Planing, Learning and Intelligent Decision Making - Homework 1

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1 Question 1

1 a)

Using $\mathcal{X}$ as the state space, $\mathcal{X} = \{A, B, C\}$, where each state represents its corresponding document.

The transition matrix, P, is given by $\begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$.

Where the first row represents the transition probabilities from state A, the second row from state B and the third row from state C. Each column represents the transition probabilities to state A, B and C, respectively.

The diagram of the Markov chain is given by

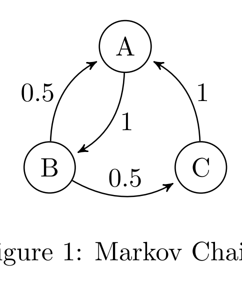

Figure 1: Markov Chain

1 b)

For state A, we have 2 possible paths to reach state A again, $A \to B \to A$ and $A \to B \to C \to A$. With the transition matrix, we can calculate the probability of each path.

Using x_t to represent the state at time t.

The probability of the first path is $P(x_1 = B|x_0 = A) * P(x_2 = A|x_1 = B) = 1 * 0.5 = 0.5$.

The probability of the second path is $P(x_1 = B|x_0 = A) * P(x_2 = C|x_1 = B) * P(x_3 = A|x_2 = C) = 1 * 0.5 * 1 = 0.5$.

Since the first path takes 2 steps and the second path takes 3 steps, $T_{AA} = 0.5 * 2 + 0.5 * 3 = 2.5$.

For state B, we have 2 possible paths to reach state B again, $B \to A \to B$ and $B \to C \to A \to B$. With the transition matrix, we can calculate the probability of each path.

The probability of the first path is $P(x_1 = A|x_0 = B) * P(x_2 = B|x_1 = A) = 0.5 * 1 = 0.5$. The probability of the second path is $P(x_1 = C|x_0 = B) * P(x_2 = A|x_1 = C) * P(x_3 = B|x_2 = A) = 0.5 * 1 * 1 = 0.5$.

Since the first path takes 2 steps and the second path takes 3 steps, $T_{BB} = 0.5 * 2 + 0.5 * 3 = 2.5$.

For state C, we have infinite possible paths to reach state C again, since the bot can stay transitioning between states A and B indefinitely.

The shortest path is $C \to A \to B \to C$, with 3 steps. The probability of this path is $P(x_1 = A|x_0 = C) * P(x_2 = B|x_1 = A) * P(x_3 = C|x_2 = B) = 1 * 0.5 * 1 = 0.5$.

The second shortest path is $C \to A \to B \to A \to B \to C$, with 5 steps. The probability of this path is $P(x_1 = A|x_0 = C) * P(x_2 = B|x_1 = A) * P(x_3 = A|x_2 = B) * P(x_4 = B|x_3 = A) * P(x_5 = C|x_4 = B) = 1 * 1 * 0.5 * 1 * 0.5 = 0.25$.

To calculate the average number of steps, we need to calculate $\sum_{n=1}^{\infty}(2n + 1)/2^n$, where $2n + 1$ is the number of steps and $1/2^n$ is the probability of the path. n represents the number of times we arrive at state B.

We can split the sum into two parts, $\sum_{n=1}^{\infty} 2n/2^n$ and $\sum_{n=1}^{\infty} 1/2^n$.

For the first sum, if we take the 2 from the denominator, we get $\sum_{n=1}^{\infty} n/2^n$, which is an Arithmetico-Geometric series. For an Arithmetico-Geometric series of the form $\sum_{n=1}^{\infty} nq^n$, the sum is given by $\frac{q}{(1-q)^2}$, if $0 < q < 1$. With this, the first sum is $2 * \frac{1/2}{(1-1/2)^2} = 2 * \frac{0.5}{0.25} = 4$.

The second sum is a geometric series, with the sum given by $\frac{1}{1-q} - 1$, if $0 < q < 1$. With this, the second sum is $\frac{1}{1-1/2} - 1 = 2 - 1 = 1$.

With this, the sum of the 2 series is $4 + 1 = 5$, and $T_{CC} = 5$.

1 c)

Knowing that μ_x is inversely proportional to T_{xx}, we can calculate μ_x for each of the states with the results from above.

$\mu_A = 1/T_{AA} = 1/2.5 = 2/5 = 0.4$
$\mu_B = 1/T_{BB} = 1/2.5 = 2/5 = 0.4$
$\mu_C = 1/T_{CC} = 1/5 = 0.2$

To get the distribution μ we must normalize the values calculated previously.

$$\mu = \begin{bmatrix}\mu_A & \mu_B & \mu_C\end{bmatrix} / (\mu_A + \mu_B + \mu_C) = \begin{bmatrix}0.4 & 0.4 & 0.2\end{bmatrix} / 1 = \begin{bmatrix}0.4 & 0.4 & 0.2\end{bmatrix}$$

For this distribution to be an invariant of the chain then $\mu_x > 0 : x \in \mathcal{X}$, such that $\sum_{x\in\mathcal{X}} \mu_x = 1$, and $\mu_x = \mu_x P$, that is $\mu_x = \sum_{y\in\mathcal{X}} \mu_y P(x|y)$ for all $x \in \mathcal{X}$. The first condition is trivial to verify, all probabilities are higher than 0. From the second and third conditions we get:

$$\begin{cases}\mu_A = \sum_{y\in\mathcal{X}} \mu_y P(A|y) \\ \mu_B = \sum_{y\in\mathcal{X}} \mu_y P(B|y) \\ \mu_C = \sum_{y\in\mathcal{X}} \mu_y P(C|y) \\ \mu_A + \mu_B + \mu_C = 1\end{cases} \Leftrightarrow \begin{cases}\mu_A = \mu_A * 0 + \mu_B * 0.5 + \mu_C * 1 \\ \mu_B = \mu_A * 1 + \mu_B * 0 + \mu_C * 0 \\ \mu_C = \mu_A * 0 + \mu_B * 0.5 + \mu_C * 0 \\ \mu_A + \mu_B + \mu_C = 1\end{cases} \Leftrightarrow \begin{cases}\mu_A = 0.5\mu_B + \mu_C \\ \mu_B = \mu_A \\ \mu_C = 0.5\mu_B \\ \mu_A + \mu_B + \mu_C = 1\end{cases}$$

$$\Leftrightarrow \begin{cases}\mu_B = 2\mu_C \\ \mu_B = \mu_A \\ \mu_C = 0.5\mu_B \\ \mu_A + \mu_B + \mu_C = 1\end{cases} \Leftrightarrow \begin{cases}\mu_B = 2\mu_C \\ \mu_B = \mu_A \\ \mu_C = 0.5\mu_B \\ 2\mu_C + 2\mu_C + \mu_C = 1\end{cases} \Leftrightarrow \begin{cases}\mu_B = 2/5 \\ \mu_A = 2/5 \\ \mu_C = 1/5 \\ \mu_C = 1/5\end{cases}$$

Which gives us the row vector $\begin{bmatrix}0.4 & 0.4 & 0.2\end{bmatrix}$, equal to the distribution obtained above, thus proving that the distribution μ is invariant of the chain.