Série 13

mercredi, 19 février 2025 14:07

1. Dériver les fonctions suivantes:

(a) $x \mapsto (1+x^2)^7$

(b) $x \mapsto \dfrac{1}{\sqrt[5]{x}}$

(c) $x \mapsto (x^2-1)^2 \cdot \sqrt{x}$

(d) $x \mapsto \dfrac{\sin(x)}{1+\cos(x)}$

(e) $x \mapsto \sin^2(x^3)$

(f) $x \mapsto \dfrac{1}{15}\cos^3(x)\left(3\cos^2(x)-5\right)$

(g) $x \mapsto \dfrac{1}{\sin(x)\cos(x)}$

(h) $x \mapsto \dfrac{\sqrt[3]{x\sqrt{8x}}}{\sqrt[4]{2x^3}}$

(i) $x \mapsto \dfrac{x^2+x+1}{x^2-x+1}$

(j) $x \mapsto \dfrac{ax+b}{cx+d}$

(k) $x \mapsto \dfrac{1}{(1+x^2)\sqrt{1+x^2}}$

(l) $x \mapsto \sqrt{x+\sqrt{x+\sqrt{x}}}$

(m) $x \mapsto \sqrt{x\sqrt{x\sqrt{x}}}$

(n) $x \mapsto \sin(\cos(x))$

(o) $x \mapsto \cot(1-2x)$

(p) $x \mapsto \dfrac{1}{3}\tan^3(x) - \tan(x) + x$

(q) $x \mapsto \arcsin\left(\dfrac{1-x^2}{1+x^2}\right)$

(r) $x \mapsto \left(\arctan\left(\dfrac{\pi}{2}\right)\right)^2$

(s) $x \mapsto \arctan\left(\dfrac{1-x}{1+x}\right)$

(t) $x \mapsto \sin(\arcsin(x))$

(u) $x \mapsto \arcsin(\sin(x))$

(v) $x \mapsto \cos^2\left(\arccos\left(\sqrt{x}\right)\right)$

a) $(1+x^2)^7$ $\quad f \to x^7 \quad g \to 1+x^2$

$7(1+x^2)^6 \cdot 2x$

$\underline{\underline{14x(1+x^2)^6}}$

b) $\dfrac{1}{\sqrt[5]{x}} \xrightarrow{\frac{1}{f(x)}} -\dfrac{\frac{1}{5}\sqrt[5]{x^{-4}}}{\sqrt[5]{x^2}}$

$\frac{1}{5}\sqrt[5]{x^{-4}} \cdot \sqrt[5]{x^{-2}} = \frac{1}{5}\sqrt[5]{x^{-6}}$

c) $(x^2-1)^2 \cdot \sqrt{x}$

$\downarrow f(x) \cdot g(x)$

$f(g(x)) \cdot g(x) + \dfrac{(x^2-1)^2}{2\sqrt{x}}$

$2(x^2-1)\cdot 2x \cdot \sqrt{x} + \dfrac{(x^2-1)^2}{2\sqrt{x}}$

$\dfrac{8x^4 - 8x^2 + x^4 - 2x^2 + 1}{2\sqrt{x}}$

$\dfrac{9x^4 - 10x^2 + 1}{2\sqrt{x}}$

d) $\dfrac{\sin(x)}{1+\cos(x)} \quad \dfrac{f(x)}{g(x)}$ $\quad +\sin(x)^2$

$\dfrac{\cos(x)\cdot(1+\cos(x)) - (-\sin(x))\cdot \sin(x)}{(1+\cos(x))^{\cancel{2}}}$

$\dfrac{\cos(x) + \cos^2(x) + 1 - \cos^2(x)}{1+\cos^2(x)}$

$\sin(x)^2 = 1 - \cos(x)^2$

$\dfrac{1}{1+\cos(x)}$

e) $\sin^2(x^3)$

$\sin(x^3)\cdot\sin(x^3)$

$(\sin(x^3))' = \cos(x^3) \cdot 3x^2$

$3x^2\cos(x^3)\cdot\sin(x^3) + 3x^2\cos(x^3)\cdot\sin(x^3)$

$6x^2\cos(x^3)\sin(x^3)$

f) $f \to \frac{1}{15}\cos^3(x)\left(3\cos^2(x) - 5\right)$

$f \to \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x)$

$-\cos^4(x)\sin(x) + \cos^2(x)\sin(x)$

$\cos^2(x)\sin(x)(-\cos^2(x)+1)$

$\underline{\underline{\cos^2(x)\sin^3(x)}}$

g) $\dfrac{1}{\sin(x)\cos(x)}$

$-\dfrac{(\sin(x)\cos(x))'}{\sin^2(x)\cos^2(x)}$

$-\dfrac{(\sin(x))'\cos(x) + \sin(x)(\cos(x))'}{\sin^2(x)\cos^2(x)}$

$-\dfrac{\cos^2(x) - \sin^2(x)}{\sin^2(x)\cos^2(x)} = \dfrac{\sin^2(x) - \cos^2(x)}{\sin^2(x)\cos^2(x)}$

h) $\dfrac{\sqrt[3]{x\sqrt{8x}}}{\sqrt[4]{2x^3}}$

$\dfrac{x^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot (2x)^{\frac{1}{6}}}{(2x^3)^{\frac{1}{4}}} = \dfrac{(2x)^{\frac{1}{3}} \cdot (2x)^{\frac{1}{6}}}{2^{\frac{1}{4}} \cdot x^{\frac{3}{4}}} = \dfrac{(2x)^{\frac{1}{2}}}{2^{\frac{1}{4}} \cdot x^{\frac{3}{4}}}$

$= \dfrac{2^{\frac{1}{2}} \cdot x^{\frac{1}{2}}}{2^{\frac{1}{4}} \cdot x^{\frac{3}{4}}} = \dfrac{2^{\frac{1}{4}}\, x^{\frac{1}{2}}}{x^{\frac{3}{4}}} = 2^{\frac{1}{4}} \cdot x^{-\frac{1}{4}}$

$f'(x) = \sqrt[4]{2} \cdot (x^{-\frac{1}{4}})' = 2^{\frac{1}{4}} \cdot -\frac{1}{4}x^{-\frac{5}{4}}$

$= -\dfrac{2^{\frac{1}{4}}}{2^2} \cdot x^{-\frac{5}{4}}$

$= -2^{-\frac{7}{4}} \cdot x^{-\frac{5}{4}}$

i) $\dfrac{x^2+x+1}{x^2-x+1}$ $\quad x^2+x+1 \Rightarrow 2x+1$ $\quad x^2-x+1 \Rightarrow 2x-1$

$\dfrac{(x^2+x+1)'(x^2-x+1) - (x^2+x+1)(x^2-x+1)'}{(x^2-x+1)^2}$

$\dfrac{(2x+1)(x^2-x+1) - (2x-1)(x^2+x+1)}{(x^2-x+1)^2}$

$\dfrac{\cancel{2x^3} - 2x^2 + 2x + \cancel{x^2} - \cancel{x} + 1 - \cancel{2x^3} - 2x^2 \cancel{- 2x} + x^2 \cancel{+ x} + 1}{(x^2-x+1)^2}$

$\dfrac{2(1-x^2)}{(x^2-x+1)^2}$

j) $\dfrac{ax+b}{cx+d}$

$\dfrac{(ax+b)'(cx+d) - (ax+b)(cx+d)'}{(cx+d)^2}$

$\dfrac{a(cx+d) - c(ax+b)}{(cx+d)^2}$

$\dfrac{\cancel{acx} + ad - \cancel{acx} - cb}{(cx+d)^2}$

$\dfrac{ad-bc}{(cx+d)^2}$

k) $\dfrac{1}{(1+x^2)\sqrt{1+x^2}}$

$\dfrac{1}{(1+x^2)\sqrt{1+x^2}} \cdot \dfrac{\sqrt{1+x^2}}{\sqrt{1+x^2}}$

$(1+x^2)^2 \Rightarrow 2(1+x^2)\cdot 2x = 4x(1+x^2)$

$\sqrt{1+x^2} \Rightarrow \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = \dfrac{x}{\sqrt{1+x^2}}$

$\dfrac{\sqrt{1+x^2}}{(1+x^2)^2}$

$\dfrac{((1+x^2)^2)'\sqrt{1+x^2} - (1+x^2)^2 \cdot (\sqrt{1+x^2})'}{(1+x^2)^4}$

$\dfrac{4x\cancel{(1+x^2)}\sqrt{1+x^2} - (1+x^2)^{\cancel{2}} \cdot \dfrac{x}{\sqrt{1+x^2}}}{(1+x^2)^{\cancel{4}3}}$

$\dfrac{4x\sqrt{1+x^2} - (1+x^2)\left(\dfrac{x}{\sqrt{1+x^2}}\right)}{(1+x^2)^3}$

$\dfrac{\dfrac{4x(1+x^2) - (x^3+x)}{\sqrt{1+x^2}}}{(1+x^2)^3}$

$\dfrac{4x + 4x^3 - x^3 - x}{\sqrt{1+x^2}\,(1+x^2)^3} = \dfrac{3x\cancel{(1+x^2)}}{\sqrt{1+x^2}\,(1+x^2)^{\cancel{3}2}} = 3x(1+x^2)^{-5/2}$

l) $\sqrt{x+\sqrt{x+\sqrt{x}}}$

$f(g(x)) \Rightarrow f'(g(x)) \cdot g(x)'$

$\frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-\frac{1}{2}} \cdot \left(x+\sqrt{x+\sqrt{x}}\right)'$

$\frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-\frac{1}{2}} \cdot \left(1+\left(\frac{1}{2}(x+\sqrt{x})^{-1/2} \cdot (x+\sqrt{x})'\right)\right)$

$\frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-\frac{1}{2}} \cdot \left(1+\left(\frac{1}{2}(x+\sqrt{x})^{-1/2} \cdot \left(1+\frac{1}{2}x^{-1/2}\right)\right)\right)$

m) $\sqrt{x\sqrt{x\sqrt{x}}}$

$x^{1/2} \cdot x^{1/4} \cdot x^{1/8}$

$x^{7/8} \Rightarrow \frac{9}{8}x^{-1/8}$

n) $\sin(\cos(x))$

$f(g(x))$

$\cos(\cos(x)) \cdot -\sin(x)$

$-\cos(\cos(x)) \cdot \sin(x)$

o) $\cot(1-2x)$

$f(g(x)) \cdot g(x)'$

$-1-\cot^2(1-2x) \cdot -2$

$2\left(1+\cot^2(1-2x)\right)$

p) $\frac{1}{3}\tan^3(x) - \tan(x) + x$

$\frac{1}{\cancel{3}}\cancel{3}\tan^2(x) \cdot (1+\tan^2(x)) - 1 - \tan^2(x) + 1$

$\cancel{\tan^2(x)} + \tan^4(x) \cancel{- 1} \cancel{- \tan^2(x)} \cancel{+ 1}$

$\tan^4(x)$

q) $\arcsin\left(\dfrac{1-x^2}{1+x^2}\right)$

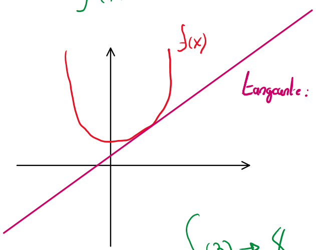
$\dfrac{1}{\sqrt{1 - \dfrac{1-x^2}{1+x^2}}} \cdot \left(\dfrac{1-x^2}{1+x^2}\right)'$

Fonction de x	Dérivée
$\alpha f(x) + \beta g(x)$ $(\alpha, \beta \in \mathbb{R})$	$\alpha f'(x) + \beta g'(x)$
$f(x) \cdot g(x)$	$f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
$f(g(x))$	$f'(g(x)) \cdot g'(x)$
$f^{-1}(x)$	$\frac{1}{f'(f^{-1}(x))}$
$\frac{1}{f(x)}$	$-\frac{f'(x)}{f^2(x)}$
$c \in \mathbb{R}$	0
x^α $(\alpha \in \mathbb{R})$	$\alpha x^{\alpha-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x)$	$-\frac{1}{\sin^2(x)} = -1 - \cot^2(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\text{arccot}(x)$	$-\frac{1}{1+x^2}$
e^x	e^x
a^x $(a > 0)$	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$ $(a > 0,\ a \neq 1)$	$\frac{1}{x \ln(a)}$

y

3. (a) Déterminer l'équation de la droite de pente 5 tangente à la parabole $y = x^2 - x + 2$.
(b) Déterminer l'équation de la normale à la courbe $x^2 + 2x - 3y = 6$ au point $x_0 = 2$.
(c) Déterminer l'équation des tangentes à la parabole $y = -x^2 + 2x - 2$ qui passent par l'origine.

a) $f'(x) = 2x - 1$ $\qquad 5 = 2x - 1 \rightarrow x = 3$



tangente : $5x + b$

$f(3) \rightarrow 8$

tangente d : $5(x-3) + 8$

d : $5x - 7$

Fonction de x	Dérivée
$\alpha f(x) + \beta g(x)$ $(\alpha, \beta \in \mathbb{R})$	$\alpha f'(x) + \beta g'(x)$
$f(x) \cdot g(x)$	$f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
$f(g(x))$	$f'(g(x)) \cdot g'(x)$
$f^{-1}(x)$	$\frac{1}{f'(f^{-1}(x))}$
$\frac{1}{f(x)}$	$-\frac{f'(x)}{f^2(x)}$
$c \in \mathbb{R}$	0
x^α $(\alpha \in \mathbb{R})$	$\alpha x^{\alpha-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x)$	$-\frac{1}{\sin^2(x)} = -1 - \cot^2(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\text{arccot}(x)$	$-\frac{1}{1+x^2}$
e^x	e^x
a^x $(a > 0)$	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$ $(a > 0,\ a \neq 1)$	$\frac{1}{x \ln(a)}$

b) $f(x) = x^2 + 2x - 3y = 6$ $\qquad x_0 = 2$

$f(x) = \frac{x^2 + 2x - 6}{3} = y$

$f'(x) = \frac{(2x+2)3}{9} = \frac{2x+2}{3}$

$f'(2) = 2$ $\qquad f(2) = \frac{4 + 4 - 6}{3} = +\frac{2}{3}$

$a = 2 \quad y_0 = \frac{2}{3}$

t : $a(x - x_0) + y_0$

$2x - 4 + \frac{2}{3}$

$2x - \frac{10}{3}$

n

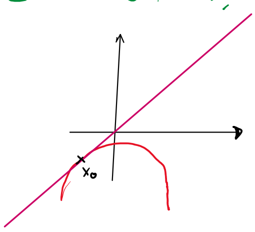
$a_t \cdot a_n = -1$

$a_n = -\frac{1}{2}$

$-\frac{1}{2}(x-2) + \frac{2}{3}$

n : $-\frac{1}{2}x + \frac{5}{3}$

c) $f(x) = -x^2 + 2x - 2$ $\qquad x_0 = ???$



d : $y = f'(x_0)(x - x_0) + y_0$

d : $f'(x_0) = \frac{y - y_0}{x - x_0}$ $\qquad \frac{y}{x}$

$a = f'(x_0) = \frac{y}{x}$

$y = f(x)$ $\qquad f'(x_0) = \frac{f(x)}{x}$

$f'(x_0) = -2x_0 + 2 = \frac{f(x_0)}{x_0}$

$-2x_0^2 + 2x_0 = -x_0^2 + 2x_0 - 2$

$-x_0^2 + 2 = 0$

$x_0 = \pm\sqrt{2}$

d : $(-2\sqrt{2} + 2)x$

$(2\sqrt{2} + 2)x$

4. Sous quel angle α les courbes suivantes se coupent-elles? Calculer $\tan(\alpha)$ sans évaluer α.

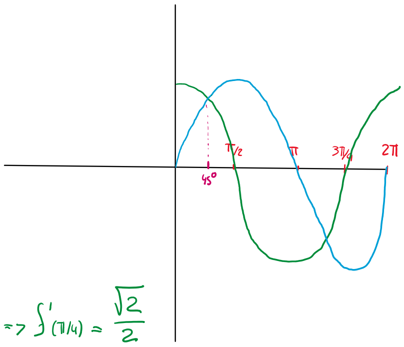
(a) $y = \sin(x)$ et $y = \cos(x)$,

(b) $y = \frac{1}{x}$ et $y = \sqrt{x}$.

a)

$f(x) = \sin(x)$

$g(x) = \cos(x)$



$f(x) = g(x)$

$x_0 = 45° \ / \ \pi/4$

$f'(x) = \cos(x) \Rightarrow f'(\pi/4) = \dfrac{\sqrt{2}}{2}$

$g'(x) = -\sin(x) \Rightarrow f'(\pi/4) = \dfrac{-\sqrt{2}}{2}$

$\alpha = \arctan\left(\dfrac{\sqrt{2}}{2}\right) - \arctan\left(\dfrac{-\sqrt{2}}{2}\right)$

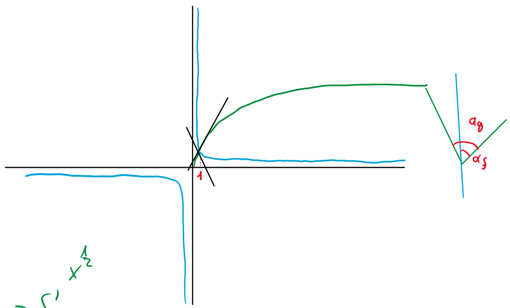
$\alpha = 35.26° - -35.26° = \underline{\underline{70.53°}}$

$\tan(\sqrt{2}/2 + \sqrt{2}/2) = \dfrac{\sqrt{2}/2 + \sqrt{2}/2}{1 - \frac{1}{2}} = 2\sqrt{2}$

b)

$f(x) = \dfrac{1}{x}$

$g(x) = \sqrt{x}$



$x_0 = 1$

$x^{\frac{1}{2}}$

$f'(x) = -\dfrac{1}{x^2} \Rightarrow f'(1) = -1$

$g'(x) = \dfrac{1}{2\sqrt{x}} \Rightarrow g'(1) = \dfrac{1}{2}$

$\alpha = \arctan\left(\frac{1}{2}\right) - \arctan(-1) = 26.57° + 45° = 71.57°$

$\tan\left(\frac{1}{2} + 1\right) = \dfrac{1/2 + 1}{1 - 1/2} = \underline{\underline{3}}$

Série 14

vendredi, 4 avril 2025 16:34

1. Etudier les fonctions suivantes et dessiner leur graphe:

(a) $f : x \mapsto \dfrac{x^2 - 12x + 27}{x^2 - 4x + 5}$

(b) $f : x \mapsto \sqrt{x^2(9 - x^2)}$

(c) $f : x \mapsto \dfrac{x^3}{6(x-2)}$

(d) $f : x \mapsto 2\cos(x) + \sin(2x)$

a) $\dfrac{(x-3)(x-9)}{x^2-4x+5}$ $D_f = \mathbb{R}$

$\Delta = b^2 - 4ac$

$16 - 20$

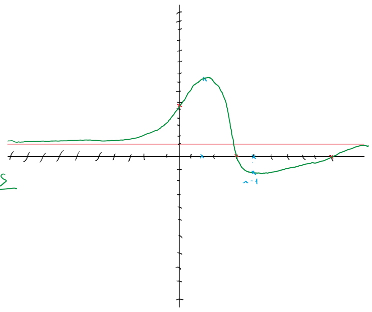
$\Delta < 0$ irréductible

Zéros: $\{3, 9\}$

I axe Y: $\{\frac{27}{5}\}$

A.H: $1 + \dfrac{-8x+22}{x^2-4x+5}$

$$\begin{array}{l|l} x^2 - 12x + 27 & x^2 - 4x + 5 \\ -x^2 + 4x - 5 & 1 \\ \hline -8x + 22 & \end{array}$$



$f'(x) = \dfrac{-8(x^2-4x+5) - (-8x+22)(2x-4)}{(x^2-4x+5)^2}$

$= \dfrac{-8x^2 + 32x - 40 + 16x^2 - 32x - 44x + 88}{(x^2-4x+5)^2}$

$= \dfrac{8x^2 - 44x + 48}{(x^2-4x+5)^2} \rightarrow \dfrac{4(2x^2 - 11x + 12)}{(x^2-4x+5)}$ $= \dfrac{4(x-\frac{3}{2})(x-4)}{(x^2-4x+5)}$

$\Delta = 121 - 96$

$= 25$

$x_{1,2} = \dfrac{11 \pm 5}{4}$

Min/Max:

$f'(x) \longrightarrow 0$

$\{\frac{3}{2}, 4\}$

x	$\frac{3}{2}$	3	4	9
f(x)	+ + + +	0 - - - - -	-	0 + + + +
f'(x)	+ 0 - - -	- - -	0 + +	+ + + + +

b) $\sqrt{x^2(9-x^2)}$ $D = [-3; +3]$

$|x|\sqrt{9-x^2}$

Zéros: $\{-3; 0; 3\}$

$f'(x) = 1\sqrt{9-x^2} + |x| \cdot \dfrac{x}{\sqrt{9-x^2}} = \dfrac{9-2x^2}{\sqrt{9-x^2}}$

$(\sqrt{9-x^2})' \Longrightarrow \dfrac{1}{2\sqrt{9-x^2}} \cdot (9-x^2)' = -\dfrac{x}{\sqrt{9-x^2}}$

Min/max

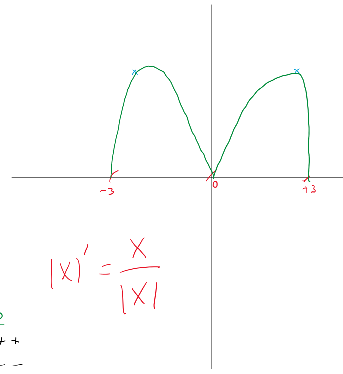
$-2x^2 + 9 = 0$

$-2x^2 = -9$

$x^2 = \frac{9}{2}$

$x = \pm\frac{3}{\sqrt{2}}$

x	-3	$-\frac{3}{\sqrt{2}}$	0	$+\frac{3}{\sqrt{2}}$	3
f(x)	$\varnothing$ // +	+ +	+ +	+ +	+ +
f'(x)	$\varnothing$ +	0 -	0 +	0 -	- -



$|x|' = \dfrac{x}{|x|}$

$\dfrac{5\pi}{6} = 150$

c) $\dfrac{x^3}{6x-12}$ $D = \mathbb{R} \setminus \{2\}$

$Z = 0$

Pas de parité

A.V $= 2$

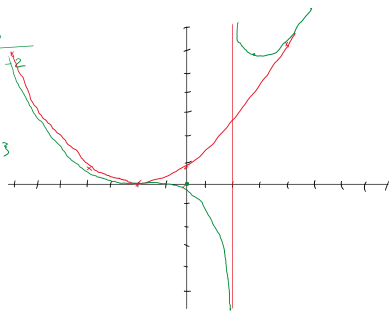
A.P $\frac{1}{6}(x+2)^2$

$$\begin{array}{l|l} x^3 & 6x - 12 \\ -x^3 + 2x^2 & \frac{1}{6}x^2 + \frac{1}{3}x + \frac{2}{3} \\ -2x^2 + 4x & \\ -4x + 8 & \end{array}$$

$f'(x) = \dfrac{3x^2(6x-12) - (6x^3)}{(6x-12)^2}$ $\frac{1}{6}(x+2)^2 + \dfrac{8}{6x-12}$

Min/Max

$= \dfrac{12x^3 - 36x^2}{36x^2 - 144x + 144} = \dfrac{x^2(x-3)}{3(x-2)^2} \Rightarrow \{0; 3\}$



x	0	2	3
f(x)	+ 0 -	$\varnothing$	+ + +
f'(x)	- 0 -	$\varnothing$	- 0 +

d) $2\cos(x) + \sin(2x)$ $D = \mathbb{R}$ $[0; 2\pi]$

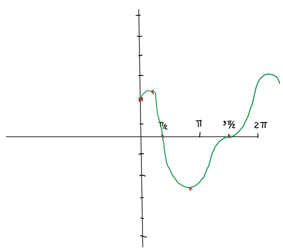
$2\cos(x) + \sin(2x) = 0$

$2\cos(x) + 2\sin(x)\cos(x) = 0$

$\cos(x)(2 + 2\sin(x)) = 0$

$Z = \{\pi/2, 3\pi/2\}$

$f(0) = 2$



$\sin^2(x) + \cos^2(x) = 1$

$\cos^2(x) = 1 - \sin^2(x)$

$f' = -2\sin(x) + 2\cos(2x)$

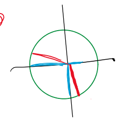
$-2\sin(x) + 2(\cos^2(x) - \sin^2(x)) = 0$

$2(1 - \sin(x) - 2\sin^2(x)) = 0$

$\sin(x) = t$

$2(1 - t - 2t^2)$

$t = -1$, $\frac{1}{2}$



$f'(x) = 0$ $\{\frac{3\pi}{2}; \frac{\pi}{6}; \frac{5\pi}{6}\}$

2. Déterminer l'équation de la tangente à la courbe $y = \dfrac{\ln(3x)}{x}$ passant par l'origine.

d: $y = f'(x_0)(x - x_0) + y_0$

d: $f'(x_0) = \dfrac{y - y_0}{x - x_0}$

$a = f'(x_0) = \dfrac{y_0}{x_0}$

$y_0 = f(x)$

$f'(x_0) = \dfrac{f(x)}{x_0}$

$f(x) = \dfrac{\ln(3x)}{x}$

$t: (0; 0)$

$t: \underset{\approx}{ax}$

d: $y = f'(x_0)(x - x_0) + y_0$

d: $f'(x_0) = \dfrac{y - y_0}{x - x_0} = a$

$a = \dfrac{0 - y_0}{0 - x_0} = \dfrac{y_0}{x_0}$

$\dfrac{y_0}{x_0} = a = f'(x_0)$

$\dfrac{f(x_0)}{x_0} = f'(x_0)$

$f'(x_0) = \dfrac{2x \cdot \ln(3x) - x}{x^2} = \dfrac{\frac{1}{x}x - \ln(3x)}{x} = \dfrac{1 - \ln(3x)}{x^2}$

$\ln(3x) = \dfrac{1}{3x} \cdot 3 = \dfrac{1}{x}$

$f'(x_0) = \dfrac{f(x)}{x}$

$\dfrac{1 - \ln(3x)}{x^2} = \dfrac{\ln(3x)}{x} \cdot \dfrac{1}{x}$

$1 - \ln(3x) = \ln(3x)$

$1 = 2\ln(3x)$

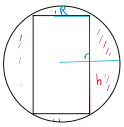
$1 = \ln(3x^2)$

$e = 3x^2$

$\pm\sqrt{\frac{e}{3}} = x$

$\dfrac{1/2}{e/9} x = \dfrac{1}{2} \cdot \dfrac{9}{e} x = \dfrac{9}{2e} x$

3. Un cylindre d'aire latérale maximale est inscrit dans une sphère de rayon r. Quelle est la hauteur du cylindre?

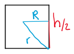


A latéral : $2\pi R \cdot h$

$V : \frac{4}{3}\pi r^3$

$V : \pi R^2 \cdot h$

$R = \sqrt{r^2 - h^2/4}$



$A = 2\pi \cdot h \cdot \sqrt{r^2 - h^2/4} = \pi h \sqrt{4r^2 - h^2}$

$A' = \pi \cdot \sqrt{4r^2 - h^2} + \pi h \cdot \frac{-2h}{2\sqrt{4r^2-h^2}}$

$= \pi \left(\sqrt{4r^2-h^2} + h \cdot \frac{-2h}{2\sqrt{4r^2-h^2}} \right)$

$= \pi \frac{8r^2 - 2h^2 - 2h^2}{2\sqrt{4r^2-h^2}} = \pi \frac{2(2r^2 - h^2)}{\sqrt{4r^2-h^2}}$

$\text{Max} = \pi \frac{2(2r^2 - h^2)}{\sqrt{4r^2 - h^2}} = 0$

$2r^2 - h^2 = 0$

$2r^2 = h^2$

$r\sqrt{2} = \underline{\underline{h}}$

Série 15
lundi, 7 avril 2025 19:22

1. Montrer que la fonction

$$f : [0,1] \to \mathbf{R},\ x \mapsto \arcsin(2x-1) + 2\arccos(\sqrt{x})$$

est une fonction constante et déterminer cette constante.

$$f' = \frac{2}{\sqrt{1-(2x-1)^2}} + 2 \cdot -\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{2}{\sqrt{-4x^2+4x}} - \frac{1}{\sqrt{x}\cdot\sqrt{1-x}}$$

$$\frac{1}{\sqrt{-x^2+x}} - \frac{1}{\sqrt{-x^2+x}} = \underline{\underline{0}}$$

$f(0) = \arcsin(-1) + 2\arccos(0) = \pi/2$ sur $[0,1]$

2. Calculer les limites suivantes:

(a) $\lim\limits_{t\to1} \dfrac{t^\alpha - t^\beta}{t^{1/\alpha} - t^{1/\beta}}$ $(\alpha \neq \beta)$

(b) $\lim\limits_{x\to0} \dfrac{(1+x)^{3/2} - (1-x)^{3/4}}{x}$

(c) $\lim\limits_{x\to0} \arcsin(x)\cdot\cot(x)$

(d) $\lim\limits_{x\to0} \dfrac{\tan(x) - \sin(x)}{x(1-\cos(x))}$

(e) $\lim\limits_{x\to0} \dfrac{a^x - 1}{b^x - 1}$ $(a, b > 0;\ a, b \neq 1)$

(f) $\lim\limits_{t\to\pi/2} \dfrac{\sin(t) + \sin(3t)}{\cos(2t)}$

(g) $\lim\limits_{x\to\infty} x\arctan\left(\dfrac{2}{x}\right)$

(h) $\lim\limits_{x\to0} \left(\dfrac{1}{x} - \dfrac{1}{\sin(x)}\right)$

(i) $\lim\limits_{x\to0} (1+2\sin(x))^{\cot(x)}$

(j) $\lim\limits_{x\to0} \dfrac{2^{1/x} - 1}{2^{1/x} + 1}$

a) "$\frac{0}{0}$"

$$\lim_{t\to1} \frac{f'(x)}{g'(x)} = \lim_{t\to1} \frac{\alpha t^{\alpha-1} - \beta t^{\beta-1}}{\frac{1}{\alpha}\sqrt[1-\alpha]{t^\alpha} - \frac{1}{\beta}\sqrt[1-\beta]{t^\beta}} = \frac{\alpha - \beta}{\frac{1}{\alpha} - \frac{1}{\beta}} = \underline{\underline{-\alpha\beta}}$$

$\frac{1}{\alpha} - \frac{\alpha}{\alpha}$

$$t^{\frac{1}{\alpha}} \ \ \frac{1}{\alpha} t^{\frac{1-\alpha}{\alpha}} = \frac{1}{\alpha}\sqrt[1-\alpha]{t^\alpha}$$

b) "$\frac{0}{0}$" $\lim\limits_{x\to0} \dfrac{(1+x)^{3/2} - (1-x)^{3/4}}{x}$

$$\lim_{x\to0} \frac{f'(x)}{g'(x)} = \lim_{x\to0}\ 1\cdot\frac{3}{2}\cdot(1+x)^{1/2} - -1\cdot\frac{3}{4(1-x)^{1/4}} = \lim_{x\to0} \frac{3\sqrt{1+x}}{2} + \frac{3}{4\sqrt[4]{1-x}} = \underline{\underline{9/4}}$$

c) $\lim\limits_{x\to0} \arcsin(x)\cdot\cot(x)$

"$0 \cdot 0$"

$$\lim_{x\to0} \frac{\arcsin(x)}{\tan(x)} = \lim_{x\to0} \frac{f'(x)}{g'(x)} = \lim_{x\to0} \frac{\frac{1}{\sqrt{1-x^2}}}{1+\tan^2(x)} = \lim_{x\to0} \frac{1+\tan^2(x)}{\sqrt{1-x^2}} = \underline{\underline{\frac{1}{1}}} = 1$$

d) "$\frac{0}{0}$" $\lim\limits_{x\to0} \dfrac{\tan(x) - \sin(x)}{x(1-\cos(x))}$

$$\lim_{x\to0} \frac{\frac{\sin(x)}{\cos(x)} - \sin(x)}{x\cdot(1-\cos(x))} = \lim_{x\to0} \frac{\sin(x)}{\cos(x)\cdot x} = \lim_{x\to0} \frac{\cos(x)}{\cos(x) + \cancel{-x\sin(x)}} = \frac{1}{1} = \underline{\underline{1}}$$

w

e) "$\frac{0}{0}$" $\lim\limits_{x\to0} \dfrac{a^x - 1}{b^x - 1} = \lim\limits_{x\to0} \dfrac{\ln(x)\cdot a^x}{\ln(x)\cdot b^x} = \lim\limits_{x\to0} \dfrac{\ln(a)\cdot e^{\ln(a^x)}}{\ln(b)\cdot e^{\ln(b^x)}} = \lim\limits_{x\to0} \dfrac{\ln(a)\cdot e^{x\ln(a)}}{\ln(b)\cdot e^{x\ln(b)}} = \dfrac{\ln(a)}{\ln(b)}$

f) $\lim\limits_{t\to\pi/2} \dfrac{\overset{1}{\sin(t)} + \overset{-1}{\sin(3t)}}{\underset{-1}{\cos(2t)}} = \dfrac{0}{-1} = 0$

g) $\lim\limits_{x\to\infty} x\arctan\left(\dfrac{2}{x}\right) = \lim\limits_{x\to\infty} \dfrac{\arctan\left(\frac{2}{x}\right)}{\frac{1}{x}}$ $\left(\frac{2}{x}\right)' = \left(2\cdot\frac{1}{x}\right)' = -\frac{2}{x^2}$ $\left(\arctan\left(\frac{2}{x}\right)\right)' = \frac{1}{1+\left(\frac{2}{x}\right)^2} \cdot -\frac{2}{x^2}$

$$\underline{\underline{H}} = \lim_{x\to\infty} \frac{\frac{1}{1+\left(\frac{2}{x}\right)^2} - \frac{2}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{4}{x^2}} = \underline{\underline{2}}$$

h) $\lim\limits_{x\to0} \dfrac{1}{x} - \dfrac{1}{\sin(x)} = \lim\limits_{x\to0} \dfrac{\sin(x) - x}{x\cdot\sin(x)} = \lim\limits_{x\to0} \dfrac{\overset{0}{\cos(x) - 1}}{\underset{1}{\sin(x) + \cos(x)}} = \underline{\underline{0}}$

$(-x^{-2})' = 2x^{-3}$

i) $\lim\limits_{x\to0} (1+2\sin(x))^{\cot(x)} = e^{\lim\limits_{x\to0} \cot(x)\cdot\ln(1+2\sin(x))} = e^{\lim\limits_{x\to0} \frac{\cos(x)\ln(1+2\sin(x))}{\sin(x)}} = e^{\lim\limits_{x\to0} \frac{-\sin(x)\ln(1+2\sin(x)) + \cos(x)\cdot\frac{1}{1+2\sin(x)}\cdot\cos(x)}{\cos(x)}} = \lim\limits_{x\to0} \dfrac{\cancel{-\sin(x)\cdot\ln(1+2\sin(x))} + \frac{\cos^2(x)}{1+2\sin(x)}}{\underset{1}{\cos(x)}} = \underline{\underline{2}}$

j) $\dfrac{(2^{1/x})'}{(2^{1/x})'} = \underline{\underline{1}}$

3. Calculer les limites suivantes:

(a) $\lim\limits_{x\to\infty} x^{1/x}$

(b) $\lim\limits_{x\to\infty} \left(\dfrac{1}{2}\ln(4x^2+1) - \dfrac{1}{3}\ln(x^3-1)\right)$

(c) $\lim\limits_{x\to0} xe^{1/x}$

(d) $\lim\limits_{x\to\infty} \dfrac{e^{\sqrt{x}}}{x^{\ln(x)}}$

a) $\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} x^0 = \underline{1}$

b) $\lim \frac{1}{6}\left(3\ln(4x^2+1) - 2\ln(x^3-1)\right) = \lim\limits_{x\to\infty} \frac{1}{6}\cdot\ln\left(\dfrac{(4x^2+1)^3}{(x^3-1)^2}\right) = \frac{1}{6}\cdot\ln\left(\lim\limits_{x\to\infty} \dfrac{(4x^2+1)^3}{(x^3-1)^2}\right) = \frac{1}{6}\cdot\ln(64) = \ln(2)$

$\lim_{x \to \infty} \quad = \lim_{x\to\infty} \quad = \ =$

b) $\lim_{x \to \infty} \frac{1}{6}\left(3\ln(4x^2+1) - 2\ln(x^3-1)\right) = \lim_{x\to\infty} \frac{1}{6} \cdot \ln\left(\frac{(4x^2+1)^3}{(x^3-1)^2}\right) = \frac{1}{6} \cdot \ln\left(\lim_{x\to\infty} \frac{(4x^2+1)^3}{(x^3-1)^2}\right) = \frac{1}{6} \cdot \ln(64) = \ln(2)$

c) $\lim_{x \to 0} x e^{1/x} \longrightarrow \lim_{x\to 0} e^{\ln(x \cdot e^{1/x})} \longrightarrow \lim_{x\to\infty} e^{\ln(x) + \frac{1}{x}} = e^{\lim_{x\to\infty} \frac{x\ln(x)}{x}} = e^{\lim_{x\to\infty} \frac{\ln(x)+1}{1}}$

Pas de limite

d) $\lim_{x \to \infty} \frac{e^{\sqrt{x}}}{x^{\ln(x)}} = \frac{e^{\sqrt{x}}}{\lim_{x\to\infty} x^{\ln(x)}} = \frac{e^{\sqrt{x}}}{\lim_{x\to\infty} e^{\ln(x^{\ln(x)})}} = \frac{e^{\sqrt{x}}}{\lim_{x\to\infty} e^{\ln(x)\ln(x)}} = \frac{e^{\sqrt{x}}}{\lim_{x\to\infty} e^{\frac{\ln(x)}{x} \cdot \frac{\ln(x)}{x}}} = e^{\sqrt{x}}$

$\left(\frac{\ln(x)}{x}\right)' = \frac{1 - \ln(x)}{x^2} \Rightarrow \lim_{x\to\infty} \left(\frac{\ln(x)}{x}\right)' = 0$

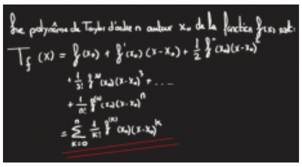
4. Déterminer le polynôme de Taylor du deuxième ordre centré en $x_0 = 0$ des fonctions suivantes:

(a) $f(x) = \sqrt{1+x}$

(b) $f(x) = \frac{1}{1+x}$

(c) $f(x) = \frac{1}{\sqrt{1+x}}$

(d) $f(x) = \ln(1+x)$



a) $1 + \frac{1}{2}x - \frac{1}{8}x^2$ ✓

$f'(x) \Rightarrow \frac{1}{2\sqrt{1+x}}$

$f''(x) \Rightarrow -\frac{\frac{1}{\sqrt{1+x}}}{4(1+x)} = -\frac{1}{4\sqrt{1+x}\,(1+x)}$

d) $\ln(1+x)$ $\quad 0 + x - \frac{x^2}{2}$

$f(x_0) = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f(x_0) = 1$

$f''(x) = -\frac{1}{(1+x)^2} = -1$

b) $\frac{1}{1+x}$ $\quad 1 - x + x^2$

$f'(x) = -\frac{1}{(1+x)^2} \Rightarrow f'(x_0) = -1$

$f''(x) = \frac{1+x+1+x}{(1+x)^4} = \frac{2+x}{(1+x)^4} \Rightarrow f''(x_0) = 2$

c) $\frac{1}{\sqrt{1+x}}$ $\quad 1 - \frac{1}{2}x + \frac{3}{8}x^2$

$f'(x) = \frac{\frac{1}{2\sqrt{1+x}}}{1+x} = \frac{1}{2\sqrt{1+x}\,(1+x)}$

$f''(x) = \frac{2\left(1 + \frac{1}{2\sqrt{1+x}}\right)}{4(1+x)^3} \Rightarrow f''(x_0) = \frac{3}{4}$

Série 16

mercredi, 9 avril 2025 12:32

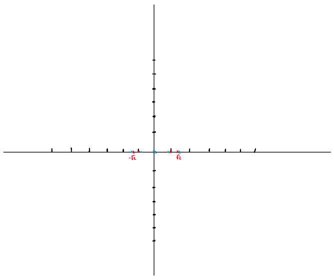
1. Etudier les fonctions suivantes:
   (a) $f : x \mapsto x^9 - 3x^7 + 2x^5$
   (b) $f : x \mapsto \dfrac{x^3}{x^2 - 3}$
   (c) $f : x \mapsto \sin(x) - \frac{1}{2}\sin(2x)$

a) $f(x) \to x^9 - 3x^7 + 2x^5$ $D = \mathbb{R}$

$f(x) = x^5(x^4 - 3x^2 + 2)$

$= x^5(x^2-2)(x^2-1)$

$= x^5(x-\sqrt{2})(x+\sqrt{2})(x+1)(x-1)$



$Z = \{0, \pm\sqrt{2}, \pm 1\}$

$f'(x) = 9x^8 - 21x^6 + 10x^4$

$= x^4(9x^4 - 21x^2 + 10)$ $\begin{cases} (9x^4 - 15x^2 - 6x^2 + 10) \\ 3x^2(3x^2-5) - 2(3x^2-5) \end{cases}$

$= x^4(3x^2-5)(3x^2-2)$

$= x^4(x\sqrt{3} - \sqrt{5})(x\sqrt{3} + \sqrt{5})(x\sqrt{3} - \sqrt{2})(x\sqrt{3} + \sqrt{2})$

$f(x) = 0$ pour $x = \{0 ; \pm\sqrt{\frac{5}{3}} ; \pm\sqrt{\frac{2}{3}}\}$

Min max

Z dérivée $\to f(x)$

$x_0, y_0 = \{\{0, 0\}, \ldots\}$

Tab des signes

. . .

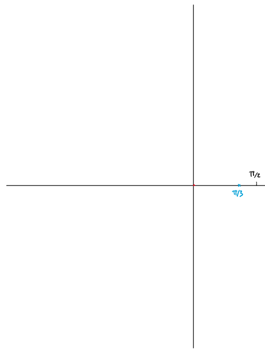
Imp + Imp = Impaire

c) $\sin(x) + \frac{1}{2}\sin(2x)$

2π périodique $[0 ; 2\pi[$

$f(x) = \sin(x)(1 + \cos(x))$

$\sin(x)^2 = 1 - \cos^2(x)$



$Z = \{0 ; \pi\}$

$f'(x) = \cos(x)(1+\cos(x)) + \sin(x)(-\sin(x))$

$= \cos(x)(1+\cos(x)) + \cos^2(x) - 1$

$= \cos(x) + \cos^2(x) + \cos^2(x) - 1$

$2\cos^2(x) + \cos(x) - 1 = 0$

$2t^2 + t - 1$

$(2t-1)(t+1) = 0$

$(2\cos(x) - 1)(\cos(x) + 1) = 0$

$Z\ \{\pi ; \frac{\pi}{3}\}$

$f''(x) = -2\sin(x)(\cos(x)+1) + (2\cos(x) - 1) - \sin x$

$-\sin(x)(2\cos(x) + 2 + 2\cos(x) - 1)$

$-\sin(x)(4\cos(x) + 1)$

2. Pour quelles valeurs des paramètres a et b la fonction

$$f : x \mapsto \frac{x^2 + ax + b}{x^2}$$

possède-t-elle le point d'inflexion $W = (2, 0)$? Déterminer pour ces valeurs de a et b les zéros, les extremums ainsi que les asymptotes de f et dessiner le graphe de f.

Fonction de x	Dérivée
$\alpha f(x) + \beta g(x)$ ($\alpha, \beta \in \mathbb{R}$)	$\alpha f'(x) + \beta g'(x)$
$f(x) \cdot g(x)$	$f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
$f(g(x))$	$f'(g(x)) \cdot g'(x)$
$f^{-1}(x)$	$\frac{1}{f'(f^{-1}(x))}$
$\frac{1}{f(x)}$	$-\frac{f'(x)}{f^2(x)}$
$c \in \mathbb{R}$	0
x^α ($\alpha \in \mathbb{R}$)	$\alpha x^{\alpha-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x)$	$-\frac{1}{\sin^2(x)} = -1 - \cot^2(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\text{arccot}(x)$	$-\frac{1}{1+x^2}$
e^x	e^x
a^x ($a > 0$)	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$ ($a > 0, a \neq 1$)	$\frac{1}{x \ln(a)}$

$f'(x) = \dfrac{(2x+a)(x^2) - (x^2+ax+b)(2x)}{x^4}$

$= \dfrac{2x^3 + ax^2 - 2x^3 - 2ax^2 - 2bx}{x^4}$

$= \dfrac{-ax - 2b}{x^3}$

$f''(x) = \dfrac{-a(x^3) - 3x^2(-ax - 2b)}{x^6}$

$= \dfrac{-ax + 3ax + 6b}{x^4}$

$0 = \dfrac{4a + 6b}{2^4}$