# Gaussian wave packet

1D $V=0$

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

$$H = \frac{p_x^2}{2m} \qquad p_x = -i\hbar \frac{\partial}{\partial x}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi$$

$$\psi(x,t) = X(x)T(t)$$

$$i\hbar XT' = -\frac{\hbar^2}{2m}X''T \quad / \frac{1}{XT}$$

$$i\hbar \frac{T'}{T} = -\frac{\hbar^2}{2m}\frac{X''}{X} = \hbar\omega$$

$$T' = \frac{1}{i}\omega T \Rightarrow \frac{dT}{dt} = -i\omega T \Leftrightarrow \frac{dT}{T} = -i\omega\, dt \Rightarrow \ln T = -i\omega t + \ln C \Rightarrow T = Ce^{-i\omega t}$$

$$C = 1 \Rightarrow T = e^{-i\omega t}$$

$$X'' + \underbrace{\frac{2m\omega}{\hbar}}_{k^2} X = 0 \qquad \omega \in \mathbb{R} \qquad E = \hbar\omega \qquad E = \frac{\hbar^2 k^2}{2m}$$

$$X(x) = Ae^{ikx} + Be^{-ikx}$$

$$\Rightarrow \psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$$

phase velocity $\frac{\omega}{k}$

$$j_x = \frac{i\hbar}{2m}\left(\psi(x)\frac{d}{dx}\psi^*(x) - \psi^*(x)\frac{d}{dx}\psi(x)\right)$$

$$= \dots$$

$$= \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)$$

$$B = 0 \ (j_x > 0)$$

$$\Rightarrow \psi(x,t) = Ae^{i(kx-\omega t)} \qquad \omega = \frac{\hbar}{2m}k^2$$

$$\psi(0,t) = Ae^{ik_0x - \frac{x^2}{2d^2}}$$

$$|\psi(0,t)|^2 = |A|^2 e^{-\frac{x^2}{d^2}}$$

$$j_x = \frac{i\hbar}{2m}\left[Ae^{ik_0x - \frac{x^2}{2d^2}} A^*\left(-ik_0 - \frac{x}{d^2}\right)e^{-ik_0x - \frac{x^2}{2d}} - A^* e^{-ik_0x - \frac{x^2}{2d^2}} A\left(ik_0 - \frac{x}{d^2}\right)e^{ik_0x - \frac{x^2}{2d^2}}\right]$$

$$= \frac{i\hbar}{2m}|A|^2 e^{\frac{x^2}{d^2}}\left(-ik_0 - \frac{x}{d^2} - ik_0 + \frac{x}{d^2}\right) = \frac{\hbar}{2mi}2ik_0|A|^2 e^{\frac{x^2}{d^2}} = \underbrace{\frac{\hbar}{m}k_0}_{v_0}|\psi(0,t)|^2$$

$$p_0 = mv_0 = \hbar k_0$$

$$\int_{-\infty}^{\infty} dx\,|\psi(0,t)|^2 = 1 \Rightarrow |A|^2 = \frac{1}{d\sqrt{\pi}}$$

$$\Psi(x,t) = c(k)\, e^{i(kx-\omega t)} \qquad \omega = \frac{\hbar}{2m}k^2$$

$$\psi(x,t) = \int_{-\infty}^{\infty} dk\, \Psi(x,t) \quad ?$$

$$\psi(x,0) = \int_{-\infty}^{\infty} dk\, C(k)\, e^{ikx} = \mathcal{F}[C(k)] \qquad \mathcal{F}[f] = \int_{\mathbb{R}^n} f(x)\, e^{i\langle x,\xi\rangle} dx$$

$$\Rightarrow C(k) = \mathcal{F}^{-1}[\psi(0,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, \psi(x,0)\, e^{-ikx} \qquad \mathcal{F}^{-1}[f] = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi)\, e^{-i\langle x,\xi\rangle} d\xi$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} dx\, e^{i(k_0-k)x - \frac{x^2}{2d^2}} = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{d^2(k_0-k)^2}{2} - \frac{d^2\left(i(k_0-k)-\frac{x}{d^2}\right)^2}{2}} dx \qquad ax - bx^2 = \frac{a^2}{4b} - \frac{1}{4b}(a-2bx)^2$$

$$= \frac{A}{2\pi} e^{-\frac{d^2}{2}(k-k_0)^2}\int_{-\infty}^{\infty} e^{-\frac{d^2\left[i(k_0-k)-\frac{x}{d^2}\right]^2}{2}} dx = \frac{Ad}{\sqrt{2\pi}} e^{-\frac{d^2}{2}(k-k_0)^2}$$

$$\psi(x,t) = \frac{Ad}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \exp\left[-\frac{1}{2}d^2(k-k_0)^2 + ikx - i\frac{\hbar t}{2m}k^2\right] \qquad \alpha = \frac{1}{2}d^2 + i\frac{\hbar t}{2m}$$

$$= \frac{Ad}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \exp\left[\frac{\beta^2}{4\alpha} - \frac{(\beta-2\alpha k)^2}{4\alpha} - \gamma\right] \qquad \beta = d^2k_0 + ix \qquad \gamma = \frac{1}{2}d^2k_0^2$$

$$= \frac{Ad}{\sqrt{2\pi}} e^{\frac{\beta^2}{4\alpha}-\gamma}\int_{-\infty}^{\infty} dk\, e^{-\frac{(\beta-2\alpha k)^2}{4\alpha}} \qquad z = \beta - 2\alpha k, \quad dz = -2\alpha\, dk$$

$$= \frac{Ad}{\sqrt{2\pi}} e^{\frac{\beta^2}{4\alpha}-\gamma}\frac{1}{2\alpha}\int_{-\infty}^{\infty} dz\, e^{-\frac{z^2}{4\alpha}} = \frac{Ad}{\sqrt{2\pi}} e^{\frac{\beta^2}{4\alpha}-\gamma}\frac{1}{2\alpha}\sqrt{4\alpha\pi} = \frac{Ad}{\sqrt{2\pi}} e^{\frac{\beta^2}{4\alpha}-\gamma}\sqrt{\frac{\pi}{\alpha}}$$

$$= \frac{Ad}{\sqrt{2\pi}}\exp\left[\frac{d^4k_0^2 + 2ixd^2k_0 - x^2 - d^4k_0^2\left(1+i\frac{\hbar t}{md^2}\right)}{2d^2\left(1+i\frac{\hbar t}{md^2}\right)}\right]\sqrt{\frac{2\pi}{d^2\left(1+i\frac{\hbar t}{md^2}\right)}}$$

$$= \frac{A}{\sqrt{1+i\frac{\hbar t}{md^2}}}\exp\left[-\frac{x^2 - 2ixd^2k_0 + i\frac{\hbar t}{m}k_0^2d^2}{2d^2\left(1+i\frac{\hbar t}{md^2}\right)}\right]$$

$$|\psi(x,t)|^2 = \frac{|A|^2}{\left[1+\left(\frac{\hbar t}{md^2}\right)^2\right]^2}\exp\left[-\frac{x^2-2ixd^2k_0+i\frac{\hbar t}{m}k_0^2d^2}{2d^2\left(1+i\frac{\hbar t}{md^2}\right)} - \frac{x^2+2ixd^2k_0-i\frac{\hbar t}{m}k_0^2d^2}{2d^2\left(1-i\frac{\hbar t}{md^2}\right)}\right]$$

$$= \frac{|A|^2}{\left[1+\left(\frac{\hbar t}{md^2}\right)^2\right]^2}\exp\left[-\frac{2\left(x^2+2xd^2k_0\frac{\hbar t}{md^2} - \frac{\hbar^2t^2}{m^2d^4}k_0^2d^4\right)}{2d^2\left[1-\left(\frac{\hbar t}{md^2}\right)^2\right]}\right]$$

$$= \frac{|A|^2}{\left[1+\left(\frac{\hbar t}{md^2}\right)^2\right]^2}\exp\left[-\frac{\left(x^2-\frac{\hbar k_0}{m}t\right)^2}{d^2\left[1-\left(\frac{\hbar t}{md^2}\right)^2\right]}\right] \qquad d' = d\left[1+\left(\frac{\hbar t}{md^2}\right)^2\right]^{\frac{1}{2}} \simeq \frac{\hbar}{md}t$$

$$\frac{\partial \Psi}{\partial x} = -\frac{2x - 2id^2k_0}{2d^2\left(1+i\frac{\hbar t}{md^2}\right)}\Psi(x,t) = ik_0\frac{1+i\frac{x}{d^2k_0}}{1+i\frac{\hbar t}{md^2}}\Psi(x,t)$$

$$j_x = \frac{i\hbar}{2m}\left(\Psi(x)\, ik_0\frac{1-i\frac{x}{d^2k_0}}{1-i\frac{\hbar t}{md^2}}\Psi^*(x,t) - \Psi^*(x,t)\, ik_0\frac{1+i\frac{x}{d^2k_0}}{1+i\frac{\hbar t}{md^2}}\Psi(x,t)\right)$$

$$= \frac{\hbar}{2mi}|\Psi(r,t)|^2\, ik_0\left(\frac{1-i\frac{x}{d^2k_0}}{1-i\frac{\hbar t}{md^2}} + \frac{1+i\frac{x}{d^2k_0}}{1+i\frac{\hbar t}{md^2}}\right)$$

$$= \frac{\hbar k_0}{2m}|\Psi(r,t)|^2\frac{2\left(1+\frac{\hbar t x}{md^4k_0}\right)}{1+\left(\frac{\hbar t}{md^2}\right)^2} \qquad v_0 = \frac{\hbar k_0}{m}$$

$$= |\Psi(r,t)|^2\, v_0\,\frac{1+\frac{\hbar t x}{md^4k_0}}{1+\left(\frac{\hbar t}{md^2}\right)^2}$$