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FINGERPRINTS OF CLASSICAL PHASE SPACE STRUCTURE IN QUANTUM CHAOS

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Chapter 1

Introduction

The behaviour of non-linear systems, at classical and quantum level, rises fundamental questions and is important in several applications. In classical mechanics, once the Hamiltonian systems become non-integrable, a sensitive dependence of the dynamics on the initial conditions manifests. This corresponds to chaotic dynamics and leads to several specific phenomena. In particular, the phase space is no longer filled by tori and depending on the energy, several trajectories will have an erratic behaviour in the available volume. An important question which is intensively debated in literature is related to the way in which the corresponding quantum system reflects this classical properties. The purpose of the present work is to explore the features of the energy levels statistics in relation with the departure from the integrable case and also as a function of excitation energy. We try to obtain a correlation between the global phase space structure in a given energy domain and the associated energy level distribution for the corresponding quantum system.

In chapter 2 we review the basic concepts from classical Hamiltonian dynamics which are useful to the characterisation of chaotic dynamics.

In chapter 3, after a brief review of the principles of quantum mechanics, we discuss the nearest neighbour distributions and their relation to the non-integrability.

Chapter 4 is devoted to a detailed description of the numerical procedures and to the analysis method of the level distributions performed with the aid of a `Python` program.

Chapter 5 presents the main results of our investigations. We discuss the possible deviations from the Wigner distribution by proposing an effective level distribution which can interpolate between the Wigner and Poissonian one. We show that the behaviour of this distribution can reflect the energy variations in phase space. Moreover, it has peculiar behaviour at the variations of the parameter with the departure from integrability.

In the last chapter, the main conclusions of our work are summarised.

Chapter 2

Classical chaos

2.1 Fundamental notions

Classical mechanics can be reformulated in a more elegant way which does not require the computation of individual forces and instead uses a single function to describe the entire system, namely the *Lagrangian*. The Lagrangian, denoted with $\mathcal{L}$, is a function of the generalised coordinates $q_1, \ldots, q_n$, their derivatives with respect to time $\dot{q}_1, \ldots, \dot{q}_n$ and optionally the time t, where n is the number of degrees of freedom.

The principles of Newtonian mechanics are replaced by the a variational principle $\delta S = 0$, where

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \, \mathrm{d}t \tag{2.1}$$

is called the *action*. This principle, also called *Hamilton's principle* or *the principle of least action* is equivalent with the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = 0,$$

where $k = 1, \ldots, n$. Thus we can obtain the equations of motion provided that we know the initial conditions $q_1(t_0), \ldots, q_n(t_0)$ and $\dot{q}_1(t_0), \ldots, \dot{q}_n(t_0)$ at $t = t_0$ by solving the set of n second order differential equations.

An other formalism for classical mechanics which is very useful in theoretical physics is the Hamiltonian formalism. In this formalism instead of the Lagrangian we use the *Hamiltonian*, $\mathcal{H}$, which is a function of the generalised coordinates $q_1, \ldots, q_n$, the generalised momenta $p_1, \ldots, p_n$ and optionally the time t. The generalised momenta are given by

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}.$$

The Hamiltonian can be obtained from the Lagrangian through a *Legendre transformation* of $\mathcal{L}$

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n, t) = \sum_{k=0}^{n} p_k \dot{q}_k - \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t). \tag{2.2}$$

The Euler-Lagrange equations are replaced by Hamilton's equations

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k}, \qquad\qquad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k}.$$

We now have $2n$ first order ordinary differential equations.

2.2 Canonical transformations

This section follows mainly the steps described in [1]. In the Hamiltonian formulation of classical mechanics *canonical transformations* represent all the transformations of the canonically conjugate variables from q and p to Q and P such that the structure of the Hamilton equations remains intact.

The Lagrange equations of a system are invariant to a transformation of the Lagrange function such that:

$$\mathcal{L}'(Q, \dot{Q}, t) = \mathcal{L}(q, \dot{q}, t) - \frac{\mathrm{d}}{\mathrm{d}t} F(q, Q, t).$$

Integrating we obtain

$$\int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t - \left(F(q, Q, t_2) - F(q, Q, t_1) \right).$$

The variation of the above equation is given by

$$\delta \int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \delta \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t - \delta F(q, Q, t_2) + \delta F(q, Q, t_1).$$

Fixing the variations at the end points to be 0, we get

$$\delta \int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \delta \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t.$$

So indeed the two Lagrangian descriptions of the system are identical since if the Hamilton principle holds in the first description it also holds in the second.

A function $F(q, Q, t)$ satisfying the above property and the condition $\frac{\partial^2 F}{\partial q \partial Q} \neq 0$ is called a *generating function*. We can classify the generating functions in four possible types as a function of their variables. The conjugate momenta corresponding to q and Q are given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \text{ and } P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}}.$$

In order to express those as a function of $F(q, Q, t)$ we use the fact that $\dot{q}$ is a cyclic coordinate for $\mathcal{L}'$

$$\frac{\partial \mathcal{L}'}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial}{\partial \dot{q}} \left(\frac{\mathrm{d}}{\mathrm{d}t} F(q, Q, t) \right) = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0,$$

where
$$\frac{\mathrm{d}}{\mathrm{d}t}F(q,Q,t) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t}.$$

Thus we obtain
$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

and
$$P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}} = -\frac{\partial F}{\partial Q}.$$

To find the Hamiltonian corresponding to the new coordinates Q and P we compute the Legendre transformation of $\mathcal{L}'$
$$\mathcal{H}'(Q,P,t) = P\dot{Q} - \mathcal{L}'(Q,\dot{Q},t)$$

Thus
$$\mathcal{H}'(Q,P,t) = P\dot{Q} - \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t} = P\dot{Q} - \mathcal{L} + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t} = p\dot{q} - \mathcal{L} + \frac{\partial F}{\partial t}$$

and so
$$\mathcal{H}'(Q,P,t) = \mathcal{H}(q(Q,P),p(Q,P),t) + \frac{\partial F(q(Q,P),Q,t)}{\partial t}. \qquad (2.3)$$

As mentioned above, there are four types of generating functions. They can be obtained by successive Legendre transformations which switch between the conjugate coordinates and they represent the same canonical transformation. We denote the above mentioned $F(q,Q,t) \equiv F_1(q,Q,t)$. The other types of generating functions will be denoted $F_2(q,P,t), F_3(p,Q,t), F_4(p,P,t)$. Thus we define

$$F_2(q,P,t) = F_1(q,Q,t) + PQ \qquad\qquad p = \frac{\partial F_2}{\partial q}, \qquad\qquad Q = \frac{\partial F_2}{\partial P}$$

$$F_3(p,Q,t) = F_1(q,Q,t) - pq \qquad\qquad q = -\frac{\partial F_3}{\partial p}, \qquad\qquad P = \frac{\partial F_3}{\partial Q}$$

$$F_4(p,P,t) = F_1(q,Q,t) - pq + PQ \qquad\qquad q = -\frac{\partial F_4}{\partial p}, \qquad\qquad Q = \frac{\partial F_4}{\partial P}.$$

The *Poisson bracket* of two arbitrary functions with respect to the canonical coordinates q and p is defined as

$$[F,G]_{q,p} = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} \right).$$

We can use the Poisson bracket to express Hamilton's equations more elegantly

$$\dot{q}_k = [q_k, H], \qquad\qquad\qquad \dot{p}_k = [p_k, H].$$

Expressed in this form, the Hamilton equations have a close resemblance to Heisenberg's form for the equations of motion in the Heisenberg picture of quantum

mechanics. The Poisson bracket structure of classical mechanics is fundamental and provides a link with quantum mechanics, being in correspondence with the commutator algebra [2].

As with the above discussion we will limit to the case of one degree of freedom, but the results are easily generalisable to n degrees of freedom.

If Q and P are a set of conjugate variables obtained by a canonical transformation from q and p, then

$$[F', G']_{Q,P} = [F, G]_{q,p},$$

where F' and G' are the transformed functions. As a consequence, if we choose $F = Q$ and $G = P$

$$[Q, P]_{Q,P} = [Q(q,p), P(q,p)]_{q,p} = 1$$

This relation represents a necessary and sufficient condition for the transformation from q, p to Q, P to be canonical. Preserving the structure of Hamilton's equations is equivalent with the invariance of the Poisson bracket and thus we can use this invariance as the definition for canonical transformations.

2.3 The Hamilton-Jacobi equation

This section will describe the Hamilton-Jacobi formalism. For further details, one can see for example [1, 3]. Instead of finding the equations of motion through solving a set of $2n$ ordinary differential equations with $2n$ dependent variables ($q_1, \ldots, q_n$ and $p_1, \ldots, p_n$) and one independent variable (t), we can use a single first order partial differential equation having $n + 1$ independent variables ($q_1, \ldots, q_n$ and t) and one dependent variable (S).

Suppose there exists a generating function S of type F_2 such that

$$\mathcal{H}' = 0.$$

From Hamilton's equations, the new coordinates Q_k and P_k, with $k = 1, \ldots, n$ will be constants, so the information about the time evolution of the system is contained in the canonical transformation itself. Thus, equation (2.3) becomes

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t) + \frac{\partial S}{\partial t} = 0 \tag{2.4}$$

which represents the *Hamilton-Jacobi equation*.

The solution $S(q_1, \ldots, q_n, P_1, \ldots, P_n, t)$ of equation (2.4) is called *Hamilton's Principal Function*. From the theory of differential equations we know that there will be $n + 1$ arbitrary constants. We can find n of these constants as $P_1, \ldots, P_n$ and the remaining constant can be incorporated in S since if S is a solution for the Hamilton-Jacobi equation, $S + const.$ is also a solution.

It is a standard convention [3] to rename the constants as follows

$$P_k \equiv \alpha_k$$

$$Q_k \equiv \beta_k = \frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial \alpha_k}.$$

These constants depend on the initial conditions $q_k(0)$ and $p_k(0)$. From

$$p_k(0) = \left.\frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial q_k}\right|_{t=0}$$

we can obtain α_ks, and β_ks are given by

$$\beta_k = \left.\frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial \alpha_k}\right|_{t=0}.$$

We can obtain the initial generalised coordinates $q_k(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, t)$ by solving

$$\beta_k = \frac{\partial S}{\partial \alpha_k}.$$

If we take the total time derivative of S

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{k=0}^{n} \frac{\partial S}{\partial q_k}\dot{q}_k + \frac{\partial S}{\partial t}$$

and use eq. (2.3)

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{k=0}^{n} p_k\dot{q}_k - \mathcal{H},$$

we recognise the Legendre transform from eq. (2.2). Thus

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mathcal{L},$$

or equivalently

$$S = \int \mathcal{L}\,\mathrm{d}t\,.$$

It follows that the generating function for this special canonical transformation is the action as defined in eq. (2.1).

If the time does not appear explicitly in the Hamiltonian, the Hamiltonian itself will be a constant of the motion and will represent the energy of the system E.

We can rewrite eq. (2.4) as follows

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t) = -\frac{\partial S}{\partial t} = E.$$

Thus we can suppose that for this case $S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)$ has the following form

$$S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) \equiv W(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n) - Et.$$

Using this expression in eq. (2.4), we obtain

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}) = E. \tag{2.5}$$

which represents the second form of the Hamilton-Jacobi equation.

The solution to this equation is called *Hamilton's Characteristic Function*. W can be expressed as

$$W = S + Et = \int (\mathcal{L} + \mathcal{H}) \, \mathrm{d}t = \int \sum_{k=0}^{n} p_k \dot{q}_k \, \mathrm{d}t = \sum_{k=0}^{n} \int p_k \, \mathrm{d}q_k \,.$$

W is a generating function of type F_2. The canonical transformation given by W is different from the one given by S, since $\mathcal{H}' = \mathcal{H} + \frac{\partial W}{\partial t} = E \neq 0$.

The new generalised coordinates Q_k are cyclic in the new Hamiltonian $\mathcal{H}' = E = \mathcal{H}(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n})$, thus the new generalised momenta care constants. While in the case of S all the generalised momenta and coordinates were constants, in the case of W all the generalised momenta are constants $\dot{P}_k = -\frac{\partial \mathcal{H}'}{\partial Q_k} = 0$, but not all the generalised coordinates are constants. If we choose to define $P_1 \equiv E$,

$$\dot{Q}_1 = \frac{\partial \mathcal{H}'}{\partial P_1} = \frac{\partial E}{\partial P_1} = 1.$$

Thus

$$Q_1 = t + const. = \frac{\partial W}{\partial P_1} = \frac{\partial W}{\partial E}.$$

So Q_1 is no longer a constant. The constant in the above equation determines the origin of the time axis. Q_1 represents the time and its conjugate coordinate is P_1, the energy.

The $n-1$ generalised momenta $P_2, \ldots, P_n$ are independent of P_1 and thus $\frac{\partial \mathcal{H}'}{\partial P_k} = 0$, where $k = 2, \ldots, n$. So the corresponding $n-1$ generalised coordinates are constants.

We can change the notation for α and β to use an index from 1 to $n-1$: $\alpha_j \equiv P_{j+1}$, $\beta_j \equiv Q_{j+1}$, where $j = 1, \ldots, n-1$. We can now express Hamilton's Characteristic Function as

$$W = W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1}).$$

From the initial generalised momenta $p_k(0)$

$$p_k(0) = \left. \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial q_k} \right|_{t=0}$$

we can obtain E and α_j. Using the initial generalised coordinates $q_k(0)$ we obtain β_j as follows

$$\beta_j = \left. \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial \alpha_j} \right|_{t=0}.$$

Thus, having E, α_j and β_j, we can obtain the time evolution for the generalised coordinates from

$$\beta_j = \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial \alpha_j}.$$

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For systems with $n > 1$ degrees of freedom, the Hamilton-Jacobi equation can be separated in n equations if we can write S as

$$S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) = \sum_{k=1}^{n} W_k(q_k, \alpha_1, \ldots, \alpha_n) - Et.$$

A simpler form of separability exists if the Hamiltonian itself is separable

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{k=1}^{n} \mathcal{H}_k(q_k, p_k).$$

In this case the second form of the Hamilton-Jacobi equation splits in n independent equations

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n) = E = \sum_{k=1}^{n} \mathcal{H}_k(q_k, p_k),$$

so

$$\mathcal{H}_k(q_k, \frac{\partial W_k}{\partial q_k}) = \alpha_k, \qquad E = \sum_{k=1}^{n} \alpha_k.$$

The Hamilton-Jacobi equation can be solved only if the variables can be separated. The separability depends both on the problem and the chosen coordinate system. This does not decrease the theoretical value of the Hamilton-Jacobi equation which provides powerful methods for finding the constants of the motion.

2.4 Action-angle variables

Going back again to a system with one degree of freedom for simplicity, we study a system which has a periodic motion. We search for a canonical transformation from q, p to a new set of canonically conjugate coordinates θ, I such that the new Hamiltonian does not depend on θ.

Since θ is an cyclic coordinate

$$\dot{I} = -\frac{\partial \mathcal{H}'}{\partial \theta} = 0,$$

I is a constant of the motion and

$$\dot{\theta} = \frac{\partial \mathcal{H}'}{\partial I} = const.$$

Thus $\theta = \frac{\partial \mathcal{H}'}{\partial I}(t - t_0) \equiv \omega(I)(t - t_0)$. θ is called the *angle variable* and I the *action variable* (not to be confused with the action).

In order to obtain this variables we will use a generating function of type F_1 denoted $W'(q, \theta)$. Since the motion is periodic in the initial coordinates, it must also be periodic in the angle variable. Thus W' is periodic in θ.

From the properties of the F_1 type generating functions

$$\mathrm{d}W' = \frac{\partial W'}{\partial q}\,\mathrm{d}q + \frac{\partial W'}{\partial \theta}\,\mathrm{d}\theta = p\,\mathrm{d}q - I\,\mathrm{d}\theta\,.$$

(we can observe that W' is related to Hamilton's Characteristic Function by a Legendre transformation)

Since the motion is periodic, if we integrate over a period q returns to the same value, while θ advances by a constant amount, which we can choose to be 2π per period.

$$\oint \mathrm{d}W' = 0 = \oint p\,\mathrm{d}q - \oint I\,\mathrm{d}\theta$$

Since I is a constant of the motion,

$$2\pi I = \oint p\,\mathrm{d}q$$

or

$$I = \frac{1}{2\pi} \oint p\,\mathrm{d}q\,. \tag{2.6}$$

Equation (2.6) can be considered the definition for the action variable, where the integral is taken around a single period of the motion (which is always a closed curve in phase space).

2.5 Integrability

A system is called *integrable* if the number of degrees of freedom is equal to the number of constants of the motion in involution which each other. Two constants of the motion are in *involution* with each other if their mutual Poisson brackets are equal to 0.

The previously mentioned constants of the motion are not required to be known in analytic form, they are only required to exist. Thus a Hamiltonian system with one degree of freedom, if the Hamiltonian function is time independent, is integrable. The conserved quantity is the energy.

An important consequence of integrability is that we can always find a canonical transformation to action-angle variables and express the new Hamiltonian as a function of the n constants of the motion given by the action variables: $\mathcal{H}' = \mathcal{H}'(I_1, \ldots, I_n)$.

For integrable systems, the n constants of the motion restrict the motion in the $2n$ dimensional phase space to a $2n - n = n$ dimensional subspace. If the motion is also periodic, then the motion is constrained to a n dimensional closed surface (with finite volume).

For an integrable system with $n = 2$, the motion in the phase space is confined to a torus. In order to understand why, we can use the previously defined action-angle variables. In this case, the motion will be described by $I_1, \theta_1, I_2, \theta_2$ and the trajectory is constrained to a 2 dimensional subspace, obtained as a direct product

of two circles, having as radii the values of the two conserved quantities, I_1 and I_2, while θ_1 and θ_2 will define the polar angles of the two circular motions. Thus, the trajectory will be confined on the surface of a torus.

To extend the discussion to systems with n degrees of freedom, we replace the 2D plane with an n dimensional cube. If we join the sides of this cube we obtain a n dimensional torus.

2.6 Chaos notions

Deterministic chaos is a feature of non-integrable systems and is characterised by high sensitivity to initial conditions: for two dynamical systems with nearly identical initial conditions, the long time evolution is uncorrelated, the motion in the two systems being very different. Poincaré was the first to bring attention to this type of behaviour when he proved that there are no analytic solutions to the 3-body problem [4]. Even if the equations of motion are deterministic, long time predictions are rendered useless since an arbitrary small variation in the initial conditions can produce very different results.

In chaotic systems, the motion is no longer constrained on tori as for the case of integrable systems. The trajectories escape from the n dimensional subspace in the $2n$ phase space space, or in a $2n - 1$ subspace if the energy is conserved.

For the following concepts we use mainly the notations in [5].

2.6.1 The Liapunov exponent

Chaotic systems exhibit a varying degree of randomness. In order to describe chaos quantitatively we introduce the *Liapunov exponent*, which is a local measure of the dispersion in phase space. The sensitivity to the initial conditions of chaotic systems can be expressed as an exponential divergence of neighbouring trajectories.

If two orbits are separated at $t = 0$ by the infinitesimal distance $\mathrm{d}x_0$, then at a later time t, the distance will be

$$\mathrm{d}x\left(t\right) \sim \mathrm{d}x\left(0\right)\mathrm{e}^{\lambda t},$$

where λ is the Liapunov exponent. If $\lambda > 1$, the motion is chaotic and λ gives the exponential growth of the initial infinitesimal separation. If the motion is bounded, $\mathrm{d}x\left(t\right)$ will grow until is comparable with the dimension of the allowed part of the phase space, from that point on varying only randomly in time. Thus the Liapunov exponent can be defined as follows

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\mathrm{d}x\left(t\right)}{\mathrm{d}x\left(0\right)} \tag{2.7}$$

For systems with discrete time evolution instead of the time we can use the number of iterations.

2.6.2 Poincaré sections

Since the phase space is $2n$ dimensional, visualising the motion can be difficult even for systems with $n = 2$ degrees of freedom. If the energy is constant, the motion will be constrained in the $2n - 1$ subspace, but even for the case of $n = 2$ the trajectory is not easy to follow in a 3D space considering that in most cases is represented on a 2D surface.

For $n = 2$ systems this difficulty can be addressed by using a slice of the phase space which reduces the dimensionality by 1. This slices are called *Poincaré sections*. If the energy is conserved, the section will be a plane.

If Σ is a Poincaré section intersecting the trajectories of the system in phase space in the points $\mathbf{x}_i$, then we can define the *Poincaré map* as a mapping $P : \Sigma \to \Sigma$ such that

$$\mathbf{x}_{i+1} = P(\mathbf{x}_i).$$

A point $\mathbf{x}^*$ is called a *fixed point* of the if

$$\mathbf{x}^* = P(\mathbf{x}^*).$$

Thus a fixed point belongs to a closed trajectory. In order to study the stability of the orbit, we can analyse the stability of the fixed point. If $\mathbf{x}^*$ is a fixed point, we consider a small perturbation in the initial conditions and we look at what happens to a point $\mathbf{x}^* + \mathbf{v}_i$ near the fixed point

$$P(\mathbf{x}^* + \mathbf{v}_i) = P(\mathbf{x}^*) + \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i + \mathcal{O}\big(\|\mathbf{v}_i\|^2\big),$$

where $\frac{\partial P}{\partial \mathbf{x}}$ is the Jacobian of the map. Assuming we can neglect $\mathcal{O}\big(\|\mathbf{v}_i\|^2\big)$, we get

$$\mathbf{x}^* + \mathbf{v}_{i+1} = \mathbf{x}^* + \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i,$$

where $\mathbf{x}^* + \mathbf{v}_{i+1} = P(\mathbf{x}^* + \mathbf{v}_i)$. Thus

$$\mathbf{v}_{i+1} = \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i, \tag{2.8}$$

also called the *linearised Poincaré map*.

In a basis given by the eigenvectors $\{\mathbf{e}_j\}$ of the Jacobian, $\mathbf{v}_i = \sum_{j=1}^{2n-1} a_j \mathbf{e}_j$, then

$$\mathbf{v}_{i+1} = \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \sum_{j=1}^{2n-1} a_j \mathbf{e}_j = \sum_{j=1}^{2n-1} \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{e}_j a_j = \sum_{j=1}^{2n-1} \lambda_j a_j \mathbf{e}_j,$$

where λ_j are the eigenvalues of the Jacobian matrix.

If we consider k iterations of eq. (2.8)

$$\mathbf{v}_{i+k} = \sum_{j=1}^{2n-1} (\lambda_j)^k a_j \mathbf{e}_j$$

If all $|\lambda_j| < 1$, then $\|\mathbf{v}_{i+k}\| \to 0$ as $k \to \infty$ and the fixed point is *linearly stable.* If $|\lambda_j| > 1$ for some j, then the fixed point is *linearly unstable.* In the case when $|\lambda_m| = 1$, where λ_m is the largest eigenvalue, we cannot decide and a nonlinear stability analysis is required.

2.7 Kolmogorov-Arnold-Moser theorem

Since most systems are non-integrable we need approximate methods. One such method is canonical perturbation theory which can provide approximate solutions when the non-integrable Hamiltonian can be broken in an integrable part and a relatively small perturbation. For a detailed treatment, see [3].

Let us consider

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H},$$

where $\mathcal{H}_0$ is the integrable part of the Hamiltonian for which the solutions of the Hamilton equations are known and $\Delta\mathcal{H}$ is the perturbation.

If we make a canonical transformation with a generating function of type F_2 such that $\mathcal{H}'_0 = 0$ we can obtain the constants of the motion for the integrable part of the Hamiltonian, Q_0 and P_0. If we use the generating function $S(q, P, t)$ to make ca canonical transformation for $\mathcal{H}$, we obtain

$$\mathcal{H}'(Q_0, P_0, t) = \Delta\mathcal{H}^0(Q_0, P_0, t).$$

Hamilton's equations read

$$\dot{Q} = \frac{\partial \mathcal{H}'}{\partial P} = \frac{\partial \Delta\mathcal{H}^0}{\partial P}, \qquad \dot{P} = -\frac{\partial \mathcal{H}'}{\partial Q} = -\frac{\partial \Delta\mathcal{H}^0}{\partial Q}.$$

In order to obtain the first order correction, we replace Q and P after derivation with Q_0 and P_0.

$$\dot{Q}_1 = \left.\frac{\partial \Delta\mathcal{H}^0}{\partial P}\right|_{\substack{Q=Q_0, \\ P=P_0}} \qquad \dot{P}_1 = -\left.\frac{\partial \Delta\mathcal{H}^0}{\partial Q}\right|_{\substack{Q=Q_0, \\ P=P_0}}$$

We can integrate the above differential equations to obtain Q_1 and P_1.

In order to obtain higher order corrections we replace in $\Delta\mathcal{H}^0$ the conjugate variables Q_0 and P_0 with Q_1 and P_1, obtaining $\Delta\mathcal{H}^1$ and the procedure repeats.

$$\dot{Q}_{i+1} = \left.\frac{\partial \Delta\mathcal{H}^i}{\partial P}\right|_{\substack{Q=Q_i, \\ P=P_i}} \qquad \dot{P}_{i+1} = -\left.\frac{\partial \Delta\mathcal{H}^i}{\partial Q}\right|_{\substack{Q=Q_i, \\ P=P_i}}$$

It is clear that if the perturbation is small enough, we can no longer apply this method and the perturbation will significantly alter the motion. Hence, an important question that arises is the stability of the perturbed solutions and the domain of applicability for the perturbation theory.

The Kolmogorov-Arnold-Moser theorem or K.A.M. theorem gives the limits of regular (or non-chaotic) motion for systems with bounded motion.

Theorem (Kolmogorov-Arnold-Moser theorem). *If the bounded motion of an integrable system given by the Hamiltonian $\mathcal{H}_0$ is disturbed by a small perturbation $\Delta\mathcal{H}$ rendering the system non-integrable and if the unperturbed motion has incommensurate frequencies $\omega_i = \dot{\theta}_i$, where θ_i are the angle variables, then the motion will remain confined to an n-torus, except for a small set of initial conditions for which the trajectories escape on the energy hyper-surface.*

The theorem tells us that the majority of the tori will survive the perturbation and the set of destroyed tori has finite volume in the phase space. A torus of the unperturbed system survives if it can be continuously deformed in a torus of the perturbed system. Furthermore, the destroyed tori have rationally related frequencies, that is there exist $m_1, \ldots, m_n \equiv \mathbf{m} \in \mathbb{Z}^n$ such that $\mathbf{m} \cdot \boldsymbol{\omega} = 0$, where $\boldsymbol{\omega} \equiv \omega_1, \ldots, \omega_n$.

Since the set of rational numbers is dense in the set of real numbers, the set of destroyed tori will be dense in the phase space. In the place of the destroyed tori chaotic motion will appear as the trajectories escape in the $2n - 1$ subspace of constant energy.

Chapter 3

Quantum Chaos

3.1 Fundamental notions

In this section will briefly present the basic principles of quantum mechanics and introduce the concepts which are required for the characterisation of energy spectra. We begin with the concept of the physical state. A *physical state* contains all the information that can be learned about the system. We associate to our physical system a *Hilbert space*, denoted with $\mathbb{H}$, which will contain all the possible states of the system. A vector in the Hilbert space will describe the state of the system (more rigorously only the direction of the vector will describe the state since by convention any two vectors that differ only by a constant describe the same physical state). In the Dirac formalism [2] the vectors in the Hilbert space corresponding to physical states are represented by *ket vectors*, denoted with $|\Psi\rangle$. The elements of the dual of the Hilbert space are called *bra vectors* and are denoted with $\langle\Psi|$. Due to the isomorphism between the Hilbert space and its dual, the bra vectors equally describe the physical state and the correspondence between the ket vectors and the bra vectors is given by dual conjugation.

The observables postulate

In quantum mechanics the properties of the physical system, called *observables* are described by linear hermitian operators. As a result of a measurement, the obtained values of the observable are among the eigenvalues of the associated hermitian operator.

The measurement postulate

Let us consider a physical system in a state described by the state vector $|\Psi\rangle \in \mathbb{H}$ and an observable described by the linear hermitian operator A. If a measurement of the observable is performed, then following the measurement the system will jump in an uncontrollable manner in one of the eigenstates of the operator associated with the measured observable and the result of the measurement will be given by the corresponding eigenvalue. The probability of obtaining a given eigenvalue is

given by the square of the absolute value of the scalar product between the final state associated with the given eigenvalue and the initial state.

Fundamental commutation relations

In analogy with classical mechanics, a quantum version of the Poisson brackets can be defined, postulating that they have the same properties as the classical Poisson brackets, namely anti-symmetry, linearity, the product rule and the Jacobi identity. These properties uniquely define the form of the quantum Poisson brackets the commutator of two observables, $[A, B] = AB - BA$, as being proportional to the quantum Poisson brackets

$$[A, B] = \mathrm{i}\hbar\{A, B\}_{QM}.$$

Thus we obtain the fundamental commutation relations in quantum mechanics

$$[Q_i, Q_j] = 0$$
$$[P_i, P_j] = 0$$
$$[Q_i, P_j] = \mathrm{i}\hbar\,\delta_{i,j}.$$

Time evolution postulate

In quantum mechanics time is just a parameter and not an observable. The time evolution can be postulated in different ways. If we consider the states as time dependent entities and observables as time independent ones, we can view the time evolution as a temporal displacement, similar to a spatial displacement. Then we can use the time evolution operator identity

$$\mathrm{i}\hbar\frac{\partial}{\partial t}U(t, t_0) = HU(t, t_0)$$

to derive the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle = H\left|\Psi\right\rangle.$$

If we consider the states as time independent and the observables as time dependent, then the time evolution of the observables will be given by

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}A(t) = [A(t), H].$$

We can observe the link between this equation and Hamilton's equations is given by the correspondence between the commutator in quantum mechanics and the Poisson bracket in classical mechanics, as mentioned in the previous chapter.

The first approach is called the Schrödinger picture of quantum mechanics, while the second is called the Heisenberg picture. There is also another formulation, namely the interaction picture also named the Dirac picture. In the following we will only consider the Schrödinger picture.

In the particular case when the Hamiltonian is time-independent, Schrödinger's equation reduces to

$$H \left| \Psi \right\rangle = E \left| \Psi \right\rangle,$$

also called the time-independent Schrödinger equation and the time evolution of the state is given by

$$\left| E(t) \right\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)H} \left| E(t_0) \right\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)E} \left| E \right\rangle.$$

Thus if the system is initially in an energy eigenstate, it will remain in the same state, having at most a phase modulation. Such states are called *stationary states.*

3.2   From symmetry to degeneracy

Let us consider the above mentioned time-independent case. If we suppose that the energy spectrum is discrete,

$$H \left| \Psi_i \right\rangle = E_i \left| \Psi_i \right\rangle.$$

If for an eigenvalue E_i repeats itself for different eigenstates $\left| \Psi_i \right\rangle$ we have what is called a *degeneracy.* In order to emphasise this, we can use a second index for the eigenvectors, $j = 1, \ldots, m$, where m is the number of times the eigenvalue repeats.

Thus for the i-th eigenvalue,

$$H \left| \Psi_{i,j} \right\rangle = E_i \left| \Psi_{i,j} \right\rangle.$$

If the Hamiltonian is invariant to a set of unitary transformations, $T^\dagger H T = H$, then this set forms a group since the transformation given by TT', with T' an arbitrary transformation from the set, will also leave the Hamiltonian invariant. Since symmetry operations form a group, the unitary operators which correspond to the symmetries of the Hamiltonian will form a group to which the Hamiltonian is invariant. Thus,

$$TH = TT^\dagger H T = TT^{-1} H T = HT,$$

or $[T, H] = 0$. We can once again observe how classical mechanics and quantum mechanics are linked through the Poisson bracket-commutator structure. In classical mechanics Noether's theorem asserts that to any symmetry of the action will correspond a constant of the motion. This will have a vanishing Poisson bracket with the Hamiltonian. For example, the invariance to rotations around a given axis will lead to angular momentum conservation.

Since the operators commute, they share a common set of eigenvectors

$$H(T \left| \Psi_i \right\rangle) = HT \left| \Psi_i \right\rangle = TH \left| \Psi_i \right\rangle = TE_i \left| \Psi_i \right\rangle = E_i(T \left| \Psi_i \right\rangle).$$

Thus if the Hamiltonian is invariant to the set of unitary transformations $\{T_j\}$, with $j = 1, \ldots, m$

$$T_j H \left| \Psi_i \right\rangle = TE_i \left| \Psi_i \right\rangle = E_i(T \left| \Psi_i \right\rangle) = E_i \left| \Psi_{i,j} \right\rangle$$

and we will not change the value of the eigenvalue E_i by applying T_j, we only transform the eigenstates $\left| \Psi_i \right\rangle$ to a linear combination $\left| \Psi_{i,j} \right\rangle$.

3.3 Level repulsion

As is the case with classical mechanics there are few situations in which the analytic solution is known. In order to find approximate solutions we can use perturbation theory if we can assume that the Hamiltonian can be viewed as to have a part for which the solutions are known and another part that can be considered a small perturbation. The following concepts are mainly following the notations from [6] and the lecture notes [7, 8].

In the framework of time-independent perturbation theory, we want to obtain an approximate solution to the problem

$$H \left| \Psi^{[j]} \right\rangle = E_j \left| \Psi^{[j]} \right\rangle \tag{3.1}$$

and we consider that the Hamiltonian can be written as $H = H_0 + \lambda V$, where the solution to the unperturbed problem

$$H_0 \left| \Psi^0_{n\alpha} \right\rangle = E^0_n \left| \Psi^0_{n\alpha} \right\rangle$$

is known and α gives the degeneracy of the n-th level.

Since the unperturbed eigenvectors form a basis in the Hilbert space, we can expand the perturbed eigenvectors in that basis.

$$\left| \Psi^{[j]} \right\rangle = \sum_{m,\beta} c^{[j]}_{m,\beta} \left| \Psi^0_{m,\beta} \right\rangle .$$

Inserting into eq. (3.1) we obtain

$$(H_0 + \lambda V) \sum_{m,\beta} c^{[j]}_{m,\beta} \left| \Psi^0_{m,\beta} \right\rangle = \sum_{m,\beta} E_j \, c^{[j]}_{m,\beta} \left| \Psi^0_{m,\beta} \right\rangle .$$

Using the solution to the unperturbed problem the above equation becomes

$$\sum_{m,\beta} (E^0_m + \lambda V) c^{[j]}_{m,\beta} \left| \Psi^0_{m,\beta} \right\rangle = \sum_{m,\beta} E_j \, c^{[j]}_{m,\beta} \left| \Psi^0_{m,\beta} \right\rangle .$$

To simplify the discussion we will now consider what happens with two unperturbed states $|\Psi^0_1\rangle$ with the energy E^0_1 and $|\Psi^0_2\rangle$ with the energy E^0_2 when $\lambda = 1$.

$$\sum_{m,\beta} (E^0_m + V) c^{[j]}_m \left| \Psi^0_m \right\rangle = \sum_{m=1,2} E_j \, c^{[j]}_m \left| \Psi^0_m \right\rangle .$$

By taking the scalar product with the state $\langle \Psi^0_k |$ and using the orthogonality relation $\langle \Psi^0_k | \Psi^0_m \rangle = \delta_{k,m}$ we obtain

$$E^0_k \, c^{[j]}_k + \sum_{m=1,2} \left\langle \Psi^0_k \right| V \left| \Psi^0_m \right\rangle c^{[j]}_m = E_j \, c^{[j]}_k,$$

or

$$c^{[j]}_k \left(E^0_k - E_j + V_{k,k} \right) + \sum_{m \neq k} V_{k,m} \, c^{[j]}_m = 0,$$

where $\langle \Psi_k^0 | V | \Psi_m^0 \rangle \equiv V_{k,m}$.

The above equation becomes

$$(E_1^0 - E_j + V_{1,1}) c_1^{[j]} + V_{1,2} c_2^{[j]} = 0, \text{ for } k = 1$$

$$V_{2,1} c_1^{[j]} + (E_2 - E_j + V_{2,2}) c_2^{[j]} = 0, \text{ for } k = 2.$$

We use the following notations:

$$H_{1,1} \equiv E_1^0 + V_{1,1}, H_{2,2} \equiv E_2^0 + V_{2,2}, H_{1,2} \equiv V_{1,2}, H_{2,1} \equiv V_{2,1}, \delta \equiv H_{1,1} - H_{2,2}, \tan \beta \equiv \frac{2|H_{1,2}|}{\delta}.$$

The above system of equations has non-trivial solutions if the determinant vanishes.

$$(H_{1,1} - E_j)(H_{2,2} - E_j) - H_{1,2}H_{2,1} = 0$$

or

$$E_j^2 - (H_{1,1} + H_{2,2})E_j + H_{1,1}H_{2,2} - H_{1,2}H_{2,1} = 0.$$

This equation has the solutions

$$E_j = \frac{(H_{1,1} + H_{2,2}) \pm \sqrt{(H_{1,1} + H_{2,2})^2 - 4(H_{1,1}H_{2,2} - H_{1,2}H_{2,1})}}{2} \qquad (3.2)$$

$$= \frac{H_{1,1} + H_{2,2}}{2} \pm \frac{1}{2}\sqrt{\delta^2 + 4|H_{1,2}|^2}. \qquad (3.3)$$

Since $(H_{1,1} - E_j) c_1^{[j]} + H_{1,2} c_2^{[j]} = 0$,

$$\frac{c_1^{[j]}}{c_2^{[j]}} = \frac{H_{1,2}}{E_j - H_{1,1}}.$$

If we rewrite eq. (3.3) as

$$E_j = \frac{1}{2}\left[H_{1,1} + H_{2,2} \mp (H_{2,2} - H_{1,1})\sqrt{1 + \frac{4|H_{1,2}|^2}{\delta^2}} \right]$$

we obtain

$$\frac{c_1^{[j]}}{c_2^{[j]}} = \frac{2H_{1,2}}{H_{2,2} - H_{1,1}}\left[1 \mp \sqrt{1 + \frac{4|H_{1,2}|^2}{\delta^2}} \right]^{-1} = -\tan \beta \left(1 \mp \sqrt{1 + \tan^2 \beta} \right).$$

This ratio can be also expressed as

$$\frac{c_1^{[j]}}{c_2^{[j]}} = \frac{-\tan \beta}{1 \mp \frac{1}{\cos \beta}} = \frac{-\sin \beta}{\cos \beta \mp 1}$$

$$= \frac{-2 \sin \frac{\beta}{2} \cos \frac{\beta}{2}}{\cos^2 \frac{\beta}{2} - \sin^2 \frac{\beta}{2} \mp \left(\cos^2 \frac{\beta}{2} + \sin^2 \frac{\beta}{2} \right)} = \begin{cases} \cot \frac{\beta}{2} \\ -\tan \frac{\beta}{2} \end{cases}.$$

Thus,

$$\left|\Psi^{[1]}\right\rangle = \cos\frac{\beta}{2}\left|\Psi_1^0\right\rangle + \sin\frac{\beta}{2}\left|\Psi_2^0\right\rangle$$

$$\left|\Psi^{[2]}\right\rangle = -\sin\frac{\beta}{2}\left|\Psi_1^0\right\rangle + \cos\frac{\beta}{2}\left|\Psi_2^0\right\rangle.$$

If the matrix elements of the interaction which mix distinct states are relatively small, that is $|H_{1,2}| \ll \delta$, $\beta \simeq 0$

$$E_{1,2} = \frac{H_{1,1} + H_{2,2}}{2} \pm \frac{\delta}{2}\sqrt{1 + \frac{4|H_{1,2}|^2}{\delta^2}}$$

Thus

$$E_1 \simeq H_{1,1} + \frac{|H_{1,2}|^2}{\delta}$$

and

$$E_2 \simeq H_{2,2} - \frac{|H_{1,2}|^2}{\delta}.$$

and the perturbed energy levels are close to the unperturbed ones. The states will also be approximatively the unperturbed ones

$$\left|\Psi^{[1]}\right\rangle \simeq \left|\Psi_1^0\right\rangle$$

$$\left|\Psi^{[2]}\right\rangle \simeq \left|\Psi_2^0\right\rangle.$$

If on the other hand, the matrix elements levels which mix distinct states are relatively strong, that is $|H_{1,2}| \gg \delta$, $\beta \simeq \frac{\pi}{2}$

$$E_{1,2} = \frac{H_{1,1} + H_{2,2}}{2} \pm \sqrt{\frac{\delta^2}{4} + |H_{1,2}|^2} \simeq \frac{H_{1,1} + H_{2,2}}{2} \pm \left(|H_{1,2}| + \frac{\delta^2}{8|H_{1,2}|}\right)$$

and

$$\left|\Psi^{[1]}\right\rangle \simeq \frac{\sqrt{2}}{2}\left|\Psi_1^0\right\rangle + \frac{\sqrt{2}}{2}\left|\Psi_2^0\right\rangle$$

$$\left|\Psi^{[2]}\right\rangle \simeq -\frac{\sqrt{2}}{2}\left|\Psi_1^0\right\rangle + \frac{\sqrt{2}}{2}\left|\Psi_2^0\right\rangle.$$

In this case we observe that if $\delta \simeq 0$, then $E_1 - E_2 \simeq 2|H_{1,2}|$. If the unperturbed energy levels were degenerated, then the perturbed energy levels will not remain so. The removal of the degeneracy is called *level repulsion*.

We can illustrate this phenomena by considering an arbitrary 2×2 hermitian matrix and computing its eigenvalues.

$$H = \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{1,2}^* & H_{2,2} \end{pmatrix}$$

The eigenvalues will be given by eq. (3.3).

The first case, $|H_{1,2}| \ll \delta$, can be viewed as the case when the off-diagonal elements are small and the matrix can be approximated with a diagonal matrix. In this case we expect that the energy levels will be close to the diagonal levels or the unperturbed levels. The perturbed eigenstates will also be approximatively equal with the unperturbed eigenstates.

The level repulsion emphasised by the second case, $|H_{1,2}| \gg \delta$, corresponds to the case when the off-diagonal elements are significant. When we expand the perturbed eigenstates in the basis given by the unperturbed eigenstates we will have significant components from each element in the basis and the new states will be a mixture of the unperturbed states. We can say that the energy levels are no longer independent. In the case when the unperturbed energy levels are degenerated, the difference between the perturbed energy levels is given by $2|H_{1,2}|$, which is a measure of the mixing of the unperturbed states.

3.4 Probability notions

Definition (Joint probability). Given two events A and B, their joint probability, $P(A \cap B)$, is the probability of the two events to occur simultaneously.

Definition (Conditional probability). The conditional probability of an event A given an event B with $P(B) > 0$, denoted $P(A \,|\, B)$ is given by

$$P(A \,|\, B) = \frac{P(A \cap B)}{P(B)}.$$

The joint probability of A and B can be expressed as $P(A \cap B) = P(A \,|\, B)P(B)$.

Definition (Continuous random variable). A continuous random variable is a function from the set of all outcomes to the set of real numbers $X : \Omega \to (R)$ such that

$$P(a \leq X(\omega) \leq b) = \int_a^b f(x)\,\mathrm{d}x\,,$$

where $f(x) \geq 0$ and $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$.

The above function f is called the *probability density function*. The probability for a continuous random variable X to take a value in the infinitesimal interval of length $\mathrm{d}x$ is given by

$$P(x \in [x, x + \mathrm{d}x]) = \int_x^{x+\mathrm{d}x} f(z)\,\mathrm{d}z \approx f(x)\,\mathrm{d}x \tag{3.4}$$

3.5 Nearest neighbour distributions

Nearest neighbour spacing distributions show how the differences between consecutive energy levels fluctuate around the average. In order to better understand this concept we shall begin with the simpler case of real random numbers as presented in [9] and then continue with a general case as in [10].

3.5.1 The nearest neighbour spacing distribution of random numbers

We consider a sequence of uniformly distributed, ordered, real, random numbers. We define the *spacing* of an ordered sequence as the sequence of differences between consecutive elements. For an interval of length $s, s > 0$, we will denote with $P(n \in s)$ the probability for the interval to contain n numbers and with $P(n \in \mathrm{d}s \,|\, m \in s)$ the conditional probability for the interval of length $\mathrm{d}s$ to contain n numbers given that the interval of length s contains m numbers.

If E is a given number in the sequence, we are interested in the probability $P(s)\,\mathrm{d}s$ to have the next number between $E + s$ and $E + s + \mathrm{d}s$. Since we are interested in the next number after E, we know that in the interval of length s there is no other number and the next number is somewhere in the infinitesimal interval $\mathrm{d}s$. Thus the joint probability of the events $1 \in \mathrm{d}s$ and $0 \in s$ is given by:

$$P(s)\,\mathrm{d}s = P(1 \in \mathrm{d}s \,|\, 0 \in s)P(0 \in s). \tag{3.5}$$

Since random numbers are not correlated, the probability of a random number to be found in the interval $\mathrm{d}s$ does not depend on the number of random numbers in s, so

$$P(1 \in \mathrm{d}s \,|\, 0 \in s) = P(1 \in \mathrm{d}s).$$

The random numbers are uniformly distributed, so their probability density function f is a constant. Hence, according to eq. (3.4), the probability of finding a number in the interval of length $\mathrm{d}s$ is given by

$$P(1 \in \mathrm{d}s) \equiv P(1 \in [0, 0 + \mathrm{d}s]) \approx f(s)\,\mathrm{d}s \sim \mathrm{d}s\,.$$

If we denote the constant probability density function with a

$$P(s)\,\mathrm{d}s = a\,\mathrm{d}s\,P(0 \in s).$$

$P(0 \in s)$ can be expressed using the complementary probability as $1 - \int_0^s P(s')\,\mathrm{d}s'$. Now we can express $P(s)\,\mathrm{d}s$ as follows:

$$P(s)\,\mathrm{d}s = a\,\mathrm{d}s\left(1 - \int_0^s P(s')\,\mathrm{d}s'\right).$$

In order to differentiate with respect to s, we will use the Leibniz rule for differentiating integrals, namely

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{G(x)}^{H(x)} F(x,t)\,\mathrm{d}t = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}\,\mathrm{d}t + F(x, H(x))\frac{\mathrm{d}H}{\mathrm{d}x} - F(x, G(x))\frac{\mathrm{d}G}{\mathrm{d}x} \tag{3.6}$$

Using this rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}P(s) = -aP(s).$$

This differential equation can be solved by separation of variables, yielding

$$P(s) = \mathcal{C}\, e^{-as}$$

In order to determine the constant $\mathcal{C}$, we use the normalisation condition for the probability density function

$$\int_{-\infty}^{\infty} P(s)\, ds = 1$$

Since $s > 0$, this reduces to

$$\int_{0}^{\infty} P(s)\, ds = \int_{0}^{\infty} \mathcal{C}\, e^{-as}\, ds = -\frac{\mathcal{C}}{a}\, e^{-as}\bigg|_{0}^{\infty} = \frac{\mathcal{C}}{a}$$

Thus $\mathcal{C} = a$ and $P(s) = a\, e^{-as}$. We can further simplify the formula if we set that the average spacing to unity. The average spacing is given by

$$\bar{s} = \int_{0}^{\infty} s P(s)\, ds = -\int_{0}^{\infty} s \frac{d}{ds}\left(e^{-as}\right) ds = \int_{0}^{\infty} e^{-as}\, ds = \frac{1}{a},$$

so setting it to unity results in $a = 1$. Thus the probability density function becomes

$$P(s) = e^{-s}. \tag{3.7}$$

This function is known as the *Poisson distribution*.

3.5.2 The Wigner distribution

We will now consider a more complicated situation, by considering the probability density function for the sequence of random numbers to be arbitrary.

We can start from eq. (3.5) since the discussion up to that point did not include any details related to the distribution of the random numbers.

In this case $P(1 \in ds \mid 0 \in s) = f_{1,0}(s)\, ds$, where $f_{n,m}(s)$ is function which describes how the probability of having n numbers in ds is influenced by the m numbers in s. Thus

$$P(s)\, ds = f_{1,0}(s)\, ds \left(1 - \int_{0}^{s} P(s')\, ds'\right)$$

By solving this integral equation, we obtain the solution

$$P(s) = \mathcal{C} f_{1,0}(s) \exp\left(-\int_{0}^{s} f_{1,0}(x)\, dx\right)$$

We observe that if we take the probability density function constant, $f_{1,0}(s) = \frac{1}{a}$, we obtain the above case of the Poisson distribution. For a probability density function, $f_{1,0}(s) = \alpha s$ we obtain

$$P(s) = \mathcal{C}\alpha s \exp\left(-\alpha \frac{s^2}{2}\right)$$

From the normalisation condition we obtain

$$\int_0^\infty P(s)\,\mathrm{d}s = \mathcal{C} = 1$$

The average spacing is given by

$$\overline{s} = \int_0^\infty s P(s)\,\mathrm{d}s = \alpha \int_0^\infty s^2 \exp\left(-\alpha \frac{s^2}{2}\right)\mathrm{d}s = \frac{1}{\sqrt{\alpha}}\sqrt{\frac{\pi}{2}}.$$

If we set the average spacing to unity, we obtain $\alpha = \frac{\pi}{2}$ and

$$P(s) = \frac{\pi}{2}s \exp\left(-\frac{\pi}{4}s^2\right)$$

This function is known as the *Wigner distribution*.

3.6   From Classical Chaos to Quantum Chaos

The classical concept of sensitivity to initial conditions loses its meaning in the quantum realm since the trajectory cannot be defined due to Heisenberg's uncertainty principle. However, there are some other ways in which we can link classically chaotic dynamics to quantum features. These bridges between classical mechanics and quantum mechanics allows us to give a meaning to quantum chaos [11].

Specifically, there are two important conjectures that allow us to connect classical systems and quantum systems as mentioned above.

The Berry-Tabor conjecture

This conjecture states that the quantum counterpart of a classically integrable system has a Poissonian nearest neighbour distribution.

The Bohigas-Gianoni-Schmit conjecture

This conjecture states that the nearest neighbour distribution of a quantum system with a classically chaotic counterpart is given by the Wigner distribution.

Chapter 4

Procedure

4.1   The Hamiltonian

In this chapter we will describe how we obtained the numerical results. We begin by computing the matrix elements of the Hamiltonian in a basis given by the eigenstates of an isotropic double harmonic oscillator. In analogy with the classical case, we consider two independent quantum numbers n_1, n_2 corresponding to the two orthogonal oscillating directions. We can define two number operators N_1, N_2 such that $N_1 |n_1, n_2\rangle = n_1 |n_1, n_2\rangle$ and $N_2 |n_1, n_2\rangle = n_2 |n_1, n_2\rangle$. We consider the creation and annihilation operators $a_1^\dagger, a_1$ and $a_2^\dagger, a_2$ such that $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$. In terms of the previously defined operators, the Hamiltonian of the isotropic double harmonic oscillator is given by

$$H_0 = \hbar\omega_0 \left(a_1^\dagger a_1 + \frac{1}{2} I + a_2^\dagger a_2 + \frac{1}{2} I \right) = \hbar\omega_0 \left(N_1 + N_2 + I \right)$$

and its eigenstates are given by

$$H_0 |n_1, n_2\rangle = \hbar\omega_0 \left(n_1 + n_2 + 1 \right) |n_1, n_2\rangle$$

Since $n_1 + n_2 = n$ can be obtained in $\sum_{i=0}^{n} i = \frac{1}{2} n(n+1)$ ways, the energy levels of the isotropic double harmonic oscillator are $\frac{1}{2} n(n+1)$ fold degenerated.

In the fundamental state $|0, 0\rangle$, with $n_1 = n_2 = 0$, the two dimensional oscillator has a *zero point motion* with the energy equal to $\hbar\omega_0$. Since we are interested in the spacings between consecutive, we can rescale the energy such that the energy of the fundamental state becomes 0. Thus the new Hamiltonian will be given by

$$H_0 = A \left(N_1 + N_2 \right),$$

where $A = \hbar\omega_0$.

We can construct the basis starting from the vacuum state $|0, 0\rangle$ by acting with the creation operators

$$a_1^{\dagger n_1} |0, 0\rangle = \sqrt{n_1!} |n_1, 0\rangle$$
$$a_2^{\dagger n_2} |0, 0\rangle = \sqrt{n_2!} |0, n_2\rangle$$

Thus, by applying the operator $a_1^{\dagger n_1} a_2^{\dagger n_2}$ we generate the basis elements ordered as follows:

$$|0,0\rangle \, |0,1\rangle \, |0,2\rangle \cdots |0,n\rangle \, |1,0\rangle \, |1,1\rangle \cdots |1,n-1\rangle \cdots |i,0\rangle \, |i,1\rangle \cdots |i,n-i\rangle \cdots |n,0\rangle \,.$$

For our investigations, the Hamiltonian is expressed as a function of the creation and annihilation operators up to fourth order terms as follows

$$
\begin{aligned}
H = {}& A \left(a_1^\dagger a_1 + a_2^\dagger a_2 \right) + \frac{B}{4} \bigg[\left(3 a_1^\dagger a_2^{\dagger 2} + 3 a_1 a_2^2 - a_1^{\dagger 3} - a_1^3 \right) \\
& + 3 \left(a_1 a_2^{\dagger 2} + a_1^\dagger a_2^2 - a_1^\dagger a_1^2 - a_1^{\dagger 2} a_1 + 2 a_1 a_2^\dagger a_2 + 2 a_1^\dagger a_2^\dagger a_2 \right) \bigg] \\
& + \frac{D}{16} \bigg[6 \left(a_1^{\dagger 2} a_1^2 + a_2^{\dagger 2} a_2^2 \right) + 2 \left(a_1^2 a_2^{\dagger 2} + a_1^{\dagger 2} a_2^2 \right) + 8 a_1^\dagger a_1 a_2^\dagger a_2 \\
& + 4 \left(a_1^\dagger a_1^3 + a_1^{\dagger 3} a_1 + a_2^\dagger a_2^3 + a_2^{\dagger 3} a_2 + a_1^2 a_2^\dagger a_2 + a_1^{\dagger 2} a_2^\dagger a_2 + a_1^\dagger a_1 a_2^2 + a_1^\dagger a_1 a_2^{\dagger 2} \right) \\
& + \left(a_1^{\dagger 4} + a_1^4 + a_2^{\dagger 4} + a_2^4 + 2 a_1^{\dagger 2} a_2^{\dagger 2} + 2 a_1^2 a_2^2 \right) \bigg] .
\end{aligned}
$$

$$(4.1)$$

The physical origin of this Hamiltonian is related to quadrupole dynamics of nuclear surfaces[12, 13]. The operators a_1 and a_2 are related to the normal modes of the quadrupole vibrations. In the case of a deformed nucleus, one mode named β-vibration corresponds to oscillations preserving the axial symmetry. For the second mode, γ-vibrations, the transversal section perpendicularly to the symmetry axis changes periodically from ellipse to circle.

The energy levels will be expressed in units of harmonic oscillator energy and therefore from here on we will consider $A = 1$. We can obtain the eigenvalues and eigenvectors of the Hamiltonian by a diagonalisation routine based on Relatively Robust Representations from Intel® Math Kernel Library [14] used via a `Python` program [15, 16, 17]Any such diagonalisation method requires a truncation of the Hilbert space which induces errors concerning the eigenvalues. This errors increase as one moves to the upper limit of the energy for a fixed dimension of the Hilbert space. Indeed we expect this energies to have more important contributions from the states that were eliminated by truncation. We tested the stability of the energy levels by comparing the results obtained for different sizes of the diagonalisation basis as is detailed in the next section.

4.2 Stability

We consider the *stable levels* to be the eigenvalues which, at a change of basis from one with a dimension of N to one with dimension $N + \Delta N$, do not change with more than a chosen threshold δ_s. In the following figure we show the variation of the energy levels when the dimension increases from $N = 120$ to $N = 140$.
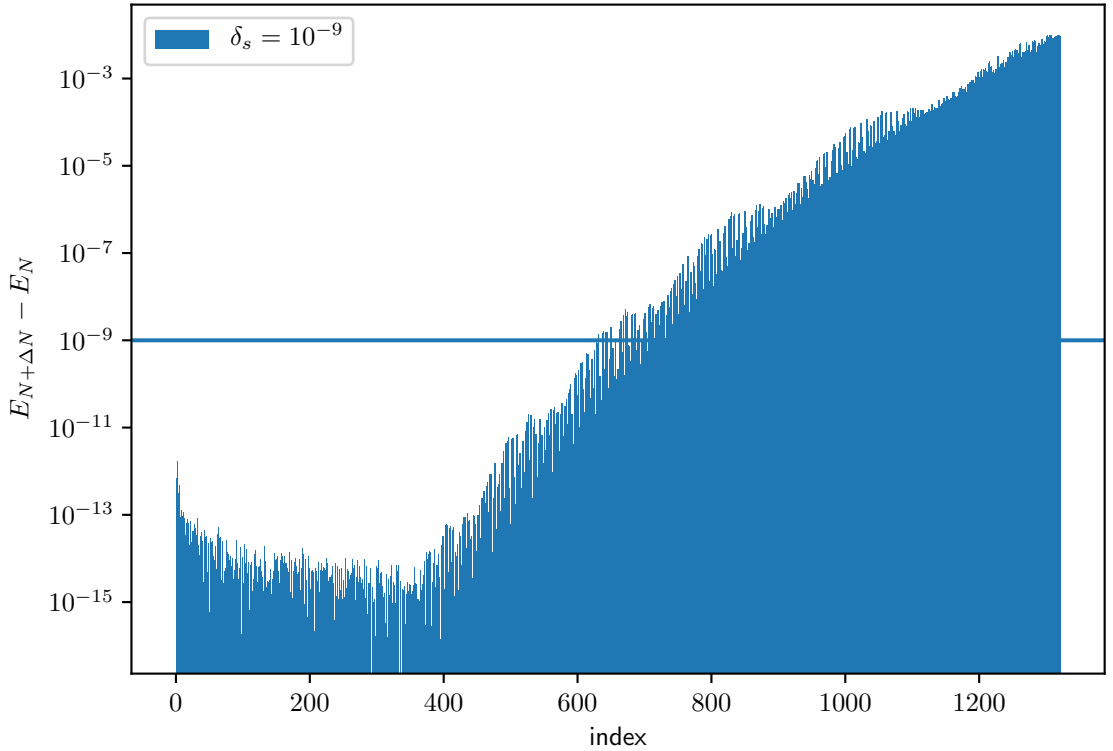
Figure 4.1: $B = 0.2, D = 0.4, N = 120$

We can observe that the first 400–600 eigenvalues have a very good stability. Thus we can choose the stability threshold for example at $\delta_s = 10^{-9}$. Qualitatively the shape of this distribution does not depend on the parameters of the Hamiltonian or the dimension of the Hilbert space because it reflects the nature of the approximation as discussed previously. For example, for $B = 0.55, D = 0.4, N = 260$ compared with $N = 280$
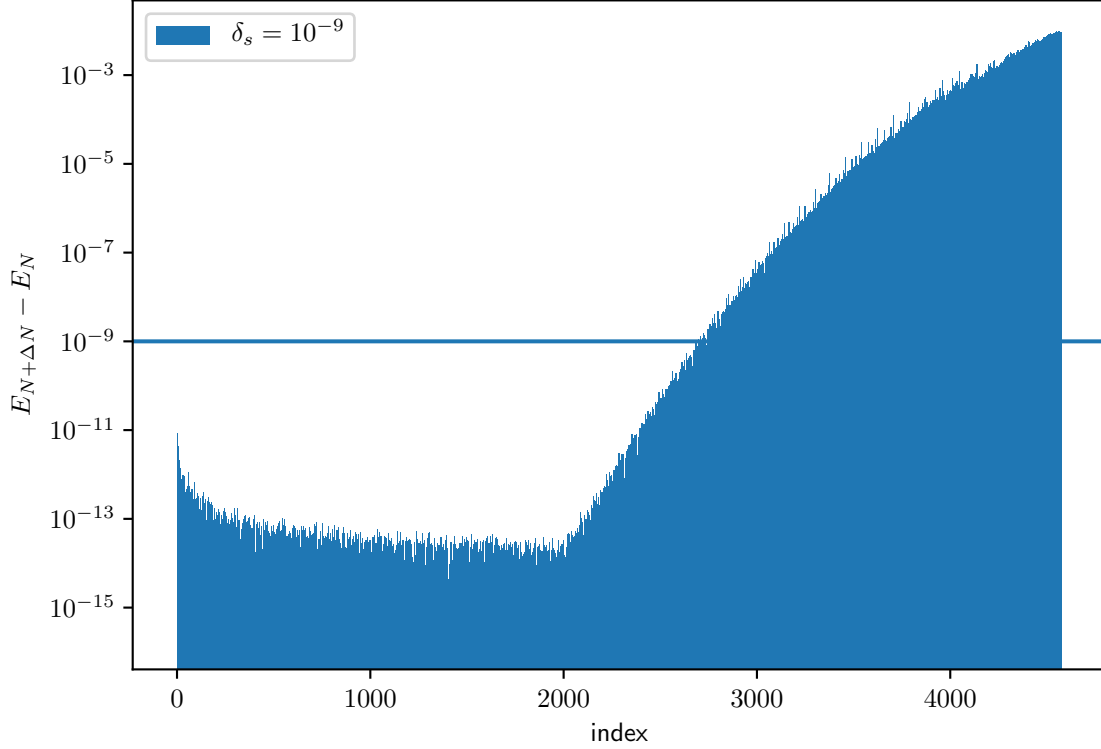
Figure 4.2: $B = 0.55, D = 0.4, N = 260$

As expected, the number of stable levels increases with the diagonalisation basis. Roughly, for a given basis size N, the first 7–8% levels differ with less than $\delta_s = 10^{-9}$ when we compare with a basis of dimension $N + \Delta N$, with $\Delta N = 20$.

4.3 Statistics

As mentioned in chapter 3, a spectrum can be characterised through the probability distribution of the nearest neighbour spacing. The *spacing* is defined as the difference between two consecutive energy levels. Similarly the *relative spacing* is defined as

$$s = \frac{E_{i+1} - E_i}{\overline{\Delta E}},$$

where $\overline{\Delta E}$ is the average spacing $\frac{E_n - E_0}{N}$.

The *nearest neighbour spacing distributions* tell us the probability $P(s)\,\mathrm{d}s$ to find a relative spacing s when we move in the spectrum obtained by diagonalisation. This probability is defined as follows

$$P(s)\,\mathrm{d}s = \frac{N_{s,s+\Delta s}}{N},$$

where $N_{s,s+\Delta s}$ is the number of levels with the relative spacing between s and $s+\Delta s$. We can also define a *cumulative probability distribution*,

$$I(s) = \sum_{s_i=0}^{s} P(s_i)\Delta s$$

4.3.1 Irreducible representations

In order to analyse the distributions of the previously obtained eigenvalues we must first take into account the possible symmetries of the system. A consistent analysis requires us to consider classes of states which transform with the same *irreducible representation*.

The Hamiltonian in eq. (4.1) has the finite $\mathcal{C}_{3v}$ symmetry group. This group has 3 irreducible representations: one bi-dimensional and two unidimensional, one symmetric and one anti-symmetric, namely $\Gamma_b, \Gamma_s, \Gamma_a$.

Separating the bi-dimensional representation

The presence of the bi-dimensional representation corresponds a two-fold degeneracy. These degeneracies can be identified by scrutinising the differences between consecutive levels $\Delta E = E_{i+1} - E_i$. An other option is to use directly the relative spacing, which is a rescaling in units of average separation. The separation of the symmetric and anti-symmetric irreducible representations will be detailed later.

In figure 4.3 we can see how ΔE varies with the index of the levels.
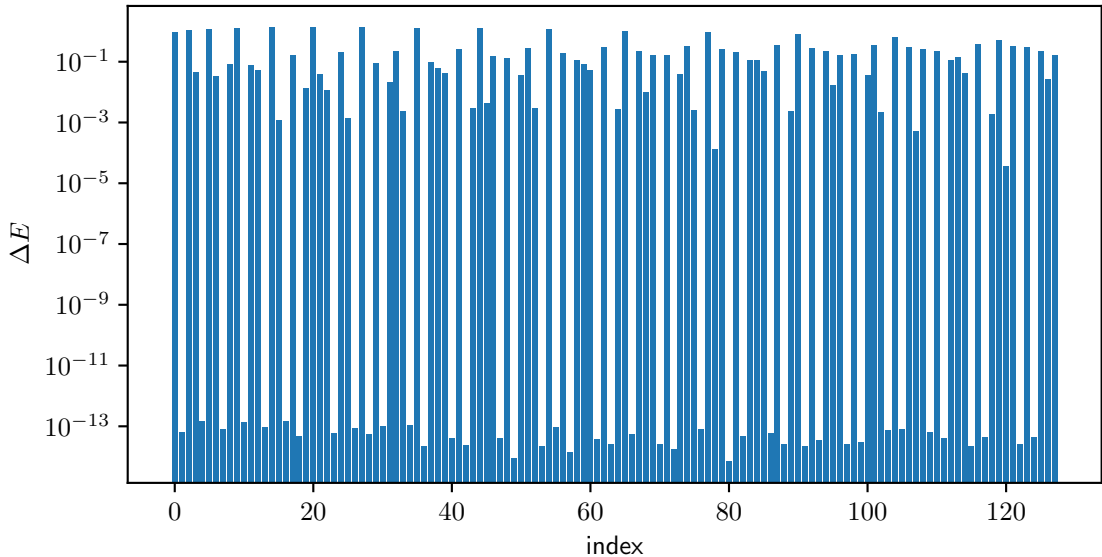
Figure 4.3: $B = 0.2, D = 0.4, N = 60$

Due to of the finite precision of the numerical implementation, the difference between two consecutive degenerate levels might not be exactly 0, its value depending on the machine precision (as it can be seen in the above figure). To take this fact into account we will consider that the levels which have s (or ΔE) greater than a chosen ε as belonging to one of the unidimensional representations. In order to choose a suitable value for ε, we use a histogram to visualise the number of levels at different spacings (see figure 4.4a).

This bimodal shape of the histogram suggests clearly the presence of the degenerate levels well separated from the rest. For some particular values for B (such as

(a) $B = 0.2, D = 0.4, N = 120$ (b) $B = 0.4, D = 0.4, N = 260$
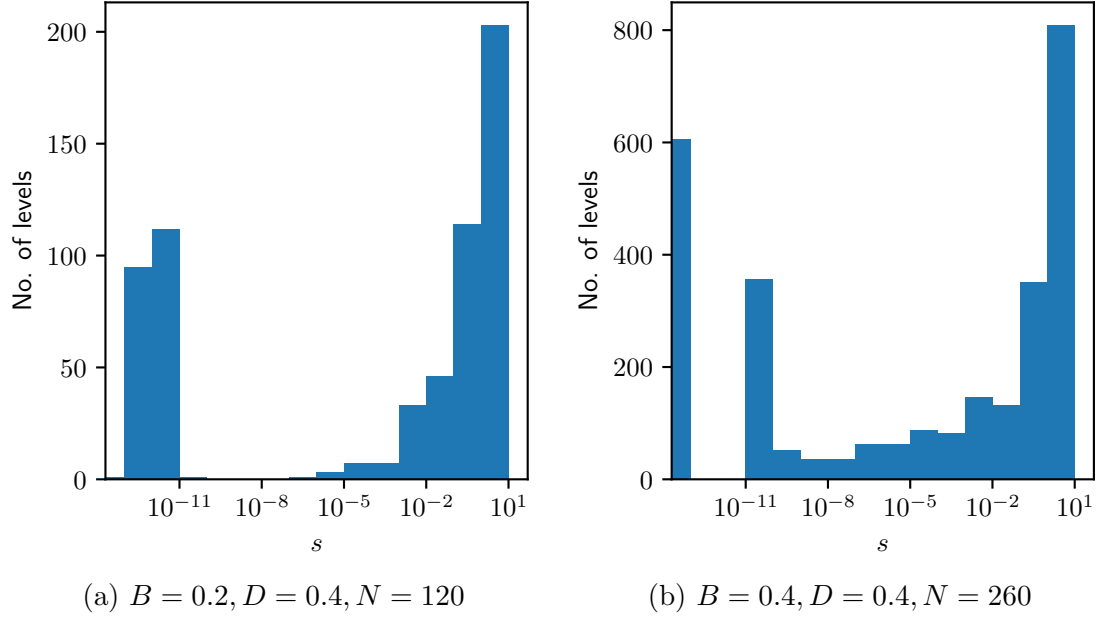
Figure 4.4: The relative spacing histograms for different parameters

$B = 0.4$) and high values for N ($N > 200$), we observed a splitting of the block corresponding to the degenerated levels in two blocks, namely one at exactly 0 and the other at very low values. (see fig. 4.4b)

If we plot the spacing as a function of the level index, we can see how each level is situated with respect to the chosen ε. Once again we can observe a clear separation of the spacings corresponding to the bi-dimensional representation (see fig. 4.5).
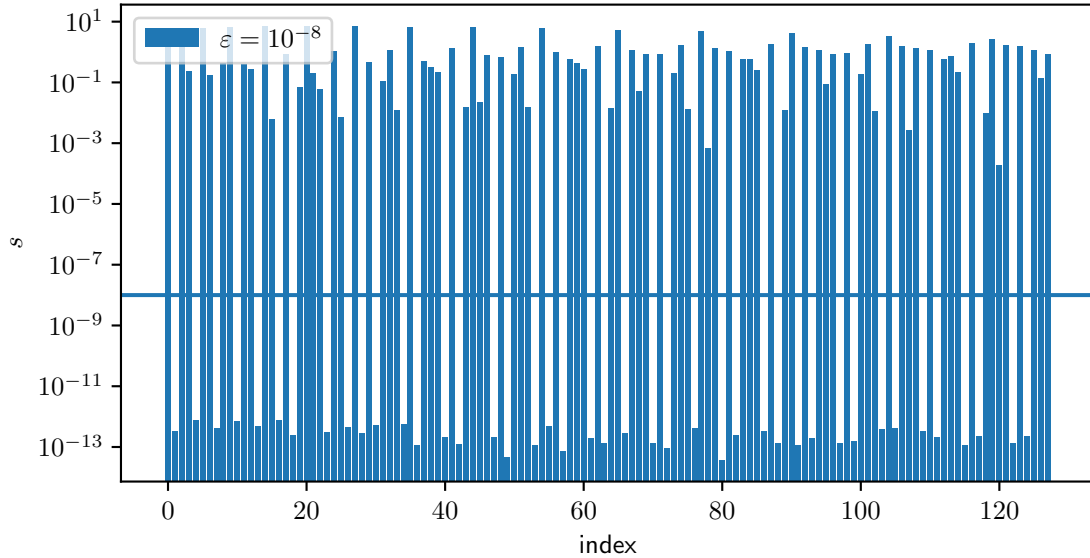
Figure 4.5: $B = 0.2, D = 0.4, N = 60$

Separating the unidimensional representations

The unidimensional representations can be distinguished from the properties of the states at reflections with respect to the Ox axis ($y \rightleftarrows -y$). We ca establish a correspondence between the symmetry of the states to this operation and the values of the quantum number n_2. The states containing dominantly oscillator ket vectors with even n_2 will transform with the symmetric representation.

The values of the quantum numbers n_1 and n_2 depend on the ordering of the basis. Because the diagonalisation algorithm returns the eigenvalues (and the corresponding eigenvectors) in ascending order, the initial ordering of the basis is lost. One method to approximate n_1 and n_2 for a given eigenvector would be to consider that their values are given by the index of the dominant coefficient. Thus, if we have the following eigenvector

$$
\begin{pmatrix}
C_{0,0} \\
C_{0,1} \\
\vdots \\
C_{0,n} \\
C_{1,0} \\
\vdots \\
C_{1,n-1} \\
\vdots \\
C_{i,0} \\
\vdots \\
C_{i,n-i} \\
\vdots \\
C_{n,0}
\end{pmatrix}
$$

and $C_{i,j}$ is the greatest coefficient, than we assign to this eigenvector the quantum numbers of the k-th element in the basis, where k is the index of the coefficient.

For example, for the simplified case of the isotropic double harmonic oscillator ($B = D = 0$) with $N = 3$, the Hamiltonian is given by

$$
H = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

The eigenvalues will be $E_i = 0, 1, 1, 2, 2, 2$ with the corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Since the basis is given by

$$|0,0\rangle \, |0,1\rangle \, |0,2\rangle \, |1,0\rangle \, |1,1\rangle \, |2,0\rangle \,,$$

the quantum numbers for the eigenvectors will be assigned as follows

$$v_1 \equiv |0,0\rangle \text{ since } k = 1$$
$$v_2 \equiv |0,1\rangle \text{ since } k = 2$$
$$v_3 \equiv |1,0\rangle \text{ since } k = 4$$
$$v_4 \equiv |0,2\rangle \text{ since } k = 3$$
$$v_5 \equiv |2,0\rangle \text{ since } k = 6$$
$$v_6 \equiv |1,1\rangle \text{ since } k = 5$$

In figures 4.6 and 4.7, respectively 4.8 and 4.9 we can see the relative spacing for each irreducible representation as a function of index and as a histogram for $B = 0.2$ and $B = 0.63$. Once the problem of selection of the states corresponding to a given irreducible representation was solved, we calculated the average spacing for different values of B (see fig. 4.10).
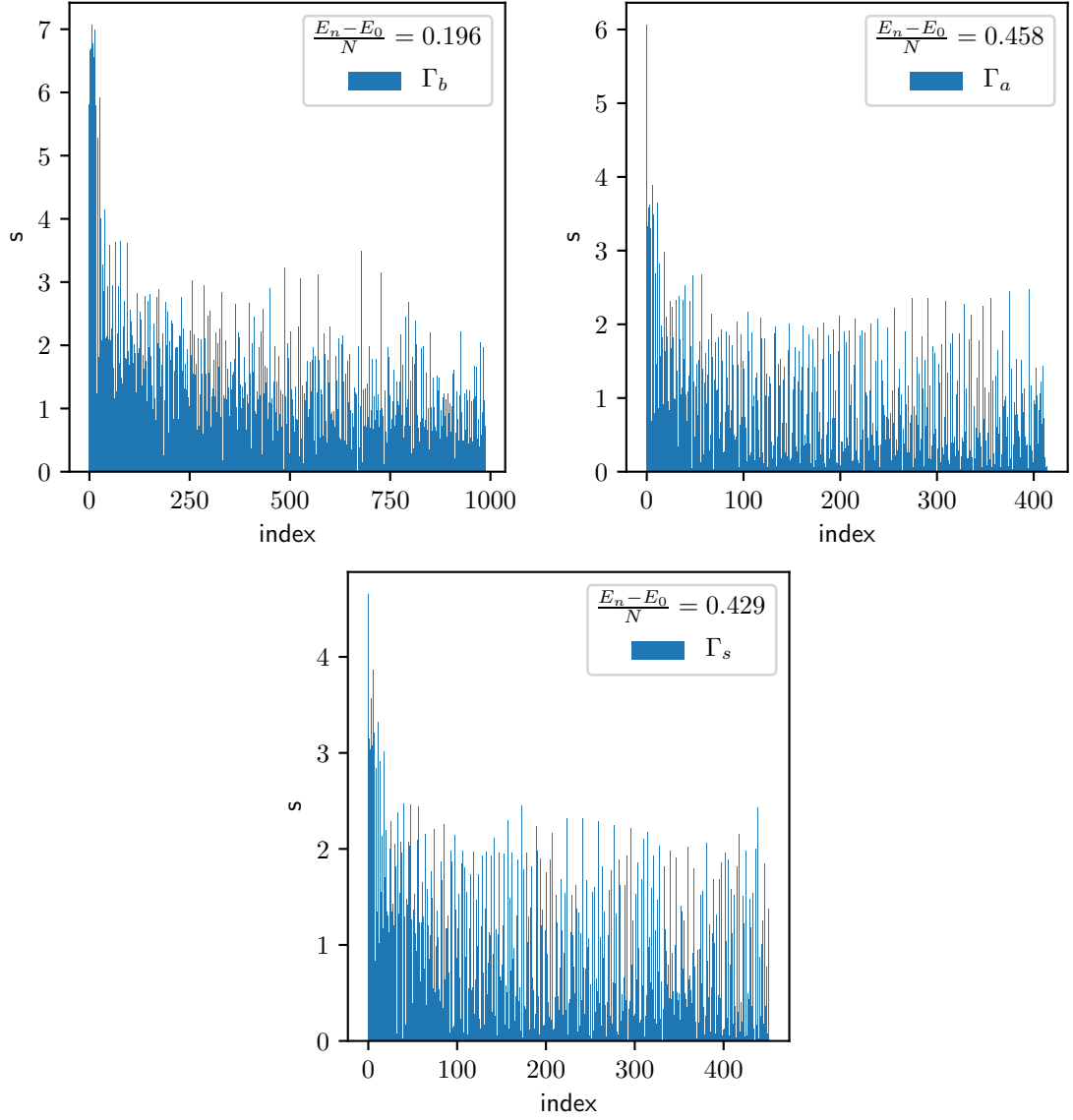
Figure 4.6: The relative spacing for each irreducible representation as a function of index for $B = 0.2, D = 0.4, N = 260$
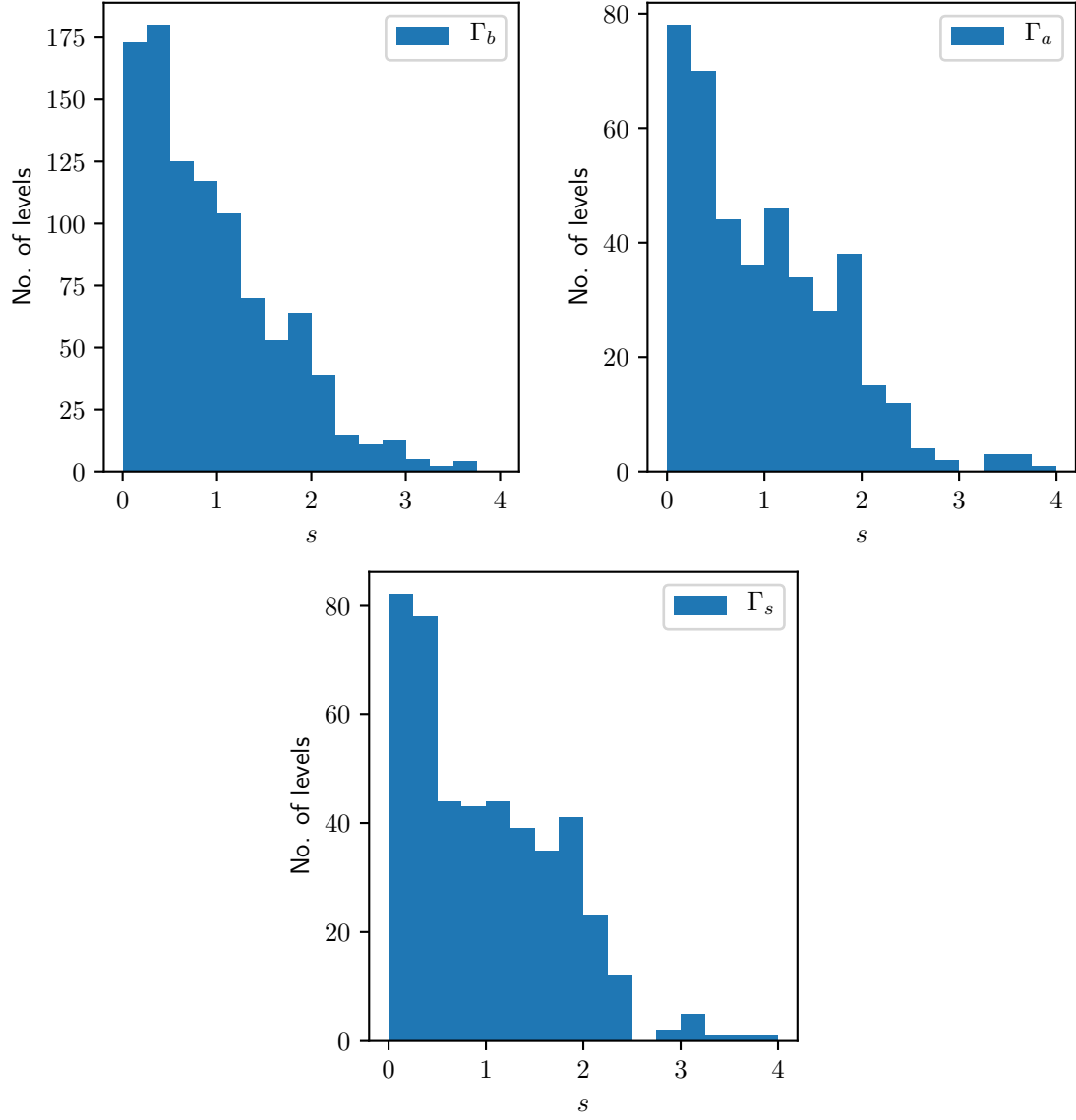
Figure 4.7: The relative spacing histogram for each irreducible representation for $B = 0.2, D = 0.4, N = 260$
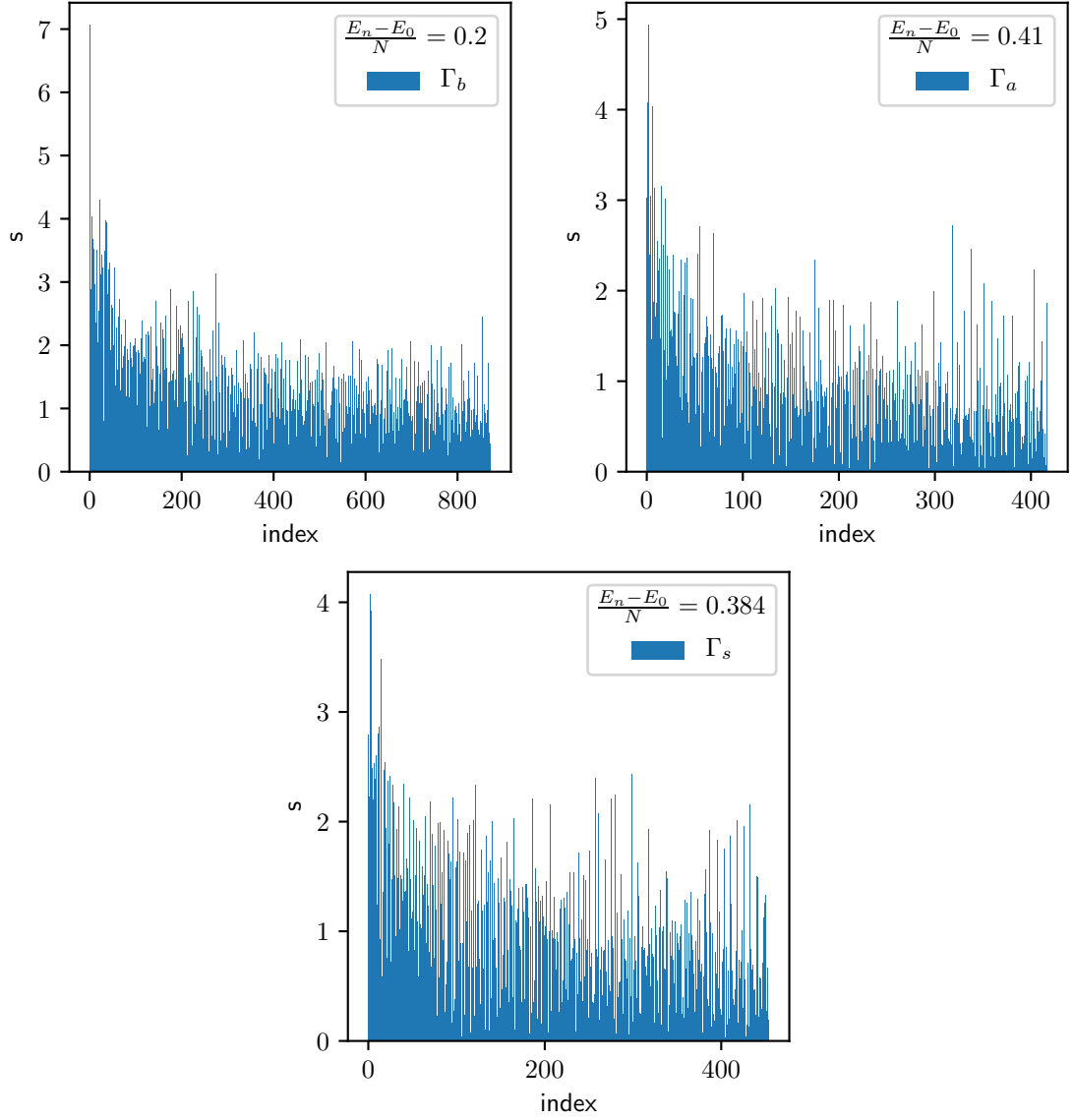
Figure 4.8: The relative spacing for each irreducible representation as a function of index for $B = 0.63, D = 0.4, N = 260$
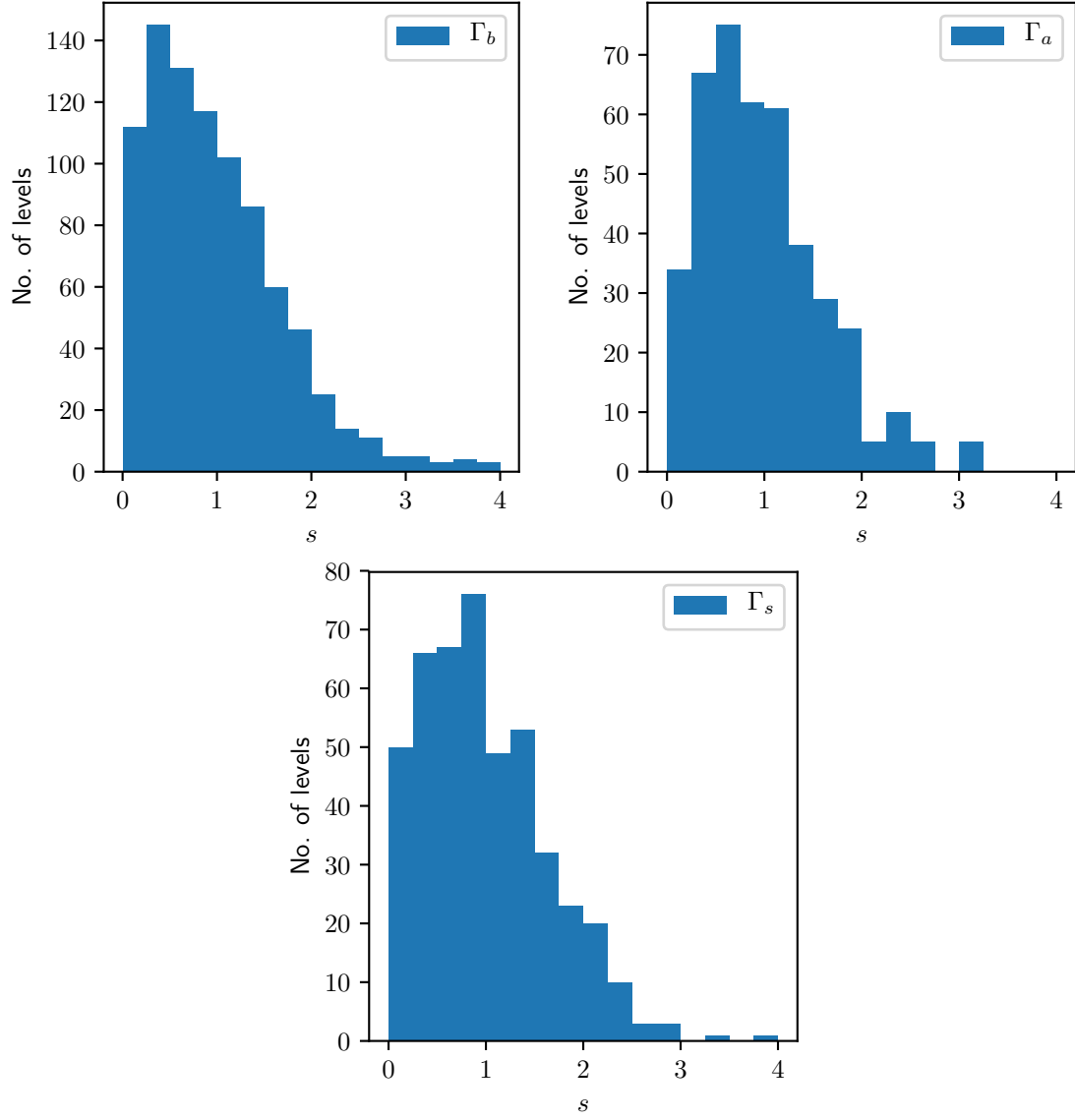
Figure 4.9: The relative spacing histogram for each irreducible representation for $B = 0.63, D = 0.4, N = 260$
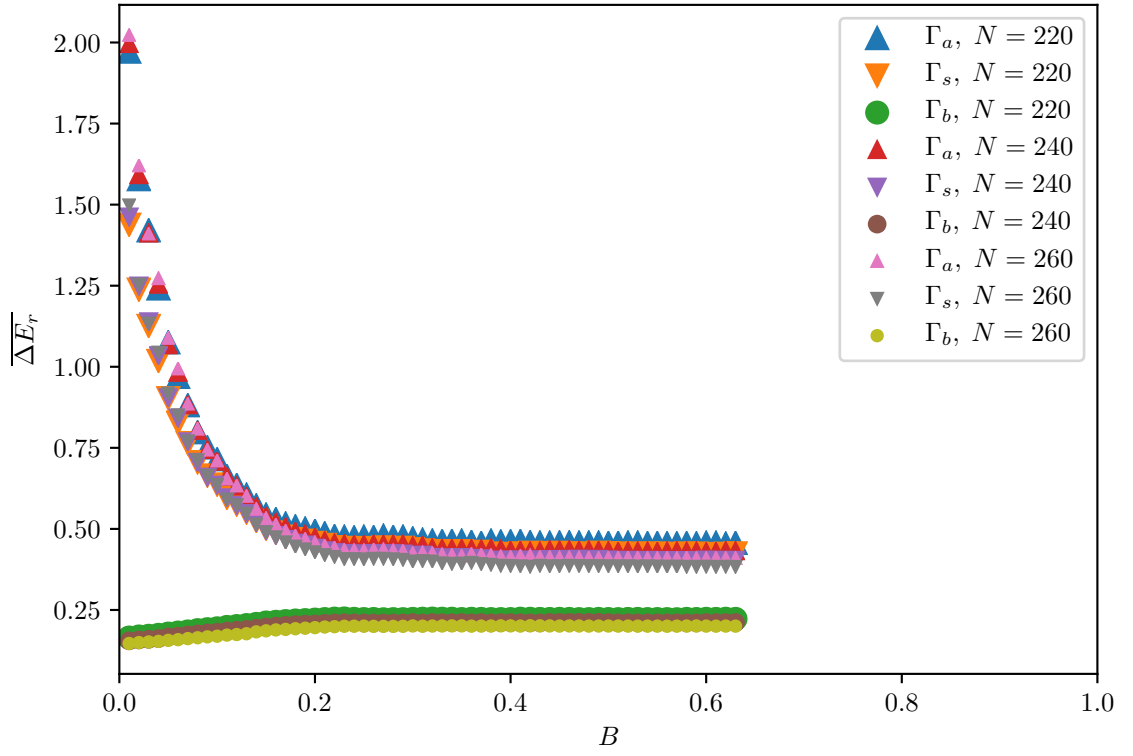
Figure 4.10: $\overline{\Delta E}$ as a function of B for each representation

4.3.2 Nearest neighbour distributions

Having separated the states with respect irreducible representations we can also build the nearest neighbour distributions. These histograms show the probability of having a given spacing between consecutive levels as a function of s.

Figures 4.11 to 4.14 show the nearest spacing distributions $P(s)$ and the cumulative distributions $I(s)$ for $B = 0.2$ and $B = 0.63$.

Taking into account the Berry-Tabor and Bohigas-Gianoni-Schmit conjectures, we compare $P(s)$ with the Poisson distribution $P_P(s) = e^{-s}$ and the Wigner distribution $P_W(s) = \frac{\pi}{2}s \exp\left(-\frac{\pi}{4}s^2\right)$.

Since $P(s)$ is a histogram, for an easier comparisom, we also computed the histograms associated with the two distributions for the same spacing.
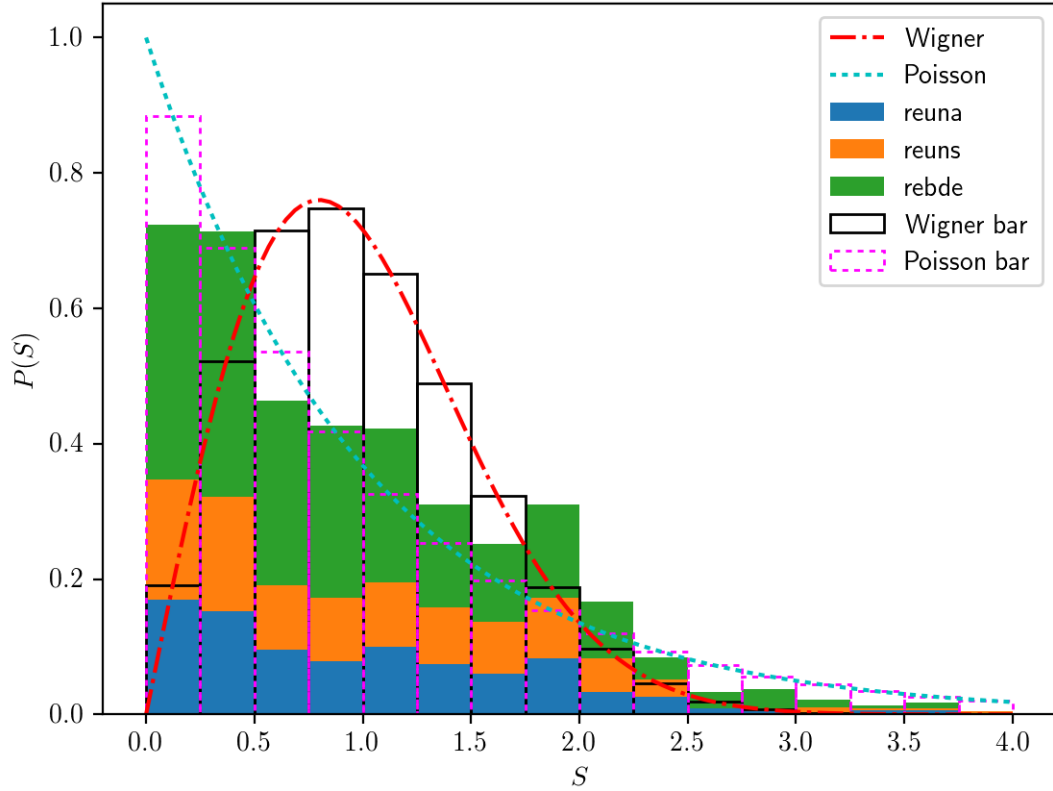
Figure 4.11: $B = 0.2, D = 0.4, N = 120$
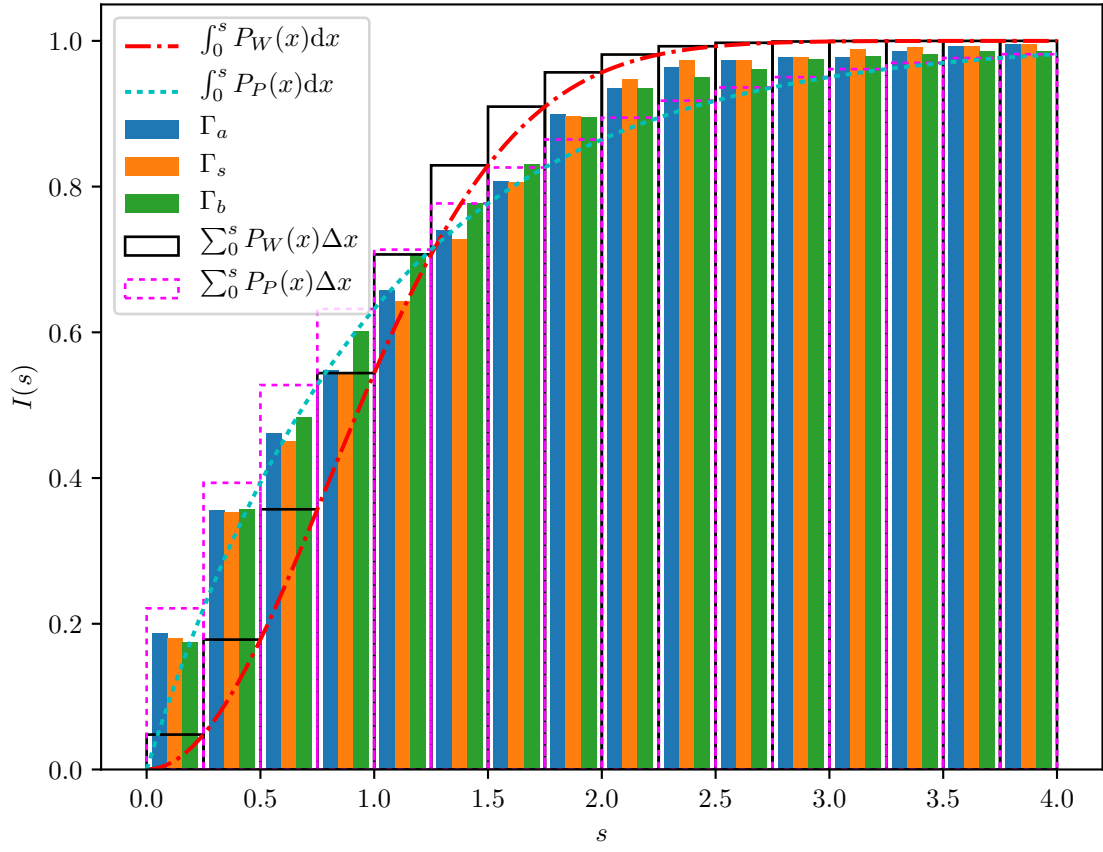
Figure 4.12: $B = 0.2, D = 0.4, N = 120$
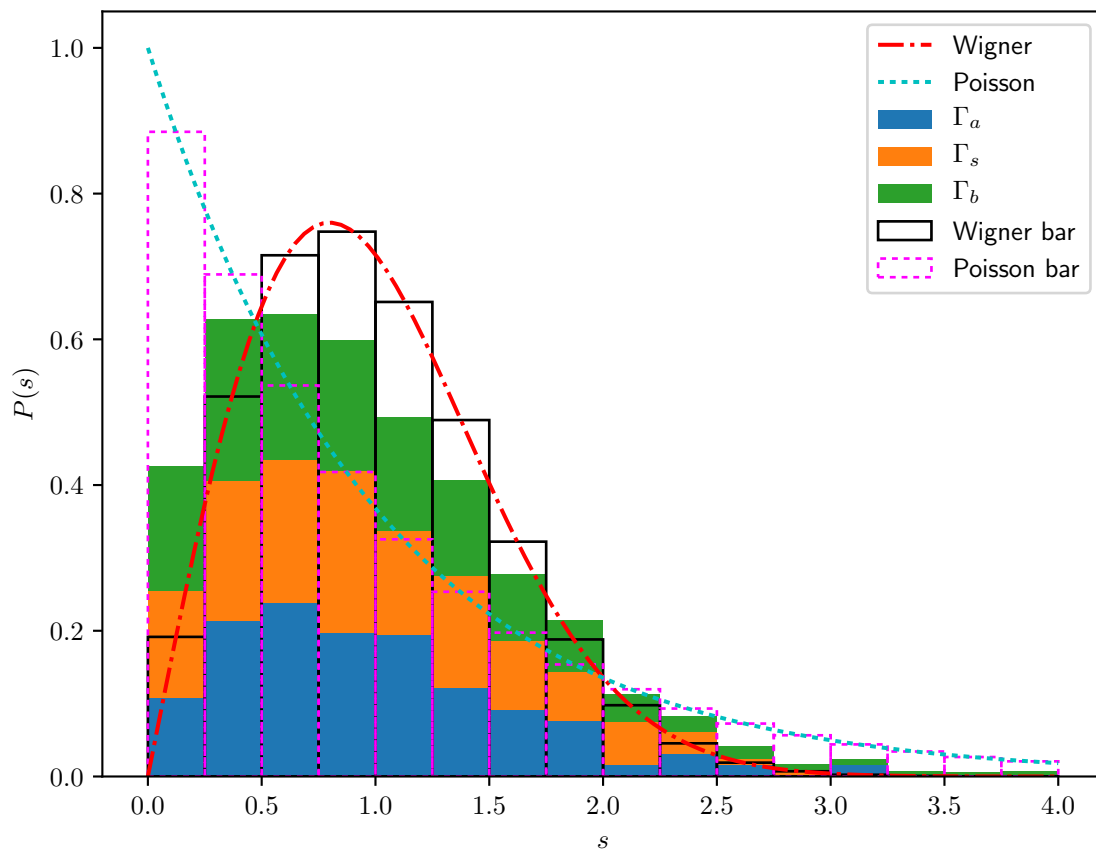
Figure 4.13: $B = 0.63, D = 0.4, N = 260$
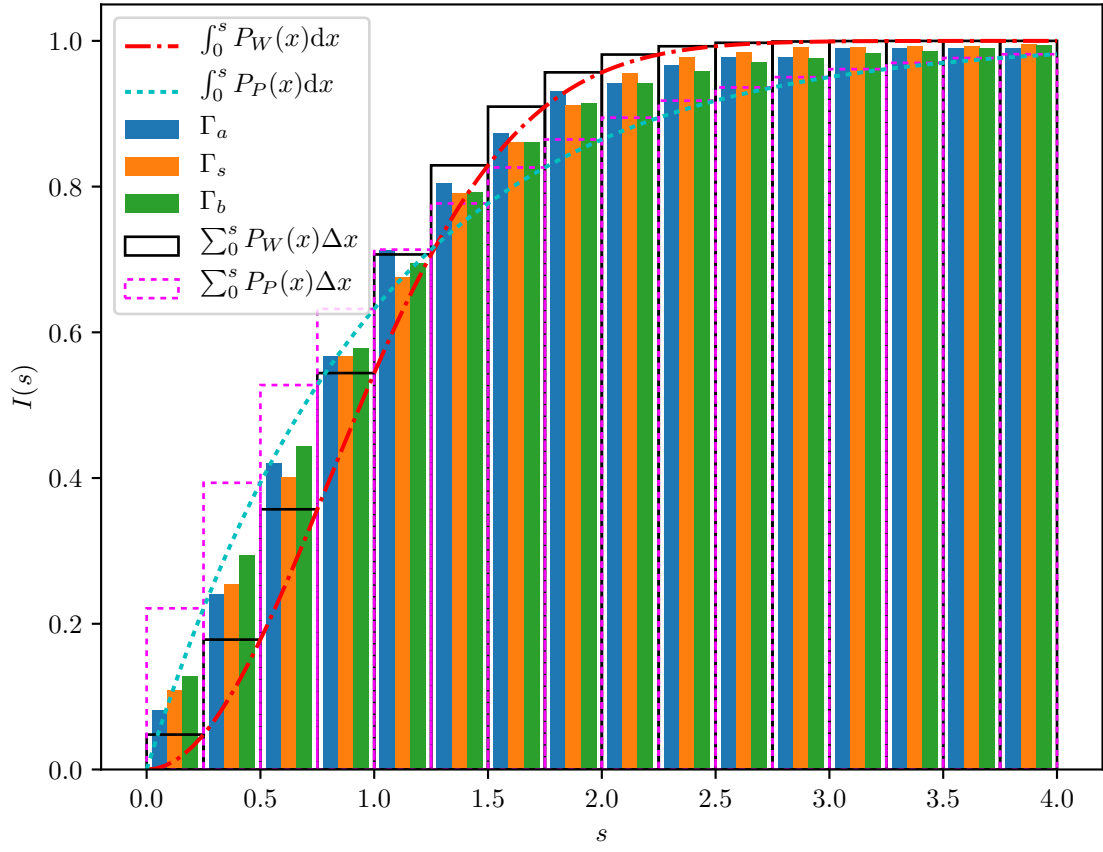
Figure 4.14: $B = 0.63, D = 0.4, N = 260$

Chapter 5

Results

For all values of the control parameter,B, we observed that the nearest neighbour distributions deviate from both the Poisson and Wigner distributions. We propose the following hypothesis: the obtained nearest neighbour distributions can be seen as a linear superposition of the Poisson and Wigner distributions:

$$P(s) = \alpha P_P(s) + (1 - \alpha)P_W(s).$$

This hypothesis is rooted in the fact that the classical counterpart of our system has an interplay between regular and chaotic motion clearly illustrated by the Poincaré sections in figures 5.1 [12]. In this way we consider that for a range of energies of the classical system, the phase space ratio of the volumes of chaotic and regular trajectories is reflected in the the superposition of Wigner and Poisson distributions.

Thus, a system with an integrable classical counterpart will have $\alpha = 1$, corresponding to the Poisson distribution, while a fully chaotic system will have $\alpha = 0$ corresponding to the Wigner distribution. For a system which has balanced contributions to the phase space volume both from the regular and chaotic trajectories, we expect the Poisson and Wigner distributions to equally contribute to $P(s)$.

For the fixed value of the non-integrability parameter of $B = 0.55$ we observe that with the increase of energy, the phase space fills with tori up to a energy of around $E = 600$ (in units of harmonic oscillator energy) and after that the chaotic trajectories begin to fill the phase space. We can say that globally, for an energy range $\Delta E \leq 600$, the phase space is characterised by an increase of regular trajectories as the energy increases. If we now consider the quantum counterpart of the system, we can use α as a measure of the closeness to the Poisson distribution. As we can see in figures 5.2 and 5.3 with the increase of the energy interval α increases from 0.493 at $\Delta E \approx 100$ to 0.876 at $\Delta E \approx 180$.

Thus, at least for a fixed value of B, we can observe a correlation between the tori volume in phase space and α, the superposition coefficient.
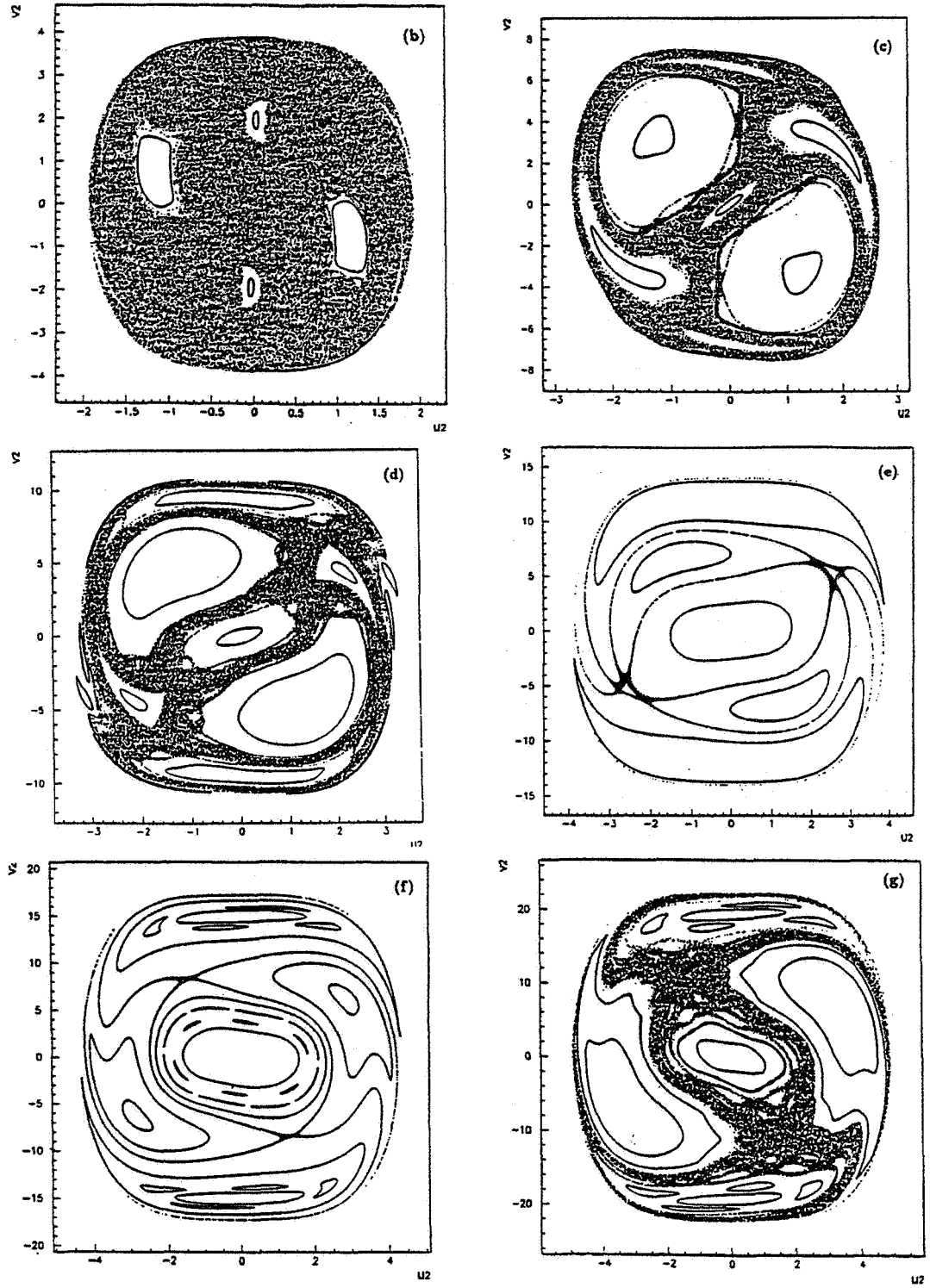
Figure 5.1: Poincaré sections for $B = 0.55$ and $E = 30A(\text{MeV})$ (b), $E = 120A(\text{MeV})$ (c), $E = 240A(\text{MeV})$ (d), $E = 400A(\text{MeV})$ (e), $E = 600A(\text{MeV})$ (f), $E = 1000A(\text{MeV})$ (g).
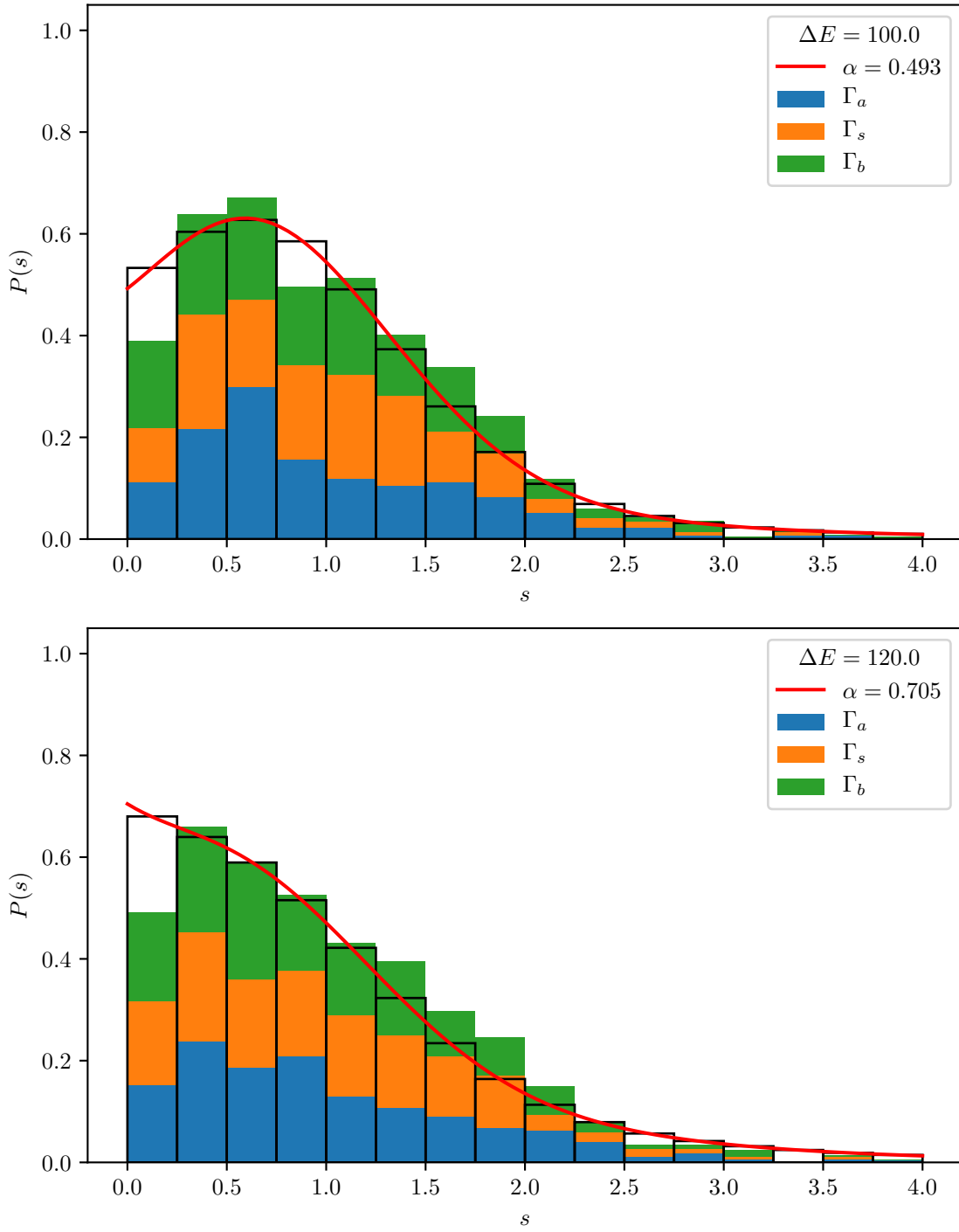
46

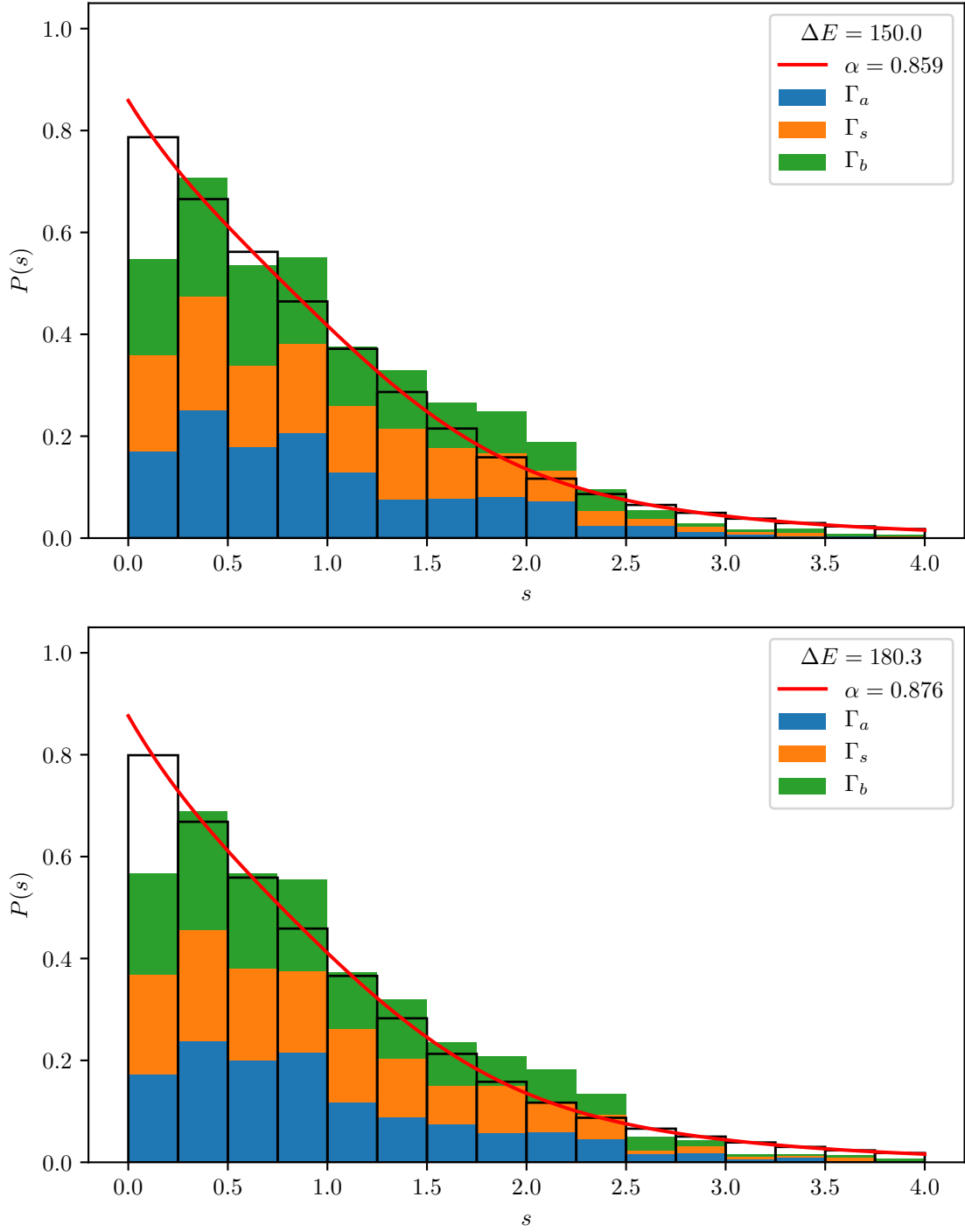Figure 5.2: $P(s)$ for $B = 0.55, D = 0.4, N = 260$ and $\Delta E_{max} = 100, 120$

Figure 5.3: $P(s)$ for $B = 0.55, D = 0.4, N = 260$ and $\Delta E_{max} = 150, 180.3$

Looking at other values for B such as $B = 0.2$ and $B = 0.63$, we observe that $P(s)$ gets closer to a Poisson distribution with the increase of energy (see figs. 5.4 to 5.7). This global feature of the system can be observed on the entire interval from $B = 0.01$ to $B = 0.63$. In order to emphasise this, we plot α as a function of ΔE (see figs. 5.8 to 5.11).

This remarkable behaviour can be viewed as consequence of the interplay of the third and fourth order terms in the Hamiltonian. The third order terms, which consist in the non-integrable part of the Hamiltonian and can be considered as contributing to the apparition of chaotic trajectories.

An other global characteristic of the system can be revealed by studying the dependence of α on the non-integrability parameter B. In figure 5.12 $\alpha(B)$ is plotted for different values of N. The value of N determines both the stability of the values and the maximum energy interval. Indeed, since our method implies the truncation of the Hilbert space, we restrict the number of considered energy levels with the stability criterion.

As with the above case of $\alpha(\Delta E)$, we can reduce the energy interval by considering only the energy levels up to a given value. In figures 5.13 and 5.14 we can see that the shape of $\alpha(B)$ for $N = 260$ remains qualitatively the same when we consider different values for the maximum energy interval. We can observe that as we increase the energy the values of α globally rise, as expected from the previous plots.

An other qualitative test of our data implies checking how the values of α change when we consider different values for N, in this case $N = 220, 240, 260$ and different energy intervals. By considering multiple values for N we can check the stability of the energy levels within a given interval provided that the respective interval is less than the maximum energy interval for all N. In figs. 5.15 to 5.17 we can see that indeed the energy levels are stable as the values of α barely change.
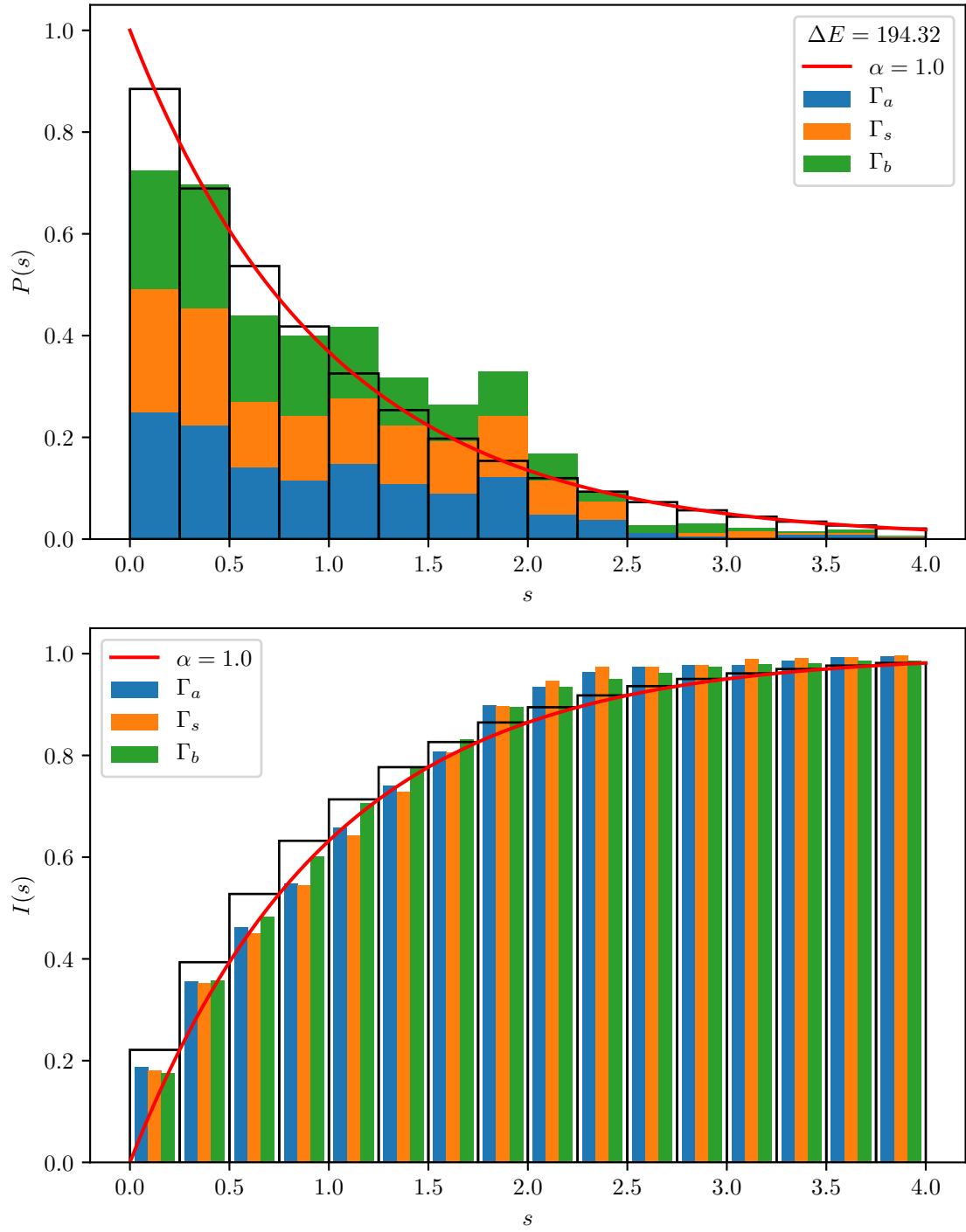
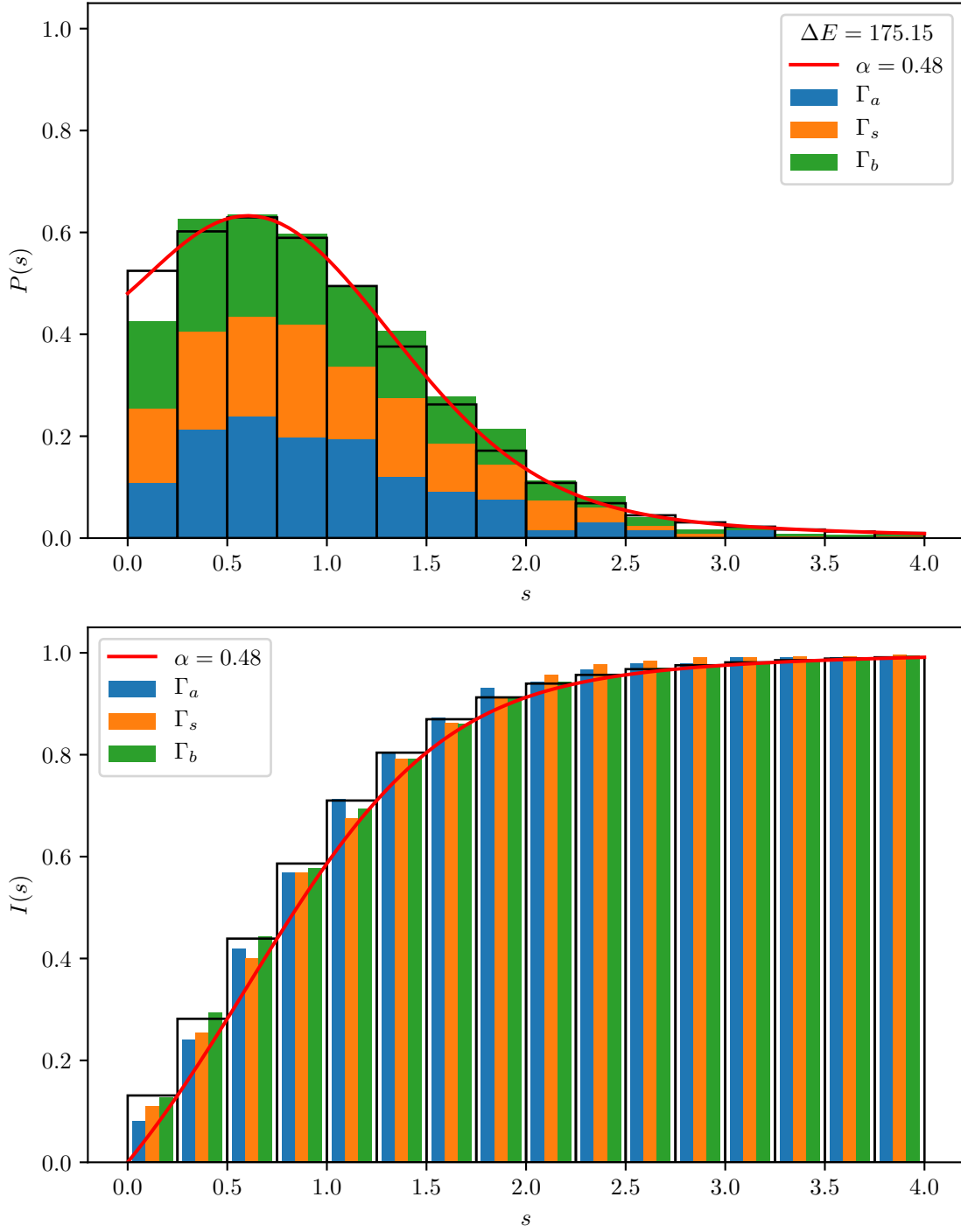Figure 5.4: $P(s), I(s)$ for $B = 0.2, D = 0.4, N = 260$

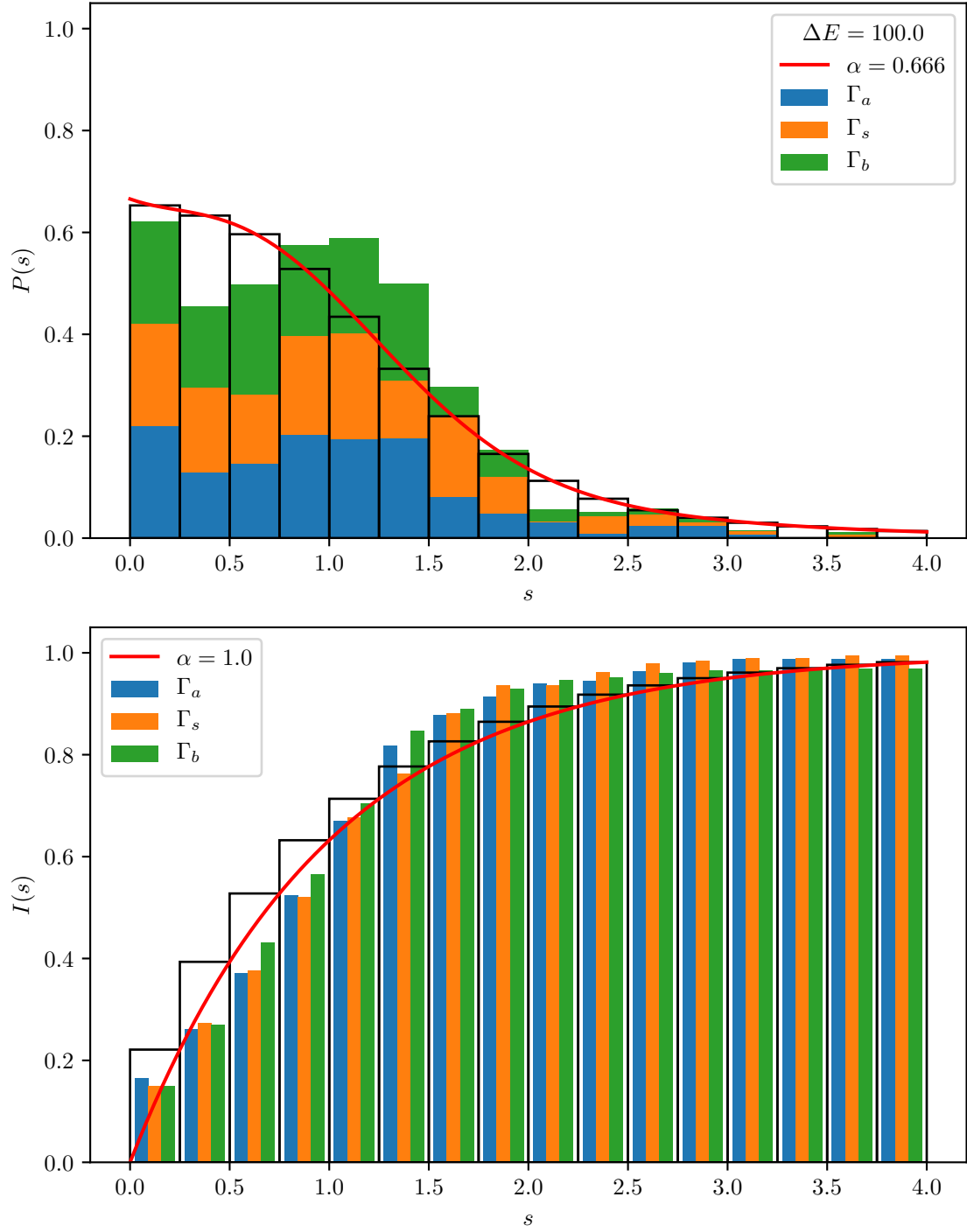Figure 5.5: $P(s), I(s)$ for $B = 0.63, D = 0.4, N = 260$

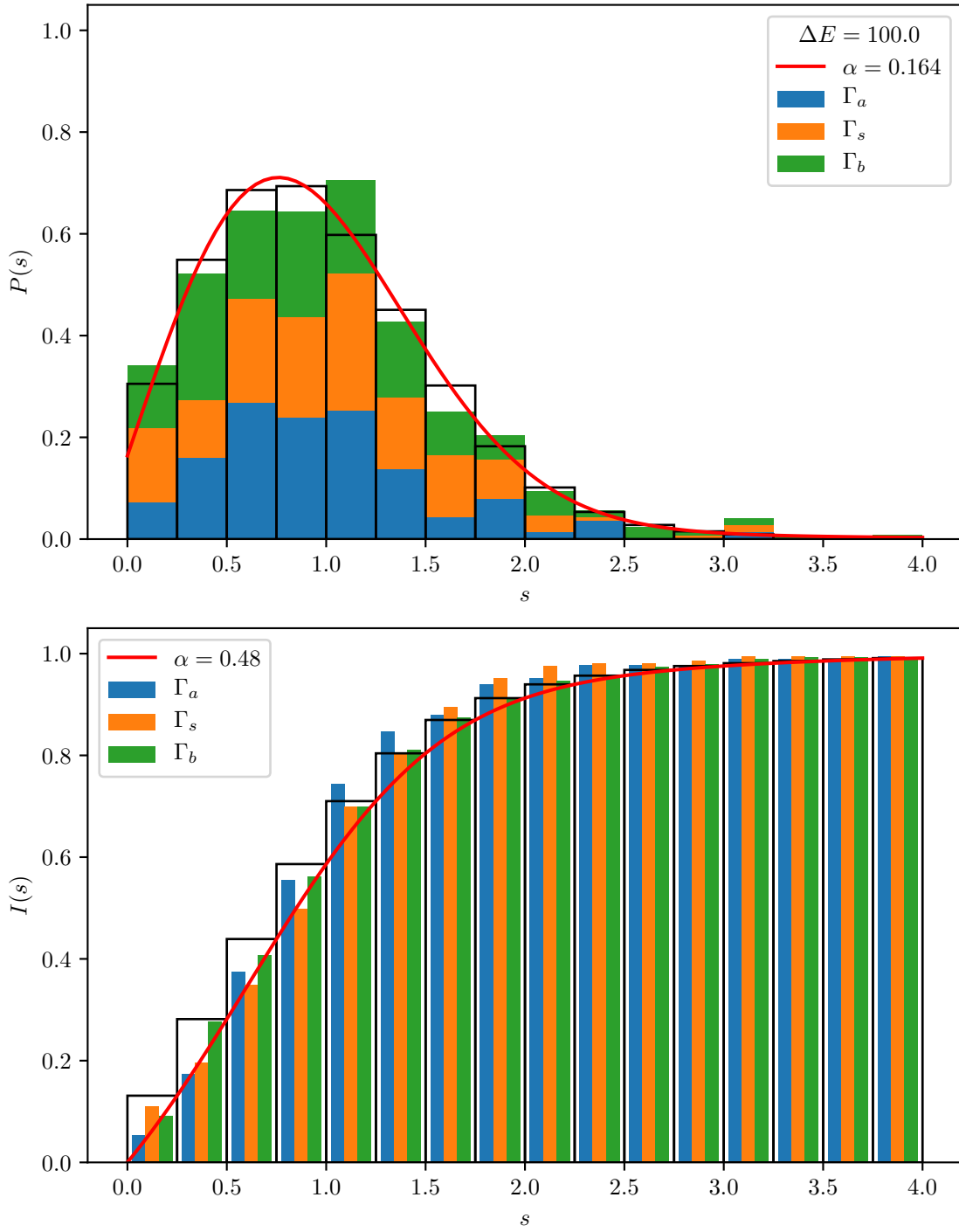Figure 5.6: $B = 0.2, D = 0.4, N = 260, \Delta E_{max} = 100$

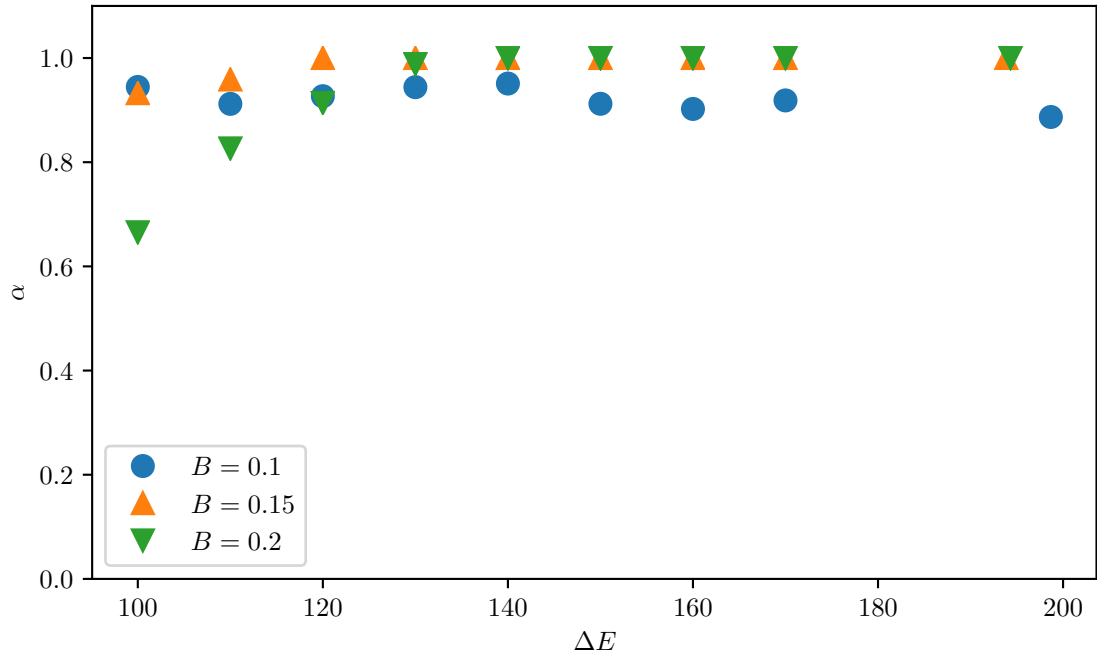Figure 5.7: $B = 0.63, D = 0.4, N = 260, \Delta E_{max} = 100$

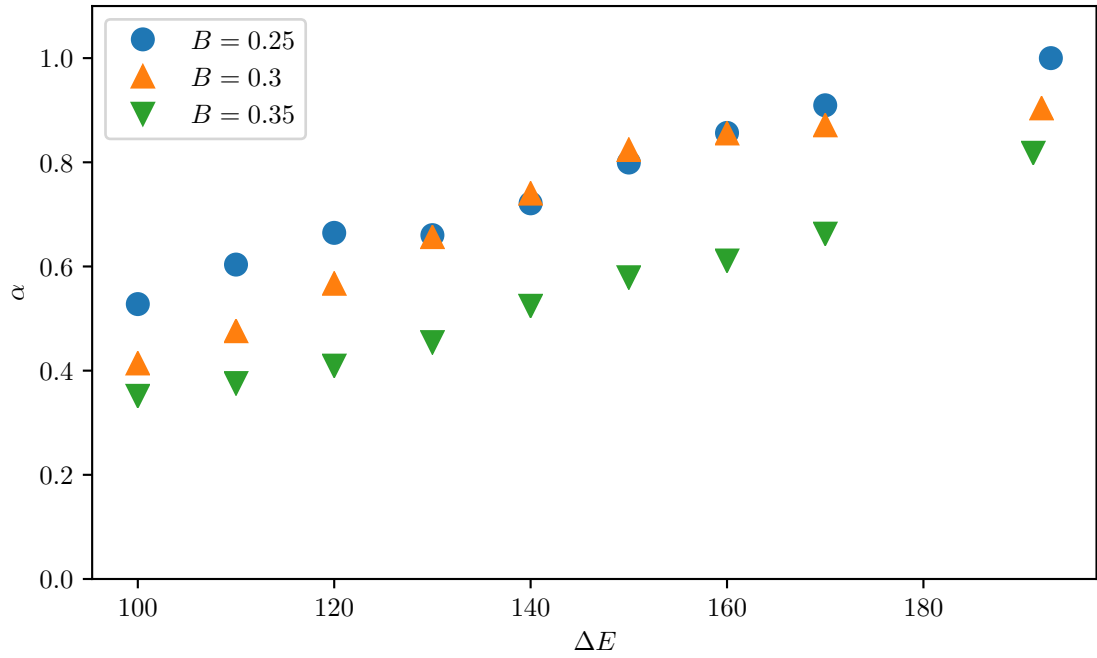Figure 5.8: $B = 0.1, 0.15, 0.2, \ N = 260$
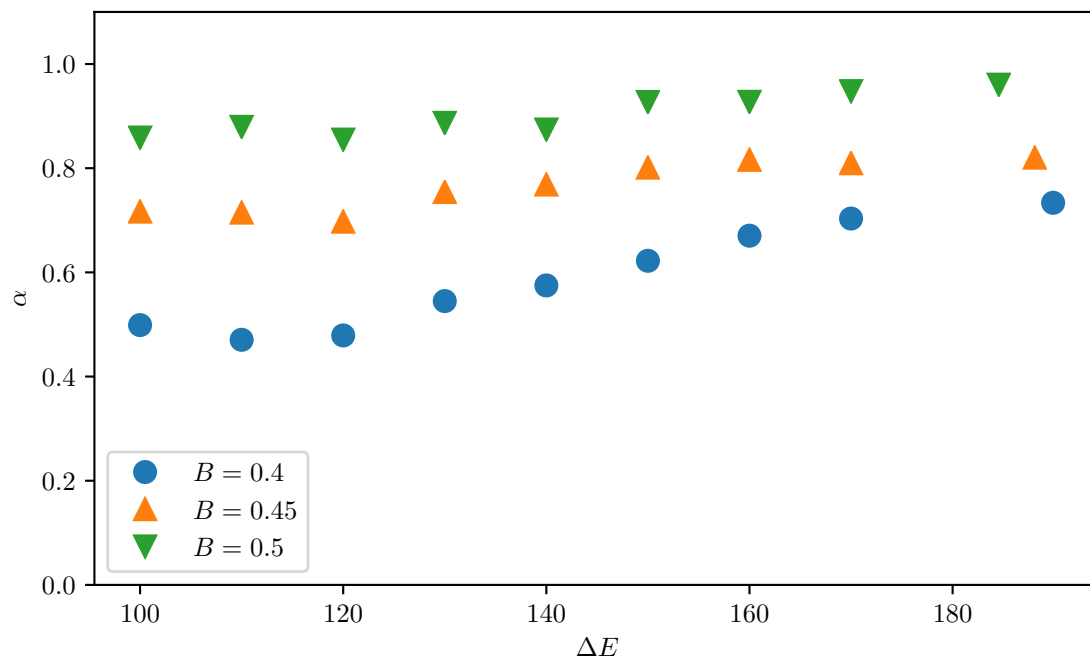

Figure 5.9: $B = 0.25, 0.3, 0.35, \ N = 260$

Figure 5.10: $B = 0.4, 0.45, 0.5, \ N = 260$

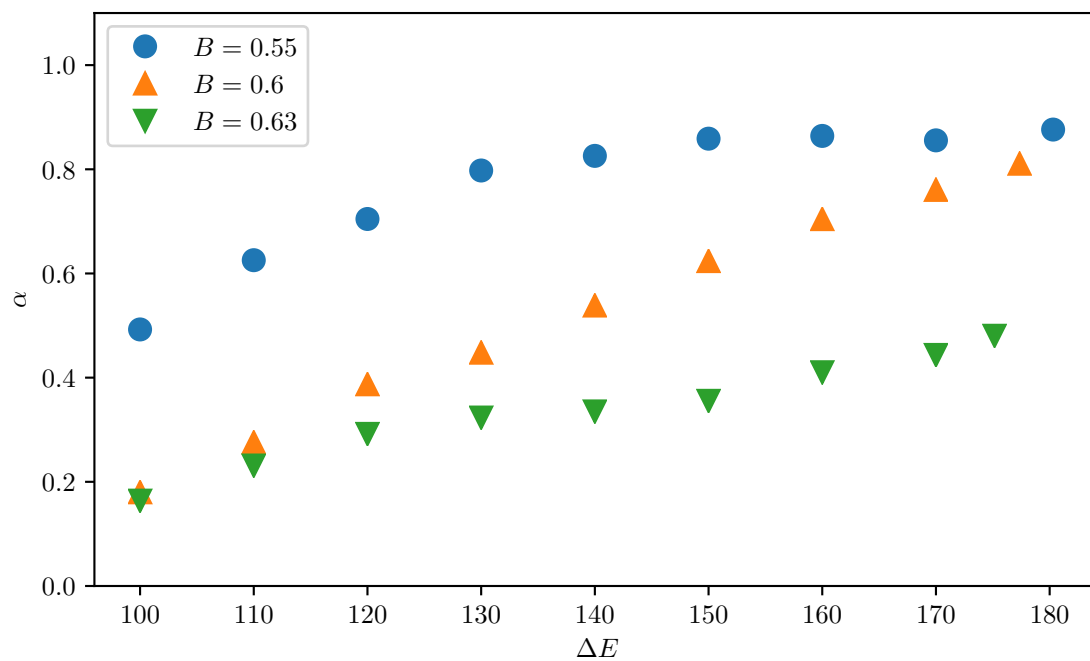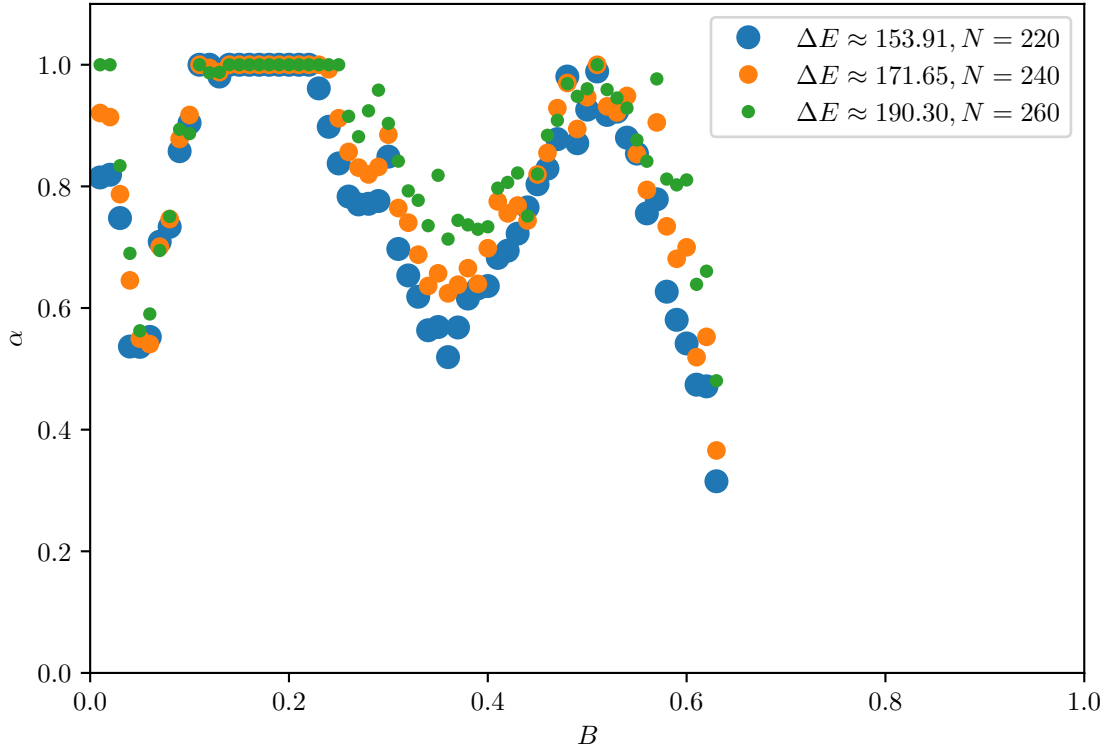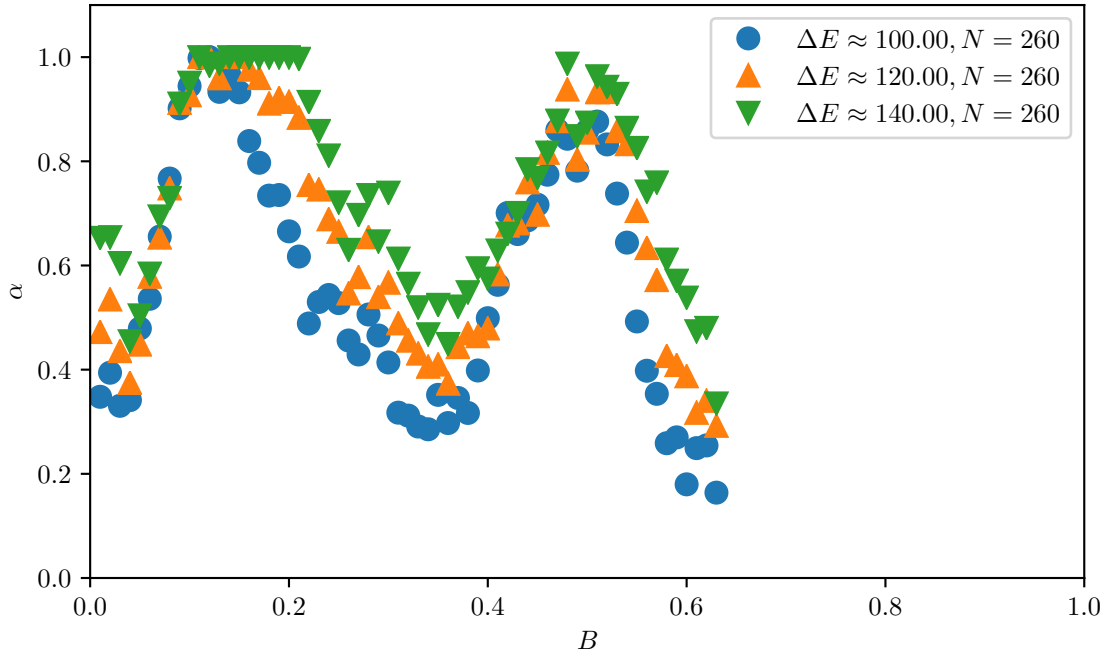Figure 5.11: $B = 0.55, 0.6, 0.63, \ N = 260$

Figure 5.12: $N = 220, 240, 260$

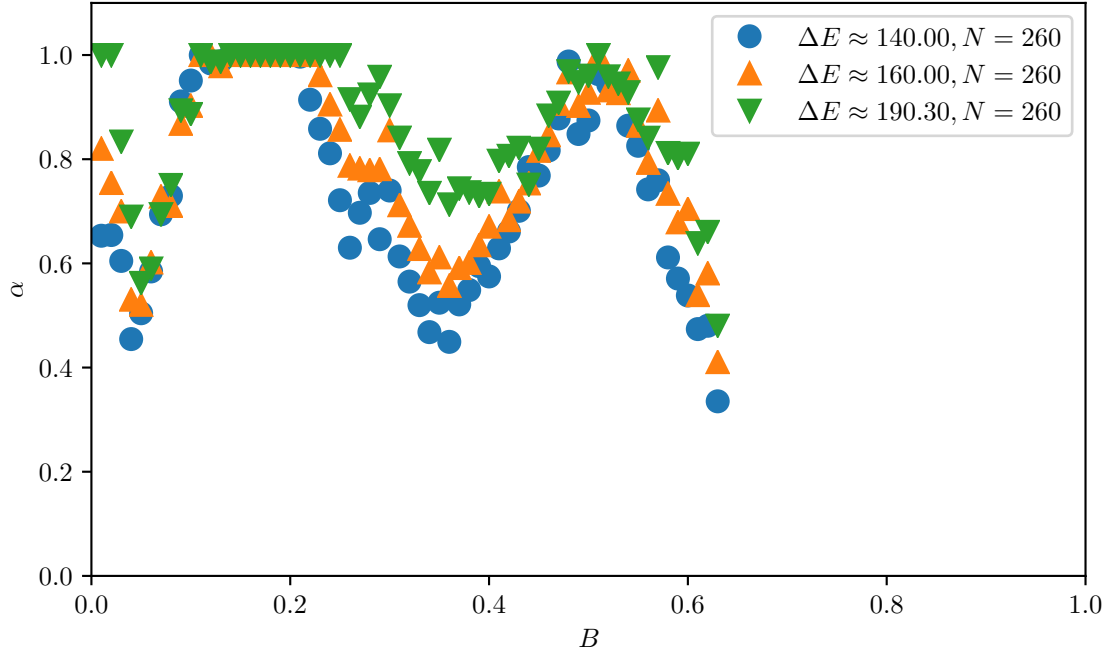Figure 5.13: $N = 260, \ \Delta E_{max} \approx 100, 120, 140$

Figure 5.14: $N = 260,\ \Delta E_{max} \approx 140, 160, 190.305$
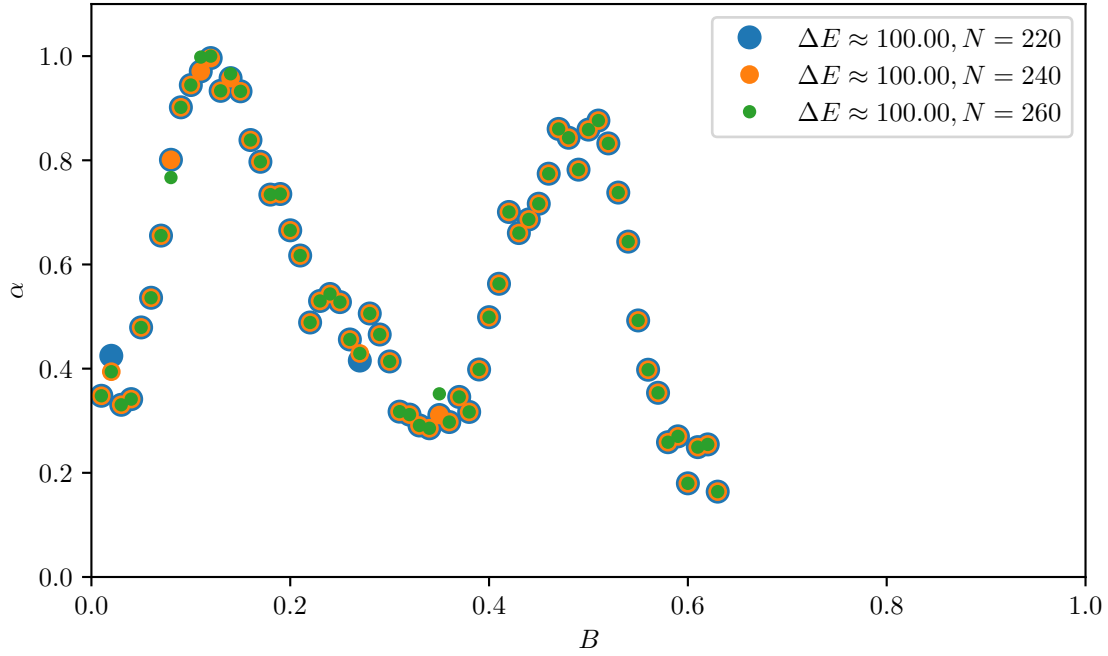

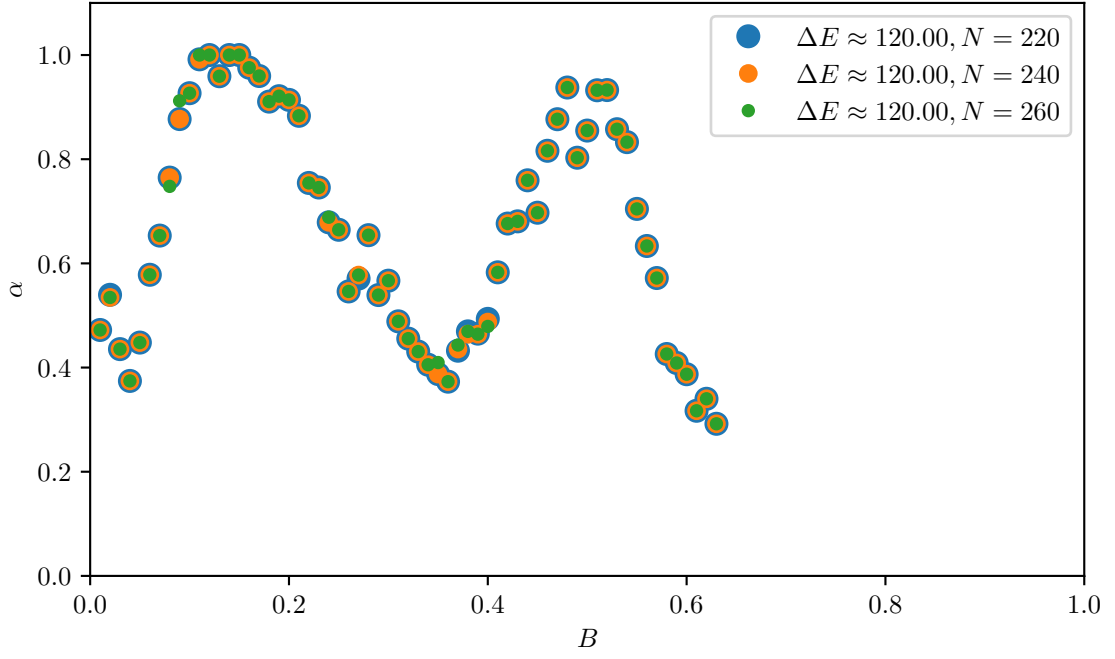Figure 5.15: $N = 220, 240, 260,\ \Delta E_{max} \approx 100$

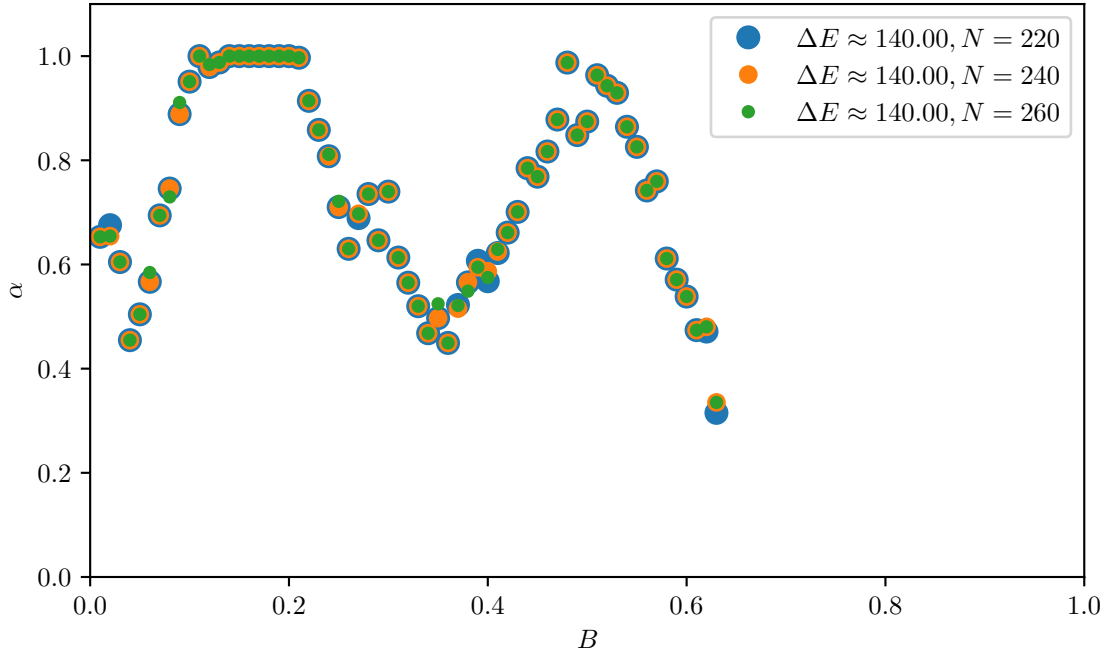Figure 5.16: $N = 220, 240, 260, \ \Delta E_{max} \approx 120$

Figure 5.17: $N = 220, 240, 260, \ \Delta E_{max} \approx 140$

Chapter 6

Conclusions

In this thesis we have studied the connection between the classical phase space structures and the spectra of the corresponding quantum system. The basic question was to explore how the relative partition between tori and chaotic trajectories in the classical phase space is reflected in the statistics of the energy levels of the quantum analogue.

Starting with a Hamiltonian that describes the dynamics of the nuclear surface in terms of quadrupole vibrations, we obtained the energy spectra through numerical diagonalisation. The Hamiltonian depends on parameters, which at classical level induce the transition from an integrable system to a non-integrable one. As a consequence, in agreement with the K.A.M. theorem, the phase space starts to be populated with chaotic trajectories which escape from the tori, filling a 3 dimensional volume. We observed that the relative weights of the tori volumes in the phase space depends in a non-trivial way with the energy and the control parameter for non-integrability. Starting from this observation we investigated systematically the properties of the energy spectra as a function of the same control parameter and in different energy ranges. Specifically, we focused on the nearest neighbour distribution and compared it with the predictions of the Poisson and Wigner distributions.

We observed that for all values of the control parameter, the distribution manifests deviations from the expected Wigner distribution characteristic to chaotic systems. We assumed that this effects are related to the significant influence of the tori at the classical level. Therefore, we proposed a distribution which is a linear superposition of the Poisson and Wigner distributions. Then, the coefficient factorising the Poisson distribution will be a measure of closeness to an integrable behaviour. Indeed, if this coefficient is equal to unity, the distribution becomes Poissonian, while when it goes to zero, it transforms Wigner one.

We noticed that when we rise the energy range for the analysis of the level distribution, this coefficient increases, reflecting the observed behaviour in classical phase space. In other words, the global structure of the classical phase space for an energy interval is reflected by the superposition fitting coefficient. More precisely, we observed that the value of this coefficient increases with the considered energy interval, while in the classical system, as we increase the energy, the volume of the regular trajectories confined on tori increases.

Furthermore, we obtained a non-trivial dependence of the superposition coefficient on the non-integrability parameter. While a greater value of this parameter is expected to make the system more chaotic, and consequently the level distribution gets closer to the Wigner one, from our analysis resulted a more intricate behaviour. The superposition coefficient decreases and increases periodically when the non-integrability parameter changes. This periodic shape shows that for specific values of the control parameter, the system will manifest a more robust integrable-like behaviour. Further work is required to better understand this aspect.

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