

# Planing, Learning and Intelligent Decision Making - Homework 2

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## 1 Question 1

### 1 a)

Using  $\mathbb{X}$  as the state space,  $\mathbb{X} = \{A, B, C\}$ .

The transition matrix is given by  $\begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$ .

Where the first row represents the transition probabilities from state A, the second row from state B and the third row from state C. Each column represents the transition probabilities to state A, B and C, respectively.

The diagram of the Markov chain is given by

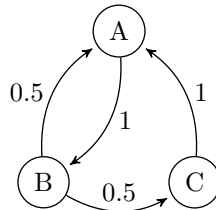


Figure 1: Markov Chain

### 1 b)

For state A, we have 2 possible paths to reach state A again,  $A \rightarrow B \rightarrow A$  and  $A \rightarrow B \rightarrow C \rightarrow A$ . With the transition matrix, we can calculate the probability of each path.

Using  $x_t$  to represent the state at time t.

The probability of the first path is  $P(x_1 = B|x_0 = A) * P(x_2 = A|x_1 = B) = 1 * 0.5 = 0.5$ .

The probability of the second path is  $P(x_1 = B|x_0 = A) * P(x_2 = C|x_1 = B) * P(x_3 = A|x_2 = C) = 1 * 0.5 * 1 = 0.5$ .

Since the first path takes 2 steps and the second path takes 3 steps,  $T_{AA} = 0.5 * 2 + 0.5 * 3 = 2.5$ .

For state B, we have 2 possible paths to reach state B again,  $B \rightarrow A \rightarrow B$  and  $B \rightarrow C \rightarrow A \rightarrow B$ . With the transition matrix, we can calculate the probability of each path.

The probability of the first path is  $P(x_1 = A|x_0 = B) * P(x_2 = B|x_1 = A) = 0.5 * 1 = 0.5$ . The probability of the second path is  $P(x_1 = C|x_0 = B) * P(x_2 = A|x_1 = C) * P(x_3 = B|x_2 = A) = 0.5 * 1 * 1 = 0.5$ .

Since the first path takes 2 steps and the second path takes 3 steps,  $T_{BB} = 0.5 * 2 + 0.5 * 3 = 2.5$ .

For state C, we have infinite possible paths to reach state C again, since it the bot can stay in states A and B indefinitely.

The shortest path is  $C \rightarrow A \rightarrow B \rightarrow C$ , with 3 steps. The probability of this path is  $P(x_1 = A|x_0 = C) * P(x_2 = B|x_1 = A) * P(x_3 = C|x_2 = B) = 1 * 0.5 * 1 = 0.5$ .

The second shortest path is  $C \rightarrow A \rightarrow B \rightarrow A \rightarrow B \rightarrow C$ , with 5 steps. The probability of this path is  $P(x_1 = A|x_0 = C) * P(x_2 = B|x_1 = A) * P(x_3 = A|x_2 = B) * P(x_4 = B|x_3 = A) * P(x_5 = C|x_4 = B) = 1 * 1 * 0.5 * 1 * 0.5 = 0.25$ .

To calculate the average number of steps, we need to calculate  $\sum_{n=1}^{\infty} (2n + 1)/2^n = 5$ .

Where  $2n + 1$  is the number of steps and  $1/2^n$  is the probability of the path. n represents the number of times we arrive at state B.

With this,  $T_{CC} = 5$ .

### 1 c)

Knowing that  $\mu_x$  is inversely proportional to  $T_{xx}$  we can calculate  $\mu_x$  for each of the states with the results from above.

$$\mu_a = 1/T_{aa} = 1/2.5 = 2/5 = 0.4$$

$$\mu_b = 1/T_{bb} = 1/2.5 = 2/5 = 0.4$$

$$\mu_c = 1/T_{cc} = 1/5 = 0.2$$

To get the distribution  $\mu$  we must normalize the values calculated previously.

$$\mu = [\mu_a \quad \mu_b \quad \mu_c] / (\mu_a + \mu_b + \mu_c) = [0.4 \quad 0.4 \quad 0.2] / 1 = [0.4 \quad 0.4 \quad 0.2]$$

For this distribution to be an invariant of the chain, then  $\mu_x > 0 : x \in \mathbb{X}$  such that  $\sum_{x \in \mathbb{X}} \mu_x = 1$ , and  $\mu_x = \mu_x P$ , that is  $\mu_x = \sum_{y \in \mathbb{X}} \mu_y P(x|y)$  for all  $x \in \mathbb{X}$ . The first condition is trivial to verify, all probabilities are higher than 0. From the second third condition we get:

$$\begin{aligned} \begin{cases} \mu_a = \sum_{y \in \mathbb{X}} \mu_y P(a|y) \\ \mu_b = \sum_{y \in \mathbb{X}} \mu_y P(b|y) \\ \mu_c = \sum_{y \in \mathbb{X}} \mu_y P(c|y) \\ \mu_a + \mu_b + \mu_c = 1 \end{cases} &\Leftrightarrow \begin{cases} \mu_a = \mu_a * 0 + \mu_b * 0.5 + \mu_c * 1 \\ \mu_b = \mu_a * 1 + \mu_b * 0 + \mu_c * 0 \\ \mu_c = \mu_a * 0 + \mu_b * 0.5 + \mu_c * 0 \\ \mu_a + \mu_b + \mu_c = 1 \end{cases} \Leftrightarrow \begin{cases} \mu_a = 0.5\mu_b + \mu_c \\ \mu_b = \mu_a \\ \mu_c = 0.5\mu_b \\ \mu_a + \mu_b + \mu_c = 1 \end{cases} \\ &\Leftrightarrow \begin{cases} \mu_b = 2\mu_c \\ \mu_b = \mu_a \\ \mu_c = 0.5\mu_b \\ \mu_a + \mu_b + \mu_c = 1 \end{cases} \Leftrightarrow \begin{cases} \mu_b = 2\mu_c \\ \mu_b = \mu_a \\ \mu_c = 0.5\mu_b \\ 2\mu_c + 2\mu_c + \mu_c = 1 \end{cases} \Leftrightarrow \begin{cases} \mu_b = 2/5 \\ \mu_a = 2/5 \\ \mu_c = 1/5 \\ \mu_c = 1/5 \end{cases} \end{aligned}$$

Which gives us the row vector  $[0.4 \ 0.4 \ 0.2]$ , equal to the distribution obtained above thus proving that the distribution  $\mu$  is invariant of the chain.