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Bayesian Nonparametric Adaptive Control of Time-varying Systems using Gaussian Processes

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Abstract— **Real-world dynamical variations make adaptive control of time-varying systems highly relevant. However, most adaptive control literature focuses on analyzing systems where the uncertainty is represented as a weighted linear combination of fixed number of basis functions, with constant weights. One approach to modeling time variations is to assume time varying ideal weights, and use difference integration to accommodate weight variation. However, this approach reactively suppresses the uncertainty, and has little ability to predict system behavior locally. We present an alternate formulation by leveraging Bayesian nonparametric Gaussian Process adaptive elements. We show that almost surely bounded adaptive controllers for a class of nonlinear time varying system can be formulated by incorporating time as an additional input to the Gaussian kernel. Analysis and simulations show that the learning-enabled local predictive ability of our adaptive controllers significantly improves performance.**

I. INTRODUCTION

Adaptive control is a well-established area within control theory. For adaptation to uncertain, nonlinear elements of a control system, the Model Reference Adaptive Control (MRAC) framework provides a means to achieve asymptotic tracking error convergence or ultimate boundedness of tracking error [2, 9, 17, 23]. In real world scenarios the uncertainty may depend on time, however, much of the published literature does not account for this case. Specific examples of such variation include networked systems where the dynamics change as agents arrive and leave the network, systems with hybrid or switched dynamics, and systems where adaptation is required to external disturbances that are a function of the operating environment. When the uncertainty is time varying, the standard MRAC adaptive law does not guarantee asymptotic convergence [25]. Extensions to handle the linear, time-varying case exist [1, 8, 13, 14, 24]. For nonlinear matched uncertainties, a difference integration based derivative-free MRAC approaches have been proposed [25, 26]. Xu's approach [25] assumes known periodicity and shows boundedness of the system states, but only for scalar systems. It can be viewed as a difference integration scheme for traditional MRAC. Yucelen's approach ([26]) does not leverage possible knowledge of known periodicity and guarantees boundedness for multi-state systems. One special case of it can be viewed as a difference integration scheme for sigma-mod based MRAC.

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Radial Basis Function (RBF) Neural Networks are a popular choice of representation for the uncertainty due to being universal function approximators, linear in the parameters. Let $z \in \mathbb{R}^n$ be the input, $\phi_i : \mathbb{R}^n \to \mathbb{R}$ be a radial basis function, and $\phi(z) = \{\phi_1(z), \ldots, \phi_l(z)\}^T$ be the output of the basis function set. For time-varying uncertainty, the modeling error can be represented by $\Delta(z) = W^{*T}(t)\phi(z)$, which then requires knowledge of- or assumptions about- the time variation of $W^*(t)$. The net results are control laws specialized to distinct cases. An alternative solution would be to incorporate time into the basis set. Doing so without incurring too high a cost in terms of the quantity, and the adaptations laws, of the basis functions then becomes a major challenge. Fortunately, nonparametric modeling methods exist which are designed specifically for this problem.

This paper introduces a nonparametric method for adapting to model-based and time-varying uncertainty. The main idea is to model the uncertainty probabilistically using Gaussian Process (GP) models [19]. The probabilistic interpretation is sensible given that sources of uncertainty include noise, servo chattering, or other non-smooth or non-deterministic (at the level of the model) effects. A GP is defined as a collection of random variables, any finite subset of which has a joint Gaussian distribution and is an example of a generative model. The advantage of GP models is that time variation of the uncertainty simply becomes a variable of the process, leading to a nonparametric model with basis functions depending on state and time. The uncertainty is then modeled as $\mathbf{W}^{*T}\Psi(x,t)$, with the interpretation that $\mathbf{W}^*$ is a vector in an infinite dimensional Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}$, and $\Psi(x,t)$ is a set of linear functionals in $\mathcal{H}$ derived from the observations. This strategy offers more flexibility in modeling and provides theoretical support from the field machine learning, where these models are employed for (online) regression and are made viable for real-time use by imposing budgeted learning via sparsification. Therefore, rather than treating uncertainty adaptation as a completely input-output driven regression problem (and possibly suffering from over-learning), we instead learn a *data-driven, probabilistic generative model* of the uncertainty. This approach also provides a means to quantify the confidence the model has in the estimate. In short, this paper shows how to incorporate GPs in an MRAC framework with time varying uncertainty, provides theoretical guarantees of stability, and presents simulations to demonstrate performance for various problems. Preliminary work regarding GP-MRAC appeared first in [3]. The work in this paper extends the theoretical and algorithmic work

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presented in a journal submission, preprint of which has been made available as a technical report [4]. The main contribution over [4] here is the incorporation of time varying uncertainty and extensive analysis and simulation study for three cases of time variations.

II. Model Reference Adaptive Control in Presence of Time Varying Uncertainty

Consider a time varying uncertain dynamical system with state vector $x(t) = [x_1^T(t), x_2^T(t)]^T \in D_x \subset \mathbb{R}^n$, such that $x_1(t) \in \mathbb{R}^{n_s}$, $x_2(t) \in \mathbb{R}^{n_s}$, and $n = 2n_s$. Let $\tau : \mathbb{R}+ :\rightarrow \mathbb{R}^m$ be an arbitrary signal capturing the time variation of the system. The following multiple-input time-varying nonlinear uncertain dynamical system is the object of investigation

$$\dot{x}_1(t) = x_2(t), \tag{1a}$$
$$\dot{x}_2(t) = f(x(t), \delta(t), \tau(t)). \tag{1b}$$

The function f is assumed to be Lipschitz continuous in $x \in D_x$ and in $\tau \in D_\tau \subset \mathbb{R}^m$, with $f(0, 0, 0) = 0$. The control inputs $\delta \in D_\delta \subset \mathbb{R}^l$ are assumed to belong to the set of admissible control inputs consisting of measurable bounded functions. Therefore, existence and uniqueness of piecewise solutions to (1) are guaranteed. Furthermore, it is assumed that an admissible control input exists that drives the system to a neighborhood of any arbitrary point in D_x from any initial condition in D_x in finite time. It is further assumed that $l \leq n_s$ (while restrictive for overactuated systems, this assumption can be relaxed through the design of appropriate control assignment [7]). The signal τ does not need to be continuous, the only restriction imposed is that it contains finitely many discontinuities over any finite time interval. The pure time-varying case where $\tau = t$ is captured by this more general formulation. Furthermore, this formulation allows for explicit incorporation of sensor measurements of τ in adaptive control design. For example, when modeling disturbances due to wind-gusts, τ could capture gust measurements, which are arbitrary functions of time.

The Approximate Model Inversion based MRAC approach used here attempts to feedback linearize the system by finding a pseudo-control input $\nu(t) \in \mathbb{R}^{n_s}$ representative of the desired acceleration of the system. If the exact plant model in (1) is known and invertible, the required control input to achieve the desired acceleration is computable by inverting the plant dynamics. However, since this usually is not the case, an approximate inversion model $\hat{f}(x, \delta)$ is employed. The inversion model is chosen to be invertible w.r.t δ, that is, the operator $\hat{f}^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^l$ is assumed to exist and assign for every unique element of $\mathbb{R}^{n+n_s}$ a unique element of $\mathbb{R}^l$. The following assumption guarantees invertibility of $\hat{f}(x, \delta)$ w.r.t. δ

Assumption 1 $\frac{\partial \hat{f}(x,\delta)}{\partial \delta}$ is continuous w.r.t δ and nonzero over $D_x \times D_\delta$.

Therefore, given a desired pseudo-control input $\nu \in \mathbb{R}^{n_s}$ a control command δ can be found by approximate dynamic inversion as follows

$$\delta = \hat{f}^{-1}(x, \nu). \tag{2}$$

Let $z = (x, \delta)$ for brevity. The use of an approximate model results in a modeling error Δ for the system,

$$\dot{x}_2 = \nu(z) + \Delta(z, \tau), \tag{3}$$

which explicitly depends on τ,

$$\Delta(z, \tau) = f(z, \tau) - \hat{f}(z). \tag{4}$$

Were the control assignment function (the mapping between control inputs to states) known and invertible with respect to δ, then an inversion model exists such that the modeling error is not dependent on the control input δ. A designer chosen reference model is used to characterize the desired response of the system

$$\begin{aligned} \dot{x}_{1_{rm}} &= x_{2_{rm}}, \\ \dot{x}_{2_{rm}} &= f_{rm}(x_{rm}, r), \end{aligned} \tag{5}$$

where $f_{rm}(x_{rm}(t), r(t))$ denote the reference model dynamics, which are assumed to be continuously differentiable in x_{rm} for all $x_{rm} \in D_x \subset \mathbb{R}^n$. The command $r(t)$ is assumed to be bounded and piecewise continuous, furthermore, f_{rm} is assumed to be such that x_{rm} is bounded for a bounded reference input.

Define the tracking error to be $e(t) = x_{rm}(t) - x(t)$, and the pseudo-control input ν to be

$$\nu = \nu_{rm} + \nu_{pd} - \nu_{ad}, \tag{6}$$

consisting of a linear feedback term $\nu_{pd} = [K_1, K_2]e$ with $K_1 \in \mathbb{R}^{n_s \times n_s}$ and $K_2 \in \mathbb{R}^{n_s \times n_s}$; a linear feedforward term $\nu_{rm} = \dot{x}_{2_{rm}}$; and an adaptive term $\nu_{ad}(z)$. Since Δ is a function of ν_{ad} as per (4), and ν_{ad} needs to be designed to cancel Δ, the following assumption needs to be satisfied:

Assumption 2 The existence and uniqueness of a fixed-point solution to $\nu_{ad} = \Delta(\cdot, \nu_{ad})$ is assumed.

Sufficient conditions for satisfying this assumption are available in [11, 27]. Assumption 2 implicitly requires that the sign of control effectiveness to be known [11].

Using (3) the tracking error dynamics can be written as

$$\dot{e} = \dot{x}_{rm} - \begin{bmatrix} x_2 \\ \nu + \Delta \end{bmatrix}. \tag{7}$$

Let $A = \begin{bmatrix} 0 & I_{n_s} \\ -K_1 & -K_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I_{n_s} \end{bmatrix}$ where $0, I \in \mathbb{R}^{n_s \times n_s}$, are the zero and identity matrices, respectively. From (6), the tracking error dynamics are then,

$$\dot{e} = Ae + B[\nu_{ad}(z) - \Delta(z, \tau)]. \tag{8}$$

The baseline full state feedback controller ν_{pd} is chosen to make A Hurwitz. Hence, for any positive definite matrix $Q \in \mathbb{R}^{n \times n}$, a positive definite solution $P \in \mathbb{R}^{n \times n}$ exists for the Lyapunov equation

$$0 = A^T P + PA + Q. \tag{9}$$

A. The parametric way of handling time variations

The typical approach to adaptive control of time varying systems assume that the uncertainty is linearly parameterized by a set of ideal time varying weights given a fixed basis set:

$$\Delta(z) = {W^*}^T(t)\phi(z) + \epsilon(z), \tag{10}$$

where $\phi(\cdot)$ is a vector of universally approximating basis functions (e.g. Radial Basis Function [12, 21]) and $\epsilon(\cdot)$ is a bounded approximation error (which vanishes when the real basis functions are known, whereby the uncertainty is said to be structured [23]). The adaptive element is chosen to be $\nu_{ad} = W^T(t)\phi(z)$ and an update law $\dot{\nu}_{ad}$ is sought. To see the issues with this formulation, let $\Delta(z,\tau) \in \mathbb{R}$, $\epsilon(z) = 0$, $\tilde{W}(t) = W(t) - W^*(t)$, and consider the Lyapunov candidate traditionally used in adaptive control: $V(e,\tilde{W}) = \frac{1}{2}(e^TPe + \tilde{W}^T\Gamma^{-1}\tilde{W})$ with $\Gamma > 0$ is a matrix of learning rates. Taking the Lie derivative of this function and substituting the traditional gradient based adaptive law $\dot{W} = -\Gamma e^TPB\phi(z)$ yields

$$\dot{V}(e,\tilde{W}) = -\frac{1}{2}e^TQe + \tilde{W}\Gamma\dot{W}^*, \tag{11}$$

which is not sign definite and requires knowledge of $\dot{W}^*(t)$ which may not be available to make any statements about stability (note that a similar analysis was presented by Xu [25] for scalar systems). To get around this issue, Xu et al. [25] and Yucelen et al. [26] have used a difference integration based approach. In this so called "derivative-free" approach the adaptive law is characterized by

$$W(t) = \alpha_1 W(t-T) - \alpha_2(t)\phi(x(t))e(t)^TPB, \tag{12}$$

where T can capture information about known periodicity. If we set $\alpha_i = 1$, $\alpha_2(t)$ to be a bounded monotone increasing function, and replace the adaptive law by a continuity guaranteeing function until $t = T$ we recover Xu's adaptive law which can be viewed as a difference integration scheme for traditional gradient based adaptive laws [25]. With $\alpha_1 < 1$, $\alpha_2 > 0$ we recover a special case of the DFMRAC law of Yucelen, which can be viewed as a difference integration scheme for σ-modification based MRAC adaptive law [10].

Both adaptive laws incorporate a discrete weight update in a continuous framework and use Lure'-Postnikov type Lyapunov candidates to get around the issue of knowing $W^*(t)$. However, these laws do not attempt to capture the uncertainty and its local time-variations. Furthermore, the associated control inputs can become quite aggressive and difficult to implement (see Section IV for numerical simulations). The next section describes a nonparametric approach to handling time variation in adaptive control.

III. A Bayesian Nonparametric Approach to Handling Time Variations

Instead of focusing on modeling the uncertainty $\Delta(z,\tau)$ as a time-dependent input-output map, we propose instead to learn a probabilistic generative model of the uncertainty. That is, it is assumed that the uncertainty can be described completely through a time varying (prior) mean function and a covariance function. Bayesian posterior updates are used to build a generative model of the uncertainty. Learning generative models is far better approach than learning NN input-output representation approach [16, 18**?**]. Given that we are attempting to learn continuous functions of time and state, Gaussian Process (GP) priors are a great fit [18].

A GP is defined as a collection of random variables such that every finite subset is jointly Gaussian. The joint Gaussian condition allows GPs to be completely characterized by their second order statistics [**?**]. The GP is a distribution over functions, that is, a draw from a GP is a function. For the sake of clarity of exposition, we will assume that $\Delta(z,\tau) \in \mathbb{R}$; the extension to the multidimensional case is straightforward. Let $m(z,\tau)$ be the mean function, and $k(\cdot,\cdot)$ be a positive definite covariance kernel, then GP is represented as

$$\Delta(z(t),\tau(t)) \sim \mathcal{GP}(m(z(t),\tau(t)), k((z(t),\tau(t)),(z(t'),\tau(t')))). \tag{13}$$

It is assumed that the mean lies in the the class of functions

$$\mathcal{G} = \left\{ g \in \mathbb{R}^{\mathcal{X}} |\ g(\cdot) = \sum_{i=1}^{\infty} \alpha_i k(z_i,\cdot) \right\} \tag{14}$$

where $\mathcal{X} = \mathbb{R}^n$, $\alpha_i \in \mathbb{R}$, $z_i \in \mathcal{X}$ and $\|g\|_{\mathcal{H}} < \infty$ s.t. $\|g(\cdot)\|^2_{\mathcal{H}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha_i\alpha_j k(z_i,z_j)$ [22]. In the following, let $\bar{z} = [z,\tau]$ for ease of exposition. Let $\bar{Z}_t = \{\bar{z}_1,\ldots,\bar{z}_t\}$ be a set of state measurements, with outputs $y_t(\bar{z}_i) = \Delta(\bar{z}_i) + \epsilon_i$, where $\epsilon_i \sim \mathcal{N}(0,\omega^2)$ is Gaussian white measurement noise. The definition of the GP above implies that the data $\{\Delta(\bar{z}_1),\ldots,\Delta(\bar{z}_t)\}$ has prior distribution $\mathcal{N}(0, K(\bar{Z}_t,\bar{Z}_t))$, where $K(\bar{Z}_t,\bar{Z}_t)$ is the Gram matrix of the elements in $\bar{Z}_t$. It can be shown that given a new input $\bar{z}_{t+1}$, the joint distribution of the data available up to t and $\bar{z}_t$ under a Gaussian prior distribution is given by

$$\begin{bmatrix} y_t \\ y_{t+1} \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} K(\bar{Z}_t,\bar{Z}_t) + \omega^2 I & k_{\bar{z}_{t+1}} \\ k^T_{\bar{z}_{t+1}} & k^*_{t+1} \end{bmatrix}\right), \tag{15}$$

where $k_{\bar{z}_{t+1}} = K(\bar{z}_{t+1},\bar{Z}_t)$ and $k^*_{t+1} = k(\bar{z}_{t+1},\bar{z}_{t+1})$. The posterior (called the predictive) distribution is obtained by conditioning the joint Gaussian prior distribution over the observation $\bar{z}_{t+1}$ to yield

$$p(y_{t+1}|\bar{Z}_t, y_t, \bar{z}_{t+1}) \sim \mathcal{N}(\hat{m}_{t+1}, \bar{\Sigma}_{t+1}), \tag{16}$$

where

$$\hat{m}_{t+1} = \alpha^T_{t+1}k_{\bar{z}_{t+1}} \tag{17}$$

$$\bar{\Sigma}_{t+1} = k^*_{t+1} - k^T_{\bar{z}_{t+1}}Ck_{\bar{z}_{t+1}} \tag{18}$$

are the estimated mean and covariance respectively, and $C := (K(\bar{Z}_t,\bar{Z}_t) + \omega^2 I)^{-1}$ and $\alpha_{t+1} := Cy_t$. Since positive definite kernels generate a mapping $\psi : \mathbb{R} \to \mathcal{H}$, where $\mathcal{H}$ is an RKHS and $k(x,y) = \langle\psi(x),\psi(y)\rangle_{\mathcal{H}}$, any set $\bar{Z}_t$ is associated with a set of elements $\bar{\psi}(Z)_t$ in $\mathcal{H}$. This can be thought of as a subspace of functions $\mathcal{F}_{\bar{Z}_t}$ in $\mathcal{H}$. In fact, the vector $\hat{m}_{t+1}$ is the set of coefficients that realizes *one* particular function from this family as an estimate of the mean.

Let $\sigma(t) \in \mathbb{N}$ denote a switching index which increments every time a data point is added or removed from basis vector set. Therefore, when the σ^{th} system is active, the mean function in (17), evaluated using the active set of basis vectors, is represented as $\hat{m}^\sigma$. This results in a model of the stochastic process $\Delta(\bar{z})$ given the sparsified set of all available data as

$$\bar{\nu}_{ad}(\bar{z}) \sim \mathcal{GP}(\hat{m}^\sigma(\bar{z}), k(\bar{z}, \bar{z}')). \tag{19}$$

The adaptive signal $\nu_{ad}(\bar{z})$ is set to be equal the estimate of the mean $\hat{m}^\sigma(\bar{z})$:

$$\nu_{ad}(\bar{z}) = \hat{m}^\sigma(\bar{z}). \tag{20}$$

Since both $\bar{Z}_t$ and y_t grow with data, computing the inverse in (17) can become computationally intractable. This is less of a problem for traditional GP regression applications, which include regression problems with finite learning samples; however, in an online setting, the cardinality of the samples need not be bounded. Therefore, in order to extend GP regression to MRAC, an online method to restrict the number of data points stored for inference is needed.

One way to restrict the number of updates to $\bar{Z}_t$ is to check to see how much the point contributes to $\mathcal{F}_{\bar{Z}_t}$ via the linear independence test. If the contribution is below a threshold γ, the point is not added to $\bar{Z}_t$. However, if a point needs to be added, and the budget has been exceeded, one of the points in $\bar{Z}_t$ needs to be deleted. In the Oldest Point scheme, the oldest point added to $\bar{Z}_t$ is deleted. In the KL scheme, the KL divergences between the GP if the point were added to $\bar{Z}_t$ and the GP's with one point removed each are computed; the point with the largest difference is then discarded [6]. For more details on this algorithm, see our preprint on GP-MRAC (with non-time varying uncertainty) [4].

A. Analysis of Stability

In this section we establish results relating to the stability of Gaussian process based MRAC using the online GP sparsification algorithm described in [4]. In particular, the boundedness of the tracking error dynamics of (8) is shown, which is represented using the GP adaptive element:

$$de = Ae\ dt + B(\epsilon_m^\sigma(\bar{z}) - G\ d\xi), \tag{21}$$

where $\nu_{ad}(\bar{z}) \sim \mathcal{GP}(\hat{m}^\sigma(\bar{z}), k(\bar{z}, \bar{z}'))$, $G = \mathbb{V}(\Delta(\bar{z}))$ is the variance of $\Delta(\bar{z})$, and $\epsilon_m^\sigma(\bar{z}) = \hat{m}^\sigma(\bar{z}) - m(\bar{z})$. Note that we have used a Wiener process representation of the GP, that is we assume that a draw from the GP $\Delta(\bar{z})$ can be modeled as $m(\bar{z}) + G\xi$. While other GP representations can be used, this approach facilitates our analysis. First it is demonstrated that due to the nonparametric nature of this algorithm, the GP never loses its local predictive ability even when the uncertainty is time varying.

Lemma 1 Let $\Delta(\bar{z})$ be represented by a Gaussian process as in (13), and $\hat{m}^\sigma(\bar{z})$ be defined as in (19). Then $c_1 := \|\Delta(\bar{z}) - \hat{m}^\sigma(\bar{z})\|$ is almost surely (a.s.) bounded for each σ.

The proof is available in [5].

The next Lemma shows that because Algorithm 1 in [4] adds or removes kernels from the basis set $\mathcal{BV}$ to keep a metric of the representation error bounded, $\epsilon_m(\bar{z})$ is also bounded even when the system is time-varying.

Lemma 2 Let $\Delta(\bar{z})$ be represented by a Gaussian process as in (13), and $\hat{m}^\sigma(\bar{z})$ be defined as in (19) inferred based on sparsified data, and let $m(\bar{z})$ be the mean of the GP without any sparsification, then $\epsilon_m(\bar{z}) := m(\bar{z}) - \hat{m}^\sigma(\bar{z})$ is bounded almost surely.

The proof is available in [5].

The boundedness of the tracking error can now be proven.

Theorem 1 Consider the time varying system in (1), the reference model in (5), the control law of (6) and (2). Let the uncertainty $\Delta(\bar{z})$ be represented by a Gaussian process as in (13), then Algorithm 1 in [4] and the adaptive signal $\nu_{ad}(\bar{z}) = \hat{m}^\sigma(\bar{z})$ guarantees that the system is mean square ultimately bounded a.s. inside a compact set.

The proof is available in [5].

IV. Application to Trajectory Tracking in Presence of Wingrock Dynamics

Modern highly swept-back or delta wing fighter aircraft are susceptible to lightly damped oscillations in roll known as "Wing Rock". Wing rock often occurs at conditions commonly encountered at landing ([20]), making precision control in presence of wing rock critical. In this section we use the GP-MRAC approach to track a sequence of roll commands in the presence of simulated wing rock dynamics. However, unlike in [4], we introduce several different classes of time variations in the wing rock dynamics. Let θ denote the roll attitude of an aircraft, p denote the roll rate and δ_a denote the aileron control input. Then a model for time varying wing rock dynamics is [15]

$$\dot{\theta} = p \tag{22}$$

$$\dot{p} = L_{\delta_a}\delta_a + \Theta(x,t), \tag{23}$$

where the generative model of $\Theta(x,t)$ is a GP, that is $\Theta(\bar{z}) \sim \mathcal{GP}(\bar{\Theta}(\bar{z}), k(\bar{z}, \bar{z}')$. The time-varying mean of this GP is

$$\bar{\Delta}(\bar{z}) = W_0^*(t) + W_1^*(t)\theta(t) + W_2^*(t)p(t) + W_3^*(t)|\theta(t)|p(t) + W_4^*(t)|p(t)|p(t) + W_5^*(t)\theta^3(t),$$

and the variance is w_n^2. The chosen inversion model has the form $\nu = \frac{1}{L_{\delta_a}}\delta_a$, with $L_{\delta_a} = 3$. Therefore, the mean of the modeling uncertainty of (4) $\bar{\Delta}(\bar{z}) = \bar{\Theta}(\bar{z})$. The states are assumed to be corrupted with Gaussian white noise. The adaptive controller uses the control law of (6). The gain for the linear part of control law (ν_{pd}) is given by $K_1 = 1.2, K_2 = 1.2$. A second order reference model with natural frequency of 1 rad/sec and damping ratio of 0.5 is chosen. The simulation uses a time-step of 0.05 sec. The maximum size of the basis set $\mathcal{BV}$ (p_{max}) was set to 100, and points were selected for storage based on both the oldest point (OP) and KL divergence (KL) schemes (see

[4]). For DF-MRAC we chose $\alpha_1 = 0.9$, $\alpha_2 = 10$, and $T = 10$ seconds (roughly equal to the largest period as suggested by [25]). In order to show the efficacy of our approach, we perform comparisons with different types of time variation by changing the weights $W^*(t)$ according to different schemes. These are a) smoothly varying, periodic functions of time b) smoothly varying periodic functions with switching and c) arbitrary non-periodic smooth functions of time. The weight variations for the schemes are depicted in Figure 1.

Figure 2 presents a comparison between the states of the system and the reference model for DF and GP-MRAC for all the three time variation schemes. It can be seen that both the KL and OP GP-MRAC outperform the DF scheme for the periodic and switching cases of time variations. This indicates that these schemes are better at handling the sudden jump in the ideal weights of the uncertainty in the first few seconds. For the smooth arbitrary time variation case, the performance of the DF and OP GP-MRAC controllers is comparable, while the KL GP-MRAC suffers slightly. This is possibly because the linear independence test employed in KL GP-MRAC does not remove older (somewhat irrelevant) points sufficiently fast due to the smooth variations in the weights, the output however is bounded. The tracking error is better visible in Figure 3. It can therefore be argued that all three algorithms give a comparable performance with respect to the tracking error metric, although DF-MRAC has poor transient performance (for the chosen parameters). Figure 4 plots the pointwise error between the mean of the true uncertainty and the adaptive element's estimate of the mean of the true uncertainty. After a poor initial transient performance for the smooth and switching schemes, DF-MRAC settles into a smaller error around zero, while the OP and KL algorithms perform very well for all time, as seen in Figures 4(a) and 4(b). In Figure 4(c) however, one can see that DF-MRAC's performance is quite poor, while both GP-MRAC controllers performs well, suggesting that the DF-MRAC adaptive element is not capturing the uncertainty. This is rather interesting, and points to the instantaneous reactive-learning nature of DF-MRAC, since for the same case, DF-MRAC did well with tracking. However, Figure 5 shows that DF-MRAC's good tracking performance comes at a much higher control cost: while the DF-MRAC control input has similar peak magnitudes, it is in general more "noisy", which is indicative of the controller taking possibly unnecessary actions (in response to the noise and time variations in the system) to reactively keep the tracking error low. On the other hand, both GP schemes have much smoother control response, indicating that its ability to learn the uncertainty results in better local prediction that pays off in control cost reduction too.

In order to quantify these algorithms' *effective* performance, a metric is needed that takes into account a) the tracking error ($e(t)$) b) the control output ($u(t)$) and c) the derivative of the control output ($\dot{u}(t)$). This is given by

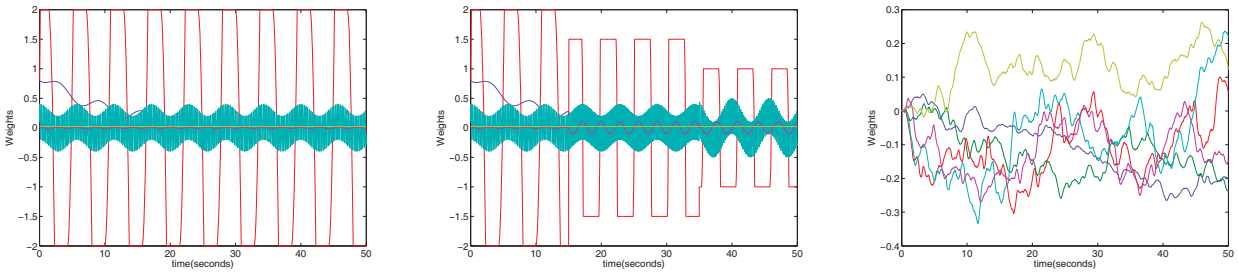
$$\mathcal{C} = \int (e^T(t)Pe(t) + u^T(t)Qu(t) + \dot{u}^T(t)R\dot{u}(t))dt, \quad (24)$$

where P, Q and R are positive weighting matrices. Furthermore, the presence of noise necessitates evaluation over multiple simulation runs. Figure 6 presents the results using (24) over fifty runs, with the lowest, highest and mean values plotted. The figures show that GP-MRAC significantly outperforms DF-MRAC over multiple-runs, with OP being the most effective scheme.

Overall, these results show that GP-MRAC is successful in capturing and (locally) adapting to time varying uncertainty with minimal prior domain knowledge. The addition of time as data to the algorithm allows it more flexibility than RBFN based DF-MRAC methods that have been proposed for adaptation to time-varying uncertainty, because it can create (and remove) basis functions that are localized in time as well as space. The ability to create and update a budgeted dictionary of bases and associated parameters to learn and predict (locally) the uncertainty is a hallmark of our nonparameteric GP-MRAC method. The results here indicate that this ability greatly helps when dealing with time-varying uncertainties. Since the goal of GP-MRAC is to model the uncertainty as accurately as possible, it is able to perform well not only in terms of the tracking error, but also in terms of control cost (since it is not forced to constantly adapt in order to reduce the tracking error). It should be noted that tuning α_1 may lead to better tracking performance by DF-MRAC, however, in our experience its ability to locally predict the uncertainty and consequently generating an efficient input in presence of noise is not comparable with GP-MRAC. Finally, in the results presented here, $\tau = t$, therefore, the KL GP-MRAC method was not able to leverage its ability to keep older centers. If τ is some other external variable that takes on recurring values, the KL scheme could well outperform the OP scheme, as is the case when the uncertainty is not time-varying [4].

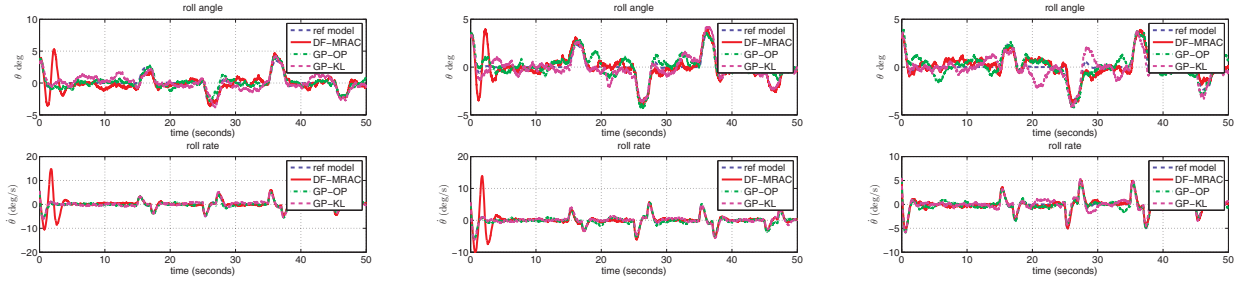
V. Conclusion

In this paper we presented a Bayesian nonparametric approach to Model Reference Adaptive Control (MRAC) of time varying systems. Gaussian Process (GP) adaptive elements were used to capture time varying uncertainties by adding a time-dependent variable to the kernel function. The performance of this adaptive control approach was shown to be superior in terms of tracking and control cost as compared to existing derivative-free approaches that have been suggested for control of time varying systems. These results bring out the benefits of using a nonparametric adaptive elements, as it is not possible to have a time dependent universally approximating basis with fixed, finite number of basis functions for systems that operate over long intervals. Furthermore, for a wide variety of possible kernel based data-driven adaptive control algorithms, the techniques presented here can be used to lay to rest the concern that learning-focused data-driven adaptive controllers can become over-burdened with irrelevant data in presence of time-variation.



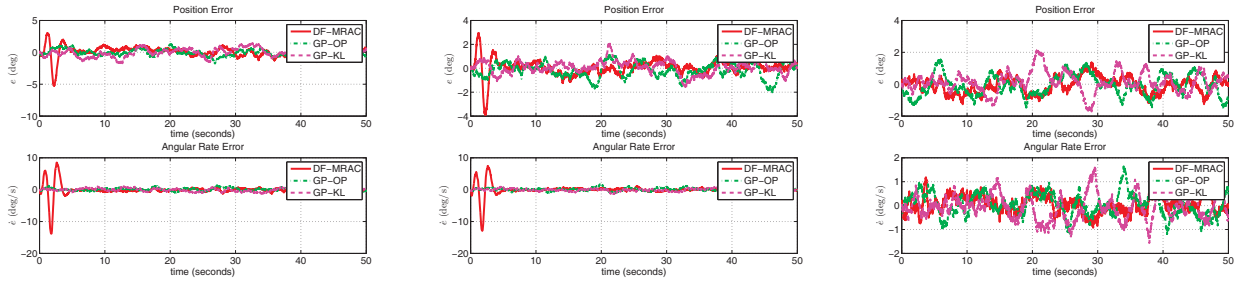
(a) Smoothly varying periodic parameters (b) Switching periodic parameters (c) Nonperiodic parameters

Fig. 1. Parameter plots



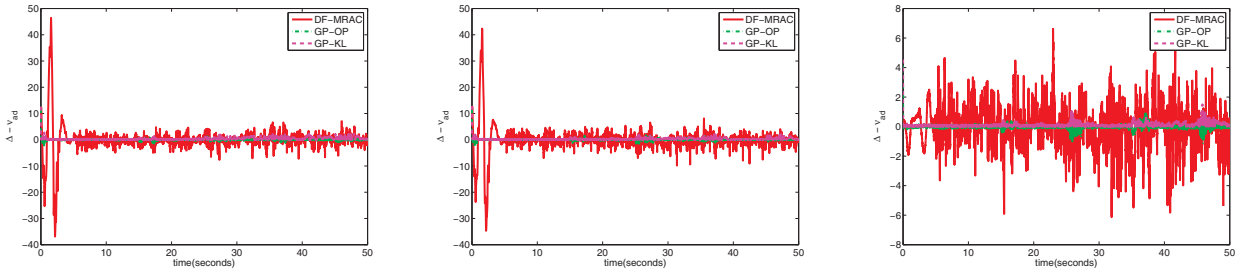
(a) Smoothly varying periodic parameters (b) Switching periodic parameters (c) Nonperiodic parameters

Fig. 2. Comparison of system states and reference model between GP-MRAC, and DF-MRAC with uniformly distributed centers over their respective domains. Note that the state measurements are corrupted with Gaussian white noise.



(a) Smoothly varying periodic parameters (b) Switching periodic parameters (c) Nonperiodic parameters

Fig. 3. Comparison of tracking error between GP-MRAC, and DF-MRAC with uniformly distributed centers over their respective domains.



(a) Smoothly varying periodic parameters (b) Switching periodic parameters (c) Nonperiodic parameters

Fig. 4. Error between the adaptive element output and the actual uncertainty. DF-MRAC's approximation with uniformly distributed centers is significantly worse than the GP-MRAC's.

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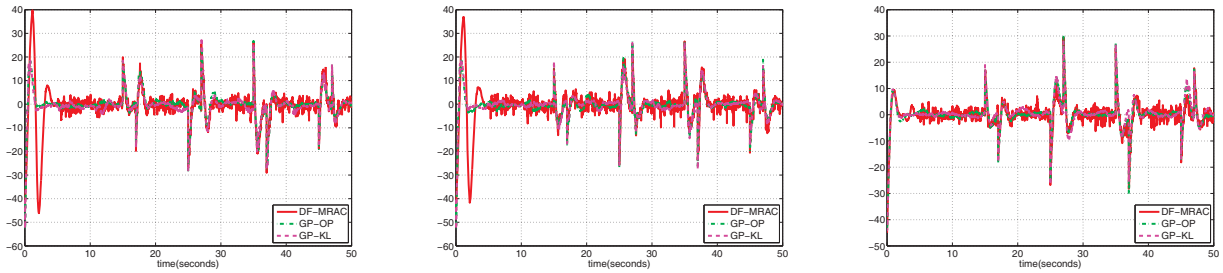
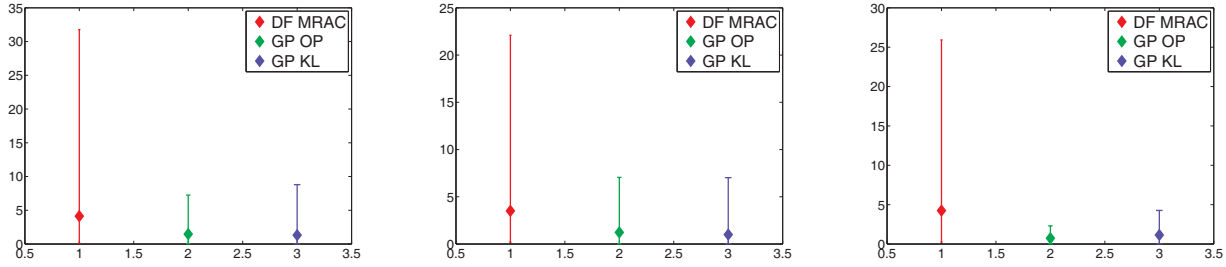


(a) Smoothly varying periodic parameters (b) Switching periodic parameters (c) Nonperiodic parameters

Fig. 5. Comparison of control output between GP-MRAC, and DF-MRAC with uniformly distributed centers over their respective domains.



(a) Smoothly varying periodic parameters (b) Switching periodic parameters (c) Nonperiodic parameters

Fig. 6. The mean and the highest value achieved of the cost metric in 24 for the three different controllers over 50 simulations runs. It can be seen that OP GP-MRAC has the best and most consistent performance of the three methods.

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