A non-semisimple Kitaev lattice model

based on joint work with Ulrich Krähmer

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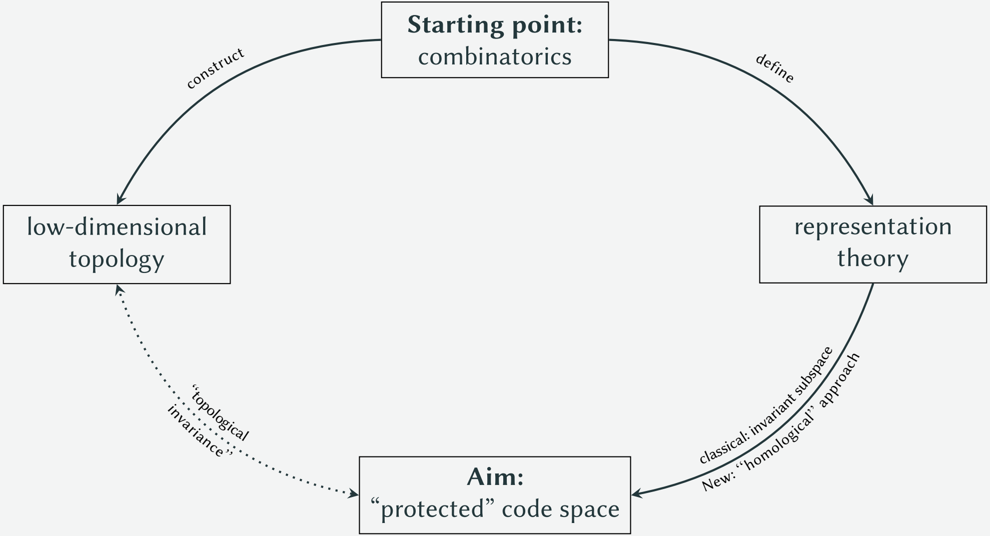
The big picture



Fault-tolerant quantum computation: Kitaev's toric code

Kitaev's construction for fault-tolerant quantum computation:

1. Consider a $k \times k$ lattice in the torus and associate the Hilbert space $\mathbb{M}_k$.
2. Specify a "Hamiltonian" $H\colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$ using "local" features of the lattice.
3. Define as a "quantum memory" the space $\mathrm{Prot}_k \stackrel{\text{def}}{=} \ker H$.

This leads to error correction/prevention implemented on a physical level:

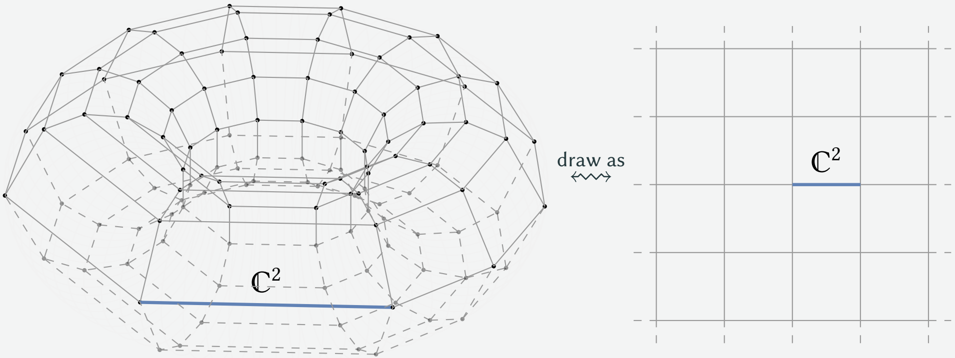
Theorem

For any $k \in \mathbb{N}$ *we have* $\dim \mathrm{Prot}_k = 4$.

$\rightsquigarrow$ For k large, most errors (modelled by linear operations on $\mathbb{M}_k$) act either trivially Prot_k or can be "detected" using various physical methods.

Kitaev’s toric code – lattices and the (extended) Hilbert space

Let $k \in \mathbb{N}$ and consider a $k \times k$ lattice embedded on a torus:

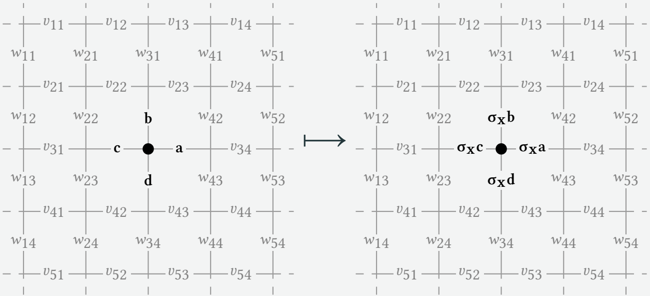


Assign to each *edge* a copy of $\mathbb{C}^2$ and set $\mathbb{M}_k = \otimes_{1 \le i,j \le k} \mathbb{C}^2 \bigotimes \otimes_{1 \le i,j \le k} \mathbb{C}^2$.

Kitaev's toric code – vertex actions

Set $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Given a *vertex* (point in the lattice) v define

$$A_v \colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$$

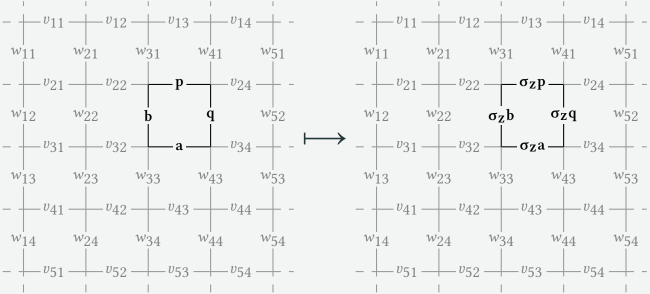


1. Gives rise to a representation $\triangleright_v \colon \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{M}_k \longrightarrow \mathbb{M}_k$.
2. Combining all "vertex actions" leads to $\triangleright \colon (\mathbb{C}\mathbb{Z}_2)^{\otimes k^2} \otimes \mathbb{M}_k \longrightarrow \mathbb{M}_k$.

Kitaev's toric code – face coactions

Set $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Given a *face* (square of the lattice) f define

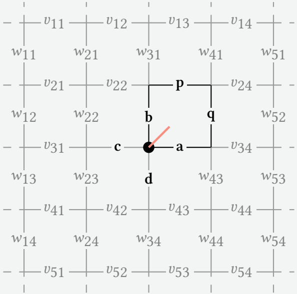
$$B_f\colon \mathbb{M}_k \longrightarrow \mathbb{M}_k$$



1. Can be interpreted as a corepresentation $\delta_f\colon \mathbb{M}_k \longrightarrow \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{M}_k$.
2. Combining all "face coaction" leads to a "global" comodule structure $\delta\colon \mathbb{M}_k \longrightarrow (\mathbb{C}\mathbb{Z}_2)^{\otimes k^2} \otimes \mathbb{M}_k$.

The local and global module structure of $\mathbb{M}_k$

Associate to each vertex the face to its "top right" $\rightsquigarrow \{vertices\} \xrightarrow{1:1} \{faces\}$.
For every such vertex-face-pair (v, f) the vertex action and face coaction turn $\mathbb{M}_k$ into a $D(\mathbb{C}\mathbb{Z}_2)$-module.



Topological invariance of the protected space

We consider $\mathbb{M}_k$ as a $D(\mathbb{C}\mathbb{Z}_2)^{\otimes k^2}$-module

Definition

The *protected space* of the Kitaev lattice model is $\mathrm{Prot}_k = \mathbb{M}_k^{\mathrm{inv}}$.

Theorem (Kitaev '97)

For any $k \in \mathbb{N}$ we have $\dim \mathrm{Prot}_k = 4$.

Proof sketch (based on Buerschaper, Mombelli, Christandl, Aguado '18).

Suppose $k \geq 1$. Define an explicit isomorphism between Prot_k and Prot_{k+1} using projectors coming from the Haar integral of $\mathbb{C}\mathbb{Z}_2$. $\square$

Works for complex semisimple Hopf algebras.

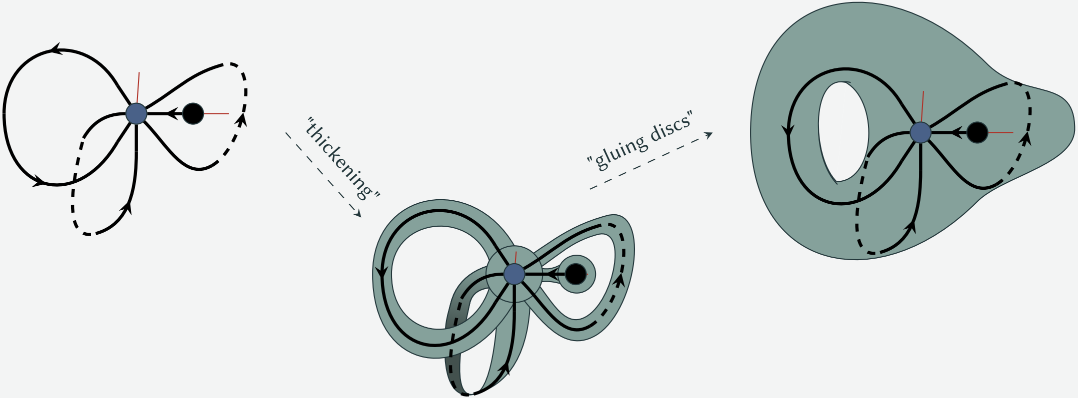
Where to go from here?

Our aim is to generalise this to finite-dimensional non-semisimple Hopf algebras. This needs three levels of generalisations:

1. lattices on oriented surfaces $\longrightarrow$ Kitaev graphs
2. regular Hopf bimodule $\longrightarrow$ involutive (twisted) Hopf bimodules
3. invariant subspaces $\longrightarrow$ bitensor products

Generalisation 1: From lattices to Kitaev graphs

Each Kitaev graph Γ gives rise to a surface with boundary Σ_Γ and a surface without boundary $\Sigma_\Gamma^{\mathrm{cl}}$:



Generalisation 1: From lattices to Kitaev graphs

Observation: We can increase the number of boundary components of Σ_Γ by gluing a certain graph $\mathbf{A}$ to Γ. The surface $\Sigma_{\Gamma\#\mathbf{A}}$ of $\Gamma\#\mathbf{A}$ has the same genus as Σ_Γ but one more boundary component.

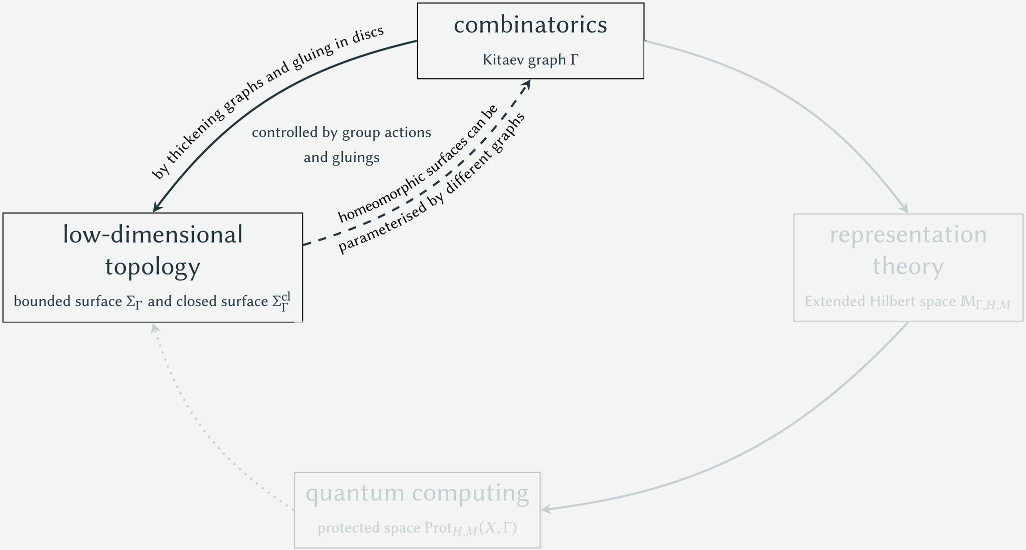
Theorem

There is a group $\mathfrak{G}$ acting on the set of Kitaev graphs such that for all graphs Γ and Δ:

$$\Sigma_\Gamma^{\mathrm{cl}} \cong \Sigma_\Delta^{\mathrm{cl}} \iff \exists a, b \in \mathbb{N}_0 \textit{ with } \Delta\#\mathbf{A}^{\#a} \in \mathfrak{G} \cdot (\Gamma\#\mathbf{A}^{\#b}).$$

The group $\mathfrak{G}$ is connected to the mapping class group of surfaces.
A better (more conceptual understanding) of $\mathfrak{G}$ would be desirable!

Generalisation 1: From lattices to Kitaev graphs



Generalisation 2: Involutive Hopf bimodules

From now on: H is a finite-dimensional complex Hopf algebra.

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Classical: Decorate each edge of a Kitaev graph with the regular bimodule-bicomodule H. Use $S\colon H \longrightarrow H$ to model the reversal of edge directions.
Problem: $S^2 = \mathrm{id}_H$ is equivalent to H semisimple.

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Definition

An *involutive Hopf bimodule* is a pair of an S^{-2}-twisted Hopf bimodule M together with an involution $\psi\colon M \longrightarrow M$ that intertwines the left and right (co)actions.

Generalisation 2: Involutive Hopf bimodules

Theorem

There is an algebra $\overleftrightarrow{B(H)}$ whose modules coincide with involutive Hopf bimodules.

Generalisation 2: Extended Hilbert space

We fix an involutive Hopf bimodule (M, ψ).

Construction

Let Γ be a Kitaev graph. The extended Hilbert space is

$$\mathbb{M}_\Gamma \stackrel{\text{def}}{=} \otimes_{e \in E_\Gamma} M.$$

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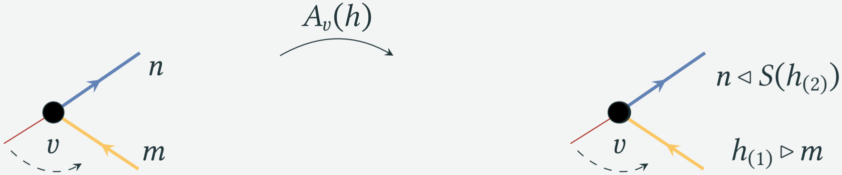
$$\mathbb{M}_\Gamma \overset{\text{def}}{=} \otimes_{e \in E_\Gamma} M.$$

For every vertex $v \in V_\Gamma$ and $f \in F_\Gamma$ define algebra maps

$$A_v \colon\ H \longrightarrow \mathrm{End}(\mathbb{M}_\Gamma), \qquad B_f \colon\ (H^*)^{\mathrm{op}} \longrightarrow \mathrm{End}(\mathbb{M}_\Gamma).$$

These maps depend on the “local” structure of Γ.

Generalisation 2: Extended Hilbert space



Generalisation 2: The extended Hilbert space as topological invariant

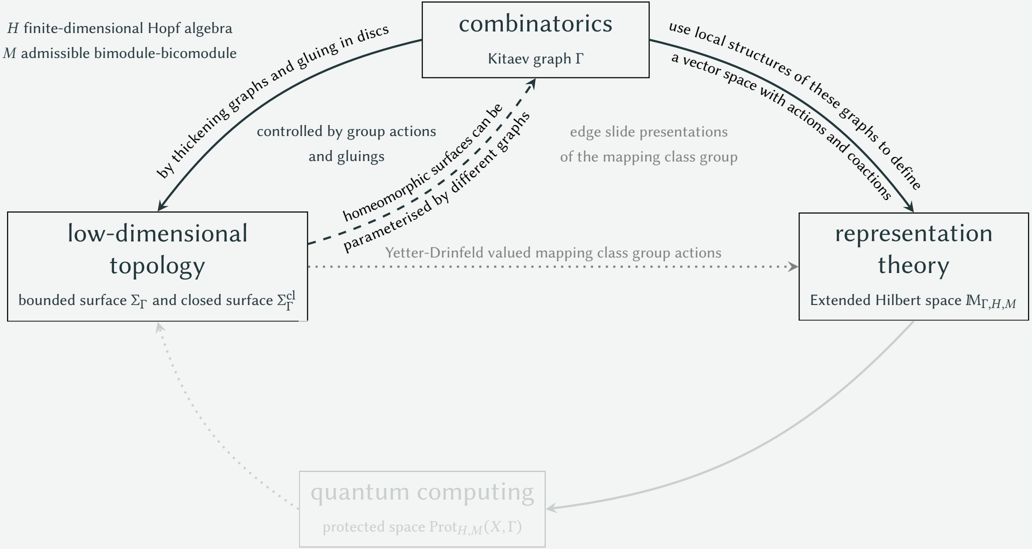
For Γ a Kitaev graph, set $H_v = H$ for each vertex $v \in V_\Gamma$ and write $\mathbb{H}_\Gamma \stackrel{\text{def}}{=} \otimes_{v \in V_\Gamma} H_v$.

Theorem

The local (co)actions turn $\mathbb{M}_\Gamma$ into an $\mathbb{H}_\Gamma$-Yetter–Drinfeld-module.
Suppose the graphs Γ and Δ satisfy $\Sigma_\Gamma \cong \Sigma_\Delta$. Then $\mathbb{M}_\Gamma$ and $\mathbb{M}_\Delta$ are isomorphic as Yetter–Drinfeld modules.

Work by Bene'10, Jackson'18, and Meusburger, Voß'21 suggest that for (certain) M there is a canonical action of the mapping class group of Σ_Γ on $\mathbb{M}_\Gamma$.

Generalisation 2: Involutive bimodules and extended Hilbert spaces



Generalisation 3: Bitensor products

Definition

Let $(X, \triangleleft, \varrho) \in \mathcal{M}_H^H$ and $(N, \triangleright, \delta) \in {}_H^H\mathcal{M}$ be right-right and left-left modules-comodules respectively.

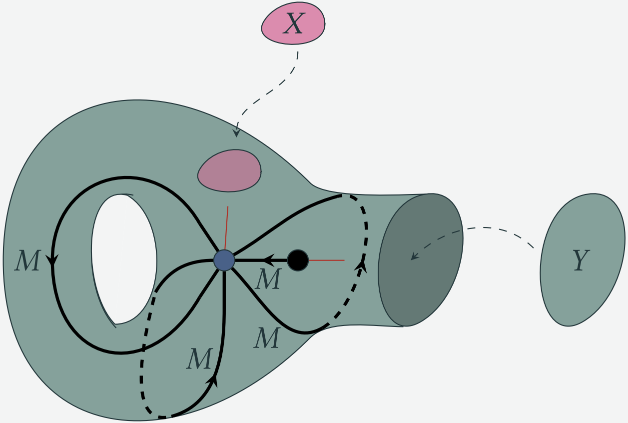
1. We write $\pi_{X,M} \colon X \otimes_{\Bbbk} M \longrightarrow X \otimes_H M \overset{\text{def}}{=} \operatorname{coker}(\triangleleft \otimes \mathrm{id}_M - \mathrm{id}_X \otimes \triangleright)$.
2. We set $\iota_{X,M} \colon X \square_H M \overset{\text{def}}{=} \ker(\varrho \otimes_{\Bbbk} \mathrm{id}_M - \mathrm{id}_X \otimes_{\Bbbk} \delta) \longrightarrow X \otimes_{\Bbbk} M$.
3. The *bitensor product* of X and M is $\mathrm{Bit}_H^H(X, M) \overset{\text{def}}{=} \operatorname{im} \iota_{X,M}\pi_{X,M}$.

If H is semisimple and $N \in {}_H^H\mathcal{YD}$, we have $\mathrm{Bit}_H^H(\Bbbk_\varepsilon^1, N) \cong \mathrm{Hom}_{D(H)}(\Bbbk_\varepsilon^1, N)$.

This map is particularly interesting in positive characteristic due to its connections with the order of S^4.

Generalisation 3: Bitensor products and closing boundaries

Let $v \in V_\Gamma$ be a vertex. It has a corresponding boundary component b of Σ_Γ. We think of $\mathrm{Bit}_{H_v}^{H_v}(X, \mathbb{M}_\Gamma)$ as closing the boundary b by gluing in a disc labelled X.



Generalisation 3: The annular graph

From now on: We assume $\dim M^{\text{coinv}} = 1$ and fix a $Y \in \mathcal{M}_H^H$ which "trivialises" the annular graph.
Note: If $\dim M^{\text{coinv}} = 1$, a suitable Y always exists.
Warning: Not all Hopf algebras admit an involutive Hopf bimodule M such that $\dim M^{\text{coinv}} = 1$.

Generalisation 3: The protected space

Definition

Let Γ be a Kitaev graph and $X \in \mathcal{M}_H^H$ and write

$$\mathbb{X}_\Gamma \stackrel{\text{def}}{=} \otimes_{v \in V_\Gamma} Z_v, \qquad \text{where } Z_v = \begin{cases} X & \text{if } v \text{ is distinguished,} \\ Y & \text{otherwise.} \end{cases}$$

We call $\mathrm{Prot}_M(X, \Gamma) \stackrel{\text{def}}{=} \mathrm{Bit}_{\mathbb{H}_\Gamma}^{\mathbb{H}_\Gamma}(\mathbb{X}_\Gamma, \mathbb{M}_\Gamma)$ the *protected space.*

Generalisation 3: Excision

Recall: Topological features of Kitaev graphs are controlled by the group $\mathfrak{G}$ and iterated gluings of graphs.

Theorem

Let Γ, Δ *be two Kitaev graphs, we have a short exact sequence*

$$0 \longrightarrow \mathrm{Prot}_M(X,\Gamma) \otimes \mathrm{Prot}_M(X',\Delta) \longrightarrow \mathrm{Prot}_M(X \otimes X', \Gamma \# \Delta) \longrightarrow \mathrm{CBit}_M(X, X', \Gamma, \Delta) \longrightarrow 0.$$

Generalisation 3:topological invariance

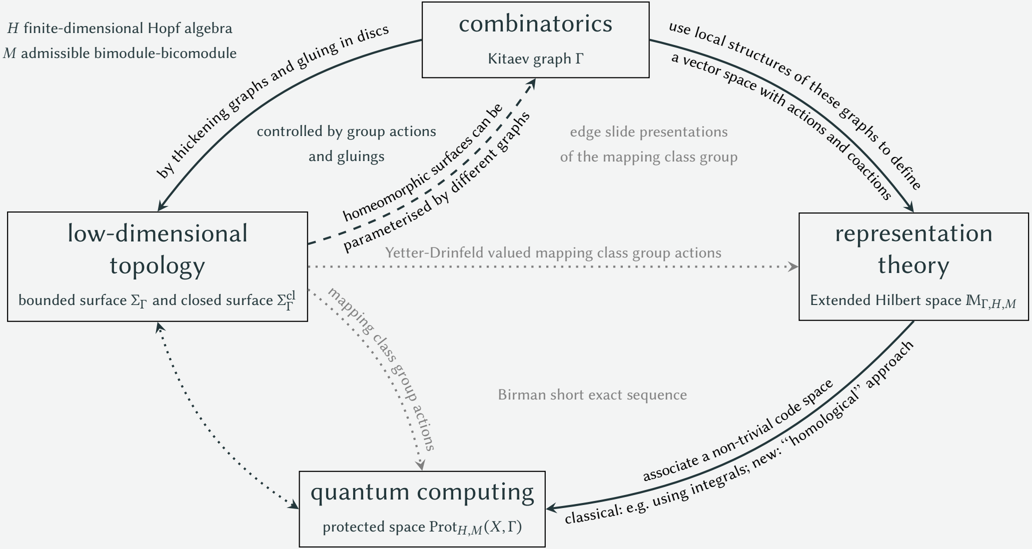
Corollary

For any Γ *we have* $\mathrm{Prot}_M(X,\Gamma) \cong \mathrm{Prot}_M(X,\Gamma\#\mathbf{A})$*. In particular, if* Γ *and* Δ *are such that* $\Sigma^{\mathrm{cl}}_{\Gamma} \cong \Sigma^{\mathrm{cl}}_{\Delta}$ *then* $\mathrm{Prot}_M(X,\Gamma) \cong \mathrm{Prot}_M(X,\Delta)$*.*

Any surface can be decomposed as a connected sum of tori.
Explicit computations and geometrical interpretation of results?

Thank you

H finite-dimensional Hopf algebra
M admissible bimodule-bicomodule



Uncovered topics

1. For group-algebras, the invariants we obtain have relations with Seifert fibered spaces.
2. "Reduction" procedure for bosonisations of Nichols algebras allows to simplify the computation.
3. Induction and restriction type identities.