Bourbaki vs Pragmatism
A methodological comparison through the multi-armed bandits problem
DRAFT

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These pages are aimed to compare two pedagogical approaches through the example of the the multi-armed bandits problem. The two approaches are here named the *Bourbachist* and *pragmatic*: the Bourbakist way is concerned with the mathematical foundation upon which a formal solution emerges from a set-theoretical axiomatic structure. On the other hand, the pragmatic approach provides an algorithmic solution underpinned by the bare minimum mathematical formalism. The manuscript ends with a critical note underlying the advantages and disadvantages of each exposition.

If the reader came across this article while searching for an introduction to the multi-armed bandit problem, he or she is referred to Section 3. The code to create the figures and to run and compare a range of solutions to the multi-armed bandit problem can be found at https://github.com/SebastianoF/multi-armed-bandits-testbed.

1 Multi-armed bandits problem

Imagine you have to repeatedly choose between K different options. All options have the same cost, though each may or may not provide you with a reward. The probability of having a reward has an *unknown* distribution, that may be different for every choice and that can vary over time.

This general setting is of concern for a wide range of problems, as choosing a medical treatment or a drug, randomising quality control of produced parts, exploring new cafes, trying a range of cars before buying one, choosing a job, and for some even a partner. In

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the algorithmic literature the setting is referred to as the *multi-armed bandits* (MAB), as it can be seen as the situation of playing repeatedly at a row of K slot machines, and after the nickname *single-armed bandit* for slot machines.

The reason why the algorithm takes its name form the gambling example and not from one of the others, more interesting and available in literature[1] is that the gambling case is the easiest to generalise and the one where the algorithmic machinery employed is also easier to visualise. So, following the tradition, imagine to be on a trip to Las Vegas with an initial amount of money to invest in a row of slot machines. Let's say you start with \$1000, and that each draw costs \$1, this gives you 1000 attempts to play and to balance an *exploration* phase with an *exploitation* phase. In the exploration phase the goal is to find an estimate for the unknown distributions of each arm. On the other hand exploitation phase aims at using the acquired knowledge to gain the highest possible reward in the face of uncertainty[2].

The goal of these pages is to provide a machinery to find a solution to the MAB problem with two different approaches:

- *Bourbakist approach.* Named after the collective pseudonym of the group of French mathematicians who have developed this approach in 1934, it has as goal a strict formalisation of the theory form a set theoretical perspective, whose conclusions depends only on a list of axioms rather than physical intuition. Historically, the initial goal of Bourbaki was to re-write one of the most widely used analysis text-book of the time, the Goursat's *Cours d'analyse mathématique*, that was considered inadequate as some of the counterexamples had been overlooked[3].

- *Pragmatic approach.* In contrast to the Bourbachist approach, the pragmatic one reduces the formalisation to its bare minimum, and it orientates the readers' efforts into finding a solution of a problem, rather than creating of a formal axiomatic theory.

The MAB introudced in this section, is formalised *à la Bourbaki* in the next one, and subsequently in the pragmatic way in Section 3.

[1] For example Bouneffouf [BR19]

[2] Thompson [Tho33] provides an early approach where the two arms are two medical treatments, Bellman [Bel56] where the problem is formulated in a Bayesian perspective for two arms, and the more recent Sutton [SB18], Chapter 1, where the problem introduces reinforcement learning.

[3] A brief history of the Bourbaki and its impact in France and the rest of the world was proposed by Marmier [Mar14]. To turn mathematics into the manipulation of axiomatic systems independent from physics, the group of mathematicians rewrote the mathematical analysis into a theory starting set theory and axioms alone, so to allow for a wider range of generalisations, while keeping possible counterexamples a bay. The same approach had been then extended from analysis to other branches of mathematics, with a taste and desire for unification. A same feature can be found in all books by Bourbaki: exercises, examples and the history motivating the given theorems are entirely omitted from the text, being considered irrelevant, or even harmful for a mathematics detached from the physical reality.

2 A Bourbakist introduction

Definition 2.1. Let $(\Omega, \mathcal{A})$ be a σ-algebra defined as a non-empty set Ω paired with a subset of its power set $\mathcal{A}$, containing the empty set and closed under numerable union and complement set. Let this be called *action space*. Let $\mathcal{I}_K = \{1, 2, \dots, K\} \subset \mathbb{N}$ be a set of indexes whose generic element k is called *arm*, by convention. Let $\mathcal{I}_T = \{1, 2, \dots, T\} \subset \mathbb{N}$ be another set whose elements are called, again conventionally, *time*. Let $\mathbb{R}_+$ be the positive real axis including the zero.

The relationship between the above defined elements are given by a function A defined as:

$$\begin{aligned} A : \mathcal{I}_T \times \mathcal{A} &\longrightarrow \mathcal{I}_K \\ (t, \omega) &\longmapsto A(t, \omega) = A_t(\omega) \end{aligned}$$

and by a function R, defined as:

$$\begin{aligned} \mathcal{R} : \mathcal{I}_T \times \mathcal{A} &\longrightarrow \mathbb{R}_+ \\ (t, \omega) &\longmapsto R(t, \omega) = \mathcal{R}_t(\omega) \end{aligned}$$

Let the former be called *action* and the latter be called *reward*. Let

$$\begin{aligned} R : \mathcal{I}_T \times \mathcal{I}_K &\longrightarrow \mathbb{R}_+ \\ (t, \omega) &\longmapsto R(t, k) = R_t(k) \end{aligned}$$

be another function, defined as the only possible function making the diagram below commutative, for each $t \in \mathcal{I}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "A_t"] \arrow[dr, "\mathcal{R}_t"'] & \mathcal{I}_K \arrow[d, "R_t"] \\ & \mathbb{R}_+ \end{tikzcd}$$

Let R_t be called again *reward*, and the difference between $\mathcal{R}$ and R will be clear from the context.
We observe that $\mathcal{R}$ maps the events of the σ-algebra, while R maps the corresponding indexes. This is an analogous of the definition of probability respect to the one of random variable and probability density function, for when the real axis is restricted to $[0, 1] \subset \mathbb{R}$.

Now consider

$$\begin{aligned} \mathcal{Q} : \mathcal{I}_T \times \mathcal{A} &\longrightarrow \mathbb{R}_+ \\ (t, \omega) &\longmapsto \mathcal{Q}(t, \omega) = \mathcal{Q}_t(\omega) \end{aligned}$$

the *estimated reward of the action ω up to time t*, for $\omega = A_t^{-1}(k)$ for a fixed $k \in \mathcal{I}_K$, with the corresponding function $Q : \mathcal{I}_T \times \mathcal{I}_K \to \mathbb{R}$. It follows that $\mathcal{Q}$ is defined as an application of the mean value in a Lebesgue space over $(\Omega, \mathcal{A})$, that is now a Borel σ-algebra[4] as:

$$\mathcal{Q}_t(\omega) = \mathbb{E}\left[R_\tau(k) \mid k = A_\tau(\omega) \;\; \forall \tau \in \mathcal{I}_t\right] \qquad \omega \in \mathcal{A} \qquad t \in \mathcal{I}_T \tag{1}$$

and therefore

$$Q_t(k) = \mathbb{E}\left[\mathcal{R}_\tau(\omega) \mid \omega = A_\tau^{-1}(k) \;\; \forall \tau \in \mathcal{I}_t\right] \qquad t \in \mathcal{I}_T \qquad k \in \mathcal{I}_K \tag{2}$$

As $\mathcal{R}$ and R did, also $\mathcal{Q}$ and Q satisfies the commutativity of a diagram analogous to the one shown above. The notation can be simplified for brevity[5] to:

$$Q_t(k) = \mathbb{E}\left[R_t(k) \mid A_t = k\right] \tag{3}$$

where the mean value is for all the time indexes up to t and where the domain values of A_t is clear from the context.

Definition 2.2. Let the *total reward* $Q_\infty : \mathcal{I}_K \to \mathbb{R}$ be the function

$$Q_\infty(k) = \mathbb{E}\left[\mathcal{Q}_t(\omega) \mid \omega = A_t^{-1}(k) \;\; \forall t \in \mathcal{I}_T\right] \qquad k \in \mathcal{I}_K \tag{4}$$

or with the simplified notation as:

$$Q_\infty(k) = \mathbb{E}\left[R_t(k) \mid A_t = k\right] \tag{5}$$

So far we have been considering the reward and the total reward for a fixed choice of k. We can vary $k \in \mathcal{I}_K$ in function of the time index. So let $\mathbf{k}$ be an element of

$$\mathcal{I}_K^T = \underbrace{\mathcal{I}_K \times \mathcal{I}_K \times \cdots \times \mathcal{I}_K}_{T\text{-times}}$$

or equivalently a function from $\mathcal{I}_T$ to $\mathcal{I}_K$. Definitions 1 and 3 are so generalised to $Q_t : \mathcal{I}_K^T \to \mathbb{R}$ for $t \in \mathcal{I}_{\le T}$, having defined $\mathcal{I}_{\le T}$ any interval of positive integers between 1 and T, and

$$Q_t(\mathbf{k}) = \mathbb{E}\left[\mathcal{R}_\tau(\omega) \mid \omega = A_\tau^{-1}(\mathbf{k}_\tau) \;\; \forall \tau \in \mathcal{I}_t\right] \qquad \mathbf{k} \in \mathcal{I}_K^T \qquad t \in \mathcal{I}_T$$

and therefore

$$Q_\infty(\mathbf{k}) = \mathbb{E}\left[\mathcal{R}_t(\omega) \mid \omega = A_t^{-1}(\mathbf{k}_t) \;\; \forall t \in \mathcal{I}_T\right] \qquad \mathbf{k} \in \mathcal{I}_K^T$$

[4] For a foundational perspective, see Bourbaki [Bou04a].

[5] The simplified notation is often the only notation appearing in engineering textbooks (e.g. Sutton [SB18]), although this would not allow the reader to understand the subtle formalisation of assigning to an event ω its index k. More tragically the simplified notation makes most of the concepts introduced so far pedantic and irrelevant.

The mean value computed with a Lebesgue measure, over the Borel space generated as the sets of images[6] $\mathcal{R}_t(\omega)$ for all $\omega \in \mathcal{A}$ can be reformulated as:

$$Q_t(\mathbf{k}) = \frac{\sum_{\tau=1}^{t} R_\tau(\mathbf{k}_\tau)\mathbf{1}_{A_\tau=\mathbf{k}_\tau}}{\sum_{\tau=1}^{t} \mathbf{1}_{A_\tau=\mathbf{k}_\tau}}$$

where $\mathbf{1}_{A_\tau=\mathbf{k}_\tau}$ equals to 1 for when the event ω corresponding to $\mathbf{k}_\tau$ is mapped exactly to $\mathbf{k}_\tau$ through A_t, and 0 for any other event. Extending the time indexes up to infinity, and to justify the notations introduced above, where we used ∞ for the finite case, we have the following theorem:

Theorem 2.1. *Given a ring of infinite cardinality to which the time index t belongs, and an Hilbert module*[7] *to which the vector $\mathbf{k}$ belongs, it follows that*

$$Q_\infty(\mathbf{k}) = \lim_{T\to\infty} Q_T(\mathbf{k})$$

Proof. Direct consequence of the definition of Q_∞ generalised to the theorem hypothesis' extended structures. $\square$

We now consider the value $\hat{k}$ that satisfies

$$\hat{k} = \arg\max_{k\in\mathcal{I}_K} Q_t(k) \qquad \forall t \in \mathcal{I}_T$$

for a constant value for each time index, as in the definition of Q_t given in 2. If we consider the possibility of varying the chosen arm k across time, and so if we are allowed to compare different images of the function A_t then $\hat{\mathbf{k}}$ is defined as

$$\hat{\mathbf{k}} = \arg\max_{\mathbf{k}\in\mathcal{I}_K^T} Q_t(\mathbf{k}) \qquad \forall t \in \mathcal{I}_T \tag{6}$$

Under the light of theorem 2.1, and with the given definitions, we can now call the so defined vector $\hat{\mathbf{k}}$ the *solution of the generalised multi-armed bandits problem.*

3 A Pragmatic (non-Bourbakist) introduction

The first problem to solve is to have a playground where a ground truth of the multi-armed bandit problem is known. With this tool, where to test a range of algorithms and parameters, we consider a benchmark dataset and we can test a range of algorithms.

[6] We consider the definition under the accordance with the axiom of choice as in the ZFC axiomatic set theory, in order to avoid *virages dangereux*. See also Bourbaki [Bou04b] and [TZ82].

[7] An algebraic structure generalising Hilbert vector spaces over the now introduced ring of time indexes. See for example [Bou87].

Creating a Benchmark

We introduce the tool in a particular case, assuming each arm normally distributed, with mean sampled from a uniform distributions in the interval $[-3, 3]$ and the standard deviation sampled from a uniform distribution on the interval $[2, 3]$. A representation of these distribution is plotted in figure 1.

If the studied case is modelled with a different distribution, what is said in this section is still applicable simply adapting this benchmark to the intended target.

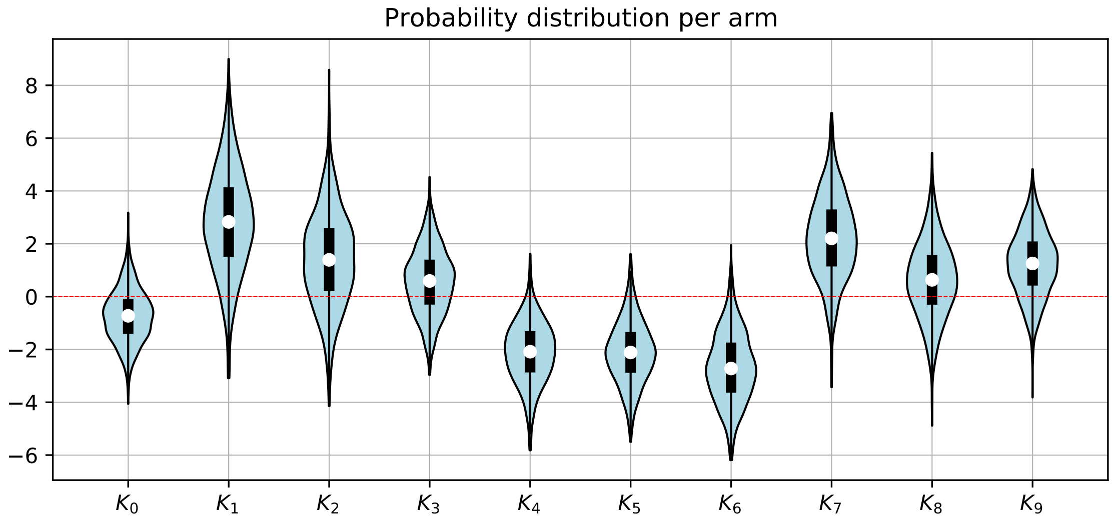

Figure 1: The distributions of the random reward for each arm K, unknown to the player, are sampled from normal distributions, with mean uniformly sampled in the interval $[-3, 3]$ and standard deviation in the interval $[1, 2]$.

Encoding the problem

The player gains a reward for after pulling an arm index k, for each time point t. We encode the rewards in a $T \times K$ *reward matrix* q. In our example $T = 1000$ is the time we will be pulling an arm and $K = 10$ is the number of arms. The element $q_{t,k}$ represents the reward collected for having pulled the arm k at the time-point t. As we can pull only one arm at a time, there is only one known value for each column of the matrix. All the other values are initialised to nan.

The multi-armed bandit is defined by two vectors μ and σ, where μ_k is the mean and σ_k is the standard deviation of the distribution of the arm k. To consider the non-stationary case, where the means and the standard deviations of the arms distributions represented in Figure 1 are not constant over time, we turns the vectors μ and σ into matrices of size $T_{\text{arm}} \times K$, providing the mean and standard deviation of the distribution of the arm k at time t.

The rewards can be accumulated for more or less numerous time points than T_{arm}, as the distributions of the arms can repeat. If $T_{\mathrm{arm}} = 1$ then the problem is stationary, and if $T = 30$ and $T_{\mathrm{arm}} = 15$ then each arm k will be looping over its parameters $\mu_{:,k}$ and $\sigma_{:,k}$ twice, according to:

$$q_{t,k} = \mathrm{norm}\left(\mu(t \bmod T_{\mathrm{arm}}, k), \sigma(t \bmod T_{\mathrm{arm}}, k)\right)$$

where $\mathrm{norm}(\mu, \sigma)$ is the normal distribution of location μ and scale σ and mod is the modulo operator.

ϵ-greedy algorithms

In the previous section we created a benchmark and encoded the multi armed bandit response/rewards into a matrix.

Now we introduce the ϵ-greedy algorithms, a class of algorithms based on the idea that if your bad luck is consistent, probably it is not just bad luck. The player using these algorithms alternates between a phase of exploration (proportional to ϵ times), and a phase of exploitation (proportional to $1 - \epsilon$ times). In the first phase, the player acquires information about the structure of the problem regardless the immediate gain, and in the second one, the information gained are exploited to maximize the reward[8].

The basic version of the ϵ-greedy algorithm (or *naive ϵ-greedy*), selects a random arm when exploring and the arm that had obtained the best reward so far in the exploitation phase. This can happen after an initial phase of pure exploration, where only random arms are pulled. An example of the application of this algorithm is shown in figure 2.

A more sophisticated version the naive ϵ-greedy is the *best reward ϵ-greedy*. In this case, instead of selecting any random arm in the exploration phase, we weight the random selection using the rewards obtained so far, excluding the arm with the highest reward. With this strategy we will be more likely to explore the arms that have provided good rewards in the past, and to refrain from spending money over an arm that have never provided any gain at all.

A third algorithm is the *least explored ϵ-greedy*. In the exploratory phase of this algorithm we increase the probability of falling over the least explored arms, instead of the one providing the highest reward. This method and the previous one are both aimed at preventing from getting stuck in a local minima, i.e. believing that the arm we are hitting is the best one, as it happens to provide a gain, though it is not the one with the optimal gain.

The fourth algorithm here presented is the *upper confidence bound ϵ-greedy algorithm* [SB18]. In this algorithm the estimated optimal arm at time t, is given by

$$\hat{k}_t = \arg\max_k \left[\hat{\mu}(k) + \lambda \sqrt{\frac{\ln(t)}{N_t(k)}} \right] \tag{7}$$

[8] An accurate historical introduction of this family of algorithms can be found at the end of chapter 1 of [SB18].

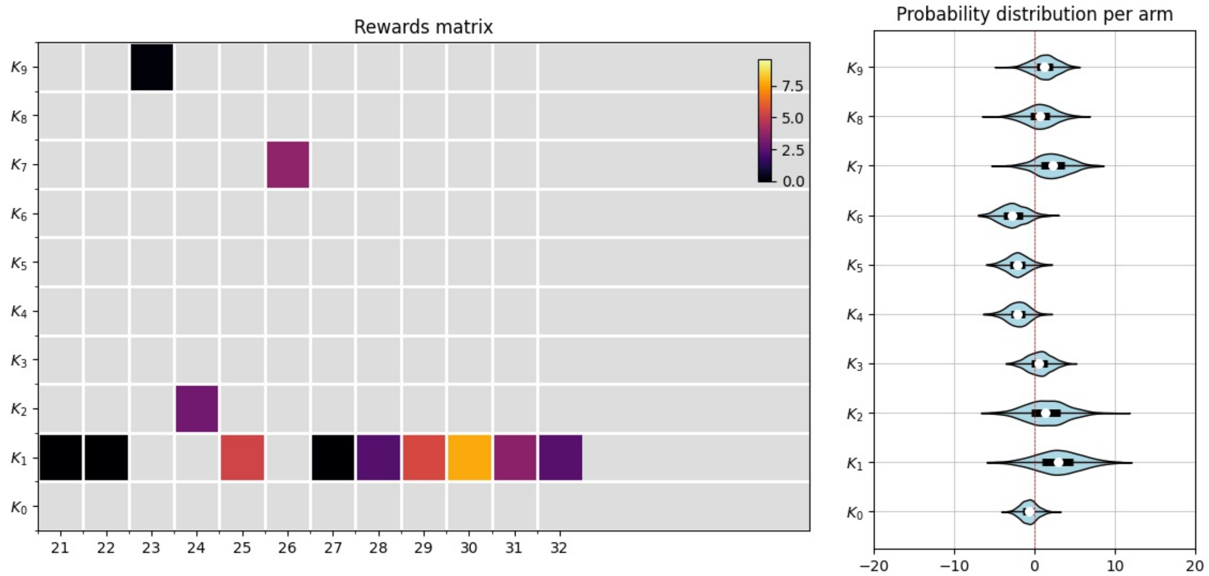

Figure 2: The reward matrix after 32 steps of a *naive* ϵ-greedy algorithm. We can see that step 23, 24 and 26 are explorations in a row of exploitation of the arm 7 that so far had produced the highest reward. To the grey colour corresponds the nan value used to initialise the reward matrix. In this instance, the reward is clipped at zero, as the loss of money is only the cost of pulling an arm at each time-point.

where $N_t(k)$ is the number of times the armed k had been pulled up to the time t, which can be easily computed from q. The regularisation term weights the uncertainty in the estimate, correcting for underestimation of the estimated mean.

Note that in this version of the game, the player will receive 0 when the sampled reward is below 0. This means that the estimated mean does not correspond to the one of the original distribution, though the result will be conservative in having an observed average close to zero for distribution whose mean is below the zero. The reader should resist the temptation of estimating the arm using the mode instead of the average in equation 7. In fact if the standard deviations are very different across arms, a very inconvenient arm, with the average below the zero, may be identified as the optimal one.

The performance of these algorithms are all depending upon the problem's distributions, and the optimal one should be found empirically via simulation. Providing a benchmarking for a general case would not be useful here. What provides the constraints to make the case less general, is based on the instance of the problem to which the algorithms are applied. A platform where to make further experiments is open-sourced and can be found at https://github.com/SebastianoF/multi-armed-bandits-testbed.

4 Comparing the two methodological approaches

In these pages, we considered the example of the multi-armed bandit problem to show two different methodological approaches towards its solution.

The Bourbaki presentation in section 2 is based on an axiomatisation of the problem and it evolves from a functional and set-theoretical perspective. It focuses on the symbolism, and according to the Bourbachist tradition, it provides no examples or algorithms, as the only connecting point between the theory and the given problem is the axsiomatic setting.

Assuming that this approach is correct (excluding typos) and coherent (excluding Goedel[9]), it is easy to see that it had the effect of turning a relatively simple problem into a complicated maze of interrelated concepts, having mainly the consequence of pushing the reader away from the practical solution.

On the contrary, the pragmatic approach of Section 3 shows that there is no need of taking a functional perspective, when a single matrix is enough. It shows that there is no need for taking several mathematical tools from other branches (Borel spaces, Lebsgue Measures...) developed in entirely different contexts to prevent counterexamples, and there is no need to extend the bibliography beyond one ot two resources. Borel spaces and any element of measure theory are not just irrelevant in aiming at a solution, they are also hindering creativity when facing a slightly different problem, as for example one with different types of distributions over time, or one where not all the arms are available at any time points.

If in doubt the hindrance of creativity, the reader is invited to continue the formalisation in this direction, to see how many pages and new definitions and diagrams are needed, if following the Bourbachist approach. We also challenge the reader to implement the code to solve the problem having only the functional and algebraic definitions at hand rather than relying on the matrix point of view.

Other examples

Certainly the example provided is biased by the fact that they are the best approximation of a Bourbaki approach and of a pragmatic approach that the author could attain. Also the reader may say, the MAB problem is not the ideal one to embed into the Bourbaki formalisation, as per se too empirical.

To answer these possibly valid points, we wish to point at the numerous cases of practical problems, whose pragmatic approach had been overly formalised[10] in a similar fashion as the one here proposed. These examples may also constitute a collection of case-studies for anyone who may believe that the author had been overzealous in writing section 2 to give on purpose a negative light upon the Bourbaki style.

[9]In the article *The ignorance of Bourbaki* [Mat92] Mathias noticed that the Bourbaki attempt of grounding the whole corpus of mathematics in an axiomatic sense have happened after, and with a conscious effort to ignore the Goedel incompleteness theorems.

[10]Or *assiomattisation*, as Bruno de Finetti [DFN08] would have said playing on the Italian word *matti*, meaning crazy.

The first most notable example belongs to the domain of optimal transport theory. From being a pragmatic tool of solve a class of optimisation problems, it had become, once in the hand of Bourbachists, a 500 pages book underpinned by a great amount of measure theory and Lebesgue spaces, perfectly irrelevant to solve any instance of an optimal transport problem, when the reader will have to face one. Comparing one of the original optimal transport theory introduction, for example the one by Hitchcock [Hit41], with the formalised version by Villani [Vil03] it is possible to see the Bourbachisation's results. The first one is easy to read, understand, implement and possibly extend in several directions by anyone having an high school mathematical education. The second one is a seemingly uncreated maze of unassailable interlinked concepts, requiring few years of academic studies only to grasp the first few pages, with no advantages in finding a numerical solution to the problem.

A second notable example of the Bourbachisation is in the domain of medical image registration. The aim of this field is to solve the problem of finding the non-rigid deformation or metamorphosis between anatomies. The problem originates from the anatomical studies of shapes growth by D'Arcy Thompson [d+42] and pragmatically extended amongst others in Modersitzki [Mod04]. The Bourbaki over-formalized branch can be found in works like [You10], where we have to wait for 12 chapters before seeing what had motivated the axiomatic mathematical theory developed until that point. In this case too, there is no explanation of what are the advantages of the axiomatic approach respect to the pragmatic one, which have appeared half a century earlier.

The reader may say, for this specific case, that the pragmatic approach [Mod04] does not use neither diffeomorphisms nor reproducing Kernel Hilbert spaces. While this is true, it is not proved either that these two mathematical devices are providing any more accurate results than their pragmatic counterparts when implemented in practice. Instead it is true that they are computationally slower and they increase the cognitive load.

The Bourbaki pattern

There are other examples in the literature where we can find a comparison between the Bourbaki / pragmatic approach, such as algebraic topology, fuzzy logic and topological data analysis. Their detachment from the practical set of problems that had them originated, had turned them into convoluted axiomatic buildings, where it is not possible to find any breakthrough result, or even practical help, in the perspective of solving the initially given problem.

It is at this point evident the existence of a pattern in the process of Bourbachisation: a problem solved by more than 50 years, and often in a very elegant and straightforward way, is taken under the wing of the academic axiomatisers and transformed into a list of definitions and theorems where no echo of the original problem motivating the theory can be found.

What is also constant in the Bourbaki pattern is that the formalised version of the practical theory adds no value to it. No new results having any practical relevance can be found in the work by Bourbaki and their followers.

There is only one notable exception to this rule that we could find. The work Mochizuki [Moc12] is claimed to have solved the *abc conjecture*. Although, worth of a pantomime, according to the mathematicians who had tried to read this work, “the proof is too impenetrable to be understood”[11].

Why is Bourbaki still around?

These pages are not written to accuse Bourbaki. Their goal is rather to try to analyse and understand the phenomenon beyond its name and their creators. It is true that, despite many have express their negative view about the Bourbachist's mathematics[12], the Bourbaki approach is still widespread if not predominant across the academic community and also greatly valued and honoured[13].

Very often, the over-axiomatization of a theory developed outside academia by scientists chasing practical problems, had received more recognitions than the original work. Highly valued and cited uber-bourbachists works are for example Grotedniek [Gro11] and the already mentioned four volumes of Mochizuki [Moc12][14].

For this reason, it is not true to claim that the Bourbachisation has no practical use. Below are itemized some points in favour of this approach, to justify the success of the Bourbaki school of thought.

- *The outcome of the Bourbaki approach is inaccessible to neophytes*[15], and it allows the existence of professors of mathematics who can not code or do not have any ability to solve any problem in practice. This seems to be the main reason that justifies the continuation of the Bourbachism, as neophytes usually are the people who have to decide where the public money goes and professors the ones who receives it.

- *The Bourbaki approach is good for the ego.* The pleasure of owning an elegant notebook filled with mathematical formulae written with a sharp pencil and a well thought after handwriting is a most sublime one. Unfortunately, after empirical evidence we can assure that this very same pleasure is a great obstacle in acquiring

[11] A statement published in Nature, Castelvecchi [Cas15].

[12] Amongst the many: Arnold [Arn98], De Finetti [DFN08], Lockhart [Loc09], the already cited Mathias and its follow up [Mat98], Velupillai [Vel12]. And even the article by Marmier [Mar14], that sees the Bourbaki movement with enthusiasm and appreciation for their work and impact in the society, begs some questions when it says “Living mathematics is being done elsewhere in the world, in connection with new problems arising from other disciplines or other human activities”.

[13] To this regard, the reader is welcome to copy-paste section 2 and to extend it, in the Bourbachist style, into an article whose title could be something like *The multi-armed bandits for the working mathematician*, to see how this would be received by the dogmatic community.

[14] It is also interesting to notice at this point the similarity between this work and the outcome of the randomized generator of maths paper at thatsmathematics.com/mathgen/.

[15] An attempt of turning an over-formalised Bourbachist theory into something accessible for the layman, can be found in Villani [Vil12]. Interestingly enough, it seems that the only way the author could popularize his theory was to throw some unexplained formulas right before focusing the attention of the reader on some entirely unrelated autobiographical events. The result is even less accessible than the original work, proving that mathematicians can also be successful surrealist artists.

knowledge. The notebook owner, when facing a problem, will inevitably tend to abandon any scientific method and will bend the problem to the behold solution. To this regard, it would be interesting to measure scientifically the dopamine level of group of mathematician trying to solve a difficult problem respect to a group tweaking axioms to make the theory more abstract.

- *Bourbachism detaches mathematics from reality*, so it makes impossible to have any objective measurement of the quality or value of the work produced, besides the readers' opinion. This is a very unscientfic feature, and again a very useful one for whoever must defeat more skilled competitors within the academic environment.

  The advocates of the superiority of *pure mathematics*[16] have even arrived at accusing the mathematics that have anything to do with reality of being more prone of making mistakes[17]. We can be reassured by the fact that under every circumstance, the reality is adamant to persist in being what it is, and it is at this point an interesting exercise to imagine someone advocating for having medicine and engineering detached from reality.

- *The Bourbaki inspired work are more generalizable.* While this appear to be true at first glance, we claim that on the contrary a Bourbaki theory is less generalisable. Simply adding a new or different assumption to the problem settings would require to re-write almost from scratch all the axiomatic building to include the new input in a generalisable manner. The pragmatic way provides handle on the problem that are easy to be tuned or re-adapted to a slightly different one.

- *Bourbachism binds the concepts in formal structures avoiding paradoxes and counterexamples.* This is a very valid point in favour of the Bourbachist method, as the concerns of mathematics with counterexample have led to various discoveries, from Galois theory, to Fractals, and the search for counterexample is a valuable tool to have a better understanding when exploring the limits of mathematical knowledge[18]. Although, the problem of the Bourbaki approach is the over-concern with counterexamples originating from theoretical considerations not related in any way to the sought solution. In section 2 we attained an algorithm that solves the problem, having never faced pathological counterexamples despite not relying on σ-algebras, Borel spaces, Lebesgue measures or even without explicit use of functions. What we came across were only some practical malices of the craft that

[16] And with another footnote, it is worth noticing here that the academic dichotomy pure/applied maths is something that can not be found in mathematics before Bourbaki. The reader is invited to classify any of the work of Archimedes, Gauss, Euler or any other great names in the history of pre-modern mathematics as *pure* or *applied* with no ambiguities.

[17] Gros [GST19] had found how even professional mathematicians can be mislead by reality. And they had leveraged on this most surprising fact, for advocating to increase the detachment. After concluding that "[...] we can't reason in a totally abstract manner", instead of suggesting to take into account the reality in the mathematical practice they suggested a move towards the opposite direction: "We have to detach ourselves from our non-mathematical intuition" [Sum19].

[18] For example Procesi [Pro77], Mandelbrot [Man83] and Gelbaum [GO03].

are never learned by whoever is limiting themselves to the Bourbaki presentation of the problem.

About this last point in particular, there is an (in)famous case where over-concerned researchers had been mislead by a misquoted counterexamples: the medical imaging paper by Lorenzi [LP13] claims that there is no bijective correspondence between the space of the tangent vector fields and the one of their integral curves. This claim is made after citing a very peculiar counterexample found in one of the Milnor research papers, involving the tangent space of the circle.

Interestingly enough the conditions for the counterexample to happen are never met in the applications presented in the paper dealing with $\mathbb{R}^3$, whose tangent space is $\mathbb{R}^3$ itself and the bijective correspondence is guaranteed by the theorem of existence and uniqueness for ODEs.

5 Conclusion

If we wanted to divide the mathematical approaches into two, we would consider the Pythagorean and the Archimedean schools of thought. The Pythagorean school, influenced by the ancient Egyptian and Persian mathematics, considers the mathematical practice as a tool to investigate and measure the reality and to find the connections between apparently distinct elements through abstraction and generalisation[19] . The Archimedean vision sees mathematics as a tool to solve very practical problems, approaching them with heuristic, algorithms and even adhoceries[20]. To neither the Pythagorean nor the Archimedean belongs the dichotomy between pure and applied mathematics, as there is no concept of *pure* in the sense of detached from the reality, being the reality always the centre of the investigation.

The Bourbachist vision does belong to any of these two mathematical schools: it is neither oriented towards finding hidden connections between apparently separated phenomenon, nor towards solving practical problems. The manipulation of abstract structures arising from the need of finding a proof that is general enough had shifted its aim: from searching for a solution, it had started wandering around the creation of abstract bazar where anything that is coherent with the initial axioms can be added, even if it does not solve problems or shows hidden connections. The unavoidable contraindication is that the abstract building are destined at becoming labyrinths without a centre, where the readers will inevitably be waisting their time.

Only from this point onwards, the subdivision between pure and applied mathematics had been required, as a line drawn between mathematical practice (either of Pythagorean or Archimedean school) and the axiomatisation.

The comparison proposed in this paper, through the multi-armed bandit problem, aims at providing an example between the pure and applied approach towards the same

[19] As for example astronomy, music and architecture [Rob95], [BM11].

[20] An example of this approach is the proof of the formula for the parabolic segment's area by Archimede [BM11] and [Str19].

problem in a very specific case. It also collects a list of analogous cases in the literature, showing the axiomatisation pattern and investigating the reasons for the success of Bourbakchism.

References

[Arn98] Vladimir I Arnol'd. On teaching mathematics. *Russian Mathematical Surveys*, 53(1):229, 1998.

[Bel56] Richard Bellman. A problem in the sequential design of experiments. *Sankhyā: The Indian Journal of Statistics (1933-1960)*, 16(3/4):221–229, 1956.

[BM11] Carl B Boyer and Uta C Merzbach. *A history of mathematics*. John Wiley & Sons, 2011.

[Bou87] Nicolas Bourbaki. Topological vector spaces, elements of mathematics. 1987.

[Bou04a] Nicolas Bourbaki. Integration. Springer Berlin, 2004.

[Bou04b] Nicolas Bourbaki. Theory of sets. Springer Berlin, 2004.

[BR19] Djallel Bouneffouf and Irina Rish. A survey on practical applications of multi-armed and contextual bandits. *arXiv preprint arXiv:1904.10040*, 2019.

[Cas15] Davide Castelvecchi. The biggest mystery in mathematics: Shinichi mochizuki and the impenetrable proof. *Nature News*, 526(7572):178, 2015.

[d+42] W Thompson d'Arcy et al. On growth and form. *On growth and form*, 1942.

[DFN08] Fulvia De Finetti and Luca Nicotra. *Bruno de Finetti: un matematico scomodo*. Belforte, 2008.

[GO03] Bernard R Gelbaum and John MH Olmsted. *Counterexamples in analysis*. Courier Corporation, 2003.

[Gro11] Alexandre Grothendieck. Some aspects of homological algebra. 2011.

[GST19] Hippolyte Gros, Emmanuel Sander, and Jean-Pierre Thibaut. When masters of abstraction run into a concrete wall: Experts failing arithmetic word problems. *Psychonomic bulletin & review*, 26(5):1738–1746, 2019.

[Hit41] Frank L Hitchcock. The distribution of a product from several sources to numerous localities. *Journal of mathematics and physics*, 20(1-4):224–230, 1941.

[Loc09] Paul Lockhart. *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form*. Bellevue literary press, 2009.

[LP13] Marco Lorenzi and Xavier Pennec. Geodesics, parallel transport & one-parameter subgroups for diffeomorphic image registration. *International journal of computer vision*, 105(2):111–127, 2013.

[Man83] Benoit B Mandelbrot. *The fractal geometry of nature*, volume 173. WH freeman New York, 1983.

[Mar14] Anne-Marie Marmier. On the idea of 'democratisation','modern mathematics' and mathematics teaching in france. *Lettera Matematica*, 2(3):139–148, 2014.

[Mat92] Adrian RD Mathias. The ignorance of bourbaki. *Mathematical Intelligencer*, 14(3):4–13, 1992.

[Mat98] ARD Mathias. Further remarks on bourbaki (a reply to criticism by professor sanford l. segal of the essay "the ignorance of bourbaki".). *Preprint*, 7, 1998.

[Moc12] Shinichi Mochizuki. Inter-universal teichmüller theory i: Construction of hodge theaters, 2012.

[Mod04] Jan Modersitzki. *Numerical methods for image registration*. Oxford University Press on Demand, 2004.

[Pro77] Claudio Procesi. *Elementi di teoria di Galois*. Decibel, 1977.

[Rob95] Gay Robins. Mathematics, astronomy, and calendars in pharaonic egypt. *Civilizations of the Ancient Near East*, 3:1799–1813, 1995.

[SB18] Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

[Str19] Steven Strogatz. *Infinite powers: How calculus reveals the secrets of the universe*. Houghton Mifflin Harcourt, 2019.

[Sum19] Article Summary. Expert mathematicians stumped by simple subtractions. https://www.sciencedaily.com/releases/2019/07/190710103231.htm, 2019.

[Tho33] William R Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3/4):285–294, 1933.

[TZ82] Gaisi Takeuti and Wilson M Zaring. Introduction to axiomatic set theory. Springer, 1982.

[Vel12] K Vela Velupillai. Bourbaki's destructive influence on the mathematisation of economics. *Economic and Political Weekly*, pages 63–68, 2012.

[Vil03] Cédric Villani. *Topics in optimal transportation*. Number 58. American Mathematical Soc., 2003.

[Vil12] C. Villani. Théorème vivant. 2012.

[You10] Laurent Younes. *Shapes and diffeomorphisms*, volume 171. Springer, 2010.