A non-semisimple Kitaev lattice model

based on joint work with Ulrich Krähmer

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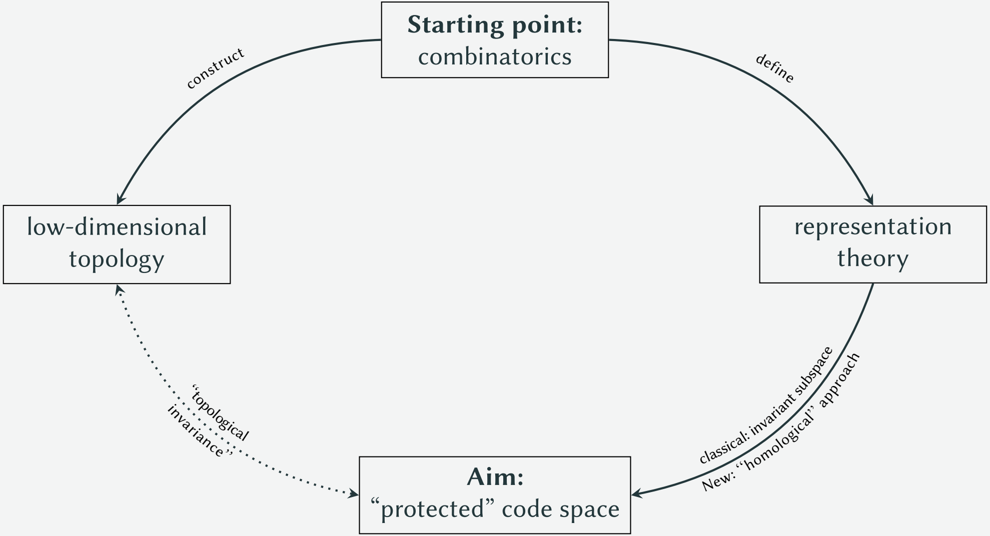
The big picture



Fault-tolerant quantum computation: Kitaev's toric code

Kitaev's construction for fault-tolerant quantum computation:

1. Consider a $k \times k$ lattice in the torus and associate the Hilbert space $\mathbb{M}_k$.
2. Specify a "Hamiltonian" $H\colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$ using "local" features of the lattice.
3. Define as a "quantum memory" the space $\mathrm{Prot}_k \stackrel{\text{def}}{=} \ker H$.

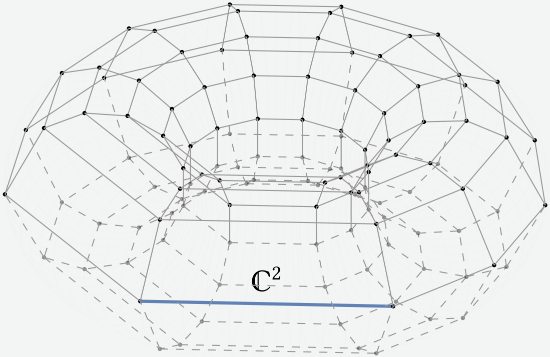
This leads to error correction/prevention implemented on a physical level:

Theorem

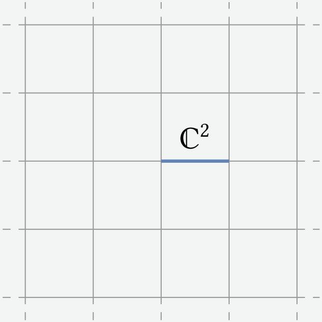
For any $k \in \mathbb{N}$ *we have* $\dim \mathrm{Prot}_k = 4$.

Kitaev’s toric code – lattices and the (extended) Hilbert space

Let $k \in \mathbb{N}$ and consider a $k \times k$ lattice embedded on a torus:



draw as

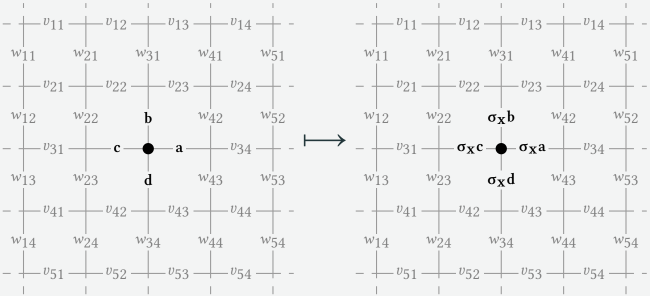


Assign to each *edge* a copy of $\mathbb{C}^2$ and set $\mathbb{M}_k = \otimes_{1\leq i,j\leq k}\mathbb{C}^2 \bigotimes \otimes_{1\leq i,j\leq k}\mathbb{C}^2$.

Kitaev's toric code – vertex actions

Set $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Given a *vertex* (point in the lattice) v define

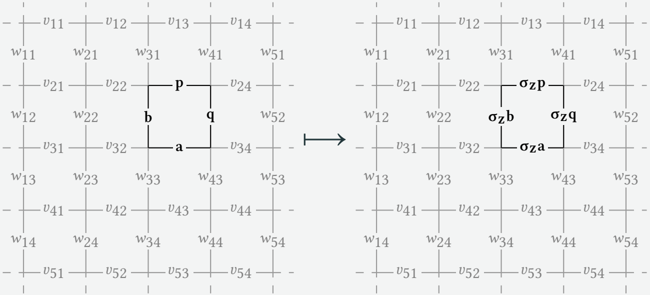
$$A_v \colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$$



1. Gives rise to a representation $\triangleright_v \colon \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{M}_k \longrightarrow \mathbb{M}_k$.
2. Combining all "vertex actions" leads to $\triangleright \colon (\mathbb{C}\mathbb{Z}_2)^{\otimes k^2} \otimes \mathbb{M}_k \longrightarrow \mathbb{M}_k$.

Set $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Given a *face* (square of the lattice) f define

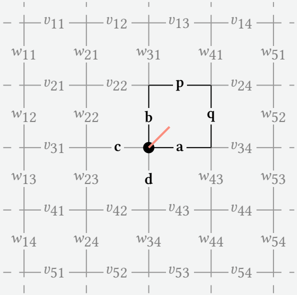
$$B_f \colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$$



1. Can be interpreted as a corepresentation $\delta_f \colon \mathbb{M}_k \longrightarrow \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{M}_k$.
2. Combining all "face coactions" leads to a "global" comodule structure $\delta \colon \mathbb{M}_k \longrightarrow (\mathbb{C}\mathbb{Z}_2)^{\otimes k^2} \otimes \mathbb{M}_k$.

The local and global module structure of $\mathbb{M}_k$

Associate to each vertex the face to its "top right" $\rightsquigarrow$ $\{\text{vertices}\} \xrightarrow{1:1} \{\text{faces}\}$.
For every such vertex-face-pair (v, f) the vertex action and face coaction turn $\mathbb{M}_k$ into a $D(\mathbb{C}\mathbb{Z}_2)$-module.



Topological invariance of the protected space

We consider $\mathbb{M}_k$ as a $D(\mathbb{C}\mathbb{Z}_2)^{\otimes k^2}$-module

Definition

The *protected space* of the Kitaev lattice model is $\mathrm{Prot}_k = \mathbb{M}_k^{\mathrm{inv}}$.

Theorem (Kitaev '97)

For any $k \in \mathbb{N}$ we have $\dim \mathrm{Prot}_k = 4$.

Proof sketch (based on Buerschaper, Mombelli, Christandl, Aguado '18).

The Haar integral of $D(\mathbb{C}\mathbb{Z}_2)$ defines a projector onto Prot_k. Taking traces yields the dimension. $\square$

Works for complex semisimple Hopf algebras.

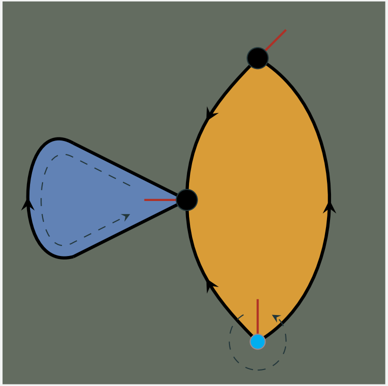
Where to go from here?

Our aim is to generalise this to finite-dimensional non-semisimple Hopf algebras.
This needs three levels of generalisations:

1. lattices on oriented surfaces $\longrightarrow$ Kitaev graphs
2. regular Hopf bimodule $\longrightarrow$ involutive (twisted) Hopf bimodules
3. invariant subspaces $\longrightarrow$ bitensor products

Generalisation 1: From lattices to Kitaev graphs

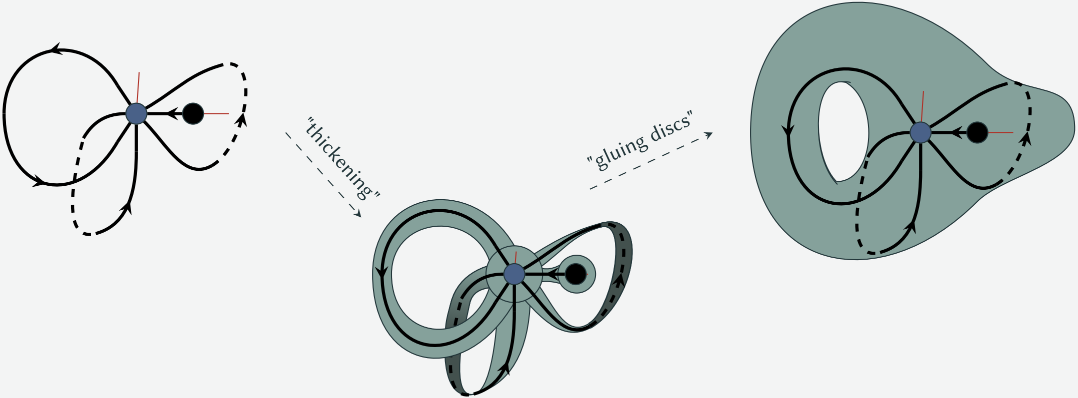
A Kitaev graph Γ is a graph such as the following:



- finite and connected
- loops and multiple edges are allowed
- edges are directed
- "edge ends" at every vertex are totally ordered
- every vertex has a unique associated adjacent face
- choice of a distinguished vertex

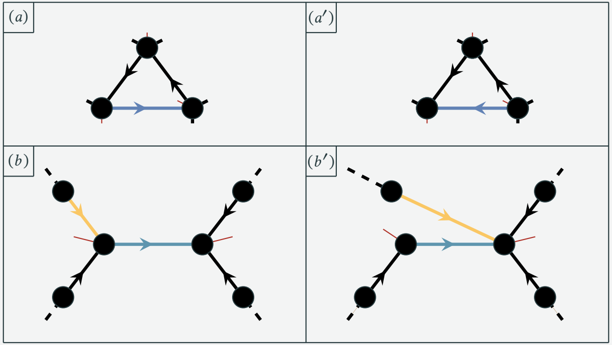
Generalisation 1: From lattices to Kitaev graphs

Each Kitaev graph Γ gives rise to a surface with boundary Σ_Γ and a surface without boundary $\Sigma_\Gamma^{\mathrm{cl}}$:



Generalisation 1: From lattices to Kitaev graphs

Construction: We define two "elementary transformations" of Kitaev graphs:



Leads to a "structure group" $\mathfrak{G}$ acting on the set of Kitaev graphs.

Generalisation 1: From lattices to Kitaev graphs

Observation: We can increase the number of boundary components of Σ_Γ without changing the genus by gluing a certain graph $\mathbf{A}$ to Γ.

Theorem

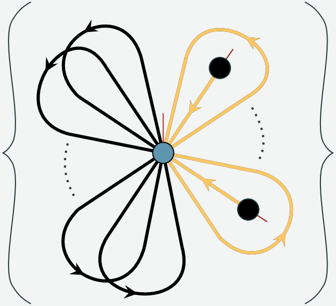
For all Kitaev graphs Γ, Δ *we have:*

$$\begin{aligned} \Sigma_\Gamma \cong \Sigma_\Delta \quad &\iff \quad \Delta \in \mathfrak{G} \bullet \Gamma \\ \Sigma_\Gamma^{\mathrm{cl}} \cong \Sigma_\Delta^{\mathrm{cl}} \quad &\iff \quad \exists a, b \in \mathbb{N}_0 \textit{ with } \Delta \# \mathbf{A}^{\# a} \in \mathfrak{G} \bullet (\Gamma \# \mathbf{A}^{\# b}). \end{aligned}$$

Generalisation 1: From lattices to Kitaev graphs

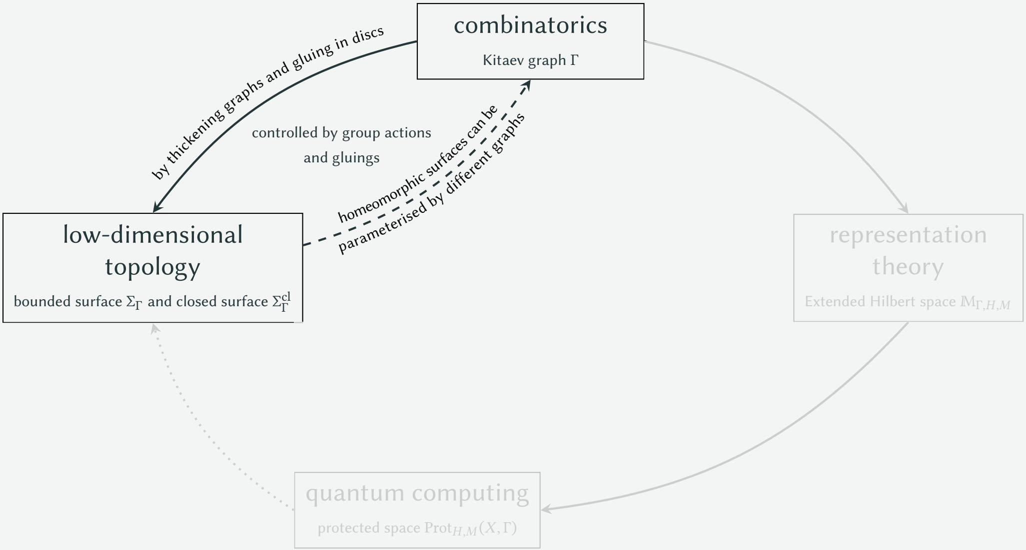
Key aspect of the proof: Each orbit of the action of $\mathfrak{G}$ contains a unique *standard graph*

this "half" of the graph controls the genus.



this "half" of the graph consists of copies of A and adds to the number of boundary components.

Generalisation 1: From lattices to Kitaev graphs



Generalisation 2: Involutive Hopf bimodules

From now on: H is a finite-dimensional Hopf algebra over an arbitrary field $\mathbb{k}$.

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Classical: Decorate each edge of a Kitaev graph with the regular bimodule-bicomodule H. Use $S\colon\ H \longrightarrow H$ to model the reversal of edge directions.
Problem: $S^2 = \mathrm{id}_H$ and $\dim H \in \Bbbk^\times$ is equivalent to H semisimple and cosemisimple.

Generalisation 2: Involutive Hopf bimodules

Definition

An *involutive Hopf bimodule* is a pair of a bimodule-bicomodule M an involution $\psi\colon M \longrightarrow M$ such that for all $g, h \in H$ and $m \in M$

$$
\begin{aligned}
(h \triangleright m \triangleleft g)_{[-1]} \otimes (h \triangleright m \triangleleft g)_{[0]} \otimes (h \triangleright m \triangleleft g)_{[1]} \\
= h_{(1)} m_{[-1]} g_{(1)} \otimes h_{(2)} \triangleright m_{[0]} \triangleleft g_{(2)} \otimes h_{(3)} m_{[1]} \mathrm{S}^{-2}(g_{(3)})
\end{aligned}
$$

$$
\psi(h \triangleright m) = \psi(m) \triangleleft \mathrm{S}(h), \qquad \psi(m)_{[-1]} \otimes \psi(m)_{[0]} = \mathrm{S}(m_{[1]}) \otimes \psi(m_{[0]}).
$$

Generalisation 2: Involutive Hopf bimodules

Key example: $p, \chi \in G(H) \times G(H^*)$ group-like and character with $\chi(p) = 1$ and

$$\chi(m_{(1)})S^2(m_{(2)})p = \chi(m_{(2)})pm_{(1)}, \qquad \text{for all } m \in H.$$

Set $M = H$ as vector space and define

$$g \triangleright m \triangleleft h = \chi^{-1}(h_{(2)})gmh_{(1)}, \quad m_{[-1]} \otimes m_{[0]} \otimes m_{[1]} = m_{(-1)} \otimes m_{(0)} \otimes m_{(1)}p,$$
$$\psi(m) = \chi(m_{(1)})p^{-1}S(m_{(2)}).$$

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$$\psi(m) = \chi(m_{(1)})p^{-1}S(m_{(2)}).$$

Observation

If we change all $S^{\mp}$ to $S^{\pm}$ in the previous definition, we obtain an embedding of stable anti-Yetter–Drinfeld modules into the category of involutive S^2-twisted Hopf bimodules.

Generalisation 2: Involutive Hopf bimodules

Theorem

There is an algebra $\overleftrightarrow{B}(H)$ with underlying vector space $\Bbbk\mathbb{Z}_2 \otimes H \otimes H^{\mathrm{op}} \otimes H^* \otimes (H^*)^{\mathrm{op}}$ whose modules coincide with involutive Hopf bimodules.

Generalisation 2: Extended Hilbert space

We fix an involutive Hopf bimodule (M, ψ).

Construction

Let Γ be a Kitaev graph. The extended Hilbert space is

$$\mathbb{M}_\Gamma \overset{\text{def}}{=} \otimes_{e \in E_\Gamma} M.$$

Generalisation 2: Extended Hilbert space

We fix an involutive Hopf bimodule (M, ψ).

Construction

Let Γ be a Kitaev graph. The ex A copy of M for every edge of Γ.

$$\mathbb{M}_\Gamma \stackrel{\text{def}}{=} \otimes_{e \in E_\Gamma} M.$$

Generalisation 2: Extended Hilbert space

We fix an involutive Hopf bimodule (M, ψ).

Construction

Let Γ be a Kitaev graph. The extended Hilbert space is

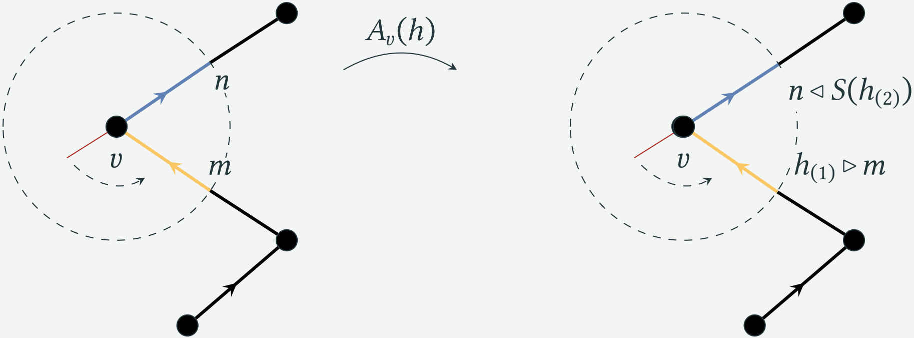
$$\mathbb{M}_\Gamma \stackrel{\text{def}}{=} \otimes_{e \in E_\Gamma} M.$$

For every vertex $v \in V_\Gamma$ and $f \in F_\Gamma$ define algebra maps

$$A_v \colon H \longrightarrow \mathrm{End}(\mathbb{M}_\Gamma), \qquad B_f \colon (H^*)^{\mathrm{op}} \longrightarrow \mathrm{End}(\mathbb{M}_\Gamma).$$

These maps depend on the “local” structure of Γ.

Generalisation 2: Extended Hilbert space



Generalisation 2: The extended Hilbert space as a Yetter–Drinfeld module

For Γ a Kitaev graph, set $H_v = H$ for each vertex $v \in V_\Gamma$ and write $\mathbb{H}_\Gamma \stackrel{\text{def}}{=} \otimes_{v \in V_\Gamma} H_v$.

Theorem

The local (co)actions turn $\mathbb{M}_\Gamma$ into an $\mathbb{H}_\Gamma$-Yetter–Drinfeld-module.

Generalisation 2: From involutive Hopf bimodules to Yetter–Drinfeld modules

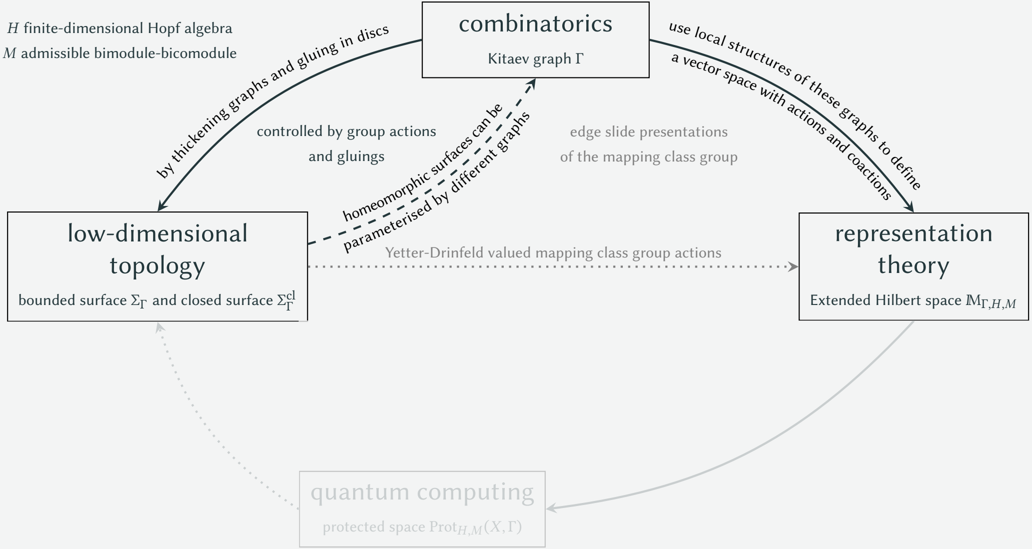
Use: The Hopf bimodule M admits a left-left Yetter–Drinfeld module structure:

$$h \bullet m = h_{(1)} \triangleright m \triangleleft S(h_{(2)}), \qquad \delta(m) = m_{[-1]} \otimes m_{[0]}, \qquad h \in H, m \in M.$$

The map $\psi \colon M \longrightarrow M$, implies that $\mathbb{M}_\Gamma$ is “locally” of the above form.

Generalisation 2: Involutive bimodules and extended Hilbert spaces

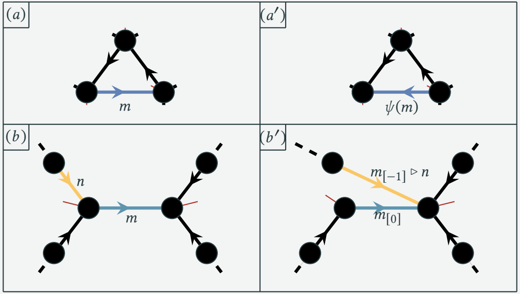
H finite-dimensional Hopf algebra
M admissible bimodule-bicomodule



Theorem

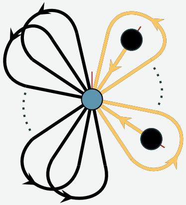
Suppose the graphs Γ and Δ satisfy $\Sigma_\Gamma \cong \Sigma_\Delta$. Then $\mathbb{M}_\Gamma$ and $\mathbb{M}_\Delta$ are isomorphic as Yetter–Drinfeld modules.

Proof strategy: Transfer the "elementary transformations" of Kitaev graphs to the level of extended Hilbert spaces.



Generalisation 3: Invariants of surfaces with boundary

Upshot: If we want to construct invariants of *closed surfaces*, it suffices to study standard graphs.



Goal: “Remove” the yellow loops.

Generalisation 3: The classical ground state

The classical protected ground state:
$\mathrm{Prot}(\Gamma) = \{m \in \mathbb{M}_\Gamma \mid \delta_f(m) = 1 \otimes m, h \bullet_v m = \varepsilon(h)m, f \in F_\Gamma, v \in V_\Gamma, h \in H\}$.
We have $\mathrm{Prot}(\Gamma) = \mathrm{im}(s\pi\iota)$, where:

$$\begin{tikzcd} \mathbb{M}_\Gamma^{\mathrm{coinv}} \arrow[r, hook, "\iota"] & \mathbb{M}_\Gamma \arrow[r, two heads, shift right, "\pi"'] & \mathbb{M}_\Gamma / \mathbb{H}_\Gamma^+ \mathbb{M}_\Gamma . \arrow[l, dashed, shift right, "s"'] \end{tikzcd}$$

Observation: The section s is given by $[m] \longmapsto \Lambda \bullet m$ for an integral $\Lambda \in \mathbb{H}_\Gamma$ with $\Lambda^2 = \Lambda$. Thus, s only exists if H is semisimple.
However: $\mathrm{im}(\pi\iota) \cong \mathrm{im}(s\pi\iota)$.

Generalisation 3: Bitensor products

Definition

Let $(X, \triangleleft, \varrho) \in \mathsf{M}_H^H$ and $(N, \triangleright, \delta) \in {}_H^H\mathsf{M}$ be right-right and left-left modules-comodules respectively.

1. We write $\pi_{X,M}\colon\ X \otimes_{\Bbbk} M \longrightarrow X \otimes_H M \stackrel{\text{def}}{=} \operatorname{coker}(\triangleleft \otimes \mathrm{id}_M - \mathrm{id}_X \otimes \triangleright)$.
2. We set $\iota_{X,M}\colon\ X \square_H M \stackrel{\text{def}}{=} \ker(\varrho \otimes_{\Bbbk} \mathrm{id}_M - \mathrm{id}_X \otimes_{\Bbbk} \delta) \longrightarrow X \otimes_{\Bbbk} M$.
3. The *bitensor product* of X and M is $\mathrm{Bit}_H^H(X, M) \stackrel{\text{def}}{=} \operatorname{im} \pi_{X,M} \iota_{X,M}$.

Generalisation 3: Bitensor products and invariant subspaces

Theorem

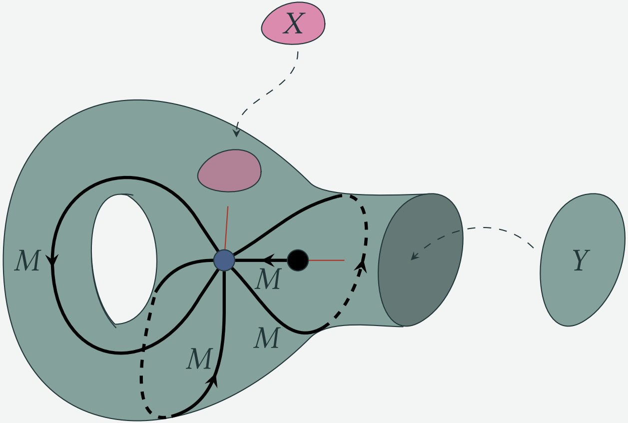
The following are equivalent:

1. *H is semisimple and counimodular, and*
2. *There is a natural isomorphism $\mathrm{Bit}_H^H(\Bbbk_\varepsilon^a, N) \cong \mathrm{Hom}_{D(H)}({}_\varepsilon^1\Bbbk, N)$, where $a \in G(H)$ is the modular element of H and $N \in {}_H\mathsf{YD}$.*

Consequence: In the complex semisimple case, our theory reduces to the classical Kitaev model if we set $M = H_{\text{reg}}$ to be the regular involutive Hopf bimodule ((co)actions are given by (co)multiplication) with the antipode as involution.

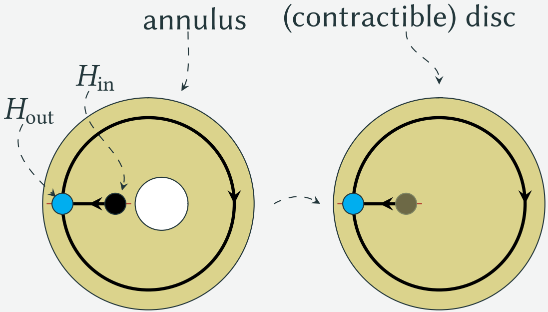
Generalisation 3: Bitensor products and closing boundaries

Let $v \in V_\Gamma$ be a vertex. It has a corresponding boundary component b of Σ_Γ. We think of $\mathrm{Bit}_{H_v}^{H_v}(X, \mathbb{M}_\Gamma)$ as closing the boundary b by gluing in a disc labelled X.



Generalisation 3: The annular graph

By closing the inner boundary component of the annular graph, we obtain a disc. We translate its contractability to the condition that the associated space is the trivial Yetter–Drinfeld module.



extended Hilbert space $\mathbb{M}_{\mathrm{A}}$ $\qquad$ $\mathrm{Bit}^{H_{\mathrm{in}}}_{H_{\mathrm{in}}}(Y, \mathbb{M}_{\mathrm{A}}) \overset{!}{=} {}^{1}_{\varepsilon}\Bbbk \in {}_{H_{\mathrm{out}}}\mathsf{YD}$

Generalisation 3: The annular graph

Lemma

The following are equivalent:

1. $M^{\mathrm{coinv}} = \{m \in M \mid m_{[-1]} \otimes m_{[0]} = 1 \otimes m\}$ *is one-dimensional, and*
2. *there exists a* $Y \in \mathsf{M}_H^H$ *such that* $\mathrm{Bit}_{H_{\mathrm{in}}}^{H_{\mathrm{in}}}(Y, \mathbb{M}_{\mathbf{A}}) \cong {}_{\varepsilon}^{1}\Bbbk$.

From now on: We assume $\dim M^{\mathrm{coinv}} = 1$ and fix a $Y \in \mathsf{M}_H^H$ which "trivialises" the annular graph.

Warning: Not all Hopf algebras admit an involutive Hopf bimodule M such that $\dim M^{\mathrm{coinv}} = 1$.

Generalisation 3: The protected space

Definition

Let Γ be a Kitaev graph and $X \in \mathsf{M}_H^H$ and write

$$\mathbb{X}_\Gamma \stackrel{\text{def}}{=} \otimes_{v \in V_\Gamma} Z_v, \qquad \text{where } Z_v = \begin{cases} X & \text{if } v \text{ is distinguished,} \\ Y & \text{otherwise.} \end{cases}$$

We call $\mathrm{Prot}_H(X, M, \Gamma) \stackrel{\text{def}}{=} \mathrm{Bit}_{\mathbb{H}_\Gamma}^{\mathbb{H}_\Gamma}(\mathbb{X}_\Gamma, \mathbb{M}_\Gamma)$ the *protected space with coefficient X*.
We refer to $\mathrm{Prot}_H(\mathbb{k}_\varepsilon^1, M, \Gamma)$ as the *ground state*.

Generalisation 3: Excision

Aim: Understand the gluing of Kitaev graphs algebraically.

Theorem

Let Γ, Δ *be two Kitaev graphs and* $X, X' \in \mathsf{M}_H^H$ *There is a space* $\mathrm{Aux}_H(X \otimes X', M, \Gamma\#\Delta)$ *with canonical embeddings and projections*

$$\begin{tikzcd}
\mathrm{Prot}_H(X, M, \Gamma) \otimes \mathrm{Prot}_H(X', M, \Delta) \arrow[dr, hook, "k"] & & \mathrm{Prot}_H(X \otimes X', M, \Gamma\#\Delta) \arrow[dl, two heads, "\pi"'] \\
& \mathrm{Aux}_H(X \otimes X', M, \Gamma\#\Delta) &
\end{tikzcd}$$

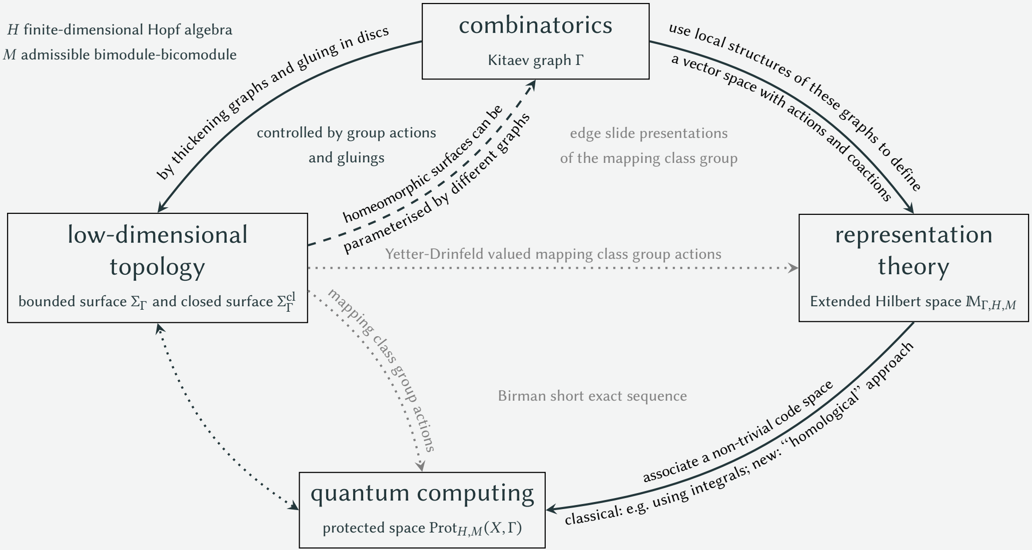
Generalisation 3: Topological invariance

Corollary

For any Γ *we have* $\mathrm{Prot}_H(X, M, \Gamma) \cong \mathrm{Prot}_H(X, M, \Gamma \# \mathbf{A})$. *In particular, if* Γ *and* Δ *are such that* $\Sigma_\Gamma^{\mathrm{cl}} \cong \Sigma_\Delta^{\mathrm{cl}}$ *then* $\mathrm{Prot}_H(X, M, \Gamma) \cong \mathrm{Prot}_H(X, M, \Delta)$.

Any surface can be decomposed as a connected sum of tori.
Explicit computations and geometrical interpretation of results?

Finally...



H finite-dimensional Hopf algebra
M admissible bimodule-bicomodule
combinatorics
Kitaev graph Γ
by thickening graphs and gluing in discs
use local structures of these graphs to define
a vector space with actions and coactions
controlled by group actions
and gluings
homeomorphic surfaces can be
parameterised by different graphs
edge slide presentations
of the mapping class group
low-dimensional
topology
bounded surface Σ_Γ and closed surface $\Sigma_\Gamma^{\mathrm{cl}}$
Yetter-Drinfeld valued mapping class group actions
representation
theory
Extended Hilbert space $\mathbb{M}_{\Gamma,H,M}$
mapping class group actions
Birman short exact sequence
associate a non-trivial code space
classical: e.g. using integrals; new: "homological" approach
quantum computing
protected space $\mathrm{Prot}_{H,M}(X,\Gamma)$

Ok great. But what do we *actually* compute?

Ok great. But what do we *actually* compute?
Let us compare our approach to the classical case.

The classical case reinterpreted

Definition

Let $g \in \mathbb{N}_0$. If $g = 0$, set $\varpi_1(\Sigma_g) \overset{\text{def}}{=} \langle x \rangle \cong \mathbb{Z}$. Otherwise, $\varpi_1(\Sigma_g)$ is generated by $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, x$ and the relations

$$[\beta_g, \alpha_g^{-1}] \dots [\beta_1, \alpha_1^{-1}] = x$$
$$\alpha_i x = x\alpha_i, \qquad \beta_i x = x\beta_i, \qquad 1 \leq i \leq g.$$

The group $\varpi_1(\Sigma_g)$ is a central extension of the fundamental group of a surface of genus g.
Such groups arise in the study of Seifert fibered spaces.

The classical case reinterpreted

Fix a finite group G.

Lemma

The space $\Omega_g \stackrel{\text{def}}{=} \operatorname{span}_{\Bbbk} \operatorname{Hom}_{\mathrm{Grp}}(\varpi_1(\Sigma_g), G)$ *becomes a* $\Bbbk G$*-Yetter–Drinfeld module by setting for all* $y, h \in G$, $f \in \operatorname{Hom}_{\mathrm{Grp}}(\varpi_1(\Sigma_g), G)$*:*

$$\delta(f) \stackrel{\text{def}}{=} f(x) \otimes f, \qquad (h \triangleright f)y = (\operatorname{ad}(h)\, f)y \stackrel{\text{def}}{=} hf(y)h^{-1}.$$

Convention: Given $p \in G$, we write ${}^{p}(\Omega_g) = \{\omega \in \Omega_g \mid \omega_{[-1]} \otimes \omega_{[0]} = p \otimes \omega\}$.d

Character varieties

Fact: If $p \in G$ is central and $\chi\colon \Bbbk G \longrightarrow \Bbbk$ is a (multiplicative) character there is a corresponding involutive Hopf bimodule $M_{(p,\chi)}$ with $\dim(M_{(p,\chi)})^{\mathrm{coinv}} = 1$.

Theorem

Let $p \in G$ be central, $\chi\colon \Bbbk G \longrightarrow \Bbbk$ a character, and Γ a Kitaev graph whose associated closed surface Σ_Γ has genus g. Then, we have:

$$\mathrm{Prot}_{M_{(p,\chi)}}(\Bbbk^1_\varepsilon, \Gamma) \cong {}^{p^{2g}}(\Omega_g) \Big/ \mathrm{span}_{\Bbbk}\{\chi^{-2g}(h)f - h \triangleright f \mid h \in G, f \in {}^{p^{2g}}(\Omega_g)\}.$$

Character varieties

Observation: G acts by conjugation on $\mathrm{Hom}_{\mathrm{Grp}}(\pi_1(\Sigma_g), G)$. Write $\mathrm{Ch}(\Sigma_g, G) \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(\Sigma_g), G)/G$ for the set of its orbits.
We have $\mathrm{Prot}_{M_{(p,\chi)}}(\mathbb{k}^1_\varepsilon, \Gamma) = \mathrm{span}_{\mathbb{k}} \mathrm{Ch}(\Sigma_g, G)$.

And that is a good point to stop.

And that is a good point to stop. But, if we still have time ...

A glimpse into small quantum groups

Let us fix a primitive N-th root of unity $q \in \mathbb{C}$ and consider the Taft algebra A_q. It has two generators h, x and the relations

$$\begin{gathered} h^N = 1, \qquad x^N = 0, \qquad hx = qxh, \\ \Delta(h) = h \otimes h, \qquad \Delta(x) = 1 \otimes x + x \otimes h, \\ S(h) = h^{N-1}, \qquad S(x) = -xh. \end{gathered}$$

Observation: $H = \operatorname{span}_{\mathbb{C}}(\{g^i \mid 0 \leq i < N\})$ is a Hopf subalgebra and the canonical embedding $\iota\colon\ H \longrightarrow A_q$ has a retract $\pi\colon\ A_q \longrightarrow H$ which is a Hopf algebra map and whose kernel is $B_q^+ \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}}(\{g^i x^{j+1} \mid 0 \leq i, j < N\}) \subset A_q$.

“Simplifying” Yetter–Drinfeld modules

Lemma

Suppose $M \in {}_{A_q}\mathsf{YD}$*. The spaces*

$$M^{\mathrm{co}H} \overset{\mathrm{def}}{=} \{m \in M \mid \delta(m) \in H \otimes M\}, \qquad B_q^+ M \overset{\mathrm{def}}{=} \mathrm{span}_{\mathbb{C}}\{a \triangleright m \mid a \in B_q^+, m \in M\}.$$

become H*-Yetter–Drinfeld modules by setting for all* $m \in M$ *and* $h \in H$*:*

$$h \diamond m = \iota(h) \bullet m, \qquad \delta(m) = \pi(m_{|-1|}) \otimes m_{|0|}.$$

Define: For $M \in {}_{A_q}\mathsf{YD}$, we set $\langle M \rangle \overset{\mathrm{def}}{=} \frac{M^{\mathrm{co}H}}{M^{\mathrm{co}H} \cap B_q^+ M} \in {}_H\mathsf{YD}$.

Inflation

Lemma

There is a functor

$$\mathrm{Inf}_H^{A_q}\colon \ \mathsf{YD}_H \longrightarrow \mathsf{Mod}_{A_q}^{A_q}, \qquad X \longmapsto X,$$

where for all $a \in A_q$ and $x \in X$ we have

$$a \bullet x = \pi(a) \diamond m, \qquad \delta(x) = \iota(x_{\{-1\}}) \otimes x_{\{0\}}.$$

Coming full circle

Theorem

Let $X \in \mathsf{YD}_H$ *and* M *an involutive* A_q*-Hopf bimodule with* $\dim M^{\mathrm{coinv}} = 1$*. For any Kitaev graph* Γ*, we have a canonical isomorphism*

$$\mathrm{Prot}_{A_q}(X, M, \Gamma) \cong \mathrm{Hom}_{D(H)}({}^{1}_{\varepsilon}\Bbbk, \langle \mathbb{M}_\Gamma \rangle).$$

Coming full circle

For $q = -1$ we have $\mathbb{C}\mathbb{Z}_2 \cong H \subset A_{-1}$.
Classical: For the torus, the only topologically protected space of the $\mathbb{Z}_2$-model is the ground state.
Question: Does A_{-1} yield "more" invariants?

Coming full circle

The action and coaction of a simple $X \in \mathsf{YD}_H$ correspond to a character $\alpha\colon \mathbb{C}\mathbb{Z}_2 \longrightarrow \mathbb{C}$ and a group-like element $k \in \mathbb{C}\mathbb{Z}_2$.

Example

There is an involutive Hopf bimodule $M_{(h,\varepsilon)}$ of A_{-1} determined by the group-like $h \in G(H)$ and the character $\varepsilon \in G(H^*)$.
Let Γ parameterise the torus and let $\mathbb{C}^k_\alpha$ be a one-dimensional right $\mathbb{C}\mathbb{Z}_2$-Yetter–Drinfeld module. We have

$$\dim \mathrm{Prot}_{A_{-1}}(\mathbb{C}^k_\alpha, M_{(h,\varepsilon)}, \Gamma) = \begin{cases} 4 & \text{if } k = 1 \text{ and } \alpha = \varepsilon, \\ 2 & \text{if } k \neq 1 \text{ and } \alpha \neq \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$