Proving the Regular Value Theorem in Lean

Nicolò Cavalleri, Michela Barbieri, Sebastian Monnet

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- Recently got a lot of attention for helping Peter Scholze check his work.
- Based on a foundational system called **type theory**.

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The implication $\mathrm{Proof}(\mathrm{p}) \to \mathrm{Proof}(\mathrm{p})$ is true for any proposition p, since we can take the identity function as our term.

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Theorem (Regular Value Theorem)

Let $f\colon M \to N$ be a smooth map from a smooth m-manifold to a smooth n-manifold, where $m > n$. Let $c \in N$ be a regular value of f with nonempty preimage.
Then $f^{-1}(c) \subseteq M$ is a smooth submanifold of M of dimension $m - n$.

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- It often leaves technical details up to the imagination.
- Since we have such strong pictures in our minds, there is usually a disconnect between our ideas and our axioms.
- When working with Lean, we need to be totally explicit about everything.

Geometry is Hard (Example)

- Suppose that we have a map $f\colon M \to N$ from an m-manifold to an n-manifold, and let $p \in M$ be a point. Consider the statement

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- Formally, this means

  " There exist open subsets $U \subseteq M$ and $V \subseteq N$ such that $p \in U$ and $f(p) \in V$, along with diffeomorphisms $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$ such that $\varphi(p) = 0$ and $\psi(f(p)) = 0$. "

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- This is not difficult in itself, but it gives some idea of the considerations involved in formalising geometry.