What is the value of

$$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$$

in the interval $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$?

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Step 1: Define the sum inside the inverse secant:

$$S=\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)$$

Step 2: Let

$$x_k=\frac{7\pi}{12}+\frac{k\pi}{2}$$

Then the term inside the sum is:

$$\sec(x_k)\sec(x_{k+1})$$

Step 3: Use the identity for secant in terms of cosine:

$$\sec(x)=\frac{1}{\cos(x)}$$

So,

$$\sec(x_k)\sec(x_{k+1})=\frac{1}{\cos(x_k)\cos(x_{k+1})}$$

Step 4: Note the periodicity of cosine with period 2π, and the increment $\frac{\pi}{2}$ in x_k. Calculate the values of $\cos(x_k)$ for $k=0,1,2,3,4$ to find a pattern.

$$\begin{aligned}x_0&=\frac{7\pi}{12}\\ x_1&=\frac{7\pi}{12}+\frac{\pi}{2}=\frac{7\pi}{12}+\frac{6\pi}{12}=\frac{13\pi}{12}\\ x_2&=\frac{7\pi}{12}+\pi=\frac{7\pi}{12}+\frac{12\pi}{12}=\frac{19\pi}{12}\\ x_3&=\frac{7\pi}{12}+\frac{3\pi}{2}=\frac{7\pi}{12}+\frac{18\pi}{12}=\frac{25\pi}{12}\\ x_4&=\frac{7\pi}{12}+2\pi=\frac{7\pi}{12}+\frac{24\pi}{12}=\frac{31\pi}{12}\end{aligned}$$

Step 5: Use cosine periodicity:

$$\cos(x+2\pi)=\cos x$$

So,

$$\cos(x_4)=\cos\left(\frac{7\pi}{12}+2\pi\right)=\cos\left(\frac{7\pi}{12}\right)$$

Similarly, for $k\geq 4$, the cosine values repeat every 4 steps.
Step 6: Calculate $\cos\left(\frac{7\pi}{12}\right)$:

$$\frac{7\pi}{12}=\pi-\frac{5\pi}{12}$$

$$\cos\left(\frac{7\pi}{12}\right)=\cos\left(\pi-\frac{5\pi}{12}\right)=-\cos\left(\frac{5\pi}{12}\right)$$

Calculate $\cos\left(\frac{5\pi}{12}\right)$:

$$\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$$

Use cosine addition formula:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Therefore,

$$\cos\left(\frac{7\pi}{12}\right) = -\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

Step 7: Calculate $\cos(x_k)$ for $k = 0, 1, 2, 3$:
- $k = 0$:

$$\cos(x_0) = \cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

- $k = 1$:

$$x_1 = \frac{13\pi}{12} = \pi + \frac{\pi}{12}$$

$$\cos\left(\pi + \frac{\pi}{12}\right) = -\cos\left(\frac{\pi}{12}\right)$$

Calculate $\cos\left(\frac{\pi}{12}\right)$:

$$\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$$

$$\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

So,

$$\cos(x_1) = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

- $k = 2$:

$$x_2 = \frac{19\pi}{12} = \pi + \frac{7\pi}{12}$$

$$\cos\left(\pi + \frac{7\pi}{12}\right) = -\cos\left(\frac{7\pi}{12}\right) = -\frac{\sqrt{2}-\sqrt{6}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

- $k = 3$:

$$x_3 = \frac{25\pi}{12} = 2\pi + \frac{\pi}{12}$$

$$\cos\left(2\pi + \frac{\pi}{12}\right) = \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Step 8: Summarize $\cos(x_k)$ for $k = 0, 1, 2, 3$:

$$\cos(x_0) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

$$\cos(x_1) = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\cos(x_2) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos(x_3) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Step 9: The pattern repeats every 4 steps:

$$\cos(x_{k+4}) = \cos(x_k)$$

Step 10: Calculate the terms $\sec(x_k)\sec(x_{k+1}) = \frac{1}{\cos(x_k)\cos(x_{k+1})}$ for $k = 0, 1, 2, 3$:
- $k = 0$:

$$\frac{1}{\cos(x_0)\cos(x_1)} = \frac{1}{\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\left(-\frac{\sqrt{6}+\sqrt{2}}{4}\right)} = \frac{1}{-\frac{(\sqrt{2}-\sqrt{6})(\sqrt{6}+\sqrt{2})}{16}} = -\frac{16}{(\sqrt{2}-\sqrt{6})(\sqrt{6}+\sqrt{2})}$$

Simplify the denominator:

$$(\sqrt{2}-\sqrt{6})(\sqrt{6}+\sqrt{2}) = \sqrt{2}\sqrt{6}+\sqrt{2}^2-\sqrt{6}^2-\sqrt{6}\sqrt{2} = (\sqrt{12}+2)-(6+\sqrt{12}) = 2-6 = -4$$

So,

$$\frac{1}{\cos(x_0)\cos(x_1)} = -\frac{16}{-4} = 4$$

- $k = 1$:

$$\frac{1}{\cos(x_1)\cos(x_2)} = \frac{1}{\left(-\frac{\sqrt{6}+\sqrt{2}}{4}\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)} = \frac{1}{-\frac{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}{16}} = -\frac{16}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$$

Simplify denominator:

$$(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2}) = 6-2 = 4$$

So,

$$\frac{1}{\cos(x_1)\cos(x_2)} = -\frac{16}{4} = -4$$

- $k = 2$:

$$\frac{1}{\cos(x_2)\cos(x_3)} = \frac{1}{\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)} = \frac{1}{\frac{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})}{16}} = \frac{16}{4} = 4$$

- $k = 3$:

$$\frac{1}{\cos(x_3)\cos(x_4)} = \frac{1}{\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)} = \frac{1}{\frac{(\sqrt{6}+\sqrt{2})(\sqrt{2}-\sqrt{6})}{16}} = \frac{16}{(\sqrt{6}+\sqrt{2})(\sqrt{2}-\sqrt{6})}$$

Note:

$$(\sqrt{6}+\sqrt{2})(\sqrt{2}-\sqrt{6}) = -(\sqrt{2}-\sqrt{6})(\sqrt{6}+\sqrt{2}) = -(-4) = 4$$

So,

$$\frac{1}{\cos(x_3)\cos(x_4)} = \frac{16}{4} = 4$$

Step 11: The terms for $k = 0, 1, 2, 3$ are:

$$4, -4, 4, 4$$

Step 12: Since the pattern repeats every 4 steps, the sequence of terms is:

$$4, -4, 4, 4, 4, -4, 4, 4, 4, -4, 4$$

for $k = 0$ to 10.

Step 13: Sum these 11 terms:

Group in cycles of 4 terms:

- First 4 terms: $4 - 4 + 4 + 4 = 8$ - Next 4 terms: $4 - 4 + 4 + 4 = 8$ - Remaining 3 terms: $4 - 4 + 4 = 4$

Total sum:

$$8 + 8 + 4 = 20$$

Step 14: The expression inside the inverse secant is:

$$\frac{1}{4} \times 20 = 5$$

Step 15: We want to find:

$$\sec^{-1}(5)$$

in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Step 16: Recall that $\sec^{-1}(x)$ is the angle θ such that $\sec\theta = x$.

Since $\sec\theta = \frac{1}{\cos\theta}$, we have:

$$\cos\theta = \frac{1}{5}$$

Step 17: Find θ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ such that $\cos\theta = \frac{1}{5}$.

$\cos\theta = \frac{1}{5} > 0$, so θ is in the first quadrant or fourth quadrant.

The interval includes the first quadrant $[0, \frac{\pi}{2}]$ and the second quadrant $[\frac{\pi}{2}, \frac{3\pi}{4}]$, and also negative angles down to $-\frac{\pi}{4}$.

Since $\cos\theta = \frac{1}{5}$ is positive, θ must be in the first quadrant.

Step 18: Calculate $\theta = \arccos\left(\frac{1}{5}\right)$.

This is the principal value of $\sec^{-1}(5)$ in the given interval.

Final answer:

$$\sec^{-1}(5) = \arccos\left(\frac{1}{5}\right)$$

Numerically,

$$\arccos\left(\frac{1}{5}\right) \approx 1.3694$$

which lies in $\left[0, \frac{\pi}{2}\right] \subset \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

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$$\boxed{\arccos\left(\frac{1}{5}\right)}$$

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Final line:

$$\arccos\left(\frac{1}{5}\right)$$