A non-semisimple Kitaev lattice model

based on joint work with Ulrich Krähmer

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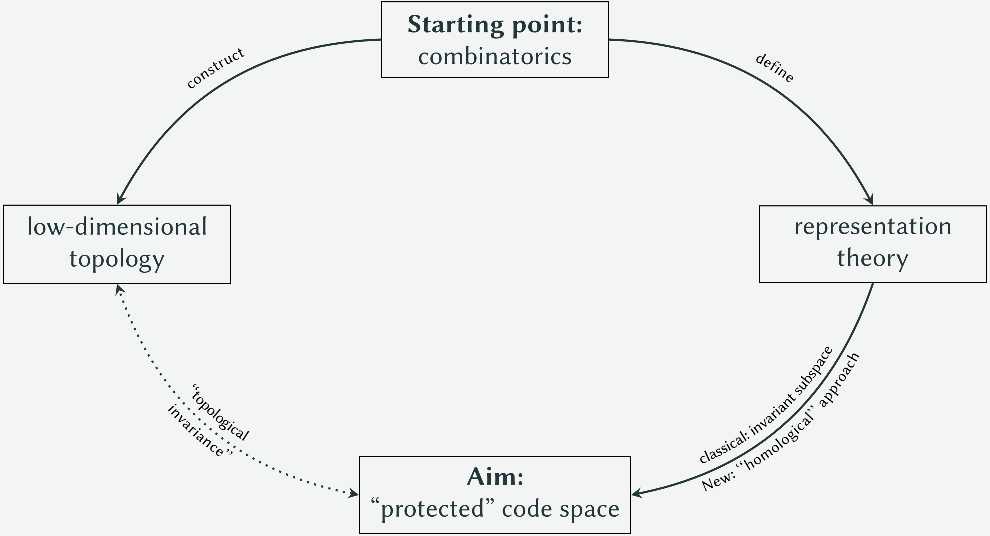
The big picture



Fault-tolerant quantum computation: Kitaev's toric code

Kitaev's construction for fault-tolerant quantum computation:

1. Consider a $k \times k$ lattice in the torus and associate the Hilbert space $\mathbb{M}_k$.
2. Specify a "Hamiltonian" $H\colon\ \mathbb{M}_k \longrightarrow \mathbb{M}_k$ using "local" features of the lattice.
3. Define as a "quantum memory" the space $\mathrm{Prot}_k \stackrel{\mathrm{def}}{=} \ker H$.

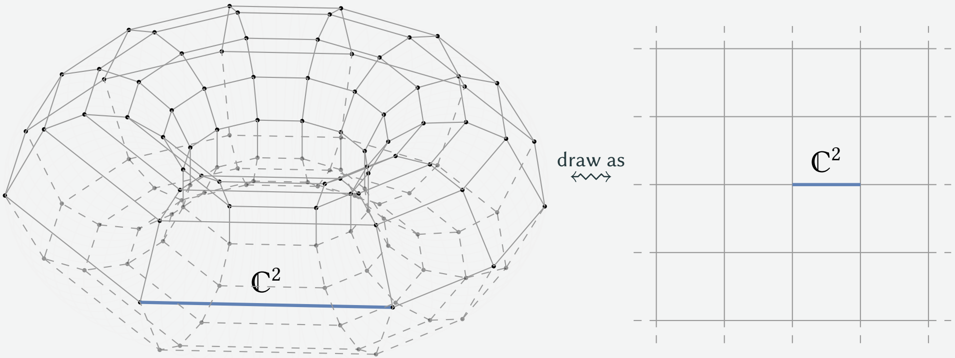
This leads to error correction/prevention implemented on a physical level:

Theorem

For any $k \in \mathbb{N}$ we have $\dim \mathrm{Prot}_k = 4$.

Kitaev's toric code – lattices and the (extended) Hilbert space

Let $k \in \mathbb{N}$ and consider a $k \times k$ lattice embedded on a torus:

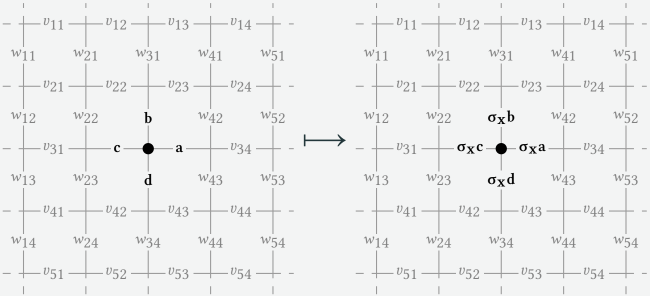


Assign to each *edge* a copy of $\mathbb{C}^2$ and set $\mathbb{M}_k = \otimes_{1 \leq i,j \leq k} \mathbb{C}^2 \bigotimes \otimes_{1 \leq i,j \leq k} \mathbb{C}^2$.

Kitaev's toric code – vertex actions

Set $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Given a *vertex* (point in the lattice) v define

$$A_v \colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$$

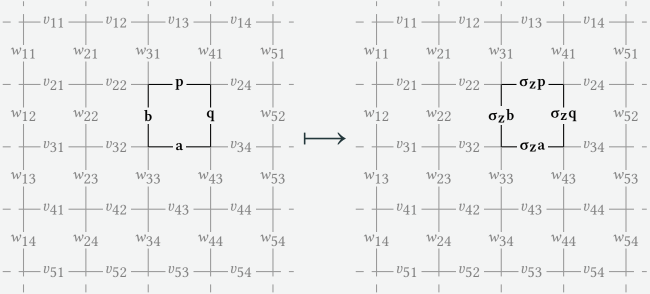


1. Gives rise to a representation $\triangleright_v \colon \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{M}_k \longrightarrow \mathbb{M}_k$.
2. Combining all "vertex actions" leads to $\triangleright \colon (\mathbb{C}\mathbb{Z}_2)^{\otimes k^2} \otimes \mathbb{M}_k \longrightarrow \mathbb{M}_k$.

Kitaev's toric code – face coactions

Set $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Given a *face* (square of the lattice) f define

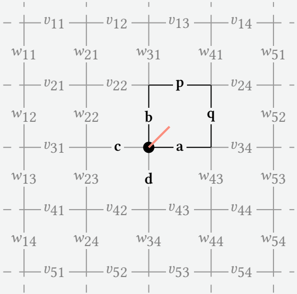
$$B_f \colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$$



1. Can be interpreted as a corepresentation $\delta_f \colon \mathbb{M}_k \longrightarrow \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{M}_k$.
2. Combining all "face coaction" leads to a "global" comodule structure $\delta \colon \mathbb{M}_k \longrightarrow (\mathbb{C}\mathbb{Z}_2)^{\otimes k^2} \otimes \mathbb{M}_k$.

The local and global module structure of $\mathbb{M}_k$

Associate to each vertex the face to its "top right" $\rightsquigarrow \{vertices\} \xrightarrow{1:1} \{faces\}$.
For every such vertex-face-pair (v, f) the vertex action and face coaction turn $\mathbb{M}_k$ into a $D(\mathbb{C}\mathbb{Z}_2)$-module.



Topological invariance of the protected space

We consider $\mathbb{M}_k$ as a $D(\mathbb{C}\mathbb{Z}_2)^{\otimes k^2}$-module

Definition

The *protected space* of the Kitaev lattice model is $\mathrm{Prot}_k = \mathbb{M}_k^{\mathrm{inv}}$.

Theorem (Kitaev '97)

For any $k \in \mathbb{N}$ *we have* $\dim \mathrm{Prot}_k = 4$.

Proof sketch (based on Buerschaper, Mombelli, Christandl, Aguado '18).

Suppose $k \geq 1$. Define an explicit isomorphism between Prot_k and Prot_{k+1} using projectors coming from the Haar integral of $\mathbb{C}\mathbb{Z}_2$. $\square$

Works for complex semisimple Hopf algebras.

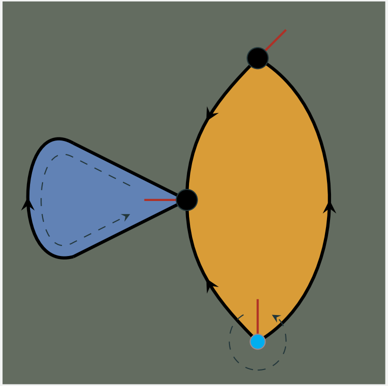
Where to go from here?

Our aim is to generalise this to finite-dimensional non-semisimple Hopf algebras. This needs three levels of generalisations:

1. lattices on oriented surfaces $\longrightarrow$ Kitaev graphs
2. regular Hopf bimodule $\longrightarrow$ involutive (twisted) Hopf bimodules
3. invariant subspaces $\longrightarrow$ bitensor products

Generalisation 1: From lattices to Kitaev graphs

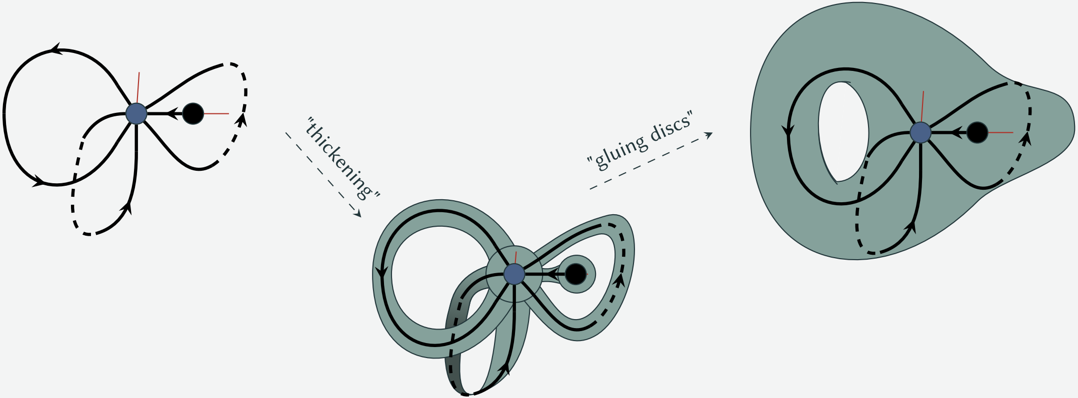
A Kitaev graph is a graph such as the following:



- finite and connected
- loops and multiple edges are allowed
- edges are directed
- "edge ends" at every vertex are totally ordered
- every vertex has a unique associated adjacent face
- choice of a distinguished vertex

Generalisation 1: From lattices to Kitaev graphs

Each Kitaev graph Γ gives rise to a surface with boundary Σ_Γ and a surface without boundary $\Sigma_\Gamma^{\mathrm{cl}}$:



Generalisation 1: From lattices to Kitaev graphs

Observation: We can increase the number of boundary components of Σ_Γ by gluing a certain graph $\mathbf{A}$ to Γ.

Theorem

There is a group $\mathfrak{G}$ acting on the set of Kitaev graphs such that for all graphs Γ and Δ:

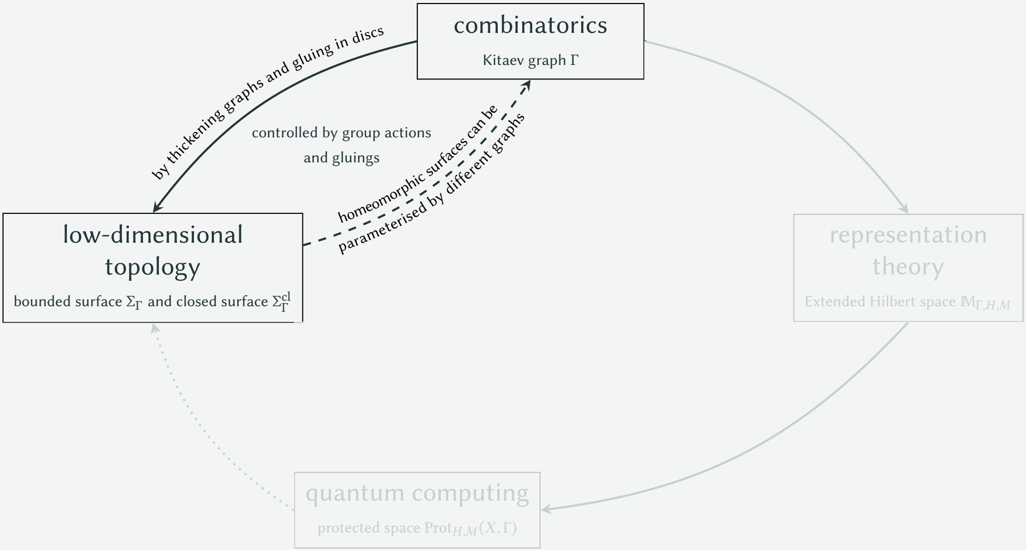
$$\Sigma_\Gamma^{\mathrm{cl}} \cong \Sigma_\Delta^{\mathrm{cl}} \iff \exists a, b \in \mathbb{N}_0 \textit{ with } \Delta \# \mathbf{A}^{\# a} \in \mathfrak{G} \cdot (\Gamma \# \mathbf{A}^{\# b}).$$

The group $\mathfrak{G}$ is connected to the mapping class group of surfaces.
A better (more conceptual understanding) of $\mathfrak{G}$ would be desirable!

Generalisation 1: supplementary material

1. We define Kitaev graphs as triples (ρ, C, pt) of a permutation $\rho \in S_\infty$, a set $C \subset \mathbb{N}$ and an element $\mathrm{pt} \in C$.
2. The group $\mathfrak{G}$ acts on these triples by either, sliding edges past each other or reversing the direction of an edge, respectively changing the labels of edges.
3. Edge slides are related to mapping class groups (via Whitehead moves).
4. This relation is worked out in detail for surfaces with one boundary component. Due to Jackson, there is a conjectural result for arbitrary surfaces.

Generalisation 1: From lattices to Kitaev graphs



Generalisation 2: Involutive Hopf bimodules

From now on: H is a finite-dimensional complex Hopf algebra.

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Classical: Decorate each edge of a Kitaev graph with the regular bimodule-bicomodule H. Use $S\colon\ H \longrightarrow H$ to model the reversal of edge directions.
Problem: $S^2 = \mathrm{id}_H$ is equivalent to H semisimple.

Generalisation 2: Involutive Hopf bimodules

Definition

An *involutive Hopf bimodule* is a pair of bimodule-bicomodule M an involution $\psi\colon M \longrightarrow M$ such that

$$\begin{aligned}(h \triangleright m \triangleleft g)_{[-1]} \otimes (h \triangleright m \triangleleft g)_{[0]} \otimes (h \triangleright m \triangleleft g)_{[1]} \\ = h_{(1)} m_{[-1]} g_{(1)} \otimes h_{(2)} \triangleright m_{[0]} \triangleleft g_{(2)} \otimes h_{(3)} m_{[1]} S^{-2}(g_{(3)})\end{aligned}$$

$$\psi(h \triangleright m) = \psi(m) \triangleleft S(h), \qquad \psi(m)_{[-1]} \otimes \psi(m)_{[0]} = S(m_{[1]}) \otimes \psi(m_{[0]}).$$

Key example: g, χ group-like and character such that $\chi(p) = 1$ and

$$\chi(m_{(1)}) S^2(m_{(2)}) p = \chi(m_{(2)}) p m_{(1)}, \qquad m \in H.$$

Set $M = H$ as vector space and define

$$g \triangleright m \triangleleft h = \chi^{-1}(h_{(2)}) g m h_{(1)}, \quad m_{[-1]} \otimes m_{[0]} \otimes m_{[1]} = m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} p.$$

Generalisation 2: Involutive Hopf bimodules

Theorem

There is an algebra $\overleftrightarrow{B(H)}$ whose modules coincide with involutive Hopf bimodules.

Idea: $\overleftrightarrow{B(H)} = \Bbbk\mathbb{Z}_2 \otimes H \otimes H^{\mathrm{op}} \otimes H^* \otimes (H^*)^{\mathrm{op}}$ as vector spaces.
As algebra it is defined as an iterated smash product.
The coinvariants of its modules are in particular anti-Yetter–Drinfeld modules.
If one replaces all copies of S by S^{-1} in the above definition, there is a functor sAYD $\longrightarrow \overleftrightarrow{B(H)}$-Mod. The stability conditions corresponds precisely to a "trivial" involution.

Generalisation 2: Extended Hilbert space

We fix an involutive Hopf bimodule (M, ψ).

Construction

Let Γ be a Kitaev graph. The extended Hilbert space is

$$\mathbb{M}_\Gamma \overset{\text{def}}{=} \otimes_{e \in E_\Gamma} M.$$

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A copy of M for every edge of Γ.

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Generalisation 2: Extended Hilbert space

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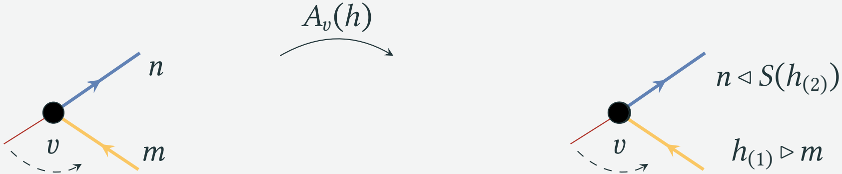
$$\mathbb{M}_\Gamma \stackrel{\text{def}}{=} \otimes_{e \in E_\Gamma} M.$$

For every vertex $v \in V_\Gamma$ and $f \in F_\Gamma$ define algebra maps

$$A_v \colon H \longrightarrow \mathrm{End}(\mathbb{M}_\Gamma), \qquad B_f \colon (H^*)^{\mathrm{op}} \longrightarrow \mathrm{End}(\mathbb{M}_\Gamma).$$

These maps depend on the "local" structure of Γ.

Generalisation 2: Extended Hilbert space



Generalisation 2: The extended Hilbert space as topological invariant

For Γ a Kitaev graph, set $H_v = H$ for each vertex $v \in V_\Gamma$ and write $\mathbb{H}_\Gamma \stackrel{\text{def}}{=} \otimes_{v \in V_\Gamma} H_v$.

Theorem

The local (co)actions turn $\mathbb{M}_\Gamma$ into an $\mathbb{H}_\Gamma$-Yetter–Drinfeld-module.
Suppose the graphs Γ and Δ satisfy $\Sigma_\Gamma \cong \Sigma_\Delta$. Then $\mathbb{M}_\Gamma$ and $\mathbb{M}_\Delta$ are isomorphic as Yetter–Drinfeld modules.

Generalisation 2: Supplementary material

To obtain the Yetter–Drinfeld module structure use: if M is a Hopf bimodule, it gives rise to a Yetter–Drinfeld module M_{YD} with underlying vector space M and (co)actions

$$h \blacktriangleright m = -h_{(1)} \triangleright m \triangleleft S(h_{(2)}), \qquad \delta(m) = m_{[-1]} \otimes m_{[0]}.$$

The definition of the local (co)actions of the extended Hilbert space is compatible with the involution (i.e. reversal of edge directions), implying that we always get locally the above module structure.

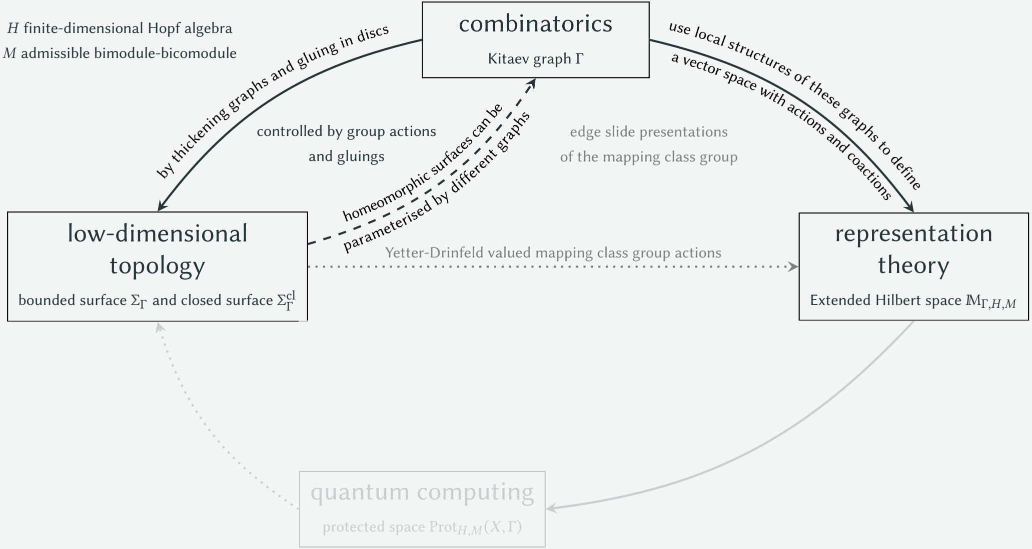
Generalisation 2: Supplementary material

To prove that extended Hilbert spaces are invariants of surfaces with boundary, it suffices to lift the action of the structure group $\mathfrak{G}$ to a Yetter–Drinfeld valued action on the space $\mathbb{M}_K = \oplus_\Gamma \mathbb{M}_\Gamma$ (the direct sum of "all" extended Hilbert spaces).
Particularly interesting: edge slides are algebraically modelled by the Yetter–Drinfeld braiding.

Generalisation 2: Involutive bimodules and extended Hilbert spaces

H finite-dimensional Hopf algebra

M admissible bimodule-bicomodule



Generalisation 3: Bitensor products

Definition

Let $(X, \triangleleft, \varrho) \in \mathcal{M}_H^H$ and $(N, \triangleright, \delta) \in {}_H^H\mathcal{M}$ be right-right and left-left modules-comodules respectively.

1. We write $\pi_{X,M} \colon X \otimes_{\mathbb{k}} M \longrightarrow X \otimes_H M \overset{\text{def}}{=} \operatorname{coker}(\triangleleft \otimes \mathrm{id}_M - \mathrm{id}_X \otimes \triangleright)$.
2. We set $\iota_{X,M} \colon X \square_H M \overset{\text{def}}{=} \ker(\varrho \otimes_{\mathbb{k}} \mathrm{id}_M - \mathrm{id}_X \otimes_{\mathbb{k}} \delta) \longrightarrow X \otimes_{\mathbb{k}} M$.
3. The *bitensor product* of X and M is $\mathrm{Bit}_H^H(X, M) \overset{\text{def}}{=} \operatorname{im} \iota_{X,M}\pi_{X,M}$.

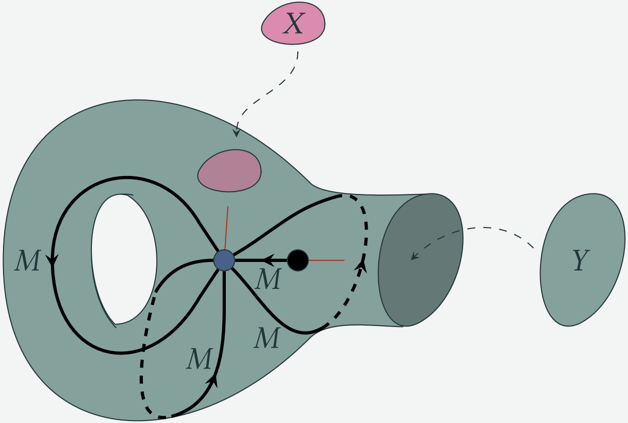
If H is semisimple and $N \in {}_H^H\mathcal{YD}$, we have $\mathrm{Bit}_H^H(\mathbb{k}_\varepsilon^1, N) \cong \mathrm{Hom}_{D(H)}(\mathbb{k}_\varepsilon^1, N)$.

Geleralisation 3: Bitensor product

In fact there is a natural isomorphism $\mathrm{Bit}_H^H(k_\varepsilon^a, -) \longrightarrow \mathrm{Hom}_{D(H)}(\mathbb{k}_\varepsilon^1, N)$, where a is the modular element of H. Its construction uses Frobenius algebra techniques. Properties of these functors provide information concerning the order of S^4. Interesting in positive characteristic.

Generalisation 3: Bitensor products and closing boundaries

Let $v \in V_\Gamma$ be a vertex. It has a corresponding boundary component b of Σ_Γ. We think of $\mathrm{Bit}_{H_v}^{H_v}(X, \mathbb{M}_\Gamma)$ as closing the boundary b by gluing in a disc labelled X.



Generalisation 3: The annular graph

From now on: We assume $\dim M^{\mathrm{coinv}} = 1$ and fix a $Y \in \mathcal{M}_H^H$ which "trivialises" the annular graph.
Note: If $\dim M^{\mathrm{coinv}} = 1$, a suitable Y always exists.
Warning: Not all Hopf algebras admit an involutive Hopf bimodule M such that $\dim M^{\mathrm{coinv}} = 1$.

Generalisation 3: The annular graph

The condition $\dim M^{\mathrm{coinv}} = 1$ means there is a one-dimensional anti-Yetter–Drinfeld module. That is, a pair in involution.

Examples of Hopf algebras without pairs in involution can be constructed as bosonisations of Nichols algebras.

The trivialisation condition for the annulus is the algebraic analogue of a disc being contractible.

Generalisation 3: The protected space

Definition

Let Γ be a Kitaev graph and $X \in \mathcal{M}_H^H$ and write

$$\mathbb{X}_\Gamma \overset{\text{def}}{=} \otimes_{v \in V_\Gamma} Z_v, \qquad \text{where } Z_v = \begin{cases} X & \text{if } v \text{ is distinguished,} \\ Y & \text{otherwise.} \end{cases}$$

We call $\mathrm{Prot}_M(X, \Gamma) \overset{\text{def}}{=} \mathrm{Bit}_{\mathbb{H}_\Gamma}^{\mathbb{H}_\Gamma}(\mathbb{X}_\Gamma, \mathbb{M}_\Gamma)$ the *protected space.*

Generalisation 3: Excision

Recall: Topological features of Kitaev graphs are controlled by the group $\mathfrak{G}$ and iterated gluings of graphs.

Theorem

Let Γ, Δ be two Kitaev graphs, we have a short exact sequence

$$0 \longrightarrow \mathrm{Prot}_M(X, \Gamma) \otimes \mathrm{Prot}_M(X', \Delta) \longrightarrow \mathrm{Prot}_M(X \otimes X', \Gamma \# \Delta) \longrightarrow \mathrm{CBit}_M(X, X', \Gamma, \Delta) \longrightarrow 0.$$

To proof the result, we have to control two things:

1. How to control bitensor products based on different algebras.
2. How to control different "coefficients" (modules-comodules) of the bitensor product.

These can be deduced from the universal property of the bitensor product.

Generalisation 3: Excision

The arrows in the above sequence are not natural as they rely on a vector space decomposition of $\mathrm{Prot}_M(X \otimes X', \Gamma \# \Delta)$.
Question: Can additional structures (such as a bilinear form on M) be used to make these maps natural?

Generalisation 3: Topological invariance

Corollary

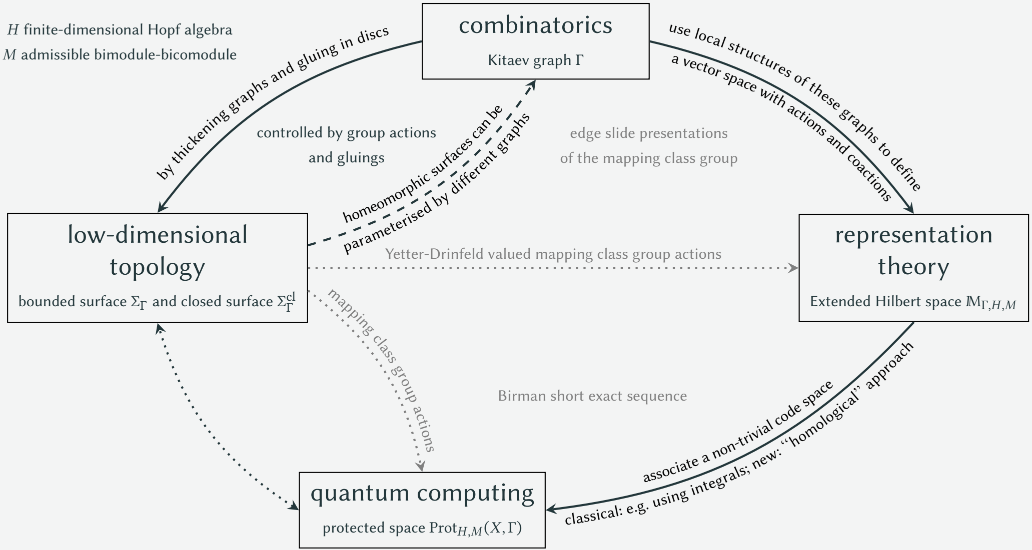
For any Γ *we have* $\mathrm{Prot}_M(X,\Gamma) \cong \mathrm{Prot}_M(X,\Gamma\#\mathbf{A})$. *In particular, if* Γ *and* Δ *are such that* $\Sigma_\Gamma^{\mathrm{cl}} \cong \Sigma_\Delta^{\mathrm{cl}}$ *then* $\mathrm{Prot}_M(X,\Gamma) \cong \mathrm{Prot}_M(X,\Delta)$.

Any surface can be decomposed as a connected sum of tori.
Explicit computations and geometrical interpretation of results?

Generalisation 3: Topological invariance

For almost tautological reasons one can show that we cannot have a topological invariant if $\dim M^{\mathrm{coinv}} > 1$.
Conversely, the main step of proving the previous result is that $\dim M^{\mathrm{coinv}} = 1$ is sufficient to trivialise the annulus.

Thank you



H finite-dimensional Hopf algebra
M admissible bimodule-bicomodule
combinatorics
Kitaev graph Γ
low-dimensional topology
bounded surface Σ_Γ and closed surface $\Sigma_\Gamma^{\mathrm{cl}}$
representation theory
Extended Hilbert space $\mathbb{M}_{\Gamma,H,M}$
quantum computing
protected space $\mathrm{Prot}_{H,M}(X,\Gamma)$
by thickening graphs and gluing in discs
use local structures of these graphs to define
a vector space with actions and coactions
controlled by group actions
and gluings
homeomorphic surfaces can be
parameterised by different graphs
edge slide presentations
of the mapping class group
Yetter-Drinfeld valued mapping class group actions
mapping class group actions
Birman short exact sequence
associate a non-trivial code space
classical: e.g. using integrals; new: "homological" approach

Uncovered topics

1. For group-algebras, the invariants we obtain have relations with Seifert fibered spaces via central extensions of fundamental groups.
2. “Reduction” procedure for bosonisations of Nichols algebras allows us to simplify the computation.
3. Induction and restriction type identities.

Uncovered topics

If we want to compute the bitensor product $\mathrm{Bit}_A^A(\Bbbk_1^\varepsilon, N)$, where

1. $A = B(V)\#H$ is a biproduct between a Nichols algebra and a semisimple Hopf algebra H and $\dim A < \infty$,
2. $\Bbbk_1^\varepsilon$ is trivial (not a necessary condition),
3. N is a A-Yetter–Drinfeld module,

we can use that we know the Jacobson radical $J(A)$. Set $N^{\mathrm{coH}} \overset{\mathrm{def}}{=} \{n \in N \mid \delta(n) \in H \otimes N\}$ and $\langle N \rangle \overset{\mathrm{def}}{=} \frac{N^{\mathrm{coH}}}{N^{\mathrm{coH}} \cap J(A)N}$, then

$$\mathrm{Bit}_A^A(\Bbbk_1^\varepsilon, N) \cong \mathrm{Bit}_H^H(\Bbbk_1^\varepsilon, \langle N \rangle).$$

Open questions

Topology:

1. Instead of generators and relations, is there a more conceptual decription of the group $\mathfrak{G}$?
2. State a presentation of mapping class groups in terms of edge slides.
3. What is the precise relation between $\mathfrak{G}$ and mapping class groups?

Hopf algebra side:

1. Investigate the extended Hilbert space for Doi-Koppinen data.
2. Investigate the extended Hilbert space for infinite-dimensional Hopf algebras.
3. Study involutive Hopf bimodules (also by extending the anti-Drinfeld double).

Bitensor product side:

1. Put additional structures on the involutive Hopf bimodule (M, ψ) to make the "Mayer–Vietrois sequence" canonical.
2. Compute the "CBit" spaces for non-trivial examples.
3. Give closed formulas for the dimensions of the protected spaces for well-behaved classes of algebras (such as Nichols algebras).
4. Study operations on the generalised protected spaces