Chapter 1

Classical chaos

1.1 Fundamental notions

Classical mechanics can be reformulated in a more elegant way which does not require the computation of individual forces and instead uses a single function to describe the entire system, namely the *Lagrangian*. The Lagrangian is a function of the generalised coordinates $q_1, \ldots, q_n$, their derivatives with respect to time $\dot{q}_1, \ldots, \dot{q}_n$ and optionally the time t, where n is the number of degrees of freedom.

The principles of Newtonian mechanics are replaced by the a variational principle $\delta S = 0$, where

$$S = \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t \tag{1.1}$$

is called the *action*. This principle, also called *Hamilton's principle* or *the principle of least action* is equivalent with the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = 0,$$

where $k = 1, \ldots, n$. Thus we have n second order differential equations.

An other formalism for classical mechanics which is very useful in theoretical physics is the Hamiltonian formalism. In this formalism instead of the Lagrangian we use the *Hamiltonian*, which is a function of the generalised coordinates $q_1, \ldots, q_n$, the generalised momenta $p_1, \ldots, p_n$ and optionally the time t. The generalised momenta is given by

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}.$$

The Hamiltonian can be obtained from the Lagrangian through a *Legendre* *transformation* of $\mathcal{L}$

$$\mathcal{H}(q_1,\ldots,q_n,p_1,\ldots,p_n,t)=\sum_{k=0}^{n}p_k\dot{q}_k-\mathcal{L}(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t). \tag{1.2}$$

The Euler-Lagrange equations are replaced by Hamilton's equations

$$\dot{q}_k=\frac{\partial\mathcal{H}}{\partial p_k}$$
$$\dot{p}_k=-\frac{\partial\mathcal{H}}{\partial q_k}$$

We now have $2n$ first order ordinary differential equations. comments?

1.2 Canonical transformations

In the Hamiltonian formulation of classical mechanics *canonical transformations* represent all the transformations of the canonically conjugate variables from q and p to Q and P such that the structure of the Hamilton equations remains intact.

The Lagrange equations of a system are invariant to a transformation of the Lagrange function such that:

$$\mathcal{L}'(Q,\dot{Q},t)=\mathcal{L}(q,\dot{q},t)-\frac{\mathrm{d}}{\mathrm{d}t}F(q,Q,t)$$

Integrating we obtain

$$\int_{t_1}^{t_2}\mathcal{L}'\,\mathrm{d}t=\int_{t_1}^{t_2}\mathcal{L}\,\mathrm{d}t-(F(q,Q,t_2)-F(q,Q,t_1))$$

The variation of the above equation is given by

$$\delta\int_{t_1}^{t_2}\mathcal{L}'\,\mathrm{d}t=\delta\int_{t_1}^{t_2}\mathcal{L}\,\mathrm{d}t-\delta F(q,Q,t_2)+\delta F(q,Q,t_1)$$

Fixing the variations at the end points to be 0, we get

$$\delta\int_{t_1}^{t_2}\mathcal{L}'\,\mathrm{d}t=\delta\int_{t_1}^{t_2}\mathcal{L}\,\mathrm{d}t$$

So indeed the two Lagrangian descriptions of the system are identical since if the Hamilton principle holds in the first description it also holds in the second.

A function $F(q,Q,t)$ satisfying the above property and the condition $\frac{\partial^2 F}{\partial q\partial Q}\neq 0$ is called a *generating function*. We can classify the generating functions in four possible types as a function of their variables. The conjugate momenta corresponding to q and Q are given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \text{ and } P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}}$$

In order to express those as a function of $F(q, Q, t)$ we use the fact that $\dot{q}$ is a cyclic coordinate for $\mathcal{L}'$

$$\frac{\partial \mathcal{L}'}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial}{\partial \dot{q}} \left(\frac{\mathrm{d}}{\mathrm{d}t} F(q, Q, t) \right) = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0,$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q, Q, t) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t}.$$

Thus we obtain

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

and

$$P = \frac{\partial \mathcal{L}'}{\partial \dot{Q}} = -\frac{\partial F}{\partial Q}$$

To find the Hamiltonian corresponding to the new coordinates Q and P we compute the Legendre transformation of $\mathcal{L}'$

$$\mathcal{H}'(Q, P, t) = P\dot{Q} - \mathcal{L}'(Q, \dot{Q}, t)$$

Thus

$$\mathcal{H}'(Q, P, t) = P\dot{Q} - \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t} = P\dot{Q} - \mathcal{L} + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t} = p\dot{q} - \mathcal{L} + \frac{\partial F}{\partial t}$$

and so

$$\mathcal{H}'(Q, P, t) = \mathcal{H}(q(Q, P), p(Q, P), t) + \frac{\partial F(q(Q, P), Q, t)}{\partial t}. \tag{1.3}$$

As mentioned above, there are four types of generating functions. They can be obtained by successive Legendre transformations which switch between the conjugate coordinates and they represent the same canonical transformation. We denote the above mentioned $F(q, Q, t) \equiv F_1(q, Q, t)$. The other types of generating functions will be denoted $F_2(q,P,t), F_3(p,Q,t), F_4(p,P,t)$. Thus we define

$$\begin{aligned}
F_2(q,P,t) &= F_1(q,Q,t) + PQ \\
p &= \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P} \\
F_3(p,Q,t) &= F_1(q,Q,t) - pq \\
q &= -\frac{\partial F_3}{\partial p}, \quad P = \frac{\partial F_3}{\partial Q} \\
F_4(p,P,t) &= F_1(q,Q,t) - pq + PQ \\
q &= -\frac{\partial F_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P}
\end{aligned}$$

The *Poisson bracket* of two arbitrary functions with respect to the canonical coordinates q and p is defined as

$$[F,G]_{q,p} = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} \right)$$

As with the above discussion we will limit to the case of one degree of freedom, but the results are easily generalisable to n degrees of freedom.

If Q and P are a set of conjugate variables obtained by a canonical transformation from q and p, then

$$[F', G']_{Q,P} = [F,G]_{q,p},$$

where F' and G' are the transformed functions. Proof? As a consequence, if we choose $F = Q$ and $G = P$

$$[Q,P]_{Q,P} = [Q(q,p), P(q,p)]_{q,p} = 1$$

This relation represents a necessary and sufficient condition for the transformation from q, p to Q, P to be canonical and is equivalent with the definition given for the canonical transformations.

An other equivalent condition would require the Jacobian matrix $\frac{\partial(Q,P)}{\partial(q,p)}$ to be that symplectic.

1.3 The Hamilton-Jacobi equation

Instead of finding the equations of motion through solving a set of $2n$ ordinary differential equations with $2n$ dependent variables ($q_1, \ldots, q_n$ and $p_1, \ldots, p_n$) and one independent variable (t), we can use a single first order partial differential equation having $n+1$ independent variables ($q_1, \ldots, q_n$ and t) and one dependent variable (S).

Suppose there exists a generating function S of type F_2 why? such that

$$\mathcal{H}' = 0.$$

From Hamilton's equations, the new coordinates Q_k and P_k, with $k = 1, \ldots, n$ will be constants, so the information about the time evolution of the system is contained in the canonical transformation itself.Thus, equation (1.3) becomes

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t) + \frac{\partial S}{\partial t} = 0 \tag{1.4}$$

which represents the *Hamilton-Jacobi equation.*

The solution $S(q_1, \ldots, q_n, P_1, \ldots, P_n, t)$ of equation (1.4) is called *Hamilton's Principal Function.* From the theory of differential equations we know that there will be $n+1$ arbitrary constants. We can find n of these constants as $P_1, \ldots, P_n$ and the remaining constant can be incorporated in S since if S is a solution for the Hamilton-Jacobi equation, $S + const.$ is also a solution.

It is a standard convention (Goldstein) to rename the constants as follows

$$\begin{aligned} P_k &\equiv \alpha_k \\ Q_k &\equiv \beta_k = \frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial \alpha_k}. \end{aligned}$$

These constants depend on the initial conditions $q_k(0)$ and $p_k(0)$. From

$$p_k(0) = \left.\frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial q_k}\right|_{t=0}$$

we can obtain α_ks, and β_ks are given by

$$\beta_k = \left.\frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial \alpha_k}\right|_{t=0}$$

We can obtain the initial generalised coordinates $q_k(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, t)$ by solving

$$\beta_k = \frac{\partial S}{\partial \alpha_k}$$

If we take the total time derivative of S

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{k=0}^{n} \frac{\partial S}{\partial q_k} \dot{q}_k + \frac{\partial S}{\partial t}$$

and use eq. (1.3)

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{k=0}^{n} p_k \dot{q}_k - \mathcal{H}$$

we recognise the Legendre transform from eq. (1.2). Thus

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mathcal{L}$$

or equivalently

$$S = \int \mathcal{L} \, \mathrm{d}t \, .$$

It follows that the generating function for this special canonical transformation is the action as defined in eq. (1.1).

If the time does not appear explicitly in the Hamiltonian, the Hamiltonian itself will be a constant of the motion and will represent the energy of the system E.

We can rewrite eq. (1.4) as follows

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t) = -\frac{\partial S}{\partial t} = E$$

Thus we can suppose that for this case $S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)$ has the following form

$$S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) \equiv W(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n) - Et$$

Using this expression in eq. (1.4), we obtain

$$\mathcal{H}(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}) = E \tag{1.5}$$

which represents the second form of the Hamilton-Jacobi equation.

The solution to this equation is called *Hamilton's Characteristic Function*. W can be expressed as

$$W = S + Et = \int (\mathcal{L} + \mathcal{H}) \, \mathrm{d}t = \int \sum_{k=0}^{n} p_k \dot{q}_k \, \mathrm{d}t = \sum_{k=0}^{n} \int p_k \, \mathrm{d}q_k$$

physical interpretation?

W is a generating function of type F_2. The canonical transformation given by W is different from the one given by S, since $\mathcal{H}' = \mathcal{H} + \frac{\partial W}{\partial t} = E \neq 0$.

The new generalised coordinates Q_k are cyclic in the new Hamiltonian $\mathcal{H}' = E = \mathcal{H}(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n})$, thus the new generalised momenta care constants. While in the case of S all the generalised momenta and coordinates were constants, in the case of W all the generalised momenta are constants $\dot{P}_k = -\frac{\partial \mathcal{H}'}{\partial Q_k} = 0$, but not all the generalised coordinates are constants. If we choose to define $P_1 \equiv E$,

$$\dot{Q}_1 = \frac{\partial \mathcal{H}'}{\partial P_1} = \frac{\partial E}{\partial P_1} = 1$$

Thus

$$Q_1 = t + const. = \frac{\partial W}{\partial P_1} = \frac{\partial W}{\partial E}.$$

So Q_1 is no longer a constant. The constant in the above equation determines the origin of the time axis. Q_1 represents the time and its conjugate coordinate is P_1, the energy.

The $n-1$ generalised momenta $P_2, \ldots, P_n$ are independent of P_1 and thus $\frac{\partial \mathcal{H}'}{\partial P_k} = 0$, where $k = 2, \ldots, n$. So the corresponding $n-1$ generalised coordinates are constants.

We can change the notation for α and β to use an index from 1 to $n-1$: $\alpha_j \equiv P_{j+1}$, $\beta_j \equiv Q_{j+1}$, where $j = 1, \ldots, n-1$. We can now express Hamilton's Characteristic Function as

$$W = W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1}).$$

From the initial generalised momenta $p_k(0)$

$$p_k(0) = \left. \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial q_k} \right|_{t=0}$$

we can obtain E and α_j. Using the initial generalised coordinates $q_k(0)$ we obtain β_j as follows

$$\beta_j = \left. \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial \alpha_j} \right|_{t=0}$$

Thus, having E, α_j and β_j, we can obtain the time evolution for the generalised coordinates from

$$\beta_j = \frac{\partial W(q_1, \ldots, q_n, E, \alpha_1, \ldots, \alpha_{n-1})}{\partial \alpha_j}$$

For systems with $n > 1$ degrees of freedom, the Hamilton-Jacobi equation can be separated in n equations if we can write S as

$$S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) = \sum_{k=1}^{n} W_k(q_k, \alpha_1, \ldots, \alpha_n) - Et$$

A simpler form of separability exists if the Hamiltonian itself is separable

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{k=1}^{n} \mathcal{H}_k(q_k, p_k)$$

In this case the second form of the Hamilton-Jacobi equation splits in n independent equations

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n) = E = \sum_{k=1}^{n} \mathcal{H}_k(q_k, p_k),$$

so

$$\mathcal{H}_k(q_k, \frac{\partial W_k}{\partial q_k}) = \alpha_k, \qquad E = \sum_{k=1}^{n} \alpha_k$$

why $E = \sum_{k=1}^{n} \alpha_k$?

The Hamilton-Jacobi equation can be solved only in some situations (only when the variables can be separated. why?), depending both on the problem and the chosen coordinate system. For orthogonal coordinates the *Staeckel conditions* (see Goldstein pp. 453) establish which kind of potentials can be separated. This does not decrease the theoretical value of the Hamilton-Jacobi equation which provides powerful methods for finding the constants of the motion.

1.4 Action-angle variables

Going back again to a system with one degree of freedom for simplicity, we study a system which has a periodic motion. We search for a canonical transformation from q, p to a new set of canonically conjugate coordinates θ, I such that the new Hamiltonian does not depend on θ.

Since θ is an ignorable coordinate

$$\dot{I} = -\frac{\partial \mathcal{H}'}{\partial \theta} = 0,$$

I is a constant of the motion and

$$\dot{\theta} = \frac{\partial \mathcal{H}'}{\partial I} = const.$$

Thus $\theta = \frac{\partial \mathcal{H}'}{\partial I}(t - t_0) \equiv \omega(I)(t - t_0)$. θ is called the *angle variable* and I the *action variable* (not to be confused with the action).

In order to obtain this variables we will use a generating function of type F_1 denoted $W'(q, \theta)$. Since the motion is periodic in the initial coordinates, it must also be periodic in the angle variable. Thus W' is periodic in θ.

From the properties of the F_1 type generating functions

$$\mathrm{d}W' = \frac{\partial W'}{\partial q}\,\mathrm{d}q + \frac{\partial W'}{\partial \theta}\,\mathrm{d}\theta = p\,\mathrm{d}q - I\,\mathrm{d}\theta\,.$$

(we can observe that W' is related to Hamilton's Characteristic Function by a Legendre transformation)

Since the motion is periodic, if we integrate over a period q returns to the same value, while θ advances by a constant amount, which we can choose to be 2π per period.

$$\oint \mathrm{d}W' = 0 = \oint p\,\mathrm{d}q - \oint I\,\mathrm{d}\theta$$

Since I is a constant of the motion,

$$2\pi I = \oint p\,\mathrm{d}q$$

or

$$I = \frac{1}{2\pi}\oint p\,\mathrm{d}q\,. \tag{1.6}$$

Equation (1.6) can be considered the definition for the action variable, where the integral is taken around a single period of the motion (which is always a closed curve in phase space).

1.5 Integrability

A system called *integrable* if the number of degrees of freedom is equal to the number of constants of the motion in involution which each other. Two constants of the motion are in *involution* with each other if their mutual Poisson brackets are equal to 1.

The previously mentioned constants of the motion are not required to be known in analytic form, they are only required to exist. Thus, for example $n = 1$ systems are always integrable if the Hamiltonian is time independent.

For integrable systems, the n constants of the motion restrict the motion in the $2n$ dimensional phase space to a $2n - n = n$ dimensional subspace. If the motion is also periodic, then the motion is constrained to a n dimensional closed surface (with finite volume).

For a system with $n = 2$ and periodic motion, the motion in the phase space is confined to a torus. In order to understand why, we can use the previously defined action-angle variables. In this case, the motion will be described by $I_1, \theta_1, I_2, \theta_2$. Since the motion is constrained to a 2 dimensional subspace, we can imagine the phase space as a plane with the Ox axis given by the direction in which θ_1 increases and with the Oy axis given by the direction in which θ_2 increases. Because the motion is periodic, we can consider that θ_i, $i = 1, 2$ takes values in $[0, 2\pi)$. Since the values for 0 and 2π are equivalent, we can join the ends of the intervals in both directions, thus forming a torus.

To extend the discussion to systems with n degrees of freedom, we replace the 2D plane with an n dimensional cube. If we join the sides of this cube we obtain a n dimensional torus.

Connection with the "hairy ball" theorem

we consider a vector field associated with the tangent in each point of the trajectory in phase space. This vector field is generated by Hamilton's equations with time as parameter(why?/how?). In the $n = 2$ case we can see that if the the shape of the closed bi-dimensional surface were a sphere, we could not "comb" the vectors without creating a singularity. Instead, if the surface is a torus, the problem disappears. This result is a consequence of the "hairy ball" theorem. This theorem states that there is no non-vanishing continuous tangent vector field on even-dimensional n-spheres. (what happens for odd n?)

+ explanation for the fact that the constants of motion must be in involution with each other

1.6 Chaos notions

1.7 Kolmogorov-Arnol'd-Moser theorem