# Chapter 1

# Theory

## 1.1 Classical Electrodynamics

Introduction stuff, cite Eisenberg and Greiner (1978).

We will begin with Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1.1a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.1c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,, \tag{1.1d}$$

which relate the electromagnetic field to sources, which must satisfy an additional equation to ensure charge conservation

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0\,. \tag{1.2}$$

As we can see above, equations (1.1c) and (1.1b) do not involve sources and thus they state the dynamical properties of the fields. Since equations (1.1a) and (1.1d) describe how the sources influence the fields, we need an additional equation to describe how the fields affect the sources

$$\mathbf{F} = \int \mathrm{d}\mathbf{r}'\, \rho(\mathbf{r}', t)\mathbf{E}(\mathbf{r}', t) + \frac{1}{c}\int \mathrm{d}\mathbf{r}'\, \mathbf{j}(\mathbf{r}', t) \times \mathbf{B}(\mathbf{r}', t)\,.$$

Maxwell's equations (1.1) relate six field quantities ($\mathbf{E}$ and $\mathbf{B}$) to four source quantities ($\rho$ and $\mathbf{j}$). This implies that there are some restrictions on the six quantities. This suggets that we can find a less redundant way to express the fields, and indeed the four quantities given by the vector potential $\mathbf{A}$ and scalar potential $\rho$ provide this representation. Equation (1.1b) implies the existence of a vector potential

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)\,. \tag{1.3}$$

Substituting (1.3) in (1.1c) we obtain

$$\boldsymbol{\nabla} \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{1.4}$$

and thus the quantity in the paranthesis can always be expressed as the gradient of a scalar field, namely the scalar potential

$$\boldsymbol{\nabla}\phi(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t) - \frac{\partial \mathbf{A}}{\partial t} \, .$$

With these considerations (1.1a) becomes

$$\boldsymbol{\nabla} \cdot \left( \boldsymbol{\nabla}\phi + \frac{\partial \mathbf{A}}{\partial t} \right) = -\frac{\rho}{\varepsilon_0}$$

or

$$\nabla^2 \phi + \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\rho}{\varepsilon_0} \tag{1.5}$$

and (1.1d)

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mu_0 \mathbf{j} - \frac{1}{c^2} \frac{\partial}{\partial t} \left( \boldsymbol{\nabla}\phi + \frac{\partial \mathbf{A}}{\partial t} \right) \, . \tag{1.6}$$

Using the following vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{1.7}$$

eq. (1.6) becomes

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j} - \frac{1}{c^2} \left( \boldsymbol{\nabla} \frac{\partial \phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right)$$

or

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} + \boldsymbol{\nabla} \left( \boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) . \tag{1.8}$$

Equations (1.5) and (1.8) were obtained by substituting the potentials obtained from the source-less equations, (1.1b) and (1.1c), into the ones with sources, (1.1a) and (1.1d). They are thus fully equivalent with Maxwell's equations (1.1) and, as we can observe, relate the four quantities given by the potentials to the four quantities for the sources. They also preserve the invariance under Lorentz transformations, with the scalar potential $\phi$ as the time-like component.

Equations (1.5) and (1.8) can be simplified by decoupling the potentials. This is possible due to the fact that potentials are not unique. To illustrate this point consider

$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \boldsymbol{\nabla}\Lambda(\mathbf{r}, t) \, .$$

This vector potential gives rise to a magnetic field

$$\boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Lambda) = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$$

equal with the original one since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\varphi) = 0$.

Similarly, for a scalar potential

$$\phi'(\mathbf{r}, t) = \phi(\mathbf{r}, t) - \frac{\partial \Lambda(\mathbf{r}, t)}{\partial t}$$

and the corresponding electric field will be

$$-\boldsymbol{\nabla}\phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial}{\partial t}\boldsymbol{\nabla}\Lambda = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}\,,$$

since the spatial and temporal derivatives commute. These kinds of transformations are called gauge transformations.

### 1.1.1 Gauge transformations

The freedom of choosing the gauge leads to the following condition satisfied by the scalar and vector potentials

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0\,,$$

called the Lorenz condition.

Indeed, if we consider a set of potentials $\mathbf{A}$ and $\phi$ that don't satisfy the condition

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \neq 0 = f(\mathbf{r}, t)\,,$$

then we can always carry out a gauge transformation to a new set of potentials $\mathbf{A}'$ and $\phi'$ that satisfy the Lorenz condition, such that

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} &= \boldsymbol{\nabla} \cdot (\mathbf{A}' - \boldsymbol{\nabla}\Lambda) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi' + \frac{\partial \Lambda}{\partial t}\right) \\ &= \boldsymbol{\nabla} \cdot \mathbf{A}' - \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = f(\mathbf{r}, t)\end{aligned}$$

or

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = \Box\,\Lambda \equiv \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} - \nabla^2\Lambda = f(\mathbf{r}, t)\,,$$

where the d'Alambertian operator is defined as

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

when choosing the Minkowski metric $(+, -, -, -)$ and

$$\boldsymbol{\nabla} \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0\,,$$

since they satisfy the Lorenz condition. The transformation we need is thus defined by the solution of $\Box\,\Lambda = f$.

Imposing the Lorenz condition on equations (1.5) and (1.4) decouples the potentials

$$\nabla^2\phi - \frac{\partial}{\partial t}\frac{1}{c^2}\frac{\partial\phi}{\partial t} = -\frac{\rho}{\varepsilon_0}$$
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}$$

yielding the simplified form of Maxwell's equations

$$\Box\,\phi = \frac{\rho}{\varepsilon_0}$$
$$\Box\,\mathbf{A} = \mu_0\mathbf{j}\,.$$

This form of Maxwell's equations preserves Lorentz invariance, as the Lorenz gauge condition can be expressed in a covariant way as the contraction of the four-vector $A \equiv (\frac{\phi}{c}, \mathbf{A})$ with the four-gradient $(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla})$.

Since the Lorenz condition doesn't fix the gauge, but only restricts us to transformations with $\Box\,\Lambda = 0$, we can impose further conditions in order to fix the gauge, but in general those will not be covariant. One such condition is given by the Coulomb gauge

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{1.9}$$

In this gauge eq. (1.5) becomes a Poisson equation for the scalar potential

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \tag{1.10}$$

with the solution given by the instantaneous Coulomb potential of the charge density in the domain $\rho(\mathbf{r}, t)$

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}', t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}'\ , \tag{1.11}$$

explaining the name of the condition (1.9).

An apparent violation of special relativity shows up in the above result which states that the scalar potential (at time $t$) is given by the instantaneous Coulomb interactions between charges (also at time $t$). The contradiction is only apparent and stems from the act that the Coulomb gauge is not Lorentz invariant.

In order to resolve the contradiction we first note that we can only observe the electric field

$$\mathbf{E}(\mathbf{r}, t) = -\boldsymbol{\nabla}\phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}\,.$$

In the Coulomb gauge, the vector potential is given by

$$\Box\,\mathbf{A} = \mu_0\mathbf{j} - \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial\phi}{\partial t}\,. \tag{1.12}$$

Considering the continuity equation (1.2) and the form of the scalar potential in eq. (1.11), the second term in eq. (1.12) becomes

$$\boldsymbol{\nabla}\frac{\partial\phi}{\partial t} = \boldsymbol{\nabla}\frac{1}{4\pi\varepsilon_0}\int\frac{\frac{\partial\rho}{\partial t}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}' = -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int\frac{\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}'\ , \tag{1.13}$$

where $\boldsymbol{\nabla}'$ denotes the derivatives with respect to $\mathbf{r}'$. Using the Helmholz decomposition we can write any sufficiently well behaved vector (the current density in this particular case) as the sum of a divergence-free (transversal) component and a curl-free (longitudinal) one:

$$\mathbf{j} = \mathbf{j}^t + \mathbf{j}^l\,,$$

where

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{j}^t &= 0\\ \boldsymbol{\nabla}\times\mathbf{j}^l &= 0\,.\end{aligned}$$

Using the vector identity (1.7) and

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$

we can write the current density as follows

$$\begin{aligned}\nabla^2\big(\mathbf{j}^t+\mathbf{j}^l\big) &= \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\mathbf{j}^l\big) - \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbf{j}^t\big)\\ \int\frac{\nabla^2\mathbf{j}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r} &= \int\frac{\boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\mathbf{j}^l\big)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}' - \int\frac{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{j}^t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}'\\ -4\pi\mathbf{j} &= \boldsymbol{\nabla}\int\frac{\boldsymbol{\nabla}\cdot\mathbf{j}^l}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}' - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int\frac{\mathbf{j}^{\mathbf{t}}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}'\end{aligned}$$

and thus we obtain the two components as

$$\begin{aligned}\mathbf{j}^t &= \frac{1}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int\frac{\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}'\\ \mathbf{j}^l &= -\frac{1}{4\pi}\boldsymbol{\nabla}\int\frac{\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}'\ .\end{aligned}$$

Comparing with (1.13) we see that

$$\frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial\phi}{\partial t} = \frac{\varepsilon_0}{c^2}\mathbf{j}^l = \mu_0\mathbf{j}^l$$

and thus the source term in eq. (1.12) can be expressed as function of the transverse current:

$$\Box\,\mathbf{A} = \mu_0(\mathbf{j}-\mathbf{j}^l) = \mu_0\mathbf{j}^t$$

and this also why the Coulomb gauge is also called the transverse gauge.

## 1.2 Electron in a Plane Wave

In this section we will consider the classical dynamics of an electron in a laser pulse following the discusion in Karsch (2018). The starting point is the equation of motion for the electron

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -e\left[\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]\ . \tag{1.14}$$

### 1.2.1 Non-relativistic treatment

### 1.2.2 Relativistic treatment

## 1.3 Particle in Cell Method

### 1.3.1 EPOCH