$$\Psi(x) = \langle x|\Psi\rangle = \frac{1}{\pi^{1/4}\sqrt{d}}\, e^{ikx - \frac{x^2}{2d^2}} \quad , d>0,\ k = \frac{p}{\hbar}$$

$$|\langle x_0|\Psi\rangle|^2 = \frac{1}{d\sqrt{\pi}}\left(e^{ikx_0 - \frac{x_0^2}{2d^2}} \cdot e^{-ikx_0 - \frac{x_0^2}{2d^2}}\right) = \frac{1}{d\sqrt{\pi}}\, e^{-\frac{x_0^2}{d^2}}$$

$$\overline{Q_x} = \langle\Psi|Q_x|\Psi\rangle = \int_{-\infty}^{\infty} x|\langle x|\Psi\rangle|^2 dx = \int_{-\infty}^{\infty} x\,\frac{1}{d\sqrt{\pi}}\, e^{-\frac{x^2}{d^2}} dx = 0$$

$$\overline{Q_x^2} = \langle\Psi|Q_x^2|\Psi\rangle = \int_{-\infty}^{\infty} x^2|\langle x|\Psi\rangle|^2 dx = \int_{-\infty}^{\infty} x^2\,\frac{1}{d\sqrt{\pi}}\, e^{-\frac{x^2}{d^2}} dx = \frac{1}{d\sqrt{\pi}}\,\frac{d^2}{2}\sqrt{\pi d^2} = \frac{d^2}{2}$$

$$\overline{P_x} = \langle\Psi|P_x|\Psi\rangle = \int_{-\infty}^{\infty}\Psi^*(x)\,\frac{\hbar}{i}\frac{d}{dx'}\Psi(x')\,dx' = \frac{1}{d\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-ikx - \frac{x^2}{2d^2}}\,\frac{\hbar}{i}\left(ik - \frac{x'}{d^2}\right) e^{ikx' - \frac{x'^2}{2d^2}}\,dx'$$

$$= \frac{\hbar}{i}\,\frac{1}{d\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{d^2}}\left(ik - \frac{x}{d^2}\right)dx = \frac{\hbar}{i}\,\frac{1}{d\sqrt{\pi}}\, ik\sqrt{\pi d^2} = \hbar k = p$$

$$\overline{P_x^2} = \langle\Psi|P_x^2|\Psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle\Psi|P_x|\Psi\rangle = -\hbar^2\int_{-\infty}^{\infty}\Psi^*(x)\,\frac{d^2}{dx^2}\Psi(x)\,dx = \frac{-\hbar^2}{d\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-ikx - \frac{x^2}{2d^2}}\,\frac{d}{dx}\left[\left(ik - \frac{x}{d^2}\right)e^{ikx - \frac{x^2}{2d^2}}\right]dx$$

$$= -\frac{\hbar^2}{d\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-ikx - \frac{x^2}{2d^2}}\left[-\frac{1}{d^2}\, e^{ikx - \frac{x^2}{2d^2}} + \left(ik - \frac{x}{d^2}\right)^2 e^{ikx - \frac{x^2}{2d^2}}\right]dx = -\frac{\hbar^2}{d\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{d^2}}\left(-\frac{1}{d^2} + k^2 + \frac{x^2}{d^4} - 2\,\frac{ikx}{d^2}\right)dx$$

$$= -\frac{\hbar^2}{d\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{d^2}}\left(-\frac{1}{d^2} - k^2 + \frac{x^2}{d^4}\right)dx = -\frac{\hbar^2}{\cancel{d\sqrt{\pi}}}\left[-\cancel{d\sqrt{\pi}}\left(k^2 + \frac{1}{d^2}\right) + \frac{1}{d^{\cancel{4}2}}\,\frac{\cancel{d^2}}{2}\,\cancel{d\sqrt{\pi}}\right] = \hbar^2\left(k^2 + \frac{1}{2d^2}\right) = p^2 + \frac{\hbar^2}{2d^2}$$

Gaussian wave packet

(Seminar)