The Market That Never Sleeps: Probabilistic Analysis of Stock and Bond Overnight Movements via Linear Algebra

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1. Introduction

This project analyzes stock and bond behavior via data from the SPDR S&P 500 ETF Trust (SPY) and the iShares 20+ Year Treasury Bond ETF (TLT), respectively. We utilized data ranging from December 2004 to April 2025.

At its core, our project researched overnight movements in the stock market, in which a given stock's price when the market opens is higher or lower than its previous closing price. We chose to investigate the relationship between overnight movement in the SPY index and the TLT index.

There were two main questions this project hoped to investigate. Firstly, we aimed to learn whether Markov chains may be used to model long-term stock market behavior. Secondly, we aimed to learn whether fitting a polynomial via least-squares approximation could provide insights on the overnight price movements of SPY over a set interval of time.

CBT, NEED TO CITE WHERE WE GOT THE DATA FROM

2. Markov Chains

2A. Background

A ***Markov chain*** is defined as an evolving system comprising of a sequence of ***stages*** where each stage is in one of a finite number of ***states***, and where the state of a given stage is only dependent on the state of the stage immediately before it. To phrase it differently, the Markov property–one of the defining features of a Markov chain–states that $P(X_{t+1} = s_j \mid X_t = s_i)$, or that the probability of a given state X at time $t+1$ is only dependent on the given state X at time t. This probability is known as a ***transition probability***.

Stochastic matrices, also commonly known as ***probability matrices***, are mathematical tools which may be used to represent Markov chains. They are square matrices of dimension n, where n is the number of possible states in the system. Each column represents a current state j, and each row represents a subsequent state i. The entries of a stochastic matrix P_{ij} are the transition probabilities corresponding to each state combination. An example stochastic matrix is displayed in **Figure A**, in which the entry P_{12} indicates that given a current state of 2, there is a probability of 0.4 (or 40%) that the next state will be 1. This gives rise to a key property of stochastic matrices: $\sum_i P_{ij} = 1$ for all j. In other words, the sum of each column must be equal to 1, as there must be a 100% chance that stage $t+1$ is in one of the n possible finite states, regardless of the state at stage t.

Figure A:

$$P = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

Any column of a probability matrix may also be isolated to form a ***state vector*** $\vec{s}$ of dimension n. The probabilities of each state at a stage m can be found by raising each component of the vector to the power of m.

$$\vec{s}^{(m)} = \begin{bmatrix} s_1^m \\ \vdots \\ s_n^m \end{bmatrix}$$

This new vector is known as the $\boldsymbol{m^{th}}$ state vector. Accordingly, raising $\vec{s}$ to the power of 0 will result in the ***initial state vector***. State vectors may also be used to investigate the long-term behavior of a system. This can be done by finding the vector $\vec{s}^{(m)}$ as m approaches infinity.

$$\lim_{m \to \infty} \vec{s}^{(m)} = \vec{s}$$

This results in what is known as a ***steady-state vector***. This steady-state vector may be found via the formula shown below, where I is an identity matrix of the same dimensions as the probability matrix P, and where $\vec{s}$ is the steady state vector. This formula is equivalent to solving a homogeneous system.

$$(I - P)\vec{s} = 0$$

2B. Application Methods

We utilized Python to create a dataframe (an object used to store tabular data) which displayed the share price of SPY and the share price of TLT at the time the markets opened and at the time the markets closed each day. **Figure 1** contains an example of this.

Figure 1:

Date	SPY Open	SPY Close	TLT Open	TLT Close
2025-04-01	557.45	560.97	107.83	108.20
2025-04-02	555.05	564.52	108.01	109.00
2025-04-03	545.11	537.70	109.25	108.73

We then used this data to calculate the share price fluctuation that took place the previous night. This was calculated as $(Open_t - Close_{t-1})/Close_{t-1}$. For example, the SPY overnight movement corresponding to April 2nd, 2025 according to **Figure 1** would be $(555.05 - 560.97)/560.97$, resulting in a movement of approximately -0.011, or -1.1%. We then normalized these movements using Z-scores. In this manner, 4 possible states were established, depending on overnight behavior for that date. These states are explained in **Figure 2**.

Figure 2:

State	**Stock Movement (Z)**	**Bond Movement (Z)**	**Interpretation**
1	≥ 0	≥ 0	Stock up, Bond up
2	≥ 0	< 0	Stock up, Bond down
3	< 0	≥ 0	Stock down, Bond up
4	< 0	< 0	Stock down, Bond down

Following the establishment of these 4 states, we once again utilized Python to create a table displaying how many times the different states "fed into" one another. This table is displayed in **Figure 3**, where each column represents the current state, and each row represents the subsequent state. For instance, entry (1, 2) of **Figure 3** indicates that there were 305 instances where a day categorized as State 2 was immediately followed by a day categorized as State 1 (i.e. Monday = State 2, Tuesday = State 1).

Figure 3:

	State 1	State 2	State 3	State 4
State 1	219	305	318	207
State 2	292	482	560	279
State 3	317	539	419	250
State 4	221	287	227	195

Following the use of **Figure 3**, we opted to scale each column such that each individual entry would be a proportion. That is, to scale entry (1, 1), we took the sum of column 1 (219, 292, 317, and 221), indicating that there were a total of 1049 days in our data which were categorized as being in State 1. Dividing entry (1, 1) by 1049, we found that a proportion of 0.2088, or 20.88% of days categorized as State 1 were immediately followed by *another* day categorized as State 1. In this manner, we reduced the entire table, as displayed in **Figure 4**.

Figure 4:

	State 1	State 2	State 3	State 4
State 1	0.2088	0.1891	0.2087	0.2223
State 2	0.2784	0.2988	0.3675	0.2997
State 3	0.3022	0.3342	0.2749	0.2685
State 4	0.2107	0.1779	0.1490	0.2095

Slightly changing this table to take the form of a matrix, we found ourselves with **Figure 5**. This was our final ***probability matrix***, or ***stochastic matrix***.

Figure 5:

$$P = \begin{bmatrix} 0.2088 & 0.1891 & 0.2087 & 0.2223 \\ 0.2784 & 0.2988 & 0.3675 & 0.2997 \\ 0.3022 & 0.3342 & 0.2749 & 0.2685 \\ 0.2107 & 0.1779 & 0.1490 & 0.2095 \end{bmatrix}$$

We then utilized this stochastic matrix, in combination with Python, to find the ***steady-state vector*** for the system. The formula used for this calculation is displayed in **Figure 6**, and the resulting steady-state vector is as shown in **Figure 7.**

Figure 6:

$$(I - P)\vec{s} = 0$$

Figure 7:

$$\vec{s} = \begin{bmatrix} 0.2050 \\ 0.3152 \\ 0.2980 \\ 0.1817 \end{bmatrix}$$

2C. Application Results

The steady-state vector found provided three primary insights.

1. $P(1+2) > P(3+4)$. That is, it is more likely that overnight stock movement will be positive rather than negative. This is intuitive, as stocks tend to rise.
2. $P(1+3) \approx P(2+4)$. Once more, this is to be expected, as 20-year bonds are usually constant/steady.
3. $P(2+3) > P(1+4)$. This is logical due to stocks and bonds historically being negatively correlated. As such, it is natural that divergence is more likely than convergence.

3. Least-Squares Approximation

3A. Background

Least-Squares Approximation, per our class, is defined as a method to fit a line through a set of data points by minimizing the sum of the squared distances between the fitted line and each data point. In linear algebra terms, least-squares approximation is seeking to find a line on $\mathbb{R}^2$, $\begin{bmatrix} a \\ b \end{bmatrix}$, that best fits a series of points $(x_1, y_1)...(x_n, y_n)$. The values a and b are the slope and intercept of the line, respectively, and the line of "best fit" is defined as the line that minimizes $s = (ax_1 + b - y_1)^2 ... + (ax_n + b - y_n)^2$, where $ax_n + b$ is the value predicted by the line, and y_n is the value of the data point. In terms more closely aligned with linear algebra, s may also be defined by the quantity below.

$$s = \left\| \begin{bmatrix} ax_1 + b \\ \vdots \\ ax_n + b \end{bmatrix} - \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\|^2$$

This implies that the below value is to be minimized.

$$\left\| \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\|^2$$

Defining $A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, this leaves us with the goal of minimizing the quantity $\left\| A\vec{v} - \vec{b} \right\|^2$. Knowing this, the establishment of three main facts must be established prior to continuing:

1. If $\vec{v}$ minimizes $\left\| A\vec{v} - \vec{b} \right\|^2$, $\vec{b} - A\vec{v}$ is normal to every vector in $im(A)$. This implies that$(A\vec{w}) \cdot (\vec{b} - A\vec{v}) = 0$, for any $\vec{w}$ in the image space of A.
2. Given two vectors in the image space of A, $\vec{w_1}$ and $\vec{w_2}$, $\vec{w_1} \cdot \vec{w_2} = \vec{w_1}^T\ \vec{w_2}$. This implies that $(A\vec{w})^T(\vec{b} - A\vec{v}) = 0 \implies \vec{w}^T A^T (\vec{b} - A\vec{v}) = 0$.
3. Given any vector $\vec{w}$ in the image space of A such that $\vec{w} \cdot \vec{w'} = 0$ for **all** $\vec{w'}$, it is implied that$\vec{w'} = \vec{w} = 0 \implies \vec{w} \cdot (A^T(\vec{b} - A\vec{v}) = 0 \implies (A^T A)\vec{v} = A^T \vec{b}$. **CBT, CHECK MATH THEORY HERE**

Thus, the ***normal equation*** to be used is defined as $(A^T A)\vec{v} = A^T \vec{b}$. This equation can be utilized to the find the vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

3B. Application Methods

We utilized Python to extracted the overnight movement data for the first 30 days of our SPY dataset, starting at December 2nd, 2004. We then smoothed this data through the use of moving averages. That is, we created a new string of data, x, such that a given entry x_i would be equal to the average of data points i through $i + 20$. In this manner, entry x_1 would be equivalent to the average overnight movement of SPY between the 1st and 21st days included in our dataset. In other words, $x_i = \sum_{i}^{i+20} (SPY\ Data)_i$. The resulting 10 data points are displayed in **Figure 8**. Note that in this case, x was more of an index regarding a given time period's average (i.e. $x = 1 \implies y =$ average of first 20 overnight price movement data points), as opposed to a value with in-context meaning.

CBT, NOMENCLATURE REGARDING MOVING AVERAGES AND THEIR RELATIONSHIP W/ LABELING INDIVIDUAL DAYS

Figure 8:

x	Date Period	Average Overnight Movement (y)
1	2004-12-22 to 2005-01-11	−0.041002
2	2004-12-23 to 2005-01-12	−0.028533
3	2004-12-24 to 2005-01-13	0.000847
4	2004-12-25 to 2005-01-14	0.009175
5	2004-12-26 to 2005-01-15	−0.006357
6	2004-12-27 to 2005-01-16	0.007206
7	2004-12-28 to 2005-01-17	0.049741
8	2004-12-29 to 2005-01-18	0.057683
9	2004-12-30 to 2005-01-19	0.024539
10	2004-12-31 to 2005-01-20	0.041740

Following this, we opted to fit a 9-degree polynomial rather than a line $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} slope \\ intercept \end{bmatrix}$. This would result in the polynomial being represented by the vector in **Figure 8**, where each entry c_i is the coefficient of the term in the polynomial x^i. In other words, the polynomial represented by **Figure 9** would follow the form of $y = c_0 + c_1 x + c_2 x^2 ... c_9 x^9$.

Figure 9:

$$\vec{v} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_9 \end{bmatrix}$$

Following this, we created a Vandermonde matrix M, whose general form is displayed in **Figure 10**.

Figure 10:

$$M = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^d \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^d \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^d \end{bmatrix}$$

For calculation purposes, this matrix M worked very similarly to the matrix $A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$ that would be generated when fitting a line. Each row of the Vandermonde matrix still corresponded to one point, with each column corresponding to a power of x. Applying this model to our data point using Python, we received the output displayed in **Figure 11**.

Figure 11:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 \\ 1 & 3 & 9 & 27 & 81 & 243 & 729 & 2187 & 6561 \\ 1 & 4 & 16 & 64 & 256 & 1024 & 4096 & 16384 & 65536 \\ 1 & 5 & 25 & 125 & 625 & 3125 & 15625 & 78125 & 390625 \\ 1 & 6 & 36 & 216 & 1296 & 7776 & 46656 & 279936 & 1679616 \\ 1 & 7 & 49 & 343 & 2401 & 16807 & 117649 & 823543 & 5764801 \\ 1 & 8 & 64 & 512 & 4096 & 32768 & 262144 & 2097152 & 16777216 \\ 1 & 9 & 81 & 729 & 6561 & 59049 & 531441 & 4782969 & 43046721 \\ 1 & 10 & 100 & 1000 & 10000 & 100000 & 1000000 & 10000000 & 100000000 \end{bmatrix}$$

We then utilized Python to calculate M^T, as displayed in **Figure 12**.

Figure 12:

$$M^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 \\ 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 \\ 1 & 16 & 81 & 256 & 625 & 1296 & 2401 & 4096 & 6561 & 10000 \\ 1 & 32 & 243 & 1024 & 3125 & 7776 & 16807 & 32768 & 59049 & 100000 \\ 1 & 64 & 729 & 4096 & 15625 & 46656 & 117649 & 262144 & 531441 & 1000000 \\ 1 & 128 & 2187 & 16384 & 78125 & 279936 & 823543 & 2097152 & 4782969 & 10000000 \\ 1 & 256 & 6561 & 65536 & 390625 & 1679616 & 5764801 & 16777216 & 43046721 & 100000000 \end{bmatrix}$$

Following this, we then used Python once again to calculate M^TM, as displayed in **Figure 13.**

Figure 13:

$$M^TM = \begin{bmatrix} 1.0\times10^{1} & 5.5\times10^{1} & 3.9\times10^{2} & 3.0\times10^{3} & 2.5\times10^{4} & 2.2\times10^{5} & 2.0\times10^{6} & 1.8\times10^{7} & 1.7\times10^{8} \\ 5.5\times10^{1} & 3.9\times10^{2} & 3.0\times10^{3} & 2.5\times10^{4} & 2.2\times10^{5} & 2.0\times10^{6} & 1.8\times10^{7} & 1.7\times10^{8} & 1.6\times10^{9} \\ 3.9\times10^{2} & 3.0\times10^{3} & 2.5\times10^{4} & 2.2\times10^{5} & 2.0\times10^{6} & 1.8\times10^{7} & 1.7\times10^{8} & 1.6\times10^{9} & 1.5\times10^{10} \\ 3.0\times10^{3} & 2.5\times10^{4} & 2.2\times10^{5} & 2.0\times10^{6} & 1.8\times10^{7} & 1.7\times10^{8} & 1.6\times10^{9} & 1.5\times10^{10} & 1.4\times10^{11} \\ 2.5\times10^{4} & 2.2\times10^{5} & 2.0\times10^{6} & 1.8\times10^{7} & 1.7\times10^{8} & 1.6\times10^{9} & 1.5\times10^{10} & 1.4\times10^{11} & 1.4\times10^{12} \\ 2.2\times10^{5} & 2.0\times10^{6} & 1.8\times10^{7} & 1.7\times10^{8} & 1.6\times10^{9} & 1.5\times10^{10} & 1.4\times10^{11} & 1.4\times10^{12} & 1.3\times10^{13} \\ 2.0\times10^{6} & 1.8\times10^{7} & 1.7\times10^{8} & 1.6\times10^{9} & 1.5\times10^{10} & 1.4\times10^{11} & 1.4\times10^{12} & 1.3\times10^{13} & 1.3\times10^{14} \\ 1.8\times10^{7} & 1.7\times10^{8} & 1.6\times10^{9} & 1.5\times10^{10} & 1.4\times10^{11} & 1.4\times10^{12} & 1.3\times10^{13} & 1.3\times10^{14} & 1.2\times10^{15} \\ 1.7\times10^{8} & 1.6\times10^{9} & 1.5\times10^{10} & 1.4\times10^{11} & 1.4\times10^{12} & 1.3\times10^{13} & 1.3\times10^{14} & 1.2\times10^{15} & 1.2\times10^{16} \end{bmatrix}$$

We found the inverse of matrix M^TM, as shown in **Figure 14**.

We substituted M and M^T in the equation $(A^TA)\vec{v} = A^T\vec{b}$ to obtain the formula $(M^TM)\vec{v} = M^T\vec{b}$. Utilizing the recently calculated matrix $(M^TM)^{-1}$, we were then able to solve for $\vec{b}$ using the formula$(M^TM)^{-1}(M^TM)\vec{v} = (M^TM)^{-1}M^T\vec{b}$, or $\vec{v} = (M^TM)^{-1}M^T\vec{b}$. The resulting vector $\vec{v}$ is displayed in **Figure 14.**

Figure 14:

$$\vec{v} = \begin{bmatrix} -5.7953\times10^{-1} \\ 1.4424\times10^{0} \\ -1.5139\times10^{0} \\ 8.1931\times10^{-1} \\ -2.4904\times10^{-1} \\ 4.4040\times10^{-2} \\ -4.4831\times10^{-3} \\ 2.4316\times10^{-4} \\ -5.4368\times10^{-6} \end{bmatrix}$$

Plugging these coefficients into our previously established polynomial of form $y = c_0 + c_1x + c_2x^2...c_9x^9$, we were left with the equation below.

Figure 15:

$$f(x) = -0.5795 + 1.4424x - 1.5139x^2 + 0.8193x^3 - 0.2490x^4 + 0.0440x^5 - 0.0045x^6 + 0.0002432x^7 - 0.000005437x^8$$

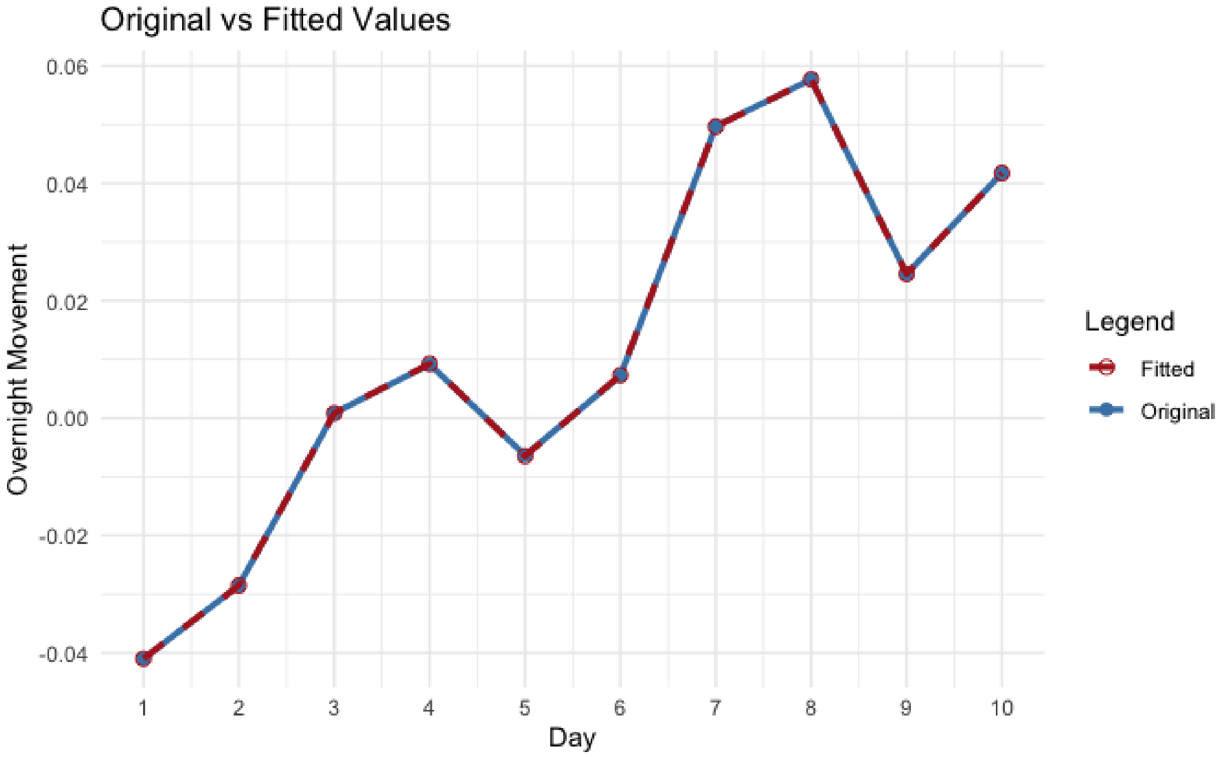
Finally, using Python one more time, we were able to use this polynomial to approximate the values of each value of x, as displayed in **Figure 16** and plotted in **Figure 17**.

CBT, UNSURE ABOUT WHETHER APPROPRIATE TO LABEL DAYS AS 1-10, OR KEEP X AS INDEX VALUES

Figure 16:

x	Original	Fitted
1	−0.041002	−0.040999
2	−0.028534	−0.028522
3	0.000848	0.000815
4	0.009175	0.009258
5	−0.006356	−0.006479
6	0.007206	0.007330
7	0.049740	0.049659
8	0.057683	0.057718
9	0.024539	0.024531
10	0.041740	0.041742

Figure 17:


Original vs Fitted Values
Overnight Movement
Day
Legend
Fitted
Original