Chapter 1

Theory

1.1 Classical Electrodynamics

Introduction stuff, cite Eisenberg and Greiner (1978).

We will begin with Maxwell's equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1.1a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.1b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.1c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \,, \tag{1.1d}$$

which relate the electromagnetic field to sources, which must satisfy an additional equation to ensure charge conservation

$$\boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0 \,.$$

As we can see above, equations (1.1c) and (1.1b) do not involve sources and thus they state the dynamical properties of the fields. Since equations (1.1a) and (1.1d) describe how the sources influence the fields, we need an additional equation to describe how the fields affect the sources

$$\mathbf{F} = \int \mathrm{d}\mathbf{r}' \, \rho(\mathbf{r}', t) \mathbf{E}(\mathbf{r}', t) + \frac{1}{c} \int \mathrm{d}\mathbf{r}' \, \mathbf{j}(\mathbf{r}', t) \times \mathbf{B}(\mathbf{r}', t) \,.$$

Maxwell's equations (1.1) relate six field quantities ($\mathbf{E}$ and $\mathbf{B}$) to four source quantities (ρ and $\mathbf{j}$). This implies that there are some restrictions on the six quantities. This suggests that we can find a less redundant way to express the fields, and indeed the four quantities given by the vector potential $\mathbf{A}$ and scalar potential ρ provide this representation. Equation (1.1b) implies the existence of a vector potential

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t) \,. \tag{1.2}$$

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Substituting (1.2) in (1.1c) we obtain

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{1.3}$$

and thus the quantity in the paranthesis can always be expressed as the gradient of a scalar field, namely the scalar potential

$$\boldsymbol{\nabla}\phi(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t) - \frac{\partial \mathbf{A}}{\partial t} \,.$$

With these considerations (1.1a) becomes

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla}\phi + \frac{\partial \mathbf{A}}{\partial t} \right) = -\frac{\rho}{\varepsilon_0}$$

or

$$\nabla^2\phi + \frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\rho}{\varepsilon_0} \tag{1.4}$$

and (1.1d)

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mu_0\mathbf{j} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\phi + \frac{\partial \mathbf{A}}{\partial t} \right) . \tag{1.5}$$

Using the following vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

eq. (1.5) becomes

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{j} - \frac{1}{c^2}\left(\boldsymbol{\nabla}\frac{\partial \phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right)$$

or

$$\nabla^2\mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \right) . \tag{1.6}$$

Equations (1.4) and (1.6) were obtained by substituting the potentials obtained from the source-less equations, (1.1b) and (1.1c), into the ones with sources, (1.1a) and (1.1d). They are thus fully equivalent with Maxwell's equations (1.1) and, as we can observe, relate the four quantities given by the potentials to the four quantities for the sources. They also preserve the invariance under Lorentz transformations, with the scalar potential ϕ as the time-like component.

Equations (1.4) and (1.6) can be simplified by decoupling the potentials. This is possible due to the fact that potentials are not unique. To illustrate this point consider

$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \boldsymbol{\nabla}\Lambda(\mathbf{r}, t) \,.$$

This vector potential gives rise to a magnetic field

$$\boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Lambda) = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$$

equal with the original one since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \varphi) = 0$.

Similarly, for a scalar potential

$$\phi'(\mathbf{r}, t) = \phi(\mathbf{r}, t) - \frac{\partial \Lambda(\mathbf{r}, t)}{\partial t}$$

and the corresponding electric field will be

$$-\boldsymbol{\nabla}\phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial}{\partial t}\boldsymbol{\nabla}\Lambda = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}\,,$$

since the spatial and temporal derivatives commute. These kinds of transformations are called gauge transformations.

1.2 Electron in a Plane Wave

In this section we will consider the classical dynamics of an electron in a laser pulse following the discusion in Karsch (2018). The starting point is the equation of motion for the electron

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -e\left[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)\right]\,. \tag{1.7}$$

1.2.1 Non-relativistic treatment

1.2.2 Relativistic treatment

1.3 Particle in Cell Method

1.3.1 EPOCH