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FINGERPRINTS OF CLASSICAL PHASE SPACE  
STRUCTURE IN QUANTUM CHAOS

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# Chapter 1

## Introduction

# Chapter 2

## Classical chaos

### 2.1 Fundamental notions

Classical mechanics can be reformulated in a more elegant way which does not require the computation of individual forces and instead uses a single function to describe the entire system, namely the *Lagrangian*. The Lagrangian, denoted with  $\mathcal{L}$ , is a function of the generalised coordinates  $q_1, \dots, q_n$ , their derivatives with respect to time  $\dot{q}_1, \dots, \dot{q}_n$  and optionally the time  $t$ , where  $n$  is the number of degrees of freedom.

The principles of Newtonian mechanics are replaced by the a variational principle  $\delta S = 0$ , where

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) dt \quad (2.1)$$

is called the *action*. This principle, also called *Hamilton's principle* or *the principle of least action* is equivalent with the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = 0,$$

where  $k = 1, \dots, n$ . Thus we can obtain the equations of motion provided that we know the initial conditions  $q_1(t_0), \dots, q_n(t_0)$  and  $\dot{q}_1(t_0), \dots, \dot{q}_n(t_0)$  at  $t = t_0$  by solving the set of  $n$  second order differential equations.

An other formalism for classical mechanics which is very useful in theoretical physics is the Hamiltonian formalism. In this formalism instead of the Lagrangian we use the *Hamiltonian*,  $\mathcal{H}$ , which is a function of the generalised coordinates  $q_1, \dots, q_n$ , the generalised momenta  $p_1, \dots, p_n$  and optionally the time  $t$ . The generalised momenta are given by

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}.$$

The Hamiltonian can be obtained from the Lagrangian through a *Legendre transformation* of  $\mathcal{L}$

$$\mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n, t) = \sum_{k=0}^n p_k \dot{q}_k - \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t). \quad (2.2)$$

The Euler-Lagrange equations are replaced by Hamilton's equations

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k}$$

We now have  $2n$  first order ordinary differential equations. comments?

## 2.2 Canonical transformations

In the Hamiltonian formulation of classical mechanics *canonical transformations* represent all the transformations of the canonically conjugate variables from  $q$  and  $p$  to  $Q$  and  $P$  such that the structure of the Hamilton equations remains intact.

The Lagrange equations of a system are invariant to a transformation of the Lagrange function such that:

$$\mathcal{L}'(Q, \dot{Q}, t) = \mathcal{L}(q, \dot{q}, t) - \frac{d}{dt} F(q, Q, t)$$

Integrating we obtain

$$\int_{t_1}^{t_2} \mathcal{L}' dt = \int_{t_1}^{t_2} \mathcal{L} dt - (F(q, Q, t_2) - F(q, Q, t_1))$$

The variation of the above equation is given by

$$\delta \int_{t_1}^{t_2} \mathcal{L}' dt = \delta \int_{t_1}^{t_2} \mathcal{L} dt - \delta F(q, Q, t_2) + \delta F(q, Q, t_1)$$

Fixing the variations at the end points to be 0, we get

$$\delta \int_{t_1}^{t_2} \mathcal{L}' dt = \delta \int_{t_1}^{t_2} \mathcal{L} dt$$

So indeed the two Lagrangian descriptions of the system are identical since if the Hamilton principle holds in the first description it also holds in the second.

A function  $F(q, Q, t)$  satisfying the above property and the condition  $\frac{\partial^2 F}{\partial q \partial Q} \neq 0$  is called a *generating function*. We can classify the generating functions in four

possible types as a function of their variables. The conjugate momenta corresponding to  $q$  and  $Q$  are given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \text{ and } P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}}$$

In order to express those as a function of  $F(q, Q, t)$  we use the fact that  $\dot{q}$  is a cyclic coordinate for  $\mathcal{L}'$

$$\frac{\partial \mathcal{L}'}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial}{\partial \dot{q}} \left( \frac{d}{dt} F(q, Q, t) \right) = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0,$$

where

$$\frac{d}{dt} F(q, Q, t) = \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial Q} \dot{Q} + \frac{\partial F}{\partial t}.$$

Thus we obtain

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

and

$$P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}} = -\frac{\partial F}{\partial Q}$$

To find the Hamiltonian corresponding to the new coordinates  $Q$  and  $P$  we compute the Legendre transformation of  $\mathcal{L}'$

$$\mathcal{H}'(Q, P, t) = P\dot{Q} - \mathcal{L}'(Q, \dot{Q}, t)$$

Thus

$$\mathcal{H}'(Q, P, t) = P\dot{Q} - \mathcal{L} + \frac{dF}{dt} = P\dot{Q} - \mathcal{L} + \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial Q} \dot{Q} + \frac{\partial F}{\partial t} = p\dot{q} - \mathcal{L} + \frac{\partial F}{\partial t}$$

and so

$$\mathcal{H}'(Q, P, t) = \mathcal{H}(q(Q, P), p(Q, P), t) + \frac{\partial F(q(Q, P), Q, t)}{\partial t}. \quad (2.3)$$

As mentioned above, there are four types of generating functions. They can be obtained by successive Legendre transformations which switch between the conjugate coordinates and they represent the same canonical transformation. We denote the above mentioned  $F(q, Q, t) \equiv F_1(q, Q, t)$ . The other types of generating functions will be denoted  $F_2(q, P, t)$ ,  $F_3(p, Q, t)$ ,  $F_4(p, P, t)$ . Thus we define

$$\begin{aligned} F_2(q, P, t) &= F_1(q, Q, t) + PQ & p &= \frac{\partial F_2}{\partial q}, & Q &= \frac{\partial F_2}{\partial P} \\ F_3(p, Q, t) &= F_1(q, Q, t) - pq & q &= -\frac{\partial F_3}{\partial p}, & P &= \frac{\partial F_3}{\partial Q} \\ F_4(p, P, t) &= F_1(q, Q, t) - pq + PQ & q &= -\frac{\partial F_4}{\partial p}, & Q &= \frac{\partial F_4}{\partial P} \end{aligned}$$

The *Poisson bracket* of two arbitrary functions with respect to the canonical coordinates  $q$  and  $p$  is defined as

$$[F, G]_{q,p} = \sum_{k=1}^n \left( \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right).$$

We can use the Poisson bracket to express Hamilton's equations more elegantly

$$\dot{q}_k = [q_k, H], \quad \dot{p}_k = [p_k, H].$$

Expressed in this form, the Hamilton equations have a close resemblance to Heisenberg's form for the equations of motion in the Heisenberg picture of quantum mechanics. The Poisson bracket structure of classical mechanics is one of the fundamental links to quantum mechanics. [1] (cite Dirac).

As with the above discussion we will limit to the case of one degree of freedom, but the results are easily generalisable to  $n$  degrees of freedom.

If  $Q$  and  $P$  are a set of conjugate variables obtained by a canonical transformation from  $q$  and  $p$ , then

$$[F', G']_{Q,P} = [F, G]_{q,p},$$

where  $F'$  and  $G'$  are the transformed functions. As a consequence, if we choose  $F = Q$  and  $G = P$

$$[Q, P]_{Q,P} = [Q(q, p), P(q, p)]_{q,p} = 1$$

This relation represents a necessary and sufficient condition for the transformation from  $q, p$  to  $Q, P$  to be canonical. Preserving the structure of Hamilton's equations is equivalent with the invariance of the Poisson bracket and thus we can use this invariance as the definition for canonical transformations.

An other equivalent condition would require the Jacobian matrix  $\frac{\partial(Q,P)}{\partial(q,p)}$  to be **symplectic**.

## 2.3 The Hamilton-Jacobi equation

Instead of finding the equations of motion through solving a set of  $2n$  ordinary differential equations with  $2n$  dependent variables ( $q_1, \dots, q_n$  and  $p_1, \dots, p_n$ ) and one independent variable ( $t$ ), we can use a single first order partial differential equation having  $n + 1$  independent variables ( $q_1, \dots, q_n$  and  $t$ ) and one dependent variable ( $S$ ).

Suppose there exists a generating function  $S$  of type  $F_2$  **why?** such that

$$\mathcal{H}' = 0.$$

From Hamilton's equations, the new coordinates  $Q_k$  and  $P_k$ , with  $k = 1, \dots, n$  will be constants, so the information about the time evolution of the system is contained in the canonical transformation itself. Thus, equation (2.3) becomes

$$\mathcal{H}(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}, t) + \frac{\partial S}{\partial t} = 0 \quad (2.4)$$

which represents the *Hamilton-Jacobi equation*.

The solution  $S(q_1, \dots, q_n, P_1, \dots, P_n, t)$  of equation (2.4) is called *Hamilton's Principal Function*. From the theory of differential equations we know that there will be  $n + 1$  arbitrary constants. We can find  $n$  of these constants as  $P_1, \dots, P_n$  and the remaining constant can be incorporated in  $S$  since if  $S$  is a solution for the Hamilton-Jacobi equation,  $S + \text{const.}$  is also a solution.

It is a standard convention (Goldstein) to rename the constants as follows

$$\begin{aligned} P_k &\equiv \alpha_k \\ Q_k &\equiv \beta_k = \frac{\partial S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)}{\partial \alpha_k}. \end{aligned}$$

These constants depend on the initial conditions  $q_k(0)$  and  $p_k(0)$ . From

$$p_k(0) = \left. \frac{\partial S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)}{\partial q_k} \right|_{t=0}$$

we can obtain  $\alpha_k$ s, and  $\beta_k$ s are given by

$$\beta_k = \left. \frac{\partial S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)}{\partial \alpha_k} \right|_{t=0}$$

We can obtain the initial generalised coordinates  $q_k(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, t)$  by solving

$$\beta_k = \frac{\partial S}{\partial \alpha_k}$$

If we take the total time derivative of  $S$

$$\frac{dS}{dt} = \sum_{k=1}^n \frac{\partial S}{\partial q_k} \dot{q}_k + \frac{\partial S}{\partial t}$$

and use eq. (2.3)

$$\frac{dS}{dt} = \sum_{k=1}^n p_k \dot{q}_k - \mathcal{H}$$

we recognise the Legendre transform from eq. (2.2). Thus

$$\frac{dS}{dt} = \mathcal{L}$$

or equivalently

$$S = \int \mathcal{L} dt.$$

It follows that the generating function for this special canonical transformation is the action as defined in eq. (2.1).

If the time does not appear explicitly in the Hamiltonian, the Hamiltonian itself will be a constant of the motion and will represent the energy of the system  $E$ .

We can rewrite eq. (2.4) as follows

$$\mathcal{H}(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}, t) = -\frac{\partial S}{\partial t} = E$$

Thus we can suppose that for this case  $S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)$  has the following form

$$S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t) \equiv W(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n) - Et$$

Using this expression in eq. (2.4), we obtain

$$\mathcal{H}(q_1, \dots, q_n, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_n}) = E \quad (2.5)$$

which represents the second form of the Hamilton-Jacobi equation.

The solution to this equation is called *Hamilton's Characteristic Function*.  $W$  can be expressed as

$$W = S + Et = \int (\mathcal{L} + \mathcal{H}) dt = \int \sum_{k=0}^n p_k \dot{q}_k dt = \sum_{k=0}^n \int p_k dq_k$$

### physical interpretation?

$W$  is a generating function of type  $F_2$ . The canonical transformation given by  $W$  is different from the one given by  $S$ , since  $\mathcal{H}' = \mathcal{H} + \frac{\partial W}{\partial t} = E \neq 0$ .

The new generalised coordinates  $Q_k$  are cyclic in the new Hamiltonian  $\mathcal{H}' = E = \mathcal{H}(q_1, \dots, q_n, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_n})$ , thus the new generalised momenta are constants. While in the case of  $S$  all the generalised momenta and coordinates were constants, in the case of  $W$  all the generalised momenta are constants  $\dot{P}_k = -\frac{\partial \mathcal{H}'}{\partial Q_k} = 0$ , but not all the generalised coordinates are constants. If we choose to define  $P_1 \equiv E$ ,

$$\dot{Q}_1 = \frac{\partial \mathcal{H}'}{\partial P_1} = \frac{\partial E}{\partial P_1} = 1$$

Thus

$$Q_1 = t + \text{const.} = \frac{\partial W}{\partial P_1} = \frac{\partial W}{\partial E}.$$



So  $Q_1$  is no longer a constant. The constant in the above equation determines the origin of the time axis.  $Q_1$  represents the time and its conjugate coordinate is  $P_1$ , the energy.

The  $n - 1$  generalised momenta  $P_2, \dots, P_n$  are independent of  $P_1$  and thus  $\frac{\partial \mathcal{H}'}{\partial P_k} = 0$ , where  $k = 2, \dots, n$ . So the corresponding  $n - 1$  generalised coordinates are constants.

We can change the notation for  $\alpha$  and  $\beta$  to use an index from 1 to  $n - 1$ :  $\alpha_j \equiv P_{j+1}$ ,  $\beta_j \equiv Q_{j+1}$ , where  $j = 1, \dots, n - 1$ . We can now express Hamilton's Characteristic Function as

$$W = W(q_1, \dots, q_n, E, \alpha_1, \dots, \alpha_{n-1}).$$

From the initial generalised momenta  $p_k(0)$

$$p_k(0) = \left. \frac{\partial W(q_1, \dots, q_n, E, \alpha_1, \dots, \alpha_{n-1})}{\partial q_k} \right|_{t=0}$$

we can obtain  $E$  and  $\alpha_j$ . Using the initial generalised coordinates  $q_k(0)$  we obtain  $\beta_j$  as follows

$$\beta_j = \left. \frac{\partial W(q_1, \dots, q_n, E, \alpha_1, \dots, \alpha_{n-1})}{\partial \alpha_j} \right|_{t=0}$$

Thus, having  $E, \alpha_j$  and  $\beta_j$ , we can obtain the time evolution for the generalised coordinates from

$$\beta_j = \frac{\partial W(q_1, \dots, q_n, E, \alpha_1, \dots, \alpha_{n-1})}{\partial \alpha_j}$$

For systems with  $n > 1$  degrees of freedom, the Hamilton-Jacobi equation can be separated in  $n$  equations if we can write  $S$  as

$$S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t) = \sum_{k=1}^n W_k(q_k, \alpha_1, \dots, \alpha_n) - Et$$

A simpler form of separability exists if the Hamiltonian itself is separable

$$\mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{k=1}^n \mathcal{H}_k(q_k, p_k)$$

In this case the second form of the Hamilton-Jacobi equation splits in  $n$  independent equations

$$\mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n) = E = \sum_{k=1}^n \mathcal{H}_k(q_k, p_k),$$

so

$$\mathcal{H}_k(q_k, \frac{\partial W_k}{\partial q_k}) = \alpha_k, \quad E = \sum_{k=1}^n \alpha_k$$

why  $E = \sum_{k=1}^n \alpha_k$ ?

The Hamilton-Jacobi equation can be solved only if the variables can be separated. The separability depends both on the problem and the chosen coordinate system. This does not decrease the theoretical value of the Hamilton-Jacobi equation which provides powerful methods for finding the constants of the motion.

## 2.4 Action-angle variables

Going back again to a system with one degree of freedom for simplicity, we study a system which has a periodic motion. We search for a canonical transformation from  $q, p$  to a new set of canonically conjugate coordinates  $\theta, I$  such that the new Hamiltonian does not depend on  $\theta$ .

Since  $\theta$  is an ignorable coordinate

$$\dot{I} = -\frac{\partial \mathcal{H}'}{\partial \theta} = 0,$$

$I$  is a constant of the motion and

$$\dot{\theta} = \frac{\partial \mathcal{H}'}{\partial I} = \text{const.}$$

Thus  $\theta = \frac{\partial \mathcal{H}'}{\partial I}(t - t_0) \equiv \omega(I)(t - t_0)$ .  $\theta$  is called the *angle variable* and  $I$  the *action variable* (not to be confused with the action).

In order to obtain these variables we will use a generating function of type  $F_1$  denoted  $W'(q, \theta)$ . Since the motion is periodic in the initial coordinates, it must also be periodic in the angle variable. Thus  $W'$  is periodic in  $\theta$ .

From the properties of the  $F_1$  type generating functions

$$dW' = \frac{\partial W'}{\partial q} dq + \frac{\partial W'}{\partial \theta} d\theta = p dq - I d\theta.$$

(we can observe that  $W'$  is related to Hamilton's Characteristic Function by a Legendre transformation)

Since the motion is periodic, if we integrate over a period  $q$  returns to the same value, while  $\theta$  advances by a constant amount, which we can choose to be  $2\pi$  per period.

$$\oint dW' = 0 = \oint p dq - \oint I d\theta$$

Since  $I$  is a constant of the motion,

$$2\pi I = \oint p \, dq$$

or

$$I = \frac{1}{2\pi} \oint p \, dq. \quad (2.6)$$

Equation (2.6) can be considered the definition for the action variable, where the integral is taken around a single period of the motion (which is always a closed curve in phase space).

## 2.5 Integrability

A system called *integrable* if the number of degrees of freedom is equal to the number of constants of the motion in involution which each other. Two constants of the motion are in *involution* with each other if their mutual Poisson brackets are equal to 0.

The previously mentioned constants of the motion are not required to be known in analytic form, they are only required to exist. Thus, for example  $n = 1$  systems are always integrable if the Hamiltonian is time independent.

An important consequence of integrability is that we can always find a canonical transformation to action-angle variables and express the new Hamiltonian as a function of the  $n$  constants of the motion given by the action variables:  $\mathcal{H}' = \mathcal{H}'(I_1, \dots, I_n)$ .

For integrable systems, the  $n$  constants of the motion restrict the motion in the  $2n$  dimensional phase space to a  $2n - n = n$  dimensional subspace. If the motion is also periodic, then the motion is constrained to a  $n$  dimensional closed surface (with finite volume).

For a system with  $n = 2$  and periodic motion, the motion in the phase space is confined to a torus. In order to understand why, we can use the previously defined action-angle variables. In this case, the motion will be described by  $I_1, \theta_1, I_2, \theta_2$ . Since the motion is constrained to a 2 dimensional subspace, we can imagine the phase space as a plane with the  $Ox$  axis given by the direction in which  $\theta_1$  increases and with the  $Oy$  axis given by the direction in which  $\theta_2$  increases. Because the motion is periodic, we can consider that  $\theta_i, i = 1, 2$  takes values in  $[0, 2\pi)$ .

Since the values for 0 and  $2\pi$  are equivalent, we can join the ends of the intervals in both directions, thus forming a torus.

To extend the discussion to systems with  $n$  degrees of freedom, we replace the 2D plane with an  $n$  dimensional cube. If we join the sides of this cube we obtain a  $n$  dimensional torus.

+ explanation for the fact that the constants of motion must be in involution with each other

Observation: If we make a canonical transformation such that  $\mathcal{H}' = 0$ , the new coordinates and momenta will be constants, so we have  $2n$  constants of the motion. If we make a canonical transformation to action-angle variables, we get  $n$  constants of the motion (the action variables). There is no contradiction since the  $2n$  constants obtained in the first case are not all in involution with each other ( $\{q_i, p_j\} = \delta_{i,j}$ ), but we can choose  $n$  of them in involution with each other.

## 2.6 Chaos notions

Deterministic chaos is a feature of non-integrable systems and is characterised by high sensitivity to initial conditions: for two dynamical systems with nearly identical initial conditions, the long time evolution is uncorrelated, the motion in the two systems being very different. Poincaré was the first to bring attention to this type of behaviour when he proved that there are no analytic solutions to the 3-body problem. (cite?) Even if the equations of motion are deterministic, long time predictions are rendered useless since an arbitrary small variation in the initial conditions can produce very different results.

In chaotic systems, the motion is no longer constrained on tori as for the case of integrable systems. The trajectories escape from the  $n$  dimensional subspace in the  $2n$  phase space, or in a  $2n - 1$  subspace if the energy is conserved.

### 2.6.1 The Liapunov exponent

Chaotic systems exhibit a varying degree of randomness. In order to describe chaos quantitatively we introduce the *Liapunov exponent*, which is a local measure of the dispersion in phase space. The sensitivity to the initial conditions of chaotic systems can be expressed as an exponential divergence of neighbouring trajectories.

If two orbits are separated at  $t = 0$  by the infinitesimal distance  $dx_0$ , then at a later time  $t$ , the distance will be

$$dx(t) \sim dx(0) e^{\lambda t},$$

where  $\lambda$  is the Liapunov exponent. If  $\lambda > 0$ , the motion is chaotic and  $\lambda$  gives the exponential growth of the initial infinitesimal separation. If the motion is bounded,  $dx(t)$  will grow until it is comparable with the dimension of the allowed part of the phase space, from that point on varying only randomly in time. Thus

the Liapunov exponent can be defined as follows

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{dx(t)}{dx(0)} \quad (2.7)$$

For systems with discrete time evolution instead of the time we can use the number of iterations.

## 2.6.2 Poincaré sections

Since the phase space is  $2n$  dimensional, visualising the motion can be difficult even for systems with  $n = 2$  degrees of freedom. If the energy is constant, the motion will be constrained in the  $2n - 1$  subspace, but even for the case of  $n = 2$  the trajectory is not easy to follow in a 3D space considering that in most cases is represented on a 2D surface.

For  $n = 2$  systems this difficulty can be addressed by using a slice of the phase space which reduces the dimensionality by 1. This slices are called *Poincaré sections*. If the energy is conserved, the section will be a plane.

If  $\Sigma$  is a surface section (Poincaré section) intersecting the trajectories of the system in phase space in the points  $\mathbf{x}_i$ , then we can define the *Poincaré map* as a mapping  $P : \Sigma \rightarrow \Sigma$  such that

$$\mathbf{x}_{i+1} = P(\mathbf{x}_i).$$

A point  $\mathbf{x}^*$  is called a *fixed point* of the if

$$\mathbf{x}^* = P(\mathbf{x}^*).$$

Thus a fixed point belongs to a closed trajectory. In order to study the stability of the orbit, we can analyse the stability of the fixed point. If  $\mathbf{x}^*$  is a fixed point, we consider a small perturbation in the initial conditions and we look at what happens to a point  $\mathbf{x}^* + \mathbf{v}_i$  near the fixed point

$$P(\mathbf{x}^* + \mathbf{v}_i) = P(\mathbf{x}^*) + \left. \frac{\partial P}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i + \mathcal{O}(\|\mathbf{v}_i\|^2),$$

where  $\frac{\partial P}{\partial \mathbf{x}}$  is the Jacobian of the map. Assuming we can neglect  $\mathcal{O}(\|\mathbf{v}_i\|^2)$ , we get

$$\mathbf{x}^* + \mathbf{v}_{i+1} = \mathbf{x}^* + \left. \frac{\partial P}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i,$$

where  $\mathbf{x}^* + \mathbf{v}_{i+1} = P(\mathbf{x}^* + \mathbf{v}_i)$ . ( $P(\mathbf{x}^* + \mathbf{v}_i) = P(\mathbf{x}^*) + P(\mathbf{v}_i)$ ?) Thus

$$\mathbf{v}_{i+1} = \left. \frac{\partial P}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{v}_i, \quad (2.8)$$

also called the *linearised Poincaré map*.

In a basis given by the eigenvectors  $\{\mathbf{e}_j\}$  of the Jacobian,  $\mathbf{v}_i = \sum_{j=1}^{2n-1} a_j \mathbf{e}_j$ , then

$$\mathbf{v}_{i+1} = \left. \frac{\partial P}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*} \sum_{j=1}^{2n-1} a_j \mathbf{e}_j = \sum_{j=1}^{2n-1} \left. \frac{\partial P}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{e}_j a_j = \sum_{j=1}^{2n-1} \lambda_j a_j \mathbf{e}_j,$$

where  $\lambda_j$  are the eigenvalues of the Jacobian matrix.

If we consider  $k$  iterations of eq. (2.8)

$$\mathbf{v}_{i+k} = \sum_{j=1}^{2n-1} (\lambda_j)^k a_j \mathbf{e}_j$$

If all  $|\lambda_j| < 1$ , then  $\|\mathbf{v}_{i+k}\| \rightarrow 0$  as  $k \rightarrow \infty$  and the fixed point is *linearly stable*. If  $|\lambda_j| > 1$  for some  $j$ , then the fixed point is *linearly unstable*. In the case when  $|\lambda_m| = 1$ , where  $\lambda_m$  is the largest eigenvalue, we cannot decide and a nonlinear stability analysis is required.

## 2.7 Kolmogorov-Arnold-Moser theorem

Since most systems are non-integrable we need approximate methods. One such method is canonical perturbation theory which can provide approximate solutions when the non-integrable Hamiltonian can be broken in an integrable part and a relatively small perturbation.

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H},$$

where  $\mathcal{H}_0$  is the integrable part of the Hamiltonian for which the solutions of the Hamilton equations are known and  $\Delta\mathcal{H}$  is the perturbation.

If we make a canonical transformation with a generating function of type  $F_2$  such that  $\mathcal{H}'_0 = 0$  we can obtain the constants of the motion for the integrable part of the Hamiltonian,  $Q_0$  and  $P_0$ . If we use the generating function  $S(q, P, t)$  to make a canonical transformation for  $\mathcal{H}$ , we obtain

$$\mathcal{H}'(Q_0, P_0, t) = \Delta\mathcal{H}^0(Q_0, P_0, t).$$

Hamilton's equations read

$$\dot{Q} = \frac{\partial \mathcal{H}'}{\partial P} = \frac{\partial \Delta\mathcal{H}^0}{\partial P} \qquad \dot{P} = -\frac{\partial \mathcal{H}'}{\partial Q} = -\frac{\partial \Delta\mathcal{H}^0}{\partial Q}$$

In order to obtain the first order correction, we replace  $Q$  and  $P$  after derivation with  $Q_0$  and  $P_0$ .

$$\dot{Q}_1 = \left. \frac{\partial \Delta \mathcal{H}^0}{\partial P} \right|_{\substack{Q=Q_0, \\ P=P_0}} \quad \dot{P}_1 = - \left. \frac{\partial \Delta \mathcal{H}^0}{\partial Q} \right|_{\substack{Q=Q_0, \\ P=P_0}}$$

We can integrate the above differential equations to obtain  $Q_1$  and  $P_1$ .

With these corrections we can construct a new generating function  $S(Q_1, P_1, t)$  (why not  $S(q, P_1, t)$ ? will this generating function be of type  $F_2$ ?) which we can use to obtain a new perturbed Hamiltonian  $\Delta \mathcal{H}^1$  (is  $\mathcal{H}''_0 = 0$ ?).

In order to obtain higher order corrections we replace in  $\Delta \mathcal{H}^0$  the conjugate variables  $Q_0$  and  $P_0$  with  $Q_1$  and  $P_1$ , obtaining  $\Delta \mathcal{H}^1$  and the procedure repeats.

$$\dot{Q}_{i+1} = \left. \frac{\partial \Delta \mathcal{H}^i}{\partial P} \right|_{\substack{Q=Q_i, \\ P=P_i}} \quad \dot{P}_{i+1} = - \left. \frac{\partial \Delta \mathcal{H}^i}{\partial Q} \right|_{\substack{Q=Q_i, \\ P=P_i}}$$

It is clear that if the perturbation is small enough, we can no longer apply this method and the perturbation will significantly alter the motion. Hence, an important question that arises is the stability of the perturbed solutions and the domain of applicability for the perturbation theory.

The Kolmogorov-Arnold-Moser theorem or KAM theorem gives the limits of regular (or non-chaotic) motion for systems with bounded motion.

**Theorem** (Kolmogorov-Arnold-Moser theorem). *If the bounded motion of an integrable system given by the Hamiltonian  $\mathcal{H}_0$  is disturbed by a small perturbation  $\Delta \mathcal{H}$  rendering the system non-integrable and if the unperturbed motion has *incommensurate* frequencies  $\omega_i = \dot{\theta}_i$ , where  $\theta_i$  are the angle variables, then the motion will remain confined to an  $n$ -torus, except for a small set of initial conditions for which the trajectories escape on the energy hypersurface.*

The theorem tells us that the majority of the tori will survive the perturbation and the set of destroyed tori has finite volume in the phase space. A torus of the unperturbed system survives if it can be continuously deformed in a torus of the perturbed system. Furthermore, the destroyed tori have rationally related frequencies, that is there exist  $m_1, \dots, m_n \equiv \mathbf{m} \in \mathbb{Z}^n$  such that  $\mathbf{m} \cdot \boldsymbol{\omega} = 0$ , where  $\boldsymbol{\omega} \equiv \omega_1, \dots, \omega_n$ .

Since the set of rational numbers is *dense* in the set of real numbers, the set of destroyed tori will be *dense in the phase space*. In the place of the destroyed tori chaotic motion will appear as the trajectories escape in the  $2n - 1$  subspace of constant energy.

# Chapter 3

## Quantum Chaos

### 3.1 Fundamental notions

In this section will briefly present the general formalism developed by P.A.M. Dirac.

The description of phenomena in quantum mechanics requires a **careful** definition of the basic concepts. Thus we begin with the concept of the physical state. A *physical state* contains all the information that can be learned about the system. We associate to our physical system a *Hilbert space*, denoted with  $\mathbb{H}$ , which will contain all the possible states of the system. Thus a vector in the Hilbert space will describe the state of the system (more rigorously only the direction of the vector will describe the state since by convention any two vectors that differ only by a constant describe the same physical state). In the Dirac formalism the vectors in the Hilbert space corresponding to physical states are represented by *ket vectors*, denoted with  $|\Psi\rangle$ . The elements of the dual of the Hilbert space are called *bra vectors* and are denoted with  $\langle\Psi|$ . Due to the isomorphism between the Hilbert space and its dual, the bra vectors equally describe the physical state and the correspondence between the ket vectors and the bra vectors is given by dual conjugation.

#### The observables postulate

In quantum mechanics the properties of the physical system, called *observables* are described by linear hermitian operators. As a result of a measurement, the measured values of the observable are among the eigenvalues of the associated hermitian operator.



## The measurement postulate

Let us consider a physical system in a state described by the state vector  $|\Psi\rangle \in \mathbb{H}$  and an observable described by the linear hermitian operator  $A$ . If a measurement of the observable is performed, then following the measurement the system will jump in an uncontrollable manner in one of the eigenstates of the operator associated with the measured observable and the result of the measurement will be given by the corresponding eigenvalue. The probability of obtaining a given eigenvalue is given by the square of the absolute value of the scalar product between the final state associated with the given eigenvalue and the initial state.

discrete / continuous discussion?

## Fundamental commutation relations?

### Time evolution postulate

In quantum mechanics time is just a parameter and not an observable. The time evolution can be postulated in different ways. If we consider the states as time dependent entities and observables as time independent ones, we can view the time evolution as a temporal displacement, similar to a spatial displacement. Then we can use the time evolution operator identity

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = H U(t, t_0)$$

to derive the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = H |\Psi\rangle.$$

If we consider the states as time independent and the observables as time dependent, then the time evolution of the observables will be given by

$$i\hbar \frac{d}{dt} A(t) = [A(t), H].$$

We can observe the link between this equation and Hamilton's equations is given by the correspondence between the commutator in quantum mechanics and the Poisson bracket in classical mechanics, as mentioned in the previous chapter.

The first approach is called the Schrödinger picture of quantum mechanics, while the second is called the Heisenberg picture. There is also another formulation, namely the interaction picture also named the Dirac picture. In the following we will only consider the Schrödinger picture.

In the particular case when the Hamiltonian is time-independent, Schrödinger's equation reduces to

$$H |\Psi\rangle = E |\Psi\rangle,$$

also called the time-independent Schrödinger equation and the time evolution of the state is given by

$$|E(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)H} |E(t_0)\rangle = e^{-\frac{i}{\hbar}(t-t_0)E} |E\rangle.$$

Thus if the system is initially in an energy eigenstate, it will remain in the same state, having at most a phase modulation. Such states are called *stationary states*.

### 3.2 From symmetry to degeneracy

Let us consider the above mentioned time-independent case. If we suppose that the energy spectrum is discrete,

$$H |\Psi_i\rangle = E_i |\Psi_i\rangle.$$

If for an eigenvalue  $E_i$  repeats itself for different eigenstates  $|\Psi_i\rangle$  we have what is called a *degeneracy*. In order to emphasise this, we can use a second index for the eigenvectors,  $j = 1, \dots, m$ , where  $m$  is the number of times the eigenvalue repeats.

Thus for the  $i$ -th eigenvalue,

$$H |\Psi_{i,j}\rangle = E_i |\Psi_{i,j}\rangle.$$

If the Hamiltonian is invariant to a set of unitary transformations,  $T^\dagger H T = H$ , then this set forms a group since the transformation given by  $TT'$ , with  $T'$  an arbitrary transformation from the set, will also leave the Hamiltonian invariant. Since symmetry operations form a group, the unitary operators which correspond to the symmetries of the Hamiltonian will form a group to which the Hamiltonian is invariant. Thus,

$$TH = TT^\dagger HT = TT^{-1}HT = HT,$$

or  $[T, H] = 0$ . We can once again observe how classical mechanics and quantum mechanics are linked through the Poisson bracket-commutator structure. In classical mechanics Noether's theorem provides constants of the motion from the symmetries of the system and those have a vanishing Poisson bracket with the Hamiltonian.

Since the operators commute, they share a common set of eigenvectors

$$H(T |\Psi_i\rangle) = HT |\Psi_i\rangle = TH |\Psi_i\rangle = TE_i |\Psi_i\rangle = E_i(T |\Psi_i\rangle).$$

Thus if the Hamiltonian is invariant to the set of unitary transformations  $\{T_j\}$ , with  $j = 1, \dots, m$

$$T_j H |\Psi_i\rangle = TE_i |\Psi_i\rangle = E_i(T |\Psi_i\rangle) = E_i |\Psi_{i,j}\rangle$$

and we will not change the value of the eigenvalue  $E_i$  by applying  $T_j$ , we only transform the eigenstates  $|\Psi_i\rangle$  to a linear combination  $|\Psi_{i,j}\rangle$ .

### 3.3 Level repulsion

As is the case with classical mechanics there are few situations in which the analytic solution is known. In order to find approximate solutions we can use perturbation theory if we can assume that the Hamiltonian can be viewed as to have a part for which the solutions are known and another part that can be considered a small perturbation.

In the framework of time-independent perturbation theory, we want to obtain an approximate solution to the problem

$$H |\Psi^{[j]}\rangle = E_j |\Psi^{[j]}\rangle \quad (3.1)$$

and we consider that the Hamiltonian can be written as  $H = H_0 + \lambda V$ , where the solution to the unperturbed problem

$$H_0 |\Psi_{n\alpha}^0\rangle = E_n^0 |\Psi_{n\alpha}^0\rangle$$

is known and  $\alpha$  gives the degeneracy of the  $n$ -th level.

Since the unperturbed eigenvectors form a basis in the Hilbert space, we can expand the perturbed eigenvectors in that basis.

$$|\Psi^{[j]}\rangle = \sum_{m,\beta} c_{m,\beta}^{[j]} |\Psi_{m,\beta}^0\rangle$$

Inserting into eq. (3.1) we obtain

$$(H_0 + \lambda V) \sum_{m,\beta} c_{m,\beta}^{[j]} |\Psi_{m,\beta}^0\rangle = \sum_{m,\beta} E_j c_{m,\beta}^{[j]} |\Psi_{m,\beta}^0\rangle.$$

Using the solution to the unperturbed problem the above equation becomes

$$\sum_{m,\beta} (E_m^0 + \lambda V) c_{m,\beta}^{[j]} |\Psi_{m,\beta}^0\rangle = \sum_{m,\beta} E_j c_{m,\beta}^{[j]} |\Psi_{m,\beta}^0\rangle.$$

To simplify the discussion we will now consider what happens with two unperturbed states  $|\Psi_1^0\rangle$  with the energy  $E_1^0$  and  $|\Psi_2^0\rangle$  with the energy  $E_2^0$  when  $\lambda = 1$ .

$$\sum_{m,\beta} (E_m^0 + V) c_m^{[j]} |\Psi_m^0\rangle = \sum_{m=1,2} E_j c_m^{[j]} |\Psi_m^0\rangle$$

By taking the scalar product with the state  $\langle \Psi_k^0 |$  and using the orthogonality relation  $\langle \Psi_k^0 | \Psi_m^0 \rangle = \delta_{k,m}$  we obtain

$$E_k^0 c_k^{[j]} + \sum_{m=1,2} \langle \Psi_k^0 | V | \Psi_m^0 \rangle c_m^{[j]} = E_j c_k^{[j]}$$

or

$$c_k^{[j]} (E_k^0 - E_j + V_{k,k}) + \sum_{m \neq k} V_{k,m} c_m^{[j]} = 0,$$

where  $\langle \Psi_k^0 | V | \Psi_m^0 \rangle \equiv V_{k,m}$ .

The above equation becomes

$$\begin{aligned} (E_1^0 - E_j + V_{1,1}) c_1^{[j]} + V_{1,2} c_2^{[j]} &= 0, \text{ for } k = 1 \\ V_{2,1} c_1^{[j]} + (E_2^0 - E_j + V_{2,2}) c_2^{[j]} &= 0, \text{ for } k = 2 \end{aligned}$$

We use the following notations:

$$H_{1,1} \equiv E_1^0 + V_{1,1}, H_{2,2} \equiv E_2^0 + V_{2,2}, H_{1,2} \equiv V_{1,2}, H_{2,1} \equiv V_{2,1}, \delta \equiv H_{1,1} - H_{2,2}, \tan \beta \equiv \frac{2|H_{1,2}|}{\delta}$$

The above system of equations has non-trivial solutions if the determinant vanishes.

$$(H_{1,1} - E_j)(H_{2,2} - E_j) - H_{1,2}H_{2,1} = 0$$

or

$$E_j^2 - (H_{1,1} + H_{2,2})E_j + H_{1,1}H_{2,2} - H_{1,2}H_{2,1} = 0.$$

This equation has the solutions

$$E_j = \frac{(H_{1,1} + H_{2,2}) \pm \sqrt{(H_{1,1} + H_{2,2})^2 - 4(H_{1,1}H_{2,2} - H_{1,2}H_{2,1})}}{2} \quad (3.2)$$

$$= \frac{H_{1,1} + H_{2,2}}{2} \pm \frac{1}{2} \sqrt{\delta^2 + 4|H_{1,2}|^2}. \quad (3.3)$$

Since  $(H_{1,1} - E_j) c_1^{[j]} + H_{1,2} c_2^{[j]} = 0$

$$\frac{c_1^{[j]}}{c_2^{[j]}} = \frac{H_{1,2}}{E_j - H_{1,1}}$$

if we rewrite eq. (3.3) as

$$E_j = \frac{1}{2} \left[ H_{1,1} + H_{2,2} \mp (H_{2,2} - H_{1,1}) \sqrt{1 + \frac{4|H_{1,2}|^2}{\delta^2}} \right]$$

we obtain

$$\frac{c_1^{[j]}}{c_2^{[j]}} = \frac{2H_{1,2}}{H_{2,2} - H_{1,1}} \left[ 1 \mp \sqrt{1 + \frac{4|H_{1,2}|^2}{\delta^2}} \right]^{-1} = -\tan \beta \left( 1 \mp \sqrt{1 + \tan^2 \beta} \right)$$

This ratio can be also expressed as

$$\begin{aligned}\frac{c_1^{[j]}}{c_2^{[j]}} &= \frac{-\tan \beta}{1 \mp \frac{1}{\cos \beta}} = \frac{-\sin \beta}{\cos \beta \mp 1} \\ &= \frac{-2 \sin \frac{\beta}{2} \cos \frac{\beta}{2}}{\cos^2 \frac{\beta}{2} - \sin^2 \frac{\beta}{2} \mp (\cos^2 \frac{\beta}{2} + \sin^2 \frac{\beta}{2})} = \begin{cases} \cot \frac{\beta}{2} \\ -\tan \frac{\beta}{2} \end{cases}\end{aligned}$$

Thus,

$$\begin{aligned}|\Psi^{[1]}\rangle &= \cos \frac{\beta}{2} |\Psi_1^0\rangle + \sin \frac{\beta}{2} |\Psi_2^0\rangle \\ |\Psi^{[2]}\rangle &= -\sin \frac{\beta}{2} |\Psi_1^0\rangle + \cos \frac{\beta}{2} |\Psi_2^0\rangle\end{aligned}$$

If the **interaction** between the energy levels is relatively small, that is  $|H_{1,2}| \ll \delta$ ,  $\beta \simeq 0$

$$E_{1,2} = \frac{H_{1,1} + H_{2,2}}{2} \pm \frac{\delta}{2} \sqrt{1 + \frac{4|H_{1,2}|^2}{\delta^2}}$$

Thus

$$E_1 \simeq H_{1,1} + \frac{|H_{1,2}|^2}{\delta}$$

and

$$E_2 \simeq H_{2,2} - \frac{|H_{1,2}|^2}{\delta}$$

and the perturbed energy levels are close to the unperturbed ones. The states will also be approximatively the unperturbed ones

$$\begin{aligned}|\Psi^{[1]}\rangle &\simeq |\Psi_1^0\rangle \\ |\Psi^{[2]}\rangle &\simeq |\Psi_2^0\rangle.\end{aligned}$$

If the **interaction** between the energy levels relatively strong, that is  $|H_{1,2}| \gg \delta$ ,  $\beta \simeq \frac{\pi}{2}$

$$E_{1,2} = \frac{H_{1,1} + H_{2,2}}{2} \pm \sqrt{\frac{\delta^2}{4} + |H_{1,2}|^2} \simeq \frac{H_{1,1} + H_{2,2}}{2} \pm \left( |H_{1,2}| + \frac{\delta^2}{8|H_{1,2}|} \right)$$

and

$$\begin{aligned}|\Psi^{[1]}\rangle &\simeq \frac{\sqrt{2}}{2} |\Psi_1^0\rangle + \frac{\sqrt{2}}{2} |\Psi_2^0\rangle \\ |\Psi^{[2]}\rangle &\simeq -\frac{\sqrt{2}}{2} |\Psi_1^0\rangle + \frac{\sqrt{2}}{2} |\Psi_2^0\rangle.\end{aligned}$$

In this case we observe that if  $\delta \simeq 0$ , then  $E_1 - E_2 \simeq 2|H_{1,2}|$ . If the unperturbed energy levels were degenerated, then the perturbed energy levels will no longer be. This phenomena is called *level repulsion*.

We can also emphasise this phenomena by considering an arbitrary  $2 \times 2$  hermitian matrix and computing its eigenvalues.

$$H = \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{1,2}^* & H_{2,2} \end{pmatrix}$$

The eigenvalues will be given by eq. (3.3).

The first case,  $|H_{1,2}| \ll \delta$ , can be viewed as the case when the off-diagonal elements are small and the matrix can be approximated with a diagonal matrix. In this case we expect that the energy levels will be close to the diagonal levels or the unperturbed levels. The perturbed eigenstates will also be approximatively equal with the unperturbed eigenstates.

When we say that the levels are weakly coupled we refer to the fact that when we expand the perturbed states in the basis given by the unperturbed states we will obtain states that can be approximated to be orthogonal, since they are approximated by the unperturbed eigenstates. Thus we can say that the energy levels are independent of each other.

The level repulsion emphasised by the second case,  $|H_{1,2}| \gg \delta$ , corresponds to the case when the off-diagonal elements are significant. When we expand the perturbed eigenstates in the basis given by the unperturbed eigenstates we will have significant components from each element in the basis and the new states will be a mixture of the unperturbed states. We can say that the energy levels are no longer independent and this is what is meant by saying that the interaction between the energy levels is strong. In the case when the unperturbed energy levels are degenerated, the difference between the perturbed energy levels is given by  $2|H_{1,2}|$ , which is a measure of the mixing of the unperturbed states.

### 3.4 Probability notions

**Definition** (Joint probability). Given two events  $A$  and  $B$ , their joint probability,  $P(A \cap B)$ , is the probability of the two events to occur simultaneously.

**Definition** (Conditional probability). The conditional probability of an event  $A$  given an event  $B$  with  $P(B) > 0$ , denoted  $P(A | B)$  is given by

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

The joint probability of  $A$  and  $B$  can be expressed as  $P(A \cap B) = P(A | B)P(B)$ .

**Definition** (Continuous random variable). A continuous random variable is a function from the set of all outcomes to the set of real numbers  $X : \Omega \rightarrow (R)$  such that

$$P(a \leq X(\omega) \leq b) = \int_a^b f(x) dx,$$

where  $f(x) \geq 0$  and  $\int_{-\infty}^{+\infty} f(x) dx = 1$ .

The above function  $f$  is called the *probability density function*. The probability for a continuous random variable  $X$  to take a value in the infinitesimal interval of length  $dx$  is given by

$$P(x \in [x, x + dx]) = \int_x^{x+dx} f(z) dz \approx f(x) dx \quad (3.4)$$

## 3.5 Nearest neighbour distributions

Nearest neighbour spacing distributions show how the differences between consecutive energy levels fluctuate around the average. In order to better understand this concept we shall begin with the simpler case of real random numbers.

### 3.5.1 The nearest neighbour spacing distribution of random numbers

We consider a sequence of uniformly distributed, ordered, real, random numbers.

We define the *spacing* of an ordered sequence as the sequence of differences between consecutive elements. The *average spacing* of the sequence is given by  $\frac{E_n - E_0}{n}$ .

discrete vs continuous random variables?

For an interval of length  $s$ ,  $s > 0$ , we will denote with  $P(n \in s)$  the probability for the interval to contain  $n$  numbers and with  $P(n \in ds | m \in s)$  the conditional probability for the interval of length  $ds$  to contain  $n$  numbers given that the interval of length  $s$  contains  $m$  numbers.

If  $E$  is a given number in the sequence, we are interested in the probability  $P(s)ds$  to have the next number between  $E + s$  and  $E + s + ds$ . Since we are interested in the next number after  $E$ , we know that in the interval of length  $s$  there is no other number and the next number is somewhere in the infinitesimal interval  $ds$ . Thus the joint probability of the events  $1 \in ds$  and  $0 \in s$  is given by:

$$P(s) ds = P(1 \in ds | 0 \in s) P(0 \in s). \quad (3.5)$$

Since random numbers are not correlated, the probability of a random number to be found in the interval  $ds$  does not depend on the number of random numbers in  $s$ , so

$$P(1 \in ds \mid 0 \in s) = P(1 \in ds).$$

The random numbers are uniformly distributed, so their probability density function  $f$  is a constant. Hence, according to eq. (3.4), the probability of finding a number in the interval of length  $ds$  is given by

$$P(1 \in ds) \equiv P(1 \in [0, 0 + ds]) \approx f(s) ds \sim ds.$$

If we denote the constant probability density function with  $a$

$$P(s) ds = a ds P(0 \in s).$$

$P(0 \in s)$  can be expressed using the complementary probability as  $1 - \int_0^s P(s') ds'$ . Now we can express  $P(s) ds$  as follows:

$$P(s) ds = a ds \left( 1 - \int_0^s P(s') ds' \right).$$

In order to differentiate with respect to  $s$ , we will use the Leibniz rule for differentiating integrals, namely

$$\frac{d}{dx} \int_{G(x)}^{H(x)} F(x, t) dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x} dt + F(x, H(x)) \frac{dH}{dx} - F(x, G(x)) \frac{dG}{dx} \quad (3.6)$$

Using this rule, we obtain

$$\frac{d}{ds} P(s) = -aP(s).$$

This differential equation can be solved by separation of variables, yielding

$$P(s) = \mathcal{C} e^{-as}$$

In order to determine the constant  $\mathcal{C}$ , we use the normalisation condition for the probability density function

$$\int_{-\infty}^{\infty} P(s) ds = 1$$

Since  $s > 0$ , this reduces to

$$\int_0^{\infty} P(s) ds = \int_0^{\infty} \mathcal{C} e^{-as} ds = -\frac{\mathcal{C}}{a} e^{-as} \Big|_0^{\infty} = \frac{\mathcal{C}}{a}$$



Thus  $\mathcal{C} = a$  and  $P(s) = a e^{-as}$ . We can further simplify the formula if we set that the average spacing to unity. The average spacing is given by

$$\bar{s} = \int_0^\infty s P(s) ds = - \int_0^\infty s \frac{d}{ds} (e^{-as}) ds = \int_0^\infty e^{-as} ds = \frac{1}{a},$$

so setting it to unity results in  $a = 1$ . Thus the probability density function becomes

$$P(s) = e^{-s}. \quad (3.7)$$

This function is known as the *Poisson distribution*.

### 3.5.2 The Wigner distribution

We will now consider a more complicated situation, by considering the probability density function for the sequence of random numbers to be arbitrary.

We can start from eq. (3.5) since the discussion up to that point did not include any details related to the distribution of the random numbers.

In this case  $P(1 \in ds | 0 \in s) = f_{1,0}(s) ds$ , where  $f_{n,m}(s)$  is function which describes how the probability of having  $n$  numbers in  $ds$  is influenced by the  $m$  numbers in  $s$ . Thus

$$P(s) ds = f_{1,0}(s) ds \left( 1 - \int_0^s P(s') ds' \right)$$

By solving ???? this integral equation, we obtain the solution

$$P(s) = \mathcal{C} f_{1,0}(s) \exp \left( - \int_0^s f_{1,0}(x) dx \right)$$

We observe that if we take the probability density function constant,  $f_{1,0}(s) = \frac{1}{a}$ , we obtain the above case of the Poisson distribution. In we consider the probability density function to be linear (why?),  $f_{1,0}(s) = \alpha s$  we obtain

$$P(s) = \mathcal{C} \alpha s \exp \left( - \alpha \frac{s^2}{2} \right)$$

From the normalisation condition we obtain

$$\int_0^\infty P(s) ds = \mathcal{C} = 1$$

The average spacing is given by

$$\bar{s} = \int_0^\infty s P(s) ds = \alpha \int_0^\infty s^2 \exp \left( - \alpha \frac{s^2}{2} \right) ds = \frac{1}{\sqrt{\alpha}} \sqrt{\frac{\pi}{2}}.$$

If we set the average spacing to unity, we obtain  $\alpha = \frac{\pi}{2}$  and

$$P(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right)$$

This function is known as the *Wigner distribution*.

## 3.6 From Classical Chaos to Quantum Chaos

In the terms of classical mechanics, such as the sensitivity to initial conditions quantum chaos does not exist because of the Heisenberg's uncertainty principle, the absence of trajectories and many other reasons. In turn, there are some other ways in which we can link classically chaotic systems to their quantum counterparts. These bridges between classical mechanics and quantum mechanics allows us to give a meaning to quantum chaos. (Berry  $\rightarrow$  quantum chaology)

There are two important conjectures that allow us to link classical systems and quantum systems as mentioned above.

### The Berry-Tabor conjecture (or theorem?)

This conjecture states that the quantum counterpart of a classically integrable system has a Poissonian nearest neighbour distribution.

### The Bohigas-Gianoni-Schmit conjecture

This conjecture states that the nearest neighbour distribution of a quantum system with a classically chaotic counterpart is given by the Wigner distribution.

# Chapter 4

## Procedure

### 4.1 The Hamiltonian

In this chapter we will describe how we obtained the numerical results. We begin by computing the matrix elements of the Hamiltonian in a basis given by the eigenstates of an isotropic double harmonic oscillator. In analogy with the classical case, we consider two independent quantum numbers  $n_1, n_2$  corresponding to the two orthogonal oscillating directions. We can define two number operators  $N_1, N_2$  such that  $N_1 |n_1, n_2\rangle = n_1 |n_1, n_2\rangle$  and  $N_2 |n_1, n_2\rangle = n_2 |n_1, n_2\rangle$ . We consider the creation and annihilation operators  $a_1^\dagger, a_1$  and  $a_2^\dagger, a_2$  such that  $N_1 = a_1^\dagger a_1$  and  $N_2 = a_2^\dagger a_2$ . In terms of the previously defined operators, the Hamiltonian of the isotropic double harmonic oscillator is given by

$$H_0 = \hbar\omega_0 \left( a_1^\dagger a_1 + \frac{1}{2}I + a_2^\dagger a_2 + \frac{1}{2}I \right) = \hbar\omega_0 (N_1 + N_2 + I)$$

and its eigenstates are given by

$$H_0 |n_1, n_2\rangle = \hbar\omega_0 (n_1 + n_2 + 1) |n_1, n_2\rangle$$

Since  $n_1 + n_2 = n$  can be obtained in  $\sum_{i=0}^n i = \frac{1}{2} n(n+1)$  ways, the energy levels of the isotropic double harmonic oscillator are  $\frac{1}{2} n(n+1)$  fold degenerated.

In the fundamental state  $|0, 0\rangle$ , with  $n_1 = n_2 = 0$ , the oscillator has the energy equal to  $\hbar\omega_0$ , having a *zero point motion* (**details / definition?**). Since we are interested (**why?**) in variations from the equilibrium state, we can rescale the potential energy such that the energy of the fundamental state becomes 0. Thus the new Hamiltonian will be given by

$$H_0 = A (N_1 + N_2),$$

where  $A = \hbar\omega_0$ .

We can construct the basis starting from the vacuum state  $|0, 0\rangle$  by acting with the creation operators

$$\begin{aligned} a_1^{\dagger n_1} |0, 0\rangle &= \sqrt{n_1!} |n_1, 0\rangle \\ a_2^{\dagger n_2} |0, 0\rangle &= \sqrt{n_2!} |0, n_2\rangle \end{aligned}$$

Thus, by applying the operator  $a_1^{\dagger n_1} a_2^{\dagger n_2}$  such that  $n_1 + n_2 = n$  (in order to obtain for each  $n$  all the states with the energy  $n \hbar \omega_0$  wrt vacuum) we obtain the basis elements ordered as follows:

$$|0, 0\rangle |0, 1\rangle |0, 2\rangle \cdots |0, n\rangle |1, 0\rangle |1, 1\rangle \cdots |1, n-1\rangle \cdots |i, 0\rangle |i, 1\rangle \cdots |i, n-i\rangle \cdots |n, 0\rangle.$$

For our investigations, the Hamiltonian is expressed as a function of the creation and annihilation operators up to fourth order terms as follows

$$\begin{aligned} H = A & \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right) + \frac{B}{4} \left[ \left( 3a_1^\dagger a_2^{\dagger 2} + 3a_1 a_2^2 - a_1^{\dagger 3} - a_1^3 \right) \right. \\ & + 3 \left( a_1 a_2^{\dagger 2} + a_1^\dagger a_2^2 - a_1^\dagger a_1^2 - a_1^{\dagger 2} a_1 + 2a_1 a_2^\dagger a_2 + 2a_1^\dagger a_2^\dagger a_2 \right) \Big] \\ & + \frac{D}{16} \left[ 6 \left( a_1^{\dagger 2} a_1^2 + a_2^{\dagger 2} a_2^2 \right) + 2 \left( a_1^2 a_2^{\dagger 2} + a_1^{\dagger 2} a_2^2 \right) + 8a_1^\dagger a_1 a_2^\dagger a_2 \right. \\ & + 4 \left( a_1^\dagger a_1^3 + a_1^{\dagger 3} a_1 + a_2^\dagger a_2^3 + a_2^{\dagger 3} a_2 + a_1^2 a_2^\dagger a_2 + a_1^{\dagger 2} a_2^\dagger a_2 + a_1^\dagger a_1 a_2^2 + a_1^\dagger a_1 a_2^{\dagger 2} \right) \\ & \left. + \left( a_1^{\dagger 4} + a_1^4 + a_2^{\dagger 4} + a_2^4 + 2a_1^{\dagger 2} a_2^{\dagger 2} + 2a_1^2 a_2^2 \right) \right]. \end{aligned} \tag{4.1}$$

The physical origin of this Hamiltonian is related to quadrupole vibrations of nuclear surfaces. (details for B, D?) (add citations) The energy levels will be expressed in units of harmonic oscillator energy and therefore from here on we will consider  $A = 1$ . We can obtain the eigenvalues and eigenvectors of the Hamiltonian by a diagonalization routine based on Relatively Robust Representations from Intel<sup>®</sup> Math Kernel Library used via a Python program(+ citations). Any such diagonalization method requires a truncation of the Hilbert space which induces errors concerning the eigenvalues. This errors increase as one moves to the upper limit of the energy for a fixed dimension of the Hilbert space. Indeed we expect this energies to have more important contributions from the states that were eliminated by truncation. We tested the stability of the energy levels by comparing the results obtained for different sizes of the diagonalization basis as is detailed in the next section.

## 4.2 Stability

We consider the *stable levels* to be the eigenvalues which, at a change of basis from one with a dimension of  $N$  to one with dimension  $N + \Delta N$ , do not change with more than a chosen threshold  $\delta_s$ . In the following figure we show the variation of the energy levels when the dimension increases from  $N = 120$  to  $N = 140$ .

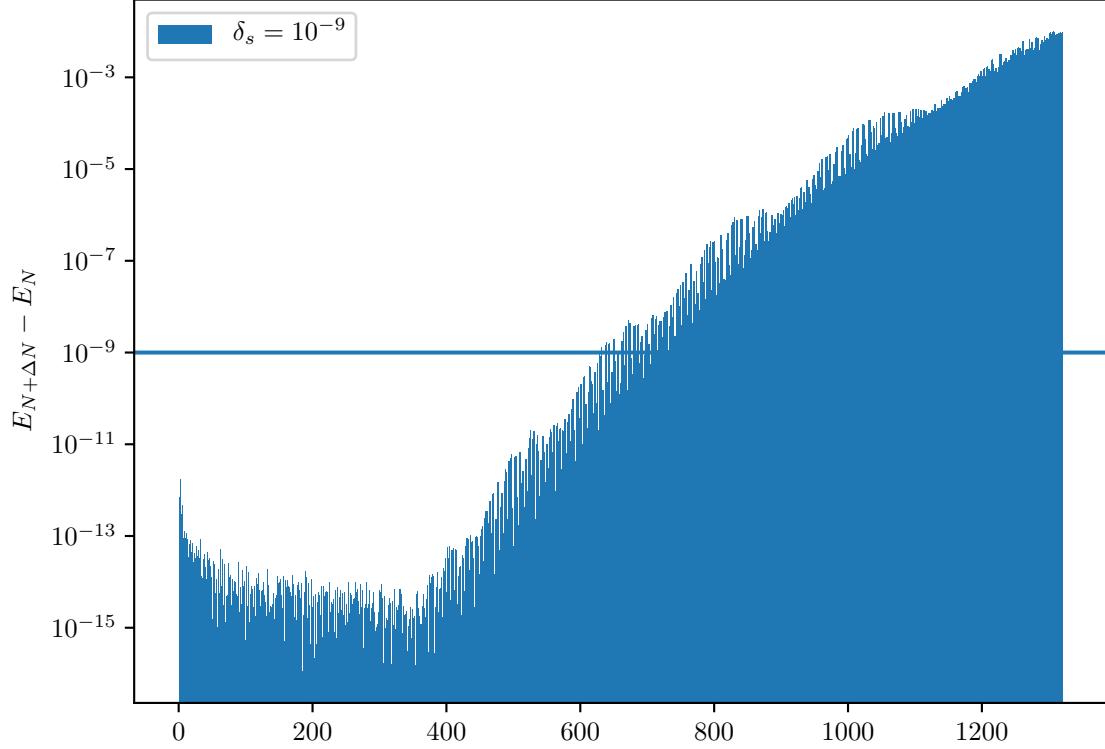


Figure 4.1:  $B = 0.2, D = 0.4, N = 120$

We can observe that the first 400–600 eigenvalues have a very good stability. Thus we can choose the stability threshold for example at  $\delta_s = 10^{-9}$ . Qualitatively the shape of this distribution does not depend on the parameters of the Hamiltonian or the dimension of the Hilbert space because it reflects the nature of the approximation as discussed previously. For example, for  $B = 0.55, D = 0.4, N = 260$  compared with  $N = 280$

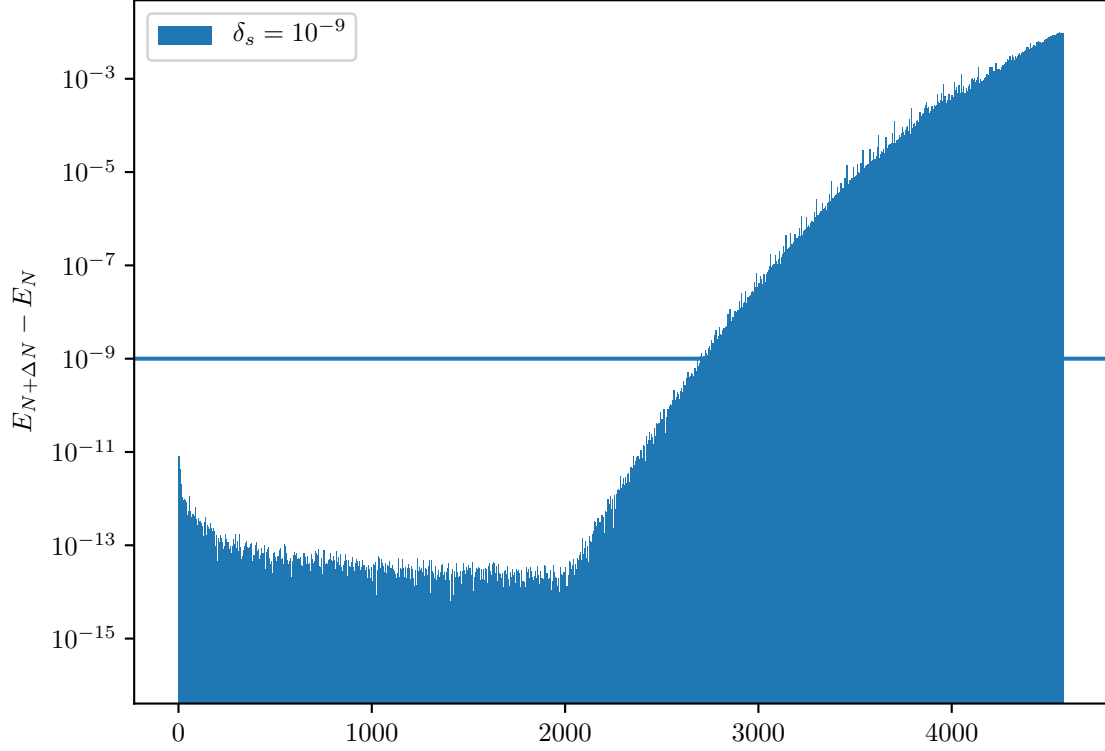


Figure 4.2:  $B = 0.55, D = 0.4, N = 260$

As expected, the number of stable levels increases with the size of the basis. Roughly, for a given basis size  $N$ , the first 7–8% levels differ with less than  $\delta_s = 10^{-9}$  when we compare with a basis of dimension  $N + \Delta N$ , with  $\Delta N = 20$ .

### 4.3 Statistics

As we mentioned in the Introduction, a spectrum can be characterised through the probability distribution of the nearest neighbour spacing. The *spacing* is defined as the difference between two consecutive energy levels. Similarly the *relative spacing* is defined as

$$s = \frac{E_{i+1} - E_i}{\overline{\Delta E}},$$

where  $\overline{\Delta E}$  is the average spacing  $\frac{E_n - E_0}{N}$ .

The *nearest neighbour spacing distributions* tell us the probability  $P(s)ds$  to find a relative spacing  $s$  when we move in the spectrum obtained by diagonalization. This probability is defined as follows

$$P(s)ds = \frac{N_{s,s+\Delta s}}{N},$$

where  $N_{s,s+\Delta s}$  is the number of levels with the relative spacing between  $s$  and  $s + \Delta s$ . We can also define a *cumulative probability distribution*,

$$I(s) = \sum_{s_i=0}^s P(s_i) \Delta s$$

### 4.3.1 Irreducible representations

In order to analyse the fluctuations of the previously obtained eigenvalues we must first take into account the symmetry of the system. The symmetries of a system can make influence the fluctuations making the given sequence of levels appear more regular than it actually is (better explanation & citation). In order to remove this influence we analyse separately each symmetry reduced subspace or equivalently each *irreducible representation* of the Hamiltonian.

If an operator  $A$  corresponds to a symmetry of the Hamiltonian, then it commutes with the Hamiltonian,  $[A, H] = 0$ . (Proof?) For the Hamiltonian in eq. (4.1) we have the following symmetries (?), which correspond to the symmetry group  $\mathcal{C}_{3v}$ . This group has 3 irreducible representations: one bi-dimensional and two uni-dimensional one symmetric and one anti-symmetric, namely  $\Gamma_b, \Gamma_s, \Gamma_a$  (notation?).

### Separating the bi-dimensional representation

The presence of the bi-dimensional representation corresponds a two-fold degeneracy which allows us to identify it by computing the differences between consecutive levels  $\Delta E = E_{i+1} - E_i$ . An other option is to use directly the relative spacing, which differs only by a constant from  $\Delta E$ , namely the average spacing. The separation of the symmetric and anti-symmetric irreducible representations will be detailed later.

In figure 4.3 we can see how  $\Delta E$  varies with the index of the levels.

Because of the finite precision of the numerical implementation, the difference between two consecutive degenerate levels might not be exactly 0, its value depending on the machine precision (as it can be seen in the above figure). To take this fact into account we will consider that the levels which have  $s$  (or  $\Delta E$ ) greater than a chosen  $\varepsilon$  belong to one of the unidimensional representations. In order to choose a suitable value for  $\varepsilon$ , we use a histogram to visualise the number of levels at different spacings (see figure 4.4a).

This bimodal shape of the histogram suggests clearly the presence of the degenerate levels well separated from the rest. For some particular values for  $B$  (such as  $B = 0.4$ ) and high values for  $N$  ( $N > 200$ ), we observed a splitting of the block corresponding to the degenerated levels in two blocks, namely one at exactly 0 and the other at low values. (see fig. 4.4b)

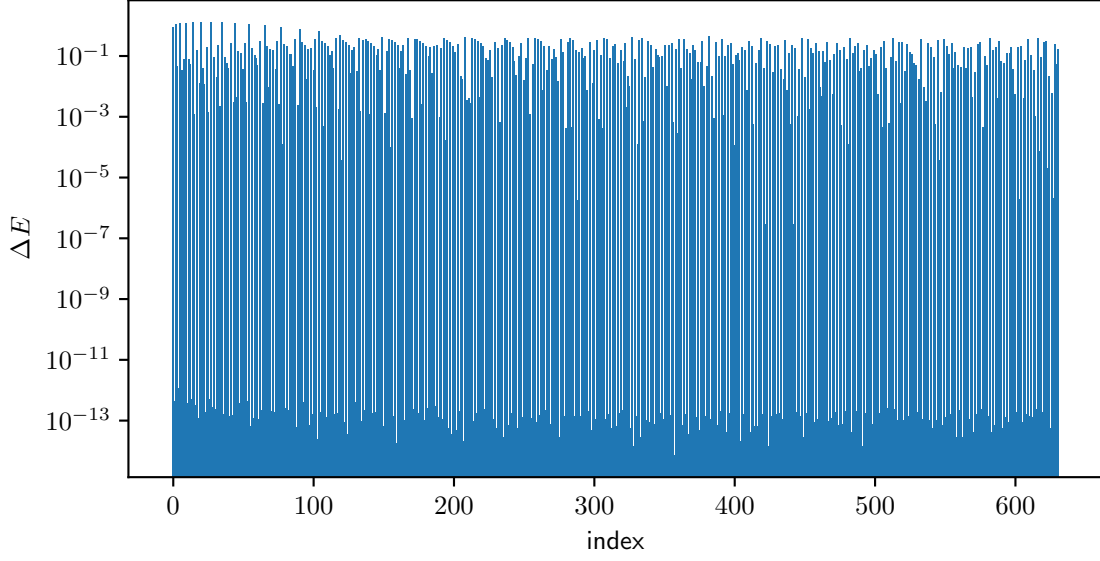
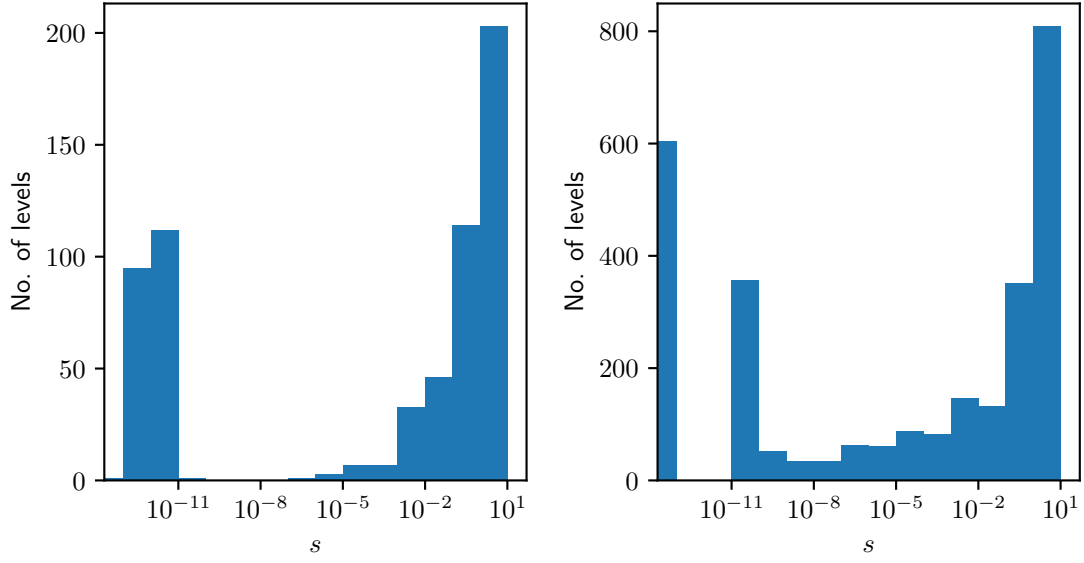


Figure 4.3:  $B = 0.2, D = 0.4, N = 120$



(a)  $B = 0.2, D = 0.4, N = 120$

(b)  $B = 0.4, D = 0.4, N = 260$

Figure 4.4: The relative spacing histograms for different parameters

If we plot the spacing as a function of the level index, we can see how each level is situated with respect to the chosen  $\varepsilon$ . Once again we can observe how the spacings corresponding to the bi-dimensional representation are separated from the rest.



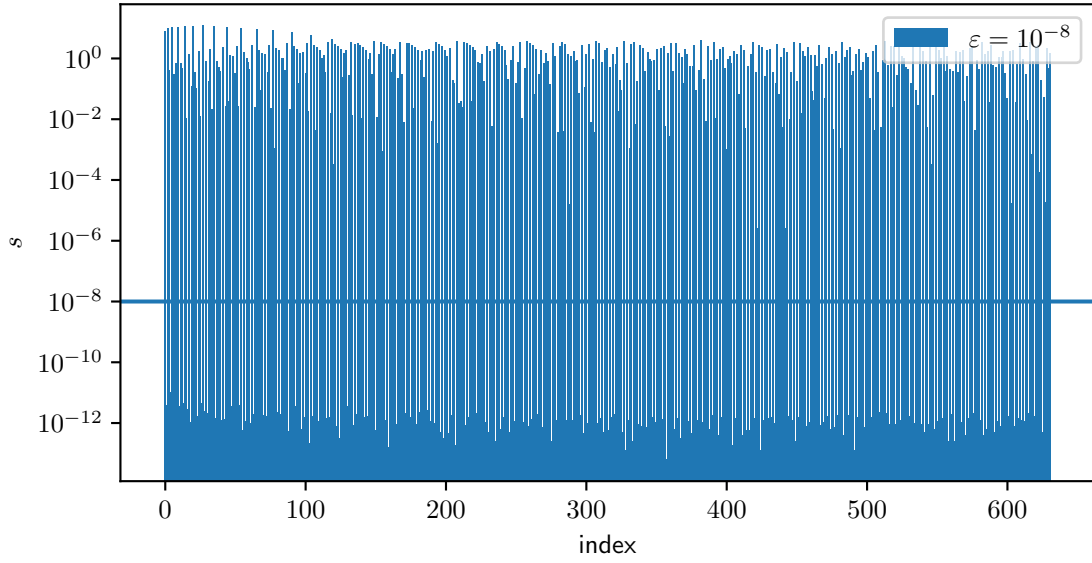


Figure 4.5:  $B = 0.2, D = 0.4, N = 120$

### Separating the unidimensional representations

In order to identify the unidimensional representations we rely on the symmetry of the potential energy operator,

...

We can observe that this operator is invariant to a reflection along the  $Ox$  axis.  $axis \rightarrow n_2$ ? For the harmonic oscillator the states with even quantum numbers have even wave functions. (proof / cite?) Thus, we can establish a correspondence (how?) between the symmetry of the representation and the parity of the quantum number  $n_2$  such that the unidimensional symmetric representation corresponds to an even  $n_2$  and the unidimensional anti-symmetric representation corresponds to an odd  $n_2$ .

The values of the quantum numbers  $n_1$  and  $n_2$  depend on the ordering of the basis. Because the diagonalisation algorithm returns the eigenvalues (and the corresponding eigenvectors) in ascending order, the initial ordering of the basis is lost. One method to approximate  $n_1$  and  $n_2$  for a given eigenvector would be to consider that their values are given by the index of the dominant coefficient.

Thus, if we have the following eigenvector

$$\begin{pmatrix} C_{0,0} \\ C_{0,1} \\ \vdots \\ C_{0,n} \\ C_{1,0} \\ \vdots \\ C_{1,n-1} \\ \vdots \\ C_{i,0} \\ \vdots \\ C_{i,n-i} \\ \vdots \\ C_{n,0} \end{pmatrix}$$

and  $C_{i,j}$  is the greatest coefficient, than we assign to this eigenvector the quantum numbers of the  $k$ -th element in the basis, where  $k$  is the index of the coefficient.

For example, for the simplified case of the isotropic double harmonic oscillator ( $B = D = 0$ ) with  $N = 3$ , the Hamiltonian is given by

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

The eigenvalues will be  $E_i = 0, 1, 1, 2, 2, 2$  with the corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Since the basis is given by

$$|0, 0\rangle |0, 1\rangle |0, 2\rangle |1, 0\rangle |1, 1\rangle |2, 0\rangle,$$

the quantum numbers for the eigenvectors will be assigned as follows

$$v_1 \equiv |0, 0\rangle \text{ since } k = 1$$

$$v_2 \equiv |0, 1\rangle \text{ since } k = 2$$

$$v_3 \equiv |1, 0\rangle \text{ since } k = 4$$

$$v_4 \equiv |0, 2\rangle \text{ since } k = 3$$

$$v_5 \equiv |2, 0\rangle \text{ since } k = 6$$

$$v_6 \equiv |1, 1\rangle \text{ since } k = 5$$

Maximum is not unique?

...

In figures 4.6 and 4.7 we can see the relative spacing for each irreducible representation as a function of index and as a histogram.

We can plot the average spacing as a function of  $B$  (see fig. 4.10) (details?)

Selection problems?

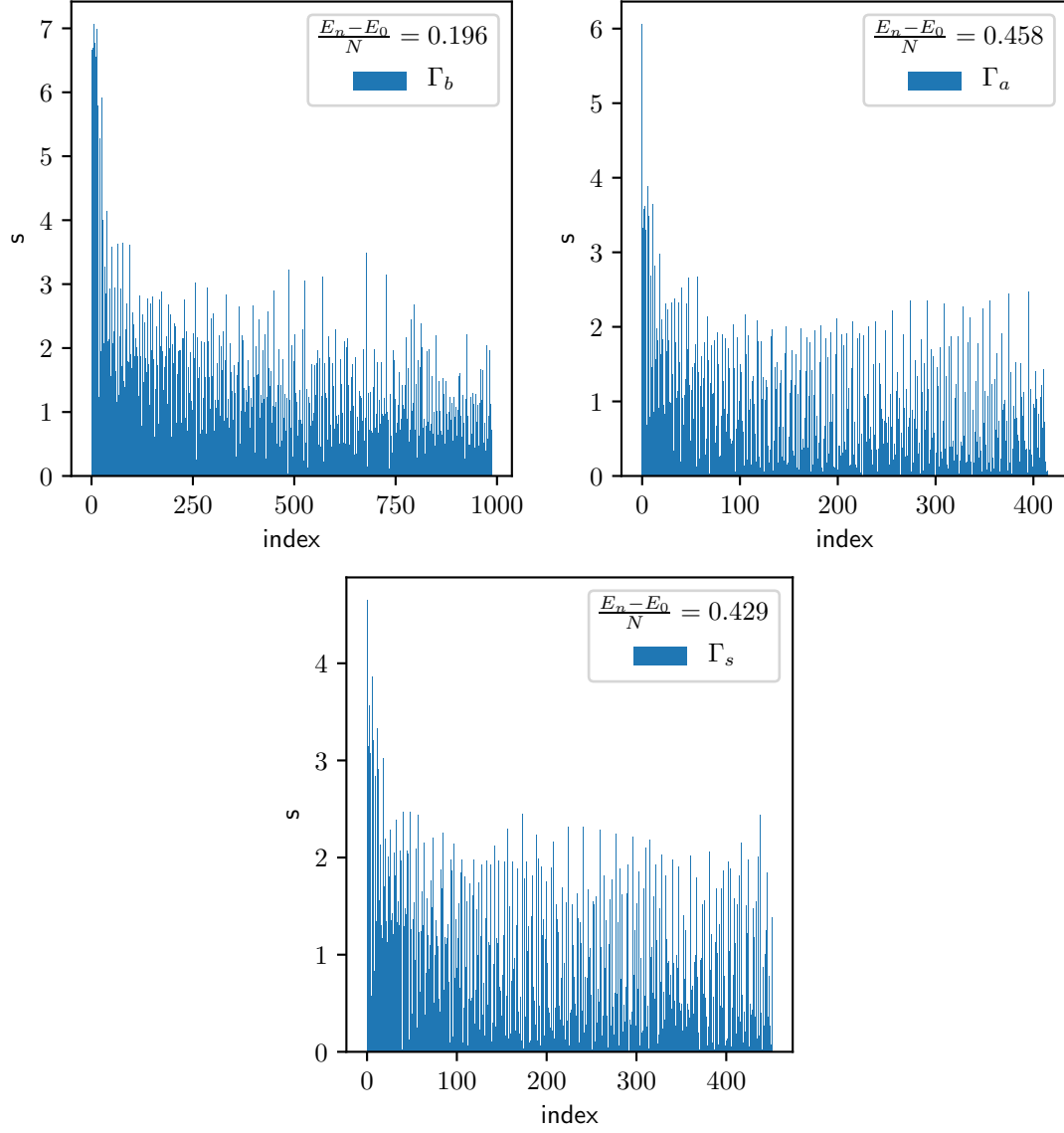


Figure 4.6: The relative spacing for each irreducible representation as a function of index for  $B = 0.2, D = 0.4, N = 260$

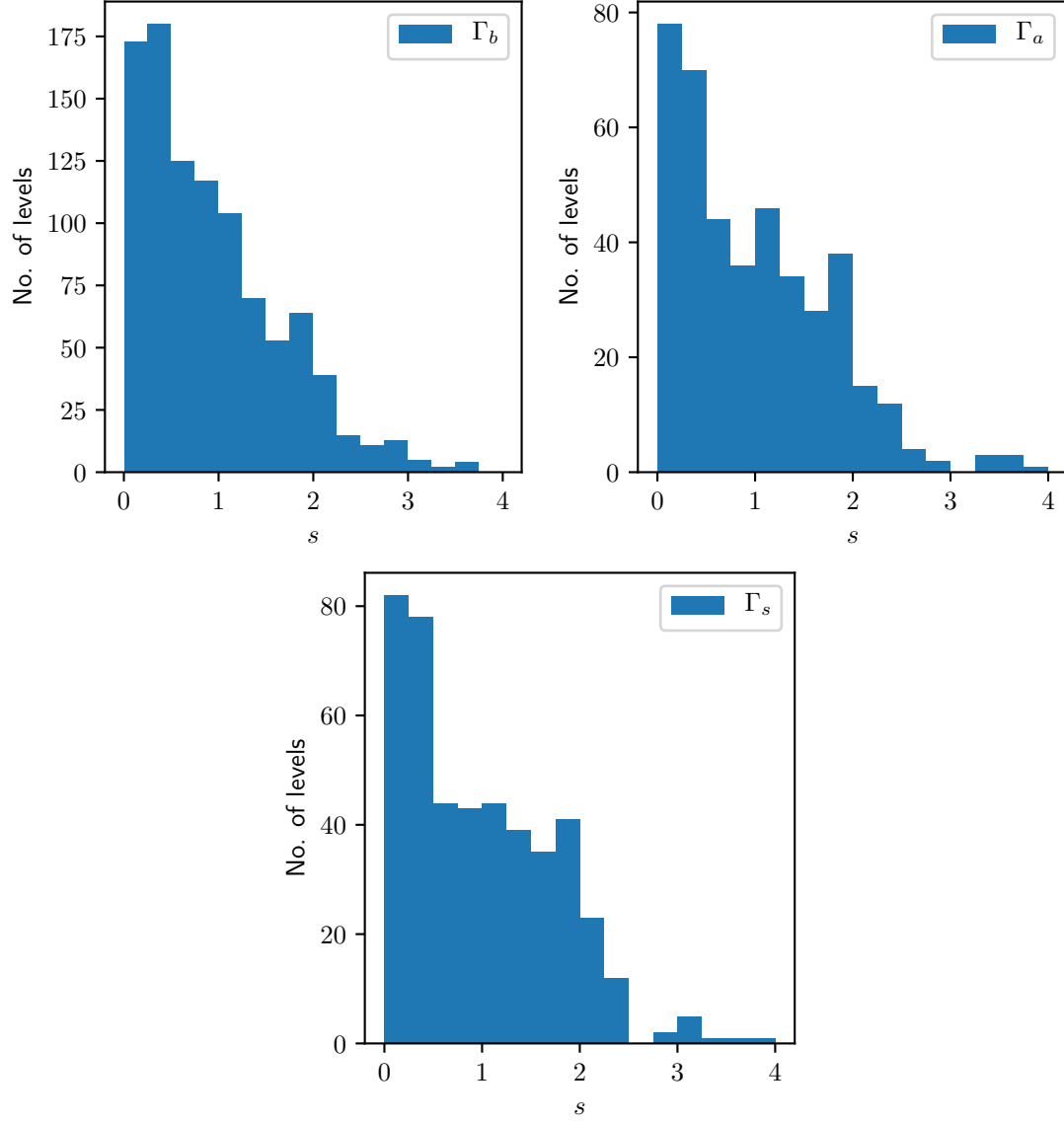


Figure 4.7: The relative spacing histogram for each irreducible representation for  $B = 0.2, D = 0.4, N = 260$

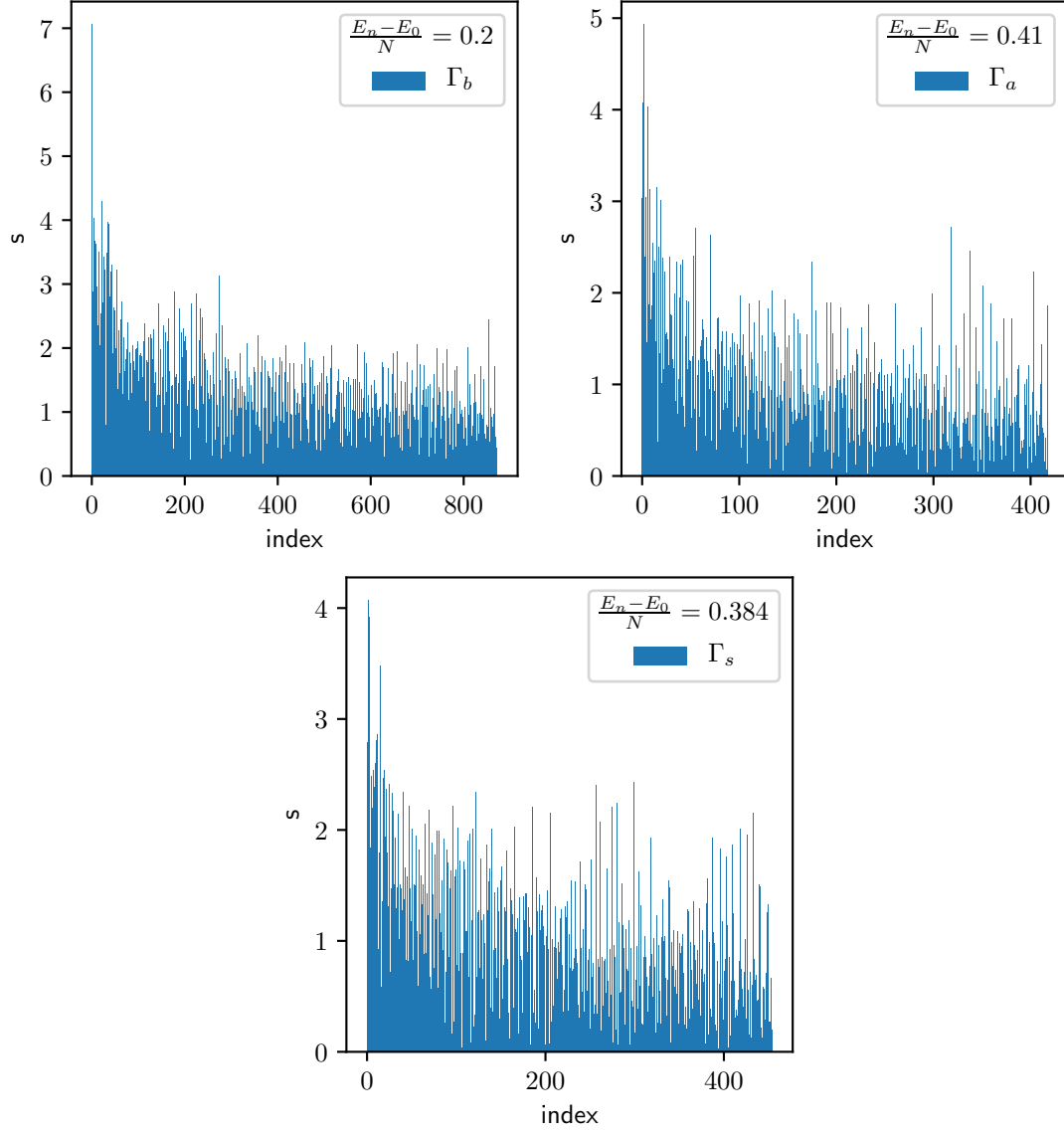


Figure 4.8: The relative spacing for each irreducible representation as a function of index for  $B = 0.63, D = 0.4, N = 260$

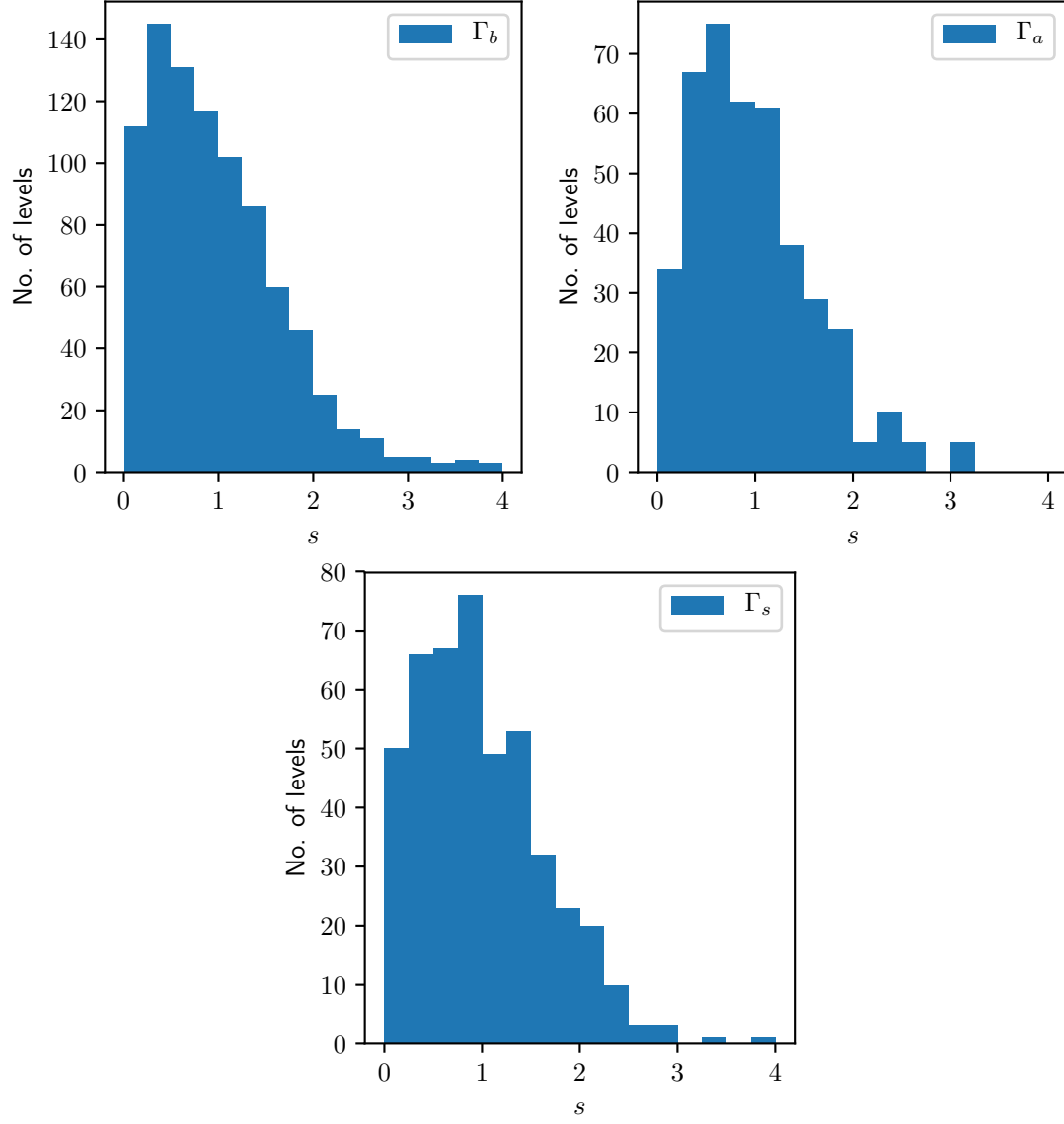


Figure 4.9: The relative spacing histogram for each irreducible representation for  $B = 0.63, D = 0.4, N = 260$

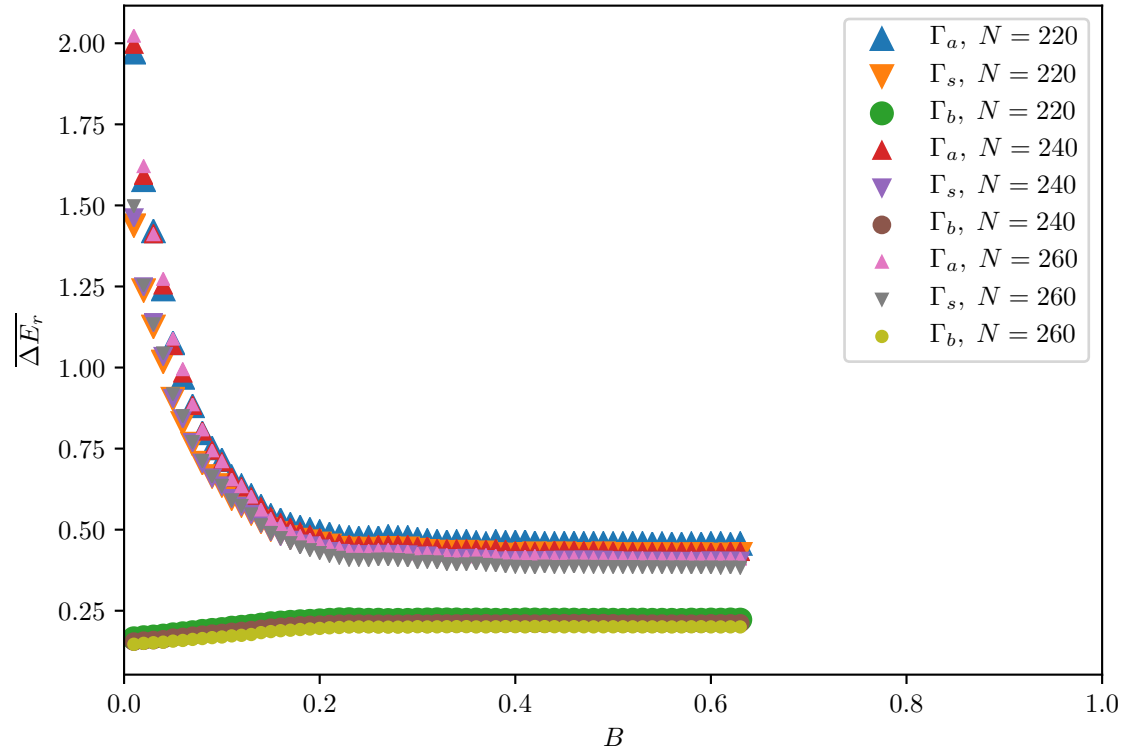


Figure 4.10:  $\overline{\Delta E}$  as a function of  $B$  for each representation



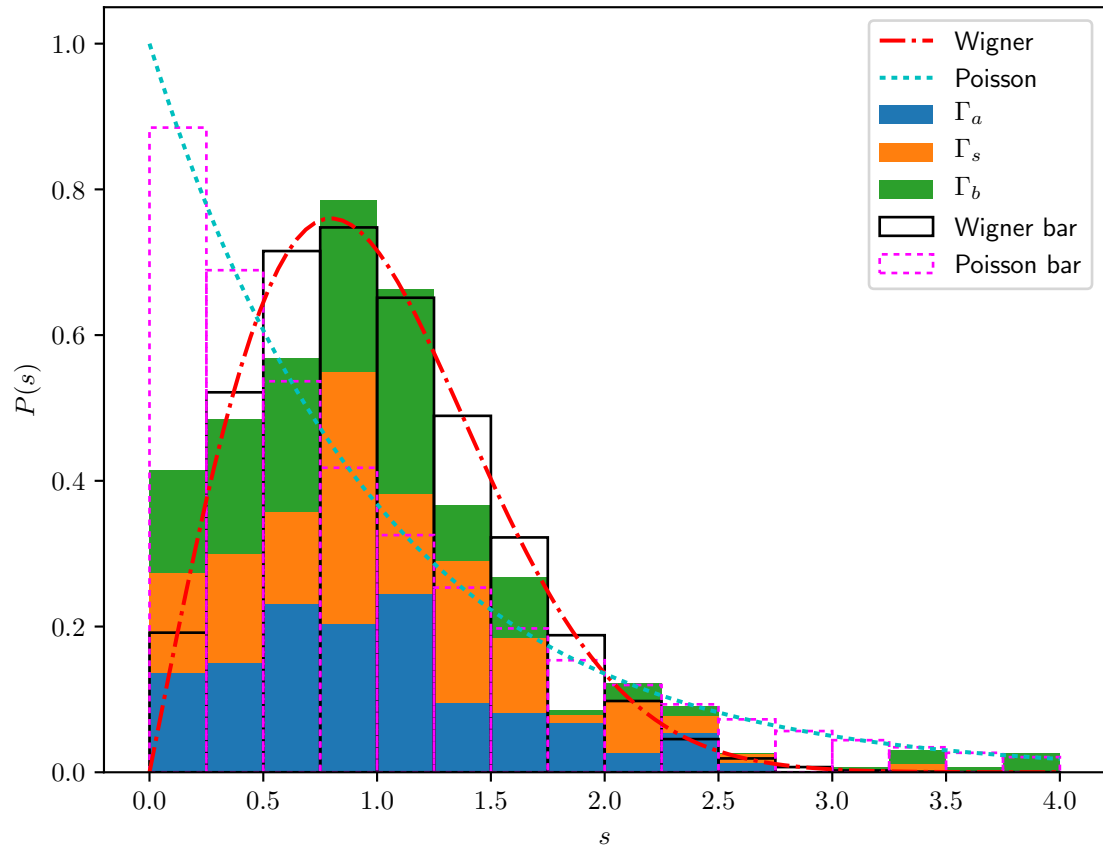


Figure 4.11:  $B = 0.2, D = 0.4, N = 120$

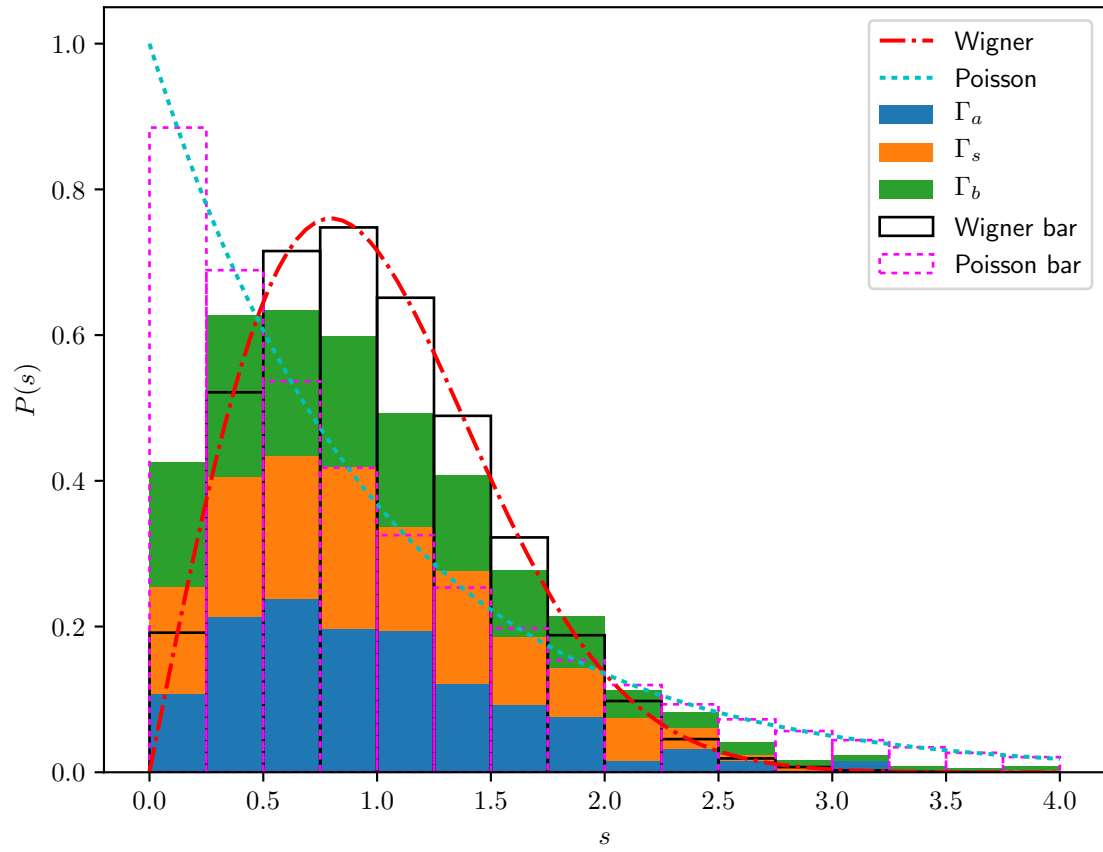


Figure 4.12:  $B = 0.63, D = 0.4, N = 260$

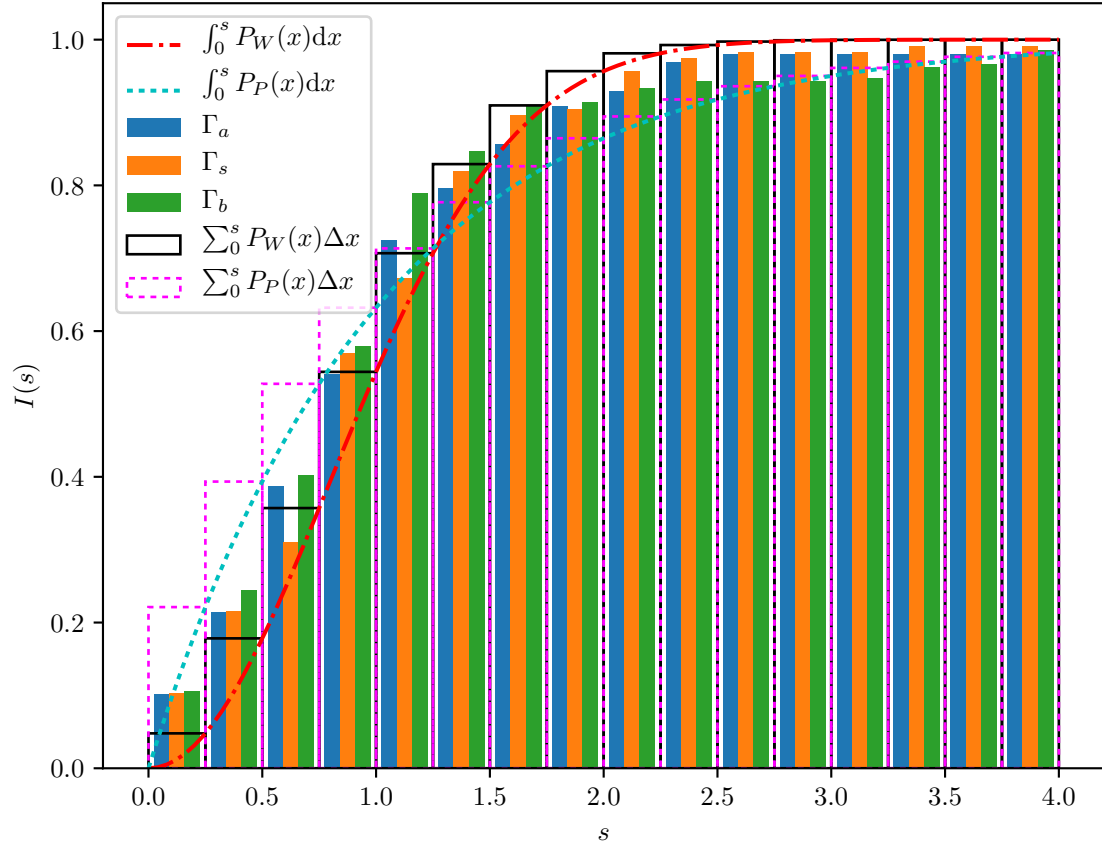


Figure 4.13:  $B = 0.2, D = 0.4, N = 120$

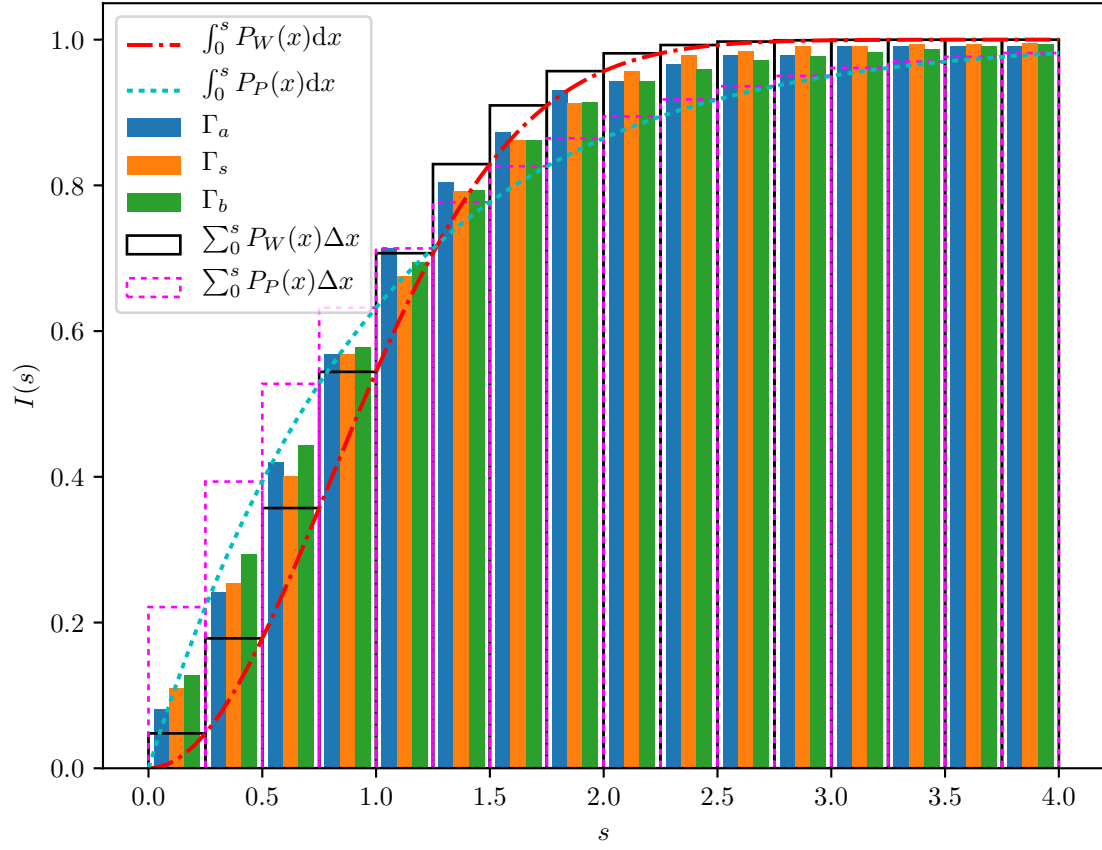


Figure 4.14:  $B = 0.63, D = 0.4, N = 260$

# Chapter 5

## Results

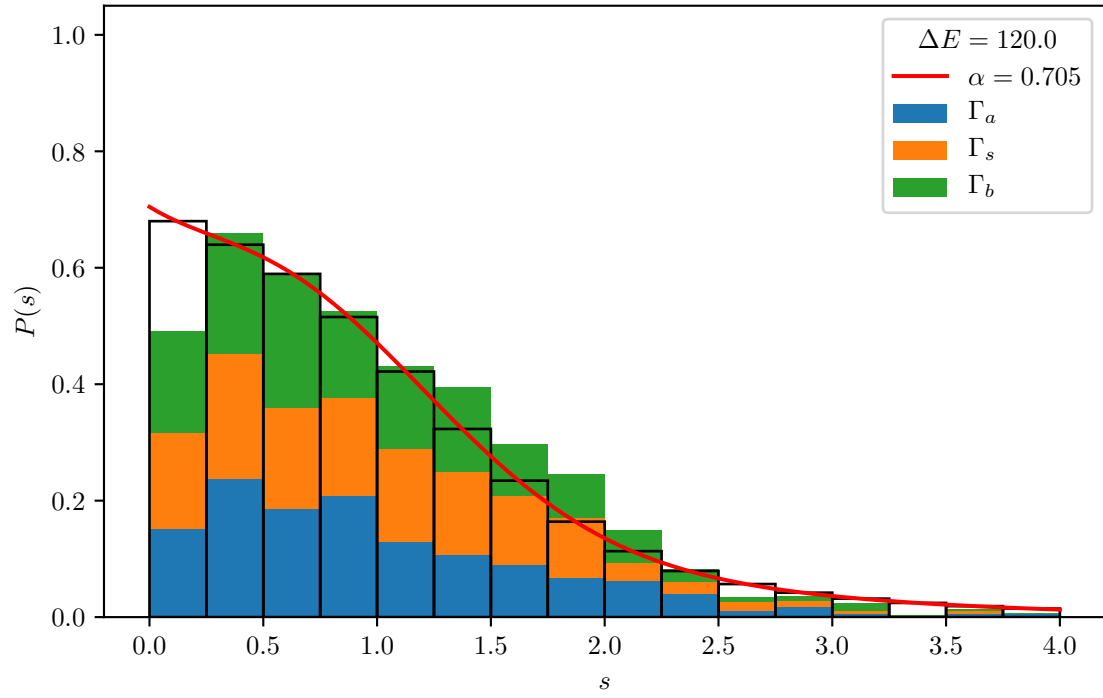
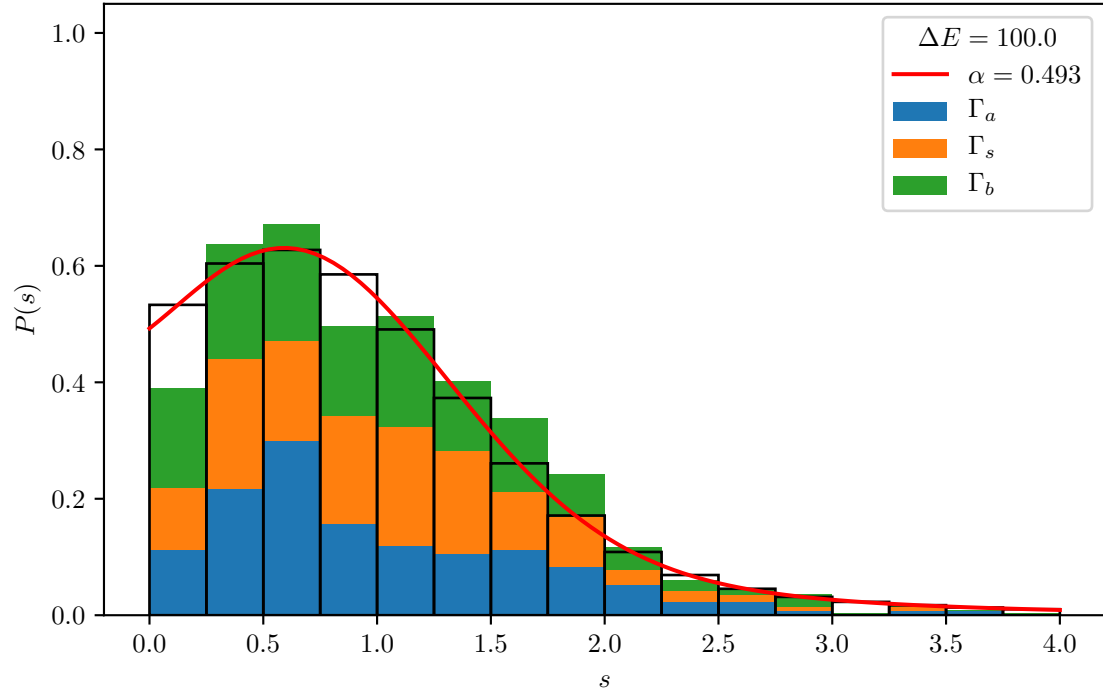


Figure 5.1:  $P(s)$  for  $B = 0.55, D = 0.4, N = 260$  and  $\Delta E_{max} = 100, 120$

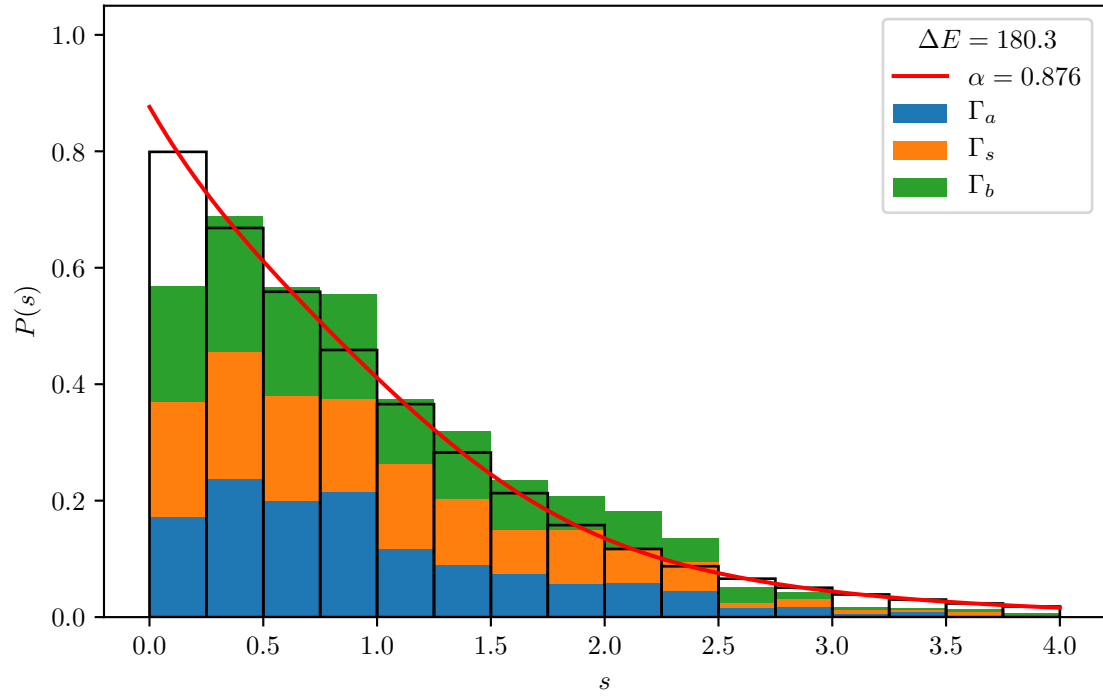
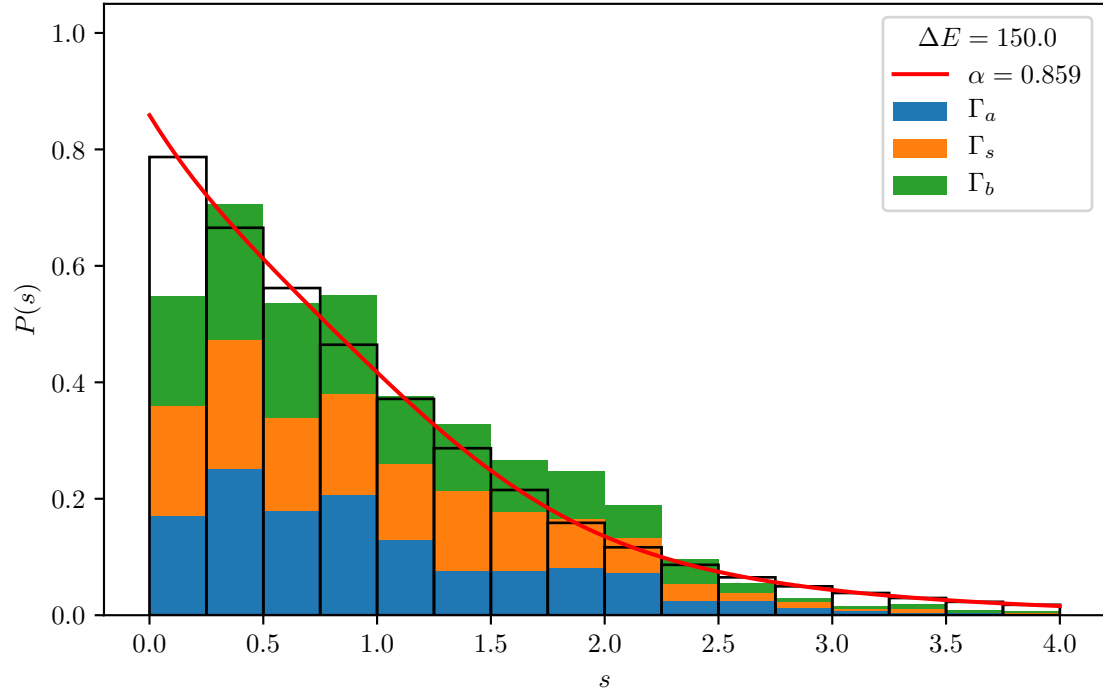


Figure 5.2:  $P(s)$  for  $B = 0.55, D = 0.4, N = 260$  and  $\Delta E_{max} = 150, 180.3$

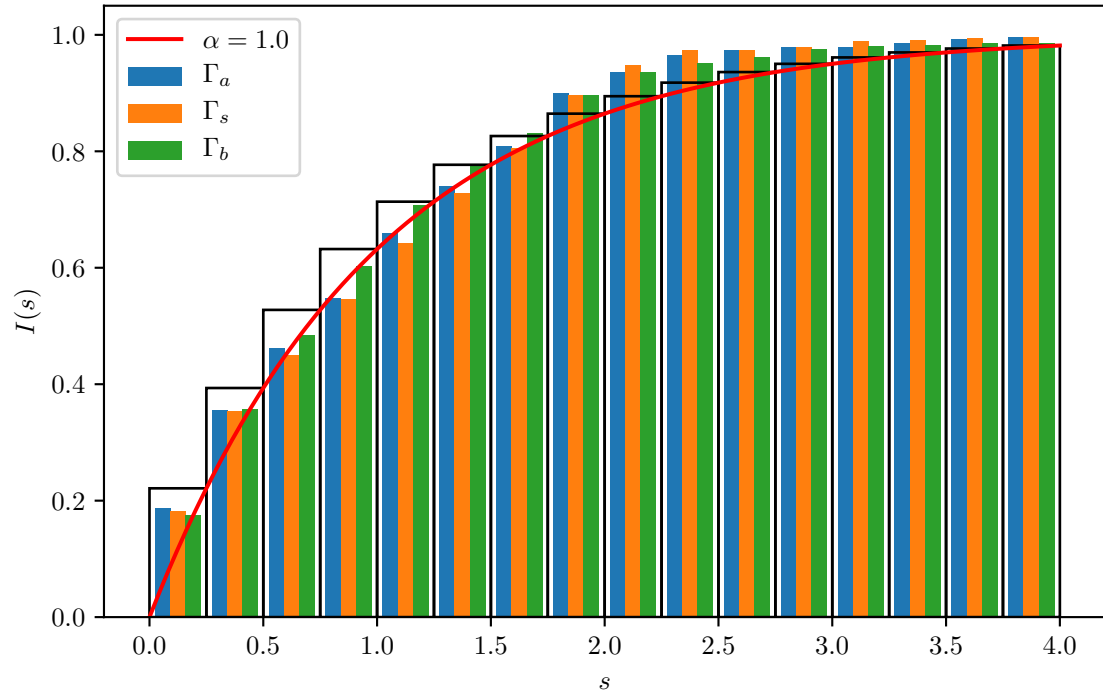
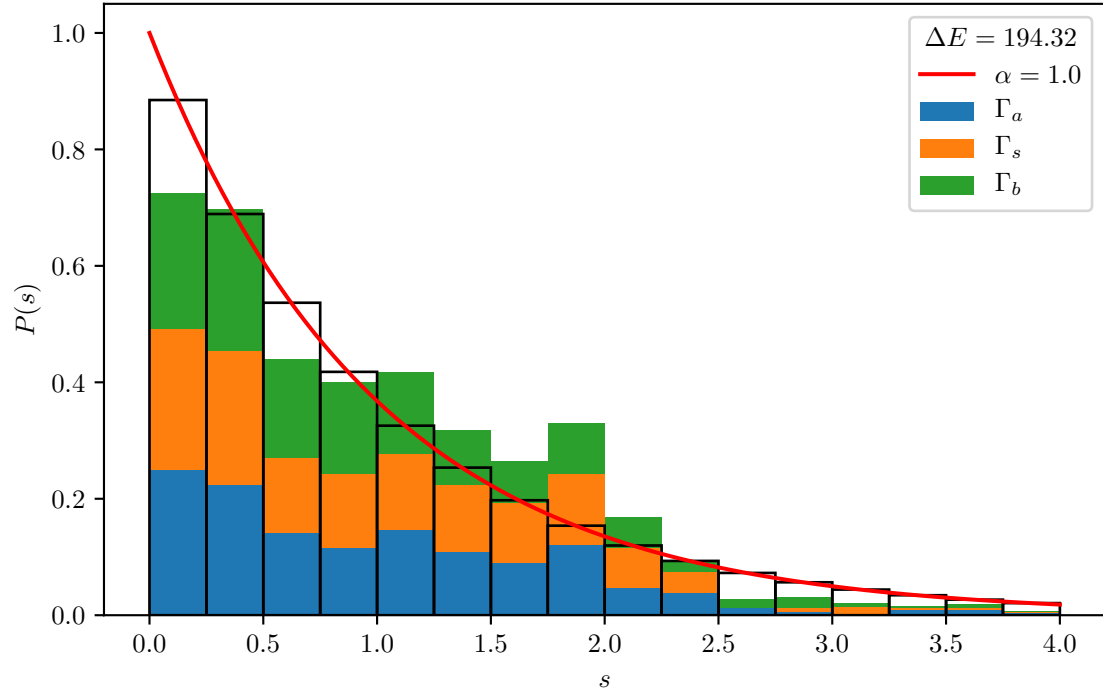


Figure 5.3:  $P(s), I(s)$  for  $B = 0.2, D = 0.4, N = 260$



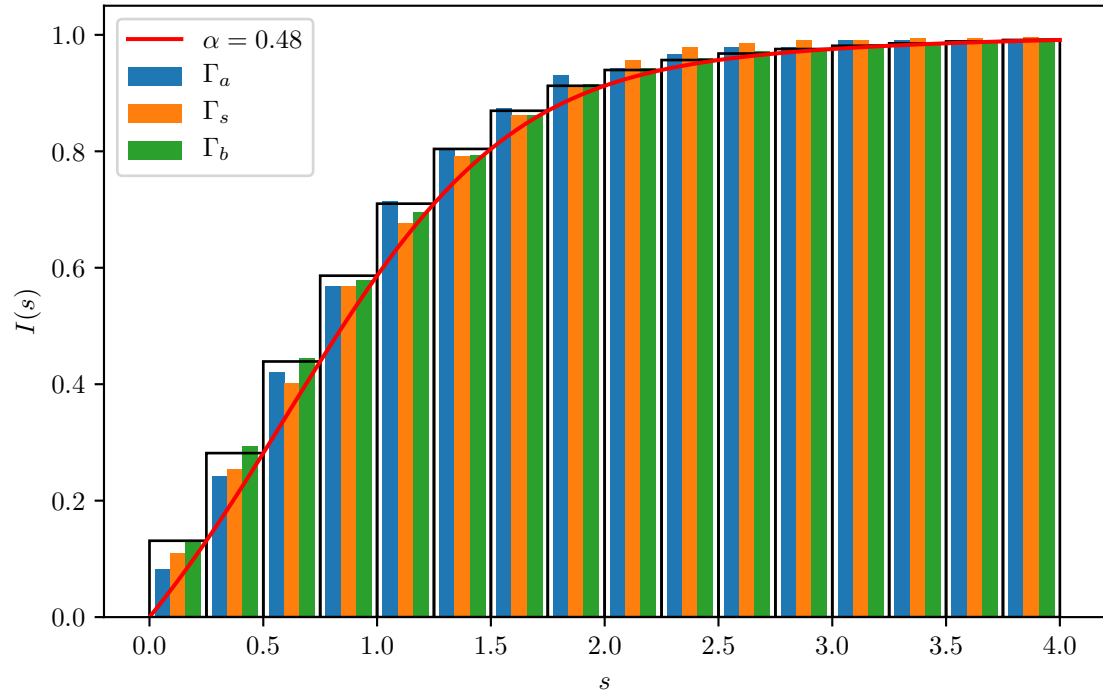
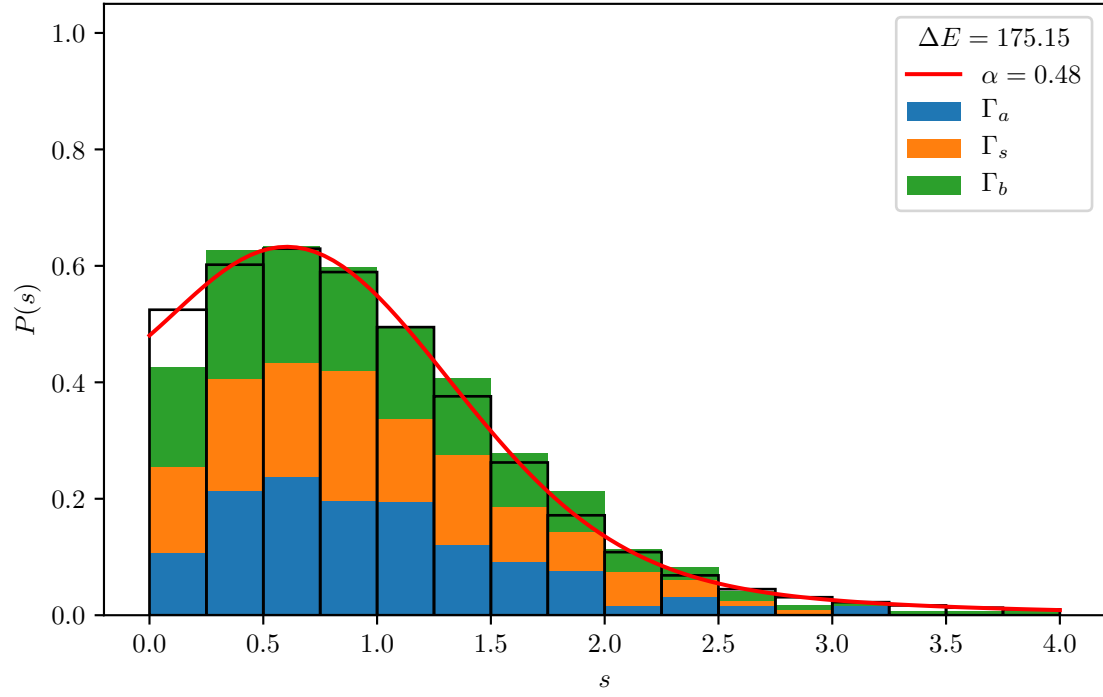


Figure 5.4:  $P(s), I(s)$  for  $B = 0.63, D = 0.4, N = 260$

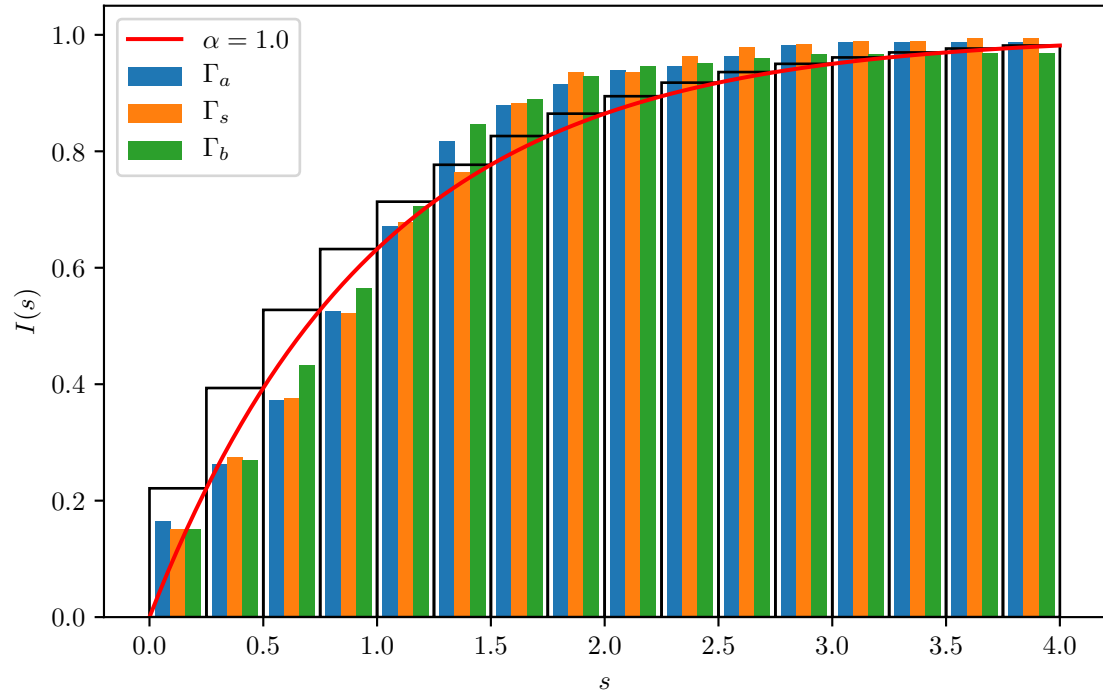
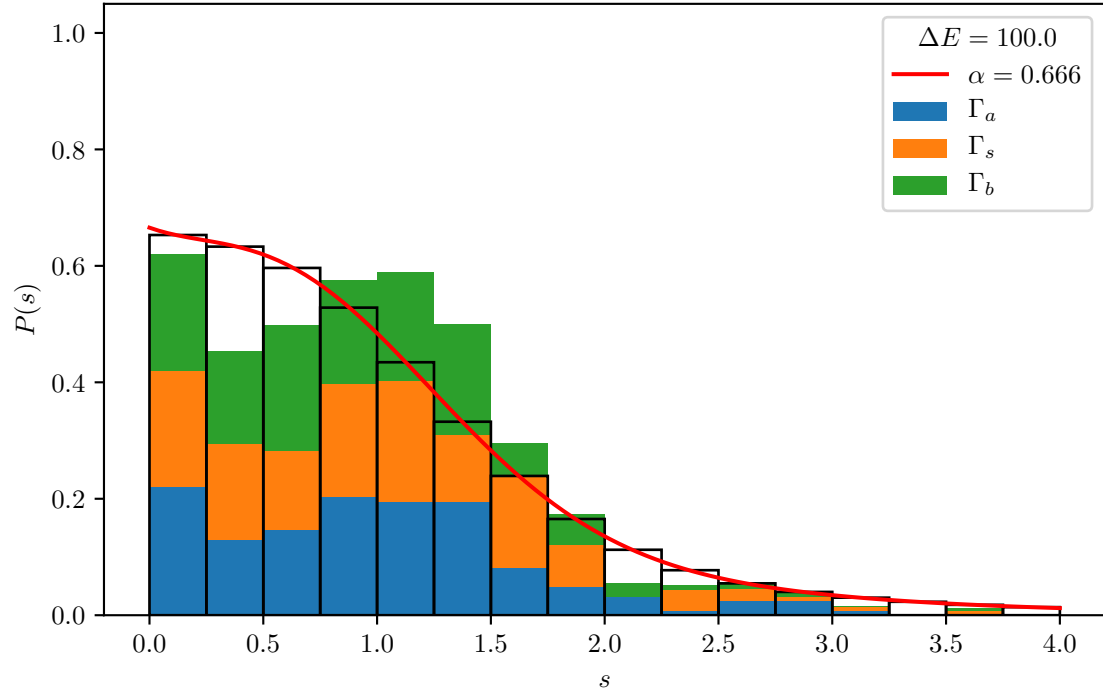


Figure 5.5:  $B = 0.2, D = 0.4, N = 260, \Delta E_{max} = 100$

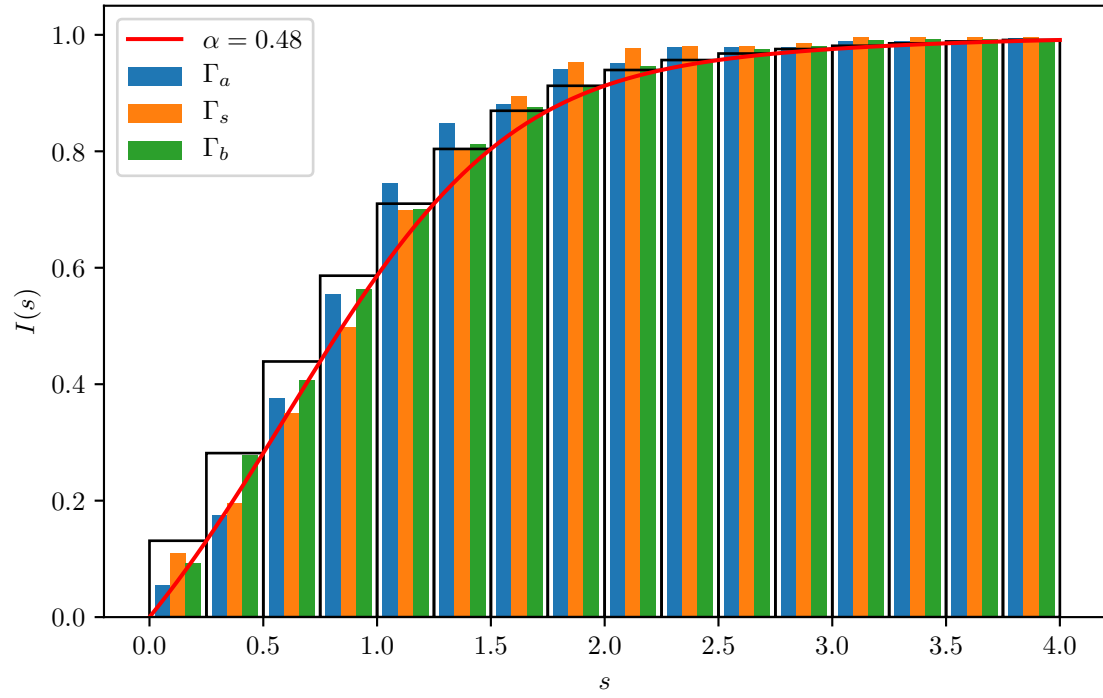
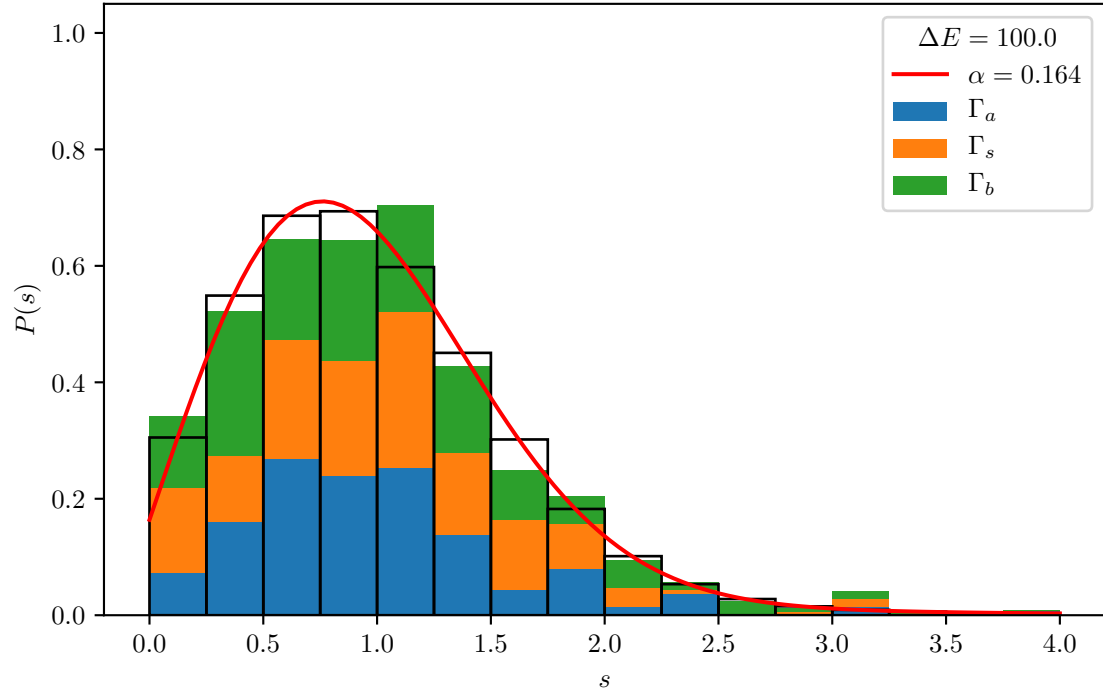


Figure 5.6:  $B = 0.63, D = 0.4, N = 260, \Delta E_{max} = 100$

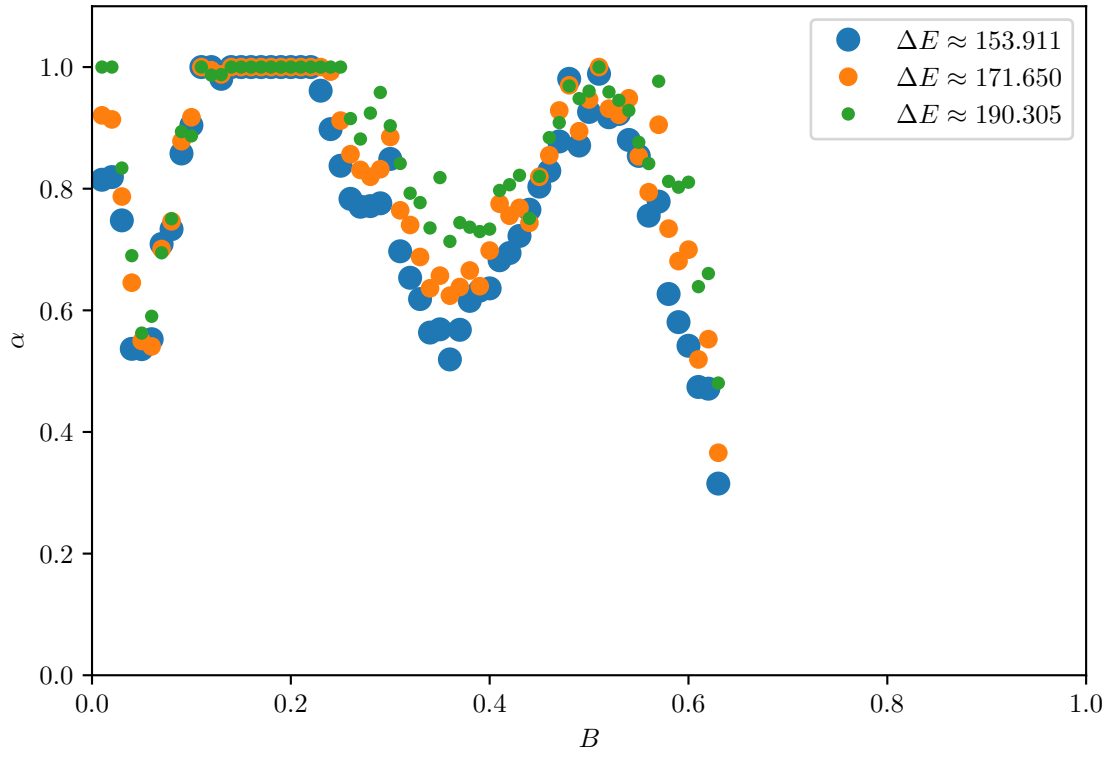


Figure 5.7:  $N = 220, 240, 260$

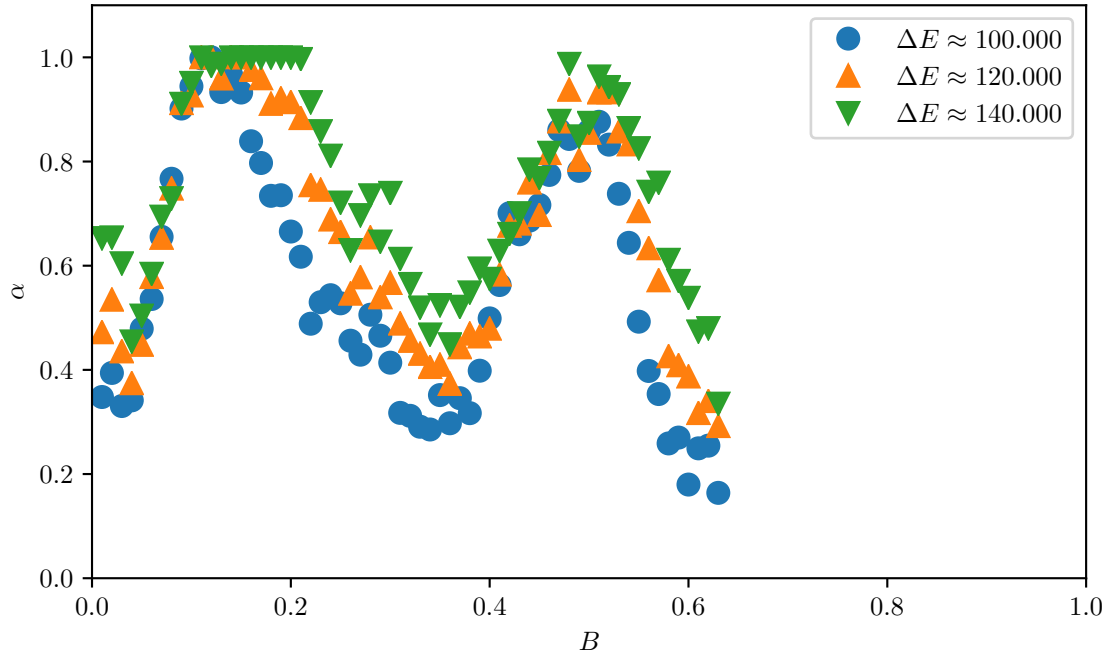


Figure 5.8:  $N = 260$ ,  $\Delta E_{max} \approx 100, 120, 140$

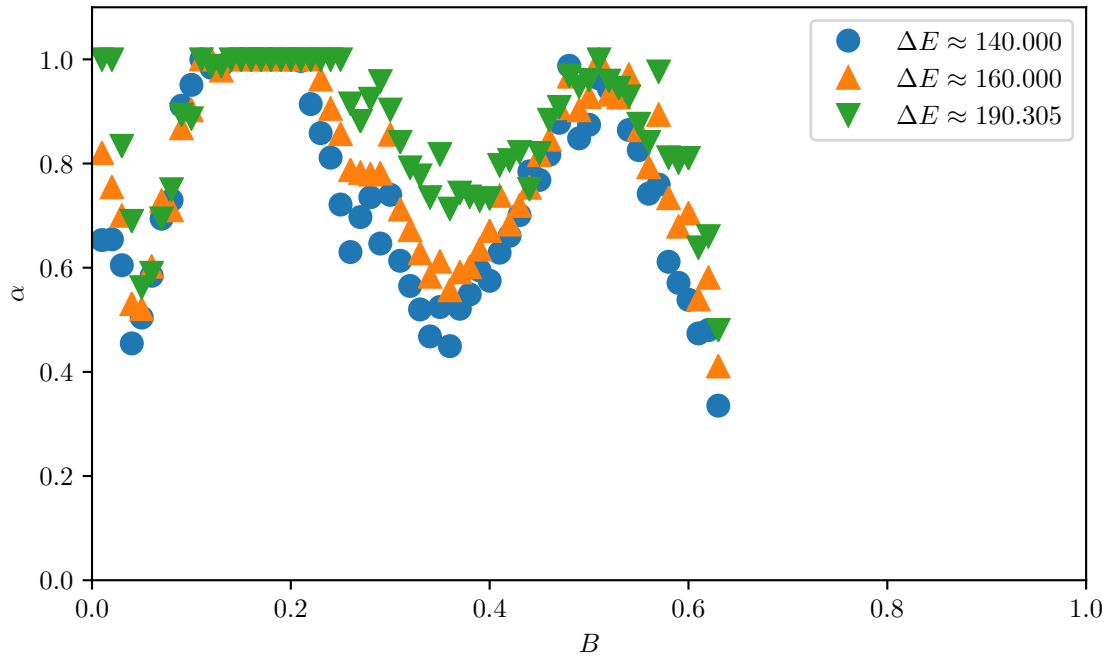


Figure 5.9:  $N = 260$ ,  $\Delta E_{max} \approx 140, 160, 190.305$

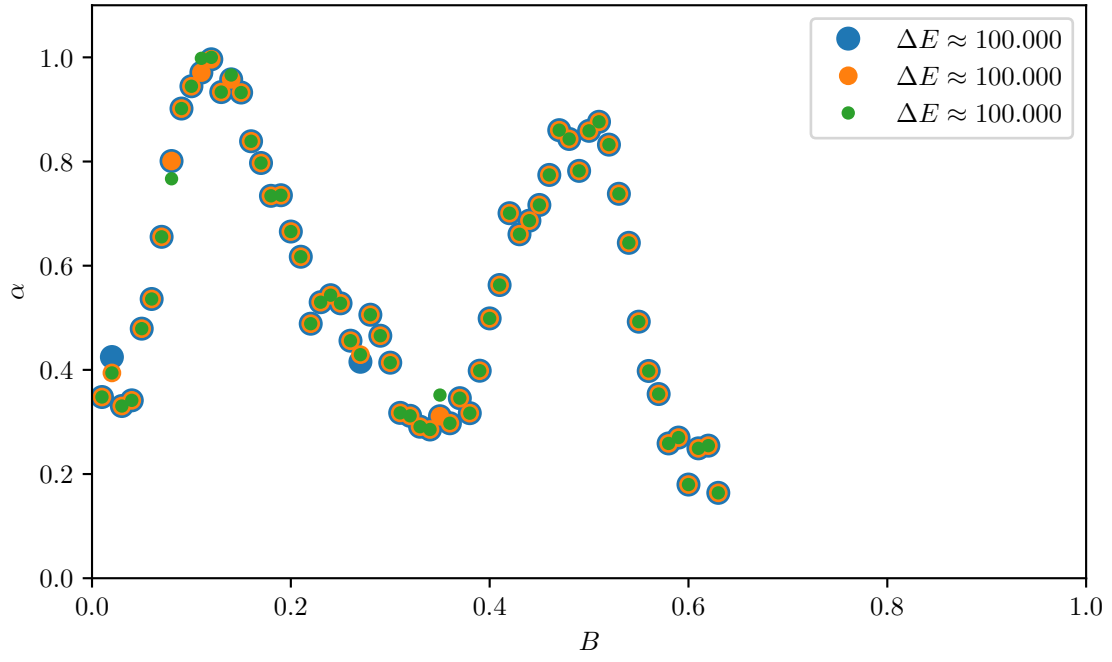


Figure 5.10:  $N = 220, 240, 260$ ,  $\Delta E_{max} \approx 100$

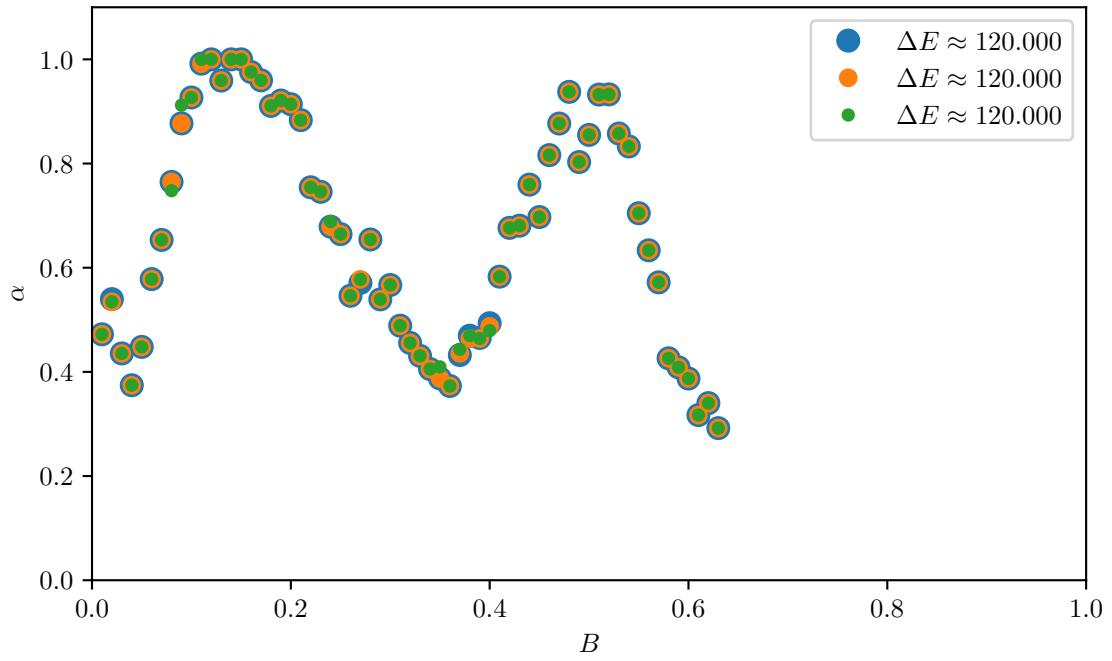


Figure 5.11:  $N = 220, 240, 260$ ,  $\Delta E_{max} \approx 120$

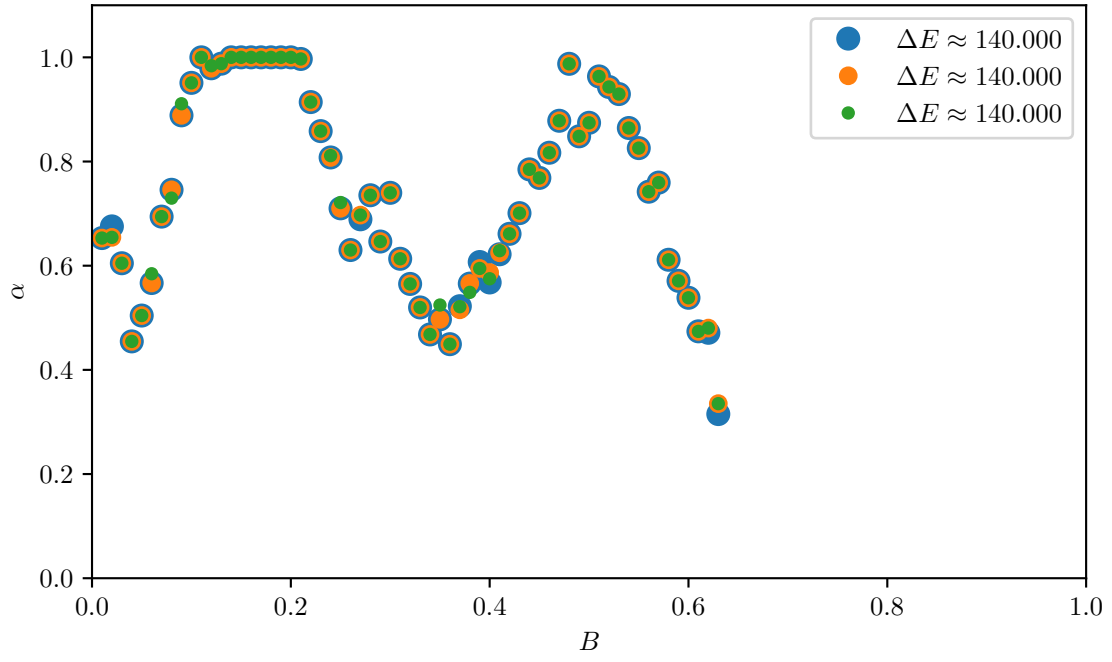


Figure 5.12:  $N = 220, 240, 260$ ,  $\Delta E_{max} \approx 140$

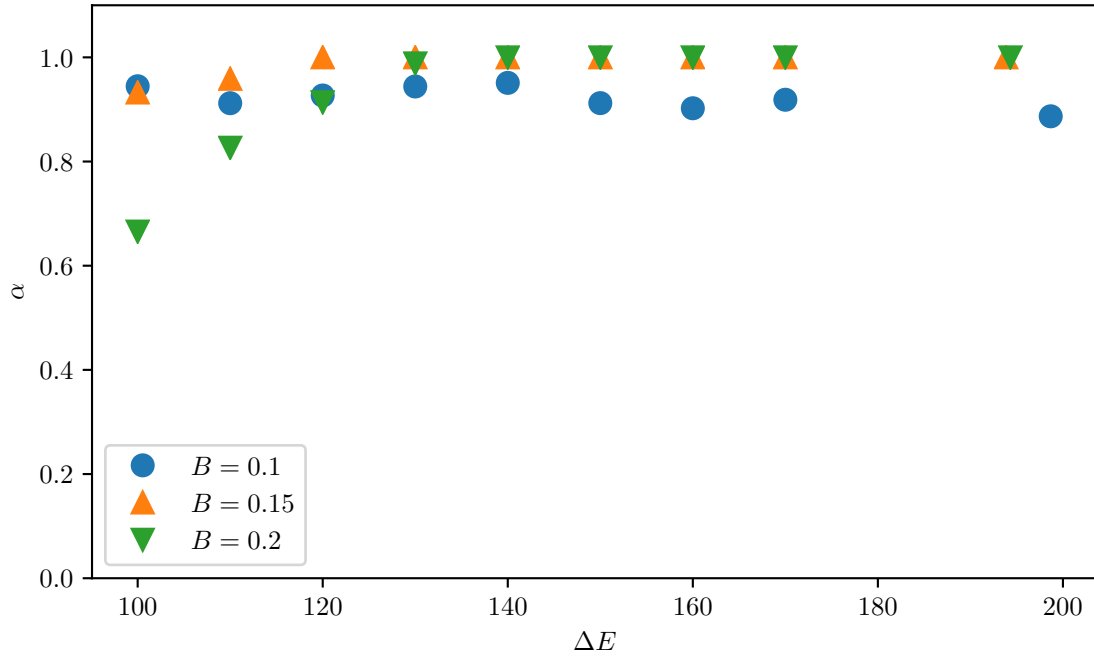


Figure 5.13:  $B = 0.1, 0.15, 0.2$ ,  $N = 260$

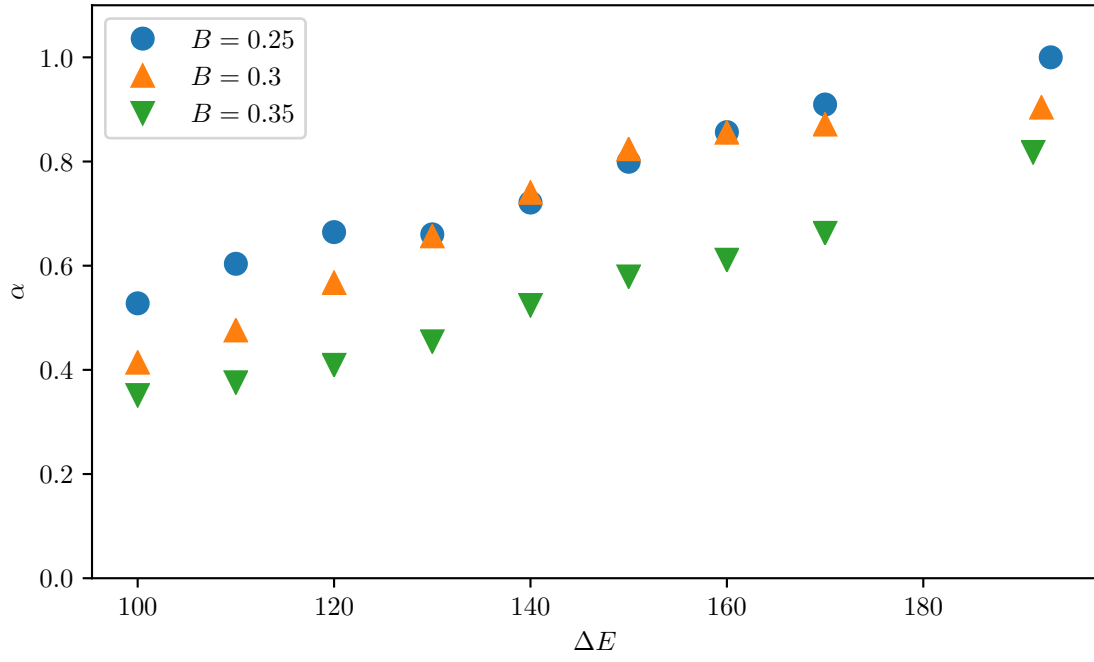


Figure 5.14:  $B = 0.25, 0.3, 0.35$ ,  $N = 260$

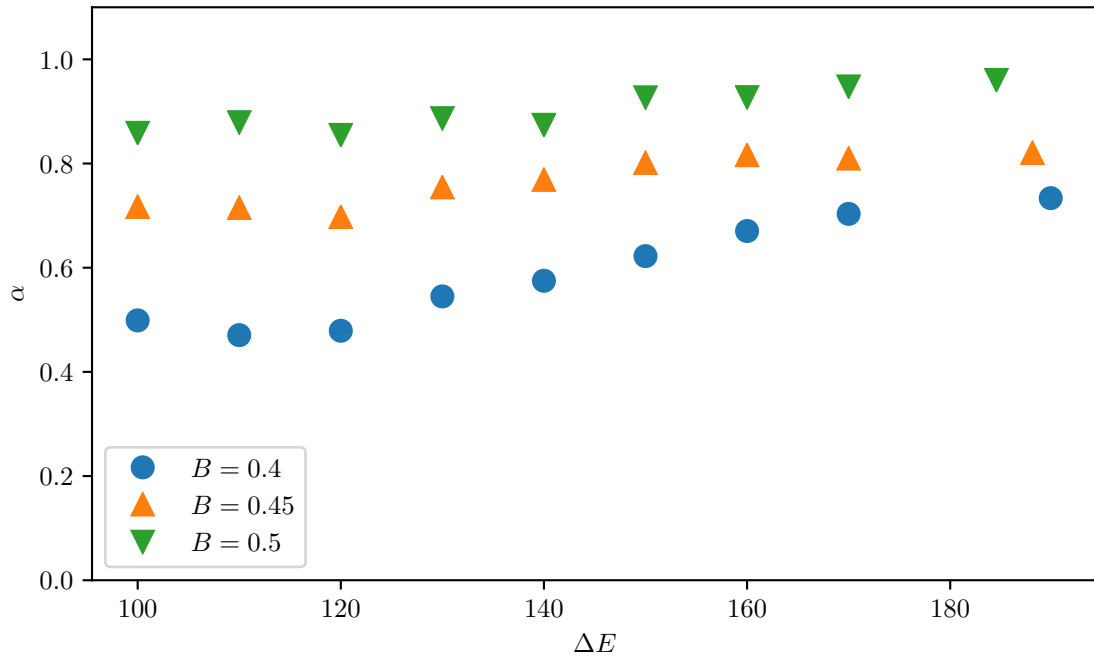


Figure 5.15:  $B = 0.4, 0.45, 0.5$ ,  $N = 260$



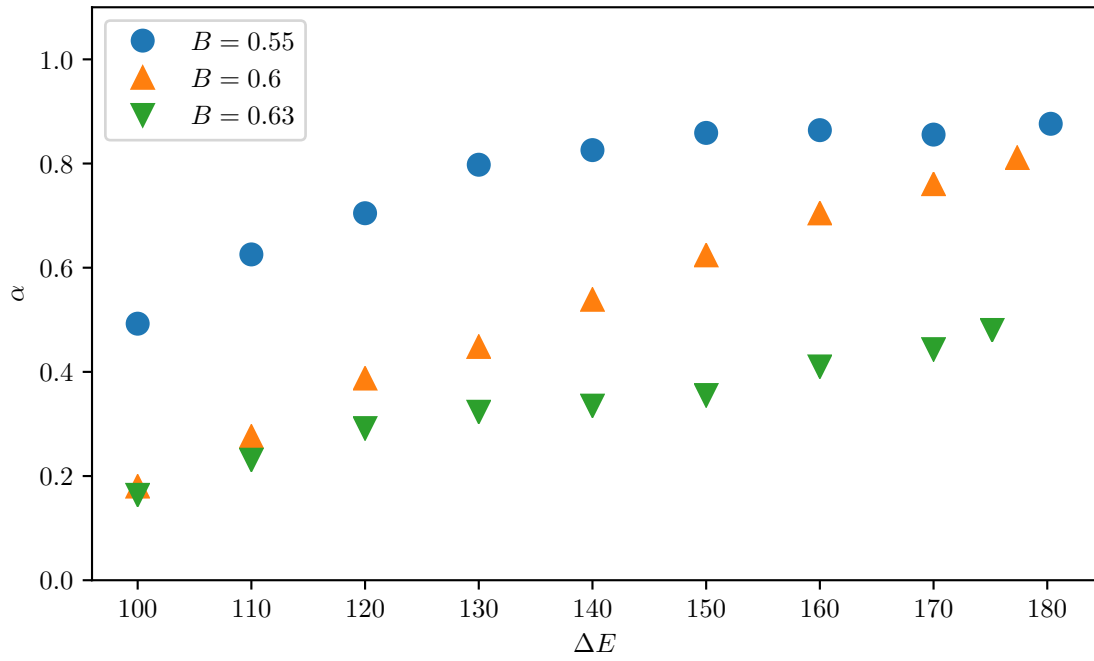


Figure 5.16:  $B = 0.55, 0.6, 0.63$ ,  $N = 260$

## Chapter 6

## Conclusions

# Bibliography

- [1] Paul Adrien Maurice Dirac. *The Principles of Quantum Mechanics 4th ed* (*The International Series of Monographs on Physics no 27*). Oxford: Clarendon Press, 1967.