

# Chapter 1

## Quantum Chaos

In the terms of classical mechanics, such as the sensitivity to initial conditions quantum chaos does not exist because of the Heisenberg's uncertainty principle, the absence of trajectories and many other reasons. In turn, there are some ways in which we can link classically chaotic systems to their quantum counterparts. These bridges between classical mechanics and quantum mechanics allows us to give a meaning to quantum chaos. (Berry  $\rightarrow$  quantum chaology)

### 1.1 Level repulsion

### 1.2 Probability notions

**Definition** (Joint probability). Given two events  $A$  and  $B$ , their joint probability,  $P(A \cap B)$ , is the probability of the two events to occur simultaneously.

**Definition** (Conditional probability). The conditional probability of an event  $A$  given an event  $B$  with  $P(B) > 0$ , denoted  $P(A|B)$  is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The joint probability of  $A$  and  $B$  can be expressed as  $P(A \cap B) = P(A|B)P(B)$ .

**Definition** (Continuous random variable). A continuous random variable is a function from the set of all outcomes to the set of real numbers  $X : \Omega \rightarrow (R)$  such that

$$P(a \leq X(\omega) \leq b) = \int_a^b f(x) dx ,$$

where  $f(x) \geq 0$  and  $\int_{-\infty}^{+\infty} f(x) dx = 1$ .

The above function  $f$  is called the *probability density function*. The probability for a continuous random variable  $X$  to take a value in the infinitesimal interval of length  $dx$  is given by

$$P(x \in [x, x + dx]) = \int_x^{x+dx} f(z) dz \approx f(x) dx \quad (1.1)$$

## 1.3 Random matrix theory

*Random matrices* are matrices which have random variables as elements, with their randomness is restricted by the symmetries of the whole matrix.

Nearest neighbour spacing distributions show how the differences between consecutive energy levels fluctuate around the average. In order to better understand this concept we shall begin with the simpler case of real random numbers.

### 1.3.1 The nearest neighbour spacing distribution of random numbers

We consider a sequence of uniformly distributed, ordered, real, random numbers.

We define the *spacing* of an ordered sequence as the sequence of differences between consecutive elements. The *average spacing* of the sequence is given by  $\frac{E_n - E_0}{n}$ .

discrete vs continuous random variables?

For an interval of length  $s$ ,  $s > 0$ , we will denote with  $P(n \in s)$  the probability for the interval to contain  $n$  numbers and with  $P(n \in ds | m \in s)$  the conditional probability for the interval of length  $ds$  to contain  $n$  numbers given that the interval of length  $s$  contains  $m$  numbers.

If  $E$  is a given number in the sequence, we are interested in the probability  $P(s)ds$  to have the next number between  $E + s$  and  $E + s + ds$ . Since we are interested in the next number after  $E$ , we know that in the interval of length  $s$  there is no other number and the next number is somewhere in the infinitesimal interval  $ds$ . Thus the joint probability of the events  $1 \in ds$  and  $0 \in s$  is given by:

$$P(s) ds = P(1 \in ds | 0 \in s) P(0 \in s). \quad (1.2)$$

Since random numbers are not correlated, the probability of a random number to be found in the interval  $ds$  does not depend on the number of random numbers in  $s$ , so

$$P(1 \in ds | 0 \in s) = P(1 \in ds).$$

The random numbers are uniformly distributed, so their probability density function  $f$  is a constant. Hence, according to eq. (1.1), the probability of finding

a number in the interval of length  $ds$  is given by

$$P(1 \in ds) \equiv P(1 \in [0, 0 + ds]) \approx f(s) ds \sim ds.$$

If we denote the constant probability density function with  $a$

$$P(s) ds = a ds P(0 \in s).$$

$P(0 \in s)$  can be expressed using the complementary probability as  $1 - \int_0^s P(s') ds'$ . Now we can express  $P(s) ds$  as follows:

$$P(s) ds = a ds \left( 1 - \int_0^s P(s') ds' \right).$$

In order to differentiate with respect to  $s$ , we will use the Leibniz rule for differentiating integrals, namely

$$\frac{d}{dx} \int_{G(x)}^{H(x)} F(x, t) dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x} dt + F(x, H(x)) \frac{dH}{dx} - F(x, G(x)) \frac{dG}{dx} \quad (1.3)$$

Using this rule, we obtain

$$\frac{d}{ds} P(s) = -aP(s).$$

This differential equation can be solved by separation of variables, yielding

$$P(s) = \mathcal{C} e^{-as}$$

In order to determine the constant  $\mathcal{C}$ , we use the normalisation condition for the probability density function

$$\int_{-\infty}^{\infty} P(s) ds = 1$$

Since  $s > 0$ , this reduces to

$$\int_0^{\infty} P(s) ds = \int_0^{\infty} \mathcal{C} e^{-as} ds = -\frac{\mathcal{C}}{a} e^{-as} \Big|_0^{\infty} = \frac{\mathcal{C}}{a}$$

Thus  $\mathcal{C} = a$  and  $P(s) = a e^{-as}$ . We can further simplify the formula if we set that the average spacing to unity. The average spacing is given by

$$\bar{s} = \int_0^{\infty} s P(s) ds = - \int_0^{\infty} s \frac{d}{ds} (e^{-as}) ds = \int_0^{\infty} e^{-as} ds = \frac{1}{a},$$

so setting it to unity results in  $a = 1$ . Thus the probability density function becomes

$$P(s) = e^{-s}. \quad (1.4)$$

This function is known as the *Poisson distribution*.

### 1.3.2 The Wigner distribution

We will now consider a more complicated situation, by considering the probability density function for the sequence of random numbers to be arbitrary.

We can start from eq. (1.2) since the discussion up to that point did not include any details related to the distribution of the random numbers.

In this case  $P(1 \in ds | 0 \in s) = f_{1,0}(s) ds$ , where  $f_{n,m}(s)$  is function which describes how the probability of having  $n$  numbers in  $ds$  is influenced by the  $m$  numbers in  $s$ . Thus

$$P(s) ds = f_{1,0}(s) ds \left( 1 - \int_0^s P(s') ds' \right)$$

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