Algèbre Linéaire 2 - Série 9

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1. $\det \text{Mat}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Si échelonné : attention opérations Gauss

(a) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 4-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}$, det : $(4-\lambda)(1-\lambda) - 6 = \lambda^2 - 5\lambda - 2$

$\Delta = 25 - 4\cdot 1 \cdot (-2) = 33$

$\Rightarrow \lambda = \dfrac{5 \pm \sqrt{33}}{2}$

E_{λ_1} : $\begin{cases} \left(4 - \left(\frac{5+\sqrt{33}}{2}\right)\right)x + 3y \\ 2x + \left(1 - \left(\frac{5+\sqrt{33}}{2}\right)\right)y \end{cases} \Bigg| \begin{matrix} -2 \\ 4 - \left(\frac{5+\sqrt{33}}{2}\right) \end{matrix} \Rightarrow 0x + 0y = 0 \Rightarrow$ (ligne redondante)

$\Rightarrow \frac{3+\sqrt{33}}{2}x + 3y = 0 \Rightarrow x = -3,\ y = \frac{3+\sqrt{33}}{2} \Rightarrow E_{\lambda_1} = \text{span}\underbrace{\begin{pmatrix} -3 \\ \frac{3+\sqrt{33}}{2} \end{pmatrix}}_{\vec{v_1}}$ dim = 1

car une seule direction (stricte)

E_{λ_2} : $\begin{cases} \left(4 - \left(\frac{5-\sqrt{33}}{2}\right)\right)x + 3y = 0 \\ 2x + \left(1 - \left(\frac{5-\sqrt{33}}{2}\right)\right)y = 0 \end{cases} \Bigg| \begin{matrix} -2 \\ 4 - \left(\frac{5-\sqrt{33}}{2}\right) \end{matrix} \Rightarrow 0x + 0y = 0 \Rightarrow$ (ligne redondante)

$\Rightarrow \frac{3-\sqrt{33}}{2}x + 3y = 0 \Rightarrow x = -3,\ y = \frac{3-\sqrt{33}}{2} \Rightarrow E_{\lambda_2} = \text{span}\underbrace{\begin{pmatrix} -3 \\ \frac{3-\sqrt{33}}{2} \end{pmatrix}}_{\vec{v_2}}$

$\Rightarrow$ A de nature diagonalisable, V une base $\{\vec{v_1}, \vec{v_2}\}$

et $D_A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, dilatation et inversion dans $\vec{v_1}$ et dilatation dans $\vec{v_2}$

(b) $\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 \Rightarrow \lambda = 1$ (mult. double)

E_1 : $\begin{cases} 0x + y = 0 \\ \cancel{0x + 0y = 0} \end{cases} \Rightarrow y = 0,\ x = \alpha \in \mathbb{R}$

$\Rightarrow E_1 = \text{span}\cancel{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}}$ $\left\{\begin{pmatrix} \alpha \\ 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\right\}$

$= \left\{\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\right\}$

$= \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ de dim. = 1

$\dim E_1 \neq \dim p(\lambda) \Rightarrow$ B pas diagonalisable

(c) $\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) + 1 = \lambda^2 - 2\lambda + 2$

$\Delta = 4 - 4\cdot 1\cdot 2 < 0 \Rightarrow p(\lambda)$ de nature quadratique

$\hookrightarrow$ C pas diagonalisable

(d) $\det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = (-\lambda)^3 = -\lambda^3 \Rightarrow \lambda = 0$ (de nature triple)

E_0 : $\begin{cases} 0x + y + 0z = 0 \\ 0x + 0y + z = 0 \\ \cancel{0x + 0y + 0z = 0} \end{cases} \Rightarrow z = 0 \Rightarrow 0x + 1y = 0 \Rightarrow E_0 = \left\{\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}\right\}$

$\Rightarrow y = 0$

$= \text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\dim E_0 \neq \dim p(\lambda) \Rightarrow$ D pas diagonalisable

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(e) $\det\begin{pmatrix}-4-\lambda & 14 & 6\\ 2 & 7-\lambda & -3\\ -12 & 36 & 16-\lambda\end{pmatrix} \xrightarrow{L_1 \leftarrow L_1+2L_2} \det\begin{pmatrix}-\lambda & -2\lambda & 0\\ 2 & -7-\lambda & -3\\ -12 & 36 & 16-\lambda\end{pmatrix} \leftarrow$

$= -2\lambda\cdot(-1)^{1+2}\cdot\det\begin{pmatrix}2 & -3\\ -12 & 16-\lambda\end{pmatrix} - \lambda\cdot(-1)^{1+1}\cdot\det\begin{pmatrix}-7-\lambda & -3\\ 36 & 16-\lambda\end{pmatrix}$

$= 2\lambda\cdot(36-2\lambda-36) - \lambda\cdot((-7-\lambda)(16-\lambda)+108)$

$= -4\lambda^2 - \lambda(\lambda^2-9\lambda-4) = -\lambda(4\lambda+\lambda^2-9\lambda-4) = -\lambda(\lambda^2-5\lambda+4) \to ?$

$= -\lambda(\lambda-4)(\lambda-1)$

$\hookrightarrow E_0: \begin{cases}-4x+14y+6z=0\\ 2x-7y-3z=0\\ -12x+36y+16z=0\end{cases} \Big|\ \begin{matrix}2L_2+L_1\\ L_3-3L_1\end{matrix} \quad \begin{cases}-4x+14y+6z=0\\ \cancel{0x+0y+0z=0}\\ 0x-6y+4z=0\end{cases} \Big|\ z=\alpha\in\mathbb{R}$

$\Rightarrow -6y=-4\alpha \Leftrightarrow y=\frac{2}{3}\alpha$

$\Rightarrow -4x+\frac{28}{3}\alpha+6\alpha=0 \Leftrightarrow -4x=-\frac{46}{3}\alpha$

$\Leftrightarrow x=\frac{23}{6}\alpha \quad \Rightarrow E_0=\operatorname{span}\begin{pmatrix}\frac{23}{6}\\ \frac{2}{3}\\ 1\end{pmatrix}$

$E_4: \begin{cases}0x+14y+6z=0\\ 2x-9y-3z=0\\ -12x+36y+12z=0\end{cases} \Leftrightarrow \begin{cases}2x-3y-3z=0\\ -12x+36y+12z=0\\ 0x+14y+6z=0\end{cases} \Big|\ 6L_1+L_2$

$\begin{cases}-12x+36y+12z=0\\ 0x+18y-6z=0\\ 0x+14y+6z=0\end{cases}$

$\begin{cases}-8x+14y+6z=0\\ 2x-11y-3z=0\\ -12x+36y+12z=0\end{cases} \Big|\ \begin{matrix}4L_2+L_1\\ 2\cdot L_3-3\cdot L_1\end{matrix} \quad \Leftrightarrow \begin{cases}-8x+14y+6z=0\\ 0x-30y-6z=0\\ \cancel{0x+30y+6z=0}\end{cases}$

$\Leftrightarrow \begin{cases}-8x+14y+6z=0\\ 0x-30y-6z=0\\ 0x+0y+27z=0\end{cases} \Leftrightarrow z=0 \Rightarrow y=0 \Rightarrow x=0$

$\Rightarrow z=\alpha\in\mathbb{R} \Rightarrow y=-\frac{1}{5}\alpha$

$\Rightarrow \cancel{E_4=\operatorname{span}\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}}$

$\Rightarrow 8x=-\frac{14}{5}\alpha+6\alpha$

$\Leftrightarrow x=\frac{2}{5}\alpha \Rightarrow E_4=\operatorname{span}\begin{pmatrix}\frac{2}{5}\\ -\frac{1}{5}\\ 1\end{pmatrix}$

$E_1: \begin{cases}-5x+14y+6z=0\\ 2x-8y-3z=0\\ -12x+36y+15z=0\end{cases} \Big|\ \begin{matrix}5L_2+2L_1\\ 5L_3-12L_1\end{matrix} \Leftrightarrow \begin{cases}-5x+14y+6z=0\\ 0x-12y-3z=0\\ \cancel{0x+12y+3z=0}\end{cases} \quad z=\alpha\in\mathbb{R}$

$\Rightarrow -12y=3\alpha \Leftrightarrow y=\frac{1}{4}\alpha \Rightarrow 5x=\frac{14}{4}\alpha+6\alpha \Leftrightarrow x=\frac{38}{20}\alpha$

$\Leftrightarrow x=\frac{19}{10}$

$\Rightarrow E_1=\operatorname{span}\begin{pmatrix}\frac{19}{10}\\ \frac{1}{4}\\ 1\end{pmatrix}$

...

2. $A=\begin{pmatrix}-4&-15\\2&7\end{pmatrix}$, $\det(A-\lambda I_n)=\det\begin{pmatrix}-4-\lambda&-15\\2&7-\lambda\end{pmatrix}=(-4-\lambda)(7-\lambda)+30$

$=\lambda^2-3\lambda+2=(\lambda-2)(\lambda-1)$

$\Rightarrow E_2:\begin{cases}-6x-15y=0\\ \cancel{2x+5y=0}\end{cases}$ $y=\alpha,\alpha\in\mathbb{R}\Rightarrow 6x=-15\alpha\Leftrightarrow x=-\frac{15}{6}\alpha$

$\hookrightarrow E_2=\text{span}\begin{pmatrix}-\frac{15}{6}\\1\end{pmatrix}\Leftrightarrow\text{span}\begin{pmatrix}-15\\6\end{pmatrix}$

$\Rightarrow E_1:\begin{cases}-5x-15y=0\\2x+6y=0\end{cases}$ $\Big|$ $5L_2+2L_1$ $\Leftrightarrow\begin{cases}-5x-15y=0,\ y=\alpha,\alpha\in\mathbb{R}\\ \cancel{0x+0y=0}\end{cases}$

$\hookrightarrow E_1=\text{span}\begin{pmatrix}-3\\1\end{pmatrix}$ $\Rightarrow 5x=-15\alpha\Leftrightarrow x=-3\alpha$

$\Rightarrow V=(E_1|E_2)$

$\Rightarrow A^{250}=V\cdot(\lambda_i)^{250}_{i\le 2}\,V^{-1}=\begin{pmatrix}-15&-3\\6&1\end{pmatrix}\cdot\begin{pmatrix}2&0\\0&1\end{pmatrix}^{250}\cdot\begin{pmatrix}-15&-3\\6&1\end{pmatrix}^{-1}$

$=\begin{pmatrix}-15&-3\\6&1\end{pmatrix}\cdot\begin{pmatrix}2^{250}&\\&1\end{pmatrix}\cdot\underbrace{\begin{pmatrix}1&3\\-6&-15\end{pmatrix}\cdot\frac{1}{\det V}}_{\text{par déf. anciens cours}}$

$\hookrightarrow\det V=-15+18=3$

$\Rightarrow V^{-1}=\frac{1}{3}\begin{pmatrix}1&3\\-6&-15\end{pmatrix}=\begin{pmatrix}\frac{1}{3}&1\\-2&-5\end{pmatrix}$

$\Rightarrow A^{250}=\begin{pmatrix}-15\cdot2^{250}&-3\\3\cdot2^{251}&1\end{pmatrix}\begin{pmatrix}\frac{1}{3}&1\\-2&-5\end{pmatrix}=\begin{pmatrix}-5\cdot2^{250}+6&-15(2^{250}-1)\\2^{251}-2&3\cdot2^{251}-5\end{pmatrix}$

3. (a) Soit s_n et t_n : $\begin{cases}s_{n+1}=(s+4t)s_n-s_n=s_n(s+4t-1)\\t_{n+1}=(s+t)t_n-t_n=t_n(s+t-1)\end{cases}$ les actuelles meurent

(b) Soit $\vec{x_n}=\begin{pmatrix}s_n\\t_n\end{pmatrix}$, $\underset{2\times1}{A\vec{x_n}}=\underset{2\times1}{\vec{x_{n+1}}}\Rightarrow A\in\text{Mat}(2\times2,\mathbb{R})$

$\Rightarrow A=\cancel{\begin{pmatrix}s+4t-1&0\\0&s+t-1\end{pmatrix}}$ $A=(f(\vec{e_1})|f(\vec{e_2}))\Rightarrow A=\begin{array}{cc}\text{1 seul S}&\text{1 seul T}\\ \end{array}\begin{pmatrix}1&1\\4&1\end{pmatrix}$

(c) i)

n	0	1	2	3
s	1	1	5	13
t	0	4	8	28

ii) $\vec{x_0}=\begin{pmatrix}1\\0\end{pmatrix}$ $\cancel{\Rightarrow\vec{x_1}=\begin{pmatrix}s+4t-1&0\\0&s+t-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=(s+4t-1)}$

$\Rightarrow\vec{x_1}=\begin{pmatrix}1\\4\end{pmatrix}$

$\vec{x_2}=\begin{pmatrix}1&1\\4&1\end{pmatrix}\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}5\\8\end{pmatrix}$, $\vec{x_3}=\begin{pmatrix}1&1\\4&1\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix}=\begin{pmatrix}13\\28\end{pmatrix}$

(d) $\vec{x_0} = \begin{pmatrix} s_0 \\ t_0 \end{pmatrix}$, $\vec{x_1} = A \cdot \vec{x_0}$, $\vec{x_2} = A \cdot \vec{x_1} = A \cdot A \cdot \vec{x_0} = A^2 \cdot \vec{x_0}$

$\Rightarrow \vec{x_n} = \underline{\underline{A^n \cdot \begin{pmatrix} s_0 \\ t_0 \end{pmatrix}}}$ (Ensuite possible de diagonaliser A)
pour simplifier puissance A^n