Solving the Euler-Lagrange Equations

Mathematical Analysis

The structure of the UAV (Unmanned Aerial Vehicle) is cross-type, where the motors are located at the tips of each arm and the coordinates coincide with their extensions.

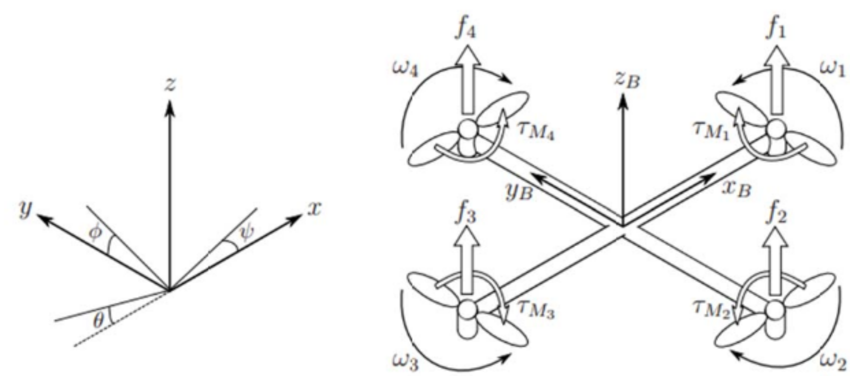

Fig 1. Marcos de referencia y sistemas de fuerzas del quadrotor

It is defined as a x_b,y_b y z_b , as the fixed coordinate system of the VANT, where x_b It is the direction in which the vehicle moves forward. The inertial coordinate system is described by the x,y,z axes which are considered fixed with respect to the ground.

The aerodynamic thrust due to the rotation of the propellers in the viscous fluid (air). Since the motors are considered to be aligned, then this force is en $\hat{z}$
$f_i = K \cdot \omega_i^2$

The total thrust is given by the following equation
$f = \sum_{i=1}^{4} f_i = k_t \cdot \sum_{i=1}^{4} \omega_i^2$

When the propellers rotate, they are subjected to frictional stress, which occurs between the air and their movement. This generates a torque in the opposite direction to the direction of rotation of the rotors, which is described with this expression:
$\tau_{M\,i} = K_d \cdot \omega^2$

The Kd coefficient has a value greater than zero and depends on the density of the air, radius and the shape of the propeller. The total moment is proportional to the thrust generated by each rotor and is generated by the imbalance of the set of forces f_2 and f_4 with f_1 and f_3. This movement is possible since pair rotors 2 and 4 rotate clockwise, and rotors 1 and 3 counterclockwise, the torque generated by aerodynamic drag is denoted by:

$$\tau_\psi = \sum_{i=1}^{4} \tau_{M\,i} = K_d \cdot \left(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2\right)$$

The corresponding moments in the direction of the body frame angles mentioned above are denoted by the following vector, where l is the distance between the rotor and the center of mass

$$\tau = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} l \cdot (f_4 - f_2) \\ l \cdot (f_3 - f_1) \\ \sum_{i=1}^{4} \tau_{Mi} \end{bmatrix}$$

The coordinates are designated:

$$\xi = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \eta = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

The complete rotation matrix of the body’s reference frame with respect to the fixed reference frame known as direct cosine matrix:

$R(\phi, \theta, \psi) = R(z, \psi) \cdot R(y, \theta) \cdot R(x, \phi)$

Kinematics

In[ ]:=

In[∘]:=
```
Rphi11 = 1;
Rphi12 = 0;
Rphi13 = 0;
Rphi21 = 0;
Rphi22 = Cos[ϕ[t]];
Rphi23 = Sin[ϕ[t]];
Rphi31 = 0;
Rphi32 = -Sin[ϕ[t]];
Rphi33 = Cos[ϕ[t]];
Rphid := {{Rphi11, Rphi12, Rphi13}, {Rphi21, Rphi22, Rphi23}, {Rphi31, Rphi32, Rphi33}}
```

In[∘]:=
```
Rphid
```

Out[∘]=
```
{{1, 0, 0}, {0, Cos[ϕ[t]], Sin[ϕ[t]]}, {0, -Sin[ϕ[t]], Cos[ϕ[t]]}}
```

In[∘]:=
```
MatrixForm[Rphid]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}[\phi[t]] & \mathrm{Sin}[\phi[t]] \\ 0 & -\mathrm{Sin}[\phi[t]] & \mathrm{Cos}[\phi[t]] \end{pmatrix}$$

In[∘]:=
```
Rphidt = Transpose[Rphid]
```

Out[∘]=
```
{{1, 0, 0}, {0, Cos[ϕ[t]], -Sin[ϕ[t]]}, {0, Sin[ϕ[t]], Cos[ϕ[t]]}}
```

In[∘]:=
```
RphiI = FullSimplify[Rphid.Rphidt]
```

Out[∘]=
```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

In[∘]:=
```
Rtheta11 = Cos[θ[t]];
Rtheta12 = 0;
Rtheta13 = -Sin[θ[t]];
Rtheta21 = 0;
Rtheta22 = 1;
Rtheta23 = 0;
Rtheta31 = Sin[θ[t]];
Rtheta32 = 0;
Rtheta33 = Cos[θ[t]];
Rthetad := {{Rtheta11, Rtheta12, Rtheta13},
  {Rtheta21, Rtheta22, Rtheta23}, {Rtheta31, Rtheta32, Rtheta33}}
```

In[∘]:=
```
MatrixForm[Rthetad]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta[t]] & 0 & -\mathrm{Sin}[\theta[t]] \\ 0 & 1 & 0 \\ \mathrm{Sin}[\theta[t]] & 0 & \mathrm{Cos}[\theta[t]] \end{pmatrix}$$

In[∘]:=
```
Rthetadt = Transpose[Rthetad]
```

Out[∘]=
```
{{Cos[θ[t]], 0, Sin[θ[t]]}, {0, 1, 0}, {-Sin[θ[t]], 0, Cos[θ[t]]}}
```

In[∘]:=
```
RthetaI = FullSimplify[Rthetad.Rthetadt]
```
Out[∘]=
```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```
In[∘]:=
```
MatrixForm[RthetaI]
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[∘]:=
```
Rpsi11 = Cos[ψ[t]]
Rpsi12 = Sin[ψ[t]];
Rpsi13  = 0;
Rpsi21 = -Sin[ψ[t]];
Rpsi22 = Cos[ψ[t]];
Rpsi23 = 0;
Rpsi31 = 0;
Rpsi32 = 0;
Rpsi33 = 1;
Rpsid := {{Rpsi11, Rpsi12, Rpsi13}, {Rpsi21, Rpsi22, Rpsi23}, {Rpsi31, Rpsi32, Rpsi33}}
```
Out[∘]=
```
Cos[ψ[t]]
```
In[∘]:=
```
MatrixForm[Rpsid]
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\psi[t]] & \mathrm{Sin}[\psi[t]] & 0 \\ -\mathrm{Sin}[\psi[t]] & \mathrm{Cos}[\psi[t]] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[∘]:=
```
Rpsidt = Transpose[Rpsid]
```
Out[∘]=
```
{{Cos[ψ[t]], -Sin[ψ[t]], 0}, {Sin[ψ[t]], Cos[ψ[t]], 0}, {0, 0, 1}}
```
In[∘]:=
```
RphiI = FullSimplify[Rphid.Rphidt]
```
Out[∘]=
```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```
In[∘]:=
```
RI1 = FullSimplify[RphiI.RthetaI]
```
Out[∘]=
```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```
In[∘]:=
```
RI = RI1.RpsiI
```
Out[∘]=
```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}.RpsiI
```
In[∘]:=
```
R2 = Rphid.Rthetad
```
Out[∘]=
```
{{Cos[θ[t]], 0, -Sin[θ[t]]}, {Sin[θ[t]] Sin[ϕ[t]], Cos[ϕ[t]], Cos[θ[t]] Sin[ϕ[t]]},
 {Cos[ϕ[t]] Sin[θ[t]], -Sin[ϕ[t]], Cos[θ[t]] Cos[ϕ[t]]}}
```

```
In[∘]:= MatrixForm[R2]

Out[∘]//MatrixForm=
( Cos[θ[t]]                  0             -Sin[θ[t]]
  Sin[θ[t]] Sin[ϕ[t]]   Cos[ϕ[t]]    Cos[θ[t]] Sin[ϕ[t]]
  Cos[ϕ[t]] Sin[θ[t]]   -Sin[ϕ[t]]   Cos[θ[t]] Cos[ϕ[t]] )

In[∘]:= Rdot1 = FullSimplify[TrigReduce[(∂t R2.Rpsid)]]

Out[∘]=
{{-Cos[ψ[t]] Sin[θ[t]] θ′[t], -Sin[θ[t]] Sin[ψ[t]] θ′[t], -Cos[θ[t]] θ′[t]},
 {Cos[θ[t]] Cos[ψ[t]] Sin[ϕ[t]] θ′[t] +
   (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]) ϕ′[t],
  Cos[θ[t]] Sin[ϕ[t]] Sin[ψ[t]] θ′[t] +
   (-Cos[ψ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[θ[t]] Sin[ψ[t]]) ϕ′[t],
  -Sin[θ[t]] Sin[ϕ[t]] θ′[t] + Cos[θ[t]] Cos[ϕ[t]] ϕ′[t]},
 {Cos[θ[t]] Cos[ϕ[t]] Cos[ψ[t]] θ′[t] +
   (-Cos[ψ[t]] Sin[θ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[ψ[t]]) ϕ′[t],
  Cos[θ[t]] Cos[ϕ[t]] Sin[ψ[t]] θ′[t] -
   (Cos[ϕ[t]] Cos[ψ[t]] + Sin[θ[t]] Sin[ϕ[t]] Sin[ψ[t]]) ϕ′[t],
  -Cos[ϕ[t]] Sin[θ[t]] θ′[t] - Cos[θ[t]] Sin[ϕ[t]] ϕ′[t]}}

In[∘]:= MatrixForm[Rdot1]

Out[∘]//MatrixForm=
(                                        -Cos[ψ[t]] Sin[θ[t]] θ′[t]
  Cos[θ[t]] Cos[ψ[t]] Sin[ϕ[t]] θ′[t] + (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]
  Cos[θ[t]] Cos[ϕ[t]] Cos[ψ[t]] θ′[t] + (-Cos[ψ[t]] Sin[θ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[ψ[t]

In[∘]:= Rdot2 = FullSimplify[TrigReduce[(R2.∂t Rpsid)]]

Out[∘]=
{{-Cos[θ[t]] Sin[ψ[t]] ψ′[t], Cos[θ[t]] Cos[ψ[t]] ψ′[t], 0},
 {-((Cos[ϕ[t]] Cos[ψ[t]] + Sin[θ[t]] Sin[ϕ[t]] Sin[ψ[t]]) ψ′[t]),
  (Cos[ψ[t]] Sin[θ[t]] Sin[ϕ[t]] - Cos[ϕ[t]] Sin[ψ[t]]) ψ′[t], 0},
 {(Cos[ψ[t]] Sin[ϕ[t]] - Cos[ϕ[t]] Sin[θ[t]] Sin[ψ[t]]) ψ′[t],
  (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]) ψ′[t], 0}}

In[∘]:= MatrixForm[Rdot2]

Out[∘]//MatrixForm=
(                 -Cos[θ[t]] Sin[ψ[t]] ψ′[t]                                   Cos[θ[t]]
  -((Cos[ϕ[t]] Cos[ψ[t]] + Sin[θ[t]] Sin[ϕ[t]] Sin[ψ[t]]) ψ′[t])   (Cos[ψ[t]] Sin[θ[t]] Sin[ϕ
   (Cos[ψ[t]] Sin[ϕ[t]] - Cos[ϕ[t]] Sin[θ[t]] Sin[ψ[t]]) ψ′[t]     (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ
```

```
In[•]:= Rdot = FullSimplify[TrigReduce[Rdot1 + Rdot2]]
Out[•]=
{{-Cos[ψ[t]] Sin[θ[t]] θ'[t] - Cos[θ[t]] Sin[ψ[t]] ψ'[t],
  -Sin[θ[t]] Sin[ψ[t]] θ'[t] + Cos[θ[t]] Cos[ψ[t]] ψ'[t], -Cos[θ[t]] θ'[t]},
 {Cos[θ[t]] Cos[ψ[t]] Sin[ϕ[t]] θ'[t] +
   (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]) ϕ'[t] -
   (Cos[ϕ[t]] Cos[ψ[t]] + Sin[θ[t]] Sin[ϕ[t]] Sin[ψ[t]]) ψ'[t],
  Cos[θ[t]] Sin[ϕ[t]] Sin[ψ[t]] θ'[t] +
   (-Cos[ψ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[θ[t]] Sin[ψ[t]]) ϕ'[t] +
   (Cos[ψ[t]] Sin[θ[t]] Sin[ϕ[t]] - Cos[ϕ[t]] Sin[ψ[t]]) ψ'[t],
  -Sin[θ[t]] Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ϕ'[t]},
 {Cos[θ[t]] Cos[ϕ[t]] Cos[ψ[t]] θ'[t] + Cos[ψ[t]] Sin[ϕ[t]] (-Sin[θ[t]] ϕ'[t] + ψ'[t]) +
   Cos[ϕ[t]] Sin[ψ[t]] (ϕ'[t] - Sin[θ[t]] ψ'[t]), Cos[θ[t]] Cos[ϕ[t]] Sin[ψ[t]] θ'[t] -
   (Cos[ϕ[t]] Cos[ψ[t]] + Sin[θ[t]] Sin[ϕ[t]] Sin[ψ[t]]) ϕ'[t] +
   (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]) ψ'[t],
  -Cos[ϕ[t]] Sin[θ[t]] θ'[t] - Cos[θ[t]] Sin[ϕ[t]] ϕ'[t]}}

In[•]:= MatrixForm[Rdot]
Out[•]//MatrixForm=
(                                                      -Cos[ψ[t]] Sin[θ[t]] θ'[t] - Cos[θ[t]]
 Cos[θ[t]] Cos[ψ[t]] Sin[ϕ[t]] θ'[t] + (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]
                 Cos[θ[t]] Cos[ϕ[t]] Cos[ψ[t]] θ'[t] + Cos[ψ[t]] Sin[ϕ[t]] (-Sin[θ[t]] ϕ'[t] +

In[•]:= R = FullSimplify[TrigReduce[R2.Rpsid]]
Out[•]=
{{Cos[θ[t]] Cos[ψ[t]], Cos[θ[t]] Sin[ψ[t]], -Sin[θ[t]]},
 {Cos[ψ[t]] Sin[θ[t]] Sin[ϕ[t]] - Cos[ϕ[t]] Sin[ψ[t]],
  Cos[ϕ[t]] Cos[ψ[t]] + Sin[θ[t]] Sin[ϕ[t]] Sin[ψ[t]], Cos[θ[t]] Sin[ϕ[t]]},
 {Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]],
  -Cos[ψ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[θ[t]] Sin[ψ[t]], Cos[θ[t]] Cos[ϕ[t]]}}

In[•]:= MatrixForm[R]
Out[•]//MatrixForm=
(                 Cos[θ[t]] Cos[ψ[t]]                              Cos[θ[t]] Sin[ψ[t]]
 Cos[ψ[t]] Sin[θ[t]] Sin[ϕ[t]] - Cos[ϕ[t]] Sin[ψ[t]]   Cos[ϕ[t]] Cos[ψ[t]] + Sin[θ[t]] Sin[
 Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]  -Cos[ψ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[

In[•]:= FullSimplify[Rdot.Transpose[R] + R.Transpose[Rdot]]
Out[•]=
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

In[•]:= MatrixForm[{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}]
Out[•]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[•]:=
```
S = MatrixForm[FullSimplify[Rdot.Transpose[R]]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & -\mathrm{Sin}[\phi[t]]\,\theta'[t] + \mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\,\psi'[t] & -\mathrm{C} \\ \mathrm{Sin}[\phi[t]]\,\theta'[t] - \mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\,\psi'[t] & 0 & \\ \mathrm{Cos}[\phi[t]]\,\theta'[t] + \mathrm{Cos}[\theta[t]]\,\mathrm{Sin}[\phi[t]]\,\psi'[t] & -\phi'[t] + \mathrm{Sin}[\theta[t]]\,\psi'[t] & \end{pmatrix}$$

In[•]:=

In[•]:=

$$\omega = \begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix} = \begin{bmatrix} \dot{\phi}(t) - \dot{\psi}(t)Sin(\theta(t)) \\ \dot{\theta}(t)Cos(\phi(t)) + \dot{\psi}(t)Sin(\phi(t))Cos(\theta(t)) \\ -\dot{\theta}(t)Sin(\phi(t)) + \dot{\psi}Cos(\phi(t))Cos(\theta(t)) \end{bmatrix}$$

Out[•]=

$$\omega = \begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix} = \begin{bmatrix} \dot{\phi}(t) - \dot{\psi}(t)Sin(\theta(t)) \\ \dot{\theta}(t)Cos(\phi(t)) + \dot{\psi}(t)Sin(\phi(t))Cos(\theta(t)) \\ -\dot{\theta}(t)Sin(\phi(t)) + \dot{\psi}Cos(\phi(t))Cos(\theta(t)) \end{bmatrix}$$

Translational Coordinates

In[•]:=
```
ftras:={{0},{0},{f}}
fresultante=Transpose[R].ftras
```

Out[•]=
```
{{f (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]])},
 {f (-Cos[ψ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[θ[t]] Sin[ψ[t]])}, {f Cos[θ[t]] Cos[ϕ[t]]}}
```

In[•]:=
```
coor := {{x[t]}, {y[t]}, {z[t]}}
velcor := {{∂t x[t]}, {∂t y[t]}, {∂t z[t]}}
```

In[•]:=

$$\texttt{cinetictras} = \frac{1}{2} * m\ (\texttt{Transpose[velcor].velcor})$$

Out[•]=

$$\left\{\left\{\frac{1}{2}\ m\ \left(x'[t]^2 + y'[t]^2 + z'[t]^2\right)\right\}\right\}$$

In[•]:=

$$\texttt{lagrangianotras} = \frac{1}{2}\ m\ \left(x'[t]^2 + y'[t]^2 + z'[t]^2\right) - m * g * z[t];$$

In[•]:=
```
EulerEquations[lagrangianotras, {x[t], y[t], z[t]}, t]
```

Out[•]=
```
{-m x''[t] == 0, -m y''[t] == 0, -m (g + z''[t]) == 0}
```

In[•]:=
```
FullSimplify[
 Solve[-m x''[t] + f (Cos[ϕ[t]] Cos[ψ[t]] Sin[θ[t]] + Sin[ϕ[t]] Sin[ψ[t]]) == 0 , x''[t]]]
```

Out[•]=

$$\left\{\left\{x''[t] \to \frac{f\ (\mathrm{Cos}[\phi[t]]\ \mathrm{Cos}[\psi[t]]\ \mathrm{Sin}[\theta[t]] + \mathrm{Sin}[\phi[t]]\ \mathrm{Sin}[\psi[t]])}{m}\right\}\right\}$$

In[∘]:=
```
FullSimplify[
 Solve[-m y''[t] + (f (-Cos[ψ[t]] Sin[ϕ[t]] + Cos[ϕ[t]] Sin[θ[t]] Sin[ψ[t]])) == 0 , y''[t]]]
```

Out[∘]=

$$\left\{\left\{y''[t] \to \frac{f\,(-\mathrm{Cos}[\psi[t]]\,\mathrm{Sin}[\phi[t]] + \mathrm{Cos}[\phi[t]]\,\mathrm{Sin}[\theta[t]]\,\mathrm{Sin}[\psi[t]])}{m}\right\}\right\}$$

In[∘]:=
```
FullSimplify[Solve[-m (g + z''[t]) + f Cos[θ[t]] Cos[ϕ[t]] == 0 , z''[t]]]
```

Out[∘]=

$$\left\{\left\{z''[t] \to -g + \frac{f\,\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]}{m}\right\}\right\}$$

According to the variational principle that governs the Euler-Lagrange equations, if the Functional (Lagrangian) is equal to the forces generated by non-conservative potentials (as is the case in this case) or to conservative potentials (and not to linear momentum as it could also be) then the result of solving this equation is three equations, one equation for each degree of freedom of the system, as seen below:
The accelerations are:

$$x'' = \frac{f}{m}\,(\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] + \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi])$$

$$y'' = \frac{f}{m}(-\mathrm{Cos}[\psi]\ \mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\psi])$$

$$z'' = \frac{f}{m}\mathrm{Cos}[\theta]\ \mathrm{Cos}[\phi] - g$$

The equations that describe the translational position of the Quad-rotor are now shown:

$$X[t] \to c_1 + t\,c_2 + \frac{f\,t^2\,\mathrm{Cos}[\phi-\psi]}{4\,m} - \frac{f\,t^2\,\mathrm{Cos}[\phi+\psi]}{4\,m} + \frac{f\,t^2\,\mathrm{Sin}[\theta-\phi-\psi]}{8\,m} + \frac{f\,t^2\,\mathrm{Sin}[\theta+\phi-\psi]}{8\,m} + \frac{f\,t^2\,\mathrm{Sin}[\theta-\phi+\psi]}{8\,m} + \frac{f\,t^2\,\mathrm{Sin}[\theta+\phi+\psi]}{8\,m}$$

$$Y[t] \to c_1 + t\,c_2 + \frac{f\,t^2\,\mathrm{Cos}[\theta-\phi-\psi]}{8\,m} + \frac{f\,t^2\,\mathrm{Cos}[\theta+\phi-\psi]}{8\,m} - \frac{f\,t^2\,\mathrm{Cos}[\theta-\phi+\psi]}{8\,m} - \frac{f\,t^2\,\mathrm{Cos}[\theta+\phi+\psi]}{8\,m} - \frac{f\,t^2\,\mathrm{Sin}[\phi-\psi]}{4\,m} - \frac{f\,t^2\,\mathrm{Sin}[\phi+\psi]}{4\,m}$$

$$Z[t] \to -\frac{g\,t^2}{2\,m} + c_1 + t\,c_2 + \frac{f\,t^2\,\mathrm{Cos}[\theta-\phi]}{4\,m} + \frac{f\,t^2\,\mathrm{Cos}[\theta+\phi]}{4\,m}$$

Rotational Coordinates

Defining the parameters of the Lagrangian as:

Out[∘]=

```
ClearAll
```

Where we define:

$\tau_{M\,i}$ the sum of the aerodynamic torques of the four rotors

$$\tau_\psi = \sum_{i=1}^{4} \tau_{M\,i} = K_d \cdot \left(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2\right)$$

"τ" as the vector of corresponding moments in the direction of the angles of the frame of the body

$$\tau = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} l\cdot(f_4 - f_2) \\ l\cdot(f_3 - f_1) \\ \sum_{i=1}^{4} \tau_{Mi} \end{bmatrix}$$

"l" is the distance between the rotor and the center of mass of the quadrotor

"f_i" is the thrust force generated by each of the rotors

"ω" as the angular velocity transformation matrix

"Ω" as the vector of angular velocities, to "mI" as the inertia tensor of the quadrotor

"η" as the rotational coordinate vector that is defined as : $\{\phi, \theta, \varphi\}^T$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\Omega = \omega_n\, \dot{\eta}$$

$$T_{\text{rot}} = \frac{1}{2}\, I\, \omega^2$$

Where I is the Inertia tensor

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Now the Euler-Lagrange equations for rotational motion are:

$$\tau = \frac{d}{\text{dt}} \left(\frac{\partial L\,(\eta,\eta')}{\partial \eta'}\right) - \frac{\partial L\,(\eta,\eta')}{\partial \eta}$$

$$\omega = \begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix} = \begin{bmatrix} \dot{\phi}(t) - \dot{\psi}(t)Sin(\theta(t)) \\ \dot{\theta}(t)Cos(\phi(t)) + \dot{\psi}(t)Sin(\phi(t))Cos(\theta(t)) \\ -\dot{\theta}(t)Sin(\phi(t)) + \dot{\psi}Cos(\phi(t))Cos(\theta(t)) \end{bmatrix}$$

Rotational coordinates

In[•]:=
```
cΩ := {ϕ'[t] - ψ'[t] Sin[θ[t]], θ'[t] Cos[ϕ[t]] + ψ'[t] Sin[ϕ[t]] Cos[θ[t]],
  -θ'[t] Sin[ϕ[t]] + ψ'[t] Cos[ϕ[t]] Cos[θ[t]]}
```

In[•]:=
```
cΩt := {{ϕ'[t] - ψ'[t] Sin[θ[t]], θ'[t] Cos[ϕ[t]] + ψ'[t] Sin[ϕ[t]] Cos[θ[t]],
   -θ'[t] Sin[ϕ[t]] + ψ'[t] Cos[ϕ[t]] Cos[θ[t]]}}
```

In[•]:=
```
minercia := {{i_xx, 0, 0}, {0, i_yy, 0}, {0, 0, i_zz}}
```

In[•]:=
$$\text{lagrangianorota} = \frac{1}{2}\,\text{cΩt.minercia.cΩ}$$

Out[•]=
$$\left\{\frac{1}{2}\left(i_{zz}\,(-\text{Sin}[\phi[t]]\,\theta'[t]+\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]\,\psi'[t])^2+i_{xx}\,(\phi'[t]-\text{Sin}[\theta[t]]\,\psi'[t])^2+i_{yy}\,(\text{Cos}[\phi[t]]\,\theta'[t]+\text{Cos}[\theta[t]]\,\text{Sin}[\phi[t]]\,\psi'[t])^2\right)\right\}$$

In[•]:=
```
Needs["VariationalMethods`"]
```

In[•]:=
$$\text{EulerEquations}\Big[\frac{1}{2}\left(i_{zz}\,(-\text{Sin}[\phi[t]]\,\theta'[t]+\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]\,\psi'[t])^2+i_{xx}\,(\phi'[t]-\text{Sin}[\theta[t]]\,\psi'[t])^2+i_{yy}\,(\text{Cos}[\phi[t]]\,\theta'[t]+\text{Cos}[\theta[t]]\,\text{Sin}[\phi[t]]\,\psi'[t])^2\right),\ \{\phi[t],\theta[t],\psi[t]\},\ t\Big]$$

Out[•]=
$$\Big\{i_{zz}\,(\text{Sin}[\phi[t]]\,\theta'[t]-\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]\,\psi'[t])\,(\text{Cos}[\phi[t]]\,\theta'[t]+\text{Cos}[\theta[t]]\,\text{Sin}[\phi[t]]\,\psi'[t])+i_{yy}\,(-\text{Sin}[\phi[t]]\,\theta'[t]+\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]\,\psi'[t])\,(\text{Cos}[\phi[t]]\,\theta'[t]+\text{Cos}[\theta[t]]\,\text{Sin}[\phi[t]]\,\psi'[t])+i_{xx}\,(\text{Cos}[\theta[t]]\,\theta'[t]\,\psi'[t]-\phi''[t]+\text{Sin}[\theta[t]]\,\psi''[t])==0,$$
$$\text{Cos}[\theta[t]]\,i_{xx}\,\psi'[t]\,(-\phi'[t]+\text{Sin}[\theta[t]]\,\psi'[t])+i_{yy}\left(\text{Sin}[2\phi[t]]\,\theta'[t]\,\phi'[t]-\text{Cos}[\theta[t]]\,\text{Cos}[2\phi[t]]\,\phi'[t]\,\psi'[t]-\text{Cos}[\theta[t]]\,\text{Sin}[\theta[t]]\,\text{Sin}[\phi[t]]^2\,\psi'[t]^2-\text{Cos}[\phi[t]]^2\,\theta''[t]-\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]\,\text{Sin}[\phi[t]]\,\psi''[t]\right)-i_{zz}\left(\text{Sin}[2\phi[t]]\,\theta'[t]\,\phi'[t]-\text{Cos}[\theta[t]]\,\text{Cos}[2\phi[t]]\,\phi'[t]\,\psi'[t]+\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]^2\,\text{Sin}[\theta[t]]\,\psi'[t]^2+\text{Sin}[\phi[t]]^2\,\theta''[t]-\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]\,\text{Sin}[\phi[t]]\,\psi''[t]\right)==0,$$
$$i_{zz}\left(-\text{Cos}[\phi[t]]\,\text{Sin}[\theta[t]]\,\text{Sin}[\phi[t]]\,\theta'[t]^2+\theta'[t]\left(\text{Cos}[\theta[t]]\,\text{Cos}[2\phi[t]]\,\phi'[t]+\text{Cos}[\phi[t]]^2\,\text{Sin}[2\theta[t]]\,\psi'[t]\right)+\text{Cos}[\theta[t]]\left(\text{Cos}[\theta[t]]\,\text{Sin}[2\phi[t]]\,\phi'[t]\,\psi'[t]+\text{Cos}[\phi[t]]\,\text{Sin}[\phi[t]]\,\theta''[t]-\text{Cos}[\theta[t]]\,\text{Cos}[\phi[t]]^2\,\psi''[t]\right)\right)+i_{xx}\,(\theta'[t]\,(\text{Cos}[\theta[t]]\,\phi'[t]-\text{Sin}[2\theta[t]]\,\psi'[t])+\text{Sin}[\theta[t]]\,(\phi''[t]-\text{Sin}[\theta[t]]\,\psi''[t]))+i_{yy}\left(\text{Cos}[\phi[t]]\,\text{Sin}[\theta[t]]\,\text{Sin}[\phi[t]]\,\theta'[t]^2+\theta'[t]\left(-\text{Cos}[\theta[t]]\,\text{Cos}[2\phi[t]]\,\phi'[t]+\text{Sin}[2\theta[t]]\,\text{Sin}[\phi[t]]^2\,\psi'[t]\right)-\text{Cos}[\theta[t]]\left(\text{Cos}[\theta[t]]\,\text{Sin}[2\phi[t]]\,\phi'[t]\,\psi'[t]+\text{Cos}[\phi[t]]\,\text{Sin}[\phi[t]]\,\theta''[t]+\text{Cos}[\theta[t]]\,\text{Sin}[\phi[t]]^2\,\psi''[t]\right)\right)==0\Big\}$$

In[ ]:=
```
VariationalD[
 1/2 (izz (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])^2 + ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])^2 +
   iyy (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t])^2), {ϕ[t], θ[t], ψ[t]}, t]
```

Out[ ]=
```
{izz (Sin[ϕ[t]] θ'[t] - Cos[θ[t]] Cos[ϕ[t]] ψ'[t])
    (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t]) +
  iyy (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])
    (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t]) +
  ixx (Cos[θ[t]] θ'[t] ψ'[t] - ϕ''[t] + Sin[θ[t]] ψ''[t]),
 Cos[θ[t]] ixx ψ'[t] (-ϕ'[t] + Sin[θ[t]] ψ'[t]) +
  iyy (Sin[2 ϕ[t]] θ'[t] ϕ'[t] - Cos[θ[t]] Cos[2 ϕ[t]] ϕ'[t] ψ'[t] - Cos[θ[t]] Sin[θ[t]]
      Sin[ϕ[t]]^2 ψ'[t]^2 - Cos[ϕ[t]]^2 θ''[t] - Cos[θ[t]] Cos[ϕ[t]] Sin[ϕ[t]] ψ''[t]) -
  izz (Sin[2 ϕ[t]] θ'[t] ϕ'[t] - Cos[θ[t]] Cos[2 ϕ[t]] ϕ'[t] ψ'[t] + Cos[θ[t]] Cos[ϕ[t]]^2
      Sin[θ[t]] ψ'[t]^2 + Sin[ϕ[t]]^2 θ''[t] - Cos[θ[t]] Cos[ϕ[t]] Sin[ϕ[t]] ψ''[t]),
 izz (-Cos[ϕ[t]] Sin[θ[t]] Sin[ϕ[t]] θ'[t]^2 + θ'[t] (Cos[θ[t]] Cos[2 ϕ[t]] ϕ'[t] +
        Cos[ϕ[t]]^2 Sin[2 θ[t]] ψ'[t]) + Cos[θ[t]] (Cos[θ[t]] Sin[2 ϕ[t]] ϕ'[t] ψ'[t] +
        Cos[ϕ[t]] Sin[ϕ[t]] θ''[t] - Cos[θ[t]] Cos[ϕ[t]]^2 ψ''[t])) +
  ixx (θ'[t] (Cos[θ[t]] ϕ'[t] - Sin[2 θ[t]] ψ'[t]) + Sin[θ[t]] (ϕ''[t] - Sin[θ[t]] ψ''[t])) +
  iyy (Cos[ϕ[t]] Sin[θ[t]] Sin[ϕ[t]] θ'[t]^2 +
     θ'[t] (-Cos[θ[t]] Cos[2 ϕ[t]] ϕ'[t] + Sin[2 θ[t]] Sin[ϕ[t]]^2 ψ'[t]) -
     Cos[θ[t]] (Cos[θ[t]] Sin[2 ϕ[t]] ϕ'[t] ψ'[t] +
        Cos[ϕ[t]] Sin[ϕ[t]] θ''[t] + Cos[θ[t]] Sin[ϕ[t]]^2 ψ''[t]))}
```

In[ ]:=

$$\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\phi}} = I_{xx}\left(\dot{\phi}-\dot{\psi}\sin\theta\right)$$

$$\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\theta}} = \dot{\theta}\left(I_{yy}\cos^2\phi+I_{zz}\sin^2\phi\right)+\dot{\psi}\left(I_{yy}\cos\phi\sin\phi\cos\theta-I_{zz}\cos\phi\sin\phi\cos\theta\right)$$

$$\begin{aligned}\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\psi}} &= -\dot{\phi}I_{xx}\sin\theta+\dot{\theta}\left(\left(I_{yy}-I_{zz}\right)\cos\phi\sin\phi\cos\theta\right)\\ &+\dot{\psi}I_{xx}\sin^2\theta+\dot{\psi}I_{yy}\sin^2\phi\cos^2\theta+\dot{\psi}I_{zz}\cos^2\phi\cos^2\theta\end{aligned}$$

Out[ ]=

$$\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\phi}} = I_{xx}\left(\dot{\phi}-\dot{\psi}\sin\theta\right)$$

Out[ ]=

$$\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\theta}} = \dot{\theta}\left(I_{yy}\cos^2\phi+I_{zz}\sin^2\phi\right)+\dot{\psi}\left(I_{yy}\cos\phi\sin\phi\cos\theta-I_{zz}\cos\phi\sin\phi\cos\theta\right)$$

Out[ ]=

$$\begin{aligned}\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\psi}} &= -\dot{\phi}I_{xx}\sin\theta+\dot{\theta}\left(\left(I_{yy}-I_{zz}\right)\cos\phi\sin\phi\cos\theta\right)\\ &+\dot{\psi}I_{xx}\sin^2\theta+\dot{\psi}I_{yy}\sin^2\phi\cos^2\theta+\dot{\psi}I_{zz}\cos^2\phi\cos^2\theta\end{aligned}$$

```
In[ ]:= dltrasphip = FullSimplify[
          D[1/2 (izz (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])^2 + ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])^2 +
               iyy (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t])^2), ϕ'[t]]]


        dltraspsYp = FullSimplify[
          D[1/2 (izz (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])^2 + ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])^2 +
               iyy (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t])^2), ψ'[t]]]
        dltrasthetap =
         FullSimplify[
          D[1/2 (izz (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])^2 + ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])^2 +
               iyy (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t])^2), θ'[t]]]

Out[ ]=
        ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])

Out[ ]=
        Sin[θ[t]] ixx (-ϕ'[t] + Sin[θ[t]] ψ'[t]) + Cos[θ[t]]
          (Cos[ϕ[t]] Sin[ϕ[t]] (iyy - izz) θ'[t] + Cos[θ[t]] (Sin[ϕ[t]]^2 iyy + Cos[ϕ[t]]^2 izz) ψ'[t])

Out[ ]=
        (Cos[ϕ[t]]^2 iyy + Sin[ϕ[t]]^2 izz) θ'[t] + Cos[θ[t]] Cos[ϕ[t]] Sin[ϕ[t]] (iyy - izz) ψ'[t]
```

In[∘]:=

$$\frac{\partial L(\eta,\dot{\eta})}{\partial \phi} = I_{yy}\left(-\dot{\psi}\dot{\theta}\cos\theta\sin^2\phi + \dot{\psi}\dot{\theta}\cos\theta\cos^2\phi + \dot{\psi}^2\sin\phi\cos\phi\cos^2\theta - \dot{\theta}^2\sin\phi\cos\phi\right)$$
$$+I_{zz}\left(-\dot{\psi}^2\sin\phi\cos\phi\cos^2\theta + \dot{\psi}\dot{\theta}\cos\theta\sin^2\phi - \dot{\psi}\dot{\theta}\cos\theta\cos^2\phi + \dot{\theta}^2\sin\phi\cos\phi\right)$$

$$\frac{\partial L(\eta,\dot{\eta})}{\partial \theta} = I_{xx}\left(-\dot{\psi}\dot{\phi}\cos\theta + \dot{\psi}^2\cos\theta\sin\theta\right) + I_{yy}\left(-\dot{\theta}\dot{\psi}\sin\phi\cos\phi\sin\theta - \dot{\psi}^2\sin^2\phi\cos\theta\sin\theta\right)$$
$$+I_{zz}\left(-\dot{\psi}^2\sin\theta\cos\theta\cos^2\phi + \dot{\psi}\dot{\theta}\sin\theta\sin\phi\cos\phi\right)$$

$$\frac{\partial L(\eta,\dot{\eta})}{\partial \psi} = 0$$

Out[∘]=

$$\frac{\partial L(\eta,\dot{\eta})}{\partial \phi} = I_{yy}\left(-\dot{\psi}\dot{\theta}\cos\theta\sin^2\phi + \dot{\psi}\dot{\theta}\cos\theta\cos^2\phi + \dot{\psi}^2\sin\phi\cos\phi\cos^2\theta - \dot{\theta}^2\sin\phi\cos\phi\right)$$
$$+I_{zz}\left(-\dot{\psi}^2\sin\phi\cos\phi\cos^2\theta + \dot{\psi}\dot{\theta}\cos\theta\sin^2\phi - \dot{\psi}\dot{\theta}\cos\theta\cos^2\phi + \dot{\theta}^2\sin\phi\cos\phi\right)$$

Out[∘]=

$$\frac{\partial L(\eta,\dot{\eta})}{\partial \theta} = I_{xx}\left(-\dot{\psi}\dot{\phi}\cos\theta + \dot{\psi}^2\cos\theta\sin\theta\right) + I_{yy}\left(-\dot{\theta}\dot{\psi}\sin\phi\cos\phi\sin\theta - \dot{\psi}^2\sin^2\phi\cos\theta\sin\theta\right)$$
$$+I_{zz}\left(-\dot{\psi}^2\sin\theta\cos\theta\cos^2\phi + \dot{\psi}\dot{\theta}\sin\theta\sin\phi\cos\phi\right)$$

Out[∘]=

$$\frac{\partial L(\eta,\dot{\eta})}{\partial \psi} = 0$$

In[∘]:=
```
dltrasphi = FullSimplify[
  D[1/2 (izz (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])^2 + ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])^2 +
      iyy (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t])^2), ϕ[t]]]
dltraspsY = FullSimplify[
  D[1/2 (izz (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])^2 + ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])^2 +
      iyy (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t])^2), ψ[t]]]
dltrastheta = FullSimplify[
  D[1/2 (izz (-Sin[ϕ[t]] θ'[t] + Cos[θ[t]] Cos[ϕ[t]] ψ'[t])^2 + ixx (ϕ'[t] - Sin[θ[t]] ψ'[t])^2 +
      iyy (Cos[ϕ[t]] θ'[t] + Cos[θ[t]] Sin[ϕ[t]] ψ'[t])^2), θ[t]]]
```

Out[∘]=

$$(i_{yy} - i_{zz})\,(-\mathrm{Sin}[\phi[t]]\,\theta'[t] + \mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\,\psi'[t])\,(\mathrm{Cos}[\phi[t]]\,\theta'[t] + \mathrm{Cos}[\theta[t]]\,\mathrm{Sin}[\phi[t]]\,\psi'[t])$$

Out[∘]=

0

Out[∘]=

$$\frac{1}{2}\,\psi'[t]\,\Big(\mathrm{Sin}[\theta[t]]\,\mathrm{Sin}[2\,\phi[t]]\,(-i_{yy} + i_{zz})\,\theta'[t] - 2\,\mathrm{Cos}[\theta[t]]\,i_{xx}\,\phi'[t] + \mathrm{Sin}[2\,\theta[t]]\,\big(i_{xx} - \mathrm{Sin}[\phi[t]]^2\,i_{yy} - \mathrm{Cos}[\phi[t]]^2\,i_{zz}\big)\,\psi'[t]\Big)$$

In[∘]:=

$$\frac{d}{dt}\left(\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\phi}}\right) \;=\; I_{xx}\left(\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\phi}\dot{\psi}\cos\theta\right)$$

Out[∘]=

$$\frac{d}{dt}\left(\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\phi}}\right) \;=\; I_{xx}\left(\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\phi}\dot{\psi}\cos\theta\right)$$

In[∘]:=

In[∘]:=
```
dtltras = FullSimplify[D[dltrasphip, t]]
dtltras1 = FullSimplify[D[dltraspsYp, t]]
dtltras2 = FullSimplify[D[dltrasthetap, t]]
```

Out[∘]=

$$i_{xx}\,(-\mathrm{Cos}[\theta[t]]\,\theta'[t]\,\psi'[t]+\phi''[t]-\mathrm{Sin}[\theta[t]]\,\psi''[t])$$

Out[∘]=

$$\begin{aligned}&-i_{zz}\Big(-\mathrm{Cos}[\phi[t]]\,\mathrm{Sin}[\theta[t]]\,\mathrm{Sin}[\phi[t]]\,\theta'[t]^2+\\&\quad\theta'[t]\,\big(\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[2\,\phi[t]]\,\phi'[t]+\mathrm{Cos}[\phi[t]]^2\,\mathrm{Sin}[2\,\theta[t]]\,\psi'[t]\big)+\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\\&\quad\;(\mathrm{Sin}[\phi[t]]\,(2\,\mathrm{Cos}[\theta[t]]\,\phi'[t]\,\psi'[t]+\theta''[t])-\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\,\psi''[t])\Big)+\\&\;i_{xx}\,(\mathrm{Cos}[\theta[t]]\,\theta'[t]\,(-\phi'[t]+2\,\mathrm{Sin}[\theta[t]]\,\psi'[t])+\mathrm{Sin}[\theta[t]]\,(-\phi''[t]+\mathrm{Sin}[\theta[t]]\,\psi''[t]))+\\&\;i_{yy}\Big(-\mathrm{Cos}[\phi[t]]\,\mathrm{Sin}[\theta[t]]\,\mathrm{Sin}[\phi[t]]\,\theta'[t]^2+\\&\quad\theta'[t]\,\big(\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[2\,\phi[t]]\,\phi'[t]-\mathrm{Sin}[2\,\theta[t]]\,\mathrm{Sin}[\phi[t]]^2\,\psi'[t]\big)+\mathrm{Cos}[\theta[t]]\\&\quad\;\mathrm{Sin}[\phi[t]]\,(\mathrm{Cos}[\phi[t]]\,(2\,\mathrm{Cos}[\theta[t]]\,\phi'[t]\,\psi'[t]+\theta''[t])+\mathrm{Cos}[\theta[t]]\,\mathrm{Sin}[\phi[t]]\,\psi''[t])\Big)\end{aligned}$$

Out[∘]=

$$\begin{aligned}&\frac{1}{2}\,(i_{yy}\,(-\mathrm{Sin}[2\,\phi[t]]\,\theta'[t]\,(2\,\phi'[t]+\mathrm{Sin}[\theta[t]]\,\psi'[t])+\theta''[t]+\\&\quad\mathrm{Cos}[2\,\phi[t]]\,(2\,\mathrm{Cos}[\theta[t]]\,\phi'[t]\,\psi'[t]+\theta''[t])+\mathrm{Cos}[\theta[t]]\,\mathrm{Sin}[2\,\phi[t]]\,\psi''[t])+\\&\;i_{zz}\,(\theta''[t]-\mathrm{Cos}[2\,\phi[t]]\,(2\,\mathrm{Cos}[\theta[t]]\,\phi'[t]\,\psi'[t]+\theta''[t])+\\&\quad\mathrm{Sin}[2\,\phi[t]]\,(\theta'[t]\,(2\,\phi'[t]+\mathrm{Sin}[\theta[t]]\,\psi'[t])-\mathrm{Cos}[\theta[t]]\,\psi''[t])))\end{aligned}$$

In[∘]:=

$$\begin{bmatrix}\frac{\dot{d}}{dt}\left(\frac{\partial \dot{L}(\eta,\acute{\dot{\eta}})}{\partial\dot{\phi}}\right)-\frac{\partial \dot{L}(\eta,\dot{\eta})}{\partial\phi}\\ \frac{d}{dt}\left(\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\theta}}\right)-\frac{\partial L(\eta,\dot{\eta})}{\partial\theta}\\ \frac{d}{dt}\left(\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\psi}}\right)-\frac{\partial L(\eta,\dot{\eta})}{\partial\psi}\end{bmatrix}=\begin{bmatrix}\tau_\phi\\ \tau_\theta\\ \tau_\psi\end{bmatrix}$$

Out[∘]=

$$\begin{bmatrix}\frac{\dot{d}}{dt}\left(\frac{\partial \dot{L}(\eta,\acute{\dot{\eta}})}{\partial\dot{\phi}}\right)-\frac{\partial \dot{L}(\eta,\dot{\eta})}{\partial\phi}\\ \frac{d}{dt}\left(\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\theta}}\right)-\frac{\partial L(\eta,\dot{\eta})}{\partial\theta}\\ \frac{d}{dt}\left(\frac{\partial L(\eta,\dot{\eta})}{\partial\dot{\psi}}\right)-\frac{\partial L(\eta,\dot{\eta})}{\partial\psi}\end{bmatrix}=\begin{bmatrix}\tau_\phi\\ \tau_\theta\\ \tau_\psi\end{bmatrix}$$

```
FullSimplify[Solve[dtltras - dltrasphi == τϕ, ϕ''[t]] /. {θ[t] → 0, ϕ[t] → 0, ψ[t] → 0}]
FullSimplify[Solve[dtltras - dltrasphi == τϕ, ϕ''[t]]]
FullSimplify[Solve[dtltras1 - dltraspsY == τψ, ψ''[t]] /. {θ[t] → 0, ϕ[t] → 0, ψ[t] → 0}]
FullSimplify[Solve[dtltras1 - dltraspsY == τψ, ψ''[t]]]
FullSimplify[Solve[dtltras2 - dltrastheta == τθ, θ''[t]]]
FullSimplify[Solve[dtltras2 - dltrastheta == τθ, θ''[t]] /. {θ[t] → 0, ϕ[t] → 0, ψ[t] → 0}]
```

Out[ ]=

$$\left\{\left\{\phi''[t] \to \frac{\tau_\phi + (i_{xx} + i_{yy} - i_{zz})\,\theta'[t]\,\psi'[t]}{i_{xx}}\right\}\right\}$$

Out[ ]=

$$\left\{\left\{\phi''[t] \to \frac{1}{i_{xx}}\left(\tau_\phi + \mathrm{Cos}[\theta[t]]\,i_{xx}\,\theta'[t]\,\psi'[t] + (i_{yy} - i_{zz})\,(-\mathrm{Sin}[\phi[t]]\,\theta'[t] + \mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\,\psi'[t])\,(\mathrm{Cos}[\phi[t]]\,\theta'[t] + \mathrm{Cos}[\theta[t]]\,\mathrm{Sin}[\phi[t]]\,\psi'[t]) + \mathrm{Sin}[\theta[t]]\,i_{xx}\,\psi''[t]\right)\right\}\right\}$$

Out[ ]=

$$\left\{\left\{\psi''[t] \to \frac{\tau_\psi + (i_{xx} - i_{yy} + i_{zz})\,\theta'[t]\,\phi'[t]}{i_{zz}}\right\}\right\}$$

Out[ ]=

$$\left\{\left\{\psi''[t] \to \left(\tau_\psi + \frac{1}{2}\left(2\,\mathrm{Cos}[\theta[t]]\,i_{xx}\,\theta'[t]\,\phi'[t] - 2\,\mathrm{Sin}[2\,\theta[t]]\,i_{xx}\,\theta'[t]\,\psi'[t] + \mathrm{Sin}[2\,\theta[t]]\,i_{yy}\,\theta'[t]\,\psi'[t] + \mathrm{Sin}[2\,\theta[t]]\,i_{zz}\,\theta'[t]\,\psi'[t] - 2\,\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[2\,\phi[t]]\,(i_{yy} - i_{zz})\,\theta'[t]\,(\phi'[t] + \mathrm{Sin}[\theta[t]]\,\psi'[t]) + \mathrm{Sin}[2\,\phi[t]]\,(i_{yy} - i_{zz})\left(\mathrm{Sin}[\theta[t]]\,\theta'[t]^2 - \mathrm{Cos}[\theta[t]]\,(2\,\mathrm{Cos}[\theta[t]]\,\phi'[t]\,\psi'[t] + \theta''[t])\right) + 2\,\mathrm{Sin}[\theta[t]]\,i_{xx}\,\phi''[t]\right)\right) \Big/ \left(\mathrm{Sin}[\theta[t]]^2\,i_{xx} + \mathrm{Cos}[\theta[t]]^2\left(\mathrm{Sin}[\phi[t]]^2\,i_{yy} + \mathrm{Cos}[\phi[t]]^2\,i_{zz}\right)\right)\right\}\right\}$$

Out[ ]=

$$\left\{\left\{\theta''[t] \to \frac{1}{2\left(\mathrm{Cos}[\phi[t]]^2\,i_{yy} + \mathrm{Sin}[\phi[t]]^2\,i_{zz}\right)}\left(2\,\tau_\theta + 2\,\mathrm{Sin}[2\,\phi[t]]\,(i_{yy} - i_{zz})\,\theta'[t]\,\phi'[t] + \mathrm{Sin}[2\,\theta[t]]\left(i_{xx} - \mathrm{Sin}[\phi[t]]^2\,i_{yy} - \mathrm{Cos}[\phi[t]]^2\,i_{zz}\right)\psi'[t]^2 + \mathrm{Cos}[\theta[t]]\,(-2\,(i_{xx} + \mathrm{Cos}[2\,\phi[t]]\,(i_{yy} - i_{zz}))\,\phi'[t]\,\psi'[t] + \mathrm{Sin}[2\,\phi[t]]\,(-i_{yy} + i_{zz})\,\psi''[t])\right)\right\}\right\}$$

Out[ ]=

$$\left\{\left\{\theta''[t] \to \frac{\tau_\theta - (i_{xx} + i_{yy} - i_{zz})\,\phi'[t]\,\psi'[t]}{i_{yy}}\right\}\right\}$$

In[ ]:=

$$\ddot{\phi} = \frac{I_{xx} + I_{yy} - I_{zz}}{I_{xx}}\dot{\psi}\dot{\theta} + \frac{\tau_\phi}{I_{xx}}$$

In[ ]:= $$\ddot{\psi} = \frac{I_{xx} - I_{yy} + I_{zz}}{I_{zz}} \dot{\theta}\dot{\phi} + \frac{\tau_{\psi}}{I_{zz}}$$

$$\ddot{\theta} = \frac{-I_{xx} - I_{yy} + I_{zz}}{I_{yy}} \dot{\psi}\dot{\phi} + \frac{\tau_{\theta}}{I_{yy}}$$

Differential equations of the 6 degrees of freedom

translational movement

$$m\, X''[t] == f\,(\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] + \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi])$$

$$m\, Y''[t] == f\,(-\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi])$$

$$m\, Z''[t] == -g + f\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]$$

Rotational movement

$$\phi''[t] \to \frac{\tau_{\phi} + (i_{xx} + i_{yy} - i_{zz})\,\theta'[t]\,\psi'[t]}{i_{xx}}$$

$$\psi''[t] \to \frac{\tau_{\psi} + (i_{xx} - i_{yy} + i_{zz})\,\theta'[t]\,\phi'[t]}{i_{zz}}$$

$$\theta''[t] \to \frac{\tau_{\theta} - (i_{xx} + i_{yy} - i_{zz})\,\phi'[t]\,\psi'[t]}{i_{yy}}$$

In[∘]:=
```
FullSimplify[
 Solve[izz (Sin[ϕ[t]] θ′[t] - Cos[θ[t]] Cos[ϕ[t]] ψ′[t]) (Cos[ϕ[t]] θ′[t] + Cos[θ[t]]
        Sin[ϕ[t]] ψ′[t]) + iyy (-Sin[ϕ[t]] θ′[t] + Cos[θ[t]] Cos[ϕ[t]] ψ′[t])
      (Cos[ϕ[t]] θ′[t] + Cos[θ[t]] Sin[ϕ[t]] ψ′[t]) +
     ixx (Cos[θ[t]] θ′[t] ψ′[t] - ϕ″[t] + Sin[θ[t]] ψ″[t]) + τϕ == 0,
   ϕ″[t]] /. {θ[t] → 0, ϕ[t] → 0, ψ[t] → 0}]
```

Out[∘]=

$$\left\{\left\{\phi''[t] \to \frac{\tau_\phi + (i_{xx} + i_{yy} - i_{zz})\, \theta'[t]\, \psi'[t]}{i_{xx}}\right\}\right\}$$

In[∘]:=
```
FullSimplify[
 Solve[izz (-Cos[ϕ[t]] Sin[θ[t]] Sin[ϕ[t]] θ′[t]^2 + θ′[t] (Cos[θ[t]] Cos[2 ϕ[t]] ϕ′[t] +
          Cos[ϕ[t]]^2 Sin[2 θ[t]] ψ′[t]) + Cos[θ[t]] (Cos[θ[t]] Sin[2 ϕ[t]] ϕ′[t] ψ′[t] +
          Cos[ϕ[t]] Sin[ϕ[t]] θ″[t] - Cos[θ[t]] Cos[ϕ[t]]^2 ψ″[t])) + ixx
      (θ′[t] (Cos[θ[t]] ϕ′[t] - Sin[2 θ[t]] ψ′[t]) + Sin[θ[t]] (ϕ″[t] - Sin[θ[t]] ψ″[t])) +
     iyy (Cos[ϕ[t]] Sin[θ[t]] Sin[ϕ[t]] θ′[t]^2 +
        θ′[t] (-Cos[θ[t]] Cos[2 ϕ[t]] ϕ′[t] + Sin[2 θ[t]] Sin[ϕ[t]]^2 ψ′[t]) -
        Cos[θ[t]] (Cos[θ[t]] Sin[2 ϕ[t]] ϕ′[t] ψ′[t] + Cos[ϕ[t]] Sin[ϕ[t]] θ″[t] +
          Cos[θ[t]] Sin[ϕ[t]]^2 ψ″[t])) + τψ == 0, ψ″[t]] /. {θ[t] → 0, ϕ[t] → 0, ψ[t] → 0}]
```

Out[∘]=

$$\left\{\left\{\psi''[t] \to \frac{\tau_\psi + (i_{xx} - i_{yy} + i_{zz})\, \theta'[t]\, \phi'[t]}{i_{zz}}\right\}\right\}$$

In[∘]:=
```
FullSimplify[
 Solve[Cos[θ[t]] ixx ψ′[t] (-ϕ′[t] + Sin[θ[t]] ψ′[t]) + iyy (Sin[2 ϕ[t]] θ′[t] ϕ′[t] -
        Cos[θ[t]] Cos[2 ϕ[t]] ϕ′[t] ψ′[t] - Cos[θ[t]] Sin[θ[t]] Sin[ϕ[t]]^2 ψ′[t]^2 -
        Cos[ϕ[t]]^2 θ″[t] - Cos[θ[t]] Cos[ϕ[t]] Sin[ϕ[t]] ψ″[t]) -
     izz (Sin[2 ϕ[t]] θ′[t] ϕ′[t] - Cos[θ[t]] Cos[2 ϕ[t]] ϕ′[t] ψ′[t] + Cos[θ[t]] Cos[ϕ[t]]^2
         Sin[θ[t]] ψ′[t]^2 + Sin[ϕ[t]]^2 θ″[t] - Cos[θ[t]] Cos[ϕ[t]] Sin[ϕ[t]] ψ″[t]) +
     τθ == 0, θ″[t]] /. {θ[t] → 0, ϕ[t] → 0, ψ[t] → 0}]
```

Out[∘]=

$$\left\{\left\{\theta''[t] \to \frac{\tau_\theta - (i_{xx} + i_{yy} - i_{zz})\, \phi'[t]\, \psi'[t]}{i_{yy}}\right\}\right\}$$