, \ldots, v_n))$$

is convex in v. But $g(v) = \Phi(v_1, \ldots, v_{k-1}, |v|, v_{k+1}, \ldots, v_n)$ and any even convex function of $t \in \mathscr{R}$ is nondecreasing for $t \geqq 0$. Hence Φ is nondecreasing in each variable.

Let Φ be convex, symmetric and nondecreasing in each variable. Let $\lambda \in [0, 1]$ and let $\boldsymbol{F}, \boldsymbol{G} \in M^{n\times n}$, $\boldsymbol{A} = \lambda \boldsymbol{F} + (1-\lambda)\boldsymbol{G}$ have singular values

$$u_1 \geqq u_2 \geqq \cdots \geqq u_n \geqq 0, \quad v_1 \geqq v_2 \geqq \cdots \geqq v_n \geqq 0, \quad a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$$

respectively. Let $\boldsymbol{c} = \lambda \boldsymbol{u} + (1-\lambda)\boldsymbol{v}$. By Lemma 5.3 $\boldsymbol{a} \in L$. Thus $\boldsymbol{a} \in L_1$, and since Φ is nondecreasing in each variable we have

$$W(\boldsymbol{A}) = \Phi(\boldsymbol{a}) \leqq \Phi(\boldsymbol{c}) \leqq \lambda\, W(\boldsymbol{F}) + (1-\lambda)\, W(\boldsymbol{G}). \quad \square$$

The geometrical basis for Theorem 5.1 is easily seen by considering the special cases $n=2$ and $n=3$ (see Fig. 2).

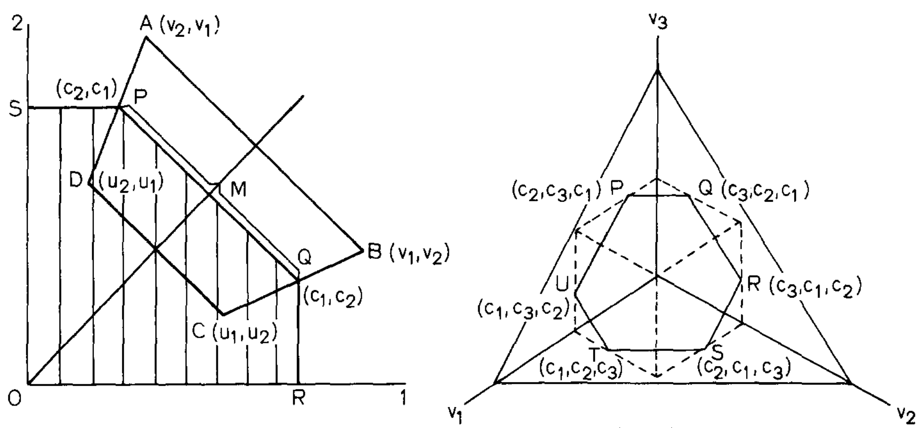


Fig. 2. (i) $n=2$ Fig. 2. (ii) $n=3$

Taking the case $n=2$ (Fig. 2 (i)) first, we see that the proof of Theorem 5.1 (i) shows that $\boldsymbol{a}$ lies on the line segment PQ (note also that the points splitting AC, BD in the ratio $\lambda : 1-\lambda$ lie on PQ) and clearly Φ is less than $\Phi(\boldsymbol{c})$ on this segment by convexity. In the case (ii) $\boldsymbol{a}$ lies in the shaded area L. If $n=3$ (Fig. 2 (ii)) the set M is the hexagonal area enclosed by the points $PQRSTU$ and lying on the plane

$$v_1 + v_2 + v_3 = c_1 + c_2 + c_3,$$

while L is the convex hull of M and its projections onto the coordinate planes, so that it is part of the cube of side c_1 shown.

We now give some sufficient condition for polyconvexity. *Let* $n=2$ *or* 3. To keep things simple we consider stored-energy functions $W(\boldsymbol{F})$ defined on sets of the form $U = \{\boldsymbol{F} \in M^{n\times n} : \det \boldsymbol{F} \in K\}$, where $K \subseteq \mathscr{R}_+$ is convex.

Theorem 5.2. *If* $n=2$ *let*

$$W(\boldsymbol{F})=\psi(v_1, v_2, v_1 v_2), \tag{5.8}$$

where v_1, v_2 *are the singular values of* $\boldsymbol{F}\in U$, *and where* $\psi: \mathscr{R}^2_+ \times K \to \mathscr{R}$ *is convex and satisfies*

(a) $\psi(x_1, x_2, \delta)=\psi(x_2, x_1, \delta)$ *for all* $x_1, x_2 \in \mathscr{R}_+$, $\delta\in K$,

(b) $\psi(x_1, x_2, \delta)$ is nondecreasing in x_1, x_2.

If $n=3$ *let*

$$W(\boldsymbol{F})=\psi(v_1, v_2, v_3, v_2 v_3, v_3 v_1, v_1 v_2, v_1 v_2 v_3), \tag{5.9}$$

where v_1, v_2, v_3 *are the singular values of* $\boldsymbol{F}\in U$, *and where* $\psi: \mathscr{R}^6_+ \times K \to \mathscr{R}$ *is convex and satisfies*

(a) $\psi(P\boldsymbol{x}, \bar{P}\boldsymbol{y}, \delta)=\psi(\boldsymbol{x}, \boldsymbol{y}, \delta)$ *for all* $P, \bar{P}\in\mathscr{P}_3$ *and all* $\boldsymbol{x}, \boldsymbol{y}\in\mathscr{R}^3_+$, $\delta\in K$,

(b) $\psi(x_1, x_2, x_3, y_1, y_2, y_3, \delta)$ *is nondecreasing in each* x_i, y_j.

Then W *is polyconvex on* U.

Proof. We give the proof for $n=3$. Define $G: E_1 \times K \to \mathscr{R}$ by

$$G(\boldsymbol{F}, \boldsymbol{A}, \delta)=\psi(v_1, v_2, v_3, a_1, a_2, a_3, \delta), \tag{5.10}$$

where v_i, a_i are the singular values of $\boldsymbol{F}, \boldsymbol{A}$. G is well-defined by (a). Clearly

$$W(\boldsymbol{F})=G(\boldsymbol{F}, \operatorname{adj}\boldsymbol{F}, \det\boldsymbol{F}) \tag{5.11}$$

for all $\boldsymbol{F}\in U$. It remains to show that G is convex. Let $\boldsymbol{F}, \boldsymbol{H}, \boldsymbol{A}, \boldsymbol{B}\in M^{3\times 3}$, $\delta, \mu\in K$, $\lambda\in[0,1]$. Let $\boldsymbol{F}, \boldsymbol{H}, \boldsymbol{A}, \boldsymbol{B}, \lambda\boldsymbol{F}+(1-\lambda)\boldsymbol{H}, \lambda\boldsymbol{A}+(1-\lambda)\boldsymbol{B}$ have singular values $v_1\geqq v_2\geqq v_3, h_1\geqq h_2\geqq h_3, a_1\geqq a_2\geqq a_3, b_1\geqq b_2\geqq b_3, u_1\geqq u_2\geqq u_3, d_1\geqq d_2\geqq d_3$, respectively. Let $\boldsymbol{w}=\lambda\boldsymbol{v}+(1-\lambda)\boldsymbol{h}$, $\boldsymbol{c}=\lambda\boldsymbol{a}+(1-\lambda)\boldsymbol{b}$. Using Lemma 5.3, Lemma 5.4, (a) and (b) we see that

$$\begin{aligned} G(\lambda\boldsymbol{F}+(1-\lambda)\boldsymbol{H}, \lambda\boldsymbol{A}+(1-\lambda)\boldsymbol{B}, \lambda\delta+(1-\lambda)\mu) &= \psi(\boldsymbol{u}, \boldsymbol{d}, \lambda\delta+(1-\lambda)\mu)\\ &\leqq \psi(\boldsymbol{w}, \boldsymbol{d}, \lambda\delta+(1-\lambda)\mu)\\ &\leqq \psi(\boldsymbol{w}, \boldsymbol{c}, \lambda\delta+(1-\lambda)\mu)\\ &\leqq \lambda\psi(\boldsymbol{v}, \boldsymbol{a}, \delta)+(1-\lambda)\psi(\boldsymbol{h}, \boldsymbol{b}, \mu)\\ &= \lambda G(\boldsymbol{F}, \boldsymbol{A}, \delta)+(1-\lambda)G(\boldsymbol{H}, \boldsymbol{B}, \mu). \end{aligned}$$

□

A special case of a function ψ satisfying the hypotheses of Theorem 5.2 for $n=3$ is

$$\psi(\boldsymbol{v}, \boldsymbol{a}, \delta)=\psi_1(\boldsymbol{v})+\psi_2(\boldsymbol{a})+\psi_3(\delta), \tag{5.12}$$

where the ψ_i are convex, and where ψ_1, ψ_2 are symmetric and nondecreasing in each variable. We use this example in Section 8.

6. Sequential Weak Continuity of Mappings on Orlicz-Sobolev Spaces

Definition 6.1. Let X and Y be real Banach spaces. A map $f: X\to Y$ is *sequentially weakly continuous* if and only if $x_r \rightharpoonup x$ in X implies $f(x_r)\rightharpoonup f(x)$ in Y.

(In general a nonlinear sequentially weakly continuous map $f: X\to Y$ is not continuous with respect to the weak topologies on X and Y (see BALL [1]).)

Sequentially weak $*$ continuous maps are defined analogously. In this section we study the case when X is an Orlicz-Sobolev space and $Y=L^1(\Omega)$. The reader unfamiliar with the theory of Orlicz and Orlicz-Sobolev spaces can replace them everywhere by the corresponding Lebesgue and Sobolev spaces, the results for which are indicated in parentheses in some of the theorems which follow. There will be no great loss of generality in doing so, at least as far as most of the examples in Section 8 are concerned. Some of the results proved here in the framework of Orlicz-Sobolev spaces are proved for ordinary Sobolev spaces in BALL [2].

We first obtain necessary conditions.

Theorem 6.1. (MORREY [1]). *Let m and n be arbitrary. Let $\psi: \mathscr{R}^m \times \mathscr{R}^n \times M^{n\times m} \to \mathscr{R}$ be continuous. Then*

$$I(\boldsymbol{u},\Omega)=\int_\Omega \psi(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x}),\nabla\boldsymbol{u}(\boldsymbol{x}))\,dx$$

is sequentially weak $*$ lower semicontinuous on $W^{1,\infty}(\Omega)$ (i.e., $\boldsymbol{u}_r \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $W^{1,\infty}(\Omega)$ implies $I(\boldsymbol{u},\Omega)\leqq \varliminf_{r\to\infty} I(\boldsymbol{u}_r,\Omega)$) if and only if $\psi(\boldsymbol{x}_0,\boldsymbol{u}_0,\cdot)$ is quasiconvex on $M^{n\times m}$ for each $\boldsymbol{x}_0\in\mathscr{R}^m$, $\boldsymbol{u}_0\in\mathscr{R}^n$.

Corollary 6.1.1. *Let $n=1$, 2 or 3 and let $\phi: M^{n\times n}\to\mathscr{R}$ be continuous. Then the map $\boldsymbol{u}\mapsto J(\boldsymbol{u},\Omega)=\int_\Omega \phi(\nabla\boldsymbol{u}(\boldsymbol{x}))\,dx$ is a sequentially weak $*$ continuous map from $W^{1,\infty}(\Omega)$ to $\mathscr{R}$ if and only if ϕ has the form* (4.2).

Proof. If the given map is sequentially weak $*$ continuous then by the theorem both ϕ and $-\phi$ are quasiconvex, so that by Corollary 4.1.1 ϕ has the form (4.2). Conversely, let ϕ have the form (4.2) and let $\boldsymbol{u}_r \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $W^{1,\infty}(\Omega)$. The sequence $\phi(\nabla\boldsymbol{u}_r(\cdot))$ is bounded in $L^\infty(\Omega)$, so that there exists a subsequence $\boldsymbol{u}_\mu$ of $\boldsymbol{u}_r$ such that $\phi(\nabla\boldsymbol{u}_\mu(\cdot)) \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega)$. Let $\alpha: \mathscr{R}^3\to\mathscr{R}$ be continuous and define $\phi_1(\boldsymbol{x},\boldsymbol{F})=\pm\phi(\boldsymbol{F})\,\alpha(\boldsymbol{x})$ so that ϕ_1 is quasiconvex. By the theorem

$$\int_\Omega \phi(\nabla\boldsymbol{u}_\mu(\boldsymbol{x}))\,\alpha(\boldsymbol{x})\,dx \to \int_\Omega \phi(\nabla\boldsymbol{u}(\boldsymbol{x}))\,\alpha(\boldsymbol{x})\,dx.$$

The arbitrariness of α implies that $\theta=\phi(\nabla\boldsymbol{u}(\cdot))$, and hence $\phi(\nabla\boldsymbol{u}_r(\cdot)) \overset{*}{\rightharpoonup} \phi(\nabla\boldsymbol{u}(\cdot))$ in $L^\infty(\Omega)$. The results follows. □

Corollary 6.1.2. *Let $n=1$, 2 or 3 and let A be an N-function* (cf. *Section* 2). *Let $\phi: M^{n\times n}\to\mathscr{R}$ be continuous and such that $\boldsymbol{u}\mapsto\phi(\nabla\boldsymbol{u}(\cdot))$ is a sequentially continuous map of $W^1L_A(\Omega)$ with the weak $*$ topology into $L^1(\Omega)$ with the weak topology. Then ϕ has the form* (4.2).

Proof. The hypotheses imply that $\boldsymbol{u}\to J(\boldsymbol{u},\Omega)$ is a sequentially weak $*$ continuous map from $W^{1,\infty}(\Omega)\to\mathscr{R}$. □

Remark: MORREY's proof of Theorem 6.1 may be easily adapted to show that if Ω is a bounded open subset of $\mathscr{R}^m$ (m arbitrary) and if $\phi: \mathscr{R}\to\mathscr{R}$, then the map $\theta\to\phi(\theta(\cdot))$ is sequentially weakly continuous from $L^p(\Omega)\to L^1(\Omega)$ if and only if ϕ is affine. For details see BALL [2].

We turn now to sufficient conditions. Our results are based on the following elementary identities for C^2 functions $\boldsymbol{u}$:

$$n=2: \ \det \nabla \boldsymbol{u} = (u^1 u^2{}_{,2})_{,1} - (u^1 u^2{}_{,1})_{,2} \tag{6.1}$$

$$n=3: \ (\operatorname{adj} \nabla \boldsymbol{u})_i^\alpha = (u^{i+2} u^{i+1}{}_{,\alpha+1})_{,\alpha+2} - (u^{i+2} u^{i+1}{}_{,\alpha+2})_{,\alpha+1} \tag{6.2}$$

$$\begin{aligned}\det \nabla \boldsymbol{u} = [u^1 (\operatorname{adj} \nabla \boldsymbol{u})_1^j]_{,j} &= [u^1 (u^2{}_{,2} u^3{}_{,3} - u^2{}_{,3} u^3{}_{,2})]_{,1} \\ &+ [u^1 (u^2{}_{,3} u^3{}_{,1} - u^2{}_{,1} u^3{}_{,3})]_{,2} + [u^1 (u^2{}_{,1} u^3{}_{,2} - u^2{}_{,2} u^3{}_{,1})]_{,3}\end{aligned} \tag{6.3}$$

(In (6.2) there is no implied summation, and the indices are to be taken modulo 3.)

Lemma 6.1.

(i) $n=2$: *If* $\boldsymbol{u} \in W^{1,2}(\Omega)$ *then* $\det \nabla \boldsymbol{u} \in L^1(\Omega)$ *and formula* (6.1) *holds in* $\mathscr{D}'(\Omega)$.

(ii) $n=3$: (a) *If* $\boldsymbol{u} \in W^{1,2}(\Omega)$ *then* $\operatorname{adj} \nabla \boldsymbol{u} \in L^1(\Omega)$ *and formula* (6.2) *holds in* $\mathscr{D}'(\Omega)$.
(b) *Let* A, B *be N-functions with* $A \succ t^2$, $A \succ \bar{B}$. *If* $\boldsymbol{u} \in W^1 E_A(\Omega)$ *and* $\operatorname{adj} \nabla \boldsymbol{u} \in E_B(\Omega)$ (e.g., $\boldsymbol{u} \in W^{1,p}(\Omega)$, $p \geqq 2$, *and* $\operatorname{adj} \nabla \boldsymbol{u} \in L^{p'}(\Omega)$) *then* $\det \nabla \boldsymbol{u} \in L^1(\Omega)$ *and formula* (6.3) *holds in* $\mathscr{D}'(\Omega)$.

Proof. (i) That $\det \nabla \boldsymbol{u} \in L^1(\Omega)$ is obvious. Formula (6.1) holds in $\mathscr{D}'(\Omega)$ if and only if

$$\int_\Omega (\det \nabla \boldsymbol{u})\, \phi \, d\boldsymbol{x} = \int_\Omega (u^1 u^2{}_{,1} \phi_{,2} - u^1 u^2{}_{,2} \phi_{,1}) \, d\boldsymbol{x} \quad \text{for all } \phi \in \mathscr{D}(\Omega). \tag{6.4}$$

But (6.4) holds trivially if $\boldsymbol{u} \in C^\infty(\Omega)$, and $C^\infty(\Omega)$ is dense in $W^{1,2}(\Omega)$ in its norm topology. Since both sides of (6.4) are continuous functions of $\boldsymbol{u} \in W^{1,2}(\Omega)$, (6.4) holds for $\boldsymbol{u} \in W^{1,2}(\Omega)$.

(ii) The proof of (a) is identical to that of (i). Let $\boldsymbol{w}$ be defined by $w^j = (\operatorname{adj} \nabla \boldsymbol{u})_1^j$. To prove (b) we first note that $u^1{}_{,j} \in E_A(\Omega)$, $w^j \in E_{\bar{A}}(\Omega)$ so that $\det \nabla \boldsymbol{u} \in L^1(\Omega)$. We next show that $\operatorname{div} \boldsymbol{w} = 0$ in a weak sense; *i.e.*,

$$\int_\Omega w^j \phi_{,j} \, d\boldsymbol{x} = 0 \quad \text{for all } \phi \in \mathscr{D}(\Omega). \tag{6.5}$$

If $\boldsymbol{u} \in C^\infty(\Omega)$, then $\operatorname{div} \boldsymbol{w} = 0$ and (6.5) holds. Since $C^\infty(\Omega)$ is dense in $W^{1,2}(\Omega)$, (6.5) holds for any $\boldsymbol{u} \in W^1 E_A(\Omega)$.

To show that (6.3) holds in $\mathscr{D}'(\Omega)$ it is thus sufficient to prove that

$$\int_\Omega u^1{}_{,j} w^j \phi \, d\boldsymbol{x} = -\int_\Omega u^1 w^j \phi_{,j} \, d\boldsymbol{x} \quad \text{for all } \phi \in \mathscr{D}(\Omega) \tag{6.6}$$

whenever $\boldsymbol{w} \in E_B(\Omega)$ and satisfies (6.5).

By the results of Donaldson & Trudinger [1, Thm. 22] there exists a sequence $\boldsymbol{u}_{(k)} \in C^\infty(\Omega)$ with $\boldsymbol{u}_{(k)} \to \boldsymbol{u}$ in $W^1 E_A(\Omega)$. Let $\rho \in \mathscr{D}(\mathscr{R}^3)$, $\rho \geqq 0$, $\int_{\mathscr{R}^3} \rho(\boldsymbol{x}) \, d\boldsymbol{x} = 1$, and define $\rho_k \in \mathscr{D}(\mathscr{R}^3)$ by $\rho_k(\boldsymbol{x}) = k \rho(k\boldsymbol{x})$. Extend $\boldsymbol{w}$ by zero outside Ω, so that $\boldsymbol{w} \in E_B(\mathscr{R}^3)$. Donaldson & Trudinger [1, Lemma 2.1] show that the convolutions $\rho_k * \boldsymbol{w}$ are in $E_B(\mathscr{R}^3)$ and $\rho_k * \boldsymbol{w} \to \boldsymbol{w}$ in $E_B(\mathscr{R}^3)$ as $k \to \infty$. Fix $\phi \in \mathscr{D}(\Omega)$. Then if k is large enough, (6.5) implies that

$$\operatorname{div}(\rho_k * \boldsymbol{w})(\boldsymbol{x}) = \int_{\mathscr{R}^3} \rho_{k,j}(\boldsymbol{x} - \boldsymbol{y}) \, w^j(\boldsymbol{y}) \, d\boldsymbol{y} = 0 \tag{6.7}$$

for all $x \in \operatorname{supp}\phi$. Therefore if $S \subseteq \mathscr{R}^3$ is an open ball containing $\operatorname{supp}\phi$, then

$$\int_\Omega u^1_{(k),j}(\rho_k * w^j)\phi\,dx = \int_S \operatorname{div}(u^1_{(k)}(\rho_k * \boldsymbol{w})\phi)\,dx - \int_\Omega u^1_{(k)}(\rho_k * w^j)\phi_{,j}\,dx$$
$$= -\int_\Omega u^1_{(k)}(\rho_k * w^j)\phi_{,j}\,dx.$$

Since $\bar{A} \prec B$ we obtain (6.6) by letting $k \to \infty$. $\square$

Remark. Note that $A \succ t^2$ if and only if $A \succ \bar{A}$. This may be proved directly from the definition of $\bar{A}$.

The functions $\det \nabla \boldsymbol{u}$ $(n=2)$, $\operatorname{adj} \nabla \boldsymbol{u}$ and $\det \nabla \boldsymbol{u}$ $(n=3)$ can be given a meaning as distributions under weaker conditions than those of Lemma 6.1. We thus *define* the distributions $\operatorname{Det} \nabla \boldsymbol{u}$ $(n=2)$, $\operatorname{Adj} \nabla \boldsymbol{u}$ and $\operatorname{Det} \nabla \boldsymbol{u}$ $(n=3)$ by

$$n=2: \quad \operatorname{Det} \nabla \boldsymbol{u} = (u^1 u^2_{,2})_{,2} - (u^1 u^2_{,1})_{,2}, \tag{6.8}$$

$$n=3: \quad (\operatorname{Adj} \nabla \boldsymbol{u})^\alpha_i = (u^{i+2} u^{i+1}_{,\alpha+1})_{,\alpha+2} - (u^{i+2} u^{i+1}_{,\alpha+2})_{,\alpha+1}, \tag{6.9}$$

$$\operatorname{Det} \nabla \boldsymbol{u} = [u^1 (\operatorname{Adj} \nabla \boldsymbol{u})^j_1]_{,j}, \tag{6.10}$$

when these distributions are meaningful. Obviously if $\boldsymbol{u}$ satisfies the hypotheses of Lemma 6.1 then these distributions may be identified with the $L^1(\Omega)$ functions $\det \nabla \boldsymbol{u}$ $(n=2)$, $\operatorname{adj} \nabla \boldsymbol{u}$ and $\det \nabla \boldsymbol{u}$ $(n=3)$ respectively.

Let A be an N-function. Following DONALDSON & TRUDINGER [1], we let

$$g_A(t) = \frac{A^{-1}(t)}{t^{1+1/n}}, \qquad t \geqq 0, \tag{6.11}$$

where A^{-1} denotes the inverse function to A on $[0, \infty)$. If A satisfies

$$\int_0^1 g_A(t)\,dt < \infty, \qquad \int_1^\infty g_A(t)\,dt = \infty, \tag{6.12}$$

then we define the N-function A^* by

$$(A^*)^{-1}(|t|) = \int_0^{|t|} g_A(s)\,ds. \tag{6.13}$$

Note that for $A(t) = |t|^p$, $1 < p < n$, we have

$$g_A(t) = t^{\frac{1}{p} - \frac{1}{n} - 1}, \qquad A^*(t) = \frac{|t|^{p'}}{p'}. \tag{6.14}$$

Lemma 6.2. *Let Ω satisfy the cone condition.*

(i) $n=2$: *Let A be an N-function satisfying either $\int_1^\infty g_A(t)\,dt < \infty$, or both* (6.12) *and $\bar{A} \prec A^*$. If $\boldsymbol{u} \in W^1 L_A(\Omega)$* (e.g., *if $\boldsymbol{u} \in W^{1,\frac{4}{3}}(\Omega)$*), *then $u^1 u^2_{,2}$ and $u^1 u^2_{,1}$ belong to $L^1(\Omega)$, so that* $\operatorname{Det} \nabla \boldsymbol{u}$ *exists as an element of $\mathscr{D}'(\Omega)$.*

(ii) $n=3$: (a) *Let A be as in* (i). *If $\boldsymbol{u} \in W^1 L_A(\Omega)$* (e.g., *if $\boldsymbol{u} \in W^{1,\frac{3}{2}}(\Omega)$*), *then $u^{i+2} u^{i+1}_{,\alpha+1}$* and *$u^{i+2} u^{i+1}_{,\alpha+2}$ belong to $L^1(\Omega)$ so that* $(\operatorname{Adj} \nabla \boldsymbol{u})^\alpha_i$ *exists as an element of $\mathscr{D}'(\Omega)$.*

(b) *Let A, B be N-functions with either A satisfying $\int_1^\infty g_A(t)\,dt<\infty$, or with A satisfying* (6.12) *and with $\bar{B}\prec A^*$. If $\boldsymbol{u}\in W^1 L_A(\Omega)$ and* $\operatorname{Adj} \nabla \boldsymbol{u}\in L_B(\Omega)$ (e.g., *if $\boldsymbol{u}\in W^{1,p}(\Omega)$*, $\operatorname{Adj} \nabla \boldsymbol{u}\in L^q(\Omega)$ *with $p>1$, $q>1$ and* $\dfrac{1}{p}+\dfrac{1}{q}\leqq\frac{4}{3}$), *then $u^1(\operatorname{Adj} \nabla \boldsymbol{u})^j_1\in L^1(\Omega)$, so that* $\operatorname{Det}\nabla\boldsymbol{u}$ *exists as an element of $\mathscr{D}'(\Omega)$.*

Proof. (i) If $\int_1^\infty g_A(t)\,dt<\infty$ then the imbedding theorem of DONALDSON & TRUDINGER [1, Thm. 3.2] implies that $u^1\in L^\infty(\Omega)$, while if A satisfies (6.12) the same theorem implies that $u^1\in L_{A^*}(\Omega)$. The result follows by Young's inequality.

(ii) This is proved similarly. □

Remark. If Ω is an arbitrary bounded open set then Lemma 6.2 holds with $L^1(\Omega)$ replaced by $L^1_{\mathrm{loc}}(\Omega)$.

The main result of this section is the following:

Theorem 6.2.

(i) $n=2$: *Let A be an N-function satisfying either $\int_1^\infty g_A(t)\,dt<\infty$, or both* (6.12) *and $\bar{A}\ll A^*$. If $\boldsymbol{u}_r \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $W^1 L_A(\Omega)$* (e.g., *if $\boldsymbol{u}_r\rightharpoonup\boldsymbol{u}$ in $W^{1,p}(\Omega)$, $p>\frac{4}{3}$*), *then* $\operatorname{Det}\nabla\boldsymbol{u}_r\to\operatorname{Det}\nabla\boldsymbol{u}$ *in $\mathscr{D}'(\Omega)$.*

(ii) $n=3$: (a) *Let A be as in* (i). *If $\boldsymbol{u}_r \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $W^1 L_A(\Omega)$* (e.g., *if $\boldsymbol{u}_r\rightharpoonup\boldsymbol{u}$ in $W^{1,p}(\Omega)$*, $p>\frac{3}{2}$) *then* $(\operatorname{Adj}\nabla\boldsymbol{u}_r)^\alpha_i\to(\operatorname{Adj}\nabla\boldsymbol{u})^\alpha_i$ *in $\mathscr{D}'(\Omega)$.*

(b) *Let A, B be N-functions with either A satisfying $\int_1^\infty g_A(t)\,dt<\infty$, or with A satisfying* (6.12) *and with $\bar{B}\ll A^*$. If $\boldsymbol{u}_r \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $W^1 L_A(\Omega)$ and* $\operatorname{Adj}\nabla\boldsymbol{u}_r \overset{*}{\rightharpoonup} \operatorname{Adj}\nabla\boldsymbol{u}$ *in $L_B(\Omega)$* (e.g., *if $\boldsymbol{u}_r\rightharpoonup\boldsymbol{u}$ in $W^{1,p}(\Omega)$*, $\operatorname{Adj}\nabla\boldsymbol{u}_r\rightharpoonup\operatorname{Adj}\nabla\boldsymbol{u}$ *in $L^q(\Omega)$ with $p>1, q>1$ and* $\dfrac{1}{p}+\dfrac{1}{q}<\frac{4}{3}$), *then* $\operatorname{Det}\nabla\boldsymbol{u}_r\to\operatorname{Det}\nabla\boldsymbol{u}$ *in $\mathscr{D}'(\Omega)$.*

Proof. (i) Fix $\phi\in\mathscr{D}(\Omega)$ and let Ω' be an open set with $\Omega\supset\Omega'\supset\operatorname{supp}\phi$ and such that the imbedding theorems of DONALDSON & TRUDINGER hold for Ω' #. Then since $\|\boldsymbol{u}_r\|_{W^1 L_A(\Omega')}$ is bounded and $\boldsymbol{u}_r\to\boldsymbol{u}$ in $L^1(\Omega')$, it follows (see KRASNOSEL'SKII & RUTICKII [1, p. 132]) that $\boldsymbol{u}_r\to\boldsymbol{u}$ in $L^\infty(\Omega')$ or $L_{\bar{A}}(\Omega')$. Therefore the Hölder inequality and the boundedness of $\|\boldsymbol{u}_r\|_{W^1 L_A(\Omega')}$ imply that

$$\int_\Omega (u^1_r u^2_{r,2}-u^1 u^2_{,2})\,\phi\,d\boldsymbol{x}=\int_{\Omega'}(u^1_r-u^1)\,u^2_{r,2}\,\phi\,d\boldsymbol{x}+\int_{\Omega'}u^1(u^2_{r,2}-u^2_{,2})\,\phi\,d\boldsymbol{x}$$

tends to zero as $r\to\infty$. Hence $u^1_r u^2_{r,2}\to u^1 u^2_{,2}$ in $\mathscr{D}'(\Omega)$. Similarly $u^1_r u^2_{r,1}\to u^1 u^2_{,1}$ in $\mathscr{D}'(\Omega)$. The result follows. The proof of (ii) is similar. □

Corollary 6.2.1.

(i) $n=2$: *Let A be as in Theorem* 6.2(i) *and let $\boldsymbol{u}\in W^1 L_A(\Omega)$. Then*

$$\operatorname{Det}\nabla\boldsymbol{u}=(u^2 u^1_{,1})_{,2}-(u^2 u^1_{,2})_{,1}\quad \text{in } \mathscr{D}'(\Omega). \tag{6.15}$$

We may take for Ω' a finite subcover by open balls of the closure of supp ϕ.

(ii) $n=3$: (a) *Let A be as in Theorem* 6.2(ii a) *and let* $\boldsymbol{u}\in W^1L_A(\Omega)$. *Then*

$$(\operatorname{Adj}\nabla\boldsymbol{u})_i^\alpha=(u^{i+1}u^{i+2}{}_{,\alpha+2})_{,\alpha+1}-(u^{i+1}u^{i+2}{}_{,\alpha+1})_{,\alpha+2}\quad in\ \mathscr{D}'(\Omega). \tag{6.16}$$

(b) *Let A,B be as in Theorem* 6.2(ii b) *and let* $\boldsymbol{u}\in W^1L_A(\Omega)$, $\operatorname{Adj}\nabla\boldsymbol{u}\in L_B(\Omega)$. *Then*

$$\operatorname{Det}\nabla\boldsymbol{u}=[u^2(\operatorname{Adj}\nabla\boldsymbol{u})_2^j]_{,j}=[u^3(\operatorname{Adj}\nabla\boldsymbol{u})_3^j]_{,j}\quad in\ \mathscr{D}'(\Omega). \tag{6.17}$$

Proof. (i) Let $\phi\in\mathscr{D}(\Omega)$ and let Ω' be an open set with $\Omega\supset\Omega'\supset\operatorname{supp}\phi$ and satisfying the segment property. Then by the results of GOSSEZ [1, Thm. 1.3] there exists a sequence $\boldsymbol{u}_r\in C^\infty(\bar{\Omega}')$ with $\boldsymbol{u}_r\overset{*}{\rightharpoonup}\boldsymbol{u}$ in $W^1L_A(\Omega')$. Clearly $(\operatorname{Det}\nabla\boldsymbol{u}_r)(\phi)=[(u_r^2u_{r,1}^1)_{,2}-(u_r^2u_{r,2}^1)_{,1}](\phi)$. Letting $r\to\infty$ we obtain from the theorem that $(\operatorname{Det}\nabla\boldsymbol{u})(\phi)=[(u^2u^1_{,1})_{,2}-(u^2u^1_{,2})_{,1}](\phi)$, and the result follows.

(ii) The proof of (a) is identical to that of (i). The proof of (b) is similar to that of Lemma 6.1 (ii b), the principal change being the use of Lemma 1.6 of GOSSEZ [1] to show that if $w\in L_B(\Omega)$, then $\rho_k*w\overset{*}{\rightharpoonup}w$ in $L_B(\Omega)$. We omit the details. □

Corollary 6.2.2.#

(i) $n=2$: *The map* $\boldsymbol{u}\mapsto\det\nabla\boldsymbol{u}$: $W^{1,p}(\Omega)\to L^{p/2}(\Omega)$ *is sequentially weakly continuous if* $p>2$.

(ii) $n=3$: (a) *The map* $\boldsymbol{u}\mapsto\operatorname{adj}\nabla\boldsymbol{u}$: $W^{1,p}(\Omega)\to L^{p/2}(\Omega)$ *is sequentially weakly continuous if* $p>2$.

(b) *The map* $\boldsymbol{u}\mapsto\det\nabla\boldsymbol{u}$: $W^{1,p}(\Omega)\to L^{p/3}(\Omega)$ *is sequentially weakly continuous if* $p>3$.

Proof. We just prove (ii b). Let $p>3$. It is clear from the Hölder inequality that if $\boldsymbol{u}\in W^{1,p}(\Omega)$, then $\det\nabla\boldsymbol{u}\in L^{p/3}(\Omega)$. Let $\boldsymbol{u}_r\rightharpoonup\boldsymbol{u}$ in $W^{1,p}(\Omega)$. Then $\operatorname{adj}\nabla\boldsymbol{u}_r$ is bounded in the reflexive space $L^{p/2}(\Omega)$, and hence by the theorem (part (ii a)) $\operatorname{adj}\nabla\boldsymbol{u}_r\rightharpoonup\operatorname{adj}\nabla\boldsymbol{u}$ in $L^{p/2}(\Omega)$. But $\det\nabla\boldsymbol{u}_r$ is bounded in the reflexive space $L^{p/3}(\Omega)$ and thus by part (ii b) of the theorem $\det\nabla\boldsymbol{u}_r\rightharpoonup\det\nabla\boldsymbol{u}$ in $L^{p/3}(\Omega)$. □

Warning. The distributions $\det\nabla\boldsymbol{u}$ ($\operatorname{adj}\nabla\boldsymbol{u}$) and $\operatorname{Det}\nabla\boldsymbol{u}$ ($\operatorname{Adj}\nabla\boldsymbol{u}$) need not be the same even if the former is a continuous function, as the following example shows.

Example 6.1, $n=2$ or 3.

Let $r=|\boldsymbol{x}|$ and let $R(r)$ be a smooth real-valued function on $(0,1]$.
Let $\Omega=\{|\boldsymbol{x}|<1\}$ and define $\boldsymbol{u}:\Omega\to\mathscr{R}^n$ by

$$\boldsymbol{u}(\boldsymbol{x})=\frac{R(r)}{r}\boldsymbol{x},\qquad r>0,\ \boldsymbol{u}(\boldsymbol{0})\ \text{arbitrary}. \tag{6.18}$$

Then for $r>0$ we have

$$u^i_{,\alpha}=\frac{R}{r}\delta^i_\alpha+\frac{rR'-R}{r^3}\delta_{\alpha\beta}x^ix^\beta \tag{6.19}$$

and

$$\det\nabla\boldsymbol{u}=\frac{R^{n-1}R'}{r^{n-1}}. \tag{6.20}$$

Note added in proof. This corollary follows from results of Y. G. RESHETNYAK [1, Thm. 4], [2, Thm. 2]. Theorem 4 in BALL [2] is also essentially a consequence of RESHETNYAK's work, to which I would have referred had I seen it in time.

Let $r \in (0,1)$, $B_r = \{|\boldsymbol{x}| < r\}$ and let $\phi \in \mathscr{D}(\Omega)$. Then

$$\int_{\Omega \setminus B_r} \phi u^i_{,\alpha} \, d\boldsymbol{x} = - \int_{\partial B_r} \phi u^i n_\alpha \, dS - \int_{\Omega \setminus B_r} \phi_{,\alpha} u^i \, d\boldsymbol{x}. \tag{6.21}$$

But

$$\int_{\partial B_r} \phi u^i n_\alpha \, dS = \int_{\partial \Omega} \phi(r\boldsymbol{x}) \, u^i(r\boldsymbol{x}) \, n_\alpha r^{n-1} \, dS.$$

Provided $r^{n-1} R(r) \to 0$ as $r \to 0$ and $\boldsymbol{u}, \nabla \boldsymbol{u} \in L^1(\Omega)$ we therefore obtain from (6.21) that

$$\int_\Omega \phi u^i_{,\alpha} \, d\boldsymbol{x} = - \int_\Omega \phi_{,\alpha} u^i \, d\boldsymbol{x},$$

where we have used the dominated convergence theorem. Thus under these conditions $\nabla \boldsymbol{u}$, as defined by (6.19), is the matrix of weak derivatives of $\boldsymbol{u}$.

In particular, letting $R(r) = 1 + r$, we find that $\boldsymbol{u} \in W^{1,p}(\Omega)$ for any $p < n$. Now let $n = 2$. Let $\psi \in C^\infty([0,1])$ take the values 1 and 0 in neighbourhoods of $r = 0$ and $r = 1$ respectively. Then $\phi(\boldsymbol{x}) \stackrel{\text{def}}{=} \psi(|\boldsymbol{x}|)$ belongs to $\mathscr{D}(\Omega)$, and

$$\int_\Omega (\det \nabla \boldsymbol{u}) \phi \, d\boldsymbol{x} = 2\pi \int_0^1 (1+r) \, \psi(r) \, dr,$$

$$\int_\Omega (u^1 u^2_{,1} \, \phi_{,2} - u^1 u^2_{,2} \, \phi_{,1}) \, d\boldsymbol{x} = -\pi \int_0^1 (1+r)^2 \, \psi'(r) \, dr.$$

Hence formula (6.1) does not hold in this case, so that $\det \nabla \boldsymbol{u} \neq \operatorname{Det} \nabla \boldsymbol{u}$. Note also that if $p < 2$ there is no sequence of $C^\infty(\bar{\Omega})$ functions $\boldsymbol{u}_r$ such that $\boldsymbol{u}_r \rightharpoonup \boldsymbol{u}$ in $W^{1,p}(\Omega)$ and $\det \nabla \boldsymbol{u}_r \rightharpoonup \det \nabla \boldsymbol{u}$ in $L^1(\Omega)$, since such a sequence would satisfy

$$\int_\Omega \det \nabla \boldsymbol{u}_r \, d\boldsymbol{x} = 4\pi,$$

whereas

$$\int_\Omega \det \nabla \boldsymbol{u} \, d\boldsymbol{x} = 3\pi.$$

When $n = 3$, a similar calculation shows that $\boldsymbol{u} \in W^{1,p}(\Omega)$ for $p < 3$, $\operatorname{adj} \nabla \boldsymbol{u} \in L^q(\Omega)$ for $q < \frac{3}{2}$, but that (6.17) does not hold.

In the above example $\operatorname{Det} \nabla \boldsymbol{u}$ has an atom at $\boldsymbol{x} = 0$. I do not known whether $\det \nabla \boldsymbol{u} = \operatorname{Det} \nabla \boldsymbol{u}$ if $\operatorname{Det} \nabla \boldsymbol{u}$ is a *function*.

7. Existence Theorems

We have already stated the result of MORREY, Theorem 6.1, which shows that for a continuous integrand quasiconvexity is necessary and sufficient for sequential weak $*$ lower semicontinuity in $W^{1,\infty}(\Omega)$. MORREY [1, 2] has also given sufficient conditions for sequential weak lower semicontinuity in $W^{1,s}(\Omega)$, $s \geqq 1$. For purposes of comparison, and for future use, we given an extension of his results due to MEYERS [1].

Theorem 7.1. *Let $f: \Omega \times \mathscr{R}^n \times M^{n \times n} \to \mathscr{R}$ be continuous, and let $f(\boldsymbol{x}, \boldsymbol{u}, \cdot)$ be quasiconvex for all $\boldsymbol{x} \in \Omega$, $\boldsymbol{u} \in \mathscr{R}^n$. Suppose there exist real constants $K_i > 0$ $(i = 1, 2)$, $s \geqq 1$, $0 < \gamma \leqq 1$ and a function $b \in L^1(\Omega)$ such that*

(i) $f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \geqq b(\boldsymbol{x})$,

(ii) $|f(\boldsymbol{x}, \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{F}+\boldsymbol{H})-f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})| \leqq K_1\{1+(|\boldsymbol{u}|+|\boldsymbol{v}|+|\boldsymbol{F}|+|\boldsymbol{H}|)^{s-\gamma}\}(|\boldsymbol{v}|+|\boldsymbol{H}|)^{\gamma}$,

(iii) $|f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{F})-f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})| \leqq K_2\{1+|\boldsymbol{F}|^s\}\, \eta(|\boldsymbol{y}|)$,

for all values of the various arguments, where $\eta: \mathscr{R}_+ \to \mathscr{R}_+$ *is a continuous increasing function with* $\eta(0)=0$. *(Here and elsewhere* $|\boldsymbol{F}|$ *denotes any fixed norm on* $\boldsymbol{F} \in M^{n \times n}$.*) Then*

$$I(\boldsymbol{u}, \Omega)=\int_{\Omega} f(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\, d\boldsymbol{x}$$

is sequentially weakly lower semicontinuous on $W^{1,s}(\Omega)$.

Remarks.

1. Let $\boldsymbol{x}_0 \in \Omega$. Then conditions (ii) and (iii) imply in particular that

$$|f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})-f(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0})| \leqq K_1[1+(|\boldsymbol{u}|+|\boldsymbol{F}|)^{s-\gamma}]\{|\boldsymbol{u}|+|\boldsymbol{F}|\}^{\gamma} \tag{7.1}$$

and

$$|f(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0})-f(\boldsymbol{x}_0, \boldsymbol{0}, \boldsymbol{0})| \leqq K_2\, \eta(|\boldsymbol{x}-\boldsymbol{x}_0|). \tag{7.2}$$

Combining (7.1) and (7.2) we see that

(i)′ $|f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})| \leqq K_0\{1+(|\boldsymbol{y}|+|\boldsymbol{F}|)^s\}$,

for all $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})$, where $K_0>0$ is a constant. Conditions (i)′, (ii) and (iii) are MEYERS' continuity and growth conditions for the function $f-b$, while (i) implies a further hypothesis of his theorem.

2. If $s \leqq n$ then the growth conditions with respect to $\boldsymbol{u}$ may be weakened by use of the Sobolev imbedding theorems; the reader is referred to MEYERS [1] for details. An extension to an Orlicz-Sobolev space setting could also probably be made.

In order to prove existence theorems by use of Theorem 7.1, it is necessary to make, in addition to conditions (i)–(iii), a coercivity assumption on f. Typically we might assume that $s>1$ and

(iv) $f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \geqq K_3|\boldsymbol{F}|^s+b(\boldsymbol{x})$, where $K_3>0$.

The conditions (i)–(iv) are extremely restrictive with regard to applications to nonlinear elasticity. Firstly, (ii) precludes any singular behaviour of f (for example (0.5)). Secondly (i)′ and (iv) together rule out integrands typified by the example

$$f(\boldsymbol{F})=|\boldsymbol{F}|^s+|\det \boldsymbol{F}|^r \tag{7.3}$$

with $nr>s$; we shall see that many such integrands belong to a physically interesting class included in our existence theorems.

Definition 7.1. Let $D \subseteq \mathscr{R}^k$ be open. A map $G_1: D \times \mathscr{R}^{\nu} \to \bar{\mathscr{R}}$ is said to be of *Carathéodory* type if

(a) for almost all $\boldsymbol{x} \in D$, $G_1(\boldsymbol{x}, \cdot)$ is continuous on $\mathscr{R}^{\nu}$, and

(b) for all $\boldsymbol{a} \in \mathscr{R}^{\nu}$, $G_1(\cdot, \boldsymbol{a})$ is measurable on D.

We shall use the following lower semicontinuity theorem, which is a special case of a result given by EKELAND & TÉMAM [1, Thm. 2.1, p. 226]. For related results see CESARI [1, 2, 3].

Theorem 7.2. *Let $G_1: \Omega \times (\mathscr{R}^n \times \mathscr{R}^\sigma) \to \bar{\mathscr{R}}$ be of Carathéodory type (Definition 7.1 with $v = n + \sigma$) and satisfy*

$$G_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) \geqq \Phi(|\boldsymbol{a}|) \tag{7.4}$$

for some N-function Φ. Suppose that $G_1(\boldsymbol{x}, \boldsymbol{u}, \cdot)$ is convex on $\mathscr{R}^\sigma$ for all $\boldsymbol{x} \in \Omega$, $\boldsymbol{u} \in \mathscr{R}^n$. Let $\boldsymbol{a}_r \rightharpoonup \boldsymbol{a}$ in $L^1(\Omega)$ and let $\{\boldsymbol{u}_r\}$ be a sequence of measurable functions with $\boldsymbol{u}_r \to \boldsymbol{u}$ almost everywhere in Ω. Then

$$\int_\Omega G_1(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \boldsymbol{a}(\boldsymbol{x}))\, d\boldsymbol{x} \leqq \varliminf_{r \to \infty} \int_\Omega G_1(\boldsymbol{x}, \boldsymbol{u}_r(\boldsymbol{x}), \boldsymbol{a}_r(\boldsymbol{x}))\, d\boldsymbol{x}. \tag{7.5}$$

We now describe the main ingredients of our existence theory, starting first with those relevant for compressible materials.

For each $\boldsymbol{x} \in \Omega$ let $W(\boldsymbol{x})$ be a nonempty convex open subset of $E = \mathscr{R}^{s(n)}$, such that for each $\boldsymbol{a} \in E$ the set $W^{-1}(\boldsymbol{a}) \overset{\text{def}}{=} \{\boldsymbol{x} \in \Omega: \boldsymbol{a} \in W(\boldsymbol{x})\}$ is measurable. We impose as a *local constraint* on our variational problem that, in a sense to be made precise later, $T(\nabla \boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x})$ almost everywhere in Ω, where we are using the notation of Section 4. In applications to nonlinear elasticity $W(\boldsymbol{x})$ will often have the form

$$W(\boldsymbol{x}) = \{\boldsymbol{a} \in E: c_1(\boldsymbol{x}, \boldsymbol{a}) < \kappa_1(\boldsymbol{x}), \kappa_2(\boldsymbol{x}) < -c_2(\boldsymbol{x}, \boldsymbol{a})\}, \tag{7.6}$$

where $c_1, c_2: \mathscr{R} \times E \to \mathscr{R}$ are of Carathéodory type and convex with respect to $\boldsymbol{a} \in E$, and where $\kappa_1, \kappa_2: \Omega \to \bar{\mathscr{R}}$ are measurable. Examples of relevant choices of c_i, κ_i $(i = 1, 2)$ are

1. c_i is arbitrary, $\kappa_1 \equiv +\infty$, $\kappa_2 \equiv -\infty$ (no constraint).

2. $(n=3) - c_2(\boldsymbol{x}, \boldsymbol{F}, \boldsymbol{A}, \delta) = \delta$, $\kappa_2 \equiv 0$ (corresponding to the continuity condition $\det \nabla \boldsymbol{u}(\boldsymbol{x}) > 0$ and an additional unilateral constraint, absent if $\kappa_1 = +\infty$, on the measure of strain $c_1(\boldsymbol{x}, \nabla \boldsymbol{u}, \operatorname{adj} \nabla \boldsymbol{u}, \det \nabla \boldsymbol{u})$ *).

We now make continuity, growth and polyconvexity hypotheses on the integrand. Because of the nature of the growth conditions, and because we wish to consider situations in which the distributions $\operatorname{adj} \nabla \boldsymbol{u}$ and $\operatorname{Adj} \nabla \boldsymbol{u}$, $\det \nabla \boldsymbol{u}$ and $\operatorname{Det} \nabla \boldsymbol{u}$ may be different, we make these hypotheses on the associated function $G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a})$ (*cf.* (4.3)).

Let $\mathscr{S} = \{(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) \in \Omega \times \mathscr{R}^n \times E: \boldsymbol{a} \in W(\boldsymbol{x})\}$. Let $G: \mathscr{S} \to \mathscr{R}$ be such that **

(H_1) *** G is continuous with respect to $\boldsymbol{u}, \boldsymbol{a}$ on $\mathscr{S}$,

(H_2) for all $\boldsymbol{u} \in \mathscr{R}^n$, $\boldsymbol{a} \in E$, $G(\cdot, \boldsymbol{u}, \boldsymbol{a})$ is measurable on $W^{-1}(\boldsymbol{a})$,

(H_3) for almost all $\boldsymbol{x} \in \Omega$, $G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) \to +\infty$ as $\boldsymbol{a} \to \partial W(\boldsymbol{x})$, the convergence being uniform with respect to $\boldsymbol{u}$ in any bounded subset of $\mathscr{R}^n$,

(H_4) (Polyconvexity) for each $\boldsymbol{x} \in \Omega$, $\boldsymbol{u} \in \mathscr{R}^n$, $G(\boldsymbol{x}, \boldsymbol{u}, \cdot)$ is convex on $W(\boldsymbol{x})$,

(H_5) (Coercivity)

* Such constraints are by no means unrealistic. A unilateral constraint at large strains might be relevant, for example, for a mixture of elastic materials with one or more constituents possessing limited extensibility (*cf.* NIEDERER [1]). See also the comments in Section 10.

** For simplicity we suppose that G is defined for all $\boldsymbol{u} \in \mathscr{R}^n$.

*** Weaker conditions are possible (*cf.* EKELAND & TÉMAM [1]).

$n=1$: there exists an N-function A, and a function $b \in L^1(\Omega)$ such that

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \geqq b(\boldsymbol{x}) + A(\boldsymbol{F}) \quad \text{for all } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \in \mathscr{S}; \tag{7.7}$$

$n=2$: there exist N-functions A, B, with A satisfying either $\int_1^\infty g_A(t)\,dt < \infty$ or both (6.12) and $\bar{A} \ll A^*$, and a function $b \in L^1(\Omega)$, such that

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \delta) \geqq b(\boldsymbol{x}) + A(|\boldsymbol{F}|) + B(\delta) \quad \text{for all } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \delta) \in \mathscr{S}; \tag{7.8}$$

$n=3$: there exist N-functions A, B, C satisfying either $\int_1^\infty g_A(t)\,dt < \infty$ or the conditions (6.12), $\bar{A} \ll A^*$ and $\bar{B} \ll A^*$, and a function $b \in L^1(\Omega)$, such that

$$\begin{aligned} &G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}, \delta) \\ &\quad \geqq b(\boldsymbol{x}) + A(|\boldsymbol{F}|) + B(|\boldsymbol{H}|) + C(\delta) \quad \text{for all } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}, \delta) \in \mathscr{S}. \end{aligned} \tag{7.9}$$

If we define $G_1 : \Omega \times (\mathscr{R}^n \times E) \to \bar{\mathscr{R}}$ by

$$\begin{aligned} G_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) &= G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) - b(\boldsymbol{x}) && \text{if } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) \in \mathscr{S} \\ &= +\infty && \text{otherwise,} \end{aligned}$$

then clearly G_1 is of Carathéodory type and satisfies (7.4) for some N-function Φ.

We define the *admissibility set* $\mathscr{A}$ by

$n=1$: $\mathscr{A} = \{\boldsymbol{u} \in W^1 L_A(\Omega) : \nabla \boldsymbol{u}(\boldsymbol{x}) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega\}$,

$n=2$: $\mathscr{A} = \{\boldsymbol{u} \in W^1 L_A(\Omega) : \operatorname{Det} \nabla \boldsymbol{u} \in L_B(\Omega), (\nabla \boldsymbol{u}(x), \operatorname{Det} \nabla \boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega\}$,

$n=3$: $\mathscr{A} = \{\boldsymbol{u} \in W^1 L_A(\Omega) : \operatorname{Adj} \nabla \boldsymbol{u} \in L_B(\Omega),\ \operatorname{Det} \nabla \boldsymbol{u} \in L_C(\Omega),\ (\nabla \boldsymbol{u}(\boldsymbol{x}),\ (\operatorname{Adj} \nabla \boldsymbol{u})(\boldsymbol{x}), \operatorname{Det} \nabla \boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega\}$.

The equivalence classes in $\mathscr{A}$ under the equivalence relation

$$\boldsymbol{u} \sim \boldsymbol{v} \quad \text{if and only if} \quad \boldsymbol{u} - \boldsymbol{v} \in W_0^1 L_A(\Omega)$$

are termed the *Dirichlet classes in* $\mathscr{A}$.

If $\boldsymbol{u} \in \mathscr{A}$ then it follows from results of EKELAND & TÉMAM [1, Prop. 1.1, p. 218] that $J(\boldsymbol{u})$ exists and is finite or $+\infty$, where

$$\begin{aligned} J(\boldsymbol{u}) &\overset{\text{def}}{=} \int_\Omega G(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\,d\boldsymbol{x} && \text{if } n=1, \\ &\overset{\text{def}}{=} \int_\Omega G(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Det} \nabla \boldsymbol{u}(\boldsymbol{x}))\,d\boldsymbol{x} && \text{if } n=2, \\ &\overset{\text{def}}{=} \int_\Omega G(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Det} \nabla \boldsymbol{u}(\boldsymbol{x}))\,d\boldsymbol{x} && \text{if } n=3. \end{aligned}$$

We are now in a position to present our first existence theorem, which includes as a special case the displacement boundary-value problem of nonlinear hyperelasticity. For $n=1$, of course, the result is well known.

Theorem 7.3. *Let G satisfy (H_1)–(H_5) above. Let $\mathscr{C}$ be a Dirichlet class in $\mathscr{A}$, and suppose that there exists $\boldsymbol{u}_1 \in \mathscr{C}$ with $J(\boldsymbol{u}_1) < \infty$. Then there exists $\boldsymbol{u}_0 \in \mathscr{C}$ that minimizes $J(\boldsymbol{u})$ in $\mathscr{C}$.*

Proof. We just give the proof for $n=3$, the other cases being easier. It is sufficient to establish the existence of a minimizer in $\mathscr{C}$ for

$$\bar{J}(\boldsymbol{u}) \stackrel{\text{def}}{=} \int_\Omega G_1(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Det} \nabla \boldsymbol{u}(\boldsymbol{x}))\, d\boldsymbol{x}. \tag{7.10}$$

$\bar{J}(\boldsymbol{u})$ is bounded below on $\mathscr{C}$. Let $\boldsymbol{u}_r$ be a minimizing sequence from $\mathscr{C}$. By (7.9) the quantities

$$\int_\Omega A(|\nabla \boldsymbol{u}_r(\boldsymbol{x})|)\, d\boldsymbol{x}, \qquad \int_\Omega B(|\operatorname{Adj} \nabla \boldsymbol{u}_r(\boldsymbol{x})|)\, d\boldsymbol{x}, \qquad \int_\Omega C(\operatorname{Det} \nabla \boldsymbol{u}_r(\boldsymbol{x}))\, d\boldsymbol{x}$$

are bounded independently of r. The Poincaré inequality for $W_0^1 L_A(\Omega)$ (Gossez [1, p. 202]) implies (*cf.* Krasnosel'skii & Ruticki [1, p. 131], Ekeland & Témam [1, p. 223]) that for a subsequence $\{\boldsymbol{u}_j\}$ we have

$$\boldsymbol{u}_j \stackrel{*}{\rightharpoonup} \boldsymbol{u}_0 \quad \text{in } W^1 L_A(\Omega), \qquad \nabla \boldsymbol{u}_j \rightharpoonup \nabla \boldsymbol{u}_0 \quad \text{in } L^1(\Omega),$$
$$\boldsymbol{u}_j \to \boldsymbol{u}_0 \quad \text{almost everywhere},$$
$$\operatorname{Adj} \nabla \boldsymbol{u}_j \stackrel{*}{\rightharpoonup} \boldsymbol{H} \quad \text{in } L_B(\Omega), \qquad \operatorname{Adj} \nabla \boldsymbol{u}_j \rightharpoonup \boldsymbol{H} \quad \text{in } L^1(\Omega),$$

and

$$\operatorname{Det} \nabla \boldsymbol{u}_j \stackrel{*}{\rightharpoonup} \delta \quad \text{in } L_C(\Omega), \qquad \operatorname{Det} \nabla \boldsymbol{u}_j \rightharpoonup \delta \quad \text{in } L^1(\Omega).$$

By Theorem 6.2(ii) $\boldsymbol{H} = \operatorname{Adj} \nabla \boldsymbol{u}_0$ and $\delta = \operatorname{Det} \nabla \boldsymbol{u}_0$. By Theorem 7.2,

$$\bar{J}(\boldsymbol{u}_0) \leqq \varliminf_{j\to\infty} \bar{J}(\boldsymbol{u}_j).$$

Since $G_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) = +\infty$ if $\boldsymbol{a} \notin W(\boldsymbol{x})$ it follows that $(\nabla \boldsymbol{u}_0(\boldsymbol{x}), \operatorname{Adj} \nabla \boldsymbol{u}_0(x), \operatorname{Det} \nabla \boldsymbol{u}_0(x)) \in W(\boldsymbol{x})$ almost everywhere. Thus $\boldsymbol{u}_0 \in \mathscr{C}$ and the result follows. □

We now give a modified version of Theorem 7.3 for the case in which $n=3$ and G is independent of δ. The proof is similar and is omitted.

Theorem 7.4. *Let $n=3$. In the definitions of $W(\boldsymbol{x})$ and $\mathscr{S}$ replace E by $E_1 = M^{3\times 3} \times M^{3\times 3}$. Let $G\colon \mathscr{S} \to \mathscr{R}$ satisfy (H_1)–(H_4) and the following hypothesis:*

(H_6) *There exist N-functions A, B with A satisfying either $\int_1^\infty g_A(t)\, dt < \infty$ or both (6.12) and $\bar{A} \ll A^*$, and a function $b \in L^1(\Omega)$ such that*

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}) \geqq b(\boldsymbol{x}) + A(|\boldsymbol{F}|) + B(|\boldsymbol{H}|) \quad \textit{for all } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}) \in \mathscr{S}.$$

Define

$$\mathscr{A} = \{\boldsymbol{u} \in W^1 L_A(\Omega) \colon \operatorname{Adj} \nabla \boldsymbol{u} \in L_B(\Omega), (\nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla \boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega\}.$$

Let

$$\hat{J}(\boldsymbol{u}) = \int_\Omega G(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla \boldsymbol{u}(\boldsymbol{x}))\, d\boldsymbol{x}. \tag{7.11}$$

Let $\mathscr{C}$ be a Dirichlet class in $\mathscr{A}$, and suppose that there exists $\boldsymbol{u}_1 \in \mathscr{C}$ with $\hat{J}(\boldsymbol{u}_1) < \infty$. Then there exists $\boldsymbol{u}_0 \in \mathscr{C}$ that minimizes $\hat{J}(\boldsymbol{u})$ in $\mathscr{C}$.

Remark. Similar modifications to Theorem 7.3 can be made when $n=2$, and when $n=3$ and G is independent of both $\boldsymbol{H}$ and δ; the details are left to the reader.

Next we give an existence theorem for the displacement boundary-value problem in three dimensions for an *incompressible* hyperelastic body.

Theorem 7.5. *Theorem* 7.4 *remains valid if* $\mathscr{A}$ *is redefined by* $\mathscr{A}=\{\boldsymbol{u}\in W^1 L_A(\Omega)$: $\operatorname{Adj} \nabla \boldsymbol{u} \in L_B(\Omega)$, $(\nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla \boldsymbol{u}(\boldsymbol{x}))\in W(\boldsymbol{x})$ *almost everywhere in* Ω, $\operatorname{Det} \nabla \boldsymbol{u}(\boldsymbol{x})=1$ *almost everywhere in* $\Omega\}$, *provided that if* $\int_1^\infty g_A(t)\,dt=\infty$, *we make the extra assumption* $\tilde{B} \ll A^*$.

Proof. Let $\{\boldsymbol{u}_r\}$ be a minimizing sequence from $\mathscr{C}$. Then $\{\operatorname{Det} \nabla \boldsymbol{u}_r\}$ is bounded in $L^\infty(\Omega)$ independently of r. As in the proof of Theorem 7.3 we may extract a subsequence $\{\boldsymbol{u}_j\}\subseteq\mathscr{C}$ with, among other properties,

$$\operatorname{Det} \nabla \boldsymbol{u}_j \xrightarrow{*} \delta \quad \text{in } L^\infty(\Omega).$$

Clearly $\delta(\boldsymbol{x})=1$ almost everywhere. By Theorem 6.2(ii), $\delta=\operatorname{Det} \nabla \boldsymbol{u}$, and the result follows. $\square$

Remark. A variety of 'weakly closed' constraints can be treated in this way. Analogous results hold for $n=2$.

For the remainder of this section we restrict our attention to the cases $n=2, 3$ and we impose growth hypotheses that are partly of polynomial type. This will enable us to work in Sobolev spaces, rather than in Orlicz-Sobolev spaces. It would be possible to extend most of our results to an Orlicz-Sobolev space setting. Such an extension would involve the use of trace theory for Orlicz-Sobolev spaces (see DONALDSON & TRUDINGER [1], FOUGÈRES [1], LACROIX [1]).

For ease of reference we now restate hypotheses (H_5) and (H_6) in modified form. Later we shall put extra restrictions on the constants appearing in these hypotheses.

(H_7) $n=2$: there exists an N-function B, real constants $K_1>0$, $K_2\geqq 0$, $\gamma>1$, $s\geqq 1$, and a function $b\in L^1(\Omega)$ such that

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \delta)\geqq b(\boldsymbol{x})+K_1|\boldsymbol{F}|^\gamma+K_2|\boldsymbol{u}|^s+B(\delta) \quad \text{for all } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \delta)\in\mathscr{S}. \tag{7.12}$$

$n=3$: there exists an N-function C, real constants $K_1>0$, $K_2\geqq 0$, $\gamma>1$, $\mu>1$, $s\geqq 1$, and a function $b\in L^1(\Omega)$ such that

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}, \delta)\geqq b(\boldsymbol{x})+K_1(|\boldsymbol{F}|^\gamma+|\boldsymbol{H}|^\mu)+K_2|\boldsymbol{u}|^s+C(\delta) \quad \text{for all } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}, \delta)\in\mathscr{S}. \tag{7.13}$$

(H_8) $n=3$: there exist constants $K_1>0$, $K_2\geqq 0$, $\gamma>1$, $\mu>1$, $s\geqq 1$, and a function $b\in L^1(\Omega)$ such that

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H})\geqq b(\boldsymbol{x})+K_1(|\boldsymbol{F}|^\gamma+|\boldsymbol{H}|^\mu)+K_2|\boldsymbol{u}|^s \quad \text{for all } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H})\in\mathscr{S}. \tag{7.14}$$

Mixed displacement traction boundary-value problems

Theorem 7.6 (*cf.* Section 1A). *Let* Ω *satisfy a strong Lipschitz condition. Let* $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$, *let* $\partial\Omega_1\cap\partial\Omega_2=\phi$, *and let* $\partial\Omega_1$ *and* $\partial\Omega_2$ *be measurable as subsets of*

$\partial\Omega$ with $\partial\Omega_1$ having positive measure. Let $\bar{\boldsymbol{u}}: \partial\Omega_1 \to \mathcal{R}^n$ be measurable and let $\bar{\boldsymbol{t}}_R \in L^\sigma(\partial\Omega_2)$ with $\sigma \geqq 1$. Let $G: \mathcal{S} \to \mathcal{R}$ satisfy hypotheses (H$_1$)-(H$_4$) *and* (H$_7$). *If $n=2$, let $\gamma > \frac{4}{3}$, $\sigma = \frac{\gamma'}{2}$ if $\gamma < 2$, $\sigma > 1$ if $\gamma = 2$, $\sigma = 1$ if $\gamma > 2$, $K_2 = 0$, and*

$$\mathcal{A} = \{\boldsymbol{u} \in W^{1,\gamma}(\Omega)\colon \operatorname{Det} \nabla\boldsymbol{u} \in L_B(\Omega), (\nabla\boldsymbol{u}(\boldsymbol{x}), \operatorname{Det} \nabla\boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega, \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ almost everywhere in } \partial\Omega_1^{\#}\}.$$

If $n=3$, let $\gamma > \frac{3}{2}$, $\frac{1}{\gamma} + \frac{1}{\mu} < \frac{4}{3}$, $\sigma = \frac{2\gamma'}{3}$ if $\gamma < 3$, $\sigma > 1$ if $\gamma = 3$, $\sigma = 1$ if $\gamma > 3$, $K_2 = 0$, and

$$\mathcal{A} = \{\boldsymbol{u} \in W^{1,\gamma}(\Omega)\colon \operatorname{Adj} \nabla\boldsymbol{u} \in L^\mu(\Omega), \operatorname{Det} \nabla\boldsymbol{u} \in L_C(\Omega), (\nabla\boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla\boldsymbol{u}(\boldsymbol{x}), \operatorname{Det} \nabla\boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega, \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ almost everywhere in } \partial\Omega_1\}.$$

Let

$$J_0(\boldsymbol{u}) = J(\boldsymbol{u}) - \int_{\partial\Omega_2} \boldsymbol{u}(\boldsymbol{x}) \cdot \bar{\boldsymbol{t}}_R(\boldsymbol{x})\, dS, \tag{7.15}$$

where the integral is defined in the sense of trace.

Suppose that there exists $\boldsymbol{u}_1 \in \mathcal{A}$ with $J_0(\boldsymbol{u}_1) < \infty$. Then there exists $\boldsymbol{u}_0 \in \mathcal{A}$ that minimizes $J_0(\boldsymbol{u})$ in $\mathcal{A}$.

Proof. We give the proof just for $n=3$.

By the trace theorems (*cf.* Morrey [2], Nečas [1]) $\bar{\boldsymbol{u}} \in L^\sigma(\partial\Omega_1)$ and there exists $k>0$ such that

$$\|\boldsymbol{u}\|_{W^{1,\gamma}(\Omega)} \geqq k \|\boldsymbol{u}\|_{L^{\sigma'}(\partial\Omega)} \quad \text{for all } \boldsymbol{u} \in W^{1,\gamma}(\Omega). \tag{7.16}$$

Since $\partial\Omega_1$ has positive measure, a result of Morrey [2, p. 82] implies that there exists $k_1 > 0$ such that

$$\int_\Omega |\boldsymbol{u}|^\gamma d\boldsymbol{x} \leqq k_1 \Big[\int_\Omega |\nabla\boldsymbol{u}|^\gamma d\boldsymbol{x} + \Big(\int_{\partial\Omega_1} |\bar{\boldsymbol{u}}|\, dS\Big)^\gamma\Big] \quad \text{for all } \boldsymbol{u} \in W^{1,\gamma}(\Omega). \tag{7.17}$$

By (H$_7$), (7.16) and (7.17) we have for arbitrary $\boldsymbol{u} \in \mathcal{A}$

$$\begin{aligned} J_0(\boldsymbol{u}) &\geqq \int_\Omega b(\boldsymbol{x})\, d\boldsymbol{x} + K_1 \int_\Omega |\nabla\boldsymbol{u}|^\gamma d\boldsymbol{x} + K_1 \int_\Omega |\operatorname{Adj} \nabla\boldsymbol{u}|^\mu d\boldsymbol{x} + \int_\Omega C(\operatorname{Det} \nabla\boldsymbol{u})\, d\boldsymbol{x} \\ &\quad - \int_{\partial\Omega_2} \boldsymbol{u}(\boldsymbol{x}) \cdot \bar{\boldsymbol{t}}_R(\boldsymbol{x})\, dS \\ &\geqq \int_\Omega b(\boldsymbol{x})\, d\boldsymbol{x} + \left(\frac{K_1}{2} - \varepsilon\right) \int_\Omega |\nabla\boldsymbol{u}|^\gamma d\boldsymbol{x} + K_1 \int_\Omega |\operatorname{Adj} \nabla\boldsymbol{u}|^\mu d\boldsymbol{x} \\ &\quad + \left(\frac{K_1}{2k_1} - \varepsilon\right) \int_\Omega |\boldsymbol{u}|^\gamma d\boldsymbol{x} - \frac{K_1}{2}\Big(\int_{\partial\Omega_1} |\bar{\boldsymbol{u}}|\, dS\Big)^\gamma + \varepsilon k^\gamma \|\boldsymbol{u}\|^\gamma_{L^{\sigma'}(\partial\Omega)} \\ &\quad - \frac{d^\gamma}{\gamma} \|\boldsymbol{u}\|^\gamma_{L^{\sigma'}(\partial\Omega)} - \frac{1}{\gamma d^{\gamma'}} \|\bar{\boldsymbol{t}}_R\|^{\gamma'}_{L^\sigma(\partial\Omega_2)} + \int_\Omega C(\operatorname{Det} \nabla\boldsymbol{u})\, d\boldsymbol{x}, \end{aligned}$$

for $\varepsilon > 0$ and $d > 0$. Choosing ε and d small enough with $d \leqq k(\varepsilon\gamma)^{1/\gamma}$ we obtain

$$J_0(\boldsymbol{u}) \geqq c + c_0 \|\boldsymbol{u}\|^\gamma_{W^{1,\gamma}(\Omega)} + K_1 \int_\Omega |\operatorname{Adj} \nabla\boldsymbol{u}|^\mu d\boldsymbol{x} + \int_\Omega C(\operatorname{Det} \nabla\boldsymbol{u})\, d\boldsymbol{x}, \tag{7.18}$$

where c and $c_0 > 0$ are constants.

\# In the sense of trace.

Let $\{\boldsymbol{u}_r\}$ be a minimizing sequence for J_0. It follows from (7.18) that a subsequence $\{\boldsymbol{u}_j\}$ satisfies

$$\begin{gathered}\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}_0 \text{ in } W^{1,\gamma}(\Omega), \quad \boldsymbol{u}_j \overset{*}{\rightharpoonup} \boldsymbol{u}_0 \text{ in } L^{\sigma}(\partial\Omega),\\ \boldsymbol{u}_j \to \boldsymbol{u}_0 \text{ almost everywhere in } \Omega \text{ and } \partial\Omega,\\ \operatorname{Adj} \nabla \boldsymbol{u}_j \rightharpoonup \operatorname{Adj} \nabla \boldsymbol{u}_0 \text{ in } L^{\mu}(\Omega), \quad \operatorname{Det} \nabla \boldsymbol{u}_j \overset{*}{\rightharpoonup} \operatorname{Det} \nabla \boldsymbol{u}_0 \text{ in } L_C(\Omega).\end{gathered}$$

For $\bar{J}$ given by (7.10) it follows that

$$\bar{J}(\boldsymbol{u}_0) \leqq \varliminf_{j\to\infty} \bar{J}(\boldsymbol{u}_j), \tag{7.19}$$

while

$$\int_{\partial\Omega_1} \boldsymbol{u}_0(\boldsymbol{x}) \cdot \bar{\boldsymbol{t}}_R(\boldsymbol{x})\,dS = \lim_{j\to\infty} \int_{\partial\Omega_1} \boldsymbol{u}_j(\boldsymbol{x}) \cdot \bar{\boldsymbol{t}}_R(\boldsymbol{x})\,dS. \tag{7.20}$$

Noting that $\boldsymbol{u}_0 = \bar{\boldsymbol{u}}$ almost everywhere in $\partial\Omega_1$ we see, as in the proof of Theorem 7.3, that $\boldsymbol{u}_0 \in \mathscr{A}$. The result follows. □

Remark. In Theorem 7.6, and in the results below, the hypotheses on $\bar{\boldsymbol{u}}$ are concealed in the assumption that $\mathscr{A}$ is nonempty.

Theorem 7.7. *If $n=2$ let $\gamma \geqq 2$, $\sigma > 1$ if $\gamma = 2$, $\sigma = 1$ if $\gamma > 2$, $K_2 = 0$. If $n=3$ let $\gamma \geqq 2$, $\frac{1}{\gamma} + \frac{1}{\mu} \leqq 1$, $\sigma = \frac{2\gamma'}{3}$ if $2 \leqq \gamma < 3$, $\sigma > 1$ if $\gamma = 3$, $\sigma = 1$ if $\gamma > 3$, $K_2 = 0$.*

Let the other hypotheses of Theorem 7.6 remain unchanged. Then Theorem 7.6 remains valid with $\operatorname{Adj} \nabla \boldsymbol{u}$, $\operatorname{Det} \nabla \boldsymbol{u}$ *replaced everywhere by* $\operatorname{adj} \nabla \boldsymbol{u}$, $\det \nabla \boldsymbol{u}$ *respectively.*

Proof. This is immediate from Lemma 6.1. □

Remark. In Theorem 7.7, and in those results below that concern the distributions $\det \nabla \boldsymbol{u}$, $\operatorname{adj} \nabla \boldsymbol{u}$ it is only necessary for G to be defined on the set $\{(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{a}) \colon \boldsymbol{x} \in \Omega, \boldsymbol{u} \in \mathscr{R}^n, \boldsymbol{a} \in \operatorname{Co}(T(M^{3\times 3}) \cap W)\}$ (see Section 4 and the remark after Example 6.1).

Next we give the analogue of Theorems 7.6 and 7.7 for incompressible materials. The proof is similar to that of Theorem 7.5 and is omitted. An analogue of Theorem 7.4 may also be simply proved.

Theorem 7.8. *Let $n=3$. Let Ω, $\partial\Omega_1$, $\partial\Omega_2$, $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{t}}_R$ be as in Theorem 7.6. In the definitions of $W(\boldsymbol{x})$ and $\mathscr{S}$ replace E by E_1. Let $G \colon \mathscr{S} \to \mathscr{R}$ satisfy hypotheses* (H_1)–(H_4) *and* (H_8).

Either let γ, μ, σ, K_2 be as in Theorem 7.6 and let

$$\mathscr{A} = \{\boldsymbol{u} \in W^{1,\gamma}(\Omega);\ \operatorname{Adj} \nabla \boldsymbol{u} \in L^{\mu}(\Omega), (\nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla \boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \textit{ almost everywhere in } \Omega,\ \boldsymbol{u} = \bar{\boldsymbol{u}} \textit{ almost everywhere in } \partial\Omega_1,\ \operatorname{Det} \nabla \boldsymbol{u}(\boldsymbol{x}) = 1 \textit{ almost everywhere in } \Omega\}$$

or let γ, μ, σ, K_2 be as in Theorem 7.7 and let

$$\mathscr{A} = \{\boldsymbol{u} \in W^{1,\gamma}(\Omega) \colon \operatorname{adj} \nabla \boldsymbol{u} \in L^{\mu}(\Omega),\ (\nabla \boldsymbol{u}(\boldsymbol{x}), \operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \textit{ almost everywhere in } \Omega,\ \boldsymbol{u} = \bar{\boldsymbol{u}} \textit{ almost everywhere in } \partial\Omega_1,\ \det \nabla \boldsymbol{u}(\boldsymbol{x}) = 1 \textit{ almost everywhere in } \Omega\}.$$

Let

$$\hat{J}_0(\boldsymbol{u}) = \hat{J}(\boldsymbol{u}) - \int_{\partial\Omega_2} \boldsymbol{u}(\boldsymbol{x}) \cdot \bar{\boldsymbol{t}}_R(\boldsymbol{x})\, dS, \tag{7.21}$$

where $\hat{J}$ is given by (7.11).

Suppose there exists $u_1 \in \mathscr{A}$ with $\hat{J}_0(\boldsymbol{u}_1) < \infty$. Then there exists $\boldsymbol{u}_0 \in \mathscr{A}$ that minimizes $\hat{J}_0(\boldsymbol{u})$ in $\mathscr{A}$.

Pure traction boundary-value problems

Theorem 7.9 (*cf.* Section 1, A1). *Let Ω satisfy a strong Lipschitz condition. Let $\bar{\boldsymbol{t}}_R \in L^\sigma(\partial\Omega)$, $\sigma \geqq 1$. Let $G: \mathscr{S} \to \mathscr{R}$ satisfy hypotheses $(H_1)-(H_4)$ and (H_7) with $K_2 > 0$. Let $\kappa = \min(\gamma, s)$.*

$n=2$: *Let* $\gamma > \frac{4}{3}$, $\sigma = \dfrac{\kappa'}{2}$ *if* $1 \leqq \kappa < 2$, $\sigma > 1$ *if* $\kappa = 2$, $\sigma = 1$ *if* $\kappa > 2$. *If* $s=1$ *let* $\|\bar{\boldsymbol{t}}_R\|_{L^\infty(\partial\Omega)} < k_0 K_2$, *where* $k_0(\Omega) > 0$ *is a certain constant. Let*

$$\mathscr{A} = \{\boldsymbol{u} \in W^{1,\gamma}(\Omega): \operatorname{Det} \nabla\boldsymbol{u} \in L_B(\Omega), (\nabla\boldsymbol{u}(\boldsymbol{x}), \operatorname{Det} \nabla\boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega\}.$$

$n=3$: *Let* $\gamma > \frac{3}{2}$, $\dfrac{1}{\gamma} + \dfrac{1}{\mu} < \dfrac{4}{3}$, $\sigma = \dfrac{2\kappa'}{3}$ *if* $\kappa < 3$, $\sigma > 1$ *if* $\kappa = 3$, $\sigma = 1$ *if* $\kappa > 3$.

If $s=1$ *let* $\|\bar{\boldsymbol{t}}_R\|_{L^\infty(\partial\Omega)} < k_0 K_2$, *where* $k_0(\Omega) > 0$ *is a certain constant. Let*

$$\mathscr{A} = \{\boldsymbol{u} \in W^{1,\gamma}(\Omega): \operatorname{Adj} \nabla\boldsymbol{u} \in L^\mu(\Omega), \operatorname{Det} \nabla\boldsymbol{u} \in L_C(\Omega), (\nabla\boldsymbol{u}(\boldsymbol{x}), \operatorname{Adj} \nabla\boldsymbol{u}(\boldsymbol{x}), \operatorname{Det} \nabla\boldsymbol{u}(\boldsymbol{x})) \in W(\boldsymbol{x}) \text{ almost everywhere in } \Omega\}.$$

Let

$$J_0(\boldsymbol{u}) = J(\boldsymbol{u}) - \int_{\partial\Omega} \boldsymbol{u}(\boldsymbol{x}) \cdot \bar{\boldsymbol{t}}_R(\boldsymbol{x})\, dS. \tag{7.22}$$

Suppose that there exists $\boldsymbol{u}_1 \in \mathscr{A}$ such that $J_0(\boldsymbol{u}_1) < \infty$. Then there exists $\boldsymbol{u}_0 \in \mathscr{A}$ that minimizes $J_0(\boldsymbol{u})$ in $\mathscr{A}$. If, in addition,

$$\left.\begin{array}{l} n=2:\ \gamma \geqq 2 \\ n=3:\ \gamma \geqq 2,\ \dfrac{1}{\gamma} + \dfrac{1}{\mu} \leqq 1 \end{array}\right\}, \tag{7.23}$$

then the result holds with Adj $\nabla\boldsymbol{u}$, Det $\nabla\boldsymbol{u}$ *replaced everywhere by* adj $\nabla\boldsymbol{u}$, det $\nabla\boldsymbol{u}$ *respectively.*

Proof. Let $n=3$. By using the hypothesis $K_2 > 0$ instead of (7.17) we obtain the *a priori* bound

$$J_0(\boldsymbol{u}) \geqq c + c_0 \|\nabla\boldsymbol{u}\|^\gamma_{L^\gamma(\Omega)} + K_1 \int_\Omega |\operatorname{Adj} \nabla\boldsymbol{u}|^\mu\, dx + \int_\Omega C(\operatorname{Det} \nabla\boldsymbol{u})\, dx + c_1 \int_\Omega |\boldsymbol{u}|^s\, dx,$$

where $c_1 > 0$. If $s=1$ then k_0 is such that $\|\boldsymbol{u}\|_{W^{1,1}(\Omega)} \geqq \dfrac{1}{k_0} \|\boldsymbol{u}\|_{L^1(\partial\Omega)}$ for all $\boldsymbol{u} \in W^{1,1}(\Omega)$. By using the Poincaré inequality (Morrey [2, p. 82]) we can complete the proof in the same way as for Theorems 6.6, 6.7. □

Theorem 7.10 (*cf.* Section 1, A2). *Let Ω satisfy a strong Lipschitz condition. Let $\bar{t}_R \in L^\sigma(\partial\Omega)$. Let $G: \mathscr{S} \to \mathscr{R}$ satisfy hypotheses $(H_1)-(H_4)$ and (H_7) with $K_2=0$.*

If $n=2$, let $\gamma>\frac{4}{3}$, $\sigma=\frac{\gamma'}{2}$ if $\gamma<2, \sigma>1$ if $\gamma=2$, $\sigma=1$ if $\gamma>2$, $\boldsymbol{e}$ be a constant vector and

$$\mathscr{A}=\{\boldsymbol{u}\in W^{1,\gamma}(\Omega)\colon \mathrm{Det}\ \nabla\boldsymbol{u}\in L_B(\Omega),$$
$$(\nabla\boldsymbol{u}(x), \mathrm{Det}\ \nabla\boldsymbol{u}(x))\in W(x) \text{ almost everywhere in } \Omega, \int_\Omega \boldsymbol{u}(x)\,dx=\boldsymbol{e}\}.$$

If $n=3$, let $\gamma>\frac{3}{2}$, $\frac{1}{\gamma}+\frac{1}{\mu}<\frac{4}{3}$, $\sigma=\frac{2\gamma'}{3}$ if $\gamma<3$, $\sigma>1$ if $\gamma=3$, $\sigma=1$ if $\gamma>3$, $\boldsymbol{e}$ be a constant vector, and

$$\mathscr{A}=\{\boldsymbol{u}\in W^{1,\gamma}(\Omega)\colon \mathrm{Adj}\ \nabla\boldsymbol{u}\in L^\mu(\Omega), \mathrm{Det}\ \nabla\boldsymbol{u}\in L_C(\Omega),$$
$$(\nabla\boldsymbol{u}(x), \mathrm{Adj}\ \nabla\boldsymbol{u}(x), \mathrm{Det}\ \nabla\boldsymbol{u}(x))\in W(x) \text{ almost everywhere in } \Omega, \int_\Omega \boldsymbol{u}(x)\,dx=\boldsymbol{e}\}.$$

Let $\rho_R\in L^{\gamma'}(\Omega)$, let $\boldsymbol{b}_0\in\mathscr{R}^n$ and let J_0 be given by

$$J_0(\boldsymbol{u})=J(\boldsymbol{u})-\int_\Omega \rho_R(x)\,\boldsymbol{b}_0\cdot\boldsymbol{u}(x)\,dx-\int_{\partial\Omega}\boldsymbol{u}\cdot\bar{t}_R\,dS. \tag{7.24}$$

Suppose there exists $\boldsymbol{u}_1\in\mathscr{A}$ with $J_0(\boldsymbol{u}_1)<\infty$. Then there exists $\boldsymbol{u}_0\in\mathscr{A}$ that minimizes $J_0(\boldsymbol{u})$ in $\mathscr{A}$.

If, in addition, we assume (7.23), *then the result holds with* Adj $\nabla\boldsymbol{u}$, Det $\nabla\boldsymbol{u}$ *replaced everywhere by* adj $\nabla\boldsymbol{u}$, det $\nabla\boldsymbol{u}$, *respectively.*

Proof. If $\boldsymbol{u}\in\mathscr{A}$, then

$$\int_\Omega\left[\boldsymbol{u}(x)-\frac{1}{m(\Omega)}\,\boldsymbol{e}\right]dx=\boldsymbol{0}.$$

Thus by a version of the Poincaré inequality (MORREY [2, p. 83]) there are constants $k_3, k_4>0$, such that

$$\int_\Omega|\boldsymbol{u}|^\gamma\,dx\leqq k_3+k_4\int_\Omega|\nabla\boldsymbol{u}|^\gamma\,dx \tag{7.25}$$

for all $\boldsymbol{u}\in\mathscr{A}$.

Applying (7.25) and the simple estimate

$$-\int_\Omega\rho_R\,\boldsymbol{b}_0\cdot\boldsymbol{u}\,dx\geqq-\left[\frac{d^\gamma}{\gamma}\,\|\boldsymbol{u}\|^\gamma_{L^\gamma(\Omega)}+\frac{1}{\gamma' d^{\gamma'}}\,\|\rho_R\,\boldsymbol{b}_0\|^{\gamma'}_{L^{\gamma'}(\Omega)}\right], \tag{7.26}$$

we again obtain the bound given for $n=3$ by (7.18). The rest of the proof follows the usual pattern. □

Remark. We re-emphasise (*cf.* Section 1, A2 and Theorem 7.13) that for nonlinear elasticity, when G is independent of $\boldsymbol{u}$, it is necessary to impose the extra condition (1.31) in order to show that $\boldsymbol{u}_0$ is in some sense a solution of the equilibrium equations.

Mixed displacement pressure boundary-value problems

For these problems we restrict our attention to the case $n=3$. The case $n=2$ can be treated similarly.

Theorem 7.11 (*cf.* Section 1B). *Let* $n=3$. *Let* Ω *satisfy a strong Lipschitz condition. Let* $\partial\Omega=\partial\Omega_1\cup\Sigma$, $\partial\Omega_1\cap\Sigma=\phi$, *with* $\partial\Omega_1$ *having positive measure as a subset of* $\partial\Omega$*. *Let* $\bar{\boldsymbol{u}}\colon\partial\Omega_1\to\mathscr{R}^3$ *be measurable, and let* $p\in W^{1,\infty}(\Omega)$. *Let* $G\colon\mathscr{S}\to\mathscr{R}$ *satisfy hypotheses* $(H_1)-(H_4)$ *and* (H_7). *Let* $K_2=0$. *If* $p(\boldsymbol{x})\equiv$ *constant almost everywhere, let* $\gamma>\frac{3}{2}$, $\dfrac{1}{\gamma}+\dfrac{1}{\mu}<\dfrac{4}{3}$. *If* $p(\boldsymbol{x})\not\equiv$ *constant almost everywhere, let* $\gamma>\frac{3}{2}$, $\dfrac{1}{\gamma}+\dfrac{1}{\mu}<1$ *and if* $\dfrac{1}{\gamma}+\dfrac{1}{\mu}=1$ *let* $\|\nabla p\|_{L^\infty(\Omega)}\leqq k_3$, *where* $k_3=k_3(K_1,K_2,\gamma,\mu,\Omega,\partial\Omega_1)$ *is a certain constant. Let*

$$\begin{aligned}\mathscr{A}=\{&\boldsymbol{u}\in W^{1,\gamma}(\Omega)\colon \text{Adj}\,\nabla\boldsymbol{u}\in L^\mu(\Omega),\ \text{Det}\,\nabla\boldsymbol{u}\in L_C(\Omega),\\ &(\nabla\boldsymbol{u}(\boldsymbol{x}),\ \text{Adj}\,\nabla\boldsymbol{u}(\boldsymbol{x}),\ \text{Det}\,\nabla\boldsymbol{u}(\boldsymbol{x}))\in W(\boldsymbol{x})\text{ almost everywhere in }\Omega,\\ &\boldsymbol{u}=\bar{\boldsymbol{u}}\text{ almost everywhere in }\partial\Omega_1\}.\end{aligned}$$

Let

$$J_1(\boldsymbol{u})=J(\boldsymbol{u})+P(\boldsymbol{u}),\tag{7.27}$$

where (*cf.* 1.35))

$$P(\boldsymbol{u})\overset{\text{def}}{=}\int_\Omega[p\,\text{Det}\,\nabla\boldsymbol{u}+\tfrac{1}{3}p_{,r}(\text{Adj}\,\nabla\boldsymbol{u})^r_k\,u^k]\,dx.\tag{7.28}$$

Suppose that there exists $\boldsymbol{u}_1\in\mathscr{A}$ *with* $J_1(\boldsymbol{u}_1)<\infty$. *Then there exists* $\boldsymbol{u}_0\in\mathscr{A}$ *that minimizes* $J_1(\boldsymbol{u})$ *in* $\mathscr{A}$.

If in addition

$$\gamma\geqq 2,\quad \frac{1}{\gamma}+\frac{1}{\mu}\leqq 1,\tag{7.29}$$

then the result holds with Adj $\nabla\boldsymbol{u}$, Det $\nabla\boldsymbol{u}$ *replaced everywhere by* adj $\nabla\boldsymbol{u}$, det $\nabla\boldsymbol{u}$ *respectively.*

Proof. It suffices to establish the bound

$$J_1(\boldsymbol{u})\geqq c+c_0\,\|\boldsymbol{u}\|^\gamma_{W^{1,\gamma}(\Omega)}+\frac{K_1}{2}\int_\Omega|\text{Adj}\,\nabla\boldsymbol{u}|^\mu\,dx+\tfrac{1}{2}\int_\Omega C(\text{Det}\,\nabla\boldsymbol{u})\,dx\tag{7.30}$$

for all $\boldsymbol{u}\in\mathscr{A}$, where c, $c_0>0$ are constants.

We use the Sobolev inequality

$$\|\boldsymbol{u}\|_{W^{1,\gamma}(\Omega)}\geqq k_2\,\|\boldsymbol{u}\|_{L^\nu(\Omega)}\qquad\text{for all }\ \boldsymbol{u}\in W^{1,\gamma}(\Omega),\tag{7.31}$$

where $k_2>0$ and $\nu=3\gamma/3-\gamma$ if $\gamma<3$, $1<\nu<\infty$ if $\gamma=3$, $\nu=\infty$ if $\gamma>3$.

* In applications we shall have $\Sigma=\bigcup_{r=2}^{M}\overline{\partial\Omega_r}$ with the $\partial\Omega_r$ as in Section 1B.

For $\boldsymbol{u} \in \mathscr{A}$, we can use (7.17), (7.31) to obtain

$$\begin{aligned} J_1(\boldsymbol{u}) \geqq & \int_\Omega b(\boldsymbol{x})\,d\boldsymbol{x} + \left(\frac{K_1}{2} - \varepsilon\right)\int_\Omega |\nabla \boldsymbol{u}|^\gamma\,d\boldsymbol{x} + K_1 \int_\Omega |\mathrm{Adj}\,\nabla \boldsymbol{u}|^\mu\,d\boldsymbol{x} + \int_\Omega C(\mathrm{Det}\,\nabla \boldsymbol{u})\,d\boldsymbol{x} \\ & + \left(\frac{K_1}{2k_1} - \varepsilon\right)\int_\Omega |\boldsymbol{u}|^\gamma\,d\boldsymbol{x} - \frac{K_1}{2}\Big(\int_{\partial\Omega_1} |\bar{\boldsymbol{u}}|\,dS\Big)^\gamma + \varepsilon k_2^\gamma\,\|\boldsymbol{u}\|^\gamma_{L^\nu(\Omega)} \\ & - \|p\|_{L^\infty(\Omega)} \int_\Omega |\mathrm{Det}\,\nabla \boldsymbol{u}|\,d\boldsymbol{x} - \tfrac{1}{3} \sum_{k,r=1}^{3} \|p_{,r}\|_{L^\infty(\Omega)} \left[\frac{d^\mu}{\mu}\,\|(\mathrm{Adj}\,\nabla \boldsymbol{u})^r_k\|^\mu_{L^\mu(\Omega)} \right. \\ & \left. + \frac{1}{\mu'\,d^\mu}\,\|u^k\|^{\mu'}_{L^{\mu'}(\Omega)}\right], \end{aligned}$$

where $\varepsilon > 0$, $d > 0$.

If p is constant, then clearly (7.30) follows. If p is not constant and $\dfrac{1}{\gamma} + \dfrac{1}{\mu} < 1$, then $\gamma > \mu'$ so that we obtain (7.30) by choosing ε and d small enough. If p is not constant and $\dfrac{1}{\gamma} + \dfrac{1}{\mu} = 1$, then $\gamma = \mu'$ and (7.30) follows similarly. □

The above proof is valid if ε is taken to be zero, but the value of k_3 so obtained is then smaller. There is no difficulty in giving the analogous results to Theorem 7.11 for the pure pressure boundary-value problem and for incompressible materials.

Solutions to the equilibrium equations

We now turn to the question of whether the minimizers whose existence we have established satisfy the corresponding Euler-Lagrange equations. There is at present no available regularity theory for our problems under acceptable hypotheses, and we therefore confine our discussion to whether the minimizers are weak solutions. Unfortunately there are technical problems associated with the two most important cases, namely (i) when the material is compressible and $W(\boldsymbol{x})$ is given by

$$W(\boldsymbol{x}) = \{\boldsymbol{a} \in E : \delta > 0\},$$

and (ii) when the material is incompressible.

We therefore consider the simpler situation when $W(\boldsymbol{x}) = E$ for all $\boldsymbol{x}$, so that there is no local constraint.

We replace (H_7) by the following stronger hypotheses on G:

(H_9) $n=2$: G is continuously differentiable in $\boldsymbol{u}$, $\boldsymbol{F}$, δ for all $\boldsymbol{x} \in \Omega$, and there exist real constants $K_1 > 0$, $K_2 \geqq 0$, $B_1 > 0$, $C_i > 0$, $\rho_i \geqq 0$, $1 < \nu \leqq \gamma$, $\tau \geqq 0$, $s \geqq 1$, and a function $b \in L^1(\Omega)$ such that

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \delta) \geqq b(\boldsymbol{x}) + K_1\,|\boldsymbol{F}|^\gamma + K_2\,|\boldsymbol{u}|^s + B_1\,|\delta|^\nu, \tag{7.32}$$

$$\left|\frac{\partial G}{\partial \boldsymbol{u}}\right| \leqq \rho_1 + C_1\,[|\boldsymbol{u}|^\tau + |\boldsymbol{F}|^\gamma + |\delta|^\nu], \tag{7.33}$$

$$\left|\frac{\partial G}{\partial \boldsymbol{F}}\right| \leqq \rho_2 + C_2[|\boldsymbol{u}|^\tau + |\boldsymbol{F}|^\gamma + |\delta|^\nu], \tag{7.34}$$

$$\left|\frac{\partial G}{\partial \delta}\right| \leqq \rho_3 + C_3\left[|\boldsymbol{u}|^\tau + |\boldsymbol{F}|^{\gamma-1} + |\delta|^{\frac{\nu}{\gamma'}}\right], \tag{7.35}$$

for all values of the arguments. If $\gamma<2$, we assume further that $\tau=\dfrac{2\gamma}{2-\gamma}$.

$n=3$: G is continuously differentiable in $\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}, \delta$ for all $\boldsymbol{x}\in\Omega$, and there exist real constants $K_1>0$, $K_2\geqq 0$, $B_1>0$, $C_i>0$, $\rho_i\geqq 0$, $1<\nu\leqq\mu\leqq\gamma$, $\tau\geqq 0$, $s\geqq 1$, and a function $b\in L^1(\Omega)$ such that

$$G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \boldsymbol{H}, \delta) \geqq b(\boldsymbol{x}) + K_1(|\boldsymbol{F}|^\gamma + |\boldsymbol{H}|^\mu) + K_2\,|\boldsymbol{u}|^s + B_1\,|\delta|^\nu, \tag{7.36}$$

$$\left|\frac{\partial G}{\partial \boldsymbol{u}}\right| \leqq \rho_1 + C_1[|\boldsymbol{u}|^\tau + |\boldsymbol{F}|^\gamma + |\boldsymbol{H}|^\mu + |\delta|^\nu], \tag{7.37}$$

$$\left|\frac{\partial G}{\partial \boldsymbol{F}}\right| \leqq \rho_2 + C_2[|\boldsymbol{u}|^\tau + |\boldsymbol{F}|^\gamma + |\boldsymbol{H}|^\mu + |\delta|^\nu], \tag{7.38}$$

$$\left|\frac{\partial G}{\partial \boldsymbol{H}}\right| \leqq \rho_3 + C_3\left[|\boldsymbol{u}|^{\frac{\tau}{\gamma'}} + |\boldsymbol{F}|^{\gamma-1} + |\boldsymbol{H}|^{\frac{\mu}{\gamma'}} + |\delta|^{\frac{\nu}{\gamma'}}\right], \tag{7.39}$$

$$\left|\frac{\partial G}{\partial \delta}\right| \leqq \rho_4 + C_4\left[|\boldsymbol{u}|^{\frac{\tau}{\gamma'}} + |\boldsymbol{F}|^{\frac{\gamma}{\mu'}} + |\boldsymbol{H}|^{\mu-1} + |\delta|^{\frac{\nu}{\mu'}}\right], \tag{7.40}$$

for all values of the arguments. If $\gamma<3$ we assume further that $\tau=\dfrac{3\gamma}{3-\gamma}$.

Let f be given by

$$\left.\begin{aligned} n=2&: \ f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) = G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \det \boldsymbol{F}) \\ n=3&: \ f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) = G(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}, \operatorname{adj} \boldsymbol{F}, \det \boldsymbol{F}) \end{aligned}\right\}. \tag{7.41}$$

Two typical results are the following:

Theorem 6.12. *In the hypotheses of Theorems* 7.7, 7.9 *replace* (H_7) *by* (H_9). *Suppose also that* (*cf.* (7.23))

$$\begin{aligned} &\text{if } n=2: \ \gamma\geqq 2, \\ &\text{if } n=3: \ \gamma\geqq 2, \quad \frac{1}{\gamma}+\frac{1}{\mu}\leqq 1. \end{aligned} \tag{7.42}$$

Then the minimizing functions $\boldsymbol{u}=\boldsymbol{u}_0$ *satisfy the Euler-Lagrange equation*

$$\int_\Omega \left[\frac{\partial f}{\partial u^i}\, v^i + \frac{\partial f}{\partial u^i_{,\alpha}}\, v^i_{,\alpha}\right] dx = \int_{\partial\Omega_2} \boldsymbol{v}\cdot\bar{\boldsymbol{t}}_R\, dS \tag{7.43}$$

for all $\boldsymbol{v}\in C^\infty(\mathscr{R}^n)$ *with* $\boldsymbol{v}|_{\partial\Omega_1}=0$.

Proof. We give the proof for the case in which $n=3$. Fix $\boldsymbol{v}\in C^\infty(\mathscr{R}^3)$ with $\boldsymbol{v}|_{\partial\Omega_1}=O$. Since $1<\nu\leqq\mu\leqq\gamma$ it follows that $\boldsymbol{u}_0+\varepsilon\,\boldsymbol{v}\in\mathscr{A}$ for any ε. Also we clearly have

$$\frac{d}{d\varepsilon}\int_{\partial\Omega_2} (\boldsymbol{u}_0+\varepsilon\,\boldsymbol{v})\cdot\bar{\boldsymbol{t}}_R\, dS = \int_{\partial\Omega_2} \boldsymbol{v}\cdot\bar{\boldsymbol{t}}_R\, dS. \tag{7.44}$$

It thus suffices to show that

$$I(\boldsymbol{u}) \stackrel{\text{def}}{=} \int_\Omega f(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\, d\boldsymbol{x} \tag{7.45}$$

is Gâteaux differentiable at $\boldsymbol{u}_0$, and that $I'(\boldsymbol{u}_0)(\boldsymbol{v})$ is given by the left-hand side of (7.43). By the dominated convergence theorem it is enough to establish the estimate

$$\left|\frac{d}{d\varepsilon} f(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}) + \varepsilon \boldsymbol{v}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}) + \varepsilon \nabla \boldsymbol{v}(\boldsymbol{x}))\right| \leqq \Theta(\boldsymbol{x}), \tag{7.46}$$

for fixed $\boldsymbol{u} \in \mathscr{A}$ with $I(\boldsymbol{u}) < \infty$, where $\Theta \in L^1(\Omega)$ and is independent of $\varepsilon \in (0, 1)$.

Carrying out the indicated differentiation in (7.46), we find from (7.41) that

$$\begin{aligned} &\left|\frac{d}{d\varepsilon} f(\boldsymbol{x}, \boldsymbol{u} + \varepsilon \boldsymbol{v}, \nabla \boldsymbol{u} + \varepsilon \nabla \boldsymbol{v})\right| \\ &\leqq c\left[\left|\frac{\partial G}{\partial \boldsymbol{u}}\right| + \left|\frac{\partial G}{\partial \boldsymbol{F}}\right| + \left|\frac{\partial G}{\partial \boldsymbol{H}}\right| |\nabla \boldsymbol{u}| + \left|\frac{\partial G}{\partial \delta}\right| |\nabla \boldsymbol{u}| + \left|\frac{\partial G}{\partial \delta}\right| |\operatorname{adj} \nabla \boldsymbol{u}|\right] \end{aligned} \tag{7.47}$$

for fixed $\boldsymbol{u} \in \mathscr{A}$ with $I(\boldsymbol{u}) < \infty$, and for all $\varepsilon \in (0, 1)$. Here and below c denotes a generic constant. The estimate (7.46) now follows from (7.37)–(7.40) and Hölder's inequality. For brevity we display the calculation only for the last term of (7.47). We have that

$$\begin{aligned} c\left|\frac{\partial G}{\partial \delta}\right| |\operatorname{adj} \nabla \boldsymbol{u}| &\leqq c\rho_4 |\operatorname{adj} \nabla \boldsymbol{u}| + cC_4\Big[|\boldsymbol{u}|^{\frac{\tau}{\gamma'}} + |\nabla \boldsymbol{u}|^{\frac{\gamma}{\mu'}} \\ &\qquad + |\operatorname{adj} \nabla \boldsymbol{u}|^{\mu - 1} + |\det \nabla \boldsymbol{u}|^{\frac{\nu}{\mu'}}\Big] |\operatorname{adj} \nabla \boldsymbol{u}| \\ &\leqq c\Big[|\operatorname{adj} \nabla \boldsymbol{u}| + |\boldsymbol{u}|^{\frac{\tau\mu'}{\gamma'}} + |\nabla \boldsymbol{u}|^{\gamma} + |\operatorname{adj} \nabla \boldsymbol{u}|^{\mu} + |\det \nabla \boldsymbol{u}|^{\nu}\Big] \\ &\leqq c\big[1 + |\boldsymbol{u}|^{\tau} + |\nabla \boldsymbol{u}|^{\gamma} + |\operatorname{adj} \nabla \boldsymbol{u}|^{\mu} + |\det \nabla \boldsymbol{u}|^{\nu}\big] \\ &\leqq \Theta_1(\boldsymbol{x}), \end{aligned}$$

for some $\Theta_1 \in L^1(\Omega)$, where we have used (7.36) and the facts that if $\gamma \geqq 3$ then $\boldsymbol{u} \in L^\infty(\Omega)$, while if $2 \leqq \gamma < 3$ then $\boldsymbol{u} \in L^\tau(\Omega)$. $\square$

Theorem 7.13 (*cf.* Section 1, A2). *In the hypotheses of Theorem* 7.10 *replace* (H_7) *by* (H_9), *let* (7.42) *hold, and let* $f = \mathscr{W}(\boldsymbol{x}, \boldsymbol{F})$ *be independent of* $\boldsymbol{u}$ *so that*

$$J_0(\boldsymbol{u}) = \int_\Omega \mathscr{W}(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x}))\, d\boldsymbol{x} - \int_\Omega \rho_R(\boldsymbol{x})\, \boldsymbol{b}_0 \cdot \boldsymbol{u}(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{u} \cdot \bar{\boldsymbol{t}}_R\, dS. \tag{7.48}$$

Suppose in addition that

$$\int_{\partial\Omega} \bar{\boldsymbol{t}}_R\, dS + \int_\Omega \rho_R\, \boldsymbol{b}_0\, d\boldsymbol{x} = 0. \tag{7.49}$$

Then the minimizing function $\boldsymbol{u} = \boldsymbol{u}_0$ *satisfies the Euler-Lagrange equations*

$$\int_\Omega \left[\rho_R\, \boldsymbol{b}_0 \cdot \boldsymbol{v} - \frac{\partial \mathscr{W}}{\partial u^i_{,\alpha}} v^i_{,\alpha}\right] d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{v} \cdot \bar{\boldsymbol{t}}_R\, dS = 0 \tag{7.50}$$

for all $\boldsymbol{v} \in C^\infty(\mathscr{R}^n)$.

Proof. Proceeding as in Theorem 7.12 we obtain (7.50) for all $\boldsymbol{v} \in C^\infty(\mathscr{R}^n)$ satisfying

$$\int_\Omega \boldsymbol{v}\, d\boldsymbol{x} = 0. \tag{7.51}$$

Let $\boldsymbol{w} \in C^\infty(\mathscr{R}^n)$ be arbitrary and set

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x}) - \frac{1}{m(\Omega)} \int_\Omega \boldsymbol{w}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{7.52}$$

Then

$$\int_\Omega \left[\rho_R\, \boldsymbol{b}_0 \cdot \boldsymbol{w} - \frac{\partial \mathscr{W}}{\partial u^i_{,\alpha}} w^i_{,\alpha} \right] d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{w} \cdot \bar{\boldsymbol{t}}_R\, dS$$
$$= \frac{1}{m(\Omega)} \int_\Omega \boldsymbol{w}(\boldsymbol{y})\, d\boldsymbol{y} \cdot \left[\int_\Omega \rho_R\, \boldsymbol{b}_0\, d\boldsymbol{x} + \int_{\partial\Omega} \bar{\boldsymbol{t}}_R\, dS \right] = 0. \quad \square$$

Remarks. A similar result can be proved for the minimizer in Theorem 7.11. We may also waive the assumption (7.42) at the expense of obtaining the Euler-Lagrange equations only in terms of the distributions Adj $\nabla \boldsymbol{u}$, Det $\nabla \boldsymbol{u}$ and with derivatives of G replacing the derivatives of f. In Case (i) above an analogous local result to Theorems 7.12, 7.13 may be proved under the *a priori* assumption that $\det \nabla \boldsymbol{u} \geqq d > 0$ locally. The details of these proofs are left to the reader.

Existence under other hypotheses

We next give a sample theorem under the hypothesis $P_{\gamma,\mu,\nu}$ for the displacement boundary-value problem for a homogeneous material. This result almost certainly has generalizations to integrands with $\boldsymbol{x}, \boldsymbol{u}$ dependence.

Theorem 7.14. *Let $n=3$. Let $U \subseteq M^{3\times 3}$ be such that* $\operatorname{Co} T(U)$ *is open. Let $\boldsymbol{u}_1 \in W^{1,\gamma}(\Omega)$, $\operatorname{adj} \nabla \boldsymbol{u}_1 \in L^\mu(\Omega)$, $\det \nabla \boldsymbol{u}_1 \in L^\nu(\Omega)$, where $\gamma \geqq 2$, $\mu > 1$, $\frac{1}{\gamma} + \frac{1}{\mu} \leqq 1$, $\nu > 1$. In the notation of Definition* 4.3 *let $g\colon U \to \mathscr{R}$ satisfy $P_{\gamma,\mu,\nu}$ at $\boldsymbol{u}_1$. Let the corresponding function G satisfy*

$$G(\boldsymbol{F}, \boldsymbol{H}, \delta) \geqq b + K_1(|\boldsymbol{F}|^\gamma + |\boldsymbol{H}|^\mu + |\delta|^\nu) \quad \textit{for all } (\boldsymbol{F}, \boldsymbol{H}, \delta) \in \operatorname{Co} T(U), \tag{7.53}$$

where b and $K_1 > 0$ are constants, and

$$G(\boldsymbol{F}, \boldsymbol{H}, \delta) \to \infty \quad \textit{as} \quad (\boldsymbol{F}, \boldsymbol{H}, \delta) \to \partial(\operatorname{Co} T(U)) \tag{7.54}$$

Let

$$\mathscr{A} = \{\boldsymbol{u} \in W^{1,\gamma}(\Omega)\colon \boldsymbol{u} - \boldsymbol{u}_1 \in W_0^{1,\gamma}(\Omega),\ \operatorname{adj} \nabla \boldsymbol{u} \in L^\mu(\Omega),$$
$$\det \nabla \boldsymbol{u} \in L^\nu(\Omega),\ \boldsymbol{\Sigma}(\boldsymbol{u}) \in \operatorname{Co} T(U) \textit{ almost everywhere in } \Omega\}.$$

Define for $\boldsymbol{u} \in \mathscr{A}$

$$I(\boldsymbol{u}) = \int_\Omega g(\nabla \boldsymbol{u})\, d\boldsymbol{x}. \tag{7.55}$$

Then if $I(\boldsymbol{u}_1) < \infty$, there exists $\boldsymbol{u}_0 \in \mathscr{A}$ that minimizes $I(\boldsymbol{u})$ in $\mathscr{A}$.

Proof. Let $\{\boldsymbol{u}_r\}$ be a minimizing sequence. By (7.53), $\{\boldsymbol{\Sigma}(\boldsymbol{u}_r)\}$ is bounded in the reflexive Banach space $\mathscr{B} = L^\gamma(\Omega)^9 \times L^\mu(\Omega)^9 \times L^\nu(\Omega)$. By the Poincaré inequality and our now standard arguments there exists a subsequence $\{u_j\}$ such that

$$u_j \rightharpoonup u_0 \quad \text{in} \quad W^{1,\gamma}(\Omega), \qquad \Sigma(u_j) \rightharpoonup \Sigma(u_0) \quad \text{in} \quad \mathcal{K}_{u_1} \subseteq \mathcal{B},$$

with $u_0 \in \mathcal{A}$. Define $G_1: E \to \bar{\mathcal{R}}$ by

$$\begin{aligned} G_1(F, H, \delta) &= G(F, H, \delta) \quad \text{if} \quad (F, H, \delta) \in \operatorname{Co} T(U) \\ &= \infty \text{ otherwise.} \end{aligned}$$

Let

$$J(\sigma) = \int_\Omega G_1(\sigma(x))\, dx. \tag{7.56}$$

Then $J: \mathcal{Cl}_{\mathcal{B}}\, \mathcal{K}_{u_1} \to \bar{\mathcal{R}}$. Since g satisfies $P_{\gamma,\mu,\nu}$ at u_1 it follows that J is convex. Applying Fatou's lemma to the integrand

$$G_1(F, H, \delta) - b - K_1(|F|^\gamma + |H|^\mu + |\delta|^\nu)$$

we see that J is (strongly) lower semicontinuous. Hence (*cf.* EKELAND & TÉMAM [1, p. 33]) J is sequentially weakly lower semicontinuous. Thus

$$J(\Sigma(u_0)) \leq \varliminf_{j \to \infty} J(\Sigma(u_j)),$$

and the proof is complete. □

The most general integrands for which our methods establish the existence of minimizers are given by the sum of polyconvex functions satisfying a suitable subset of hypotheses (H_1)–(H_9), quasiconvex functions satisfying the hypotheses of Theorem 6.1, and, where appropriate, functions satisfying condition $P_{\gamma,\mu,\nu}$ as in the above theorem. By suitably combining the growth conditions of each of the terms in this sum various existence theorems may be given. At present both the scarcity of examples of quasiconvex functions that are not polyconvex and the abundance of physically useful polyconvex functions make these theorems of little interest. We therefore leave the routine formulation of the results to the reader.

8. Applications to Specific Models of Elastic Materials

Many forms of the stored-energy function have been proposed for nonlinear elastic materials, particularly for various rubbers. An excellent review of the literature can be found in the papers of OGDEN [2, 3]. We now examine the extent to which these models satisfy the hypotheses of our existence theorems. By concentrating on a certain class of models below we do not mean to imply that other models are inferior for empirical reasons. Neither should the omission of a model from the discussion be construed as suggesting that it fails to satisfy our existence hypotheses. Our purpose is simply to indicate the flexibility of the hypotheses to serve for a variety of stored-energy functions and also to discuss the position occupied with respect to these hypotheses by certain well known models.

We assume that $n=3$ unless otherwise stated, and consider for simplicity only isotropic materials, for which the stored-energy function has the form (see (1.15))

$$\mathcal{W}(x, F) = \Phi(x, v_1, v_2, v_3). \tag{8.1}$$

For compressible materials, in the case when the local constraint set $W(\boldsymbol{x})$ has the form

$$W(\boldsymbol{x})=E_1\times K(\boldsymbol{x}), \tag{8.2}$$

with $K(\boldsymbol{x})\subseteq\mathscr{R}$ nonempty, open and convex, it follows from Theorem 4.3 that $\operatorname{Co} U(\boldsymbol{x})=W(\boldsymbol{x})$, where $U(\boldsymbol{x})=\{\boldsymbol{F}\in M^{3\times 3}: \det \boldsymbol{F}\in K(\boldsymbol{x})\}$. Necessary and sufficient conditions for $\mathscr{W}(\boldsymbol{x},\cdot)$ to be polyconvex on $U(\boldsymbol{x})$ are given by Theorem 4.4. For simplicity we shall assume in the compressible case that $K(\boldsymbol{x})=\{\delta>0\}$ (continuity condition), while in the incompressible case $W(\boldsymbol{x})=E_1$ (no local constraint).

We consider a modification of a class of stored-energy functions introduced by OGDEN [2, 3]. For $\alpha\geqq 1$ let

$$\psi(\alpha)=v_1^\alpha+v_2^\alpha+v_3^\alpha-3, \qquad \chi(\alpha)=(v_2\,v_3)^\alpha+(v_3\,v_1)^\alpha+(v_1\,v_2)^\alpha-3. \tag{8.3}$$

Consider first incompressible materials, and let

$$\mathscr{W}(\boldsymbol{x},\boldsymbol{F})=B(\boldsymbol{x})+\sum_{i=1}^{M}a_i(\boldsymbol{x})\,\psi(\alpha_i)+\sum_{i=1}^{N}c_i(\boldsymbol{x})\,\chi(\beta_i), \tag{8.4}$$

where $\alpha_1\geqq\cdots\geqq\alpha_M\geqq 1$, $\beta_1\geqq\cdots\geqq\beta_N\geqq 1$, and where B, a_i, c_i are functions in $L^1(\Omega)$ satisfying

$$a_i(\boldsymbol{x})\geqq k>0, \qquad c_i(\boldsymbol{x})\geqq k>0, \qquad \text{for almost all } \boldsymbol{x}\in\Omega \tag{8.5}$$

for some constant k.

By Theorem 5.2 $\mathscr{W}(\boldsymbol{x},\boldsymbol{F})$ is polyconvex on $U(\boldsymbol{x})$. Since $v_1^\alpha+v_2^\alpha+v_3^\alpha$ is a continuous function of $\boldsymbol{F}$ # it follows that

$$v_1^\alpha+v_2^\alpha+v_3^\alpha\geqq d(\alpha)\,|\boldsymbol{F}|^\alpha \tag{8.6}$$

for some constant $d(\alpha)>0$. Similarly

$$(v_2\,v_3)^\alpha+(v_3\,v_1)^\alpha+(v_1\,v_2)^\alpha\geqq e(\alpha)\,|\operatorname{adj}\boldsymbol{F}|^\alpha, \qquad e(\alpha)>0. \tag{8.7}$$

It follows from (8.4)-(8.7) that $\mathscr{W}$ considered as a function of $\boldsymbol{x}, \boldsymbol{F}, \operatorname{adj}\boldsymbol{F}$ satisfies hypotheses (H_1)-(H_4), and (H_8) with

$$\gamma=\alpha_1, \qquad \mu=\beta_1, \qquad K_2=0. \tag{8.8}$$

Thus if

$$\alpha_1>\tfrac{3}{2}, \qquad \frac{1}{\alpha_1}+\frac{1}{\beta_1}<\tfrac{4}{3}, \tag{8.9}$$

we obtain from Theorems 7.8 and the analogues of Theorems 7.9–7.11 for incompressible materials the existence of minimizers for the various boundary-value problems in terms of the distributions $\nabla\boldsymbol{u}$, $\operatorname{Adj}\nabla\boldsymbol{u}$. These minimizers satisfy the incompressibility condition $\operatorname{Det}\nabla\boldsymbol{u}=1$ almost everywhere. Note that in order to obtain existence for variable pressures p in the mixed displacement pressure problem we require (Theorem 7.11) either

$$\frac{1}{\alpha_1}+\frac{1}{\beta_1}<1 \tag{8.10}$$

Because, for example, it is a finite-valued convex function.

or

$$\frac{1}{\alpha_1}+\frac{1}{\beta_1}=1, \qquad \|\nabla p\|_{L^\infty(\Omega)}<k_3(a_i, c_i, \alpha_i, \beta_i, \Omega, \partial\Omega_1). \tag{8.11}$$

If

$$\alpha_1 \geqq 2, \qquad \frac{1}{\alpha_1}+\frac{1}{\beta_1}\leqq 1, \tag{8.12}$$

then we have the stronger forms of the existence theorems with the incompressibility condition $\det \nabla \boldsymbol{u}=1$ holding almost everywhere.

As a special case we consider the inhomogeneous *Mooney-Rivlin* material, for which $B\equiv 0$, $M=N=1$, $\alpha_1=\beta_1=2$, so that

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{F})=a_1(\boldsymbol{x})(I_{\boldsymbol{B}}-3)+c_1(\boldsymbol{x})(II_{\boldsymbol{B}}-3). \tag{8.13}$$

Clearly (8.12) is satisfied so that the Mooney-Rivlin material is included in the existence theory. Note that in the mixed displacement pressure problem the critical case (8.11) applies.

The incompressible *Neo-Hookean* material

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{F})=\alpha_1(\boldsymbol{x})\,(I_{\boldsymbol{B}}-3) \tag{8.14}$$

is not covered by the theorems. To illustrate this point let us consider a single term stored-energy function

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{F})=a_1(\boldsymbol{x})\,\psi(\alpha) \tag{8.15}$$

with a_1 satisfying (8.5). To prove existence by our methods under the incompressibility constraint $\operatorname{Det}\nabla\boldsymbol{u}=1$ ($\det\nabla\boldsymbol{u}=1$) it is necessary that if $\boldsymbol{u}_r \rightharpoonup \boldsymbol{u}$ in $W^{1,\alpha}(\Omega)$, then $\operatorname{Det}\nabla\boldsymbol{u}_r \rightharpoonup \operatorname{Det}\nabla\boldsymbol{u}$ in $\mathscr{D}'(\Omega)$ ($\det\nabla\boldsymbol{u}_r \to \det\nabla\boldsymbol{u}$ in $\mathscr{D}'(\Omega)$). We must thus have (*cf.* the methods of Section 6) $\alpha>\frac{9}{4}$ ($\alpha\geqq 3$). Note, however, that we do get existence theorems for the Neo-Hookean material in two dimensions.

OGDEN fitted a stored-energy function with three terms ($M=2$, $N=1$) of the form (8.4) to data of TRELOAR for homogeneous vulcanized rubber†. The values of the various constants obtained were

$$\begin{aligned}
&\alpha_1=5.0, \quad \alpha_2=1.3, \quad \beta_1=2,\\
&a_1=2.4\times 10^{-3}, \quad a_2=4.8, \quad c_1=0.05\,\mathrm{kg\,cm^{-2}},\\
&B=0.
\end{aligned} \tag{8.16}$$

Clearly (8.12) is satisfied. Furthermore, since $\alpha_1>3$ the minimizers $\boldsymbol{u}_0$ of the various boundary-value problems belong to $C(\bar{\Omega})$; indeed by the results of MORREY [2] (see also FRIEDMAN [1] and NEČAS [1]) we have in this case $\boldsymbol{u}_0\in C^{0,\frac{2}{5}}(\bar{\Omega})$, the space of Hölder continuous functions on $\bar{\Omega}$ with exponent $\frac{2}{5}$. Note that the imbedding theorems do not imply that $\boldsymbol{u}_0\in C(\bar{\Omega})$ for the Mooney-Rivlin material.

For compressible materials OGDEN [3] considered the effect of adding a term $\Gamma(\det\boldsymbol{F})$ to (8.4)#. Suppose that

$$\left.\begin{aligned}
&\Gamma(t)\geqq C(t) \quad \text{for all } t>0, \quad \text{where } C \text{ is an } N\text{-function},\\
&\Gamma \text{ is convex on } (0,\infty), \quad \Gamma(t)\to\infty \quad \text{as} \quad t\to 0+.
\end{aligned}\right\} \tag{8.17}$$

† For related experimental work see TRELOAR [1].

OGDEN retained the values of the constants $B, \alpha_1, \alpha_2, a_1, a_2, c_1$, but for incompressible materials replaced the term $c_1\,\chi(2)$ by $c_1[v_1^{-2}+v_2^{-2}+v_3^{-2}-3]$. These terms are identical if $v_1 v_2 v_3=1$, and since $v_1 v_2 v_3$ is in practice very close to 1 the alteration is insignificant for experimental correlations.

Then the modified stored-energy function satisfies (H_1)-(H_4) and (H_7) with $\gamma=\alpha_1$, $\mu=\beta_1$, $K_2=0$, and thus satisfies the hypotheses of our existence theorems under the conditions (8.9)-(8.12).

It is clear that a wide variety of stored-energy functions having the form (5.11) (with $\boldsymbol{x}$-dependence if necessary) can be treated by our theory in an way analogous to that for the models discussed above. We end this section by exhibiting such a stored-energy function, which satisfies the hypotheses of Theorem 7.3 but not those of Theorem 7.6, *etc.* (for $\partial\Omega_2=\phi$), and thus requires the Orlicz-Sobolev space apparatus. Our example is of a stored-energy function with slow growth. For functions of very fast growth the Orlicz-Sobolev space setting would also be necessary for any proof that the Euler-Lagrange equations are satisfied.† We need two lemmas:

Lemma 8.1. *Let C, D be N-functions. Then*

$$\frac{C^{-1}(s)}{D^{-1}(s)}\to 0 \quad \text{as } s\to\infty \tag{8.18}$$

if and only if $D\ll C$.

Proof. We just prove the 'only if' part, the 'if' part being easier. Set $s=C(\lambda t)$ for $\lambda=0$. Then by (8.18) $\lambda t/D^{-1}(C(\lambda t))\to 0$ as $t\to\infty$. Hence $t\leqq D^{-1}(C(\lambda t))$ for t large enough. By the convexity of D we have for t large enough

$$\frac{D(t)}{C(\lambda t)}\leqq\frac{t}{D^{-1}(C(\lambda t))}=\frac{1}{\lambda}\,\frac{C^{-1}(C(\lambda t))}{D^{-1}(C(\lambda t))}\to 0 \quad \text{as } t\to\infty. \ \square$$

Lemma 8.2. *Let g, k be non-negative continuous functions on $\mathscr{R}_+$ such that $k(s)\to 0$ as $s\to\infty$ and such that*

$$\int_0^\infty g(t)\,dt=\infty.$$

Let

$$\theta(s)=\frac{\int_0^s k(t)\,g(t)\,dt}{\int_0^s g(t)\,dt}$$

Then $\theta(s)\to 0$ as $s\to\infty$.

Proof. This follows immediately from L'Hôpital's rule. $\square$

Now let A be an N-function with principal part

$$A(t)=t^{\frac{3}{2}}\log t. \tag{8.19}$$

Let

$$\mathscr{W}(\boldsymbol{F})=a\,A(v_1+v_2+v_3)+c\,A(v_2 v_3+v_3 v_1+v_1 v_2)+\Gamma(v_1 v_2 v_3) \tag{8.20}$$

† In order to give a comprehensive description of anisotropy and inhomogeneity in rod theories ANTMAN [8] uses more general Orlicz-Sobolev spaces than are used in this work. The methods described here probably extend to these spaces, with a consequent broadening in the applicability of the existence theorems presented here.

where Γ is as in (8.17) and a, c are positive constants. Since $A \not\succ t^{\frac{3}{2}+\varepsilon}$ for any $\varepsilon>0$ Theorem 7.6 does not apply. To show that (H_5) is satisfied, so that Theorem 7.3 may be applied, we must prove that $\bar{A} \ll A^*$. Let $B(t)=t^{\frac{3}{2}}$. Then $B \prec A$ so that $\bar{A} \prec \bar{B} \sim t^3 \sim B^*$. It is therefore sufficient to prove that $B^* \ll A^*$. We have that

$$g_A(t)=\frac{A^{-1}(t)}{t^{\frac{4}{3}}}=\frac{k(t)}{t^{\frac{2}{3}}}, \tag{8.21}$$

where

$$k(t) \stackrel{\text{def}}{=} \frac{1}{(\log A^{-1}(t))^{\frac{2}{3}}} \to 0 \quad \text{as } t \to \infty.$$

Also $g_B(t)=\frac{1}{t^{\frac{2}{3}}}$, and $\int_0^\infty g_A(t)\,dt=\int_0^\infty g_B(t)\,dt=\infty$. Therefore

$$\frac{A^{*-1}(s)}{B^{*-1}(s)}=\frac{\int_0^s k(t)\,g_B(t)\,dt}{\int_0^s g_B(t)\,dt}. \tag{8.22}$$

By Lemmas 8.1 and 8.2 we deduce $B^* \ll A^*$ as required.

9. An Example of Nonuniqueness; Buckling of a Rod

In this section we establish nonuniqueness for the mixed displacement zero traction boundary-value problem corresponding to buckling of a rod of a homogeneous, incompressible, Mooney-Rivlin material having a uniform cross-section. We do this by exhibiting an admissible displacement field with total energy lower than that of the trivial solution, and then applying Theorem 7.8 to ensure the existence of a nontrivial minimizer for the total energy. Under suitable conditions a similar but more complex analysis can be carried out for incompressible rods consisting of material not of Mooney-Rivlin type. The extension to compressible materials, however, is not so easy because an explicit trivial solution is not available.

In the stress-free reference configuration the rod occupies the region

$$\Omega=D \times (0,l), \qquad l>0, \tag{9.1}$$

where the cross-section D is a nonempty bounded open set in $\mathscr{R}^2$ satisfying a strong Lipschitz condition. We suppose that the density in the reference configuration is a constant $\rho_R>0$, and that the plane $x_2=0$ contains the line of centroids of the rod in the reference configuration, so that

$$\int_D x_2\,dS=0, \qquad dS=dx_1\,dx_2. \tag{9.2}$$

Let $\partial\Omega_1=\bar{D} \times \{0,l\}$, $\partial\Omega_2=\partial D \times (0,l)$. Let $A=\int_D dS$ be the area of D. For $\lambda>0$ we consider the equilibrium mixed boundary-value problem with boundary conditions

$$\begin{aligned} \boldsymbol{u}&=\bar{\boldsymbol{u}} \quad \text{on } \partial\Omega_1, \\ \boldsymbol{t}_R&=\boldsymbol{0} \quad \text{on } \partial\Omega_2, \end{aligned} \tag{9.3}$$

where $\bar{\boldsymbol{u}}: \bar{\Omega} \to \mathscr{R}^3$ is given by

$$\bar{\boldsymbol{u}}(\boldsymbol{x}) = (\lambda^{-\frac{1}{2}} x_1, \lambda^{-\frac{1}{2}} x_2, \lambda x_3). \tag{9.4}$$

The stored-energy function has the form (*cf.* 8.11))

$$\mathscr{W}(\boldsymbol{F}) = a(I_{\boldsymbol{B}} - 3) + c(II_{\boldsymbol{B}} - 3), \tag{9.5}$$

where $a > 0$, $c > 0$ are constants. In the notation of Theorem 7.8 we set

$$\gamma = \mu = 2, \qquad K_2 = 0, \qquad W(\boldsymbol{x}) = E_1 = M^{3\times 3} \times M^{3\times 3}, \tag{9.6}$$

$$\mathscr{A} = \{\boldsymbol{u} \in W^{1,2}(\Omega)\colon \operatorname{adj} \nabla \boldsymbol{u} \in L^2(\Omega),\ \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ almost everywhere in } \partial\Omega_1,$$
$$\det \nabla \boldsymbol{u} = 1 \text{ almost everywhere in } \Omega\},$$

$$\hat{J}(\boldsymbol{u}) = \int_\Omega \mathscr{W}(\nabla \boldsymbol{u}(\boldsymbol{x}))\, d\boldsymbol{x}. \tag{9.7}$$

From (9.4) we obtain

$$\nabla \bar{\boldsymbol{u}} = \operatorname{diag}(\lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}}, \lambda), \qquad \det \nabla \bar{\boldsymbol{u}} = 1, \tag{9.8}$$

and

$$\hat{J}(\bar{\boldsymbol{u}}) = [a(2\lambda^{-2} + \lambda^2 - 3) + b(2\lambda + \lambda^{-2} - 3)]\, A\, l < \infty. \tag{9.9}$$

It is easily shown that $\bar{\boldsymbol{u}}$ satisfies the equilibrium equations and boundary conditions for a suitable hydrostatic pressure. By Theorem 7.8 (with $\boldsymbol{u}_1 = \bar{\boldsymbol{u}}$) there exists $\boldsymbol{u}_0 \in \mathscr{A}$ that minimizes $\hat{J}(\boldsymbol{u})$ in $\mathscr{A}$.

It thus remains to construct a function $\boldsymbol{u} \in \mathscr{A}$ with[#]

$$\hat{J}(\boldsymbol{u}) < J(\bar{\boldsymbol{u}}). \tag{9.10}$$

We perform this construction in a manner reminiscent of derivations of rod theories in engineering. Let y_0, θ_0 be real valued functions with $y_0 \in C^2([0, l])$, $\theta_0 \in C^2([0, l])$ and

$$y_0(0) = y_0(l) = 0, \qquad \theta_0(0) = \theta_0(l) = \theta_0'(l) = \theta_0'(l) = 0. \tag{9.11}$$

Let $\varepsilon > 0$, $y = \varepsilon y_0$, $\theta = \varepsilon \theta_0$, and define

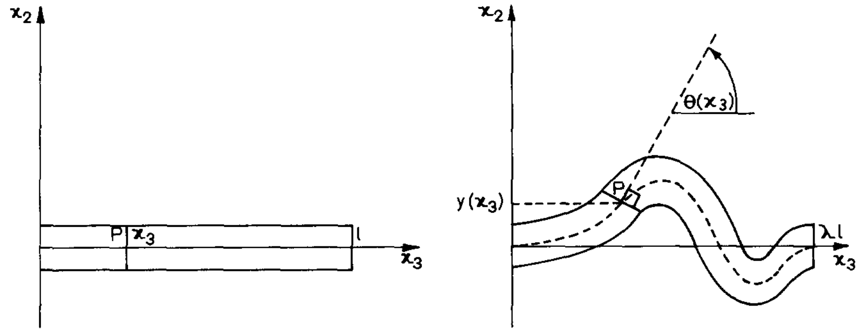
$$\boldsymbol{u}_\varepsilon(\boldsymbol{x}) = (x_1\, g(x_2, x_3),\ y(x_3) + \lambda^{-\frac{1}{2}} x_2 \cos\theta(x_3),\ \lambda x_3 - \lambda^{-\frac{1}{2}} x_2 \sin\theta(x_3)), \tag{9.12}$$

where g is a function to be chosen. $\boldsymbol{u}_\varepsilon$ represents a deformation in which points $(0, 0, x_3)$ are mapped to $(0, y(x_3), \lambda x_3)$, and in which cross-sections normal to the x_3-axis in the reference configuration stay plane and are so inclined that their normals remain parallel to the $x_2 x_3$ plane and make an angle $\theta(x_3)$ with the x_3-axis (Fig. 3).

From (9.12) we obtain

$$\nabla \boldsymbol{u}_\varepsilon = \begin{pmatrix} g & x_1 g_{,2} & x_1 g_{,3} \\ 0 & \lambda^{-\frac{1}{2}} \cos\theta & y' - \lambda^{-\frac{1}{2}} x_2 \sin\theta \cdot \theta' \\ 0 & -\lambda^{-\frac{1}{2}} \sin\theta & \lambda - \lambda^{-\frac{1}{2}} x_2 \cos\theta \cdot \theta' \end{pmatrix}. \tag{9.13}$$

[#] Note that we do not have to satisfy the zero traction condition on $\partial\Omega_2$ because our existence theorem incorporates this as a natural boundary condition.



(i) Reference configuration (ii) Deformed configuration

Fig. 3

Therefore

$$\det \nabla \boldsymbol{u}_\varepsilon = g[\lambda^{\frac{1}{2}} \cos\theta + \lambda^{-\frac{1}{2}} y' \sin\theta - \lambda^{-1} x_2 \theta']. \tag{9.14}$$

Since $\lambda > 0$ the expression in brackets in (9.14) is positive for all $\boldsymbol{x} \in \Omega$ if ε is small enough. Thus we may choose

$$g = [\lambda^{\frac{1}{2}} \cos\theta + \lambda^{-\frac{1}{2}} y' \sin\theta - \lambda^{-1} x_2 \theta']^{-1}, \tag{9.15}$$

whence

$$\det \nabla \boldsymbol{u}_\varepsilon = 1 \quad \text{for all } \boldsymbol{x} \in \Omega. \tag{9.16}$$

It follows from (9.11)–(9.16) that $\boldsymbol{u}_\varepsilon \in \mathscr{A}$ for ε small enough. Let

$$\boldsymbol{B} = \nabla \boldsymbol{u}_\varepsilon \nabla \boldsymbol{u}_\varepsilon{}^T. \tag{9.17}$$

A routine but tedious calculation shows that

$$\begin{aligned}
B_1^1 &= \lambda^{-1} + 2\varepsilon\lambda^{-\frac{5}{2}} x_2 \theta_0' + \varepsilon^2 [\lambda^{-1}\theta_0^2 - 2\lambda^{-2}\theta_0 y_0' + (3x_2^2 + x_1^2)\lambda^{-4}\theta_0'^2 + \lambda^{-4} x_1^2 x_2^2 \theta_0''^2] \\
&\quad + o(\varepsilon^2), \\
B_2^2 &= \lambda^{-1} + \varepsilon^2 [y_0'^2 - \lambda^{-1}\theta_0^2] + o(\varepsilon^2), \\
B_3^3 &= \lambda^2 - 2\varepsilon\lambda^{\frac{1}{2}} x_2 \theta_0' + \varepsilon^2 [\lambda^{-1}\theta_0^2 + \lambda^{-1} x_2^2 \theta_0'^2] + o(\varepsilon^2), \\
B_2^1 &= B_1^2 = \varepsilon\lambda^{-\frac{5}{2}} x_1 \theta_0' + o(\varepsilon), \\
B_3^1 &= B_1^3 = \varepsilon\lambda^{-1} x_1 x_2 \theta_0'' + o(\varepsilon), \\
B_3^2 &= B_2^3 = \varepsilon[\lambda y_0' - \lambda^{-1}\theta_0] + o(\varepsilon),
\end{aligned} \tag{9.18}$$

and hence that

$$\begin{aligned}
I_{\boldsymbol{B}} &= 2\lambda^{-1} + \lambda^2 + 2\varepsilon x_2 \theta_0' (\lambda^{-\frac{5}{2}} - \lambda^{\frac{1}{2}}) + \varepsilon^2 [\lambda^{-1}\theta_0^2 + ((3x_2^2 + x_1^2)\lambda^{-4} + \lambda^{-1} x_2^2)\theta_0'^2 \\
&\quad + x_1^2 x_2^2 \lambda^{-4} \theta_0''^2 - 2\lambda^{-2} y_0' \theta_0 + y_0'^2] + o(\varepsilon^2), \\
II_{\boldsymbol{B}} &= \lambda^{-2} + 2\lambda + 2\varepsilon x_2 \theta_0' (\lambda^{-\frac{7}{2}} - \lambda^{-\frac{1}{2}}) + \varepsilon^2 [\lambda^{-2}\theta_0^2 + ((x_2^2 + x_1^2)\lambda^{-2} + 3x_2^2 \lambda^{-5})\theta_0'^2 \\
&\quad + x_1^2 x_2^2 \lambda^{-5} \theta_0''^2 - 2\lambda^{-3}\theta_0 y_0' + \lambda^{-1} y_0'^2] + o(\varepsilon^2),
\end{aligned} \tag{9.19}$$

where $o(\varepsilon)$ denotes a function of $\boldsymbol{x}$ such that $o(\varepsilon)/\varepsilon \to 0$ uniformly in $\bar{\Omega}$ as $\varepsilon \to 0$.

From (9.2), (9.5), (9.7) we obtain (with the standard meaning for $o(\varepsilon^2)$)

$$\hat{J}(\boldsymbol{u}_\varepsilon) = \hat{J}(\bar{\boldsymbol{u}}) + \varepsilon^2 \int_0^l [d_0 \theta_0^2 + d_1 \theta_0'^2 + d_2 \theta_0''^2 - d_3 \theta_0 y_0' + d_4 y_0'^2]\, dx_3 + o(\varepsilon^2), \tag{9.20}$$

where

$$\begin{aligned} d_0 &= A\lambda^{-2}(a\lambda+c), \\ d_1 &= a[(3k_2+k_1)\lambda^{-4}+k_2\lambda^{-1}]+c[(k_1+k_2)\lambda^{-2}+3k_2\lambda^{-5}], \\ d_2 &= k_3\lambda^{-5}(a\lambda+c), \\ d_3 &= 2A\lambda^{-3}(a\lambda+c), \\ d_4 &= A\lambda^{-1}(a\lambda+c), \end{aligned} \tag{9.21}$$

and

$$k_1=\int_D x_1^2\,dS, \qquad k_2=\int_D x_2^2\,dS, \qquad k_3=\int_D x_1^2 x_2^2\,dS. \tag{9.22}$$

The Euler-Lagrange equations corresponding to the quadratic part of (9.20) are

$$2d_4 y_0' - d_3\theta_0 = \gamma, \tag{9.23}$$

$$d_2\theta_0'''' - d_1\theta_0'' + d_0\theta_0 - \frac{d_3}{2}y_0' = 0, \tag{9.24}$$

where γ is a constant. Setting $\gamma=0$ and combining (9.23), (9.24) we obtain

$$\theta_0'''' - \alpha_1\theta_0'' + \alpha_2\theta_0 = 0, \tag{9.25}$$

where

$$\alpha_1=\frac{d_1}{d_2}, \qquad \alpha_2=\frac{d_0-d_3^2/4d_4}{d_2}=\frac{A(\lambda^3-1)}{k_3}. \tag{9.26}$$

With the above as motivation we seek a solution, antisymmetric about $x_3=\frac{l}{2}$, to the equation

$$\theta_0'''' - \alpha_1\theta_0'' + \bar\alpha_2\theta_0 = 0 \tag{9.27}$$

subject to boundary conditions

$$\theta_0=\theta_0'=0 \quad \text{at } x_3=0, l, \tag{9.28}$$

for some $\bar\alpha_2>\alpha_2$. If θ_0 is such a solution and

$$y_0(x_3) \stackrel{\text{def}}{=} \frac{d_3}{2d_4}\int_0^{x_3}\theta_0(s)\,ds \tag{9.29}$$

then y_0 satisfies (9.11) and we have from (9.20), (9.26)–(9.29) that

$$\hat J(\boldsymbol{u}_\varepsilon)=\hat J(\bar{\boldsymbol{u}})+\varepsilon^2 d_2(\alpha_2-\bar\alpha_2)\int_0^l \theta_0^2\,dx_3+o(\varepsilon^2). \tag{9.30}$$

It follows that for ε small enough $\hat J(\boldsymbol{u}_\varepsilon)<\hat J(\bar{\boldsymbol{u}})$ as required.

To solve (9.27), (9.28) we first note that for all $\lambda\in[0,1]$ we have

$$\alpha_1 \geqq \frac{3ck_2}{k_3(a+c)}>0, \qquad \alpha_2\leqq 0. \tag{9.31}$$

Set

$$\bar\alpha_2=\alpha_2+\tau, \qquad 0<\tau<\tfrac14\left(\frac{3ck_2}{k_3(a+c)}\right)^2. \tag{9.32}$$

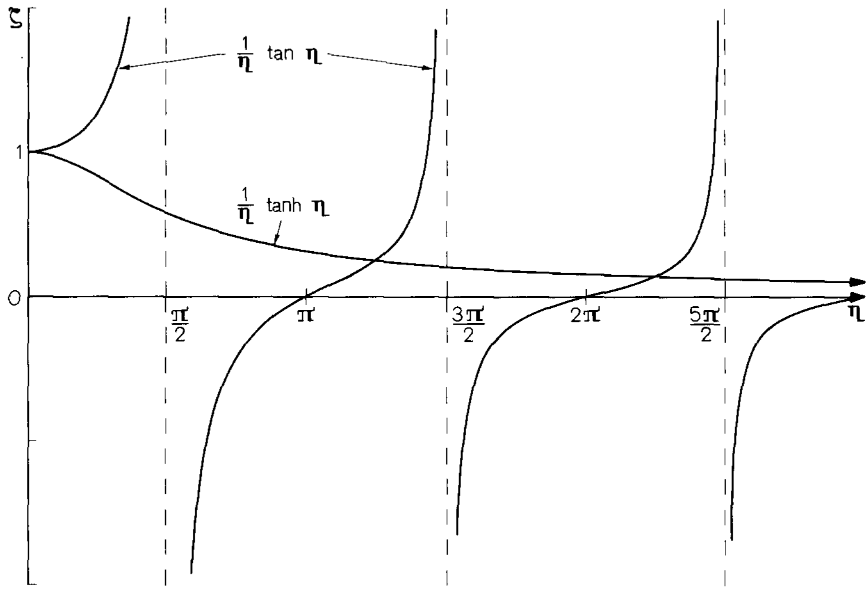


Fig. 4

The indicial equation for (9.27) is

$$m^4 - \alpha_1 m^2 + \bar{\alpha}_2 = 0 \tag{9.33}$$

with roots $m = \pm\kappa, \pm i\mu$, where

$$\kappa^2 = \frac{\alpha_1 + \sqrt{\alpha_1^2 - 4\bar{\alpha}_2}}{2}, \quad \mu^2 = \frac{\sqrt{\alpha_1^2 - 4\bar{\alpha}_2} - \alpha_1}{2}, \tag{9.34}$$

and $\kappa > \mu > 0$. (κ, μ are real by (9.31), (9.32).) Hence

$$\theta_0(x_3) = A_1 \sinh\kappa\left(x_3 - \frac{l}{2}\right) + A_2 \sin\mu\left(x_3 - \frac{l}{2}\right) \tag{9.35}$$

will satisfy (9.27), (9.28) provided that

$$\frac{2}{\mu l}\tan\frac{\mu l}{2} = \frac{2}{\kappa l}\tanh\frac{\kappa l}{2}. \tag{9.36}$$

Also θ_0 given by (9.35) is antisymmetric about $x_3 = \frac{l}{2}$. The graphs of the functions $\zeta = \frac{1}{\eta}\tan\eta$, $\zeta = \frac{1}{\eta}\tanh\eta$ are sketched in Fig. 4. It is easy to prove the indicated monotonicity properties.

Let κ_0, κ_1 be the values of $\left(\frac{\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}}{2}\right)^{\frac{1}{2}}$ at $\lambda = 0, 1$, respectively, so that

$$\kappa_0 = \left(\frac{3k_2}{2k_3} + \sqrt{\frac{9k_2^2}{4k_3^2} + \frac{A}{k_3}}\right)^{\frac{1}{2}}, \quad \kappa_1 = \left(\frac{4k_2 + k_1}{k_3}\right)^{\frac{1}{2}}. \tag{9.37}$$

It is not hard to show that κ and μ are continuous functions of λ in $[0,1]$, and that κ is not constant in $[0,1]$. Therefore there exists an interval in the range of $\kappa\colon [0,1]\to \mathscr{R}_+$ with length $\delta>0$. If $\kappa_0 \neq \kappa_1$ then we may take

$$\delta=|\kappa_0-\kappa_1|+r(\tau), \tag{9.38}$$

where $r(\tau)\to 0$ as $\tau\to 0$.

From Figure 4 it is thus clear that if

$$\frac{\delta l}{2}>2\pi, \tag{9.39}$$

then there exists $0<\lambda<1$ such that (9.36) is satisfied. The corresponding θ_0 is the function required. To satisfy (9.39) one need only choose l large enough. Thus *sufficiently long rods of arbitrary cross-section will exhibit nonuniqueness for some* $\lambda\in(0,1)$. (An obvious refinement of this argument shows that if $0<\lambda_0<1$ then for l large enough there will be nonuniqueness for some λ with $\lambda_0<\lambda<1$.)

Usually $\kappa_0\neq\kappa_1$ so that by choosing $\tau>0$ small enough we get nonuniqueness for some $\lambda\in(0,1)$ whenever

$$|\kappa_0-\kappa_1|>\frac{4\pi}{l}. \tag{9.40}$$

This is a condition expressed entirely in terms of l and the cross-sectional parameters k_1, k_2, k_3 and A.

Example. Let D be the disc $x_1^2+x_2^2<a^2$. Then

$$k_1=k_2=\frac{\pi a^4}{4}, \qquad k_3=\frac{\pi a^6}{24}, \qquad A=\pi a^2 \tag{9.41}$$

so that

$$\kappa_0=\frac{1}{a}\sqrt{9+\sqrt{105}}, \qquad \kappa_1=\frac{1}{a}\sqrt{30}. \tag{9.42}$$

Condition (9.40) therefore becomes

$$\frac{a}{l}<v, \qquad v\doteqdot 0.09. \tag{9.43}$$

The condition (9.40) is somewhat crude; indeed it is possible that no such condition is necessary. Improved estimates, and lower bounds on the supremum of $0<\lambda<1$ for which nonuniqueness occurs, may be obtained by more detailed calculations based on Figure 4. I have not included these results since they are messy and since my method is severely limited in scope due to the type of trial deformation considered in (9.12). Even in situations where we envisage nonuniqueness occurring by Euler buckling, the deformation of cross-sections implied by (9.12) is unrealistic.

There are numerous formal stability calculations in the literature for rods in tension or compression, and for other problems in three dimensional nonlinear elasticity. For the most part these calculations are based on the theory of small deformations superposed upon large; the status of this theory with respect to nonlinear elasticity has yet to be established. The reader is referred for details and references to FOSDICK & SHIELD [1], HOLDEN [1], KNOPS & WILKES [1], SENSENIG [1], WESOŁOWSKI [1], WILKES [1]. Nonuniqueness for the pure traction boundary-value problem of a rectangular block of Neo-Hookean material loaded uniformly on each face has been established explicitly by RIVLIN [1, 2, 3]. For various one and two-dimensional rod and shell theories rigorous proofs of non-uniqueness have been given by ANTMAN [2, 3, 6].

10. Concluding Remarks

The main implication of this work for constitutive inequalities is that the quasiconvexity condition (and in particular the Legendre-Hadamard condition) is consistent with realistic models of hyperelastic solids. In the one-dimensional case, when convexity and quasiconvexity of $\mathscr{W}(x, \cdot)$ are the same, Theorem 3.2 shows that the *existence* of $C^1(\bar{\Omega})$ minimizers for various homogeneous displacement boundary-value problems implies that $\mathscr{W}$ is quasiconvex, while the same result holds in three dimensions if Ω is a cube. If $\mathscr{W}$ is not quasiconvex then minimizers may exist that are not C^1. Some examples in one dimension are discussed by ERICKSEN [3]. It should be noted that we have not proved that $C^1(\bar{\Omega})$ minimizers exist in general for displacement boundary-value problems when $\partial\Omega$ is suitably regular under any reasonable hypotheses on $\mathscr{W}$.

The existence theorems proved in this article take the form that existence is established for a given material for *all* suitable boundary data. In general such unqualified existence is not to be expected for real materials, since rupture will occur under extreme conditions of deformation. We may also not be interested in solutions having at some points deformation gradients that lie outside the range in which the material behaves elastically. One way of partially circumventing these difficulties is to choose the local constraint set $W(\boldsymbol{x})$ so as to prohibit such behaviour, and then to check *a posteriori* whether the minimizer $\boldsymbol{u}_0$ is such that $\nabla \boldsymbol{u}_0(\boldsymbol{x}) \in \partial W(\boldsymbol{x})$ for any $\boldsymbol{x}$. One would then like *a priori* conditions on the size of the boundary data to prevent this happening. The derivation of any such conditions would require delicate estimates. The reader is referred to the papers by ERICKSEN [4] and KNOWLES & STERNBERG [1] for further discussion of some of these points.

In general weak lower semicontinuity will not hold if the quasiconvexity or polyconvexity hypotheses are replaced by a hypothesis of convexity of the function $\mathscr{W}$ restricted to positive definite symmetric tensors U. It is nevertheless instructive to see how an attempt to establish lower semicontinuity in this case breaks down. The difficulty is that if $\boldsymbol{u}_r \rightharpoonup \boldsymbol{u}$ in the Sobolev space $W^{1,\gamma}(\Omega)$, then the weak limit in $L^1(\Omega)$ of the sequence $\boldsymbol{U}_r = \sqrt{\nabla \boldsymbol{u}_r^T \nabla \boldsymbol{u}_r}$, will not necessarily be $\sqrt{\nabla \boldsymbol{u}^T \nabla \boldsymbol{u}}$, and indeed may not arise from any displacement. This is because $\sqrt{\nabla \boldsymbol{u}^T \nabla \boldsymbol{u}}$ is not of the form (4.2) and hence not sequentially weakly continuous. The difficulty is also connected with the nonlinearity of the Riemann-Christoffel tensor based on $\boldsymbol{C}$. Similar mathematical problems arise from attempts to establish existence under the COLEMAN & NOLL condition [1], or HILL's inequalities [2, 3]. These conditions do not imply the Legendre-Hadamard condition. The COLEMAN & NOLL condition cannot apply to all hyperelastic materials because it is violated for nearly incompressible materials such as rubber (see HILL [2], OGDEN [1, 3], RIVLIN [2], SIDOROFF [1]).

For bodies that are not homeomorphic to an open ball the various minimizers whose existence we have established may represent deformations topologically isolated from those desired on physical grounds†. One might require, for example, all admissible deformations to be accessible from a given deformation by a homotopy of globally invertible configurations. In this article we have not studied such global constraints (although they may be of a weakly closed type), but have concentrated on local constraints such as the local invertibility condition $\det \nabla \boldsymbol{u} > 0$. Local invertibility is a relatively weak requirement; indeed a hollow sphere may be everted without violation of the condition in any intermediate deformation (SMALE [1]).#

Finally I remark on the implications of the results of Section 6 for theories of elasticity incorporating pointwise constraints on the deformation gradient $\boldsymbol{F}$. These results suggest strongly that the only nontrivial homogeneous constraints giving rise to a well posed theory have the form (see (4.2))

$$\phi(\boldsymbol{F}) = A + B_i^\alpha F_\alpha^i + C_i^\alpha (\operatorname{adj} \boldsymbol{F})_\alpha^i + D \det \boldsymbol{F} = 0, \tag{10.1}$$

where $A, B_i^\alpha, C_i^\alpha, D$ are constants. It is not hard to show that the only *objective* constraints of this form (*i.e.*, ϕ satisfying $\phi(\boldsymbol{QF}) = \phi(\boldsymbol{F})$ for all orthogonal $\boldsymbol{Q}$) are those with $B_i^\alpha = C_i^\alpha = 0$, so that $\det \boldsymbol{F}$ is specified. In particular, as we have seen, the incompressibility condition $\det \boldsymbol{F} = 1$ gives rise to a well posed theory. Note, however, that the constraint of inextensibility (TRUESDELL & NOLL [1, p. 72]) is not included. It seems possible, therefore, that solutions do not in general exist for boundary-value problems of inextensible elasticity, and that a higher order theory is required to make such constraints well behaved.

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† For bodies that are homeomorphic to an open ball the same may occur for mixed problems; the situation for displacement boundary-value problems is unclear.

\# A visualization of the eversion due to SHAPIRO can be found in PHILLIPS [1].

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