Chapter 1

Classical chaos

1.1 Fundamental notions

Classical mechanics can be reformulated in a more elegant way which does not require the computation of individual forces and instead uses a single function to describe the entire system, namely the *Lagrangian*. The Lagrangian, denoted with $\mathcal{L}$, is a function of the generalised coordinates $q_1, \ldots, q_n$, their derivatives with respect to time $\dot{q_1}, \ldots, \dot{q_n}$ and optionally the time t, where n is the number of degrees of freedom.

The principles of Newtonian mechanics are replaced by the a variational principle $\delta S = 0$, where

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \ldots, q_n, \dot{q_1}, \ldots, \dot{q_n}, t)\, \mathrm{d}t \tag{1.1}$$

is called the *action*. This principle, also called *Hamilton's principle* or *the principle of least action* is equivalent with the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q_k}} = 0,$$

where $k = 1, \ldots, n$. Thus we can obtain the equations of motion provided that we know the initial conditions $q_1(t_0), \ldots, q_n(t_0)$ and $\dot{q_1}(t_0), \ldots, \dot{q_n}(t_0)$ at $t = t_0$ by solving the set of n second order differential equations.

An other formalism for classical mechanics which is very useful in theoretical physics is the Hamiltonian formalism. In this formalism instead of the Lagrangian we use the *Hamiltonian*, $\mathcal{H}$, which is a function of the generalised coordinates $q_1, \ldots, q_n$, the generalised momenta $p_1, \ldots, p_n$ and optionally the time t. The generalised momenta are given by

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q_k}}.$$

The Hamiltonian can be obtained from the Lagrangian through a *Legendre transformation* of $\mathcal{L}$

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n, t) = \sum_{k=0}^{n} p_k \dot{q}_k - \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t). \tag{1.2}$$

The Euler-Lagrange equations are replaced by Hamilton's equations

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k}, \qquad\qquad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k}$$

We now have $2n$ first order ordinary differential equations. comments?

1.2 Canonical transformations

In the Hamiltonian formulation of classical mechanics *canonical transformations* represent all the transformations of the canonically conjugate variables from q and p to Q and P such that the structure of the Hamilton equations remains intact.

The Lagrange equations of a system are invariant to a transformation of the Lagrange function such that:

$$\mathcal{L}'(Q, \dot{Q}, t) = \mathcal{L}(q, \dot{q}, t) - \frac{\mathrm{d}}{\mathrm{d}t} F(q, Q, t)$$

Integrating we obtain

$$\int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t - (F(q, Q, t_2) - F(q, Q, t_1))$$

The variation of the above equation is given by

$$\delta \int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \delta \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t - \delta F(q, Q, t_2) + \delta F(q, Q, t_1)$$

Fixing the variations at the end points to be 0, we get

$$\delta \int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \delta \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t$$

So indeed the two Lagrangian descriptions of the system are identical since if the Hamilton principle holds in the first description it also holds in the second.

A function $F(q, Q, t)$ satisfying the above property and the condition $\frac{\partial^2 F}{\partial q \partial Q} \neq 0$ is called a *generating function*. We can classify the generating functions in four

possible types as a function of their variables. The conjugate momenta corresponding to q and Q are given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \text{ and } P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}}$$

In order to express those as a function of $F(q, Q, t)$ we use the fact that $\dot{q}$ is a cyclic coordinate for $\mathcal{L}'$

$$\frac{\partial \mathcal{L}'}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial}{\partial \dot{q}} \left(\frac{\mathrm{d}}{\mathrm{d}t} F(q, Q, t) \right) = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0,$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q, Q, t) = \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial Q} \dot{Q} + \frac{\partial F}{\partial t}.$$

Thus we obtain

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

and

$$P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}} = -\frac{\partial F}{\partial Q}$$

To find the Hamiltonian corresponding to the new coordinates Q and P we compute the Legendre transformation of $\mathcal{L}'$

$$\mathcal{H}'(Q, P, t) = P\dot{Q} - \mathcal{L}'(Q, \dot{Q}, t)$$

Thus

$$\mathcal{H}'(Q, P, t) = P\dot{Q} - \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t} = P\dot{Q} - \mathcal{L} + \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial Q} \dot{Q} + \frac{\partial F}{\partial t} = p\dot{q} - \mathcal{L} + \frac{\partial F}{\partial t}$$

and so

$$\mathcal{H}'(Q, P, t) = \mathcal{H}(q(Q, P), p(Q, P), t) + \frac{\partial F(q(Q, P), Q, t)}{\partial t}. \tag{1.3}$$

As mentioned above, there are four types of generating functions. They can be obtained by successive Legendre transformations which switch between the conjugate coordinates and they represent the same canonical transformation. We denote the above mentioned $F(q, Q, t) \equiv F_1(q, Q, t)$. The other types of generating functions will be denoted $F_2(q, P, t), F_3(p, Q, t), F_4(p, P, t)$. Thus we define

$$\begin{aligned}
&F_2(q, P, t) = F_1(q, Q, t) + PQ && p = \frac{\partial F_2}{\partial q}, && Q = \frac{\partial F_2}{\partial P} \\
&F_3(p, Q, t) = F_1(q, Q, t) - pq && q = -\frac{\partial F_3}{\partial p}, && P = \frac{\partial F_3}{\partial Q} \\
&F_4(p, P, t) = F_1(q, Q, t) - pq + PQ && q = -\frac{\partial F_4}{\partial p}, && Q = \frac{\partial F_4}{\partial P}
\end{aligned}$$

The *Poisson bracket* of two arbitrary functions with respect to the canonical coordinates q and p is defined as

$$[F, G]_{q,p} = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right).$$

We can use the Poisson bracket to express Hamilton's equations more elegantly

$$\dot{q}_k = [q_k, H], \qquad \dot{p}_k = [p_k, H].$$

Expressed in this form, the Hamilton equations have a close resemblance to Heisenberg's form for the equations of motion in the Heisenberg picture of quantum mechanics. The Poisson bracket structure of classical mechanics is one of the fundamental links to quantum mechanics. (cite Dirac).

As with the above discussion we will limit to the case of one degree of freedom, but the results are easily generalisable to n degrees of freedom.

If Q and P are a set of conjugate variables obtained by a canonical transformation from q and p, then

$$[F', G']_{Q,P} = [F, G]_{q,p},$$

where F' and G' are the transformed functions. As a consequence, if we choose $F = Q$ and $G = P$

$$[Q, P]_{Q,P} = [Q(q,p), P(q,p)]_{q,p} = 1$$

This relation represents a necessary and sufficient condition for the transformation from q, p to Q, P to be canonical. Preserving the structure of Hamilton's equations is equivalent with the invariance of the Poisson bracket and thus we can use this invariance as the definition for canonical transformations.

An other equivalent condition would require the Jacobian matrix $\frac{\partial(Q,P)}{\partial(q,p)}$ to be symplectic.

1.3 The Hamilton-Jacobi equation

Instead of finding the equations of motion through solving a set of $2n$ ordinary differential equations with $2n$ dependent variables ($q_1, \ldots, q_n$ and $p_1, \ldots, p_n$) and one independent variable (t), we can use a single first order partial differential equation having $n+1$ independent variables ($q_1, \ldots, q_n$ and t) and one dependent variable (S).

Suppose there exists a generating function S of type F_2 why? such that

$$\mathcal{H}' = 0.$$

From Hamilton's equations, the new coordinates Q_k and P_k, with $k = 1, \ldots, n$ will be constants, so the information about the time evolution of the system is contained in the canonical transformation itself.Thus, equation (1.3) becomes

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t) + \frac{\partial S}{\partial t} = 0 \tag{1.4}$$

which represents the *Hamilton-Jacobi equation.*

The solution $S(q_1, \ldots, q_n, P_1, \ldots, P_n, t)$ of equation (1.4) is called *Hamilton's Principal Function.* From the theory of differential equations we know that there will be $n + 1$ arbitrary constants. We can find n of these constants as $P_1, \ldots, P_n$ and the remaining constant can be incorporated in S since if S is a solution for the Hamilton-Jacobi equation, $S + const.$ is also a solution.

It is a standard convention (Goldstein) to rename the constants as follows

$$\begin{aligned} P_k &\equiv \alpha_k \\ Q_k \equiv \beta_k &= \frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial \alpha_k}. \end{aligned}$$

These constants depend on the initial conditions $q_k(0)$ and $p_k(0)$. From

$$p_k(0) = \left.\frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial q_k}\right|_{t=0}$$

we can obtain α_ks, and β_ks are given by

$$\beta_k = \left.\frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial \alpha_k}\right|_{t=0}$$

We can obtain the initial generalised coordinates $q_k(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, t)$ by solving

$$\beta_k = \frac{\partial S}{\partial \alpha_k}$$

If we take the total time derivative of S

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{k=0}^{n} \frac{\partial S}{\partial q_k}\dot{q_k} + \frac{\partial S}{\partial t}$$

and use eq. (1.3)

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{k=0}^{n} p_k\dot{q_k} - \mathcal{H}$$

we recognise the Legendre transform from eq. (1.2). Thus

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mathcal{L}$$

or equivalently

$$S = \int \mathcal{L}\, \mathrm{d}t\,.$$

It follows that the generating function for this special canonical transformation is the action as defined in eq. (1.1).

If the time does not appear explicitly in the Hamiltonian, the Hamiltonian itself will be a constant of the motion and will represent the energy of the system E.

We can rewrite eq. (1.4) as follows

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t) = -\frac{\partial S}{\partial t} = E$$

Thus we can suppose that for this case $S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)$ has the following form

$$S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) \equiv W(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n) - Et$$

Using this expression in eq. (1.4), we obtain

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}) = E \tag{1.5}$$

which represents the second form of the Hamilton-Jacobi equation.

The solution to this equation is called *Hamilton's Characteristic Function*. W can be expressed as

$$W = S + Et = \int (\mathcal{L} + \mathcal{H})\, \mathrm{d}t = \int \sum_{k=0}^{n} p_k \dot{q}_k\, \mathrm{d}t = \sum_{k=0}^{n} \int p_k\, \mathrm{d}q_k$$

physical interpretation?

W is a generating function of type F_2. The canonical transformation given by W is different from the one given by S, since $\mathcal{H}' = \mathcal{H} + \frac{\partial W}{\partial t} = E \neq 0$.

The new generalised coordinates Q_k are cyclic in the new Hamiltonian $\mathcal{H}' = E = \mathcal{H}(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n})$, thus the new generalised momenta care constants. While in the case of S all the generalised momenta and coordinates were constants, in the case of W all the generalised momenta are constants $\dot{P}_k = -\frac{\partial \mathcal{H}'}{\partial Q_k} = 0$, but not all the generalised coordinates are constants. If we choose to define $P_1 \equiv E$,

$$\dot{Q}_1 = \frac{\partial \mathcal{H}'}{\partial P_1} = \frac{\partial E}{\partial P_1} = 1$$

Thus

$$Q_1 = t + const. = \frac{\partial W}{\partial P_1} = \frac{\partial W}{\partial E}.$$

So Q_1 is no longer a constant. The constant in the above equation determines the origin of the time axis. Q_1 represents the time and its conjugate coordinate is P_1, the energy.

The $n-1$ generalised momenta $P_2, \ldots, P_n$ are independent of P_1 and thus $\frac{\partial \mathcal{H}'}{\partial P_k} = 0$, where $k = 2, \ldots, n$. So the corresponding $n-1$ generalised coordinates are constants.

We can change the notation for α and β to use an index from 1 to $n-1$: $\alpha_j \equiv P_{j+1},\ \beta_j \equiv Q_{j+1}$, where $j = 1, \ldots, n-1$. We can now express Hamilton's Characteristic Function as

$$W = W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1}).$$

From the initial generalised momenta $p_k(0)$

$$p_k(0) = \left. \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial q_k} \right|_{t=0}$$

we can obtain E and α_j. Using the initial generalised coordinates $q_k(0)$ we obtain β_j as follows

$$\beta_j = \left. \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial \alpha_j} \right|_{t=0}$$

Thus, having E, α_j and β_j, we can obtain the time evolution for the generalised coordinates from

$$\beta_j = \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial \alpha_j}$$

For systems with $n > 1$ degrees of freedom, the Hamilton-Jacobi equation can be separated in n equations if we can write S as

$$S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) = \sum_{k=1}^{n} W_k(q_k, \alpha_1, \ldots, \alpha_n) - Et$$

A simpler form of separability exists if the Hamiltonian itself is separable

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{k=1}^{n} \mathcal{H}_k(q_k, p_k)$$

In this case the second form of the Hamilton-Jacobi equation splits in n independent equations

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n) = E = \sum_{k=1}^{n} \mathcal{H}_k(q_k, p_k),$$

so

$$\mathcal{H}_k(q_k, \frac{\partial W_k}{\partial q_k}) = \alpha_k, \qquad E = \sum_{k=1}^{n} \alpha_k$$

why $E = \sum_{k=1}^{n} \alpha_k$?

The Hamilton-Jacobi equation can be solved only if the variables can be separated. The separability depends both on the problem and the chosen coordinate system. This does not decrease the theoretical value of the Hamilton-Jacobi equation which provides powerful methods for finding the constants of the motion.

1.4 Action-angle variables

Going back again to a system with one degree of freedom for simplicity, we study a system which has a periodic motion. We search for a canonical transformation from q, p to a new set of canonically conjugate coordinates θ, I such that the new Hamiltonian does not depend on θ.

Since θ is an ignorable coordinate

$$\dot{I} = -\frac{\partial \mathcal{H}'}{\partial \theta} = 0,$$

I is a constant of the motion and

$$\dot{\theta} = \frac{\partial \mathcal{H}'}{\partial I} = const.$$

Thus $\theta = \frac{\partial \mathcal{H}'}{\partial I}(t - t_0) \equiv \omega(I)(t - t_0)$. θ is called the *angle variable* and I the *action variable* (not to be confused with the action).

In order to obtain this variables we will use a generating function of type F_1 denoted $W'(q, \theta)$. Since the motion is periodic in the initial coordinates, it must also be periodic in the angle variable. Thus W' is periodic in θ.

From the properties of the F_1 type generating functions

$$\mathrm{d}W' = \frac{\partial W'}{\partial q}\,\mathrm{d}q + \frac{\partial W'}{\partial \theta}\,\mathrm{d}\theta = p\,\mathrm{d}q - I\,\mathrm{d}\theta\,.$$

(we can observe that W' is related to Hamilton's Characteristic Function by a Legendre transformation)

Since the motion is periodic, if we integrate over a period q returns to the same value, while θ advances by a constant amount, which we can choose to be 2π per period.

$$\oint \mathrm{d}W' = 0 = \oint p\,\mathrm{d}q - \oint I\,\mathrm{d}\theta$$

Since I is a constant of the motion,

$$2\pi I = \oint p \, \mathrm{d}q$$

or

$$I = \frac{1}{2\pi} \oint p \, \mathrm{d}q \, . \tag{1.6}$$

Equation (1.6) can be considered the definition for the action variable, where the integral is taken around a single period of the motion (which is always a closed curve in phase space).

1.5 Integrability

A system called *integrable* if the number of degrees of freedom is equal to the number of constants of the motion in involution which each other. Two constants of the motion are in *involution* with each other if their mutual Poisson brackets are equal to 0.

The previously mentioned constants of the motion are not required to be known in analytic form, they are only required to exist. Thus, for example $n = 1$ systems are always integrable if the Hamiltonian is time independent.

An important consequence of integrability is that we can always find a canonical transformation to action-angle variables and express the new Hamiltonian as a function of the n constants of the motion given by the action variables: $\mathcal{H}' = \mathcal{H}'(I_1, \ldots, I_n)$.

For integrable systems, the n constants of the motion restrict the motion in the $2n$ dimensional phase space to a $2n - n = n$ dimensional subspace. If the motion is also periodic, then the motion is constrained to a n dimensional closed surface (with finite volume).

For a system with $n = 2$ and periodic motion, the motion in the phase space is confined to a torus. In order to understand why, we can use the previously defined action-angle variables. In this case, the motion will be described by $I_1, \theta_1, I_2, \theta_2$. Since the motion is constrained to a 2 dimensional subspace, we can imagine the phase space as a plane with the Ox axis given by the direction in which θ_1 increases and with the Oy axis given by the direction in which θ_2 increases. Because the motion is periodic, we can consider that θ_i, $i = 1, 2$ takes values in $[0, 2\pi)$.

Since the values for 0 and 2π are equivalent, we can join the ends of the intervals in both directions, thus forming a torus.

To extend the discussion to systems with n degrees of freedom, we replace the 2D plane with an n dimensional cube. If we join the sides of this cube we obtain a n dimensional torus.

+ explanation for the fact that the constants of motion must be in involution with each other

Observation: If we make a canonical transformation such that $\mathcal{H}' = 0$, the new coordinates and momenta will be constants, so we have $2n$ constants of the motion. If we make a canonical transformation to action-angle variables, we get n constants of the motion (the action variables). There is no contradiction since the $2n$ constants obtained in the first case are not all in involution with each other ($\{q_i, p_j\} = \delta_{i,j}$), but we can choose n of them in involution with each other.

1.6 Chaos notions

Deterministic chaos is a feature of non-integrable systems and is characterised by high sensitivity to initial conditions: for two dynamical systems with nearly identical initial conditions, the long time evolution is uncorrelated, the motion in the two systems being very different. Poincaré was the first to bring attention to this type of behaviour when he proved that there are no analytic solutions to the 3-body problem. (cite?) Even if the equations of motion are deterministic, long time predictions are rendered useless since an arbitrary small variation in the initial conditions can produce very different results.

In chaotic systems, the motion is no longer constrained on tori as for the case of integrable systems. The trajectories escape from the n dimensional subspace in the $2n$ phase space space, or in a $2n-1$ subspace if the energy is conserved.

1.6.1 The Liapunov exponent

Chaotic systems exhibit a varying degree of randomness. In order to describe chaos quantitatively we introduce the *Liapunov exponent*, which is a local measure of the dispersion in phase space. The sensitivity to the initial conditions of chaotic systems can be expressed as an exponential divergence of neighbouring trajectories.

If two orbits are separated at $t = 0$ by the infinitesimal distance $\mathrm{d}x_0$, then at a later time t, the distance will be

$$\mathrm{d}x\,(t) \sim \mathrm{d}x\,(0)\,\mathrm{e}^{\lambda t},$$

where λ is the Liapunov exponent. If $\lambda > 1$, the motion is chaotic and λ gives the exponential growth of the initial infinitesimal separation. If the motion is bounded, $\mathrm{d}x\,(t)$ will grow until is comparable with the dimension of the allowed part of the phase space, from that point on varying only randomly in time. Thus

the Liapunov exponent can be defined as follows

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\mathrm{d}x\,(t)}{\mathrm{d}x\,(0)} \tag{1.7}$$

For systems with discrete time evolution instead of the time we can use the number of iterations.

1.6.2 Poincaré sections

Since the phase space is $2n$ dimensional, visualising the motion can be difficult even for systems with $n = 2$ degrees of freedom. If the energy is constant, the motion will be constrained in the $2n - 1$ subspace, but even for the case of $n = 2$ the trajectory is not easy to follow in a 3D space considering that in most cases is represented on a 2D surface.

For $n = 2$ systems this difficulty can be addressed by using a slice of the phase space which reduces the dimensionality by 1. This slices are called *Poincaré sections*. If the energy is conserved, the section will be a plane.

If Σ is a surface section(Poincaré section) intersecting the trajectories of the system in phase space in the points $\mathbf{x}_i$, then we can define the *Poincaré map* as a mapping $P : \Sigma \to \Sigma$ such that

$$\mathbf{x}_{i+1} = P(\mathbf{x}_i).$$

A point $\mathbf{x}^*$ is called a *fixed point* of the if

$$\mathbf{x}^* = P(\mathbf{x}^*).$$

Thus a fixed point belongs to a closed trajectory. In order to study the stability of the orbit, we can analyse the stability of the fixed point. If $\mathbf{x}^*$ is a fixed point, we consider a small perturbation in the initial conditions and we look at what happens to a point $\mathbf{x}^* + \mathbf{v}_i$ near the fixed point

$$P(\mathbf{x}^* + \mathbf{v}_i) = P(\mathbf{x}^*) + \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i + \mathcal{O}\big(\|\mathbf{v}_i\|^2\big),$$

where $\frac{\partial P}{\partial \mathbf{x}}$ is the Jacobian of the map. Assuming we can neglect $\mathcal{O}\big(\|\mathbf{v}_i\|^2\big)$, we get

$$\mathbf{x}^* + \mathbf{v}_{i+1} = \mathbf{x}^* + \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i,$$

where $\mathbf{x}^* + \mathbf{v}_{i+1} = P(\mathbf{x}^* + \mathbf{v}_i)$.$(P(\mathbf{x}^* + \mathbf{v}_i) = P(\mathbf{x}^*) + P(\mathbf{v}_i)?)$ Thus

$$\mathbf{v}_{i+1} = \left.\frac{\partial P}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i, \tag{1.8}$$

also called the *linearised Poincaré map*.

In a basis given by the eigenvectors $\{\mathbf{e}_j\}$ of the Jacobian, $\mathbf{v}_i = \sum_{j=1}^{2n-1} a_j \mathbf{e}_j$, then

$$\mathbf{v}_{i+1} = \frac{\partial P}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\mathbf{x}^*} \sum_{j=1}^{2n-1} a_j \mathbf{e}_j = \sum_{j=1}^{2n-1} \frac{\partial P}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\mathbf{x}^*} \mathbf{e}_j a_j = \sum_{j=1}^{2n-1} \lambda_j a_j \mathbf{e}_j,$$

where λ_j are the eigenvalues of the Jacobian matrix.

If we consider k iterations of eq. (1.8)

$$\mathbf{v}_{i+k} = \sum_{j=1}^{2n-1} (\lambda_j)^k a_j \mathbf{e}_j$$

If all $|\lambda_j| < 1$, then $\|\mathbf{v}_{i+k}\| \to 0$ as $k \to \infty$ and the fixed point is *linearly stable*. If $|\lambda_j| > 1$ for some j, then the fixed point is *linearly unstable*. In the case when $|\lambda_m| = 1$, where λ_m is the largest eigenvalue, we cannot decide and a nonlinear stability analysis is required.

1.7 Kolmogorov-Arnold-Moser theorem

Since most systems are non-integrable we need approximate methods. One such method is canonical perturbation theory which can provide approximate solutions when the non-integrable Hamiltonian can be broken in an integrable part and a relatively small perturbation.

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H},$$

where $\mathcal{H}_0$ is the integrable part of the Hamiltonian for which the solutions of the Hamilton equations are known and $\Delta\mathcal{H}$ is the perturbation.

If we make a canonical transformation with a generating function of type F_2 such that $\mathcal{H}'_0 = 0$ we can obtain the constants of the motion for the integrable part of the Hamiltonian, Q_0 and P_0. If we use the generating function $S(q, P, t)$ to make ca canonical transformation for $\mathcal{H}$, we obtain

$$\mathcal{H}'(Q_0, P_0, t) = \Delta\mathcal{H}^0(Q_0, P_0, t).$$

Hamilton's equations read

$$\dot{Q} = \frac{\partial \mathcal{H}'}{\partial P} = \frac{\partial \Delta\mathcal{H}^0}{\partial P} \qquad\qquad \dot{P} = -\frac{\partial \mathcal{H}'}{\partial Q} = -\frac{\partial \Delta\mathcal{H}^0}{\partial Q}$$

In order to obtain the first order correction, we replace Q and P after derivation with Q_0 and P_0.

$$\dot{Q}_1 = \left.\frac{\partial \Delta\mathcal{H}^0}{\partial P}\right|_{\substack{Q=Q_0,\\P=P_0}} \qquad \dot{P}_1 = -\left.\frac{\partial \Delta\mathcal{H}^0}{\partial Q}\right|_{\substack{Q=Q_0,\\P=P_0}}$$

We can integrate the above differential equations to obtain Q_1 and P_1.

With these corrections we can construct a new generating function $S(Q_1, P_1, t)$ (why not $S(q, P_1, t)$? will this generating function be of type F_2?) which we can use to obtain a new perturbed Hamiltonian $\Delta\mathcal{H}^1$ (is $\mathcal{H}''_0 = 0$?).

In order to obtain higher order corrections we replace in $\Delta\mathcal{H}^0$ the conjugate variables Q_0 and P_0 with Q_1 and P_1, obtaining $\Delta\mathcal{H}^1$ and the procedure repeats.

$$\dot{Q}_{i+1} = \left.\frac{\partial \Delta\mathcal{H}^i}{\partial P}\right|_{\substack{Q=Q_i,\\P=P_i}} \qquad \dot{P}_{i+1} = -\left.\frac{\partial \Delta\mathcal{H}^i}{\partial Q}\right|_{\substack{Q=Q_i,\\P=P_i}}$$

It is clear that if the perturbation is small enough, we can no longer apply this method and the perturbation will significantly alter the motion. Hence, an important question that arises is the stability of the perturbed solutions and the domain of applicability for the perturbation theory.

The Kolmogorov-Arnold-Moser theorem or KAM theorem gives the limits of regular (or non-chaotic) motion for systems with bounded motion.

Theorem (Kolmogorov-Arnold-Moser theorem)**.** *If the bounded motion of an integrable system given by the Hamiltonian $\mathcal{H}_0$ is disturbed by a small perturbation $\Delta\mathcal{H}$ rendering the system non-integrable and if the unperturbed motion has incommensurate frequencies $\omega_i = \dot{\theta}_i$, where θ_i are the angle variables, then the motion will remain confined to an n-torus, except for a small set of initial conditions for which the trajectories escape on the energy hypersurface.*

The theorem tells us that the majority of the tori will survive the perturbation and the set of destroyed tori has finite volume in the phase space. A torus of the unperturbed system survives if it can be continuously deformed in a torus of the perturbed system. Furthermore, the destroyed tori have rationally related frequencies, that is there exist $m_1, \ldots, m_n \equiv \mathbf{m} \in \mathbb{Z}^n$ such that $\mathbf{m} \cdot \boldsymbol{\omega} = 0$, where $\boldsymbol{\omega} \equiv \omega_1, \ldots, \omega_n$.

Since the set of rational numbers is dense in the set of real numbers, the set of destroyed tori will be dense in the phase space. In the place of the destroyed tori chaotic motion will appear as the trajectories escape in the $2n-1$ subspace of constant energy.