Deep Learning - Homework 2

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1 Question 1

1.1 Question 1.1

Using a single attention head, the output Z is given by:

$$Z = \text{softmax}\left(\frac{QK^T}{\sqrt{d_k}}\right)V \tag{1}$$

The complexity of computing this Z is the complexity of computing all the matrix multiplications and the softmax.

Starting with the dimensions of each matrix, we have: $Q \in \mathbb{R}^{L\times D}, K \in \mathbb{R}^{L\times D}, V \in \mathbb{R}^{L\times D}$

The complexity of computing a matrix product AB where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ is $O(mnp)$. We can prove this since to compute the element c_{ij} of the matrix $C = AB$ we need to compute the dot product of the i-th row of A with the j-th column of B, and this dot product has complexity $O(n)$, because it is the sum of n products of real numbers. Since we need to compute mn elements of C, the complexity of computing C is $O(mnp)$.

Let's consider $P = softmax\left(\frac{QK^T}{\sqrt{d_k}}\right)$. With this P we can compute $Z = PV$.

In our case, we first need to compute QK^T and since $Q \in \mathbb{R}^{L \times D}$ and $K^T \in \mathbb{R}^{D \times L}$, the complexity of this operation is $O(L^2 D)$.

Then, we need to divide each element of QK^T by $\sqrt{d_k}$, which has complexity $O(L^2)$. Finally, we need to compute the softmax of each row of the matrix $QK^T/\sqrt{d_k}$, which has complexity $O(L^2)$. The complexity of computing P is $O(L^2 D + L^2 + L^2) = O(L^2 D)$. Since $P = softmax\left(\frac{QK^T}{\sqrt{d_k}}\right)$, $P \in \mathbb{R}^{L \times L}$.

Lastly we need to compute the matrix product PV. Since $P \in \mathbb{R}^{L \times L}$ and $V \in \mathbb{R}^{L \times D}$, the complexity of this operation is $O(L^2 D)$.

With this the final complexity of computing Z is $O(L^2 D)$, since the complexity of computing P is $O(L^2 D)$ and the complexity of computing PV is $O(L^2 D)$.

Let's consider the numebr of hidden units (D) is fixed, so the complexity of computing Z is $O(L^2)$.

This may cause a problem for long sequences of text, since the complexity of computing Z is $O(L^2)$, where L is the length of the sequence. This means that the complexity of computing Z is quadratic in the length of the sequence, which is not good for long sequence inputs.

1.2 Question 1.2

For this exercise we will use the McLaurin series expansion of the exponential function to approximate the softmax and reduce the computational computational complexity. The McLaurin series expansion of the exponential function is given by:

$exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

First, considering $exp(t) \approx 1 + t + \frac{t^2}{2}$, we want to create a feature map $\phi : \mathbb{R}^D \to \mathbb{R}^M$ such that, for arbitrary $q \in \mathbb{R}^D$ and $k \in \mathbb{R}^D$ we have $exp(q^T k) \approx \phi(q)^T \phi(k)$.

With this, we want to find a mapping ϕ such that: $\phi(q)^T \phi(k) = 1 + q^T k + \frac{(q^T k)^2}{2}$

The first two terms of the series are trivial, since we can define $\phi(q) = [1, q_1, \ldots, q_n]$. For the third term, we need to decompose the square of the dot product of q and k into a sum of products of the elements of q and k.

For vectors x and z with the same dimension n, we have that $x^T z = \sum_{i=1}^{D} x_i z_i$. Now for the square of the doct product we have:

$$
\begin{aligned}
&(x^T z)^2 = (x_1 z_1 + \cdots + x_n)(x_1 z_1 + \cdots + x_n z_n) = \\
&(x_1 z_1)^2 + 2x_1 z_1 x_2 z_2 + \cdots + 2x_1 z_1 x_n z_n + \\
&+ (x_2 z_2)^2 + \cdots + x_2 z_2 x_n z_n + \\
&\cdots + \\
&(x_n z_n)^2 = \\
&\sum_{i=1}^{n} (x_i)^2 (z_i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i z_i x_j z_j
\end{aligned}
$$

This way we can define $\phi(x) = [1, x_1, \ldots, x_n, \frac{1}{\sqrt{2}} x_1^2, x_1 x_2, \ldots, x_1 x_n, \frac{1}{\sqrt{2}} x_2^2, \ldots, x_2 x_n, \ldots, x_{n-1} x_n, \frac{1}{\sqrt{2}} x_n^2]$

With this $exp(q^T k) \approx \phi(q)^T \phi(k)$, and we can use this to approximate the softmax function.

In terms of dimensionality, if the vector x has dimension D, the first two terms of the series will have dimension $D + 1$. From the third term, we can see that the number of terms will be $\sum_{i=1}^{D} i = \frac{D(D+1)}{2}$. With this, for $K = 2$, the vector $\phi(x)$ will have dimension $M = 1 + D + \frac{D(D+1)}{2}$.

Now we want to acess what would be the dimensionality of the feature space M if we used the McLaurin series with $K \geq 3$ terms. For this, we have to acess the dimensionality of each term.

According to the multinomial theorem:
$(x_1 + x_2 + \cdots + x_D)^K = \sum_{k_1+k_2+\cdots+k_D=K, k_1,k_2,\ldots,k_D \geq 0} \binom{n}{k_1,k_2,\ldots,k_D} \prod_{t=1}^{D} x_t^{k_t}$
where $\binom{n}{k_1,k_2,\ldots,k_D} = \frac{n!}{k_1!k_2!\ldots k_D!}$

According to this theorem, the number of multinomial coefficients is given by $\binom{K+D-1}{D-1}$, and thus, the number of terms in the expansion for the $K-th$ is $\binom{K+D-1}{D-1}$.

With this, for $K \geq 3$, the dimensionality of the feature space will be:
$M = \sum \binom{K+D-1}{D-1}$

1.3 Question 1.3

In the previous exercise, we defined the feature map $\phi : \mathbb{R}^{\mathbb{D}} \to \mathbb{R}^{\mathbb{M}}$, such that $exp(q^T k) \approx \phi(q)^T \phi(k)$. Now let's consider the mapping Φ where $\Phi(X)$, which results in a matrix whose rows are $\phi(x_i)$, where x_i is the i-th row of X.

Our goal is to show that the self-attention operation can be approximated as $Z \approx D^{-1}\Phi(Q)\Phi(K)^T V$, where $D = Diag(\Phi(Q)\Phi(K)^T 1_L)$.

Looking at the original self-attention operation, we can see that the diffence is that now we want to approximate $softmax(QK^T)$ as $D^{-1}\Phi(Q)\Phi(K)^T$.

Considering $softmax(QK^T)_{ij} = \frac{exp(q_i^T k_j)}{\sum_{l=1}^{L} exp(q_i^T k_l)}$, since we want $softmax(QK^T) \approx D^{-1}\Phi(Q)\Phi(K)^T$, we can see that D^{-1} will correspond to the denominator of the softmax operation, and $\Phi(Q)\Phi(K)^T$ will correspond to the numerator.

Let's focus on the $\Phi(Q)\Phi(K)^T$ part. We know that this is the same as:

$$\Phi(Q)\Phi(K)^T = \begin{bmatrix} ---\phi(q_1)--- \\ ---\phi(q_2)--- \\ \vdots \\ ---\phi(q_L)--- \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ \phi(k_1) & \phi(k_2) & \cdots & \phi(k_L) \\ | & | & \cdots & | \end{bmatrix}$$

Since $\phi(q_i)$ is spread along the i-th row, for the (i,j)-th element of the matrix $\Phi(Q)\Phi(K)^T$ we have that cell i,j is equal to $phi(q_i)_1\phi(k_j)_1 + phi(q_i)_2\phi(k_j)_2 + \cdots + phi(q_i)_M\phi(k_j)_M = \phi(q_i)^T\phi(k_j)$.

With this we can see that $\Phi(Q)\Phi(K)^T$ is a matrix whose (i,j)-th element is $\phi(q_i)^T\phi(k_j)$.

We know that $\phi(q)^T\phi(k) \approx exp(q^T k)$, so we can see that $\Phi(Q)\Phi(K)^T$ is a matrix whose (i,j)-th element is an approximation $exp(q_i^T k_j)$.

Now let's look at D. We know that $D = Diag(\Phi(Q)\Phi(K)^T 1_L)$. Since $\Phi(Q)\Phi(K)^T$ is a matrix whose (i,j)-th element is an approximation $exp(q_i^T k_j)$, and 1_L is a vector of ones, the product $\Phi(Q)\Phi(K)^T 1_L$ will be a vector whose i-th element is an approximation of $\sum_{j=1}^{L} exp(q_i^T k_j)$, since it is the sum of the i-th row of $\Phi(Q)\Phi(K)^T$.

With this, we can see that $D \in \mathbb{R}^{L \times L}$ is a diagonal matrix whose (i,i)-th element is an approximation of $\sum_{j=1}^{L} exp(q_i^T k_j)$. Since this is a diagonal matrix, and the inverse of a diagonal matrix is a diagonal matrix whose (i,i)-th element is the inverse of the (i,i)-th element we can see that D^{-1} is a diagonal matrix whose (i,i)-th element is an approximation of $\frac{1}{\sum_{j=1}^{L} exp(q_i^T k_j)}$.

Now that we have D^{-1} and $\Phi(Q)\Phi(K)^T$, we can see that $D^{-1}\Phi(Q)\Phi(K)^T$ is a matrix whose (i,j)-th element is an approximation of the softmax operation, which is what we wanted to show. With this, we can see that $Z = softmax(QK^T)V \approx D^{-1}\Phi(Q)\Phi(K)^T V$.

1.4 Question 1.4

2 Question 2

2.1 Question 2.1

After running the code, the best configuration was for the learning rate of 0.01. The following plots were generated:

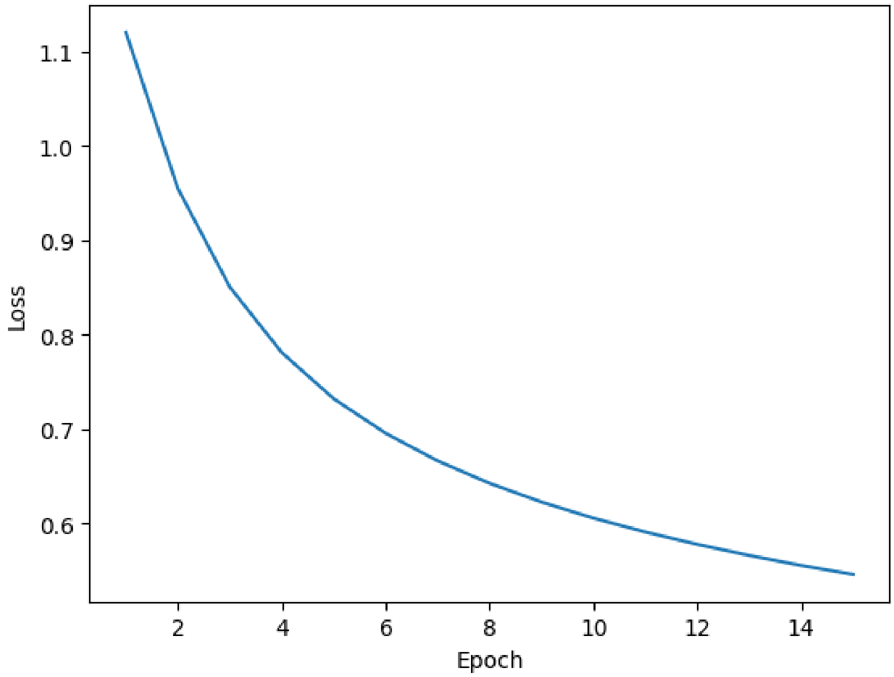

Figure 1: Training loss for $\eta = 0.01$

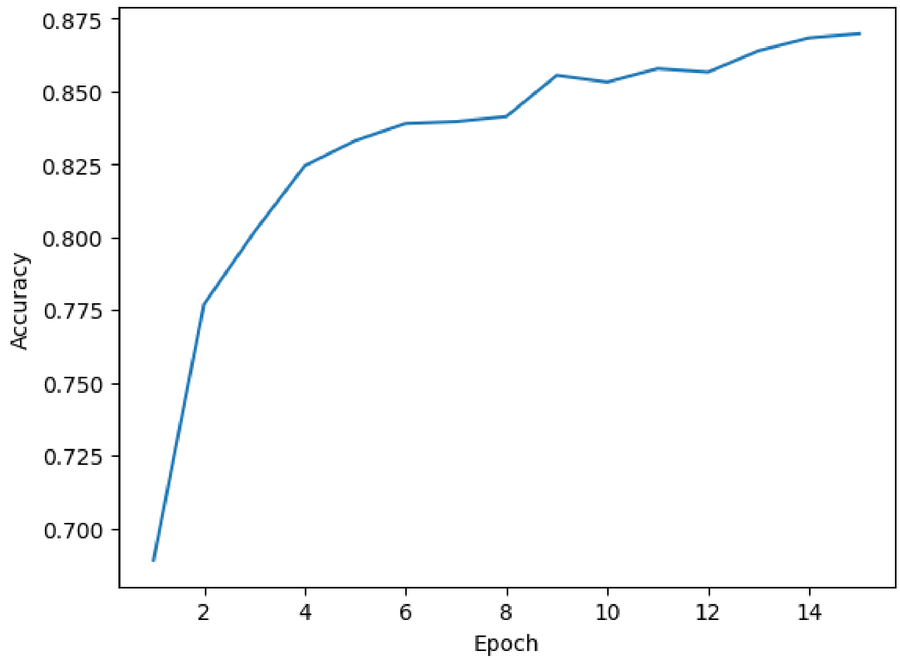

Figure 2: Validation accuracy for $\eta = 0.01$

The final test accuracy was 0.8280.

2.2 Question 2.2

The performance of this network was slightly worse than the previous one, having achieved a final test accuracy of 0.8147.

2.3 Question 2.3

Both network present the same number of parameters, 224892. The difference in performance between the two networks, resides in the use of max pooling layers. Max pooling can help the network focusing on the most important features, making the network more robust to small changes in the input. Furthermore, max pooling can also help with overfitting. In our case, the use of max pooling layers helped the network to achieve a better test accuracy results.

3 Question 3

3.1 Question 3.1

After running the code, the following plots were generated:

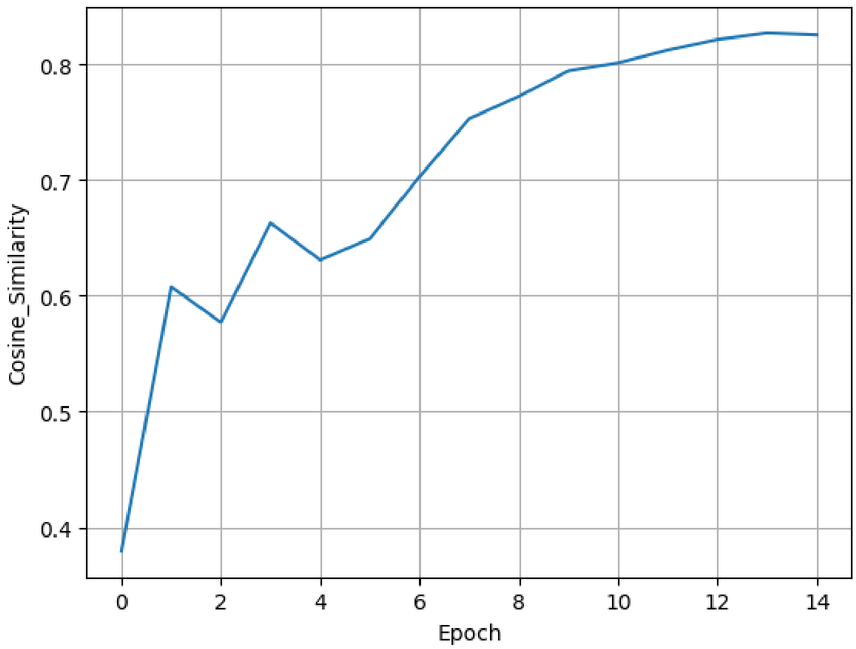

Figure 3: Cosine similarity

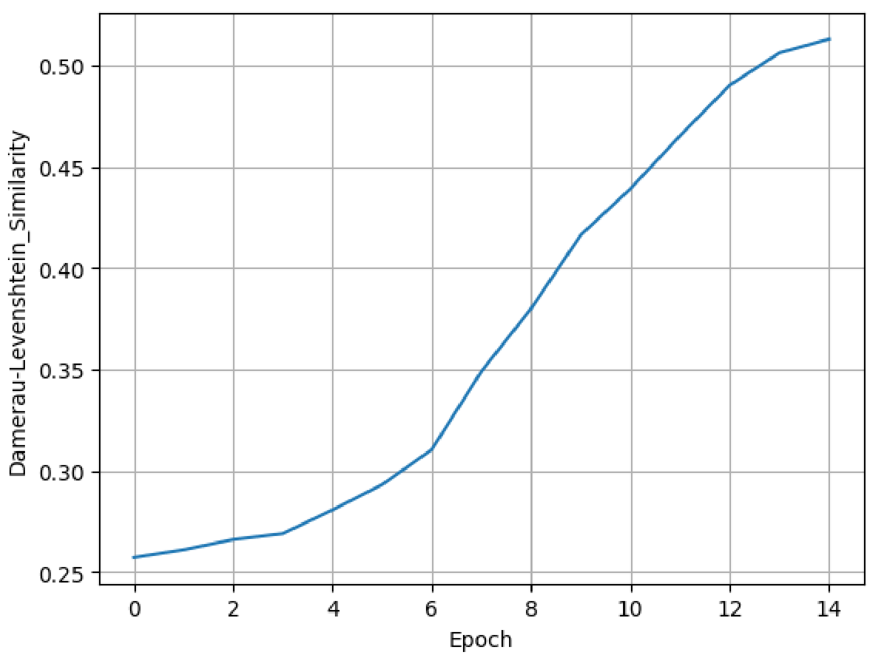

Figure 4: Damereau-Levenshtein similarity

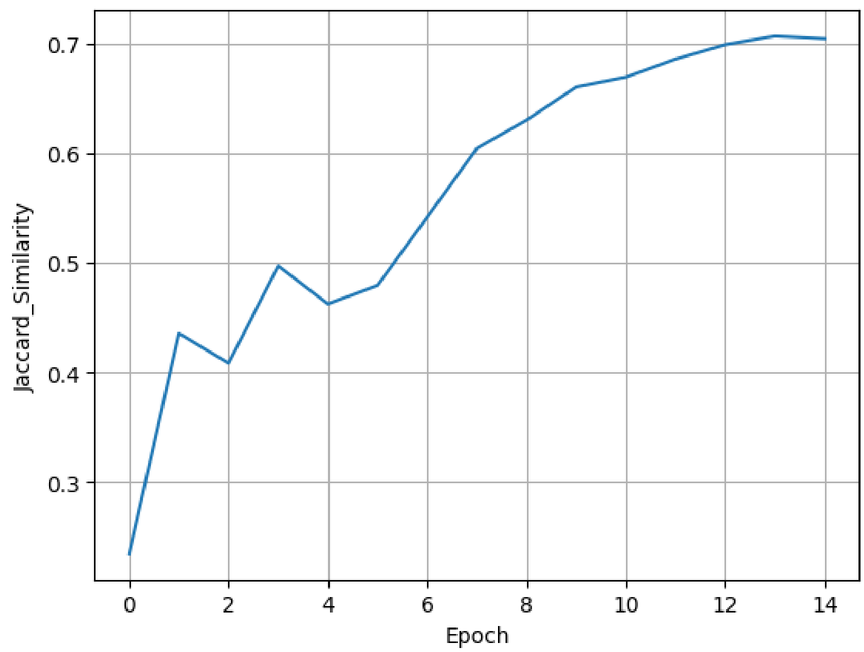

Figure 5: Jaccard similarity

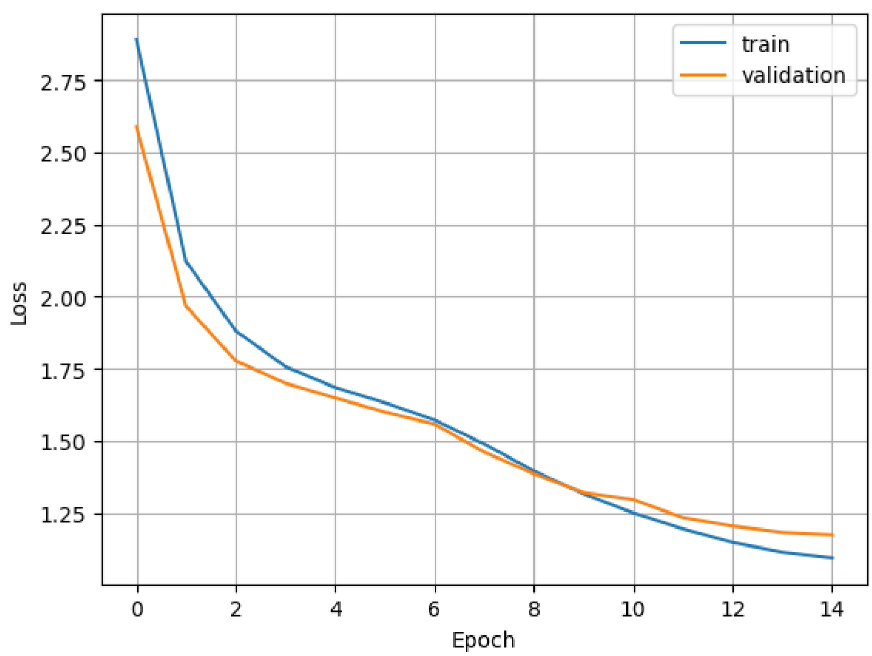

Figure 6: Validation and training loss

In the test set, the jaccard similarity was 0.715, the cosine similarity was 0.832 and the damereau-levenshtein similarity was 0.509. The final test loss was 1.183.

3.2 Question 3.2

3.3 Question 3.3

3.4 Question 3.4

4 Credits

5 Sources

- What is a max pooling layer in CNN?