The Log-composition of Stationary Velocity Fields in Diffeomorphic Image Registration

Sebastiano Ferraris

Universtiy College London
Medical Physiscs and Biomedical Engineering

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Supervisor
Tom Vercauteren

Co-Supervisor
Marc Modat

Abstract

Medical imaging employs techniques and tools belonging to several branches of mathematics and physics that, when applied to morphometric and statistical study of biological shape variability, are collected under the name of *computational anatomy.* One of its critical tool, image registration, is widely used in both academical studies and applications, and continuously challenges researchers to enhance accuracy, improve reliability and reduce the time of the computations. The use of diffeomorphisms in image registration and the concomitant introduction of the log-euclidean framework to compute statistics, provide an interesting options to model the organs' deformations and to quantify the variations of anatomies. One of the challenge of this setting are the numerical computation of the Lie exponential of stationary velocity fields (SVF), the Lie logarithm of the corresponding transformation and their combinations as they appear in the BCH formula: concept at the core of the underpinning theory of the *log-demons* algorithm.

The necessity of finding fast numerical computation techniques of the BCH formula gave birth to the concept of log-composition presented in this thesis within some strategies for its computation.

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Chapter 1

Introduction to Diffeomorphic Image Registration

The series is divergent, therefore we may be able
to do something with it.
- Oliver Heaviside

1.1 Toward an ill-posed Problem

The process of determining correspondences between two or more images acquired from patients scans is a challenging task that has seen the application of a growing number of mathematical theories contributing to its solution.
The challenge and the concomitant difficulties in approaching the problem is a consequence of the fact that dealing with image registration problem means dealing with an ill-posed problem. Transformations between anatomies are not unique, and the impossibility to recover spatial or temporal evolution of an anatomical transformation from temporally isolated images, makes any validation a difficult, if not an impossible task. In addition each situation inevitably leads to consider some prior knowledge within the initial data, that may affect the problems' parameters and chose some constraints, that, of course, impact dramatically the range of possible results.
It is often the practical situation that provides the hint in choosing the optimal constraints, but it almost never provides enough information to reduce the large amount of options involved. A wide range of variants in methodologies and approaches to solve the registration problem has been thus proposed in the last decades: a quick glance to Google scholar reveals about 1200000 papers in *medical image registration* (55% of the whole *image registration* resources). Surveys in medical image registration can be found in [SDP13], [ZF03].

1.1.1 Examples of applications

One of the main application of image registration is in the domain of brain imaging: this tool can be used to examine differences between subjects and distinguish biological features between subjects (cross-sectional studies) or to compare different acquisition of the same subject, before and after a surgery or after a fixed period of time (longitudinal studies). In both cases an accurate comparison between images and the parameters of the transformation involved may result in a quantification of anatomical variability and in a better understanding of the patients' features. For example, brain atrophy is considered a biomarker to diagnose

the Alzheimer disease and to analyze its evolution; most of the algorithms and techniques involved in the atrophy measurement requires longitudinal or cross-sectional scans to be aligned, and so are directly affected by the solution of the registration algorithm [FF97], [GWRNJ12], [PCL+15].
Also when dealing with motion correction, if the sequence of images is affected by the motion of cardiac pulses or respiratory cycles, registration algorithms are often used for the realignment. For example, in lungs radiotherapy, lungs' motion is directly computed using a registration algorithm, and it is related with the respiratory surrogate signal. The correspondence model is then used to direct the X-ray or electrons beam on the cancerous tissue, minimizing the damage on the unaffected cells [MHSK], [MHM+11].
Another application is to compose together several images to obtain a bigger picture (mosaicing): in this case image registration is used to align images in the overlapping regions [VPM+06], [Sze94].

1.1.2 Diffeomorphisms in medical imaging: State of the Art

In the attempt to classify image registration algorithms, one of the most relevant feature that distinguish them is the choice of the family to whom the transformation belongs and its parametrization. Since anatomies are in a continuous process of modification over time, in general without any variation in the topological features, the use of diffeomorphisms to model transformations of organs appears as one of the most natural. Not accidentally diffeomorphisms are as well an important class of solutions of partial and ordinary differential equation aimed to model the dynamics of fluids.
In the development of diffeomorphic image registration, we can broadly identify some steps that led to the concept of log-composition presented in this research:

1981-1996 ▷ The use of diffeomorphisms in medical image registration starts from the research of a solution to partial differential equations: deformations are modeled as the consequent effect of two balancing forces applied to an elastic body [Bro81] or to conserve the energy momentum [CRM96]. In this early stage, diffeomorphisms are the domain of the solution to differential equation, and are not considered with their Lie group structure.

1998-2004 ▷ Based on the concept of attraction, the Demons algorithm [Thi98], [PCA99] propose the computation of the transformation between images in an iterative framework, where the update of the transformation at each step is parametrized with a vector field that is optimized at each step. This vector field is defined as the set of vectors (demons) that moves one image into the other.
Here diffeomorphisms are not directly involved and the vectors at each voxel are considered as independent elements. In the same year of [Thi98], the set of diffeomorphism was taken into account in image matching and computational anatomy, not only as the set of solutions of some family of differential equations, but with its tangent space [DGM98, Tro98, GM98].

2005-2006 ▷ The almost concomitant appearance of the Large Deformation Diffeomorphic Metric Mapping (LDDMM) [BMTY05] and the further investigation on the tangent space to the Lie group of diffeomorphisms as the space where to perform statistics (the so called log-Euclidean framework) [ACPA06b, AFPA06] keep the valuable approach in using diffeomorphisms as Lie group and to consider them with their Lie algebra, to model the little deformation in the tangent space as well as to rely on this normed space in computing the distance between transformations.

2007-2013 ▷ The LDDMM revealed all the opportunities provided by differential geometry in considering tangent vectors to the space of transformation in a framework for the computation of image registration. In this setting, the tangent vector field comes from the solution of the ODE that models the transformations and it consists of the set of the non-stationary vector field (also time varying vector field or TVVF). After the log-Euclidean framework [ACPA06b] aimed at the computation of statistics of diffeomorphisms, the set of tangent vector field is restricted to the time-independent vector field (also stationary velocity field or SFV); the same restriction was subsequently considered in some further improvements of LDDMM (DARTEL, Stationary LDDMM [Ash07], [HBO07]). Log-Euclidean framework brought new life also to the Demons algorithm, that become, in 2007, the diffeomorphic demons [VPPA07]. Subsequent approaches that involves the symmetrization of the energy function and a different norm (local correlation coefficient instead of L^2) are proposed in symmetric log-demons [VPPA08] and LCC-demons [LAF+13] respectively.

1.1.3 Using Diffeomorphisms: Utility and Liability

If the images that has to be registered are images of the same subject with no scaling shrinking or any other deformation than decentralization, then the set of transformations can be bonded to the rigid body transformations group $SE(3)$. In this case, the registration algorithm will be suitable for example to compensate the motion in a rapid sequence of scans, or if some little differences has happened, to compare them in longitudinal and cross sectional scans. If we need to compare images before and after a locally non-isometric deformation has occurred (i.e. distances are not preserved locally) but the topology is preserved (i.e. there are no new holes between the scenes represented by different images), then the group $SE(3)$ is not versatile enough to model the transformation. An alternative model may involve the group of transformations defined by the set of diffeomorphism Diff. This appears particularly appealing in medical application, since the transformations that occurs between internal organs in general preserve the topology.
On the other side their mathematical formalization as Lie group, is on the other hand not of immediate understanding, and it is still an open field of research.
Attempt to provide this object some handles for easy manipulation was done for the first time in 1966 by Vladimir Arnold [Arn66] (see also [Arn98], more readable for non-French speakers): to solve differential equation in hydrodynamic, Diff is considered as a Lie group possessing a Lie algebra. This assumption is not formally prosecuted in accordance to the problem-oriented nature of this paper. Subsequent steps in the exploration of the set of diffeomorphisms as a Lie group can be found in [MA70, EM70, Omo70, Les83]. A survey on early development of infinite dimensional Lie group can be found in [Mil84], while more recent results and applications on diffeomorphisms has been published in [OKC92, BHM10, Sch10, BBHM11].

Considering *Diff* as a differentiable manifold involves the idea of having it locally in correspondence with some generalized “infinite-dimensional euclidean”space. Attempt to set this correspondence showed that for some infinite-dimensional group the transition functions are smooth over Banach spaces [KW08]. This led to the idea of Banach Manifolds. Unfortunately the group of diffeomorphisms do not belongs to the category of Banach manifold but requires a more generals space on which the transition map are smooth: the Frechet spaces. Here, important theorems from analysis, as the inverse function theorem, or the main results from the Lie group theory in a finite dimensional settings, as Lie correspondence theorems do not holds anymore.
These difficulties led some researchers in approaching the set of diffeomorphisms from other

perspectives: for example, instead of treating *Diff* as a group equipped with differential structures it is seen as a quotient of other well behaved group [Woj94].
Without denying the importance of fundamentals and underestimating the doors research in this domain may open, we will approach the matter in as similar way of what has been done in set theory: we will use a *naive approach* to infinite dimensional Lie group, where the fundamental definition of infinite dimensional Lie group is a generalization of the finite dimensional case left more to the intuition than to a robust formalization. We work then mostly on finite dimensional settings, relying on important theorems and available close forms, and we will extend methods and results developed here in the infinite case -clearly - with proper precautions.

Another limitation that the reader should be aware of do not comes down to the theoretical difficulties of handling diffeomorphisms, but from the necessity of deal with discrete images and softwares. Two subset of some space have the same topology if exists an homeomorphism between them, but this analytical definition do not holds if the objects involved are considered in a discretized space. Separated subset remains separated until their distance is less than the size of a voxel for a significant region; if this happen, even with a homeomorphic underpinning model, the discretization process do not preserve the topology.

1.2 Image Registration Framework

1.2.1 Introductory Definitions

A *d-dimensional image* is a continuous function from a subset Ω of the coordinate space $\mathbb{R}^d$ (having in mind particular cases $d = 2, 3$) to the set of real numbers $\mathbb{R}$. Given two of them, $F : \Omega_F \to \mathbb{R}$ and $M : \Omega_M \to \mathbb{R}$, called respectively *fixed image* and *moving image*, the *image registration problem* consists in the investigation of features and parameters of the transformation function

$$\begin{aligned} \varphi : \mathbb{R}^d \supseteq \Omega_F &\longrightarrow \Omega_M \subseteq \mathbb{R}^d \\ \mathbf{x} &\longmapsto \varphi(\mathbf{x}) \end{aligned}$$

such that for each point $\mathbf{x} \in \Omega_F$ the element $M(\varphi(\mathbf{x}))$ and $F(\mathbf{x})$ are as closed as possible according to a chosen measure of similarity. The function defined as the composition of the moving image after the transformation, $M \circ \varphi$, is called *warped image*.

The underpinning idea can be represented by the following diagram, where φ is the solution that makes f the identity function:

$$\begin{tikzcd} \Omega_F \arrow[r, "\varphi"] \arrow[d, "F"] & \Omega_M \arrow[d, "M"] \\ \mathbb{R} \arrow[r, dashed, "\text{f}"] & \mathbb{R} \end{tikzcd}$$

If $\Omega_F \neq \Omega_M$, it is always possible to apply an homeomorphism to transform them into a common domain Ω, called *background space*, on which both of the images are defined.

The definition of image registration problem proposed here, leaves two key degrees of freedom in searching for a solution: the transformation's domain (also called *deformation model*), and the metric to measure the similarity between images.
Once these are chosen, they can be used as constituent of an *image registration framework*: we define it as an iterative process that, at each step provides a new function φ that approaches f to the identity. Each iteration involves the optimization of a function that measure the similarity between the fixed image and the warped image computed at the previous step. To refine the energy function, the metric can be considered with an additive regularization term, that introduces a constraint based on prior knowledge about the searched solution:

$$\mathcal{E}(F, M, \varphi) = \mathrm{Sim}(F, M, \varphi) + \mathrm{Reg}(\varphi) \tag{1.1}$$

where Sim is a function that measure the similarity, while Reg is a regularization term. The optimization algorithm on which the framework is based and the resampling strategy - process of resize the image from one dimension to another - provide additional degrees of freedom in defining a framework for an algorithm aimed to solve image registration problem.

1.2.2 Iterative Registration Algorithm

The definition of registration problem and the iterative framework described above, raise several issues. For example there are no reasons to believe that such a correspondence is unique and that there is at least one of them whose behaviour corresponds to a reasonable biological transformation between anatomies. One way to deal with this problem is to add some constraints on the transformation φ, such that it models realistic changes that can occur in biological tissues. The kind and quality of the constraints are one of the features that distinguish one registration algorithm from the other.
The image registration framework here presented can be see as a electronic device with 5 knobs, each with its range:

$$\begin{aligned}
\{\varphi\} &\in \{\text{Transformations}\} \\
\mathrm{Sim} &\in \{\text{Similarity measures}\} \\
\mathrm{Reg} &\in \{\text{Regularization Terms}\} \\
\mathrm{Opt} &\in \{\text{Optimization techniques}\} \\
\mathrm{Res} &\in \{\text{Resampling techniques}\}
\end{aligned}$$

Under the hood of this ideal device we may see an engine that can be schematically represented as in figure 1.1.
Modulating on the value of each knob, changing for example the set of transformation or the resampling technique, we change between the possible registration algorithms that generally falls in this framework.

Far from being a complete overview of all of the possible frameworks, it does not take into account the fact that each version or implementation inevitably involves different needs and consequent challenges. Solutions found for each case may fall outside this simplification scheme: for example the parametrization of the transformations (or the deformation field's update) at each iterative step do not appear in this picture, even though is a fundamental feature.
In the following subsections we will going from the generalized framework to some specific algorithms. We are interested in particular in the parametrization of the *diffeomorphisms* (bijective differentiable maps with differentiable inverse [Lee12]) of the latest important algorithms: the LDDMM and the diffeomorphic Demons.

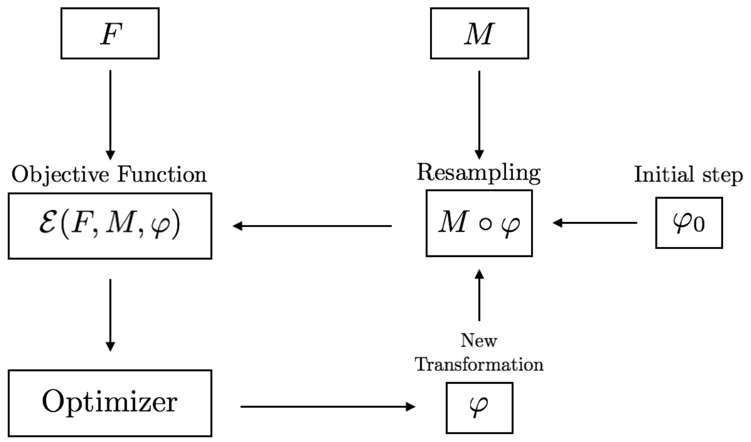

Figure 1.1: Image registration framework scheme.

1.2.3 LDDMM: Classic, Shooting and Stationary

As previously done in the elastic registration [Bro81], the LDDMM framework [BMTY05] originates by considering motion between images as the motion of a fluid, and utilizes ODE from fluid dynamics to compute the deformation between fixed and moving. The discretized vector fields that can be stored in the computers' memory, are 5-dimensional matrix on a regular lattice L over the background space Ω. The standard structure, used by most of the available software for image manipulation (Nibabel, py-NIfTI, niftilib, ... see [Mod09]) is

$$M = M(x_i, y_j, z_k, t, d) \qquad (i, j, k) \in L, \quad t \in T \quad d = 1, 2, 3 \tag{1.2}$$

where (x_i, y_j, z_k) are discrete position of the lattice L, t is the time parameter in a discretized domain T and d is index of the coordinate axis. So, the *tangent vector* $\mathbf{v}_\tau(x_i, y_j, z_k)$ at time t, has coordinates defined by

$$\mathbf{v}_t(x_i, y_j, z_k) = (M(x_i, y_j, z_k, t, 1), M(x_i, y_j, z_k, t, 2), M(x_i, y_j, z_k, t, 3))$$

Consider the set of homeomorphisms $\mathrm{Hom}(\Omega)$ (continuous function from the background space Ω to itself with continuous inverse) that act on the set of images from the background space $\mathcal{I}_\Omega$:

$$\begin{aligned} \mathrm{Hom}(\Omega) \times \mathcal{I}_\Omega &\longrightarrow \mathcal{I}_\Omega \\ (\varphi, F) &\longmapsto F \circ \varphi^{-1} \end{aligned}$$

its orbit, given an image F and a subgroup of the homeomorphisms $\mathbb{G} \subseteq \mathrm{Hom}(\Omega)$, consists in the set of the images having the same topology of F:

$$\mathcal{E}_{\mathbb{G}}(F) = \{F \circ \varphi^{-1} \mid \varphi \in \mathbb{G}\}$$

The similarity term in the LDDMM is the L^2 norm (see [SS09], chapter 4) between the moving image and the fixed image in the same orbit:

$$\mathrm{Sim}(F, M, \varphi) = \frac{1}{\sigma^2} ||F(\varphi^{-1}) - M||^2_{L^2}$$

while the regularization term, that provides the optimal φ at each step, is defined on the norm of the velocity vector field tangent to the transformation.
Let $\mathcal{V}(\Omega)$ be the set of all of the vector field over Ω. A generic time varying vector field (TVVF) is the continuously differentiable map defined as

$$\begin{aligned} V : [0,1] &\longrightarrow \mathcal{V}(\Omega) \\ t &\longmapsto V_{(t)} : \Omega \longrightarrow \mathbb{R}^d \\ &\qquad\qquad \mathbf{x} \longmapsto V_{(t,\mathbf{x})} \end{aligned}$$

Once initial conditions are given, at each TVVF, corresponds a unique time varying (or non-stationary) homomorphisms defined by the following ODE

$$\frac{d\phi_t(\mathbf{x})}{dt} = V_{(t,\mathbf{x})} \tag{1.3}$$

where

$$\begin{aligned} \phi : [0,1] &\longrightarrow \mathrm{Hom}(\Omega) \\ t &\longmapsto \phi_t : \Omega \longrightarrow \Omega \\ &\qquad\qquad \mathbf{x} \longmapsto \phi_t(\mathbf{x}) \end{aligned}$$

The transformation φ between fixed and moving images ($F \circ \varphi^{-1} = M$), can be defined by the couple $(V_{(t)}, \phi_t)$, such that, for $t = 0$, $\phi_t = e$, identity of the group of homeomorphisms, and for $t = 1$, $\phi_t = \varphi$ is the sought homomorphism.

$$\varphi := \phi_1 = \phi_0 + \int_0^1 V^{(t)}(\phi)dt$$

In consequence of this, the function ϕ, paired with the vector field $V_{(t)}$, defines a continuous sequence of homorphisms in the same orbit of the fixed image $\mathcal{E}_{\mathrm{Hom}(\Omega)}(F)$.
To obtain an efficient algorithm and a meaningful constraint on the resulting transformation, it is reasonable to consider ϕ_t as the shortest path between the identity and φ, so to have $V_{(t)}$ as the one that minimize the distance between transformations:

$$l = \inf_{V^{(t)}\ :\ \dot{\phi}_t(\mathbf{x}) = V^{(t)}(\mathbf{x})} \int_0^1 ||V^{(t)}||^2_{L^2} dt$$

Ending points of path on the set of diffeomorphisms, whose tangent vector field (that varies over time), are used as regularization term:

$$\mathrm{Reg}(F, M, \varphi) = \int_0^1 ||LV^{(t)}||^2_{L^2} dt \qquad \dot{\phi}_t(\mathbf{x}) = V^{(t)}(\mathbf{x}) \quad \phi_0 = e \quad \phi_1 = \varphi$$

Were L is a linear operator that can be dependent on some parameters that add additional constraint to the algorithm limiting the length of the speed of the transformation at each step. With more details, it is defined as $L = (\alpha\vec{\nabla}^2 + \gamma)$ for α and γ real parameters and $\vec{\nabla}^2$ the Laplace operator. From the differential equation 1.3, and in consequence of the definition of φ the energy function 1.1 become

$$\mathcal{E}(F, M, \varphi) = \int_0^1 ||LV^{(t)}||^2_{L^2} dt + \frac{1}{\sigma^2}||F(\varphi^{-1}) - M||^2_{L^2}$$

And so the optimization algorithm, at each step of the registration will look for

$$\hat{V} = \underset{V^{(t)} \,:\, \dot{\phi}_t(\mathbf{x}) = V^{(t)}(\mathbf{x})}{\operatorname{argmin}} \int_0^1 ||LV^{(t)}||_{L^2}^2 dt + \frac{1}{\sigma^2} ||F(\varphi^{-1}) - M||_{L^2}^2$$

Each transformation involved in the optimization algorithm are discretized time varying velocity fields; the update at each step is given by

$$\mathbf{v}_{k+1} = \mathbf{v}_k - \epsilon \nabla(\Delta\mathcal{E})$$

where $\mathbf{v}_k$ is the k-th step of the approximation of the velocity vector field V discretized according to 1.2, while $\Delta\mathcal{E}$ is the discretized version of the energy function and ϵ is the gradient descent step size.

A direct upgrade of the classical LDDMM performs the optimization on the geodesic flows, defined by a set of Hamiltonian equation (Shooting LDDMM [VRRC12]).
In this algorithm the iterative evolution of the deformation field, solution of the optimization algorithm, is regularized with the constraint imposed by an additional scalar field called *initial momentum*. As proved by the authors, the evaluation of this constraint at each step provides geodesics flows of homeomorphisms, but it is computationally expensive. Using the log-Euclidean framework presented in [AFPA06], the algorithm proposed in [Ash07] uses a constraint based on the stationarity of the involved velocity field. Instead of considering time varying velocity fields constrained by a set of Hamiltonian equations, the domain of vector field is reduced to the stationary, which is an -almost- equivalent constraint, that considerably reduces the computational complexity. Resulting algorithm, the DARTEL (Diffeomorphic Anatomical Registration using Exponentiated Lie Algebra), was published in contemporary with [HBO07] based on the same concept of the parametrization of geodesics path of diffeomorphisms with stationary velocity fields, and with [VPPA07] that uses SVF in the Demons framework instead on the LDDMM.

1.2.4 Demonology: Classic, Additive, Diffeomorphic, Log and Symmetric

The first demons-based algorithm in image registration was proposed by [Thi98] in analogy with the Maxwell's demon in thermodynamics. This early version - often called *classic demons* - do not involves diffeomorphisms. The floating image is deformed with a vector field resulting from the computation of the optical flow regularized by a gaussian filter at each step. The optical flow is based on the idea that a voxel in the moving image is attracted by some force to all the points in the fixed with similar intensity.
Let $\mathbf{x}$ be a point in the background space Ω, the unknown vector field $V : \Omega \to \mathbb{R}^d$ is the function that at each voxel satisfies:

$$V(\mathbf{x}) \cdot \vec{\nabla} F(\mathbf{x}) = M(\mathbf{x}) - F(\mathbf{x}) \tag{1.4}$$

whose solution provides the update of the deformation field at each step.

The final deformation, solution to the registration problem is obtained composing at each step the previous transformation with an update: let $\{T_k\}_{k=1}^N$ be the sequence of deformation and let δT_k be the update at the step k. Then they can be expressed as the identity plus the displacement field:

$$T_k(\mathbf{x}) = \mathbf{x} + V_k(\mathbf{x})$$
$$\delta T_k(\mathbf{x}) = \mathbf{x} + \delta V_k(\mathbf{x})$$

And the k-th deformation is computed by composition as:

$$\begin{aligned} T_{k+1}(\mathbf{x}) &:= (\delta T_k \circ T_k)(\mathbf{x}) \\ &= \mathbf{x} + \delta V_k(\mathbf{x}) + V_k(\mathbf{x} + \delta V_k(\mathbf{x})) \end{aligned}$$

Since the third addend is close to $V_k(\mathbf{x})$, some implementation - as for example ITK - considered only the sum between V_{k+1} and V_k in the computation of the update:

$$\begin{aligned} T_{k+1}(\mathbf{x}) &:= (\delta T_k + T_k)(\mathbf{x}) \\ &= \mathbf{x} + V_k(\mathbf{x}) + \delta V_k(\mathbf{x}) \end{aligned}$$

Demons algorithms with this implementation are often called *additive demons*.

In the paper that presents the PASHA demons [CBD+03], authors underline the fact that the scalar product 1.4 underpinning the classic demon consists in the minimization of a local energy function, one per volxel. They reformulate the algorithm using a global energy function, aimed to make the method easier to be analyzed and compared with others. With some modifications, the Classic demons algorithm is accompanied back to the framework presented in the previous section, having a global energy function which optimization provides at each step the update of the transformation.
Again the PASHA algorithm does not involve any diffeomorphism. The solution is smoothed with the widespread stratagem of applying a Gaussian filter G at each of the global transformation $\mathbf{v}_k$ involved in the registration:

$$V_{k+1}(\mathbf{x}) := G_1(V_k(\mathbf{x}) + G_2(\delta V_k(\mathbf{x}))$$

If G_1 is omitted the model is sometime called *fluid*, while if G_2 is omitted is called *elastic*.

As in the LDDMM case, diffeomorphisms were introduced within the demons algorithm (*diffeomorphic demons* [VPM+06]) after the introduction of the log-Euclidean framework [AFPA06]. To each stationary velocity field $V \in \mathcal{V}(\Omega)$ is associated a diffeomorphisms φ by the ODE $d\varphi/dt = V_{(t,p)}$.
Using Lie theory, SVF are elements of the *Lie algebra* - usually denoted with $\mathfrak{g}$ - while the set of diffeomorphisms are elements of the Lie group - denoted with $\mathbb{G}$ -.
Roughly speaking, the lie algebra $\mathfrak{g}$ is the tangent space (as local linear approximation of a manifold) to the lie group $\mathbb{G}$, and these two spaces are in local correspondence thanks to two crossing-structure functions: the *Lie exponential* and the *Lie logarithm*. *Lie exponential* maps vector fields on the corresponding Lie group elements, while the *Lie logarithm* - inverse of the Lie exponential under some condition [Lee12]- maps each diffeomorphisms in the correspondent tangent vector field:

$$\varphi = \exp(V) \quad V = \log(\varphi) \qquad\qquad \varphi \in \mathbb{G} \quad V \in \mathfrak{g}$$

In this settings, the update can not be computed simply with a sum of vector fields, since it must reflect the composition of diffeomorphisms defined by the corresponding vector fields.

It has been made several approaches to face the problem of the computation of the update: diffeomorphic demons compute each new diffeomorphisms as the composition between the diffeomorphism obtained at the previous step φ_k with the exponential of the SVF $\delta\mathbf{v}_k$ obtained by the optimization algorithm :

$$\varphi_{k+1} := \varphi_k \circ \exp(\delta\mathbf{v}_k)$$

while in the subsequent log-demon [VPPA08] the composition is performed in the tangent space toward exponential and logarithm functions

$$\mathbf{v}_{k+1} := \log(\exp(\mathbf{v}_k) \circ \exp(\delta\mathbf{v}_k)) \tag{1.5}$$

For this last computation, another treasure from the theory of Lie group has been stolen: the BCH formula. Defined as the solution for $\mathbf{z}$ of the equation

$$\exp(\mathbf{z}) = \exp(\mathbf{x}) \circ \exp(\mathbf{y})$$

As we will see in this research, it involves an infinite series of nested Lie bracket, that do not makes its computation as straightforward as we wish. This thesis is devoted to the research of some numerical method for its computation and the last section of this chapter is aimed to introduce the definition of log-composition, whose numerical approximation will help in the computation of equation 1.5.

1.3 The Log-composition of SVF in the Diffeomorphisms Group

Every non-rigid registration algorithm requires to be implemented and to work with discretized images. The nature of the computers' memory prevent from the possibility of storing the continuous fluid transformations that solves the differential equations of the LDDMM or any of the diffeomorphisms resulting from the demons: the only thing that we can do, unfortunately, is to store the discretized vector fields and resampling them with the images using one of the available techniques.
When relying on diffeomorphisms, we still have to consider discretized vector fields our objects, but we need the Lie group of diffeomorphisms as support to compute their composition 1.5. This operation of composition is baptized under the name of *log-composition* and it is defined as

$$\mathbf{x} \oplus \mathbf{y} := \log(\exp(\mathbf{x}) \circ \exp(\mathbf{y})) \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathfrak{g}$$

The main aim of this research is to find and compare numerical ways to compute it, and so to have a method for the computation of the update 1.5.

It is worthed to notice that a fast log-composition is not useful only for the diffeomorphic demons. It may be used to solve some of the problems that rely on Lie groups and need to be implemented in a computer. In medical imaging it can be used for

1. Diffeomorphic demon, and symmetric diffeomorphic demon [VPM+06, VPPA08].
2. Fast computation of the logarithm [BO08].
3. Calculus on diffusion tensor [AFPA06].
4. Compute the discrete ladder for Parallel Transport in Transformation Groups [LP14a].

The next chapter is aimed to the formal definition of the log-composition, underpinned with the tools from differential geometry theory and to present two new numerical technique to compute it.

Chapter 2

Tools from Differential Geometry

People know or dimly perceive, that if thinking is not kept pure and keen, if spirit's contemplation do not holds, even mechanics of automobiles and ships will soon cease to run. Even engineer's slide rule, computations of banks and stock exchanges will wonder aimlessly for the lost of authority, and chaos will ensue.

-Hermann Hesse, *Magister Ludi*

2.1 A Lie Group Structure for the Set of Transformation

We consider every group $\mathbb{G}$ as a group of transformations acting on $\mathbb{R}^d$, having in mind the particular case $d = 2, 3$ for 2-dimensional or 3-dimensional images. We will focus out attention to transformations defined by matrices or diffeomorphism. Other than group they also have the structure of Lie group: they are considered with a maximal atlas that makes them differentiable manifold, in which the composition of two transformation and the inverse of each transformation are well defined differentiable maps:

$$\begin{aligned} \mathbb{G} \times \mathbb{G} &\longrightarrow \mathbb{G} \\ (x, y) &\longmapsto xy^{-1} \end{aligned}$$

Differential geometry is in general a technique to use the well known calculus features and operators on spaces different from the usual $\mathbb{R}^n$. Adding the differentiable structure to a group of transformations gives us new handles to hold and manipulate them: in particular provides the opportunity to define a tangent space to each point of the group (and so a fiber bundle), a space of vector fields, a set of flows and one parameter subgroup as well as other features that enrich this structure. The abstract idea of vector field over a manifold will be concretized for image registration introducing the concepts of *displacement field*, *deformation field* and *velocity field (stationary or time varying)* that will be there presented. Due to space limitations we will refer to [DCDC76], [Lee12] for the definitions and concepts of differential geometry and [dCV92] for definition and concepts of Riemannian geometry.

2.2 Lie Exponential, Lie logarithm, Lie log-composition and the BCH formula

Let $\mathbf{v}$ be an element in the tangent space fo the Lie group $\mathbb{G}$ indicated with $\mathfrak{g}$. The *Lie exponential* is defined as

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow \mathbb{G} \\ \mathbf{v} &\longmapsto \exp(\mathbf{v}) = \gamma(1) \end{aligned}$$

where $\gamma : [0,1] \to \mathbb{G}$ is the unique one-parameter subgroup of $\mathbb{G}$ having $\mathbf{v}$ as its tangent vector at the identity ([dCV92], [EMP06], [AFPA06]). It satisfies the following properties:

1. $\exp(t\mathbf{v}) = \gamma(t)$.
2. $\exp(\mathbf{v}) = e$ if $\mathbf{v} = \mathbf{0}$.
3. $\exp(\mathbf{v}) \circ \exp(-\mathbf{v}) = e$
4. The exponential function satisfies the one parameter subgroup property:
$$\exp((t+s)\mathbf{v}) = \gamma(t+s) = \gamma(t) \circ \gamma(s) = \exp(t\mathbf{v}) \exp(s\mathbf{v})$$
5. $\exp(\mathbf{v})$ is invertible and $(\exp(\mathbf{v}))^{-1} = \exp(-\mathbf{v})$.
6. exp is a diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathfrak{g}$ to a neighborhood of e in $\mathbb{G}$.

The neighborhoods of $\mathbb{G}$ and of $\mathfrak{g}$ such that the last property holds, are called *internal cut locus* of $\mathbb{G}$ and $\mathfrak{g}$ respectively. The *cut locus* is the boundary of the internal cut locus.
When we deal with a matrix Lie group of dimension n, we have the following remarkable properties:

1. for all $\mathbf{v}$ in a matrix Lie algebra $\mathfrak{g}$:
$$\exp(\mathbf{v}) = \sum_{k=0}^{\infty} \frac{\mathbf{v}^k}{k!}$$
2. If $\mathbf{u}$ and $\mathbf{v}$ are commutative then $\exp(\mathbf{u}+\mathbf{v}) = \exp(\mathbf{u})\exp(\mathbf{v})$.
3. If $\mathbf{c}$ is an invertible matrix then $\exp(\mathbf{c}\mathbf{v}\mathbf{c}^{-1}) = \mathbf{c}\exp(\mathbf{v})\mathbf{c}^{-1}$.
4. $\det(\exp(\mathbf{v})) = \exp(\mathrm{trace}(\mathbf{v}))$
5. For any norm, $||\exp(\mathbf{v})|| \leq \exp(||\mathbf{v}||)$.
6. $\exp(\mathbf{u}+\mathbf{v}) = \lim_{m\to\infty}(\exp(\frac{\mathbf{v}}{m})\exp(\frac{\mathbf{v}}{m}))^m$
7. If $\exp(\mathbf{w}) = \exp(\mathbf{u}) \circ \exp(\mathbf{v})$ then $\exp(-\mathbf{w}) = \exp(-\mathbf{v}) \circ \exp(-\mathbf{u})$.
8. For ad adjoint map in the Lie algebra we have $\exp(\mathrm{Ad}_{\mathbf{u}}\mathbf{v}) = \mathrm{Ad}_{\mathbf{u}}\exp(\mathbf{v})$

The idea of defining an inverse of the Lie exponential leads to the idea of the Lie logarithm, defined

$$\begin{aligned} \log : \mathbb{G} &\longrightarrow \mathfrak{g} \\ p &\longmapsto \log(p) = \mathbf{v} \end{aligned}$$

where $\mathbf{v}$ is the tangent vector having p as it exp.

If $\mathbb{G}$ is a matrix Lie group of dimension n, the following properties hold:

1. for all $\mathbf{v}$ in the matrix Lie algebra $\mathfrak{g}$:

$$\log(\mathbf{v}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(\mathbf{v}-I)^k}{k!}$$

   where I is the identity matrix.

2. For any norm, and for any $n \times n$ matrix $\mathbf{c}$, exists an α such that

$$||\log(I+\mathbf{c}) - \mathbf{c}|| \leq \alpha||\mathbf{c}||^2$$

3. For any $n \times n$ matrix $\mathbf{c}$ and for any sequence of matrix $\{\mathbf{d}_j\}$ such that $||\mathbf{d}_j|| \leq \alpha/j^2$ it follows:

$$\lim_{k\rightarrow\infty}\big(I + \frac{\mathbf{c}}{k} + \mathbf{d}_k\big)^k = \exp\left(\mathbf{c}\right)$$

Here we may see the beginning of the problem we have to deal with for the rest of the research, when passing from the finite dimensional case to the infinite dimensional case.
The domain of the logarithm is the matrix Lie group in which only the composition is defined. Nevertheless it is possible to compute $I + \mathbf{c}$, and this still make sense (and satisfy remarkable properties) when applied to the log. In addition the domain of the exponential is the matrix Lie algebra, but the exponential can be nevertheless applied to any matrix. *This can be done thanks to the fact that for matrices, $\mathfrak{g}$ and $\mathbb{G}$ are subset of a bigger algebra, the algebra of invertible matrix.* In this structure the operation of sum is still defined over the group that admits only compositions, and infinite series are doors to pass from the structure of group to the algebra and vice versa. When presenting the rigid body transformation in chapter 3 we will see another couple of access doors based on numerical approximations.

When dealing with diffeomorphism in practical application we have to deal with a theoretical and a practical issues: on one side the the operations in Lie algebra are not compatible with the composition in the Lie group: expression as $e + \mathbf{v}$ contains a sum that is undefined (and therefore meaningless) in the Lie group structure. On the other side the (extremely) continuous nature of diffeomorphisms is not compatible with the (extremely) discrete nature of computers. It is not possible to implement something that maintains any of the property of diffeomorphism in a computer. The only options we have for practical implementations are the vector fields discretized on a d dimensional grid. In the paper of Arsigny [ACPA06a], scaling and squaring and inverse scaling and squaring are proposed for the computation of exponential and logarithm respectively; they transform a discretized vector field in another discretized vector field, while theoretical domain and codomain of these transformations are radically different. In addition, as shown in [HOP08] for some parameters of Stationary LDDMM and the diffeomorphic Demons the diffeomorphisms involved do not preserve the signs of the Jacobian determinant, and therefore are not diffeomorphisms anymore.
A different situation arises when dealing with classic Lie group: their matrix representation is not only ideal for implementation, but the Lie algebra $\mathfrak{g}$ of a classic Lie group $\mathbb{G}$ can be still represented with a set of matrices whose sum and products are compatible with the composition, since $\mathfrak{g}$ and $\mathbb{G}$ are both subset of a bigger algebra of the general linear group $GL(n)$ [KKK08]. case the exponential and logarithm have domain and codomain in the appropriate restriction of $GL(n)$.

The passage from finite dimensional case of matrices to infinite dimensional case of diffeomorphisms requires a way to represent diffeomorphisms having only discrete vector fields to deal with, where operations within Lie algebra and Lie group are not anymore compatible. The strategy here proposed is to define an operation in the Lie algebra Lie algebra $\mathfrak{g} = \mathcal{V}(\Omega)$ that reflects the properties of the composition in the corresponding Lie group structure $\mathbb{G} = \mathrm{Diff}(\Omega)$.

The Lie Log-composition (Lie to distinguish it from the Affine Log-composition of the next section) is defined here as the inner binary operation on the Lie algebra that reflects the composition on the lie group:

$$\begin{aligned} \oplus : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (\mathbf{v}_1, \mathbf{v}_2) &\longmapsto \mathbf{v}_1 \oplus \mathbf{v}_2 = \log(\exp(\mathbf{v}_1) \circ \exp(\mathbf{v}_2)) \end{aligned}$$

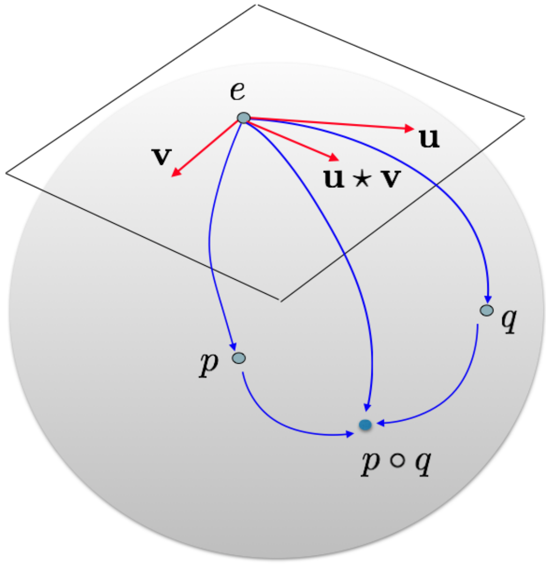

Figure 2.1: graphical visualization of the Lie log-composition.

Following properties holds for the Lie log-composition:

1. $\mathfrak{g}$ with the Lie log-composition $\oplus$ is a local topological non-commutative group (local group for short): if $C_{\mathfrak{g}}$ is the internal cut locus of $\mathfrak{g}$ then:

   (a) $(\mathbf{u}_1 \oplus \mathbf{u}_2) \oplus \mathbf{u}_3 = \mathbf{u}_1 \oplus (\mathbf{u}_2 \oplus \mathbf{u}_3)$ for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in $C_{\mathfrak{g}}$.

   (b) $\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in $C_{\mathfrak{g}}$.

   (c) $\mathbf{u} \oplus (-\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u}$ in $C_{\mathfrak{g}}$.

2. For all t, s real, such that $(t+s)\mathbf{u}$ is in $C_{\mathfrak{g}}$,

   $$(t\mathbf{u}) \oplus (s\mathbf{u}) = (t+s)\mathbf{u}$$

   And in particular, if the Lie algebra $\mathfrak{g}$ has dimension 1 the local group structure is compatible with the additive group of the vector space $\mathfrak{g}$.

We refer to $(\mathfrak{g}, \oplus)$ as the Lie Log-group. Additional observation of this algebraic structure in the particular case of diffeomorphisms, are contained in the next chapter.

To compute the log-composition there is a formula, the BCH (for Lie group [Hal15], general case [Woj94], applied to medical imaging [VPPA08]), that provides the exact solution to the Log-composition.

$$BCH(\mathbf{u},\mathbf{v}) = \mathbf{u}+\mathbf{v}+\frac{1}{2}[\mathbf{u},\mathbf{v}]+\frac{1}{12}([\mathbf{u},[\mathbf{u},\mathbf{v}]]+[\mathbf{v},[\mathbf{v},\mathbf{u}]])-\frac{1}{24}[\mathbf{v},[\mathbf{u},[\mathbf{u},\mathbf{v}]]]+...$$

This expansion will provide the most immediate way to obtain a numerical computation of the log-composition.

2.3 Affine Exponential, Affine logarithm and affine log-composition

Considering a Lie Group $\mathbb{G}$ with a connection ∇ (that provides geodesics and curvature over manifold on which no Riemannian metric has been defined, see [dCV92]), the vector field $\nabla_U(V)$ associates at each point of the manifold the projection on the tangent plane of the covariant derivative of U in the direction of V.
First consequence of the definition of the connection is the possibility of define *geodesics* between points p and q of the manifold. A geodesic is a curve γ such that:

$$\gamma : [0,1] \longrightarrow \mathbb{G} \qquad \gamma(0) = p,\ \gamma(1) = p,\ \nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

Note that in this case the concept of geodesic did not involves any metric defined on the surface of the manifold. If also a Riemannian metric is defined on $\mathbb{G}$, then geodesics defined by the metric coincides with the geodesics defined by the connection only for the particular Levi-Civita connection (see [dCV92]). A connection is said to be left invariant if it is closed for left invariant vector fields, i.e. if for any $V, W \in Left\mathcal{V}(\mathbb{G})$ their connection $\nabla_U V$ is still left invariant.

Second straight consequence of the definition of connection allows us to define a new kind of exponential from the Lie algebra to the Lie group that relies on the concept of geodesics. This time the tangent plane that defines the Lie algebra is considered at the point p of the Lie group and $\mathbf{v} \in T_p\mathbb{G} \simeq \mathfrak{g}$ is a tangent vector at the point p:

$$\begin{aligned} \exp : \mathbb{G} \times \mathfrak{g} &\longrightarrow \mathbb{G} \\ (p,\mathbf{v}) &\longmapsto \exp_p(\mathbf{v}) = \gamma(1;p,\mathbf{v}) \end{aligned}$$

The curve $\gamma(t;p,\mathbf{v}) = \gamma(t)$ on $\mathbb{G}$ is the unique one with the following features:

$$\gamma(0) = p \qquad \dot{\gamma}(0) = \mathbf{v} \qquad \nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

This second kind of exponential is called affine exponential.
The following properties hold:

1. If ∇ is a Cartan connection then $\exp_e$ and exp coincides.
2. For all p in $\mathbb{G}$, $\mathbf{v} \in T_p\mathbb{G}$ and t real

$$\exp_p(t\mathbf{v}) = \gamma(t;p,\mathbf{v})$$

3. Given $\mathbf{u} \in T_e\mathbb{G}$, $\mathbf{v} \in T_{\exp_e(\mathbf{u})}\mathbb{G}$, exists a $\mathbf{w} \in T_e\mathbb{G}$ such that

$$\exp_e(\mathbf{w}) = \exp_{\exp_e(\mathbf{u})}(\mathbf{v}) \circ \exp_e(\mathbf{u})$$

4. If V is a unitary left-invariant vector field, then for $V_e \in T_e\mathbb{G}$

$$\exp_e(2V_e) = \exp_{\exp_e(V_e)}(V_{\exp_e(V_e)}) \circ \exp_e(V_e)$$

Last two properties provides the intuitive idea that it is possible to move on the fiber bundle of the Lie group transporting in some sense a tangent vector defined at the identity on another tangent space. Certainly the Lie group possess a unique Lie algebra, as the tangent space at some point (the group's identity by convention), but two different tangent space (so two times the same Lie algebra structure) may not be oriented in the same way.

To approach the inverse of the affine exponential we consider the affine logarithm:

$$\begin{aligned} \log : \mathbb{G} \times \mathbb{G} &\longrightarrow T_p\mathbb{G} \simeq \mathfrak{g} \\ (p, q) &\longmapsto \log_p(q) = \mathbf{v} \end{aligned}$$

Where $\mathbf{v}$ is the vector at the tangent plane defined at p such that the curve on $\mathbb{G}$ with the following features

$$\gamma(0) = p \qquad \gamma(1) = q \qquad \nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

has as its tangent in p the vector $\mathbf{v}$.

Any Lie group $\mathbb{G}$ considered with a left-invariant connection ∇ can be equipped with a metric, based on the elements of its tangent space and on the log, for example:

$$\text{dist}(x, y) := ||\log_e(x^{-1} \circ y)|| \qquad \forall x, y \in \mathbb{G}$$

This metric is not necessarily coincident with the Riemannian one.

Is now time to extend the definition of Log-composition, with the definition of affine logarithm and affine exponential. The first step is to extend the definition of internal cut locus of the Lie algebra, even when not centered at the zero. If the tangent space is not considered at e of $\mathbb{G}$ but at some general point p, we still have a diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathfrak{g}$ to a neighborhood of p in $\mathbb{G}$. The internal cut locus of $\mathfrak{g}$ this time is based on p and it is denoted with $C_{\mathfrak{g}}(p)$.
Given a point p_1 and a vector $\mathbf{v}_1$ on its tangent space $T_{p_1}\mathbb{G}$, the *affine Log-composition* is defined as the operation $\tilde{\oplus}$ over the fiber bundle of $\mathbb{G}$ such that

$$\begin{aligned} \cdot \,\tilde{\oplus}\, \mathbf{v}_1 : T_{\exp_{p_1}(\mathbf{v}_1)}\mathbb{G} &\longrightarrow T_{p_1}\mathbb{G} \\ \mathbf{v}_2 &\longmapsto \mathbf{v}_2 \,\tilde{\oplus}\, \mathbf{v}_1 = \log_{p_1}(\exp_{\exp_{p_1}(\mathbf{v}_1)}(\mathbf{v}_2) \circ \exp_{p_1}(\mathbf{v}_1)) \end{aligned}$$

Note that not necessarily $\mathbf{v}_1 \,\tilde{\oplus}\, \mathbf{v}_2$ is a vector belonging to the internal cut locus based on the starting point p_1.

2.4 Parallel Transport: Definition and Properties

In this section we introduce the concept of parallel transport for a generic Lie group $\mathbb{G}$. On this definition, again borrowed from differential geometry (for introduction and general definition: [MTW73], [Kne51], [KMN00]. For medical imaging applications [LAP11], [LP13] [LP14b], [PL+11]), relies an important method for the computation of the Log-composition.

Definition 2.4.1. Let $\mathbb{G}$ be a finite dimensional connected Lie group defined with a connection ∇. Given $p, q \in \mathbb{G}$ and $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma(0) = p$ and $\gamma(1) = q$, then the vector $V_p \in T_p\mathbb{G}$, belonging to some vector field V is parallel transported along γ up to $T_q\mathbb{G}$ if V satisfies

$$\forall t \in [0,1] \qquad \nabla_{\dot{\gamma}} V_{\gamma(t)} = 0$$

The parallel transport is the function that maps V_p from $T_p\mathbb{G}$ to $T_q\mathbb{G}$ along γ:

$$\begin{aligned} \Pi(\gamma)_p^q : T_p\mathbb{G} &\longrightarrow T_q\mathbb{G} \\ V_p &\longmapsto \Pi(\gamma)_p^q(V_p) = V_q \end{aligned}$$

In the next properties we explore how did parallel transport and affine exponential behave when expressed as a composition and when there are changing in the signs.

Property 2.4.1 (Inversion)**.** $\mathbb{G}$ Lie group, ∇ connection, $p, q \in \mathbb{G}$. Given γ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\mathbf{u} \in T_p\mathbb{G}$, we have:

1. $\Pi(\gamma)_p^q(-\mathbf{u}) = -\Pi(\gamma)_p^q(\mathbf{u}))$
2. $q = \exp_p(\mathbf{u}) \iff p = \exp_q(-\Pi(\gamma)_p^q(\mathbf{u}))$

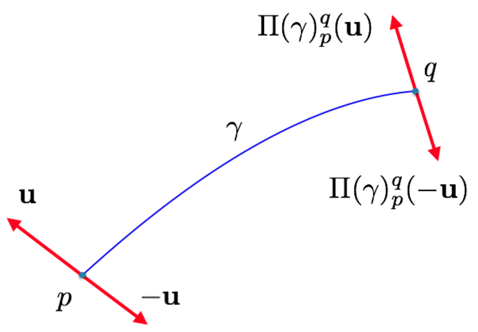

Figure 2.2: First inversion property.

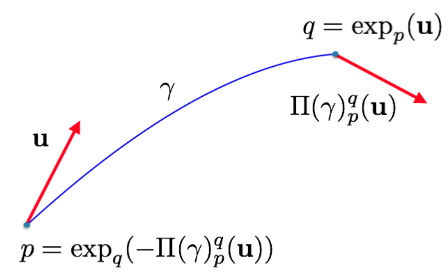

Figure 2.3: Second inversion property.

Property 2.4.2 (change of signs of the composition for affine exponential)**.** $\mathbb{G}$ Lie group, ∇ connection, $p, q \in \mathbb{G}$, $\mathbf{u} \in T_p\mathbb{G}$, $\mathbf{v} \in T_q\mathbb{G}$ and $q = \exp_p(\mathbf{u})$. Let β be the tangent curve to $\mathbf{u}$ at p such that $\beta(1) = q$ and $r = \exp_b(\mathbf{v})$. Given $\mathbf{w} \in T_p\mathbb{G}$ such that

$$\exp_p(\mathbf{w}) = \exp_q(\mathbf{v}) \circ \exp_p(\mathbf{u})$$

Then

$$\exp_p(-\mathbf{w}) = \exp_{\beta(-1)}(-\Pi(\beta)_q^{\beta(-1)}(\mathbf{v})) \circ \exp_p(-\mathbf{u})$$

Lemma 2.4.1. Let $\mathbb{G}$ be a finite dimensional connected Lie group defined with a Cartan connection ∇ and $\mathbf{u}$ tangent vector in $T_e\mathbb{G}$. Let γ be a geodesic defined on $\mathbb{G}$ such that $\gamma(0) = e$, $\dot{\gamma}(0) = \mathbf{u}$ and $p = \gamma(1)$, point in the Lie group. Let β be the curve over $\mathbb{G}$ defined as $\beta(t) = p \circ \gamma(t)$, then the two following conditions hold:

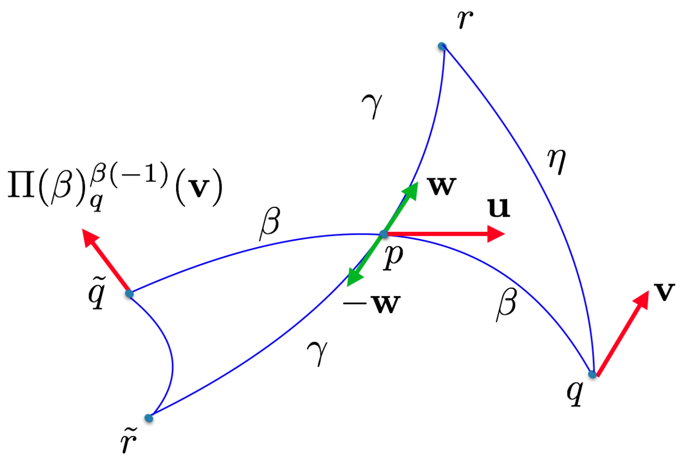

Figure 2.4: Change of sign property.

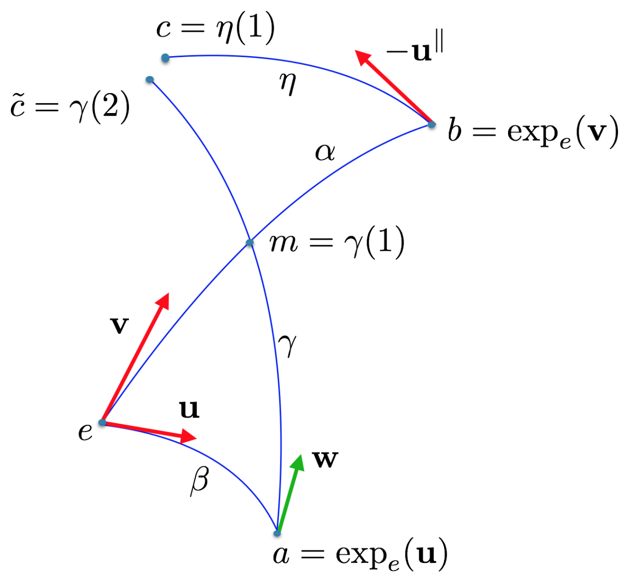

Figure 2.5: Pole ladder applied to parallel transport.

1. If ∇ is a Cartan connection then β is a geodesic.
2. For $\mathbf{u}_p := D(L_p)_e(\mathbf{u}) \in T_p\mathbb{G}$, push forward of the left-translation:

$$\exp_p(t\mathbf{u}_p) = p \circ \exp_e(tD(L_{p^{-1}})_p(\mathbf{u}_p)) = p \circ \exp_e(t\mathbf{u}) \tag{2.1}$$

The following theorem is an application of the pole ladder [LAP11] for the computation of the exponential that will underpin one of the numerical methods for the computation of the log-composition.

Theorem 2.4.1. Let $\mathbb{G}$ be a finite dimensional connected Lie group defined with a Cartan connection ∇. If, for each couple of linearly independent vectors $\mathbf{u}, \mathbf{v} \in T_e\mathbb{G}$, we consider the following elements:

$$a = \exp_e(\mathbf{u}) \qquad b = \exp_e(\mathbf{v})$$
$$\mathbf{v}_a^{\|} = \Pi(\gamma)_e^a(\mathbf{v})$$
$$\gamma : [0,1] \to \mathbb{G} \quad \gamma(0) = e \quad \dot{\gamma}(0) = \mathbf{u}$$

$$\mathbf{v}^{\|} := D(L_{a^{-1}})_e(-\Pi(\alpha)_a^e(\mathbf{v}_a^{\|}))$$

Then it follows that if $\mathbf{u}$ is in the open ball $B(\mathbf{0}, \epsilon)$, then exists a $\tilde{\epsilon} > 0$ such that $\mathbf{u}_e^{\|}$ belongs to $B(\mathbf{0}, \tilde{\epsilon})$, and exists a $\delta(\epsilon + \tilde{\epsilon}) > 0$ such that the inequality

$$||\log(\exp_e(\mathbf{v}^{\|})) - \log(\exp_e\big(\frac{\mathbf{u}}{2}\big) \circ \exp_e(\mathbf{v}) \circ \exp_e\big(-\frac{\mathbf{u}}{2}\big))|| < \delta(\epsilon + \tilde{\epsilon})$$

holds.

The last statement can be rewritten as the approximation:

$$\exp_e(\mathbf{v}^{\|}) \simeq \exp_e\big(\frac{\mathbf{u}}{2}\big) \circ \exp_e(\mathbf{v}) \circ \exp_e\big(-\frac{\mathbf{u}}{2}\big) \tag{2.2}$$

that will turn out to be the main tool for the computation of the log-composition using parallel transport.

When considering the equation 2.1, we use implicitly the formula for the change of base for affine exponential and logarithm [APA06]. It is in fact possible, using the derivative of the left-translation L_p, to bring back the exp and the log functions based at the point p of the manifold to the exp and the log evaluated at the identity using the following formulas:

$$\log_p(q) = DL_p(e)\log_e(q) \tag{2.3}$$
$$\exp_p(\mathbf{u}) = p \circ \exp_e(DL_p(e)^{-1}\mathbf{u}) \tag{2.4}$$

Further theoretical developments are beyond the aim of this research, but the reader can refer to the bibliography, or to the last version of *some notes on differential geometry for diffeomorphic image registration* on github [**?**], where proofs and examples are provided, and relations between equations 2.3 and parallel transport are investigated.

2.5 Numerical Computations of the Log-composition

In this section we provide explicit formulas for the computation of the log composition, using the tools introduced in the previous sections.

2.5.1 Truncated BCH formula for the Log-composition

To compute the Lie log-composition, literature provides the BCH formula (for matrices [Hal15], general case: [KO89] , [Ser09]), defined as the solution of the equation $exp(\mathbf{w}) = \exp(\mathbf{u}) \circ \exp(\mathbf{v})$, for $\mathbf{u}$ and $\mathbf{v}$ in the Lie algebra $\mathfrak{g}$:

$$BCH(\mathbf{u},\mathbf{v}) = \mathbf{u} + \mathbf{v} + \frac{1}{2}[\mathbf{u},\mathbf{v}] + \frac{1}{12}([\mathbf{u},[\mathbf{u},\mathbf{v}]] + [\mathbf{v},[\mathbf{v},\mathbf{u}]]) - \frac{1}{24}[\mathbf{v},[\mathbf{u},[\mathbf{u},\mathbf{v}]]] + ...$$

It consists of an infinite series of Lie bracket whose asymptotic behaviour cannot be predicted only from the coefficient of each nested Lie bracket term. In practical applications it can be computed using its *approximation of degree k*, defined as the sum of the BCH terms having no more than k nested Lie bracket. For example:

$$\begin{aligned}
BCH^0(\mathbf{u},\mathbf{v}) &= \mathbf{u} + \mathbf{v} \\
BCH^1(\mathbf{u},\mathbf{v}) &= \mathbf{u} + \mathbf{v} + \frac{1}{2}[\mathbf{u},\mathbf{v}] \\
BCH^2(\mathbf{u},\mathbf{v}) &= \mathbf{u} + \mathbf{v} + \frac{1}{2}[\mathbf{u},\mathbf{v}] + \frac{1}{12}([\mathbf{u},[\mathbf{u},\mathbf{v}]] + [\mathbf{v},[\mathbf{v},\mathbf{u}]])
\end{aligned}$$

These numerical approximations of the log-composition $\mathbf{u} \oplus \mathbf{v}$ can be considered as a first step but are not satisfactory since they do not provide any condition to consider the error carried by each term.

2.5.2 Taylor Expansion for the Log-composition

A more sophisticated numerical method for the computation of the log-composition that relies on the Taylor expansion, that holds only for matrix Lie groups.
As shown in the appendix of [KO89] the terms of the BCH can be recollected using the Hausdorff method: each of the therm containing the n-th power of the vector $\mathbf{v}$ are collected together in the formal series V^n. Therefore

$$BCH(\mathbf{u},\mathbf{v}) = \mathbf{u} + V^1\mathbf{v} + V^2\mathbf{v} + V^3\mathbf{v} + \cdots$$

Given the adjoint map:

$$\begin{aligned}
ad_{\mathbf{u}} : \mathfrak{g} &\longrightarrow \mathfrak{g} \\
\mathbf{v} &\longmapsto ad_{\mathbf{u}}\mathbf{v} := [\mathbf{u},\mathbf{v}]
\end{aligned}$$

and the multiple adjoint maps, defined as:

$$\mathrm{ad}^n_{\mathbf{u}}\mathbf{v} := [\underbrace{\mathbf{u},[\mathbf{u},...[\mathbf{u}}_{\text{n-times}},\mathbf{v}]...]]$$

$$\mathrm{ad}^{-n}_{\mathbf{u}}\mathbf{v} := [[...[\mathbf{v},\underbrace{\mathbf{u}]...],\mathbf{u}}_{\text{n-times}}] = (-1)^n \mathrm{ad}^n_{\mathbf{u}}\mathbf{v}$$

it can be proved that the operator V_1 that when applied to $\mathbf{v}$ provides the linear part of $\mathbf{v}$ in the BCH results in

$$V_1 = \frac{\mathrm{ad}^{-1}_{\mathbf{u}}}{\exp(\mathrm{ad}_{\mathbf{u}}) - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!}\mathrm{ad}^{-n}_{\mathbf{u}} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\mathrm{ad}^n_{\mathbf{u}}$$

Where $\{B_n\}_{n=0}^{\infty}$ is the sequence of the second-kind Bernoulli number. If first-kind Bernoulli number are used, then each term of the summation must be multiplied for $(-1)^n$, as did for example in [KO89]. The denominator is defined within the structure of the formal power series ring [MT13].
In conclusion, the log-composition can expressed as:

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \frac{\mathrm{ad}_{\mathbf{u}}^{-1}}{\exp(\mathrm{ad}_{\mathbf{u}}) - 1}\mathbf{v} + \mathcal{O}(\mathbf{v}^2)$$

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \sum_{n=0}^{\infty} \frac{B_n}{n!}\mathrm{ad}_{\mathbf{u}}^{n}\mathbf{v} + \mathcal{O}(\mathbf{v}^2) \tag{2.5}$$

that will turn out to be an important tool for the computation of the log-composition in the finite dimensional case.

2.5.3 Parallel Transport for the Log-composition

To obtain a numerical computation for the log-composition using parallel transport, three assumptions has to be made:

1. If $\mathbf{v}^{\parallel}$ is defined as in theorem 2.4.1, then exists an $\hat{\epsilon} > 0$ such that

$$||\mathbf{u} \oplus \mathbf{v} - (\mathbf{u} + \mathbf{v}^{\parallel})|| < \hat{\epsilon}$$

2. If the vector $\mathbf{w} \in \mathfrak{g}$ is small enough, then:

$$\exp(\mathbf{u}) \simeq e + \mathbf{u}$$

The second condition is valid if the hypothesis of proposition 8.6 pag. 163 [You10] holds. A deepening in this direction is not in the scope of this research. For our purposes we will consider $\mathbf{w}$ small enough to make the approximation questioned here reasonable.

From these assumptions and from equation 2.2 it follows that

$$\begin{aligned} \mathbf{u} \oplus \mathbf{v} &\simeq \mathbf{u} + \mathbf{v}^{\parallel} \\ e + \mathbf{v}^{\parallel} &\simeq \exp_e\big(\frac{\mathbf{u}}{2}\big) \circ \exp_e(\mathbf{v}) \circ \exp_e\big(-\frac{\mathbf{u}}{2}\big) \end{aligned}$$

Therefore

$$\mathbf{u} \oplus \mathbf{v} \simeq \mathbf{u} + \exp_e\big(\frac{\mathbf{u}}{2}\big) \circ \exp_e(\mathbf{v}) \circ \exp_e\big(-\frac{\mathbf{u}}{2}\big) - e \tag{2.6}$$

This is the third numerical method for the computation of the log-composition explored in this thesis.
We have to notice that when we apply it on the infinite dimensional case, the approximation 2.6 holds under the following additional assumption:

3. Theorem 2.4.1 holds when the Lie group is infinite dimensional.

whose eventual confirmation is at the moment not known to the author. We assume it is true in coherence with what has been said in the introduction, section 1.1.3.

Chapter 3

Spatial Transformations for the Computations of the Log-composition: SE(2) and SVF

Every working mathematician knows that if one does not control oneself (best of all by examples), then after some ten pages half of all the signs in formulae will be wrong and twos will find their way from denominators into numerators.
-V.I. Arnold

In the previous chapter we have introduced some essential mathematical tools for the numerical computation of the log-composition. They can be applied for several groups of transformation. In this research we show how they behave on two selected groups:

SE(2) - The group of rigid body transformation of the plane (any combination of rotations, translations) is the optimal playground to test the introduced numerical methods. Here all the closed form can be derived analytically, and therefore a ground truth is always available for comparisons. A representation of this Lie group as a subgroup of the real linear group $GL(2)$, with corresponding Lie algebra will be provided, with all of the closed form of the numerical computations presented in this research.

SVF - As a subset of the group of Diffeomorphisms. why this, why only subset, the strategy to compute the ground truth.

The core object of the theory, exponential and logarithmic map, as well as pole ladder, are obviously different for each manifold and each metric, if there is one: practical implementations have to be determined case by case.
This chapter is aimed to go through the details of the generalized theory for the cases of SE(2) and the Lie group of diffeomorphisms parametrized with stationary velocity fields.

3.1 The Group of Rigid Body Transformations

A rigid body transformation in a normed vector space is a transformation that preserves distances. The set of rigid body transformations is constructed as any combination of rotations,

translations and reflection, and forms the euclidean group $E(2)$. For 2d rigid registration usually reflections are not required and so we restrict our attention to the special euclidean group $SE(2)$. We are interested in two things about them: their expression in matrix form, and the Lie group and the Lie algebra structures involved.
We denote denoted with $M_3(\mathbb{R})$ the set of all of the 3×3 matrices with real entries . Its subset, defined by all the matrices with non-zero determinant, and thus by all the invertible matrices, is denoted with $GL_3(\mathbb{R})$. A *matrix group* is any proper or improper subgroup of $GL_3(\mathbb{R})$. The group of 2d rigid body transformation

$$\mathbb{G} = \{(\theta, tx, ty) \mid \theta \in [0, 2\pi), tx, ty \in \mathbf{R}^2\}$$

using matrices, so as a subgroup of $GL_3(\mathbb{R})$. Rotation in the plane can be expressed using matrix of the orthogonal group $SO(2)$, linear subgroup of $GL_2(\mathbb{R})$, so that rotations' actions on planes' points are simply defined as a product:

$$SO(2) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

To include the translation we can add its $(tx, ty)^T$ parameter to the action of the rotation over the initial point $(x_i, y_i)^T$ to obtain the transformed $(x_t, y_t)^T$. So each element of the group $\mathbb{G}$ act over $\mathbf{R}^2$ as

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \begin{pmatrix} tx \\ ty \end{pmatrix}$$

Another way to express rigid body transformation group's elements is to include the translation in a bigger matrix, subgroup (not linear, since the translation is not linear) of $GL_3(\mathbb{R})$. This is defined as the group $SE(2)$:

$$SE(2) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) & t_x \\ \sin(\theta) & \cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in [0, 2\pi), (tx, ty) \in \mathbf{R}^2 \right\}$$

Expressed in this way the matrices act on the point of the plane represented as the elements of the vector space $\{1\} \times \mathbf{R}^2$.
The passage between the restricted form $\mathbb{G}$ and $SE(2)$ is defined by the injection:

$$\begin{aligned} \rho_{\mathbb{G}} : \mathbb{G} &\longrightarrow SE(2) \\ (\theta, tx, ty) &\longmapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) & tx \\ \sin(\theta) & \cos(\theta) & ty \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We are now interested the Lie algebra of the Lie group $SE(2)$. It is defined as:

$$\mathfrak{se}(2) = \left\{ \begin{pmatrix} 0 & -\theta & dt_x \\ \theta & 0 & dt_y \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in [0, 2\pi), (tx, ty) \in \mathbf{R}^2 \right\}$$

Expressing $r \in SE(2)$ as:

$$\mathbf{r} = \begin{pmatrix} R(\theta) & t \\ 0 & 1 \end{pmatrix} \qquad R(\theta) \in SO(2) \quad t \in \mathbb{R}^2$$

for t plane translation and $R(\theta)$ in $SO(2)$, then the element of the Lie algebra can be expressed as:

$$d\mathbf{r} = \begin{pmatrix} dR(\theta) & dt \\ 0 & 1 \end{pmatrix} \qquad R(\theta) \in SO(2) \quad t \in \mathbb{R}^2$$

Both $SE(2)$ and $\mathfrak{se}(2)$ are in bijective correspondence with $\mathbb{G}$, and both are subset of the bigger algebra of, The algebra $\mathfrak{se}(2)$ do not form a group with the operation of composition, but it is provided with the lie bracket defined by the commutator:

$$[d\mathbf{r}, d\mathbf{s}] = d\mathbf{r}d\mathbf{s} - d\mathbf{s}d\mathbf{r}$$

The Lie logarithm between Lie group and Lie algebra is given by:

$$\begin{gathered} \log : \mathfrak{se}(2) \longrightarrow SE(2) \\ \begin{pmatrix} R(\theta) & t \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} dR(\theta) & dt \\ 0 & 1 \end{pmatrix} \end{gathered}$$

Where

$$dR(\theta) = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

and $dt = L(\theta)t$ for

$$L(\theta) = \frac{\theta}{2} \begin{pmatrix} \frac{\sin(\theta)}{1-\cos(\theta)} & 1 \\ -1 & \frac{\sin(\theta)}{1-\cos(\theta)} \end{pmatrix}$$

The inverse function, Lie exponential is given by:

$$\begin{gathered} \exp : SE(2) \longrightarrow \mathfrak{se}(2) \\ \begin{pmatrix} dR(\theta) & dt \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} R(\theta) & t \\ 0 & 1 \end{pmatrix} \end{gathered}$$

where $t = L(\theta)^{-1}dt$ for

$$L(\theta)^{-1} = \frac{1}{\theta} \begin{pmatrix} \sin(\theta) & -(1-\cos(\theta)) \\ (1-\cos(\theta)) & \sin(\theta) \end{pmatrix}$$

The proposed exponential function is not well defined over all $\mathfrak{se}(2)$.
In fact the elements of $\mathbb{G}$ can be identified with no risk with their matrices, while the same thing do not happen for the element of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. If we formalize the passage between $\mathfrak{g}$ and $\mathfrak{se}(2)$ with the function:

$$\begin{gathered} \rho_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{se}(2) \\ (\theta, dtx, dty) \longmapsto \begin{pmatrix} 0 & -\theta & dtx \\ \theta & 0 & dty \\ 0 & 0 & 1 \end{pmatrix} \end{gathered}$$

it is not an injection if we do not restrict its domain. In addition, given two elements (θ_0, dtx_0, dty_0) and (θ_1, dtx_1, dt_1) in $\mathfrak{g}$, with $\theta_1 \neq 0$, we have that for each $k \in \mathbb{Z}$, if

$$\theta_0 = \theta_1 + 2k\pi$$

and

$$(dtx_0, dty_0) = \frac{\theta_0}{\theta_1}(dtx_1, dty_1)$$

then

$$\exp(\theta_0, dtx_0, dty_0) = \exp(\theta_1, dtx_1, dty_1)$$

The exponential is then well defined only on the quotient of $\mathfrak{g}$ over the relation $\sim$, defined by

$$(\theta_0, dtx_0, dty_0) \sim (\theta_1, dtx_1, dt_1) \iff \exp(\theta_0, dtx_0, dty_0) = \exp(\theta_1, dtx_1, dt_1)$$

The quotient set $\mathfrak{g}/\sim$ coincides the neighborhood U of the identity on which the function $\rho_{\mathfrak{g}}$ becomes an injection

$$\rho_{\mathfrak{g}/\sim} : \mathfrak{g} \longrightarrow \mathfrak{se}(2)$$

and exp is a bijection having log as its inverse. What said so far can be summarize in the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[rrr, "\rho_{\mathfrak{g}}"] \arrow[dr, "\pi"] & & & \mathfrak{se}(2) \\
& \mathfrak{g}/\sim \arrow[r, "\rho_{\mathfrak{g}/\sim}"] \arrow[dl, "exp"] & U \subset \mathfrak{se}(2) \arrow[ur, hook] \arrow[dr, "exp"] & \\
\mathbb{G} \arrow[uu, "\log"] \arrow[rrr, "\rho_{\mathbb{G}}"] & & & SE(2) \arrow[uu, "\log"']
\end{tikzcd}$$

We can see that the function $\rho_{\mathfrak{g}}$ is the inverse of a restriction of the general vectorization function that aligns column vector in a single vector. This will be particularly useful for our purposes.
xxx this part must be set after subsection 2.5 is done, to avoid repetitions and circular properties!
We can see that the function $\rho_{\mathfrak{g}}$ is the inverse of a restriction of the general vectorization function that aligns column vector in a single vector:

$$\begin{aligned}
\text{Vect} : M_3(\mathbb{R}) &\longrightarrow \mathbb{R}^{3\times 3} \\
[A_1 | A_2 | A_3] &\longmapsto (A_1^t, A_2^t, A_3^t)
\end{aligned}$$

Thanks to this adjoint action can be defined as an action over The vectorization, in combination with Lie bracket, Kronecker product, adjoint action and adjoint map, satisfies the following properties:

- $\text{Vect}([M, X]) = (I \otimes M - M^t \otimes I)\text{Vect}(X)$
- $\text{Vect}([X, M]) = (M^t \otimes I - I \otimes M)\text{Vect}(X)$

These are still valid for its restriction

$$\begin{aligned}\mathrm{Vect}^{\sim} : M_3(\mathbb{R}) &\longrightarrow \mathbb{R}^3 \\ [A_1|A_2|A_3] &\longmapsto (a_{2,1}, a_{3,1}, a_{3,2})\end{aligned}$$

that respects the Lie group operations between the restricted representation $\mathfrak{g}$ and the matrix representation $SE(2)$:

and will be used in the next subsection to compute the log composition.

In the finite-dimensional case, investigate here the log-composition can be computed with a close formula:

$$d\mathbf{r}_1 \star d\mathbf{r}_2 = \log(\exp(d\mathbf{r}_1) \circ \exp(d\mathbf{r}_2))$$

which results

$$d\mathbf{r}_1 \star d\mathbf{r}_2 = xxx\text{On some lost paper... to be computed again!}$$

Log and Exp Approximations for Small Rotations

Computations of logarithm and exponential obtained so far are a consequence of these formula:

$$\exp(\mathbf{r}) = \sum_{k=0}^{\infty} \frac{\mathbf{v}^k}{k!} \qquad \log(\mathbf{r}) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(\mathbf{v} - I)^k}{k!}$$

Remarkably, infinite series of elements of a group (whose sum is not even defined within the group structure) is an element into an associated algebra, while another infinite series of matrices of the algebra appears to be the natural way to going backward. A second door to passing from one structure to the other, when $\mathbf{r}$ is little appears to be the following approximation:

$$\exp(\mathbf{r}) \simeq I + \mathbf{r} \qquad \log(d\mathbf{r}) \simeq d\mathbf{r} - I$$

In fact for little θ, $\sin(\theta) \simeq \theta$, $\cos(\theta) \simeq 0$ and $L(\theta)^{-1} \simeq I$.
xxx this may deserve an investigation about the errors in the approximations error!

3.2 The Set of Stationary Velocity Fields

Corollary 3.2.1. Using the If, with previous notations, the condition (1) is an approximation

$$\exp_C(\frac{\mathbf{k}}{2}) = \exp(\xi) \circ \exp_M(\frac{\mathbf{k}}{2})$$

for some ξ in $\mathfrak{g}$ such that $\parallel \xi \parallel < \delta$ then the approximation has error

$$O(\parallel \delta\mathbf{u}^{\parallel} \parallel^2) + O(\parallel \mathbf{u} + \delta\mathbf{u} \parallel^3) + \text{xxx something that must be investigated depending on } \delta$$

Chapter 4

Log-computation Algorithm using Log-composition

We believe that we know something about the things themselves when we speak of trees, colors, snow, and flowers; and yet we possess nothing but metaphors for things — metaphors which correspond in no way to the original entities.
-Nietzsche, *On Truth and Lies in extra-moral sense.*

zz write chapter intro.

The problem of the computation of the logarithm computation can be stated as follows: given $p \in \mathbb{G}$ the goal is to find $\mathbf{u}$ such that $\exp(\mathbf{u})$ is the best possible approximation of p. This chapter is devote to the numerical computation of the logarithm, using an iterative algorithm based on the Log-composition. In this context each of the presented techniques are suitable to perform this computation.

Before we need to introduce the space of the approximation of the Lie algebra and the Lie group.

4.1 Space of approximations

We emphasize the fact that if $\mathfrak{g}$ and $\mathbb{G}$ are subset of a bigger algebra, then exp and log can be considered as infinite series. Remarkable consequence is the approximation of $\exp(\mathbf{v})$ with $1+\mathbf{v}$ if the transformation $\mathbf{v}$ is small. This approximation is the base of what follows in this chapter.
In parallel with the log-composition $\mathbf{v}_1 \oplus \mathbf{v}_2 = \log(\exp(\mathbf{v}_1) \circ \exp(\mathbf{v}_2))$ we define two approximating functions:

$$\begin{aligned} \mathrm{app} : \mathfrak{g} &\longrightarrow \mathfrak{g}^{\sim} \\ \mathbf{u} &\longmapsto \exp(\mathbf{u}) - 1 \end{aligned}$$

$$\begin{aligned} \mathrm{App} : \mathbb{G} &\longrightarrow \mathbb{G}^{\sim} \\ \exp(\mathbf{u}) &\longmapsto 1 + \mathbf{u} \end{aligned}$$

Where $\mathfrak{g}^{\sim}$ is a space of approximations of elements of $\mathfrak{g}$, and $\mathbb{G}^{\sim}$ is a space of approximations of elements in $\mathbb{G}$ (xxx that requires some more investigations and formal definition in

conjunction with truncated series).
Consequence of this definition is the fact that

$$\mathbf{u} \simeq \mathrm{app}(\mathbf{u}) \qquad \exp(\mathbf{u}) \simeq \mathrm{App}(\exp(\mathbf{u}))$$

The two following straightforward properties, that holds for all $\mathbf{u}, \mathbf{v}$ in the Lie algebra

1. $\mathbf{u} = \mathbf{v} \oplus (-\mathbf{v} \oplus \mathbf{u})$
2. $\mathrm{app}(\mathbf{v} \oplus \mathbf{u}) = \exp(\mathbf{v}) \exp(\mathbf{u}) - 1 \in \mathfrak{g}^{\sim}$

lead us to consider the algorithm presented in [BO08], here called log-computation, with a new reformulation.

4.2 The Log-computation Algorithm using Log-composition

If the goal is to find $\mathbf{u}$ when its exponential is known, we can consider the sequence transformations $\{\mathbf{u}_j\}_{j=0}^{\infty}$ that approximate $\mathbf{u}$ as consequence of

$$\mathbf{u} = \mathbf{u}_j \oplus (-\mathbf{u}_j \oplus \mathbf{u}) \Longrightarrow \mathbf{u} \simeq \mathbf{u}_j \oplus \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u})$$

This suggest that a reasonable approximation for the $(j+1)$-th element of the series can be defined by

$$\mathbf{u}_{j+1} := \mathbf{u}_j \oplus \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u})$$

If we chose the initial value $\mathbf{u}_0$ to be zero, then the algorithm presented in [BO08] become:

$$\begin{cases} \mathbf{u}_0 = 0 \\ \mathbf{u}_{j+1} = \mathbf{u}_j \oplus \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u}) \end{cases} \tag{4.1}$$

Each strategy that we have examined to compute the Lie composition, become a numerical method for the computation of the logarithm.

4.2.1 BCH Strategy

At each step, we compute the approximation $\mathbf{v}_{j+1}$ with the k-th truncation of the BCH formula:

$$\begin{cases} \mathbf{u}_0 = 0 \\ \mathbf{u}_{j+1} = \mathrm{BCH}^k(\mathbf{u}_j, \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u})) \end{cases} \tag{4.2}$$

thus, for the first degree we have

$$\begin{aligned} \mathrm{BCH}^1(\mathbf{u}_j, \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u})) &= \mathbf{u}_j + \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u}) \\ &= \mathbf{u}_j + \exp(-\mathbf{u}_j) \exp(\mathbf{u}) - 1 \end{aligned}$$

For the second degree we have:

$$\begin{aligned} \mathrm{BCH}^2(\mathbf{u}_j, \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u})) &= \mathbf{u}_j + \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u}) + \frac{1}{2}[\mathbf{u}_j, \mathrm{app}(-\mathbf{u}_j \oplus \mathbf{u})] \\ &= \mathbf{u}_j + \exp(-\mathbf{u}_j) \exp(\mathbf{u}) - 1 + \\ &+ \frac{1}{2}(\mathbf{u}_j(\exp(-\mathbf{u}_j)\exp(\mathbf{u}) - 1) - (\exp(-\mathbf{u}_j)\exp(\mathbf{u}) - 1)\mathbf{u}_j) \end{aligned}$$

Theorem 4.2.1 (Bossa)**.** The iterative algorithm (4.2) converges to $\mathbf{v}$ with error $\delta_n \in \mathbb{G}$, where

$$\delta_n := \log(\exp(\mathbf{v}) \circ \exp(-\mathbf{v}_n)) \in O(||p - e||^{2^n})$$

4.2.2 Parallel Transport Strategy

If we use the parallel transport for the computation of the log-composition, we obtain:

$$\begin{cases} \mathbf{u}_0 = \mathbf{0} \\ \mathbf{u}_t = \mathbf{u}_{t-1} + \exp(-\frac{\mathbf{u}_{t-1}}{2}) \circ \exp(\delta\mathbf{u}_{t-1}) \circ \exp(\frac{\mathbf{u}_{t-1}}{2}) - e \end{cases} \tag{4.3}$$

4.2.3 Symmetrization Strategy

The algorithm for the computation of the group logarithm can be improved considering a symmetric version of the underpinning strategy. In this version we use the first order approximation of the BCH formula (see equation (4.6) in the following proof), compensating with the fact that the symmetrization should decrease the error involved. It gives birth to the following algorithm:

$$\begin{cases} \mathbf{v}_0 = \mathbf{0} \\ \mathbf{v}_{t+1} = \mathbf{v}_t + \frac{1}{2}(\tilde{\delta}\mathbf{v}_t^L + \tilde{\delta}\mathbf{v}_t^R) \end{cases} \tag{4.4}$$

Where $\tilde{\delta}\mathbf{v}_t^R = \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) - e$ and $\tilde{\delta}\mathbf{v}_t^L = \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) - e$.

Proof. To show why it works we remind that the starting point was

$$p = \exp(\mathbf{v}) = \exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0)$$

where $\exp(\delta\mathbf{v}_0) = \exp(-\mathbf{v}_0) \circ p$.
An equivalent starting point would have been $\exp(\mathbf{v}) = \exp(\delta\mathbf{v}) \circ \exp(\mathbf{v}_0)$ for $\exp(\delta\mathbf{v}) = p \circ \exp(-\mathbf{v}_0)$.
This idea leads to the definition of

$$\begin{aligned} \exp(\delta\mathbf{v}_t^R) &:= p \circ \exp(-\mathbf{v}_t) = \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) \\ \exp(\delta\mathbf{v}_t^L) &:= \exp(-\mathbf{v}_t) \circ p = \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) \end{aligned}$$

It follows that

$$\begin{aligned} \exp(\mathbf{v}) &= \exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0^R) \\ \exp(\mathbf{v}) &= \exp(\delta\mathbf{v}_0^L) \circ \exp(\mathbf{v}_0) \end{aligned}$$

Using $\exp(\delta\mathbf{v}_t^R) \approx e + \delta\mathbf{v}_t^R$ and $\exp(\delta\mathbf{v}_t^L) \approx e + \delta\mathbf{v}_t^L$ we can use the following approximation to define the symmetric algorithm:

$$\begin{aligned} \exp(\delta\mathbf{v}_t^R) &= \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) \\ e + \tilde{\delta}\mathbf{v}_t^R &= \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) \\ \tilde{\delta}\mathbf{v}_t^R &= \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) - e \end{aligned}$$

$$\begin{aligned} \exp(\delta\mathbf{v}_t^L) &= \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) \\ e + \tilde{\delta}\mathbf{v}_t^L &= \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) \\ \tilde{\delta}\mathbf{v}_t^L &= \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) - e \end{aligned}$$

Which gives birth to iterative algorithm, for a given initial value V_0:

$$\begin{cases} \mathbf{v}_0 \\ \mathbf{v}_{t+1} = \mathrm{BCH}(\mathbf{v}_t, \tilde{\delta}\mathbf{v}_t^R) \end{cases} \quad \begin{cases} \mathbf{v}_0 \\ \mathbf{v}_{t+1} = \mathrm{BCH}(\tilde{\delta}\mathbf{v}_t^L, \mathbf{v}_t) \end{cases} \tag{4.5}$$

If follows that

$$\mathbf{v}_{t+1} = \frac{1}{2}(\mathrm{BCH}(\tilde{\delta}\mathbf{v}_t^L, \mathbf{v}_t) + \mathrm{BCH}(\mathbf{v}_t, \tilde{\delta}\mathbf{v}_t^R))$$

Taking the first order approximation of the BCH formula:

$$BCH(\tilde{\delta}\mathbf{v}_t^L, \mathbf{v}_t) \approx \tilde{\delta}\mathbf{v}_t^L + \mathbf{v}_t \quad (4.6)$$

$$BCH(\mathbf{v}_t, \tilde{\delta}\mathbf{v}_t^R) \approx \mathbf{v}_t + \tilde{\delta}\mathbf{v}_t^R \quad (4.7)$$

we get

$$\mathbf{v}_{t+1} = \mathbf{v}_t + \frac{1}{2}(\tilde{\delta}\mathbf{v}_t^L + \tilde{\delta}\mathbf{v}_t^R)$$

□

We observe that the symmetric approach do not requires to use the BCH formula at each passage, having considered the approximation at the first order of the BCH.
We conclude with a formula that relates $\tilde{\delta}\mathbf{v}_t^L$ with $\tilde{\delta}\mathbf{v}_t^R$:

Theorem 4.2.2. Be $\tilde{\delta}\mathbf{v}_t^R = \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) - e$ and $\tilde{\delta}\mathbf{v}_t^L = \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) - e$ as before, then

$$\delta\mathbf{v}_t^L \approx \exp(-\mathbf{v}_t) \circ \delta\mathbf{v}_t^R \circ \exp(\mathbf{v}_t)$$

Proof. Since $\exp(\mathbf{v}_t) \circ \exp(\delta\mathbf{v}_t^R) \approx exp(\delta\mathbf{v}_t^L) \circ \exp(\mathbf{v}_t)$ it follows

$$\exp(\delta\mathbf{v}_t^R) = \exp(-\mathbf{v}_t) \circ \delta\mathbf{v}_t^L \circ \exp(\mathbf{v}_t)$$

Using $\exp(\delta\mathbf{v}_t^R) = e + \delta\mathbf{v}_t^R$ and $\exp(\delta\mathbf{v}_t^L) = e + \delta\mathbf{v}_t^L$ we get

$$e + \delta\mathbf{v}_t^R = \exp(-\mathbf{v}_t) \circ (e + \delta\mathbf{v}_t^L) \circ \exp(\mathbf{v}_t)$$

$$\delta\mathbf{v}_t^R = \exp(-\mathbf{v}_t) \circ \delta\mathbf{v}_t^L \circ \exp(\mathbf{v}_t)$$

□

Chapter 5

Experimental Results

"A victory is twice itself when the achiever brings home full numbers."
Much ado about nothing, *Leonato, scene 1.*

5.1 Group composition at the Service of Image Registration

5.2 Experimental Results

5.2.1 Logarithm Computation using Log-composition

5.3 Further Research and Conclusion

Considering only the results, this one-year research can be considered much ado about nothing, but...

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