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FREE LATTICES[1]

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1. Introduction

A **lattice** (sometimes called a structure) is a partially ordered set of elements each two of which have a greatest lower and a least upper bound, denoted $A \cap B$ and $A \cup B$, read "A meet B" and "A join B." Postulates[2] for partial ordering are that $\leqq$, defined between some pairs of elements, satisfy

(1) $\qquad A \leqq A$ for all A;

(2) $\qquad$ if $A \leqq B$ and $B \leqq A$, then $A = B$;

(3) $\qquad$ if $A \leqq B$ and $B \leqq C$, then $A \leqq C$.

A lattice is said to be **generated** by a set of elements X_i (the "generators") if it consists of the X_i and their finite combinations by $\cap$ and $\cup$—e.g., $X_1 \cup X_2$, $\{[X_1 \cup X_2] \cap X_3\} \cup X_1$—sometimes known as "lattice polynomials."

However, these polynomials do not all constitute distinct elements; for instance, $X_i \cap X_i = X_i$ in any lattice by definition of $\cap$; furthermore, in a specific lattice we may have additional rules of equality; e.g. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The **free lattice** generated by the X_i is[3] a lattice generated by them in which there are no laws of equality except those derivable from the postulates for a lattice. This is the most general lattice generated by the X_i, in the sense that every other can be obtained from it by a homomorphism—determined by the additional rules of equality.

We are concerned here with the internal structure of a free lattice. Given two polynomials, how can we find which of $\leqq$, $=$, $\geqq$ (if any) hold between them in the free lattice (equivalently, in all lattices)? This question is answered in §2. In §3 we show that given a polynomial, there is a shortest polynomial

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[2] Equivalently, we may postulate for a lattice the identities $L1$: $X \cup X = X \cap X = X$; $L2$: $X \cup Y = Y \cup X, X \cap Y = Y \cap X$; $L3$: $X \cup (Y \cup Z) = (X \cup Y) \cup Z$, $X \cap (Y \cap Z) = (X \cap Y) \cap Z$; $L4$: $X \cup (X \cap Y) = X \cap (X \cup Y) = X$, note the theorem that $X = X \cap Y$ if and only if $Y = X \cup Y$, and define $X \leqq Y$ as equivalent to $X = X \cap Y$ or $Y = X \cup Y$. Cf. Ore, *On the Foundation of Abstract Algebra I*, Annals of Math., 36 (1935), p. 409.

[3] Its existence is guaranteed by a theorem of universal algebra; cf. G. Birkhoff, *On the Structure of Abstract Algebras*, Proc. Camb. Phil. Soc., 31 (1935), pp. 440–1. Compare free groups.

equal to it in the free lattice (which can be found by a definite procedure), and in §4 it is shown that certain elements cover others.

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2. Conditions for $A \leqq B$ in a free lattice

Given two elements A and B, we should like to know: is $A \leqq B$ in the free lattice? Sometimes this is obviously true; e.g. $X_1 \leqq X_1 \cup X_2$. Sometimes we can show it false, for by the abovementioned homomorphism, $A \leqq B$ in the free lattice implies $A \leqq B$ in every lattice with the same generators. Hence if we can exhibit a specific lattice where $A \nleqq B$, then $A \nleqq B$ in the free lattice. But this is a trial and error method; we should like some definite procedure for settling the question.

THEOREM 1. *In the free lattice generated by a set of elements* X_i,

(4) $X_i \leqq X_j$ if and only if $i = j$;

(5) *recursively,* $A \leqq B$ *if and only if one or more of the following hold:*

(a) $A \equiv A_1 \cup A_2$ *where* $A_1 \leqq B$ *and* $A_2 \leqq B$,

(b) $A \equiv A_1 \cap A_2$ *where* $A_1 \leqq B$ *or* $A_2 \leqq B$,

(c) $B \equiv B_1 \cup B_2$ *where* $A \leqq B_1$ *or* $A \leqq B_2$,

(d) $B \equiv B_1 \cap B_2$ *where* $A \leqq B_1$ *and* $A \leqq B_2$.

NOTE. In ($5a$) it is permissible that A_1 be itself a join; etc.

We see that this is the sort of condition desired, for, given A and B, we need look only at B and part of A, and A and part of B. Since the elements are the *finite* combinations of the generators, this process will eventually end with (4).

PROOF. These conditions are obviously sufficient; to prove them necessary we proceed by a series of definitions and lemmas.

DEFINITION OF $\subset$. **(6)** $X_i \subset X_j$ if and only if $i = j$; **(7)** $A \subset B$ if and only if one or more of ($5a$–d) are true with $\leqq$ replaced by $\subset$.

(8) DEFINITION. $A \supset B$ if and only if $B \subset A$.

NOTE. The set of definitions (6)–(8) is self-dual; i.e., if $\cap$ and $\cup$, $\subset$ and $\supset$ are interchanged the set remains the same. In particular, ($7a$) and ($7d$) are dual, ($7b$) and ($7c$) are dual, and (6) is self-dual. This property will enable us to omit many cases in proofs, where we need only make these same changes throughout.

(9) DEFINITION. $A \cong B$ if and only if $A \subset B$ and $A \supset B$.

(10) DEFINITION. The **length** of A, denoted $L(A)$, is the total number of X's appearing in A, counting repetitions; e.g., $L(X_1) = 1$, $L(X_1 \cup X_1) = 2$, $L(\{[X_1 \cup X_2] \cap X_3\} \cup X_1) = 4$.

With $\leqq$ as inclusion relation and $=$ as equality the combinations of the X_i form the free lattice; we now show that they also form a lattice with $\subset$ and $\cong$ in these roles; cf. (16).

(11) LEMMA. **(a)** $A \subset A$, **(b)** $A \subset A \cup B$, **(c)** $A \supset A \cap B$ *for all* A, B. *Thus* $A \cup B$ *is an upper bound to* A *and* B *under* $\subset$.

PROOF by induction on $L(A)$. ($11a$) is true for $L(A) = 1$, by (6). If ($11a$) is true for $L(A) \leqq m$, then so are ($11b$, c) by ($7c$, b). Hence ($11a$) holds for

$L(A) = m + 1$, for say $A \equiv A_1 \cup A_2$ (dually if $A \equiv A_1 \cap A_2$); then $A_1 \subset A$, $A_2 \subset A$ by induction; $\therefore A \equiv A_1 \cup A_2 \subset A$ by (7a).

(12) Lemma. **(a)** *$A_1 \cap A_2 \subset X_j$ if and only if $A_1 \subset X_j$ or $A_2 \subset X_j$.*
(b) *$X_j \subset B_1 \cup B_2$ if and only if $X_j \subset B_1$ or $X_j \subset B_2$.*
(c) *$A_1 \cup A_2 \subset B$ if and only if $A_1 \subset B$ and $A_2 \subset B$.*
(d) *$A \subset B_1 \cap B_2$ if and only if $A \subset B_1$ and $A \subset B_2$.*
(e) *$A_1 \cap A_2 \subset B_1 \cup B_2$ if and only if $A_1 \cap A_2 \subset B_1$ or $A_1 \cap A_2 \subset B_2$ or $A_1 \subset B_1 \cup B_2$ or $A_2 \subset B_1 \cup B_2$.*

Proof. "If" is obviously true by (7). "Only if" is proved by induction on $m = L(A) + L(B)$. It is true for $m \leqq 2$ vacuously. Assume (12) true for $m \leqq k - 1$; then for $m = k$,

(12a) $A_1 \cap A_2 \subset X_j$. Then $A_1 \subset X_j$ or $A_2 \subset X_j$, as desired, by (7b), the only part of (7) which applies.

(12c) $A_1 \cup A_2 \subset B$. *Case* 1. $A_1 \cup A_2 \subset X_j$. Then $A_1 \subset X_j$, $A_2 \subset X_j$ by (7a). *Case* 2. $A_1 \cup A_2 \subset B_1 \cup B_2$. Then by (7$a$, c), (i) $A_1 \subset B$ and $A_2 \subset B$ or (ii) $A_1 \cup A_2 \subset B_1$ or (iii) $A_1 \cup A_2 \subset B_2$. If (ii) $A_1 \cup A_2 \subset B_1$, then $A_1 \subset B_1$ and $A_2 \subset B_1$ by induction (12c); $\therefore A_1 \subset B_1 \cup B_2$, $A_2 \subset B_1 \cup B_2$ by (7c), as desired. Likewise if (iii) holds the lemma does, and if (i), then the proof is immediate. *Case* 3. $A_1 \cup A_2 \subset B_1 \cap B_2$. Then ($i$) $A_1 \subset B_1 \cap B_2$ and $A_2 \subset B_1 \cap B_2$ or (ii) $A_1 \cup A_2 \subset B_1$ and $A_1 \cup A_2 \subset B_2$, by (7a, d). If (i), Q.E.D. If (ii), then $A_1 \subset B_1$, $A_2 \subset B_1$, $A_1 \subset B_2$, $A_2 \subset B_2$ by induction (12c); $\therefore A_1 \subset B_1 \cap B_2$, $A_2 \subset B_1 \cap B_2$ by (7d).

(12b, d) dually.

(12e) $A_1 \cap A_2 \subset B_1 \cup B_2$. Then $A_1 \subset B_1 \cup B_2$ or $A_2 \subset B_1 \cup B_2$ or $A_1 \cap A_2 \subset B_1$ or $A_1 \cap A_2 \subset B_2$ by (7b, c). Q.E.D.

(13) Lemma. *If $A \subset B$ and $B \subset C$, then $A \subset C$.*

Proof by induction on $m = L(A) + L(B) + L(C)$. True for $m = 3$ by (6). Induction:

Case 1. *Non-meet* $\subset B \subset C$.

Case 1a. $X_i \subset X_j \subset C$. $\therefore i = j$ by (6). $\therefore X_i \equiv X_j$. $\therefore X_i \subset C$.

Case 1b. $X \subset B_1 \cup B_2 \subset C$. $\therefore X \subset$ some B_i by (12b). $B_i \subset C$ by (12c). $\therefore X \subset C$ by induction.

Case 1c. $X_i \subset B_1 \cap B_2 \subset X_j$. Part of dual of case 1$b$.

Case 1d. $X \subset B_1 \cap B_2 \subset C_1 \cup C_2$.

$\therefore B_1 \cap B_2 \subset$ some C_i	or some $B_i \subset C_1 \cup C_2$ by (12e).
$\therefore X \subset C_i$ by induction.	$X \subset B_i$ by (12d).
$\therefore X \subset C_1 \cup C_2$ by (12b) or (7c).	$\therefore X \subset C_1 \cup C_2$ by induction.

Case 1e. $A_1 \cup A_2 \subset B \subset C$. $\therefore A_i \subset B$ (all i) by (12c). $\therefore A_i \subset C$ (all i) by induction. $\therefore A_1 \cup A_2 \subset C$ by (12c).

Case 1f. $X \subset B_1 \cap B_2 \subset C_1 \cap C_2$. Part of dual of case 1$e$.

Case 2. *Meet* $\subset B \subset C$.

Case 2a. $A_1 \cap A_2 \subset B \subset$ non-join. Part of dual of case 1.

Case 2*b*. $A_1 \cap A_2 \subset X \subset C_1 \cup C_2$. $\therefore$ some $A_i \subset X$ by (12*a*). $\therefore A_i \subset C_1 \cup C_2$ by induction. $\therefore A_1 \cap A_2 \subset C_1 \cup C_2$ by (12*e*)

Case 2*c*. $A_1 \cap A_2 \subset B_1 \cup B_2 \subset C_1 \cup C_2$.

$\therefore A_1 \cap A_2 \subset$ some B_i	or some $A_i \subset B_1 \cup B_2$ by (12*e*).
$B_i \subset C_1 \cup C_2$ by (12*c*).	$\therefore A_i \subset C_1 \cup C_2$ by induction.
$\therefore A_1 \cap A_2 \subset C_1 \cup C_2$ by induction.	$\therefore A_1 \cap A_2 \subset C_1 \cup C_2$ by (12*e*).

Case 2*d*. $A_1 \cap A_2 \subset B_1 \cap B_2 \subset C_1 \cup C_2$. Dual of case 2*c*.

(14) LEMMA. *$A \cup B$ is the least upper bound to A and B under $\subset$; dually for $A \cap B$.*

For it is *an* upper bound by (11), and if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$ by (7*a*).

(15) LEMMA. *$\cong$ is an equality relation.*[4]

(16) LEMMA. *The finite combinations of the X_i by $\cap$ and $\cup$ form a lattice, generated by the X_i, with $\subset$ as inclusion relation and $\cong$ as equality and $A \cup B$, $A \cap B$ as least upper, greatest lower bounds to A and B.*

PROOF. (1), (2), (3) are satisfied [(11), (9), (13)], and $A \cup B$ is least upper bound by (14).

PROOF OF THEOREM 1. Any other lattice generated by the X_i is a homomorphic image of the free lattice, hence $\leqq$ in a free lattice is sufficient for $\subset$. But by definition of $\cup$ as least upper bound and induction, $\subset$ is sufficient for $\leqq$ and hence $\cong$ for $=$. Therefore $\subset$ and $\leqq$ are equivalent in the free lattice. Theorem 1 then follows from (6) and (7).

Now denote $A_1 \cup A_2 \cup \cdots \cup A_n$ by $\sum_1^n A_i$, $A_1 \cap \cdots \cap A_n$ by $\prod_1^n A_i$, or simply $\sum A_i$, $\prod A_i$ if there is no confusion.

(17) COROLLARY. **(a)** *$\prod A_i \subset X_j$ if and only if some $A_i \subset X_j$.*
(b) *$X_i \subset \sum B_j$ if and only if $X_i \subset$ some B_j.*
(c) *$\sum A_i \subset B$ if and only if every $A_i \subset B$.*
(d) *$A \subset \prod B_j$ if and only if $A \subset$ every B_j.*
(e) *$\prod A_i \subset \sum B_j$ if and only if $\prod A_i \subset$ some B_j or some $A_i \subset \sum B_j$.*
(f) *$A_i \subset \sum A_j$, $B_i \supset \prod B_j$ all i.*
(g) *$\sum A_i$ is the least upper bound to $A_1, \cdots, A_n$ under $\subset$.*

PROOF. By repetition of (12), (11).

To answer "Is $A = B$?," apply (2) and theorem 1. Conditions (12) or (17) are more convenient in practise than (5). (17*b*) is like the condition that a prime number divide a product.

3. Canonical forms

Having thus found one collection of elements equal to each other, and likewise other collections, we should like to choose as canonical forms one element from each collection.

[4] Cf. Schröder, *Algebra der Logik*, vol. 1, p. 184, or MacNeille, *Partially Ordered Sets*, Trans. Am. Math. Soc., 42 (1937), pp. 416–60, or it may be readily verified directly.

THEOREM 2. *Of all the elements equal in a free lattice, there is one of shortest length, unique except for commutativity and associativity.*

PROOF. We show by induction on $k = L(A) + L(B)$ that if $A \cong B$, $A \not\equiv B$, then $C\, \exists$: $A \cong B \cong C$, and $L(C) < L(A)$ or else $L(C) < L(B)$. Then if the theorem were false, say A and B were alleged to be both of the shortest length, we could find a still shorter element unless $A \equiv B$! It is true vacuously for $k = 2$. Induction:

Case 1. $\sum A_i \cong \sum B_i$. Here, but not in the previous section, we assume A_i and B_i not themselves joins. $\sum A_i \subset \sum B_i$ by (9). $\therefore A_i \equiv \prod_j a_j^i \subset \sum B_n$ (all i) by (17*c*). $\therefore$ by (17*e*), for any i, (i) some $a_p^i \subset \sum B_j$ or (ii) $A_i \subset$ some B_r. Note that (ii) holds if $A_i \equiv X_s$, by (17*b*). Similarly, for any i, (iii) some $b_p^i \subset \sum A_j$ or (iv) $B_i \subset$ some A_r.

If (i) holds for some i, then $a_p^i \subset \sum B_j \subset \sum A_j$ by (9). $A_j \subset \sum A_n$, $j \neq i$. $\therefore a_p^i \cup \sum_{j \neq i} A_j \subset \sum A_j$. Also $A_i \subset a_p^i$, so $\sum A_j \subset a_p^i \cup \sum_{j\neq i} A_j$; $\therefore a_p^i \cup \sum_{j\neq i} A_j \cong \sum A_j$ and the theorem holds in this case; likewise if (iii) holds for some i.

Otherwise, (ii) and (iv) hold for all i. $\therefore$ for all i, $A_i \subset$ some $B_{f(i)}$, $B_i \subset$ some $A_{g(i)}$ [f, g need not be single-valued]. If $i\, \exists$: $g[f(i)] \neq i$, then $A_i \subset B_{f(i)} \subset A_j$ ($j \neq i$), $\sum_{n\neq i} A_n \cong \sum A_n$, and the theorem holds; similarly if $f[g(i)] \neq i$. If not, then $f[g(i)] = i$, $g[f(i)] = i$ for all i. $\therefore A_i \cong B_{f(i)}$, $B_i \cong A_{g(i)}$ all i. But $\sum A_i \not\equiv \sum B_i$ by hypothesis, $\therefore p\, \exists$: $A_p \not\equiv B_{f(p)}$ or $B_p \not\equiv A_{g(p)}$, say the former. $\therefore$ by induction, $D\, \exists$: $A_p \cong B_{f(p)} \cong D$, $L(D) < L(A_p)$ or $L(D) < L(B_{f(p)})$, say the former. Then $D \cup \sum_{n\neq p} A_n \cong \sum A_n$ and the theorem holds.

Case 2. $\prod A_i \cong \prod B_i$. Dual.

Case 3. $\sum A_i \cong \prod B_j$. $\therefore \sum A_i \subset \prod B_j$; hence every $A_i \subset \prod B_j$ and $\sum A_i \subset$ every B_j. Also $\prod B_j \subset \sum A_i$,

$$\therefore \prod B_j \subset \text{some } A_p \qquad \text{or some } B_p \subset \sum A_i .$$

$$A_p \subset \prod B_j \text{ by above.} \qquad \sum A_i \subset B_p \text{ by above.}$$

$$A_p \cong \prod B_j \cong \sum A_i . \qquad \therefore B_p \cong \sum A_i \cong \prod B_j . \quad \text{Q.E.D.}$$

We take this unique shortest form as the canonical form.

(18) COROLLARY 1. $A \equiv \sum A_i \equiv \sum_i (\prod_j a_j^i) \cup \sum_{i \in E} X_i$ (E any subset of $1, \cdots, n$) is canonical if and only if (a) no $a_p^i \subset \sum A_n$ and (b) no $A_i \subset \sum_{n\neq i} A_n$ and (c) every A_i is canonical. Dually for $\prod A_i$.

Corollary 1 follows from the proof of theorem 2. From case 3 we might also require: no $A_p \cong \sum A_n$, but this is included in (b).

If $\sum A_i$ is not canonical, we can find the canonical element equal to it, for if (a) is false, then $a_p^i \cup \sum_{j\neq i} A_j \cong \sum A_j$, if ($b$) is false, then $\sum_{n\neq i} A_n \cong \sum A_n$, if ($c$) is false then replace it by its canonical form; in any case we get a shorter, equal element, to which we can again apply the process.

We can now build up a diagram of the free lattice step by step, starting with the shortest elements. We could not hope to get it all at once since there are infinitely many distinct elements if there are more than two generators.[5]

(19) COROLLARY 2. If $\sum A_i \cong \sum B_i$, and $\sum A_i$ is canonical, then $(a) B_i \subset \sum A_i$, all i, and (b) given $i, j \ni: A_i \subset B_j$.

PROOF. (a) follows from (12c), and if $A_i \equiv X$, then (b) follows from (12c, b). Otherwise $A_i \equiv \prod_j a_j^i \subset \sum B_i$ by (12c). $\therefore$ by (12e), either $A_i \subset$ some B_j as desired, or some $a_j^i \subset \sum B_i \cong \sum A_i$ and then by (18a) $\sum A_i$ is not canonical contrary to hypothesis.

4. Covering theorems

DEFINITION. A **covers** B if $A > B$ and no $C \ni: A > C > B$.

We present some scattered results on the existence of such pairs of elements. Denote the free lattice of n generators by F_n. In this section (but not previously) we assume n finite.

THEOREM 3. *In the free lattice of n generators, if* $A > (\sum_{i \neq p} X_i) \cap (\sum_{i \neq q} X_i)$, *then* $A = \sum_i^n X_i$ or $\sum_{i \neq p} X_i$ *or* $\sum_{i \neq q} X_i$.

We first prove a

LEMMA. *In* F_n, $A \equiv \sum A_i > \sum_{i \neq p} X_i$ *if and only if given* $i, j \ni: A_j \geqq X_i$.

PROOF OF LEMMA by induction on $L(A)$. Obvious for $L(A) \leqq n$. Suppose $L(A) = k > n$, obviously redundant terms having been omitted; e.g., $X_1 \cup X_1$ contracted to X_1. Also, by (17b), we may suppose no A_i is itself a join. Then some A_i has at least two factors, say $A_1 \equiv \prod a_j$. Then

$$\sum_{i \neq p} X_i < \sum A_i \leqq a_s \cup \sum_{i \neq 1} A_i \qquad \text{(all } s\text{)}.$$

By induction, given $i, j \ni: j^{\text{th}}$ term $\geqq X_i$. If for some s this term is an A_i, then the lemma holds; otherwise $a_s \geqq X_i$ for all s, and $A_1 \geqq X_i$.

COROLLARY 1. *In* F_n, $\sum A_i = \sum_i^n X_i$ *if and only if given* $i, j \ni: A_j \geqq X_i$. $\sum A_i = \sum_{i \neq p} X_i$ *if and only if given* i $(i \neq p)$, $j \ni: A_j \geqq X_i$, *but no* $A_j \geqq X_p$.

COROLLARY 2. *In* F_n, $\sum_1^n X_i$ *covers* $\sum_{i \neq p} X_i$.

COROLLARY 3. *In* F_n, *every* $A \neq \sum_1^n X_i$ *is* $\leqq \sum_{i \neq p} X_i$ *for some* p. Proved by induction.

The rest of theorem 3 is now proved in much the same way.

COROLLARY 4. *If* F_n, $\sum_{i \neq p} X_i$ *covers* $(\sum_{i \neq p} X_i) \cap (\sum_{i \neq q} X_i)$, *any* q.

COROLLARY 5. *In* F_n, *if* $A > X_p$, *then* $A \geqq X_p \cup (\prod_{i \neq p} X_i)$.

THEOREM 4. $\sum_1^n X_i$ *covers* $\sum_{i \neq p} X_i$ *which in turn covers* $(\sum_{i \neq p} X_i) \cap (\sum_{i \neq q} X_i)$, *and* $X_p \cup (\prod_{i \neq p} X_i)$ *covers* X_p, *in any lattice generated by the* X_i *in which these elements are distinct.*

For any other lattice is a homomorphic image of the free one.

The duals of these results are of course likewise true.

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[5] Cf. G. Birkhoff, *op. cit.*, p. 451. The distinctness of his elements in the free lattice can also be shown by (12). Note that, taking every sixth element, we get an infinite "chain" of distinct elements.