Cyclic duality for slice 2-categories

Ulrich Krähmer (with John Boiquaye and Philipp Joram)

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I: Singular homology and the Dold-Kan correspondence

Homology

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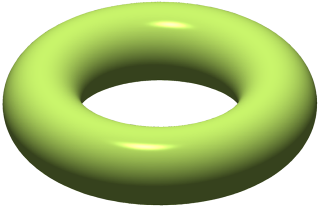
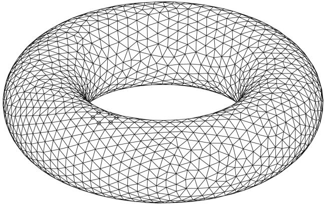
$$H_0(X) \cong \mathbb{K}, \quad H_1(X) \cong \mathbb{K}^2, \quad H_2(X) \cong \mathbb{K}, \quad H_n(X) = 0, \quad n > 2.$$



Triangulations

1. A **triangulation** of X is a decomposition into homeomorphic copies of **standard** n**-simplices** (points, edges, triangles, tetrahedra...)

$$\Delta_n \stackrel{\text{def}}{=} \{(x_0, \ldots, x_n) \in [0,1]^{n+1} \mid \sum_i x_i = 1\}.$$



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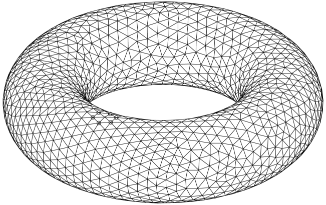

2. Let S_n^{tri} be the set of n-simplices in X, enumerate $S_0^{\mathrm{tri}} = \{1, \ldots, N\}$, and identify $s \in S_n^{\mathrm{tri}}$ with its $n+1$ corners, $s = \{s_0, \ldots, s_n\}$.

The boundary map b

1. Let $C_n^{\mathrm{tri}} \stackrel{\mathrm{def}}{=} \mathbb{K}^{S_n^{\mathrm{tri}}}$ be the vector space with basis S_n^{tri} and define

$$b \stackrel{\mathrm{def}}{=} \sum_{i=0}^{n} (-1)^i \partial_i \colon C_n^{\mathrm{tri}} \to C_{n-1}^{\mathrm{tri}},$$

where ∂_i is the (linear extension of the) map that assigns to a simplex its i-th face,

$$\partial_i(\{s_0, \dots, s_n\}) \stackrel{\mathrm{def}}{=} \{s_0, \dots, \widehat{s_i}, \dots, s_n\}.$$

The homology

1. These maps turn the sequence C^{tri} into a **chain complex**

$$\partial_i \partial_{j+1} = \partial_j \partial_i \text{ for } i \leqslant j \quad \Rightarrow \quad bb = 0.$$

2. The homology of the triangulated space X is

$$H(X) \stackrel{\mathrm{def}}{=} H(C, b) \stackrel{\mathrm{def}}{=} \ker b / \mathrm{im}\, b.$$

Singular chains

1. If you can't or don't want to fix a triangulation, let S_n^{sing} be the set of **singular simplices** in $X \stackrel{\mathrm{def}}{=}$ continuous maps $s \colon \Delta_n \to X$, and define $\partial_i \colon S_n^{\mathrm{sing}} \to S_{n-1}^{\mathrm{sing}}$ by

$$(\partial_i s)(x_0, \ldots, x_{n-1}) \stackrel{\mathrm{def}}{=} s(x_0, \ldots, x_i, 0, x_{i+1}, \ldots, x_{n-1}).$$

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2. You also get a chain complex that computes the same homology $H(X)$. This fits into a powerful general theory: $X \mapsto S^{\mathrm{sing}}$ is a **simplicial functor $\mathbf{Top} \to \mathbf{Set}$**.

Simplicial objects

Definition

The objects in the **simplicial category** $\boldsymbol{\Delta}$ are the ordered sets

$$[m] \stackrel{\text{def}}{=} \{0, \ldots, m\}, \quad m \geqslant 0.$$

Morphisms $f \colon [m] \to [n]$ are nondecreasing maps: $i \leqslant j \Rightarrow f(i) \leqslant f(j)$.

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A functor $S\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is a **simplicial object** in $\mathcal{C}$.

One usually writes $S_n \stackrel{\text{def}}{=} S([n])$.

Face and degeneracy maps in $\boldsymbol{\Delta}$

1. $\boldsymbol{\Delta}$ is generated by the **faces** $d_i \colon [m-1] \to [m]$ and **degeneracies** $s_j \colon [m+1] \to [m]$, $i, j = 0, \ldots, m$.

$d_2 \colon [3] \to [4]$

$$
\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & 2 \arrow[dr] & 3 \arrow[dr] & \\
0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

$s_2 \colon [5] \to [4]$

$$
\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[dl] & 4 \arrow[dl] & 5 \arrow[dl] \\
0 & 1 & 2 & 3 & 4 &
\end{tikzcd}
$$

Face and degeneracy maps on S

1. If $S\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is a simplicial object, the d_i, s_j induce morphisms $\partial_i\colon S_m \to S_{m-1}$ and $\sigma_j\colon S_m \to S_{m+1}$.
2. In particular, a simplicial vector space becomes as above a chain complex with boundary maps

$$b = \sum (-1)^i \partial_i.$$

The Dold-Kan correspondence

1. Every simplicial vector space C decomposes as chain complex as

$$C_n = \bigcap_{i<n} \ker\, \partial_i \oplus \sum_{i \geqslant 0} \mathrm{im}\ \sigma_i .$$

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2. $(\sum_{i\geqslant 0} \mathsf{im}\ \sigma_i, b)$ has trivial homology, so
$$(C, b) \overset{\text{quasi}}{\cong} (N(C), b) \overset{\text{def}}{=} C / \sum_{i \geqslant 0} \mathsf{im}\ \sigma_i.$$

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Theorem (Dold-Kan)

N is an equivalence simplicialal vector spaces $\simeq$ chain complexes.

II: Cyclic homology and the Dwyer-Kan correspondence

Duplicial and paracyclic objects

Definition

The **paracyclic category** Λ^∞ has the same objects as $\mathbf{\Delta}$. Morphisms $f\colon [m] \to [n]$ are nondecreasing maps $\mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(j+m+1) = f(j)+n+1.$$

Adding the condition $f(0) \geqslant 0$ yields the **duplicial category K**.

We view $\mathbf{\Delta}$ as subcategory of $\mathbf{K}$ by extending a nondecreasing map $[m] \to [n]$ using the above periodicity condition to a map $\mathbb{Z} \to \mathbb{Z}$.

The extra degeneracies

1. $\mathbf{K}$ is generated by $\mathbf{\Delta}$ and the maps $t\colon [m] \to [m], i \mapsto i+1$:

$$\begin{tikzcd}
\dots & -1 \arrow[dr] & 0 \arrow[dr] & 1 \arrow[dr] & 2 \arrow[dr] & 3 \arrow[dr] & 4 \arrow[dr] & \dots & \dots \\
\dots & \dots & 0 & 1 & 2 & 3 & 4 & 5 & \dots
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\dots & \dots & 0 & 1 & 2 & 3 & 4 & 5 & \dots
\end{tikzcd}$$

2. Alternatively, one can add to $\mathbf{\Delta}$ the **extra degeneracies** $s_{-1} \stackrel{\text{def}}{=} ts_m\colon [m+1] \to [m]$ as $t = s_{-1}d_{m+1}$. Here is $m = 3$:

$$\begin{tikzcd}
\dots & -2 \arrow[drr] & -1 \arrow[dr] & 0 \arrow[dr] & 1 \arrow[dr] & 2 \arrow[dr] & 3 \arrow[dr] & 4 \arrow[d] & 5 \arrow[d] & \dots \\
\dots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \dots
\end{tikzcd}$$

The Dwyer-Kan correspondence

1. A duplicial vector space C becomes a **mixed complex** (chain and cochain) with coboundary map

$$c \stackrel{\text{def}}{=} \sum_{j=-1}^{n} (-1)^{j+1} \sigma_j \colon C_n \to C_{n+1}, \quad cc = 0.$$

Theorem (Dwyer-Kan)

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Cyclic homology

1. Define $B\colon C_n \to C_{n+1}$ by

$$B \stackrel{\mathrm{def}}{=} c(1 + q + q^2 + \cdots + q^n), \quad q := 1 - bc.$$

Then $qc = c \Rightarrow BB = 0$.

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2. The **(periodic) cyclic homology** $HP(C)$ of (C, b, c) is the homology of the $\mathbb{Z}_2$-graded chain complex

$$T_0 \overset{b+B}{\longleftrightarrow} T_1 \,, \qquad T_i \stackrel{\mathrm{def}}{=} \bigoplus_{j \in \mathbb{N}} C_{2j+i}/\mathrm{im}\,(bB + Bb).$$

Commutative algebra $\rightsquigarrow$ noncommutative geometry

Theorem (Hochschild, Kostant, Rosenberg, Rinehart)

If A is a unital associative $\mathbb{K}$-algebra, then $C_n = A^{\otimes_{\mathbb{K}} n+1}$ is a duplicial vector space via

$$\partial_i(f_0 \otimes \cdots \otimes f_n) \stackrel{\text{def}}{=} \begin{cases} f_0 \otimes \cdots \otimes f_i f_{i+1} \otimes \cdots \otimes f_n, & i < n, \\ f_n f_0 \otimes \cdots \otimes f_{n-1}, \end{cases}$$

$$\sigma_j(f_0 \otimes \cdots \otimes f_n) = f_0 \otimes \cdots \otimes f_{j-1} \otimes 1 \otimes f_{j+1} \otimes f_n.$$

If A is the coordinate ring of a smooth affine variety X and $\mathbb{Q} \subseteq \mathbb{K}$, then $HP(C)$ is the (algberaic) De Rham cohomology of X.

III: Cyclic duality for slice 2-categories

Cyclic duality

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Theorem (Connes, Dwyer-Kan)

There are isomorphisms $\mathbf{\Lambda}^\infty \cong (\mathbf{\Lambda}^\infty)^{\mathrm{op}}, \mathbf{K} \cong \mathbf{K}^{\mathrm{op}}$ *with* $[m]^\circ \stackrel{\mathrm{def}}{=} [m]$ *and* $f^\circ \colon [n] \to [m]$ *given for* $f \colon [m] \to [n]$ *by* $f^\circ(i) \stackrel{\mathrm{def}}{=} \max\{j \mid -f(-j) \leqslant i\}$.

2. Elmendorf: With $a \perp b \stackrel{\mathrm{def}}{=} a \geqslant -b$, this means $(k = -j)$

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Question

So $\mathbf{K} \cong \mathbf{K}^{\mathrm{op}}, \boldsymbol{\Lambda}^{\infty} \cong (\boldsymbol{\Lambda}^{\infty})^{\mathrm{op}}$, but $\boldsymbol{\Delta} \not\cong \boldsymbol{\Delta}^{\mathrm{op}}$. Why?

(2,1)-categories

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1. Let $\mathcal{C}$ be a strict (2,1)-category all of whose 1-cells are monic.
2. Our motivating example is $\mathcal{C} = \mathbf{Mfld}^d$. Here:
3. Objects $T \in \mathbf{Mfld}^d_0$ are d-dimensional compact manifolds,
4. 1-cells $f\colon X \to Y$ are embeddings (injective smooth immersions),
5. 2-cells $[\phi]\colon f \Rightarrow g$ are isotopy classes of isotopies

$$\phi\colon [0,1] \times X \to Y, \quad \phi(0,-) = f, \quad \phi(1,-) = g.$$

6. Example: f, g could be two knots (embeddings of $S^1 \times D^2$ into $S^3 = \mathbb{R}^3 \cup \{\infty\}$) and ϕ deforms them into each other.

The slice category

1. A **subobject** of $T \in \mathcal{C}_0$ is an isomorphism class $[x]$ of an object

$$x\colon X \to T$$

in the **slice category** of $\mathcal{C}$ over T, that is, in the preorder of all 1-cells in $\mathcal{C}$ with codomain T with the preorder relation

$$x \leq y :\Leftrightarrow \exists f \in \mathcal{C}_1 : x = yf$$

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2. Example: In $\mathcal{C} = \mathbf{Mfld}^d$, a subobject is a submanifold $X \subseteq T$.

The slide 2-category

1. Same objects as the slice category,
2. 1-cells $x \to y$ are 2-cells $\phi\colon x \Rightarrow z$ in $\mathcal{C}$ with $z = yf \leq y$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & \arrow[d, phantom, "\Nearrow"] & Y \arrow[dl, "y"] \\ & T\,. & \end{tikzcd}$$

3. Example: If $X, Y \subseteq T$ are submanifolds, ϕ is a smooth family $\{X_t \subseteq T\}_{t\in[0,1]}$ of submanifolds with $X_0 = X$, $X_1 \subseteq Y$.

2-cells and the homotopy category

1. 2-cells between $\phi\colon x \Rightarrow yf$ and $\psi\colon x \Rightarrow yg$ are 2-cells ξ in $\mathcal{C}$ with

$$\psi = y\xi \circ \phi,$$

   where $y\xi$ is the horizontal composition of id_y with ξ.
2. So: 2-cells in $\mathcal{C}/T$ deform the target object $[z]$ of ϕ inside $[y]$.

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2. So: 2-cells in $\mathcal{C}/T$ deform the target object $[z]$ of ϕ inside $[y]$.
3. In the **homotopy category** $\mathrm{ho}(\mathcal{C}/T)$ such final perturbations of $[z]$ in $[y]$ get identified.
4. Example: A 2-cell between families $\{X_t\}, \{\tilde{X}_t\}$ of submanifolds is an extension of X_t to $t \in [1,2]$ with $X_t \subseteq Y$ for $t \geqslant 1$ and $X_2 = \tilde{X}_1$, which after reparametrisation becomes isotopic to $\{\tilde{X}_t\}$.

An answer to our question - informal version

1. After adding initial and terminal objects, $\mathbf{\Delta}$ and $\mathbf{K}$ are (skeletal subcategories of) $\mathrm{ho}(\mathbf{Mfld}^1/T)$ with $T = [0,1]$ resp. $T = S^1$.
2. The $m+1$ elements of $[m]$ become replaced by m intervals in T (tubular neighbourhoods if you want).

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Theorem (Boiquaye-Joram-K)

$\mathrm{ho}(\mathbf{Mfld}^d/T) \simeq \mathrm{ho}(\mathbf{Mfld}^d/T)^{\mathrm{op}}$ if $\partial T = \varnothing$, with $X^\circ = \overline{T \setminus X}$. The dual of a morphism also inverts time, $\{X_t^\circ\} = \{\overline{T \setminus X_{1-t}}\}$.

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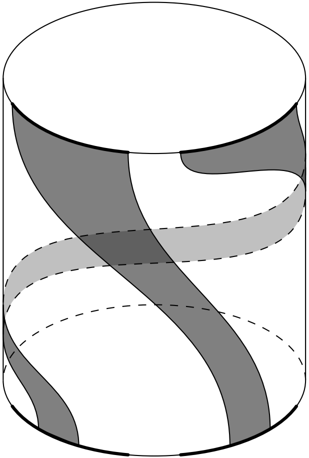
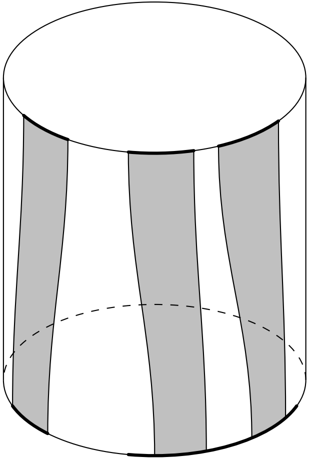
Theorem (Boiquaye-Joram-K)

$\mathrm{ho}(\mathbf{Mfld}^d/T) \simeq \mathrm{ho}(\mathbf{Mfld}^d/T)^{\mathrm{op}}$ *if* $\partial T = \varnothing$*, with* $X^\circ = \overline{T \setminus X}$*. The dual of a morphism also inverts time,* $\{X_t^\circ\} = \{\overline{T \setminus X_{1-t}}\}$*.*

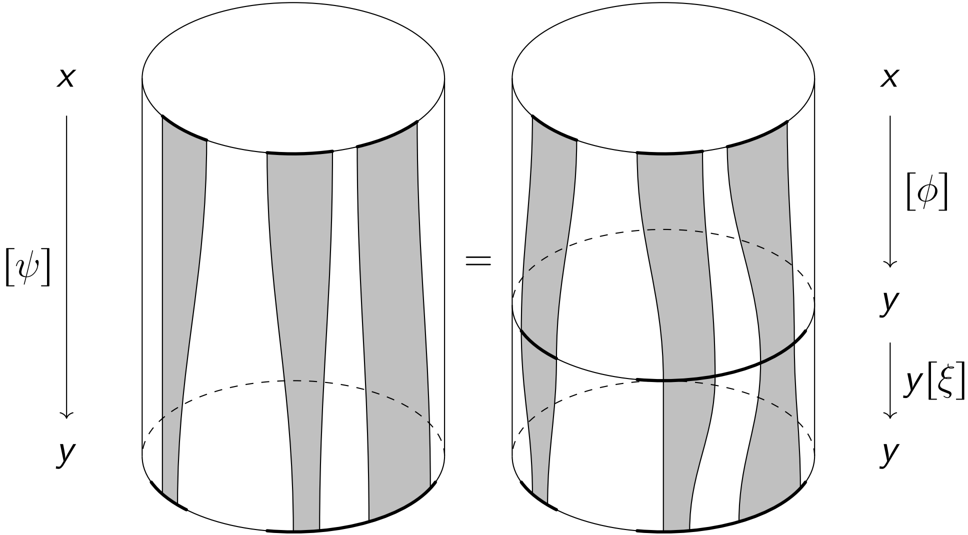
3. So $[m]^\circ = [m]$ is in a sense a coincidence!

$t\colon [1] \to [1]$ and $s_1\colon [2] \to [1]$





The action of 2-cells



More precisely

1. Assume there is a self-duality of the slice category $((\mathcal{C}/T)_0, \leq)$ that is equivariant with respect to the natural action of the automorphism group $\mathrm{Aut}(T)$ of T, i.e. a map $\circ\colon (\mathcal{C}/T)_0 \to (\mathcal{C}/T)_0$, $x \mapsto x^\circ$ with

$$[x^{\circ\circ}] = [x], \quad [(gx)^\circ] = [g(x^\circ)], \quad x \leq y \Leftrightarrow y^\circ \leq x^\circ$$

for all $x, y \in \mathcal{C}/T_0$ and $g \in \mathrm{Aut}(T)$.

2. In $\mathcal{C} = \mathbf{Mfld}^d$, x is the embedding of a submanifold $X \subseteq T$ and on the slice 1-category we have an obvious duality in which x° embeds the closure of $T \backslash X$ into T.

More precisely

1. Such a duality on $\mathcal{C}/T$ defines a subrelation $\lll$ of $\leq$ which is an $\mathrm{Aut}(T)$-cosieve in $((\mathcal{C}/T)_0, \leq)$ (is closed under $\mathrm{Aut}(T)$-action and under postcomposition with any 1-cell).

$$u \lll v :\Leftrightarrow \forall \xi \colon x \Rightarrow z\, \exists \gamma \colon \mathrm{id}_T \Rightarrow g : t(x) = s(v^\circ) \Rightarrow \gamma x = x, \gamma v^\circ x = v^\circ \xi.$$

2. In $\mathcal{C} = \mathbf{Mfld}^d$, $u \lll v$ means that the submanifold $[u]$ is not just contained in the submanifold $[v]$, but is contained in ints interior.

More ingredients

1. Our main result states that if all $x \in (\mathcal{C}/T)_0$ satisfy a strong form of the **homotopy extension property** and admit an abstract version of **tubular neighbourhoods**, then $\circ$ extends to $\mathrm{ho}(\mathcal{C}/T)$.

More ingredients

1. Our main result states that if all $x \in (\mathcal{C}/T)_0$ satisfy a strong form of the **homotopy extension property** and admit an abstract version of **tubular neighbourhoods**, then $\circ$ extends to $\mathrm{ho}(\mathcal{C}/T)$.
2. We abbreviate
$$G \stackrel{\mathrm{def}}{=} \bigcup_{g \in \mathrm{Aut}(T)} \mathcal{C}_2(\mathrm{id}_T, g).$$
This is a group under horizontal composition, the **source group** of the **automorphism 2-group Aut**(T) of T.

Main theorem

Theorem

Assume $\lll$ *is an* $\mathrm{Aut}(T)$*-cosieve in* $(\mathcal{C}/T_0, \leq)$ *with*

1. $\mathrm{id}_T^\circ \lll \mathrm{id}_T^\circ$,
2. *for all* $f, h \colon X \to Y$, $y \colon Y \to T$, *and* $\phi \colon yf \Rightarrow yh$, *we have*
$$(\exists \xi \colon f \Rightarrow h : \phi = y\xi) \quad \Leftrightarrow \quad (\forall u \lll y^\circ \, \exists \gamma \in G : \gamma u = u, \gamma y f = \phi)$$
3. *for all* $u \lll y^\circ, v \lll y^\circ$ *there exists* $\tau \colon \mathrm{id}_T \Rightarrow t$ *in* G *and* $r \lll y$ *with*
$$\tau u = u, \ \tau v = v, \ [tr] = [y].$$

Then $\circ$ *lifts to an* $\mathrm{Aut}(T)$*-equivariant self-duality on* $\mathrm{ho}(\mathcal{C}/T)$.