6D Pose Uncertainty in Robotic Perception

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Abstract. Robotic perception is fundamental to important application areas. In the Joint Research Project DESIRE, we develop a robotic perception system with the aim of perceiving and modeling an unprepared kitchen scenario with many objects. It relies on the fusion of information from weak features from heterogenous sensors in order to classify and localize objects. This requires the representation of wide spread probability distributions of the 6D pose.

In this paper we present a framework for probabilistic modeling of 6D poses that represents a large class of probability distributions and provides among others the operations of fusion of estimates and uncertain propagation of estimates.

The orientation part of a pose is described by a unit quaternion. The translation part is described either by a 3D vector (when we define the probability density function) or by a purely imaginary quaternion (which leads to a prepresentation of a transform by a dual quaternion). A basic probability density function over the poses is defined by a tangent point on the 3D sphere (representing unit quaternions), and a 6D Gaussian distribution over the product of the tangent space of the sphere and of the space of translations. The projection of this Gaussian induces a distribution over 6D poses.

One such base element is called a Projected Gaussian. The set of Mixtures of Projected Gaussians can approximate the probability density functions that arise in our application, is closed under the operations mentioned above and allows for an efficient implementation.

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1 Introduction

As a basic tool in robotic perception, probability density functions of 6D poses need to be represented. In order to be able to represent and process weak information from imperfect sensors, widely spread densities need to be covered by the representation and the inference mechanisms.

The more critical part in the representation of a rigid transform is the rotation. The requirements concerning the parameterization of the rotation are contradictory, but our design goal is to satisfy them as well as possible:

- Unique: There should be only one representation for each orientation
- Minimal: The rotation should be represented with few parameters
- Composable: There should be an easy way to derive the parameters of the composed rotation from the parameters of two rotations in the composition
- Smooth: The rotation should be an at least continuous, or better still a differentiable function of the parameters.
- Distance and area preserving: Properties like areas or distances in the parameter space should be preserved under rigid transform. This is important when we deal with probability density functions over the rotations or transforms.

The formalism for the probability density function of the 6D poses should satisfy the following properties:

- Coordinate System Independent: A coordinate change should only change the arguments to the pdf, not the structure or the parameters of the pdf.
- Information Fusion: The formalism supports the fusion of two probability density informations (for example maximum likelihood estimation).
- Information Propagation: The formalism supports the propagation of uncertain information (i.e. a pose estimate) through an uncertain transform.
- The representation of the pdf uses not too many parameters, much fewer than for example a particle set.

Since each position and orientation w.r.t a given coordinate system is the result of a translation and a rotation. *Position* and *translation* can be and will be used synonymously in this paper, as well as *orientation* and *rotation*. Also, *pose* and *(rigid) transform* are used synomymously.

In Section 2 we will recapitulate various approaches to the parametrization of rigid transforms and corresponding probability density functions. None of them fulfills all requirements listed above, but they provide ingredients to our synthesis. In Section 3 we will present our approach to probability density functions over rigid transforms. In Section 4 we will recollect the presented system and indicate directions of future work.

2 Previous Work

The representation of rigid transforms, and especially of orientation, in 3D is a central issue in a variety disciplines of arts, science and engineering, and contributions from various disciplines are available.

The most popular representations of a 3D rotation are rotation matrix, Euler angles, Rodrigues vector and unit quaternions. For rotation matrices, renormalization is difficult, Euler angles are not invariant under transforms and have singularities, and Rodrigues vectors do not allow for an easy composition algorithm.

Stuelpnagel [9] points out that unit quaternions are a suitable representation of rotations in 3D with few parameters, but does not provide probability distributions.

Choe [7] represents the probability distribution of rotations via a projected Gaussian on a tangent space. However, he only deals with concentrated distributions, and he does not take translations into account.

Goddard and Abidi [4, 5] use dual quaternions for motion tracking. They also capture the correlation between rotation and translation. The probability distribution over the parameters of the state model is a uni-modal normal distribution. This is an appropriate model if the initial estimate is sufficiently certain, and if the information that is to be fused to the estimate is sufficiently well focused. Dual quaternions provide a closed form for the composition of rigid transforms, similar to the transform matrix in homogeneous coordinates (see also Kavan et al. [13]).

Antone [6] suggests to use the Bingham distribution in order to represent weak information. However, he does not give a practical algorithm for fusion of information or propagation of uncertain information. Also, Love [10] states that the renormalization of the Bingham distribution is computationally expensive. Furthermore, it is not (yet) clear to us how the Bingham distribution for rotations could be extended to rigid transforms.

Mardia et al. [12] use a mixture of bivariate von Mises distributions. They fit the mixture model to a data set using the EM algorithm. This allows for modelling widely spread distributions. However, they do not treat translations.

In general, the Jacobian is used to propagate the covariance matrix of a random variable through a non-linear function. Kraft et al. [11] use an unscented Kalman Filter - this technique could be applied also in our setting. However, it would have to be extended to the mixture distributions.

From the analysis of the previous work, we synthesize our approach as follows: We use unit quaternions to represent rotations in 3D, and dual quaternions to obtain a concise algebraic description of rigid transforms and their composition. The base element of a probability distribution over the rigid transforms is a Gaussian in the 6D tangent space, characterized by the tangent point to the unit quaternions and the mean and the covariance of the distribution. Such a base element is called a Projected Gaussian. We use mixtures of Projected Gaussians to reach the necessary expressive power of the framework.

3 Pose Uncertainty by Mixtures of Projected Gaussian Distributions

We assume that the quaternion as such is sufficiently well known to the reader. In order to clarify our notation, at first some basics are restated.

3.1 Quaternions

Let $\mathbb{H}$ be the quaternions, i.e $\mathbb{H} = \{q|q = a+ib+jc+kd\}$, where a is the real part of the quaternion, and the vector $v = (b,c,d)$ is the imaginary part. The imaginary units $\{i,j,k\}$ have the properties $i^2 = j^2 = k^2 = ijk = -1, ij = k, jk = i, ki = j$. The quaternions can be identified with $\mathbb{R}^4$ via the coefficients, $q = a+ib+jc+kd \sim (a,b,c,d)$. The norm of a quaternion is defined as $\|q\|^2 = a^2+b^2+c^2+d^2$, the conjugate of a quaternion as $q^* = a-ib-jc-kd$ .With the above properties of quaternions we have $\|q\|^2 = q*q^*$.

Analogously to the way that unit complex numbers $z = \cos(\phi)+i\sin(\phi) = e^{i\phi}$ represent rotations in 2D via the formula $p_{\text{rot}} = zp$ for any point $p \in \mathbb{C}$, unit quaternions represent rotations in 3D.

A point (p_1, p_2, p_3) in 3D is represented as the purely imaginary quaternion $p = ip_1 + jp_2 + kp_3$; a rotation around the unit 3D axis v by the rotation angle θ is given by the quaternion
$q = \cos(\theta/2) + \sin(\theta/2)(iv_1 + jv_2 + kv_3)$.

The rotated point is obtained as $p_{\text{rot}} = q*p*q^*$. Clearly, q and $-q$ represent the same rotation, so the set U of unit quaternions is a double coverage of the special orthogonal group SO(3) of rotations in 3D.

The set U of unit quaternions is identified with the 3-dimensional unit sphere S_3 in $\mathbb{R}^4$, and probability density functions on U are defined by probability density functions on S_3.

3.2 Base Element

For a sufficiently expressive set of probability density functions on the rotations we choose a mixture of base elements.

Each base element is obtained by projecting a Gaussian distribution defined on a tangent space onto the sphere of unit quaternions. This technique is illustrated in Figure 1 for the example of a 1-dimensional unit sphere in $\mathbb{R}^2$. Note that the peaks are lower due to renormalization.

Definition 1: Let S_3 be the 3-dimensional unit sphere in $\mathbb{R}^4$ and r_0 be an arbitrary point on S_3. Further, let $T(r_0) \sim \mathbb{R}^3$ be the 3-dimensional tangent space to the sphere S_3 at the point r_0, with a local coordinate system that has the point q_0 as origin. Further, let $\mathcal{N}(\mu, \Sigma)$ be a Gaussian distribution on $T_R(r_0)$ and the corresponding probability density function be p_T. With the 2-valued central

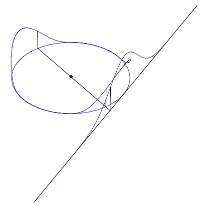

Fig. 1 A base element on the unit circle, obtained by projecting a Gaussian on a tangent line.

projection $\Pi_{r_0} : T_R(r_0) \longrightarrow S_3$ (Figure 1 illustrates how a probability density function is induced on the unit sphere $S_1 \subset \mathbb{R}^2$), a density function is given on S_3 by $p_S(r) := \frac{1}{C} p_T\left(\Pi_{r_0}{}^{-1}(r)\right)$ with $C = \int_S p_T\left(\Pi_{r_0}{}^{-1}(r)\right) dr$. The set of these pdfs p_S is called set of Rotational Projected Gaussians or RPG. The subset of pdfs for which $\mu = 0$ in the corresponding Gaussian on the tangent space T_R is denoted as RPG_0. Note that this definition is not valid for points $r^\perp \in S_3$ that are orthogonal to r_0. $p_S\left(r^\perp\right) := 0$ is the continuous completion.

In practice, the RPG is represented by its tangent point and the basis of the tangent space, and by the parameters of the corresponding Gaussian distribution: $p_S \sim \mathcal{N}\left(T_R(r_0), \mu, \Sigma\right)$. Note that the same distribution can be represented by RPGs with antipodal tangent points.

3.3 *Pose Uncertainty*

The pose uncertainty is modeled along the lines of the rotation uncertainty by including the translation.

Definition 2: Let SE(3) be the group of rigid transforms in $\mathbb{R}^3$, the rotation represented by a unit quaternion or equivalently a point on S_3 and the translation by a vector in $\mathbb{R}^3$, SE$(3) \sim S_3 \times \mathbb{R}^3$. Let $x = (r_0, t)$ be a transform (or pose), with rotation r_0 and translation t. The tangent space to x is given by $T(r_0) := T_R(r_0) \times \mathbb{R}^3 \sim \mathbb{R}^6$, where $T_R(r_0) \sim \mathbb{R}^3$ is the tangent space to the rotation part. Let $\mathcal{N}(\mu, \Sigma)$ be a Gaussian distribution on $T_R(r_0)$. With the 2-valued mapping

$\Pi_{(r_0,t)} : T(r_0) \longrightarrow S_3 \times \mathbb{R}^3$,

$\Pi_{(r_0,t)}(y_1, y_2, y_3, y_4, y_5, y_6) = \left(\Pi_{(r_0)}(y_1, y_2, y_3), (y_4, y_5, y_6)\right)$

a density function is given on $S_3 \times \mathbb{R}^3$ by $p(r,t) := \frac{1}{C} p\left(\Pi_{(r_0,t)}{}^{-1}(r,t)\right)$ with

$C = \int_{S_3 \times \mathbb{R}^3} p\left(\Pi_{(r_0,t)}{}^{-1}(r,t)\right) drdt$. The set of these pdfs p is called set of Projected Gaussians or PG. The subset of pdfs for which $\mu_1 = \mu_2 = \mu_3 = 0$ in the corresponding Gaussian on the tangent space is referenced as PG_0.

Note that $S_3 \times \mathbb{R}^3$ is a double coverage of SE(3) in the same way that S_3 is a double coverage of SO(3). Note also that we do not shift the origin with respect to the position part of the pose. Again, in practice the PG is represented by the tangent space as $p \sim \mathcal{N}\left(T(r_0), \mu, \Sigma\right)$.

3.4 Fusion of Projected Gaussians

In analogy to the fusion of Gaussian pdfs pertaining to the same phenomenon, we now describe the fusion of two PGs. The approach is to find a common tangent space that can represent both of the original PGs reasonably well. A detailed analysis based on the approximation theory of the MPG framework exceeds the scope of this paper - a valid heuristic for PG_0 type distributions is to fuse PGs if the angle between the tangent points is less than 15°, or, equivalently, larger than 165°. Below we describe the fusion process for PG in general - in practice we use PG_0.

Let $p_1 \sim \mathcal{N}(T(r_{0,1}), \mu_1, \Sigma_1)$ and $p_2 \sim \mathcal{N}(T(r_{0,2}), \mu_2, \Sigma_2)$ be two pose pdfs with $\cos(r_{0,1} \cdot r_{0,2}) \geq 0.966$ (if $\cos(r_{0,1} \cdot r_{0,2}) \leq -0.966$, use $-r_{0,2}$ instead of $r_{0,2}$, the rest is unchanged).

1. Select $r_{0,3} = \frac{1}{\|r_{0,1}+r_{0,2}\|}(r_{0,1} + r_{0,2})$ as the first tangent point for a common tangent space $T(r_{0,3})$. The basis of the space can be selected arbitrarily (we use a random basis).
2. Restate p_1 in $T(r_{0,3})$: We define the transfer function $f_{1,3} : T(r_{0,1}) \longrightarrow T(r_{0,3})$ by $f_{1,3}(y) := \Pi_{(r_{0,3},t)}{}^{-1}\left(\Pi_{(r_{0,1},t)}(y)\right)$, and the Jacobian of this transfer function at the mean value $\mu_1$of the original distribution p_1: $J_{1,3} = \left.\frac{\partial f_{1,3}}{\partial y}\right|_{\mu_1}$. The statistical moments of the distribution p_1 represented in $T(r_{0,3})$ are then estimated as $\mu_{3,1} = f_{1,3}(\mu_1)$ and $\Sigma_{3,1} = J_{1,3} \cdot \Sigma_1 \cdot J_{1,3}{}^T$, so $p_{3,1} \sim \mathcal{N}(T(r_{0,3}), \mu_{3,1}, \Sigma_{3,1})$.
3. Restate p_2 in $T(r_{0,3})$ as $p_{3,2} \sim \mathcal{N}(T(r_{0,3}), \mu_{3,2}, \Sigma_{3,2})$. Note that while this is technically well defined even for large angle difference and wide spread distributions, it only makes sense for rather small angle differences and concentrated distributions. If wide distributions are needed, we use mixtures (Section 3.6).
4. Fuse $p_{3,1}$and $p_{3,2}$: These pdfs are now stated in the same $\mathbb{R}^6$, so the fused pdf is $p_3 \sim \mathcal{N}(T(r_{0,3}), \mu_3, \Sigma_3)$, with the parameters $\Sigma_3 = \left(\Sigma_{3,1}{}^{-1} + \Sigma_{3,2}{}^{-1}\right)^{-1}$and $\mu_3 = (\Sigma_{3,1} + \Sigma_{3,2})^{-1} \cdot (\Sigma_{3,2} \cdot \mu_{3,1} + \Sigma_{3,1} \cdot \mu_{3,2})$. The resulting probability density function on $S_3 \times \mathbb{R}^3$ needs to be normalized according to definition 2.
5. Generally, $\mu_3 \neq 0$. Since it is advantageous to refrain to base elements of type PG_0, p_3 is restated according to step 2, with the new tangent point $r_{0,4} = \Pi_{(r_{0,3},t)}(\mu_3)$. Finally, the resulting base element is renormalized.

3.5 Composition of Transforms

It is required to model uncertain transforms of uncertain poses, for example if a sensor is mounted on a mobile robot and the pose estimate is needed in world coordinates. In our framework, transforms and poses are represented as dual quaternions in order to calculate the probability distribution function of the composition, see Goddard [4] for more detail.

3.5.1 Dual Quaternions

A dual quaternion $\mathbf{q}_1 = q_{1,1} + eq_{1,2}$ is composed of two quaternions q_1 and q_2 and the dual number e, with $e \cdot e = 0$. Summation of dual quaternions is per component, $\mathbf{q}_1 + \mathbf{q}_2 = (q_{1,1} + eq_{1,2}) + (q_{2,1} + eq_{2,2}) = (q_{1,1} + q_{2,1}) + e\,(q_{1,2} + eq_{1,1})$.

The product of dual quaternions is

$$\mathbf{q}_1 * \mathbf{q}_2 = (q_{1,1} + eq_{1,2}) * (q_{2,1} + eq_{2,2}) = (q_{1,1} * q_{2,1}) + e\,(q_{1,2} * q_{2,1} + q_{1,1} * q_{2,2}).$$

The conjugate of a dual quaternion is $\mathbf{q}^* = {q_1}^* + e{q_1}^*$.

Let q_r be the rotation unit quaternion, and $q_t = [0, t_1, t_2, t_3]$ be the quaternion derived from the translation components, then the dual quaternion $\mathbf{q} = q_r + e0.5 q_t * q_r$ represents the transform. A point (p_1, p_2, p_3) is embedded into the dual quaternions as $\mathbf{p} = [1,0,0,0] + e\,[0, p_1, p_2, p_3]$, and with this convention the rotation and translation is $\mathbf{q} * \mathbf{p} * \mathbf{q}^*$. The composition of two transforms, or of a transform and a pose, is represented by the product of the dual quaternions,

$$\hat{\mathbf{p}} = \mathbf{q}_2 * \mathbf{q}_1 * \mathbf{p} * {\mathbf{q}_1}^* * {\mathbf{q}_2}^* = \mathbf{q}_2 * \mathbf{q}_1 * \mathbf{p} * (\mathbf{q}_2 * \mathbf{q}_1)^*.$$

The composition function g will be used to derive the covariance:

$$\mathbf{q}_3 = g\,(\mathbf{q}_1, \mathbf{q}_2) = g_T\,(y_1, y_2),$$

where the y_i are the 6-dimensional vectors on the corresponding tangent spaces, and g and g_T are related via the central projections $\Pi_{(r_{0,1},t)}$ and $\Pi_{(r_{0,2},t)}$.

3.5.2 Calculation of Composition

This algebraic formulation justifies to set the tangent point of the composed base element to $r_{0,3} = r_{0,2} * r_{0,1}$, which for PG_0 is also the mean value. For base elements in $\mathrm{PG} \setminus \mathrm{PG}_0$, the mean values in the describing pdfs need to be projected to $S_3 \times \mathbb{R}^3$, then propagated and projected back to the tangent space.

The Jacobian of g_T is used to derive the covariance matrix of the base element describing the composition. With $J_C = \left.\frac{\partial g_T}{\partial(y_1, y_2)}\right|_{(0,0)}$ and $\Sigma_C = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$ the resulting covariance matrix of the composition is $\Sigma_3 = J_C \cdot \Sigma_C \cdot {J_C}^T$.

3.6 *Mixture of Projected Gaussians*

As stated above, a precondition for the fusion PG base elements is that their tangent points are sufficiently close to each other and that they are sufficiently well concentrated. For this reason, widely spread probability density functions should not be modeled in a single base element.

Instead, we use a mixture of PG or PG_0 base elements. Thus let $p_i \in \mathrm{PG}$ or $p_i \in \mathrm{PG}_0$ be base elements, then the set of Mixtures of Projected Gaussians MPG or Mixtures of Projected Gaussians with zero mean MPG_0 is defined as

$\left\{p_m = \frac{1}{n}\sum_{i=1}^{n}\pi_i p_i \,|0 \leq \pi_i \leq 1, \sum_{i=1}^{n}\pi_i = 1\right\}$. The techniques of fusion and composition carry over to mixtures in a similiar way they work for mixtures of Gaussians [1].

Let $p_{m,1}, p_{m,2} \in$ MPG, $p_{m,1} = \frac{1}{n}\sum_{i=1}^{n}\pi_{1,i}p_{1,i}$, $p_{m,2} = \frac{1}{l}\sum_{j=1}^{l}\pi_{2,j}p_{2,j}$. The base elements of the fused mixture are obtained from fusing the base elements of the original mixtures: $p_{m,3} = f\left(p_{m,1}, p_{m,2}\right) = C \cdot \sum_{i,j=1}^{n,l}\lambda_{i,j}\cdot\pi_{1,i}\cdot\pi_{2,j}\cdot f\left(p_{1,i}, p_{2,j}\right)$, with a normalizing constant $C = \left(\sum_{i,j=1}^{n,l}\lambda_{i,j}\cdot\pi_{1,i}\cdot\pi_{2,j}\right)^{-1}$. The weights $\pi_{1,i}$ and $\pi_{2,j}$ are those of the prior mixture.

The plausibility is composed of two factors, $\lambda_{i,j} = \alpha_{i,j}\cdot\delta_{i,j}$.

The factor $\alpha_{i,j} = e^{-a\arccos\left(\left(r_{0,1,i}\cdot r_{0,2,j}\right)^2\right)}$ says whether the mixture elements can share a tangent space and thus probably pertain to the same cases in the mixture. The angle distance is controlled by the factor a. Plausible results were obtained with $a = 5$, but this lacks a rigid mathematical justification.

The factor $\delta_{i,j} = \left(\mu_{3,1,i} - \mu_{3,2,j}\right)\cdot\left(\Sigma_{3,1,i} + \Sigma_{3,2,j}\right)^{-1}\cdot\left(\mu_{3,1,i} - \mu_{3,2,j}\right)^T$ is the Mahalanobis distance of the mean values and covariances transported to the common tangent space. It expresses that even if the mixture elements could share a tangent space, they could still not be compatible.

The composition carries over in a similar manner.

$$p_{m,3} = g\left(p_{m,1}, p_{m,2}\right) = C \cdot \sum_{i,j=1}^{n,l}\pi_{1,i}\cdot\pi_{2,j}\cdot g\left(p_{1,i}, p_{2,j}\right),$$

with $C = \left(\sum_{i,j=1}^{n,l}\pi_{1,i}\cdot\pi_{2,j}\right)^{-1}$. In this case, there is no question of whether two base elements could apply at the same time, since the two probability distributions are assumed to be independent, so the factor $\lambda_{i,j}$ is omitted.

Note that in both cases the individually fused or combined resulting base elements are assumed to be renormalized.

4 Conclusion and Outlook

In this paper we present the framework of Mixtures of Projected Gaussians that allows for modelling a large variety of possible probability distribution functions of 6D poses. In contrast to a particle filter approach, much fewer parameters are needed to describe the distribution. Like particle filter appraoches, it allows for classical probabilistic inference rules like the Bayes update.

The MPG representation of probability density functions is part of our overall architecture for robotic perception [3]. In this larger framework we also use particle filter representations [2]. We can transform the probability density models between the different representation forms. Currently, we use rejection sampling in order to sample from MPG distributions to obtain particle sets, and we use a variant of the EM algorithm in order to estimate MPG parameters from sample sets.

The operations of fusion, propagation or multiplication of MPG distributions generally result in a large number of mixture elements. However, many of them have practically zero weight, while others are approximately identical. The approach for identifying doublettes described in [1] will be carried over to the MPG. Similar components are merged, negligible ones are omitted and the weights are renormalized.

The algorithms for probabilistic inference (fusion, propagation, multiplication) are fully implemented in *Mathematica* and ported to C. The single time consuming step is the integration in renormalizing the components. Here, we hope to find a quadrature function that takes full advantage of the special structure of the integrand to speed up the processing.

The covariance matrices are currently estimated using the Jacobian of the non-linear transforms. These estimates could be improved by using the unscented estimation technique (see Julier and Uhlmann [14])

In this paper we focus on the perception of static objects. The MPG framework can be extended to the dynamic case as well, following concepts by Goddard [4] and by Brox et al. [8].

Further work needs to be done on the analysis of the errors made in the approximations by mixtures of projected Gaussians. This includes also the selection of an appropriate statistical error measure, which in turn might very well depend on the application.

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