Toward a BCH-free Computation of the Composition of Stationary Velocity Fields

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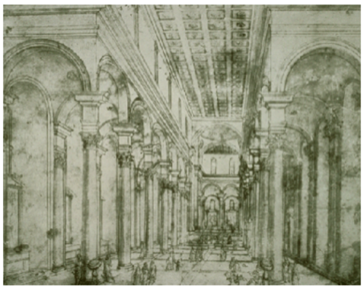
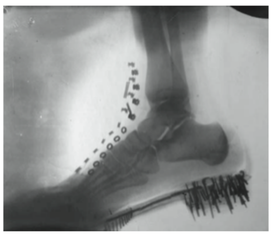
Introduction

Since the early anatomical studies, the reproduction of images has been one of the main tool to investigate structures and differences of organs and organisms. Better ways to represent reality have undoubtedly led to better ways to understand it, and the work of pioneers in each of the discipline sharing the common need of precise and detailed images, were aided by available techniques offered in every epoch. At the dawn, Artists and Anatomists not



surprisingly could have not be separated, and every enhancement made in one field followed immediately one in the other. Although, whatever name every age gave to disciplines, the origin of every improvement in the field of images has its roots in the study of Geometry.
The first and most significant example is the powerful idea of perspective. Fixing a point from where to start infinitely parallel straight lines, leads, 16 centuries after their formalizations, Brunelleschi to revolutionize the world's perception in artists' minds. Two centuries later, when the inventors' work took the name of Science, the study of magnifying lenses and their combination, made available new explorable territories. Here again the Geometry couldn't have more important role in the systematic study of the lights deformation effect through conic-shaped glass.
Growing number of needs grows skills and techniques: artist, scientist and anatomist could

not stay anymore below the same person's name, but what has remained constant is that every new discovery and the improvement in one field became an immediate advancement in others. With the first stages of photographic techniques in the early '800, in parallel with the Young double slit experiment and its Fresnel interpretation, scientific community bring new attention to the shape of light. In late years of the same century the elegant Maxwell unification and its formalization by Heaviside determined the geometrical parameters of light. The discovery of X-ray by the first Nobel prize Roentgen and its refinement by Tesla is considered [Bra08] the birth certificate of Medical Imaging. Domain destined to walk hand by hand with physics and engineering and to be eventually the most collaborative and crossing discipline ever seen with a great number of others evolving domain. The transition from pencil's scientists and painting's artists to photographic equipment that characterize this period didn't affected only this newborn branch. Instruments brings literally light on the severe limits of human eye capabilities: what was hidden reality become visible.
Just after the formalization of electric and magnetic field's shape, and the consequent mastery of electrical forces, discoveries of photoelectric effect opened new fields in the Physics of matter. Contemporary reformulation of diffusion processes, provided what will be angular theories in the construction of fundamental equipment for medical imaging.
In the wake of the third industrial revolution, electronic engineering, with its radio circuits, triods and valves destined to became transistor, provided each time better instrument to medical science where the acquisition and manipulation of patient images is a thriving theme. Thanks to philanthropic, scientific and economic interest gravitating around health care, the complicated interaction between sciences, physics and medicine get smoothed leading to achievement as radiography, ultrasound, thermography, magnetic resonance, optical fiber, nuclear medicine, confocal microscopy just to name a few. Technologies aimed to visualize the interior of a body always attained from several parts to reach their aims.
When photography get rid of photosensitive films to welcome digital sensors, images become stored in byte's grids with consequent powerful manipulation possibilities. The translation of acquired data from several sources not always or entirely compatible with the human sight to a suitable form meets new challenges. Among the various possibility offered by image processing, two tools to reveal internal features and compare anatomies in images analysis are mostly utilized in medicine: segmentation and registration. Both aimed to investigate patients' anatomies and physiologies, segmentation consists in the enhance contours, detect edges and to reveal hidden structure. Registration is the process of determining correspondences between one or more images acquired from patients scans.
This Master Thesis deals with diffeomorphic image registration and, letting aside dribs and drabs history of science, still a few words about the state of the art are needed to complete its introduction.

Toward a Solution of an Ill Posed Problem

Dealing with image registration means searching for a solution to an ill posed problem: transformations between anatomies are not unique, and the impossibility of recover the spatial or temporal evolution of a anatomical transformation from temporally isolated images, makes any validation a difficult, if not impossible task. Among all of the possible voxelwise mapping that transform one image into another one, clinical interest may vary according to the requirements of each specific case[1].
In brain imaging, for example, registration is performed to examine differences between subjects to distinguish healthy from sick patients and for a better understanding of the disease's

[1] A recent survey in medical image registration can be found in [SDP13].

feature. Another case may require to compare different acquisition of the same subject, before and after a surgery or after a fixed period of time: parameters and features of the transformation are completely different. Also the correction of motion distortion in image's acquisition phase, the mosaicing of several images or the construction of the model of cardiac and respiratory motion may require customized image registration techniques.
Any approach is therefore varied and flexible: this led to a wide range of tools that has been proposed by researchers in the last decades[2].
There are several features that distinguish image registration's algorithm, but the most relevant is the choice of the family to whom the transformation belongs. Since anatomies appears to transform continuously over time, without any variation in the topological features, the use of diffeomorphisms as transformation appears one of the most natural.
The continuous nature of these functions appears to be in contrast with the discrete nature of voxel images as well as with any computer's parametrization ability.
The approach of modeling with richer structure for simpler elements is actually very common even in less sophisticated math: for example when measuring the diagonal of a 1 meters side table. The decimal unlimited non periodical $\sqrt{2}$ doesn't help until we don't consider it as an answer belonging to a larger-than-reality mathematical structure: computations and theorem (as the Pythagorean theorem) are well defined and meaningful.
Back to images, the simple structure of a raster image as 3-dimensional matrix is really limited if compared to the continuous object they represent. Modeling with continuous function provides a structure that reflects the object's topology. In addition enable us to apply mathematical features from differential geometry and dynamical system theory.
The first idea of using smooth and continuous function for image registration goes back to the idea of using the Navier-Cauchy partial differential equation to model the deformation of images as two balancing forces applied to an elastic body [Bro81]. The solution's domain restriction to diffeomorphisms for the solution of the Lagrange transport equation for medical imaging registration appears in [DGM98a] and [Tro], and with his many variants is an active subject of study for research in mathematic applied to medical imaging. An important framework for the computation of image registration of diffeomorphism is provided by the Large Deformation Diffeomorphic Metric Mapping (LDDMM). Here diffeomorphisms are parametrized as ending point of integral curves of vector field on the Lie group of diffeomoprhism equipped with a Riemannian metric [DGM98b], [BMTY05]. Solid mathematical foundations are payed in term of computational complexity. Different parametrization of diffeomorphisms, as Stationary Velocity Fields [ACPA06] has been embedded in the LDDMM framework, to reduce computational time. This approach gives birth to the DARTEL [Ash07] and the Stationary LDDMM [HBO07]. Starting from the Tririon's DEMON algorithm [Thi98], a different framework for diffeomorphic image registration was presented as Diffeomorphic Daemon [VPPA07] and the LCC-daemon [LAF$^+$13] . A comparison between stationary LDDMM and Diffeomorphic Demons with emphasis in both theoretical and practical aspects can be found in [HOP08].
The theme of diffeomorphism do not recurs only in medical application but it is a continuously improved subject of research also theoretical studies as [Mil84], [BBHM11], [BHM10] or studies applied to other domain of science [Arn] [OKC92]. Geometry remains an important underpinning structure for many achievement in medical imaging and still advanced research in geometry is actively used by this one.
Image registration do not involves only the construction of a transformation. The necessity of having statistics to infer the variability of anatomical structure bring the field of

[2] A quick glance to Google scholar reveals about 1200000 papers in *medical image registration* (55% of the whole *image registration* resources).

Patter Analysis from Machine Vision to Medical Imaging, with the name of Computational Anatomy (cite survey computational anatomy). Statistics on diffeomorphisms are naturally introduced relying on Lie algebras and introducing the log-euclidean framework; proposed for the first time in 2004 [AFPA06] followed by an update of 2006 , as a faster improvement to the affine invariant Riemannian approach. It has found successful applications in many domains of medical imaging (in vivo mosaicing [Ver08], brain Alzheimer detection [Lor12], cardiac image analysis [MPS+11], mandible imaging using polyaffine registration [CSR11]) and has been continuously improved.
Sono quindi molteplici le facce da studiare e dalle quali attingere che l'uso dei diffeomorfismi comporta:

la ricerca teorica, il loro uso nelle applciazioni pratiche e il loro sudio nell'analisti statistica dell'evoluzione delle deformazioni anatomiche.

Aim of this thesis is to present numerical methods to compute the composition fo diffeomorphisms in the log-euclidean framework. Questa tesi sviluppa il tema particolare della loro composizione, usando lo sviluppo in serie di taylor, le accelerazioni delle serie numeriche, e il parallel transport, tantando di tenere conto degli aspetti teorici e pratici della questione.

Thesis' Organization and No(ta)tions

The first chapter about the general framework in image registration and parametrization of diffeomorphisms for LDDMM and SVF is followed by three distinct part. The first one presents basic tools of differential geometry and parallel transport, with emphasis on the computational side. The second part is about the principal objects utilized to explore new numerical techniques and to compare them with the one currently utilized. The last part is devoted to the results for the computation on synthetic dataset and on patient images.

Chapter 1: introduction of the registration framework. After a first section about the main definitions and concept used throughout the thesis, we present the main feature of the general framework used to perform registration. Particular attention is given to the pros ad cons of using diffeomorphism as set of transformation between anatomies and some considerations about the current methods currently used for the implementation: the LDDMM and SVF.

Chapter 2: an in deep of the possibilities of set of transformations when provided by mathematical structure of Lie group. Main mathematical elements and tools from Lie group theory directly involved in the image registration techniques are formally defined with a particular attention to flows, left translation, push forward, Lie logarithm and Lie exponential. We define as well the concept of Log-composition around which the research gravitates: it originates form the BCH formula, presented here as well as the first more immediate way to compute the Log-composition. The second way to compute it is provided by the Taylor expansion, presented in the last section.

Chapter 3: BCH and Taylor expansion are tow possibility to compute the Log-composition. A third one presented in this chapter originates by a geometrical approach and it is given by the parallel transport. The first and the second sections are devoted to present the theoretical tools to define formally the parallel transport. Last section is about two strategies to compute the parallel transport without involving the Christoffel symbols: the Schild's Ladder and the Pole Ladder.

Chapter 4: ...

Chapter 5 and 6: validity of results in the Log-composition computation are tested with two groups of transformations commonly used in medical image registration: the group of rigid body transformation and the group of diffeomorphisms (expressed in the application as the set of Stationary Velocity Fields). These chapters are aimed to present them in details and they are oriented to the application.

Chapter 7: this is the central part of the research. The Log-composition is analyzed as s valuable tool in image registration, within the framework presented in chapter 1. A summary of the methods for its computation is presented as possible numerical approximation to be utilized for image registration: BCH formula, Taylor expansion, parallel transport and Accelerating Convergences series.

Chapter 8: the algorithm for the Lie-logarithm computation presented in *A new algorithm for the computation of the group logarithm of diffeomorphism* [BO08] gravitates around the BCH formula. If reformulated with the Log-composition each of its numerical approximation is a valid tool to improve its performance. Of particular interests are the methods that avoid the computation of the BCH formula on which the algorithm was initially based.

Chapter 9: is devoted to experimental results. Performance of the Log-composition applied to rigid body transformation and diffeomorphisms are separately computed and compared. In addition a version of NiftyReg based on various we present the results of the numerical methods presented in the previous section, on synthetic data as well as on clinical data within a version of the LCC-Demons customized with parallel transport.

Chapter 10 we draw the conclusion of what has been done so far (with a shameless and challenging emphasis of what is missing and what is still to be done).

Contents

Chapter 1

Image Registration Framework

Every working mathematician knows that if one does not control oneself (best of all by examples), then after some ten pages half of all the signs in formulae will be wrong and twos will find their way from denominators into numerators.
-V.I. Arnold

1.1 Introductory Definitions

We define a *d-dimensional image* as a continuous function from a subset Ω of the coordinate space $\mathbb{R}^d$ (having in mind particular cases $d = 2, 3$) to the set of real numbers $\mathbb{R}$. Given two of them $T : \Omega_T \to \mathbb{R}$ and $R : \Omega_R \to \mathbb{R}$, called respectively *target image* and *reference image*, the *image registration problem* consists in the investigation of features and parameters of the transformation function

$$\begin{aligned} \varphi : \mathbb{R}^d \supseteq \Omega_R &\longrightarrow \Omega_T \subseteq \mathbb{R}^d \\ \mathbf{x} &\longmapsto \varphi(\mathbf{x}) \end{aligned}$$

such that for each point $\mathbf{x} \in \Omega_t$ the element $T(\varphi(\mathbf{x}))$ and $R(\mathbf{x})$ are closed as possible according to a chosen metric. The underpinning idea can be represented by the following diagram, where φ is the solution that makes f the identity function.

$$\begin{tikzcd} \Omega_R \arrow[r, "\varphi"] \arrow[d, "R"] & \Omega_T \arrow[d, "T"] \\ \mathbb{R} \arrow[r, dashed, "\mathrm{f}"] & \mathbb{R} \end{tikzcd}$$

This definition leave two degrees of freedom in searching for a solution: the domain of the transformation (also called deformation model), and the metric to measure the similarity between images.
Once these are chosen, they can be used as constituent of an *image registration framework*: an iterative process that at each step provides a new function φ that approaches f to the identity (figure 1.1). This framework can provide additional degrees of freedom: the metric

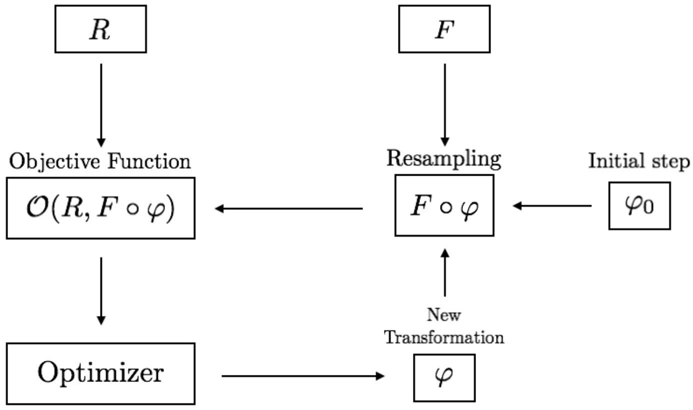


Figure 1.1: Generic image registration framework scheme.

can be considered with an additive regularization term, that introduces a constraint based on prior knowledge about the searched solution:

$$\mathcal{O}(R, F \circ \varphi) = \mathrm{Sim}(R, F, \varphi) + \mathcal{R}(\varphi)$$

where Sim is a function to measure the similarity while $\mathcal{R}$ is the regularization term. In addition optimization algorithm on which the optimizer is based and the resampling (the process of resize the image from one dimension to another) strategy can be defined over several possibilities. This generic scheme is far from being fixed and among all of the possible degrees of freedom there can be several variants (for a survey in image registration see for example [SDP13], [HBHH01]). Moreover each of the implementation currently available may have been performed in different ways according to the authors need and perspective.
In this thesis φ will be bounded to the group of rigid transformation or in the set of bijective continuous function with continuous inverse (i.e. diffeomorphisms). Using diffeomorphisms with the underpinning Lie algebra and Lie group theory was proposed for the first time in 2006 (Arsigny et Al [AFPA06]), as a faster improvement to the affine invariant Riemannian approach, has found successful applications in many domains of medical imaging (in vivo mosaicing [Ver08], brain Alzheimer detection [Lor12], cardiac image analysis [MPS$^+$11], mandible imaging using polyaffine registration [CSR11]) and has been continuously improved since its introduction (Ashburner [Ash07], Vercauteren [VPPA08], Lorenzi [LP13]). In this setting, computations in the Lie group of diffeomorphism are made in its Lie algebra, in which it is possible to apply easily calculus and statistics.

1.2 Iterative Registration Algorithm

1.3 Using Diffeomorphisms: Utility and Liability

1.3.1 Parametrization of Diffeomorphisms: LDDMM and SVF

1.3.2 Composition of Diffeomorphisms: the BCH formula

Among multiple approaches in image registration, the use of diffeomorphism

Under this light, the transitions from the Lie group to Lie algebra (the Lie logarithm) and its return (the Lie exponential) become fundamental tools. The main topic of this research is the evaluation of the operation that we define here as *Lie algebra Group Composition*, or simply *Group Composition*, defined as the vector $\mathbf{w}$ in the Lie algebra $\mathfrak{g}$ that reflects the composition in the Lie group of two vectors $\mathbf{u}, \mathbf{v}$ in the same tangent space:

$$\mathbf{w} = \log(\exp(\mathbf{u}) \circ \exp(\mathbf{v})) \qquad \forall \mathbf{u}, \mathbf{v} \in \mathfrak{g}$$

The role that the Group Composition plays in the diffeomorphic image registration is presented in the next subsection and explored in a formal way in section **??**.

The definition of the registration problem and the iterative algorithm described raise several issues. For example there are no reasons to believe that such a correspondence is unique and that there is at least one of them whose behaviour corresponds to a reasonable biological transformation between anatomies. Some constraints on p must be defined in order to have a transformation that models realistic changes that can occur in biological tissues. The kind of constraint that we will use in this research is to give some limitation to set (and the consequent mathematical structure) in which the transformation p lives.
In the finite dimensional case a big family of transformation suitable for p is given by the category of the complete subgroup of $\mathrm{GL}(d)$, called Matrix Lie group, or Classical Lie group. One of the Matrix Lie group explored in this research is the special euclidean group $\mathrm{SE}(d)$, or group of rigid transformations. A little increase in the number of degrees of freedom leads to the group of the affine transformations, where scaling and shearing have been added to the possible transformations.
The idea of maximize the number of degrees of freedom in a transformation that maintains anatomical features between images, led to consider the Lie group of diffeomorphism Diff. A diffeomorphism p can be expressed as the sum of the identity function with a differentiable transformation depending on $\mathbf{x}$. It is possible to write:

$$p(\mathbf{x}) = \mathbf{x} + \gamma(\mathbf{x})$$

With this notation the vector $\dot{\gamma}(\mathbf{x})$ is the speed[1] of each point from its original to a new position due to the application of p.
In the iterative diffeomorphic registration algorithm (in which the structure is defined by regularization and similarity), to obtain the update at each step, it is required to apply consecutively two diffeomorphisms p_i and $\hat{p}_{i+1}$ and consider the resulting composition as the required update. As a consequence, the corresponding vector of the transformation p_{i+1} in the tangent space of the Lie group can be computed, given two vectors in the tangent bundle, with an operation called here *the group composition in the Lie algebra*.
In [BHO07] the BCH formula appears as an improvement of the scaling and squaring algorithm to compute the logarithm of a matrix. As noticed in the same paper, the BCH formula is used in a infinite dimensional settings, while its validity has been proven only for finite dimensional Lie group. In addition while composing the logarithm of a composition of two exponentials, the initial tangent vectors belongs to the same tangent space. This is not the case while composing p_i with $\hat{p}_{i+1}$ the tangent vector that defines $\hat{p}_{i+1}$ is an element of the tangent plane to p_i.
Is it possible to reach a proper computation thanks to the definition of Affine exponential, that differs from the Lie exponential used in [BHO07] .

The BCH formula provides an expression of the Group Composition in the finite dimensional case expressed as an infinite series of nesting commutators. The difficulties involved

[1] as we will see, a feasible way to express a set of diffeomorphisms is as the solution of the differential equation defined by the velocities of the transformation.

in dealing with this formula, as well as its non completely proper use in the infinite dimensional setting, gave birth to some methodologies and approaches, whose investigation is the main goal of this research. One of these approaches, involves the Taylor expansion and has a computable form in the finite dimensional. A geometrical approach that holds also in the infinite dimensional case is the parallel transport. It can be considered as a natural approach to evaluate the group composition with offset. It consists in the transport of the vector $\mathbf{v}$ along the geodesic having $\mathbf{u}$ as tangent vector such that during the transport it maintains the condition of parallelism. The resulting vector $\mathbf{v}^{\parallel}$ in the tangent plane of $\mathbf{u}$ can be then summed with $\mathbf{v}$ with similar results to the direct application of the BCH. Its evaluation involves the Schild's ladder and the pole ladder. Both have already found practical application in medical imaging [LP14], [LP13]. Parallel transport is necessarily related with the definition of the connection. Choosing among possible connections become a crucial feature, whose consequences in the corresponding realization in the transformation of images requires further studies.

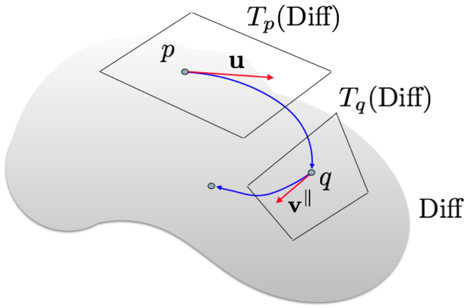


Figure 1.2: Group composition with offset using parallel transport.

The lack of a close form of any kind and the theoretical difficulties inherent to the investigation of infinite dimensional group of diffeomorphism makes an experimental approach meaningful.

Part I

Tools

Chapter 2

A Lie Group Structure for the Set of Transformation

Every working mathematician knows that if one does not control oneself (best of all by examples), then after some ten pages half of all the signs in formulae will be wrong and twos will find their way from denominators into numerators.
-V.I. Arnold

We consider every group $\mathbb{G}$ as a group of transformation acting on $\mathbb{R}^d$, having in mind the particular case $d = 2, 3$ for 2-dimensional or 3-dimensional images. We will focus out attention to transformations defined by matrices or diffeomorphism. Other than group they also have the structure of Lie group: they are considered with a maximal atlas that makes them differentiable manifold, in which the composition of two transformation and the inverse of each transformation are well defined differentiable maps:

$$\begin{aligned} \mathbb{G} \times \mathbb{G} &\longrightarrow \mathbb{G} \\ (x, y) &\longmapsto xy^{-1} \end{aligned}$$

Differential geometry is in general a technique to use the well known calculus features and operators on spaces different from the usual $\mathbb{R}^n$. Adding the differentiable structure to a group of transformations gives us new handles to hold them: in particular provides the opportunity to define a tangent space to each point of the group (and so a fiber bundle), a space of vector fields, a set of flows and one parameter subgroup as well as other features that enrich this structure. The abstract idea of vector field over a manifold will be concretized for image registration introducing the concepts of *displacement field*, *deformation field* and *velocity field (stationary or time varying)* that will be there presented. Avoid pedantry is as important as to avoid confusions on notations and definitions, therefore it is necessary to call back a few concepts from differential geometry tailored for rigid-body and diffeomorphic image registration, before getting into the heat of the applications.

2.1 Velocity Vector Fields and Flows

Let $\gamma(t)$ be a (continuous) path over a Lie group $\mathbb{G}$, such that $t \in (-\eta, \eta) \subseteq \mathbb{R}$ and $\gamma(0) = p$. If (C, ψ) is a local chart, neighborhood of p, the tangent vector of γ at the point p can be expressed as

$$\mathbf{u} = \frac{d}{dt}(\psi \circ \gamma)(t)\Big|_{t=0}$$

For different choice of γ passing through p, we obtain different tangent vectors.

$$\begin{tikzcd}
& C \subseteq \mathbb{G} \arrow[dr, "\psi"] & \\
\mathbb{R} \arrow[ur, "\gamma"] \arrow[rr, "\psi \circ \gamma"] & & \mathbb{R}^n
\end{tikzcd}$$

It can be proved that the set of all of the tangent vector at the point p defines a vector space: the tangent space at p, indicated with $T_p\mathbb{G}$. It can be proved that this construction do not depend on the local chart's choice.
Taking into account the disjoint union of all of the tangent spaces of $\mathbb{G}$ we obtain the tangent bundle $T\mathbb{G}$; it can be proven that it is, in its turn, a differentiable manifold.
Be $\mathbb{G}$ n-dimensional Lie group. A *vector field* over $\mathbb{G}$ is a function that assigns at each point p of $\mathbb{G}$, a tangent vector V_p in the tangent space $T_p\mathbb{G}$, such that V_p is differentiable respect to p.
If $(C; x_1, \dots, x_n) = (C, \psi)$ is a local chart of $\mathbb{G}$, neighborhood of p, then V_p can be expressed locally as:

$$V_p = \sum_{i=1}^{n} v_i(p) \frac{\partial}{\partial x_i}\Big|_p \qquad\qquad v_i \in \mathcal{C}^\infty(C)$$

Using the Einstein summation convention V_p is sometime expressed as $V_p = v^i(p)\partial_i\big|_p$. The smooth functions v_i define the vector fields in the base $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. The idea of expressing the elements of the base in terms of differential operator reveals the possibility to consider each vector field as a directional derivative over the algebra of smooth functions defined on the manifold.
The set of all vector field over M, indicated with $\mathcal{V}(M)$, is a real vector space and a module over $\mathcal{C}^\infty(M)$:

$$\begin{aligned}
&(V+W)_p = V_p + W_p && \forall V, W \in \mathcal{V}(M)\\
&(aV)_p = aV_p && \forall a \in \mathbb{R}\\
&(fV)_p = f(p)V_p && \forall V \in \mathcal{V}(M) \quad \forall f \in \mathcal{C}^\infty(M)
\end{aligned}$$

Moreover $\mathcal{V}(M)$ acts over $\mathcal{C}^\infty(M)$ as follows

$$\begin{aligned}
\mathcal{V}(M) \times \mathcal{C}^\infty(M) &\longrightarrow \mathcal{C}^\infty(M)\\
(V, f) &\longmapsto Vf : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}\\
&\qquad\qquad\qquad p \longmapsto (Vf)(p) = V_p f
\end{aligned}$$

In the local chart the real number $V_p f$ is given by

$$(Vf)(p) = V_p f = \sum_{j=1}^{n} v_j(p) \frac{\partial f}{\partial x_j}\Big|_p$$

and represents the directional derivative of f along the vector $V_p \in T_pM$.
If $(C_2; y_1, \dots, y_n)$ is another local chart, $p \in C_2$, then the change of coordinates can be expressed as follows:

$$V_p = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} v_i(p) \frac{\partial y_j}{\partial x_i} \Big|_p \Big) \frac{\partial}{\partial y_j} \Big|_p$$

A vector field can be time-dependent if each of its vectors varies smoothly within a parameter t, otherwise it is time-independent. In this case a continuous function over the set of times T is defined:

$$\begin{aligned} \mathbb{R} \supseteq T &\longrightarrow \mathcal{V}(\mathbb{G}) \\ t &\longmapsto V^{(t)} : \mathbb{G} \longrightarrow T\mathbb{G} \\ &\qquad\qquad p \longmapsto V_p^{(t)} \end{aligned}$$

where $V_p^{(t)}$ has local coordinates

$$V_p^{(t)} = \sum_{i=1}^{n} v_i(p,t) \frac{\partial}{\partial x_i} \Big|_p \qquad v_i \in \mathcal{C}^\infty(C \times T)$$

Let V a vector field over a differentiable manifold $\mathbb{G}$, an *integral curve* of V is given by

$$c : (a,b) \longrightarrow \mathbb{G} \quad \text{such that} \quad \dot{c}(t) = V_{c(t)} \in T_{c(t)}\mathbb{G} \ \forall t \in (a,b)$$

To get the equations of the integral curves, we consider the local expression

$$V = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \qquad\qquad v_i \in \mathcal{C}^\infty(C)$$

and the unknown curve in the same local chart

$$c(t) = (c_1, c_2, \dots, c_n) \qquad \dot{c}(t) = \sum_{i=1}^{n} \frac{dc_i(t)}{dt} \frac{\partial}{\partial x_i} \Big|_{c(t)}$$

Imposing the condition $\dot{c}(t) = V_{c(t)}$ we get:

$$\sum_{i=1}^{n} \frac{dc_i(t)}{dt} \frac{\partial}{\partial x_i} \Big|_{c(t)} = \sum_{i=1}^{n} v_i(c(t)) \frac{\partial}{\partial x_i} \Big|_{c(t)}$$

For a given point of the manifold, and considering the integral curves passing for this point we obtain the initial condition $c(0) = p$ for a Cauchy problem :

$$\begin{cases} \frac{dc_i(t)}{dt} = v_i(c_1, t_2, \dots, c_n) \\ c_i(0) = p_i \end{cases} \tag{2.1}$$

Thanks to the Cauchy theorem it has a unique solution $\gamma(t)$. The unique integral curve passing through p when $t = 0$ is noted by $c^{(p)}(t)$.
Integral curves can be divided in 2 classes: the one whose domain can be extended to the whole real line $\mathbb{R}$ (in this case V is called *completely integrable vector field*) and the one

whose domain is a strict subset of $\mathbb{R}$. We reminds that the *flow* of the vector field V is the defined as:

$$\begin{aligned} \Phi_V : S \times \mathbb{G} &\longrightarrow \mathbb{G} \\ (t,p) &\longmapsto \Phi_V(t,p) = c^{(p)}(t) \end{aligned}$$

where $S = \mathbb{R}$ or $S \subset \mathbb{R}$ if V is or is not respectively completely integrable. Fixing the point p, the flow become simply the integral curve passing through p; keep t fixed and letting p varying over the manifold, we get the position of each point on the manifold subject to the vector field V at the time t. This last idea gives raise to the *one-parameter subgroup*:

$$\forall p \in M \qquad \Phi_V(t,p) = \varphi_t \qquad G = \{\varphi_t \ : \ t \in S\}$$

$$\begin{aligned} G \times G &\longrightarrow G \\ (\varphi_{t_1}, \varphi_{t_2}) &\longmapsto \varphi_{t_1+t_2} \end{aligned}$$

Despite the name, the fact that G forms a group is less important[1] than considering the compatibility between a sum on the real line and a product between functions. This property will be largely used when dealing with Lie logarithms. In general a continuous function

$$f : \mathbb{R} \supseteq (-\eta, \eta) \longrightarrow \mathbb{G} \qquad f(0) = p$$

satisfies the *one parameter subgroup property* if $f(t+s) = f(t)f(s)$ where the last multiplication is the composition on the group.

2.2 Push-forward, Left, Right and Adjoint Translation

Given two Lie group $\mathbb{G}$ and $\mathbb{H}$ linked by the differentiable map $F : \mathbb{G} \to \mathbb{H}$, then the *push forward* at the point p is defined as the covariant operator

$$\begin{aligned} (F_\star)_p : T_p\mathbb{G} &\longrightarrow T_{F(p)}\mathbb{H} \\ V_p &\longmapsto (F_\star V_p) : \mathcal{C}^\infty(\mathbb{H}) \longrightarrow \mathbb{R} \\ &\qquad\qquad f \longmapsto (F_\star V_p)(f) = V_p(f \circ F) = v(p)^i \partial_i (f \circ F)\big|_p \end{aligned}$$

When the point p is implicit by the context it will be omitted: namely $(F_\star)_p = F_\star$.

In general the push forward gives the right to the vector field V defined over $\mathbb{G}$ to act as a derivative on another manifold $\mathbb{H}$. Push forward is well defined since a vector field is completely determined by its action over $\mathcal{C}^\infty(\mathbb{H})$. It can be proved that it is linear, satisfies the Leibnitz rules, and $(G \circ F)_\star = G_\star \circ F_\star$; moreover, the push forward of the identity is the identity map between vector spaces, and if F is a diffeomorphism, $F_\star$ is an isomorphism of vector spaces.
The *pull-back*, is defined on the dual space of $\mathbb{G}$ and $\mathbb{H}$ as the contravariant operator of the push forward[2]:

$$\begin{aligned} F^\star : \mathcal{C}^\infty(\mathbb{H}) &\longrightarrow \mathcal{C}^\infty(\mathbb{G}) \\ f &\longmapsto F^\star f := f \circ F \end{aligned}$$

[1] In this context: from a group theory point of view the action of the group $(\mathbb{R}, +)$ over the manifold has as its orbits, the set of disjoints integral curves.

[2] Push-forward is defined between vector spaces, pull-back between space of functions and $V_p(F^\star f) = (v(p)^i \partial_i\big|_p)(f \circ F) = v(p)^i \partial_i (f \circ F)\big|_p = V_p(f \circ F)$.

The following diagram relates pull-back and push-forward:

$$
\begin{tikzcd}
T_p\mathbb{G} \arrow[r, "F_\star"] \arrow[d, dashed] & T_p\mathbb{H} \arrow[d, dashed] & \\
\mathbb{G} \arrow[r, "F"] \arrow[rr, bend right, "F^\star f"'] & \mathbb{H} \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}
$$

Here we restrict our attention at the case $\mathbb{G} = \mathbb{H}$, thus F is a map that moves the points of $\mathbb{G}$ smoothly. Each element p of a Lie group $\mathbb{G}$ defines three maps on $\mathbb{G}$:

1. *left-translation*:

$$
\begin{aligned}
L_p : \mathbb{G} &\longrightarrow \mathbb{G} \\
q &\longmapsto pq
\end{aligned}
$$

2. *right-translation*:

$$
\begin{aligned}
R_p : \mathbb{G} &\longrightarrow \mathbb{G} \\
q &\longmapsto qp
\end{aligned}
$$

3. *adjoint map*

$$
\begin{aligned}
\mathrm{Ad}_p : \mathbb{G} &\longrightarrow \mathbb{G} \\
q &\longmapsto pqp^{-1}
\end{aligned}
$$

The push forward for the vector field V at the point q are given by:

1. *left-translation*:

$$(L_p)_\star V_q f = V_q(f \circ L_p) = \sum_{i=1}^{n} v_i(q) \frac{\partial f \circ L_p}{\partial x_i} \Big|_q = \sum_{i=1}^{n} v_i(q) \frac{\partial f}{\partial x_i} \Big|_{pq}$$

2. *right-translation*:

$$(R_p)_\star V_q f = V_q(f \circ R_p) = \sum_{i=1}^{n} v_i(q) \frac{\partial f \circ R_p}{\partial x_i} \Big|_q = \sum_{i=1}^{n} v_i(q) \frac{\partial f}{\partial x_i} \Big|_{qp}$$

3. *adjoint map*

$$(\mathrm{Ad}_p)_\star V_q f = V_q(f \circ \mathrm{Ad}_p) = \sum_{i=1}^{n} v_i(q) \frac{\partial f \circ \mathrm{Ad}_p}{\partial x_i} \Big|_q = \sum_{i=1}^{n} v_i(q) \frac{\partial f}{\partial x_i} \Big|_{pqp^{-1}}$$

We note that in each expression the coefficient $v_i(q)$ remains the same even if the partial derivative is not applied at the point q. Therefore the linear combination of the constant coefficients $v_i(q)$ can be considered as a scalar product with the elements of the base applied at the function f. Left and right translation of the vector $\mathbf{u}$ can be expressed as scalar product with the *differential*, equivalent concept as the push forward, that emphasizes the scalar product implied in the definition:

$$\begin{aligned}(DL_p)_q : T_q\mathbb{G} &\longrightarrow T_{pq}\mathbb{G}\\ \mathbf{u} &\longmapsto (DL_p)_q \cdot \mathbf{u}\end{aligned}$$

$$\begin{aligned}(DR_p)_q : T_q\mathbb{G} &\longrightarrow T_{qp}\mathbb{G}\\ \mathbf{u} &\longmapsto (DR_p)_q \cdot \mathbf{u}\end{aligned}$$

where $(DL_p)_q$, $(DR_p)_q$ are properly defined vectors that can be expressed local coordinates as follow

$$(DL_p)_q = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big|_{pq} \qquad (DR_p)_q = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big|_{qp}$$

Or equivalently linear operators defined as:

$$\begin{aligned}(DL_p)_q : \mathcal{C}^{\infty}(M) &\longrightarrow \mathbb{R}\\ f &\longmapsto (DL_p)_q(f) = \frac{\partial f}{\partial x_i}\Big|_{pq}\end{aligned}$$

$$\begin{aligned}(DR_p)_q : \mathcal{C}^{\infty}(M) &\longrightarrow \mathbb{R}\\ f &\longmapsto (DR_p)_q(f) = \frac{\partial f}{\partial x_i}\Big|_{qp}\end{aligned}$$

A change of notation $V_q = \mathbf{u}$ makes push-forward and differential strikingly equivalent. This holds also for the generic map F:

$$(DF)_q(f) = \sum_{i=1}^{n} \frac{\partial f \circ F}{\partial x_i}\Big|_q \qquad (DF)_q(f) \cdot \mathbf{u} = \sum_{i=1}^{n} u_i \frac{\partial f \circ F}{\partial x_i}\Big|_q$$

The subscript q in $(DL_p)_q$ can be omitted when the tangent space of $\mathbf{u}$ is clear by the context.

A vector field V defined over a manifold is *left-invariant* if it is invariant for each left translation. It means that $(L_q)_\star V_p = V_p$ for any choice of p and q. If we consider all of the possible push forward of the left translation applied to a single tangent vector at the origin $\mathbf{v}$ of $T_e\mathbb{M}$ we have a unique left-invariant vector field defined as $\mathbf{v}^L$ such that

$$\mathbf{v}^L_q := (L_q)_\star \mathbf{v} \qquad \forall q \in M$$

Vice versa every left-invariant vector field V is uniquely represented by V_e. The set of all of the left-invariant vector fields form a linear subspace of the space of the vector field, indicated with left$\mathcal{V}(M)$. This can be easily proved by:

$$(L_g)_\star(aV + bW) = a(L_g)_\star V + b(L_g)_\star W \qquad \forall V, W \in \mathcal{V}(\mathbb{G}) \quad \forall a, b \in \mathbb{R}$$

In fact for each $h \in \mathbb{G}$ and for each $f \in \mathcal{C}^{\infty}(\mathbb{G})$ the linearity property holds:

$$\begin{aligned}(L_g)_\star(aV_h + bW_h)f &= (aV_h + bW_h)(f \circ L_g)\\ &= aV_h(f \circ L_g) + bW_h(f \circ L_g)\\ &= a(L_g)_\star V_h f + b(L_g)_\star W_h f\end{aligned}$$

The linearity property leads to the definition of the group of homomorphism over $\mathbb{G}$. It is the set of all the Lie group homomorphism from $\mathbb{R}$ to $\mathbb{G}$:

$$Hom(\mathbb{R}, \mathbb{G}) = \{\varphi : \mathbb{R} \to \mathbb{G} \mid \varphi(a+b) = \varphi(a) \circ \varphi(b) \quad \forall a, b \in \mathbb{R}\}$$

Tangent spaces, flows, one-parameter subgroup and Lie group homomorphisms are bounded together by the following remarkable result, which is a most important precondition for the definition of the Lie group exponential, and so deserve to be written in form of a lemma and formally proved.

Lemma 2.2.1. Let $\mathbb{G}$ be a Lie group. For each $\mathbf{v}$ in the tangent space $T_e\mathbb{G}$, exists a unique homomorphism $\gamma_{\mathbf{v}}$ in $Hom(\mathbb{R}, \mathbb{G})$ (or equivalently a function satisfying the one-parameter subgroup property) such that

$$\dot{\gamma}_{\mathbf{v}}(0) = \mathbf{v}$$

Proof. The homomorphism $\gamma_{\mathbf{v}}$ coincides with the integral curve Φ of the left invariant vector field generated by $\mathbf{v}$ passing through the identity. Its uniqueness is then a consequence of the Cauchy theorem. The same theorem also specifies the existence for a small enough neighbour $(-\eta, \eta) \subset \mathbb{R}$. To extend the solution to the whole $\mathbb{R}$ it is enough to consider that $\gamma_{\mathbf{v}}(t+s) = \gamma_{\mathbf{v}}(t)\gamma_{\mathbf{v}}(s)$ for each $s, t \in (-\eta, \eta)$:

$$\gamma_{\mathbf{v}}(t+s) = \Phi(t+s, e) = \Phi(t, \gamma_{\mathbf{v}}(s)) = \gamma_{\mathbf{v}}(t)\gamma_{\mathbf{v}}(s)$$

$\square$

We observe that $\gamma_{\mathbf{v}}$ is exactly the one parameter subgroup of $\mathbf{v}^L$ defined above, and then we can write $\gamma_{\mathbf{v}}(t) = \Phi(t, e) = \varphi_e(t)$.

We conclude this paragraph remembering the definition of the Lie algebra of a Lie group that take into account every feature so far introduced:

Definition 2.2.1. Given a Lie group $\mathbb{G}$, its Lie algebra $\mathfrak{g}$ is defined as:

1. The vector space $T_e\mathbb{G}$ of all of the tangent vector at the identity (or at any other point of the manifold): $\mathfrak{g} := T_e\mathbb{G}$.

2. The set of the left invariant vector Field over $\mathbb{G}$: $\mathfrak{g} := \text{left}\mathcal{V}(\mathbb{G})$.

3. The set of all of the flows passing through e: $\mathfrak{g} := \{\Phi(e, t) : t \in S \subseteq \mathbb{R}\}$.

4. The set of homomorphism $Hom(\mathbb{R}, \mathbb{G})$.

The Lie algebra can be also defined independently from a Lie group as a vector space endowed with Lie bracket (bilinear form, antisymmetric, that satisfies the Jacobi identity). In the finite dimensional case given a Lie algebra $\mathfrak{g}$ it can be proved that exists always a Lie group $\mathbb{G}$ such that $\mathfrak{g}$ is the Lie algebra defined over $\mathbb{G}$. This property (third Lie theorem) do not holds anymore infinite dimensional Lie algebra of diffeomorphisms.

2.3 Lie Exponential, Lie logarithm and the Log-composition

Let $\mathbf{v}$ be an element in the tangent space $\mathfrak{g}$ and $V \in \text{left}\mathcal{V}(\mathbb{G})$ the unique vector field defined by $\mathbf{v}$ over a local coordinate system around the origin. Let Φ be the flow associated with the vector field and $\gamma(t)$ the unique integral curve of v passing through the identity of the group. The *Lie exponential* is defined as

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow \mathbb{G} \\ \mathbf{v} &\longmapsto \exp(\mathbf{v}) = \gamma(1) \end{aligned}$$

It satisfies the following properties:

1. $\exp(V) = e$ if $V = 0$.
2. $\exp(V)$ is invertible and $(\exp(V))^{-1} = \exp(-V)$.
3. $\exp((a+b)V) = \exp(aV)\exp(bV)$.
4. If U and V are commutative then $\exp(V+U) = \exp(V)\exp(U)$.
5. For any norm $||\exp(V)|| \leq \exp(||V||)$.

The inverse of the Lie exp is defined by

$$\begin{aligned} \log : \mathbb{G} &\longrightarrow \mathfrak{g} \\ p &\longmapsto \log(p) = \mathbf{v} \end{aligned}$$

where $\mathbf{v}$ is the tangent vector having p as it exp. The following properties holds:

1. ....

2.4 BCH formula for the Computation of Log-composition

Let $\mathbf{u}, \mathbf{v}$ be element of the Lie algebra. The solution $\mathbf{w}$ of the equation $exp(\mathbf{w}) = \exp(\mathbf{u}) \circ \exp(\mathbf{v})$ is provided by the BCH formula, when $\mathbf{v}$ is small:

$$BCH(\mathbf{u},\mathbf{v}) = \mathbf{u} + \mathbf{v} + \frac{1}{2}[\mathbf{u},\mathbf{v}] + \frac{1}{12}([\mathbf{u},[\mathbf{u},\mathbf{v}]] + [\mathbf{v},[\mathbf{v},\mathbf{u}]]) - \frac{1}{24}[\mathbf{v},[\mathbf{u},[\mathbf{u},\mathbf{v}]]] + ...$$

....Nested Lie bracket can be written using multiple composition of the *adjoint action* of $\mathbf{u}$ on the Lia algebra:

$$\begin{aligned} ad_{\mathbf{u}} : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ \mathbf{v} &\longmapsto ad_{\mathbf{u}} = [\mathbf{u},\mathbf{v}] \end{aligned}$$

With this operator we can write

$$[\underbrace{\mathbf{u},[\mathbf{u},...[\mathbf{u}}_{\text{n-times}},\mathbf{v}]...]] = ad_{\mathbf{u}}^n(\mathbf{v})$$

In the appendix of ....Klarsfeld adjoint action are used to provide an expansion of the BCH formula. This can be rewritten as

$$BCH(\mathbf{u},\mathbf{v}) = \mathbf{u} + \frac{ad_{\mathbf{u}}\exp(ad_{\mathbf{u}})}{\exp(ad_{\mathbf{u}}) - 1}\mathbf{v} + O(\mathbf{v}^2)$$

The functional applied to $\mathbf{v}$ can be rewritten as

$$\frac{ad_{\mathbf{u}}\exp(ad_{\mathbf{u}})}{\exp(ad_{\mathbf{u}}) - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} ad_{\mathbf{u}}^n$$

Where $\{B_n\}$ is the sequence of the second-kind Bernoulli number[3].

[3]If first-kind Bernoulli number is used then each term of the summation must be multiplied for $(-1)^n$, as did for example in ....Klarsfeld.

2.5 Taylor Expansion for the Computation of Log-composition

Chapter 3

Parallel Transport Tool

In this section we present parallel transport for the finite dimensional Lie group and we make the assumption that obtained results hold in the infinite dimensional case.

3.1 Connections and Geodesics

.... definition of connection and geodesics. Properties that concern us.

3.2 Affine exp and Affine log

Any Lie group $\mathbb{G}$ considered with a left-invariant connection ∇ can be equipped with a metric based on the elements of its tangent space:

$$\mathrm{dist}(x, y) := ||\log_e(x^{-1} \circ y)|| \qquad \forall x, y \in \mathbb{G}$$

Where the log function is the *Affine logarithm*, inverse function of the the *Affine exponential*:

$$\begin{aligned} \log : \mathbb{G} \times \mathbb{G} &\longrightarrow T\mathbb{G} \\ (p, q) &\longmapsto \log_p(q) = V_p \end{aligned} \qquad\qquad \begin{aligned} \exp : \mathbb{G} \times T\mathbb{G} &\longrightarrow T\mathbb{G} \\ (p, q) &\longmapsto \exp_p(q) = \gamma(1) \end{aligned}$$

for V is the tangent vector field of the geodesic γ drawn on $\mathbb{G}$, passing through p and q,

$$\gamma(0) = p, \quad \gamma(1) = q, \quad \nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

and having V_p as its tangent vector at p:

$$\begin{aligned} \log_p(q) = V \;\; &\Leftrightarrow \;\; V_{\gamma(t)} = \dot{\gamma}(t) \;\; \forall t \in [0, 1] \\ \exp_p(q) = \gamma(1) \;\; &\Leftrightarrow \;\; \dot{\gamma}(p) = V_p \end{aligned}$$

The left-invariant condition over ∇ has some consequences:

1. the vector field V resulting by the application of the log function is left-invariant, so it is an element of the Lie algebra $\mathfrak{g}$ and uniquely determined by V_{e}.
2. Affine exp and log and coincides with Lie exp and log.
3. V is a complete vector field.

3.3 Parallel Transport through Examples

Definition 3.3.1. Let $\mathbb{G}$ be a finite dimensional connected Lie group defined with a connection ∇. Given $a, b \in \mathbb{G}$ and $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma(0) = a$ and $\gamma(1) = b$, then the vector $X_a \in T_a\mathbb{G}$, belonging to some vector field X is parallel transported along γ up to $T_b\mathbb{G}$ if for all $t \in [0,1]$ $\nabla_{\dot{\gamma}} X_{\gamma(t)} = 0$.
The parallel transport is the function that maps X_a from $T_a\mathbb{G}$ to $T_b\mathbb{G}$ along γ:

$$\begin{aligned} \Pi(\gamma)_a^b : T_p\mathbb{G} &\longrightarrow T_b\mathbb{G} \\ X_a &\longmapsto \Pi(\gamma)_a^b(X_p) = Y_b \end{aligned}$$

Theorem 3.3.1 (Inversion)**.** $\mathbb{G}$ Lie group, ∇ connection, $a, b \in \mathbb{G}$. Given γ such that $\gamma(0) = a$, $\gamma(1) = b$ and $\dot{\gamma}(0) = \mathbf{u} \in T_a\mathbb{G}$, we have:

$$\Pi(\gamma)_a^b(-\mathbf{u}) = -\Pi(\gamma)_a^b(\mathbf{u}) \tag{3.1}$$

$$a = \exp_b(\mathbf{u}) \iff b = \exp_a(-\Pi(\gamma)_b^a(\mathbf{u})) \tag{3.2}$$

How does the change of sign behave when the Lie group exponential is expressed as a composition is explored in the next property:

Theorem 3.3.2 (Inversion for composition, Lie exponential)**.** $\mathbb{G}$ Lie group, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in T_e\mathbb{G}$. If

$$\exp(\mathbf{w}) = \exp(\mathbf{u}) \circ \exp(\mathbf{v})$$

then

$$\exp(-\mathbf{w}) = \exp(-\mathbf{v}) \circ \exp(-\mathbf{u})$$

In the next property we explore how does behave the affine exponential expressed as a composition when changed of sign:

Theorem 3.3.3 (Inversion for composition, affine exponential)**.** $\mathbb{G}$ Lie group, ∇ connection, $a, b \in \mathbb{G}$, $\mathbf{u} \in T_a\mathbb{G}$, $\mathbf{v} \in T_b\mathbb{G}$. Let β be the tangent curve to $\mathbf{u}$ at a and $c = \exp_b(\mathbf{v})$. Given $\mathbf{w} \in T_c\mathbb{G}$ such that

$$\exp_a(\mathbf{w}) = \exp_b(\mathbf{v}) \circ \exp_a(\mathbf{u})$$

Then

$$\exp_a(-\mathbf{w}) = \exp_{\tilde{b}}(-\Pi(\beta)_b^{\tilde{b}}(\mathbf{v})) \circ \exp_a(-\mathbf{u})$$

where $\tilde{b}$ is the affine exponential of $-\mathbf{u}$ or the element $\beta(-1)$.

3.4 Change of Base Formulas with and without Parallel Transport

Using the derivative of the left-translation L_p it is possible to bring back the exp and the log function based at the point p of the manifold to the one evaluated at the identity using the following formulas:

$$\begin{aligned} \log_p(q) &= DL_p(e)\log_e(q) \\ \exp_p(\mathbf{u}) &= p \circ \exp_e(DL_p(e)^{-1}\mathbf{u}) \end{aligned}$$

....Proof and examples..

3.5 Parallel Transport in Practice: Schild's Ladder and Pole Ladder

....

Chapter 4

Accelerating Convergences Series

Part II

Studied Objects

Chapter 5

The Group of Rigid Body Transformations

People know or dimly perceive, that if thinking is not kept pure and keen, if spirit's contemplation do not holds, even mechanics of automobiles and ships will soon cease to run. Even engineer's slide rule, computations of banks and stock exchanges will wonder aimlessly for the lost of authority, and chaos will ensue.
-Hermann Hesse, *Magister Ludi*

In general a matrix Lie group is any complete subgroup of $GL(n, \mathbb{R})$. It is a particular finite dimensional Lie Group whose Lie algebra are subalgebra of the same bigger algebra that contains the Lie Group.

Property 5.0.1. Be $\mathbb{G}$ a matrix Lie group.

a) If $\mathbb{G} = GL(n, \mathbb{R})$ then $T_e\mathbb{G} = M(n, \mathbb{R})$.

b) If $\mathbb{G} \subseteq GL(n, \mathbb{R})$ then $T_e\mathbb{G} \subseteq M(n, \mathbb{R})$.

Proof. since det is a continuous function we have that

$$\begin{gathered} \forall X \in M(n, \mathbb{R}) \;\; \exists \eta > 0 \text{ such that } \forall t \in (-\eta, \eta) \\ det(I + tX) \neq 0 \end{gathered}$$

where I is the matrix identity. If we consider the path

$$\begin{aligned} \gamma : [0, 1] &\longrightarrow GL(n, \mathbb{R}) \\ t &\longmapsto I + tX \end{aligned}$$

as the path joining I and X, it follows

$$\frac{d}{dt}(I + tX) \Big|_{t=0} = X \in M(n, \mathbb{R})$$

$\square$

Be V in $\mathfrak{g}$, Lie algebra of the matrix Lie group $\mathbb{G}$. For matrices the Lie exponential map coincides with the Taylor expansion of the exponential[1] having V as argument:

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow \mathbb{G} \\ V &\longmapsto \exp(V) := \sum_{j=0}^{\infty} \frac{V^j}{j!} \end{aligned}$$

....Presentation of the matrix form of the rigid body transformations, close form for lie bracket, exp and log.
....Vectorization. Literature of rigid body transformation in image registration.
.... Prop: The element $\exp(tV)$ is smooth in $\mathbb{R}^{n^2}$ for all $V \in M(n, \mathbb{R})$ .... Prop: If C is an invertible element in then $\exp(CVC^{-1}) = C\exp(V)C^{-1}$.

[1] This is possible only when the Lie group and its algebra are subsets of the same bigger algebra as happens in the case of matrix Lie group (proof can be found in [Jr] or [Hal]).

Chapter 6

The Set of Stationary Velocity Fields

Accurate reckoning: the entrance into knowledge of all existing things and all obscure secrets.
- Ahmes, 1800 B.C. Quoted in A.B.Chase, Rhind Mathematical Papyrus (Reston Va. 1967).

The set of diffeomorphisms can be seen as an infinite dimensional Lie group. For these reasons .... we reduce the set of transformation to the SVF. This has the following positive consequences .... and it has been applied in .... . Nevertheless reducing the set of transformation to the SVF has bring new issues and challenges .... - limitations -

6.1 Set and Set Only

Let Ω be an open connected subset of $\mathbb{R}^d$ containing the origin. We define Diff(Ω) the infinite dimensional Lie group of diffeomorphism over Ω with neutral element e:

$$\text{Diff} := \{f : \mathbb{R}^d \longrightarrow \mathbb{R}^d \mid \text{ diffeomorphism } \}$$

.... Definition of Banach and Frechet infinite dimensional Lie group.

6.2 Some Tools for the Infinite Dimensional Case

It can be proved that the Lie algebra of Diff($\mathbb{R}^d$) is isomorphic to the Lie algebra of the vector field over $\mathbb{R}^d$.

$$Lie(\text{Diff}(\mathbb{R}^d)) = \mathcal{V}(\mathbb{R}^d) \tag{6.1}$$

To Visualize the meaning of this isomorphism we can consider the following diagram:

$$
\begin{tikzcd}
 & T_{\mathbf{0}}\mathbb{R}^D \arrow[r, "\rho_\star"] \arrow[d] & T_{\rho(\mathbf{0})}\mathbb{R}^D \arrow[d] \\
\mathbb{R} \arrow[ur, dashed, "\gamma'(0)"] \arrow[urr, dashed, "(\rho\circ\gamma)'(0)"] \arrow[r, "\gamma"] & \mathbb{R}^D \arrow[r, "\rho"] \arrow[dr, dashed, "f\circ\rho"] & \mathbb{R}^D \arrow[d, "f"] \\
 & & \mathbb{R}
\end{tikzcd}
$$

where $(\rho_\star)_\mathbf{0}$ is the push forward of ρ, defined as follows:

$$
\begin{aligned}
\rho_\star : T_\mathbf{0}\mathbb{R}^D &\longrightarrow T_{F(\mathbf{0})}\mathbb{R}^D \\
\mathbf{v} &\longmapsto \rho_\star\mathbf{v} : \mathcal{C}^\infty(\mathbb{R}^D) \longrightarrow \mathbb{R} \\
& \qquad\qquad f \longmapsto \rho_\star\mathbf{v}f := \mathbf{v}(f\circ\rho)
\end{aligned}
$$

We can consider the first floor of the diagram as the group of diffeomorphism of $\mathbb{R}^d$ and the second floor of the diagram as the algebra of the continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$. For $\rho \in \mathrm{Diff}(\mathbb{R}^D)$ and $\gamma : (-\eta, \eta) \to \mathbb{R}^D$ such that $\gamma(0) = \mathbf{0}$, then $(\rho\circ\gamma)'(0)$ belongs to $T_{\rho(\mathbf{0})}\mathbb{R}^D$ and $(\rho_\star)$ is a continuous function from $\mathbb{R}^D$ to $\mathbb{R}^D$ and belongs to $\mathrm{Lie}(\mathrm{Diff}(\mathbb{R}^D))$.

Lemma 6.2.1 (existence)**.** Be $p \in \mathrm{Diff}(\mathbb{R}^d)$, then exists a $\mathbf{v}$ in the Lie algebra $\mathcal{V}(\mathbb{R}^d)$, such that

$$||p - \exp(V)|| < \delta$$

for some δ and for some metric in $\mathrm{Diff}(\mathbb{R}^d)$.

Proof. Investigate a proof to define δ. □

Lemma 6.2.2 (identity lemma)**.** Be $p \in \mathrm{Diff}(\mathbb{R}^d)$, such that $p = \exp(\mathbf{v})$ for some $\mathbf{v} \in \mathfrak{g}$. Then

$$\exp(-\mathbf{v})\circ p = p\circ\exp(-\mathbf{v}) = e$$

Proof.

$$
\begin{aligned}
\exp(-\mathbf{v})\circ p = \exp(-\mathbf{v})\circ\exp(\mathbf{v}) = \varphi_1\varphi_{-1} = \varphi_0 = e \\
p\circ\exp(-\mathbf{v}) = \exp(\mathbf{v})\circ\exp(-\mathbf{v}) = \varphi_{-1}\varphi_1 = \varphi_0 = e
\end{aligned}
$$

□

Property 6.2.1. If $\mathbf{v}$ is close to the origin $\exp(\mathbf{v})$ can be numerically approximated with:

$$\exp(\mathbf{v}) = e + \mathbf{v}$$

Proof. .... !! □

....Exp and Log function in the infinite dimension case

...we can not express the exp function using the Taylor expansion in the infinite dimensional Lie group Diff(Ω). We define it as an unknown function with some features related to the 1-parameter subgroup structure over $\mathbb{G}$: We define exp function as

$$\begin{aligned} exp : \mathfrak{g} &\longrightarrow \mathbb{G} \\ \mathbf{v} &\longmapsto \exp(\mathbf{v}) := \gamma(1) \end{aligned}$$

The following properties are satisfied:

1. exp is well defined and surjective (at least near 0).
2. If $\exp(\mathbf{v}) = \gamma(1)$ then $\exp(t\mathbf{v}) = \gamma(t)$.
3. It satisfies the one parameter subgroup property.
4. It satisfies the differential equation

$$\frac{d}{dt}\exp(t\mathbf{v})\Big|_{t=0} = \mathbf{v}$$

We observe that exp respects the one parameter subgroup structure of the Lie group $\mathbb{G}$: stretching the tangent vector $\mathbf{v}$ by a parameter t, the same stretch is reflected in $\exp(V)$ along the same integral curve.
In addition exp respect the 1-parameter subgroup structure:

$$\exp((t+s)\mathbf{v}) = \varphi_{t+s} = \gamma(t+s) \qquad \forall t, s \in \mathbb{R}$$

moreover, if two elements p_1, p_2 of $\mathbb{G}$ belongs to integral curve passing in e of the same integral curve defined by $\mathbf{v}$, their log function are a vectors having the same direction:

$$p_1 = \gamma(t_1),\ p_2 = \gamma(t_2) \Rightarrow \exists \mathbf{v} \in \mathfrak{g},\ \lambda \in \mathbb{R} \mid \log(p_1) = \mathbf{v},\ \log(p_2) = \lambda\mathbf{v}$$

It follows that for a fixed $t \in \mathbb{R}$ and $\gamma(t) = \exp(\mathbf{v})$ for some $\mathbf{v}$ in $left\mathfrak{X}(\mathbb{G})$, then $\gamma(1) = \exp(\frac{1}{t}\mathbf{v})$.

.... Issue related to the image of exp for stationary velocity fields in the finite dimensional case. define $\mathrm{Diff}_s(\Omega)$ as the subset of Diff(Ω) defined by the images of exp from the tangent space to the Lie group.

.... We define *log* function as

$$\begin{aligned} log : \mathbb{G} &\longrightarrow \mathfrak{g} \\ g &\longmapsto log(g) \end{aligned}$$

such that for p in $\mathbb{G}$ we have $\exp(log(p)) = p$ when $log(p)$ is defined.

6.3 SVF in Practice

Three main definitions around which the whole theory of diffeomorphic image registration gravitate are introduced in this section.
.... Define here displacement and deformation.

....the set of time dependent spatial transformation. We can express it as the set of continuous functions from Ω to $\mathbb{R}^d$ depending on a real parameter in $T \subseteq \mathbb{R}$:

$$\mathcal{V}_T(\mathbb{R}^d) = \mathcal{V}_T := \{V : \Omega \times T \longrightarrow \mathbb{R}^d \mid \text{ continuous } \}$$

its elements are called *time varying velocity field* (TVVF) and can be expressed as

$$V(\mathbf{x}, t) = \sum_{i=1}^{d} v_i(\mathbf{x}, t) \frac{\partial}{\partial x_i} \Big|_{\mathbf{x}} \qquad v_i \in \mathcal{C}^\infty(\Omega \times T)$$

In case $V(\mathbf{x}, t) = V(\mathbf{x}, s)$ for all s, t real, then V is a *stationary velocity field* (SVF), and the set of the stationary velocity field, second item presented in this section, is defined as

$$\mathcal{V}(\mathbb{R}^d) = \mathcal{V} := \{V : \Omega \longrightarrow \mathbb{R}^d \mid \text{ continuous } \}$$

Their elements can be expressed as

$$V(\mathbf{x}) = V_{\mathbf{x}} = \sum_{i=1}^{d} v_i(\mathbf{x}) \frac{\partial}{\partial x_i} \Big|_{\mathbf{x}} \qquad v_i \in \mathcal{C}^\infty(\Omega)$$

While $\mathcal{V}$ and $\mathcal{V}_T$ are Lie algebra, Diff is a Lie group with the operation of composition.

If we imagine a particle starting at the point $\mathbf{x}$ of $\Omega \subseteq \mathbb{R}^d$ at time 0, with velocity vector for each instant of time given by $V(\mathbf{x}, t)$, then its trajectory $\gamma = \gamma(t)$ is determined by the ODE:

$$\frac{d\gamma}{dt} = V(\mathbf{x}, t)$$

In case $V(\mathbf{x}, t)$ is a stationary velocity field the equation is *stationary* or *autonomous*.

....if $V^{(t)} = V^{(s)}$ for all s, t real, then we call this vector field *stationary velocity field* (SVF), otherwise are called *time varying velocity field* (TVVF).

....The set of the SVF can be expressed as

$$\text{SVF} := \{\varphi_t(e) \mid t \in \mathbb{R}, \dot{\varphi}_t(e) = V_{\varphi_t(e)}, V \in \mathfrak{V}(\Omega)\}$$

(note that in this way V is not an element of the Lie algebra!! We should have said $V \in left\mathfrak{V}(\text{Diff})$).

....We know that SVF are geodesics-complete if a norm over diff is defined, while SVF are not complete. $\varphi_t(e)$ do not spans Diff i.e. for each point of Diff may not always pass an integral curve of a left-invariant vector field over Diff. We will consider only the element of Diff of the form $\varphi_t(e)$. We assume also that each vector field is complete. (THIS MUST BE INVESTIGATED LATER!)

....Thanks to the Dini theorem we have that a SVF can be considered locally as an element of a local expression of Diff. Moreover to each spatial transformation vector field corresponds an element of the one parameter subgroup of local transformation over $\mathbb{R}^2$ (not sure...).

Part III

Applications

Chapter 7

Numerical Approximation of the Log-Composition

We believe that we know something about the things themselves when we speak of trees, colors, snow, and flowers; and yet we possess nothing but metaphors for things — metaphors which correspond in no way to the original entities.
-Nietzsche, *On Truth and Lies in extra-moral sense.*

... Group composition as a need from diffeomorphic image registration.
....It is of fundamental importance to have the possibility to going from an element of a group of spatial transformations to a tangent space, in which each vector corresponds to the tangent vector field that this transformation causes on the space. It makes possible to lean on the group structure a structure of vector space, which implies the possibility to compute statistics on the group of transformation as well as compose velocity fields in the tangent space passing through the corresponding transformation (both of them are made possible thanks to the local bijection between the Lie group and the Lie algebra).

7.1 Group composition at the Service of Image Registration

We define the group composition the inner binary operation $\star$ in $\mathfrak{g}$ such that

$$\begin{aligned} \star : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (\mathbf{v}_1, \mathbf{v}_2) &\longmapsto \mathbf{v}_1 \star \mathbf{v}_2 = \log(\exp(\mathbf{v}_1) \circ \exp(\mathbf{v}_2)) \end{aligned}$$

(we remember that $\circ$ is the operation of Lie group, i.e. the composition of diffeomorphism and exp and log with no subscript are the Lie exponential and Lie logarithm).
The *offset group composition* is defined as the inner binary operation $\tilde{\star}$ over the $\mathbb{G}$ fiber bundle such that

$$\begin{aligned} \tilde{\star} : T_{p_1}\mathbb{G} \times T_{p_2}\mathbb{G} &\longrightarrow T_{p_1}\mathbb{G} \\ (\mathbf{v}_1, \mathbf{v}_2) &\longmapsto \mathbf{v}_1 \,\tilde{\star}\, \mathbf{v}_2 = \log_{p_1}(\exp_{p_2}(\mathbf{v}_2) \circ \exp_{p_1}(\mathbf{v}_1)) \end{aligned}$$

where $p_1 \in \mathbb{G}$, $p_2 = \exp_{p_1}(\mathbf{v}_1)$ and Affine exponential and Affine logarithm are considered. Posing $p_3 = \exp_{p_2}(\mathbf{v}_2)$ the second offset group composition is the tangent vector in p_1 that

corresponds to the composition between p_1 and p_2, i.e. $\log_{p_1}(p_3 \circ p_2)$.
The BCH formula is the solution to the first kind group composition. It can be practically computed using its *approximation of degree* k defined as the sum of the BCH terms having no more than k nested Lie bracket. For example:

$$
\begin{aligned}
BCH^0(\mathbf{u},\mathbf{v}) &= \mathbf{u}+\mathbf{v} \\
BCH^1(\mathbf{u},\mathbf{v}) &= \mathbf{u}+\mathbf{v}+\frac{1}{2}[\mathbf{u},\mathbf{v}] \\
BCH^2(\mathbf{u},\mathbf{v}) &= \mathbf{u}+\mathbf{v}+\frac{1}{2}[\mathbf{u},\mathbf{v}]+\frac{1}{12}([\mathbf{u},[\mathbf{u},\mathbf{v}]]+[\mathbf{v},[\mathbf{v},\mathbf{u}]])
\end{aligned}
$$

These numerical approximations of the group composition leave the difficulty of managing the problem of the error carried by each term. In some cases the increase of the degree of the BCH approximation do not necessarily implies a decrease in error:
.... Add an example in which this happens.

We present other ways to compute the Lie group composition in the following subsections.

7.2 Accelerating convergences applied to the BCH formula

....

7.3 Taylor expansion to compute the group composition

....

7.4 Parallel transport to compute the group composition

....Here will be made the strong assumption according to which the parallel transport defined for the finite dimensional case, works also in the infinite dimensional case.

Lemma 7.4.1. $\mathbb{G}$ Lie group, ∇ connection, $a \in \mathbb{G}$, $\mathbf{u} \in T_e\mathbb{G}$. Let γ be a curve defined on $\mathbb{G}$ such that $\gamma(0) = e$, $\gamma(1) = a$, $\dot\gamma(0) = \mathbf{u}$. Let β be the curve over $\mathbf{G}$ defined as $\beta(t) = a \circ \gamma(t)$, then the two following conditions hold:

1. If ∇ is a Cartan connection then β is a geodesic.
2. For $\mathbf{u}_a := D(L_a)_e(\mathbf{u}) \in T_a\mathbb{G}$:

$$\exp_a(t\mathbf{u}_b) = b \circ \exp_e(tD(L_{a^{-1}})_a(\mathbf{u}_a)) = b \circ \exp_e(t\mathbf{u}) \tag{7.1}$$

Theorem 7.4.1. Let $\mathbb{G}$ be a finite dimensional connected Lie group defined with a Cartan connection ∇. If, for each couple of linearly independent vectors $\mathbf{u}, \mathbf{v} \in T_e\mathbb{G}$, we consider the following elements:

$$
\begin{gathered}
a = \exp_e(\mathbf{u}) \qquad b = \exp_e(\mathbf{v}) \\
\mathbf{u}^{\parallel} = \Pi(\alpha)_e^b(\mathbf{u}) \\
\gamma : [0,1] \to \mathbb{G} \quad \gamma(0) = e \quad \dot\gamma(0) = \mathbf{v}
\end{gathered}
$$

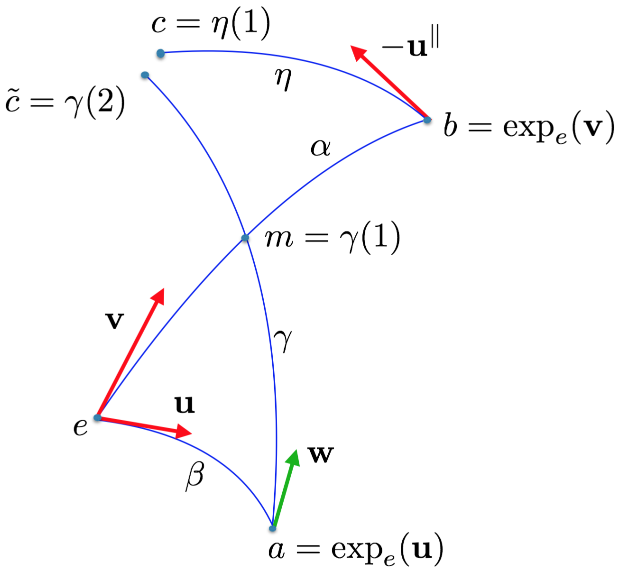


Figure 7.1: Pole ladder applied to parallel transport.

Then, for $\mathbf{u}_e^{\|} := D(L_{b^{-1}})_e(-\Pi(\alpha)_a^b(\mathbf{u}))$, the approximation

$$\exp_e(\mathbf{u}_e^{\|}) \simeq \exp_e\big(\frac{\mathbf{v}}{2}\big) \circ \exp_e(\mathbf{u}) \circ \exp_e\big(-\frac{\mathbf{v}}{2}\big)$$

holds.

Proof. As a consequence of the construction we have the following considerations:

$$\begin{aligned}
\gamma(t) = \exp(t\mathbf{w}) &= a \circ \exp_e(D(L_{ba^{-1}})_e(t\mathbf{w})) = \exp_e(\mathbf{u}) \circ \exp_e(D(L_{a^{-1}})_e(t\mathbf{w})) \\
m = \alpha(\frac{1}{2}) &= \exp_e\big(\frac{\mathbf{v}}{2}\big) = \gamma(1) = \exp_a(\mathbf{w}) \\
\exp_e(D(L_{a^{-1}})_e(\mathbf{w})) &= \exp_e(-\mathbf{u}) \circ \exp_e\big(\frac{\mathbf{v}}{2}\big)
\end{aligned}$$

Let η be the integral curve of $-\Pi(\alpha)_a^b(\mathbf{u})$ starting at b. If $c := \eta(1)$ and $\tilde{c} := \gamma(1)$, then on one side we have:

$$\begin{aligned}
\tilde{c} = \gamma(1) = \exp_a(2\mathbf{w}) &= a \circ \exp_e(D(L_{a^{-1}})_e(2\mathbf{w})) \\
&= \exp_e(\mathbf{u}) \circ \exp_e(D(L_{a^{-1}})_e(\mathbf{2w})) \\
&= \exp_e(\mathbf{u}) \circ \exp_e(2D(L_{a^{-1}})_e(\mathbf{w})) \\
&= \exp_e(\mathbf{u}) \circ \big(\exp_e(D(L_{a^{-1}})_e(\mathbf{w}))\big)^2 \\
&= \exp_e(\mathbf{u}) \circ \big(\exp_e(-\mathbf{u}) \circ \exp_e(\frac{\mathbf{v}}{2})\big)^2 \\
&= \exp_e\big(\frac{\mathbf{v}}{2}\big) \circ \exp_e(-\mathbf{u}) \circ \exp_e\big(\frac{\mathbf{v}}{2}\big)
\end{aligned}$$

On the other side:

$$\begin{aligned} c = \eta(1) = \exp_b(-\mathbf{u}^{\parallel}) &= b \circ \exp_e(D(L_{b^{-1}})_e(-\mathbf{u}^{\parallel})) \\ &= \exp_e(\mathbf{v}) \circ \exp_e(D(L_{b^{-1}})_e(-\mathbf{u}^{\parallel})) \\ &= \exp_e(\mathbf{v}) \circ \exp_e(-\mathbf{u}_e^{\parallel}) \end{aligned}$$

where $D(L_{b^{-1}})_e(\mathbf{u}^{\parallel})$ has been written $\mathbf{u}_e^{\parallel}$ for brevity. If we consider $c \simeq \tilde{c}$ it follows that:

$$\exp_e\Big(\frac{\mathbf{v}}{2}\Big) \circ \exp_e(-\mathbf{u}) \circ \exp_e\Big(\frac{\mathbf{v}}{2}\Big) \simeq \exp_e(\mathbf{v}) \circ \exp_e(-\mathbf{u}_e^{\parallel})$$

which implies

$$\exp_e(-\mathbf{u}_e^{\parallel}) \simeq \exp_e(-\mathbf{v}) \circ \exp_e\Big(\frac{\mathbf{v}}{2}\Big) \circ \exp_e(-\mathbf{u}) \circ \exp_e\Big(\frac{\mathbf{v}}{2}\Big)$$
$$\exp_e(-\mathbf{u}_e^{\parallel}) \simeq \exp_e\Big(-\frac{\mathbf{v}}{2}\Big) \circ \exp_e(-\mathbf{u}) \circ \exp_e\Big(\frac{\mathbf{v}}{2}\Big)$$

As a consequence of property of the signs inversion it follows that

$$\exp_e(\mathbf{u}_e^{\parallel}) \simeq \exp_e\Big(\frac{\mathbf{v}}{2}\Big) \circ \exp_e(\mathbf{u}) \circ \exp_e\Big(-\frac{\mathbf{v}}{2}\Big)$$

□

Corollary 7.4.2. ....
If, with previous notations, the condition (1) is an approximation

$$\exp_C(\frac{\mathbf{k}}{2}) = \exp(\xi) \circ \exp_M(\frac{\mathbf{k}}{2})$$

for some ξ in $\mathfrak{g}$ such that $\| \xi \| < \delta$ then the approximation has error

$$O(\| \delta\mathbf{u}^{\parallel} \|^2) + O(\| \mathbf{u} + \delta\mathbf{u} \|^3) + \text{...something that must be investigated depending on } \delta$$

Chapter 8

Numerical Approximation to Compute the Lie logarithm

8.1 Exponential and Logarithm Approximation with Truncated Power Series

8.2 A reformulation of the Bossa Algorithm using Log-composition

Having explored some methods to evaluate the BCH formula we can use them to find an approximation of the logarithm function of an element of $\mathbb{G}$. Given $p \in \mathbb{G}$ the goal is to find $\mathbf{u}$ such that $\exp(\mathbf{u})$ is the best possible approximation of p.

The first way to find a solution is via the algorithm presented in [BO08], it reduces the problem of the computation of logarithm to the problem of the computation of the group composition.
If $p = \exp(\mathbf{v})$ for any $\mathbf{v} \in \mathfrak{g}$, near the identity we can write:

$$\begin{aligned} p = \exp(\mathbf{v}) &= (\exp(\mathbf{v}) \circ \exp(-\mathbf{v})) \circ \exp(\mathbf{v}) \\ &= \exp(\mathbf{v}) \circ (\exp(-\mathbf{v}) \circ p) \\ &= \exp(\mathbf{v}) \circ \exp(\delta \mathbf{v}) \\ &\approx \exp(\mathbf{v}) \circ \exp(\tilde{\delta} \mathbf{v}) \end{aligned}$$

Where $\tilde{\delta}\mathbf{v}$, as we are going to see, turns out to be $\exp(-\mathbf{v}) \circ p - e$, for e identity transformation of the Lie group. The iterative algorithm is then

$$\begin{cases} \mathbf{v}_0 = 0 \\ \mathbf{v}_n = \mathrm{BCH}^k(\mathbf{v}_{n-1}, \tilde{\delta}\mathbf{v}_{n-1}) \end{cases} \tag{8.1}$$

for some degree k of approximation.

$$\tilde{\delta}\mathbf{v}_{n-1} = \exp(-\mathbf{v}_{n-1}) \circ \Phi - e$$

Proof. Let $\mathbf{v}_0$ be an element of $\mathfrak{g}$ in some sense close to $\mathbf{v}$ then:

$$p = \exp(\mathbf{v}) = \exp(\mathbf{v}_0) \circ (\exp(-\mathbf{v}_0) \circ p)$$

We define $\delta\mathbf{v}_0 \in \mathfrak{g}$ as $\delta\mathbf{v}_0 = \exp(-\mathbf{v}_0) \circ p$. Then

$$\begin{aligned} p &= \exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0) \\ \exp(V) &= \exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0) \\ \mathbf{v} &= \log(\exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0)) \\ \mathbf{v} &\simeq \mathrm{BCH}^k(\mathbf{v}_0, \delta\mathbf{v}_0) \end{aligned}$$

We approaching the tangent vector $\mathbf{v}$ using an iterative algorithm based on the BCH formula and the lemma 6.2.1.

$$\exp(\delta\mathbf{v}_0) \approx e + \delta\mathbf{v}_0 \Longrightarrow \delta\mathbf{v}_0 \approx \exp(\delta\mathbf{v}_0) - e$$

Having $\mathbf{v}_0$ as our initial value we define

$$\tilde{\delta}\mathbf{v}_0 := \exp(\delta\mathbf{v}_0) - e$$

Using $p = \exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0)$ we can say that $\exp(\delta\mathbf{v}_0) = \exp(-\mathbf{v}_0) \circ p$ and then

$$\tilde{\delta}\mathbf{v}_0 = \exp(-\mathbf{v}_0) \circ p - e$$

just by definition. Since p is known we can start our successive approximation, and if we set $\mathbf{v}_0 = \mathbf{0}$ we end up with the iterative algorithm (8.1). $\square$

Theorem 8.2.1 (Bossa)**.** The iterative algorithm (8.1) converges to $\mathbf{v}$ with error $\delta_n \in \mathbb{G}$, where

$$\delta_n := log(\exp(\mathbf{v}) \circ \exp(-\mathbf{v}_n)) \in O(||p - e||^{2^n})$$

8.3 Parallel Transport Strategy

We can use the consideration of the previous section to evaluate the BCH formula in this context, and get an approximation for the evaluation of the Log function.

$$\tilde{\delta}\mathbf{v}_{t-1} = \exp(-\mathbf{v}_{t-1}) \circ p - e$$

we get the iterative algorithm

$$\begin{cases} \mathbf{v}_0 = \mathbf{0} \\ \mathbf{v}_t = \mathbf{v}_{t-1} - \exp(-\frac{\mathbf{v}_{t-1}}{2}) \circ \exp(\delta\mathbf{v}_{t-1}) \circ \exp(\frac{\mathbf{v}_{t-1}}{2}) + e \end{cases} \tag{8.2}$$

8.4 Symmetrization Strategy

The algorithm for the computation of the group logarithm can be improved considering a symmetric version of the underpinning strategy 8.1. In this version we use the first order approximation of the BCH formula (see equation (8.5) in the following proof), compensating with the fact that the symmetrization should decrease the error involved. It gives birth to the following algorithm:

$$\begin{cases} \mathbf{v}_0 = \mathbf{0} \\ \mathbf{v}_{t+1} = \mathbf{v}_t + \frac{1}{2}(\tilde{\delta}\mathbf{v}_t^L + \tilde{\delta}\mathbf{v}_t^R) \end{cases} \tag{8.3}$$

Where $\tilde{\delta}\mathbf{v}_t^R = \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) - e$ and $\tilde{\delta}\mathbf{v}_t^L = \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) - e$.

Proof. To show why it works we remind that the starting point was

$$p = \exp(\mathbf{v}) = \exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0)$$

where $\exp(\delta\mathbf{v}_0) = \exp(-\mathbf{v}_0) \circ p$.
An equivalent starting point would have been $\exp(\mathbf{v}) = \exp(\delta\mathbf{v}) \circ \exp(\mathbf{v}_0)$ for $\exp(\delta\mathbf{v}) = p \circ \exp(-\mathbf{v}_0)$.
This idea leads to the definition of

$$\begin{aligned}\exp(\delta\mathbf{v}_t^R) &:= p \circ \exp(-\mathbf{v}_t) = \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t)\\ \exp(\delta\mathbf{v}_t^L) &:= \exp(-\mathbf{v}_t) \circ p = \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v})\end{aligned}$$

It follows that

$$\begin{aligned}\exp(\mathbf{v}) &= \exp(\mathbf{v}_0) \circ \exp(\delta\mathbf{v}_0^R)\\ \exp(\mathbf{v}) &= \exp(\delta\mathbf{v}_0^L) \circ \exp(\mathbf{v}_0)\end{aligned}$$

Using $\exp(\delta\mathbf{v}_t^R) \approx e + \delta\mathbf{v}_t^R$ and $\exp(\delta\mathbf{v}_t^L) \approx e + \delta\mathbf{v}_t^L$ we can use the following approximation to define the symmetric algorithm:

$$\begin{aligned}\exp(\delta\mathbf{v}_t^R) &= \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t)\\ e + \tilde{\delta}\mathbf{v}_t^R &= \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t)\\ \tilde{\delta}\mathbf{v}_t^R &= \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) - e\end{aligned}$$

$$\begin{aligned}\exp(\delta\mathbf{v}_t^L) &= \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v})\\ e + \tilde{\delta}\mathbf{v}_t^L &= \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v})\\ \tilde{\delta}\mathbf{v}_t^L &= \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) - e\end{aligned}$$

Which gives birth to iterative algorithm, for a given initial value V_0:

$$\begin{cases}\mathbf{v}_0\\ \mathbf{v}_{t+1} = \mathrm{BCH}(\mathbf{v}_t, \tilde{\delta}\mathbf{v}_t^R)\end{cases} \quad \begin{cases}\mathbf{v}_0\\ \mathbf{v}_{t+1} = \mathrm{BCH}(\tilde{\delta}\mathbf{v}_t^L, \mathbf{v}_t)\end{cases} \tag{8.4}$$

If follows that

$$\mathbf{v}_{t+1} = \frac{1}{2}(\mathrm{BCH}(\tilde{\delta}\mathbf{v}_t^L, \mathbf{v}_t) + \mathrm{BCH}(\mathbf{v}_t, \tilde{\delta}\mathbf{v}_t^R))$$

Taking the first order approximation of the BCH formula:

$$BCH(\tilde{\delta}\mathbf{v}_t^L, \mathbf{v}_t) \approx \tilde{\delta}\mathbf{v}_t^L + \mathbf{v}_t \tag{8.5}$$

$$BCH(\mathbf{v}_t, \tilde{\delta}\mathbf{v}_t^R) \approx \mathbf{v}_t + \tilde{\delta}\mathbf{v}_t^R \tag{8.6}$$

we get

$$\mathbf{v}_{t+1} = \mathbf{v}_t + \frac{1}{2}(\tilde{\delta}\mathbf{v}_t^L + \tilde{\delta}\mathbf{v}_t^R)$$

□

We observe that the symmetric approach do not requires to use the BCH formula at each passage, having considered the approximation at the first order of the BCH.
We conclude with a formula that relates $\tilde{\delta}\mathbf{v}_t^L$ with $\tilde{\delta}\mathbf{v}_t^R$:

Theorem 8.4.1. Be $\tilde{\delta}\mathbf{v}_t^R = \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) - e$ and $\tilde{\delta}\mathbf{v}_t^L = \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) - e$ as before, then

$$\delta\mathbf{v}_t^L \approx \exp(-\mathbf{v}_t) \circ \delta\mathbf{v}_t^R \circ \exp(\mathbf{v}_t)$$

Proof. Since $\exp(\mathbf{v}_t) \circ \exp(\delta\mathbf{v}_t^R) \approx exp(\delta\mathbf{v}_t^L) \circ \exp(\mathbf{v}_t)$ it follows

$$\exp(\delta\mathbf{v}_t^R) = \exp(-\mathbf{v}_t) \circ \delta\mathbf{v}_t^L \circ \exp(\mathbf{v}_t)$$

Using $\exp(\delta\mathbf{v}_t^R) = e + \delta\mathbf{v}_t^R$ and $\exp(\delta\mathbf{v}_t^L) = e + \delta\mathbf{v}_t^L$ we get

$$\begin{aligned} e + \delta\mathbf{v}_t^R &= \exp(-\mathbf{v}_t) \circ (e + \delta\mathbf{v}_t^L) \circ \exp(\mathbf{v}_t) \\ \delta\mathbf{v}_t^R &= \exp(-\mathbf{v}_t) \circ \delta\mathbf{v}_t^L \circ \exp(\mathbf{v}_t) \end{aligned}$$

□

8.5 Symmetric-Parallel Transport Strategy

If we are not satisfied to having take only the firs order approximation of the BCH in the equation (8.5) we use at this stage the parallel transport in the method presented in this section. Going back to the algorithm 8.3 we can apply to

$$\mathbf{v}_{t+1} = \frac{1}{2}(\mathrm{BCH}(\tilde{\delta}\mathbf{v}_t^L, \mathbf{v}_t) + \mathrm{BCH}(\mathbf{v}_t, \tilde{\delta}\mathbf{v}_t^R))$$

the parallel transport to get

$$\begin{aligned} \mathbf{v}_{t+1} &= \frac{1}{2}((\tilde{\delta}\mathbf{v}_t^L)^{\parallel} + \mathbf{v}_t + \mathbf{v}_t + (\tilde{\delta}\mathbf{v}_t^R)^{\parallel}) \\ &= 2\mathbf{v}_t + \frac{1}{2}((\tilde{\delta}\mathbf{v}_t^L)^{\parallel} + (\tilde{\delta}\mathbf{v}_t^R)^{\parallel}) \end{aligned}$$

Applying the definition of parallel transport we get

$$(\tilde{\delta}\mathbf{v}_t^L)^{\parallel} + (\tilde{\delta}\mathbf{v}_t^R)^{\parallel} = \exp(-\frac{\mathbf{v}_t}{2}) \circ (\tilde{\delta}\mathbf{v}_t^L + \tilde{\delta}\mathbf{v}_t^R) \circ \exp(\frac{\mathbf{v}_t}{2})$$

where

$$\begin{aligned} \tilde{\delta}\mathbf{v}_t^L &= \exp(\mathbf{v}) \circ \exp(-\mathbf{v}_t) - e \\ \tilde{\delta}\mathbf{v}_t^R &= \exp(-\mathbf{v}_t) \circ \exp(\mathbf{v}) - e \end{aligned}$$

Then a new improvement of the algorithm **??** is

$$\begin{cases} \mathbf{v}_0 = 0 \\ \mathbf{v}_t = 2\mathbf{v}_{t-1} + \frac{1}{2}(\exp(-\frac{\mathbf{v}_{t-1}}{2}) \circ (\tilde{\delta}\mathbf{v}_{t-1}^L + \tilde{\delta}\mathbf{v}_{t-1}^R) \circ \exp(\frac{\mathbf{v}_{t-1}}{2})) \end{cases} \tag{8.7}$$

(This must be investigated!)

Chapter 9

Experimental Results

9.0.1 Toy Examples to Compare the Log-Compositions

9.0.2 NiftyReg Applications

9.0.3 BCH-free Computation of the Lie Logarithm

Chapter 10

Further Research and Conclusion

There, I've done my best! If this won't suit I shall have to wait till I can do better.
-Jo

Appendix A

Appendix: NiftyReg and NiftyBit

This is a description of the tools you used to make your thesis. It helps people make future documents, reminds you, and looks good.

(example) This document was set in the Times Roman typeface using LaTeX and BibTeX, composed with a text editor.

Bibliography

[ACPA06] Vincent Arsigny, Olivier Commowick, Xavier Pennec, and Nicholas Ayache. Statistics on diffeomorphisms in a log-euclidean framework. In *1st MICCAI Workshop on Mathematical Foundations of Computational Anatomy: Geometrical, Statistical and Registration Methods for Modeling Biological Shape Variability*, pages 14–15, 2006.

[AFPA06] Vincent Arsigny, Pierre Fillard, Xavier Pennec, and Nicholas Ayache. Log-Euclidean metrics for fast and simple calculus on diffusion tensors. *Magnetic Resonance in Medicine*, 56(2):411–421, August 2006.

[Arn]

[Ash07] J. Ashburner. A fast diffeomorphic image registration algorithm. *NeuroImage*, 38(1):95–113, 2007.

[BBHM11] MARTIN BAUER, MARTINS BRUVERIS, PHILIPP HARMS, and PETER W MICHOR. Geodesic distance for right invariant sobolev metrics of fractional order on the diffeomorphism group. 2011.

[BHM10] Martin Bauer, Philipp Harms, and Peter W Michor. Sobolev metrics on shape space of surfaces in n-space. *Arxiv preprint arXiv:1009.3616*, 2010.

[BHO07] Matias Bossa, Monica Hernandez, and Salvador Olmos. Contributions to 3D diffeomorphic atlas estimation: application to brain images. *Medical image computing and computer-assisted intervention : MICCAI ... International Conference on Medical Image Computing and Computer-Assisted Intervention*, 10:667–674, 2007.

[BMTY05] M Faisal Beg, Michael I Miller, Alain Trouvé, and Laurent Younes. Computing large deformation metric mappings via geodesic flows of diffeomorphisms. *International journal of computer vision*, 61(2):139–157, 2005.

[BO08] Matias Bossa and Salvador Olmos. A New Algorithm for the Computation of the Group Logarithm of Diffeomorphisms. In Xavier Pennec and Sarang Joshi, editors, *Second International Workshop on Mathematical Foundations of Computational Anatomy - Geometrical and Statistical Methods for Modelling Biological Shape Variability*, New York, USA, 2008.

[Bra08] William G Bradley. History of medical imaging. *Proceedings of the American Philosophical Society*, pages 349–361, 2008.

[Bro81] Chaim Broit. *Optimal Registration of Deformed Images.* PhD thesis, Philadelphia, PA, USA, 1981. AAI8207933.

[CSR11] X. Pennec C. Seiler and M. Reyes. Geometry-aware multiscale image registration via obbtree-based polyaffine log-demons, 2011.

[DGM98a] Paul Dupuis, Ulf Grenander, and Michael I. Miller. Variational problems on flows of diffeomorphisms for image matching, 1998.

[DGM98b] Paul Dupuis, Ulf Grenander, and Michael I Miller. Variational problems on flows of diffeomorphisms for image matching. *Quarterly of applied mathematics*, 56(3):587, 1998.

[Hal] Brian Hall. Lie groups, lie algebras, and representations: An elementary introduction.

[HBHH01] Derek L G Hill, Philipp G Batchelor, Mark Holden, and David J Hawkes. Medical image registration. *Physics in Medicine and Biology*, 46(3):R1, 2001.

[HBO07] Monica Hernandez, Matias N Bossa, and Salvador Olmos. Registration of anatomical images using geodesic paths of diffeomorphisms parameterized with stationary vector fields. In *Computer Vision, 2007. ICCV 2007. IEEE 11th International Conference on*, pages 1–8. IEEE, 2007.

[HOP08] Monica Hernandez, Salvador Olmos, and Xavier Pennec. Comparing algorithms for diffeomorphic registration: Stationary lddmm and diffeomorphic demons. In *2nd MICCAI workshop on mathematical foundations of computational anatomy*, pages 24–35, 2008.

[Jr] Alexander Kirillow Jr. Introduction to lie groups and lie algebras.

[LAF+13] Marco Lorenzi, Nicholas Ayache, Giovanni B Frisoni, Xavier Pennec, and Alzheimer's Disease Neuroimaging Initiative. LCC-Demons: a robust and accurate symmetric diffeomorphic registration algorithm. *NeuroImage*, 81:470–483, 2013.

[Lor12] Marco Lorenzi. *Deformation-based morphometry of the brain for the development of surrogate markers in Alzheimer's disease.* Ph.d. thesis, University of Nice Sophia Antipolis, December 2012.

[LP13] Marco Lorenzi and Xavier Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. *International Journal of Computer Vision*, 105(2):111–127, November 2013.

[LP14] Marco Lorenzi and Xavier Pennec. Discrete Ladders for Parallel Transport in Transformation Groups with an Affine Connection Structure. In Frank Nielsen, editor, *Geometric Theory of Information*, Signals and Communication Technology, pages 243–271. Springer, 2014.

[Mil84] J Milnor. Remarks on infinite-dimensional lie groups, in 'relativity, groups and topology, ii'(les houches, 1983), 1007–1057, 1984.

[MPS+11] Tommaso Mansi, Xavier Pennec, Maxime Sermesant, Hervé Delingette, and Nicholas Ayache. iLogDemons: A demons-based registration algorithm for tracking incompressible elastic biological tissues. *International Journal of Computer Vision*, 92(1):92–111, 2011.

[OKC92] V Yu Ovsienko, BA Khesin, and Yu V Chekanov. Integrals of the euler equations of multidimensional hydrodynamics and superconductivity. *Journal of Soviet Mathematics*, 59(5):1096–1101, 1992.

[SDP13] A. Sotiras, C. Davatzikos, and N. Paragios. Deformable medical image registration: A survey. *Medical Imaging, IEEE Transactions on*, 32(7):1153–1190, July 2013.

[Thi98] J-P Thirion. Image matching as a diffusion process: an analogy with maxwell's demons. *Medical image analysis*, 2(3):243–260, 1998.

[Tro]

[Ver08] Tom Vercauteren. Image registration and mosaicing for dynamic in vivo fibered confocal microscopy, 2008.

[VPPA07] Tom Vercauteren, Xavier Pennec, Aymeric Perchant, and Nicholas Ayache. Non-parametric diffeomorphic image registration with the demons algorithm. In *Medical Image Computing and Computer-Assisted Intervention–MICCAI 2007*, pages 319–326. Springer, 2007.

[VPPA08] Tom Vercauteren, Xavier Pennec, Aymeric Perchant, and Nicholas Ayache. Symmetric log-domain diffeomorphic registration: A demons-based approach. In Dimitris N. Metaxas, Leon Axel, Gabor Fichtinger, and Gábor Székely, editors, *Medical Image Computing and Computer-Assisted Intervention - MICCAI 2008, 11th International Conference, New York, NY, USA, September 6-10, 2008, Proceedings, Part I*, volume 5241 of *Lecture Notes in Computer Science*, pages 754–761. Springer, 2008.