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A Closed-Form GARCH Option Valuation Model

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This paper develops a closed-form option valuation formula for a spot asset whose variance follows a GARCH(p, q) process that can be correlated with the returns of the spot asset. It provides the first readily computed option formula for a random volatility model that can be estimated and implemented solely on the basis of observables. The single lag version of this model contains Heston's (1993) stochastic volatility model as a continuous-time limit. Empirical analysis on S&P500 index options shows that the *out-of-sample* valuation errors from the single lag version of the GARCH model are substantially lower than the *ad hoc* Black–Scholes model of Dumas, Fleming and Whaley (1998) that uses a separate implied volatility for each option to fit to the smirk/smile in implied volatilties. The GARCH model remains superior even though the parameters of the GARCH model are held constant and volatility is filtered from the history of asset prices while the *ad hoc* Black–Scholes model is updated every period. The improvement is largely due to the ability of the GARCH model to simultaneously capture the correlation of volatility with spot returns and the path dependence in volatility.

Since Black and Scholes (1973, henceforth BS) and Merton (1973) originally developed their option valuation formulas, researchers have developed option valuation models that incorporate stochastic volatility (see Heston (1993) and the references therein). The two types of volatility models have been continuous-time stochastic volatility models and discrete-time Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models. A related class of volatility models is the implied binomial tree or the deterministic volatility models of Derman and Kani (1994), Dupire (1994), and Rubinstein

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(1994) in which the spot volatility is a function of the current asset price and time only.

This paper presents an option formula for a stochastic volatility model with Generalized Autoregressive Conditional Heteroskedasticity (GARCH). The new formula describes option values as functions of the current spot price and the observed path of historical spot prices. It captures both the stochastic nature of volatility and correlation between volatility and spot returns. On a daily frequency the model is numerically close to the continuous-time stochastic volatility model of Heston (1993), but much easier to apply with available data. Our empirical analysis on S&P 500 index options shows that the out-of-sample valuation errors from the GARCH model are much lower than those from other models, including heuristic rules that are used by market makers to fit to the variations in implied volatilities across strike prices and maturities. The GARCH model successfully predicts out-of-sample option prices because it exploits the correlation of volatility with the path of stock returns. In addition to improving the prediction of volatility, the correlation parameter induces strike price and maturity patterns across option values such as the pronounced smirk in implied volatilities in the index options market.

Continuous-time stochastic volatility models [for example Heston (1993)] are difficult to implement and test. Although these models assume that volatility is observable, it is impossible to exactly filter a volatility variable from discrete observations of spot asset prices in a continuous-time stochastic volatility model. Consequently it is not possible to compute out-of-sample options valuation errors from the history of asset returns. Also the unobservability of volatility implies that one has to use implied volatilities computed from option prices to value other options. Holding the model parameters constant through time [as in Bates (1996, 1999) and Nandi (1998)], this approach requires estimating numerous implied volatilities from options records, one for every date and is computationally very burdensome in a long time series of options records. Another alternative is to estimate all parameters (including volatility) daily from the cross-section of observed option prices as in Bakshi, Cao and Chen (1997) or directly using the BS implied volatility from a particular option/options as a proxy for the unobserved spot volatility as in Knoch (1992). However, using implied volatilities to value an option requires the use of other contemporaneous options that may not always be feasible if one does not have reliable option prices such as in cases of thinly traded or illiquid markets.

In contrast to the continuous-time models, GARCH models have the inherent advantage that volatility is readily observable from the history of asset prices. As a result, a GARCH option model allows one to value an option using spot volatilities computed directly from the history of asset returns without necessarily using the implied volatilities inferred from other contemporaneous options. Thus it is possible to value an option solely on the basis of observables because the parameters of the valuation formula can be readily estimated from the discrete observations of asset prices. If a closed-form solution were available, a GARCH model would enable one to readily combine the cross-sectional information in options with the information in the time series of the underlying asset. Since volatility is a readily computed function of the history of asset prices, only a finite number of parameters need to be estimated irrespective of the length of the time series, thus considerably simplifying the estimation procedure.

Unfortunately, existing GARCH models do not have closed-form solutions for option values. These models are typically solved by simulation [Engle and Mustafa (1992), Amin and Ng (1993), Duan (1995)] that can be slow and computationally intensive for empirical work. More recently, Duan, Gauthier and Simonato (1999) provide a series approximation and Ritchken and Trevor (1999) provide a lattice approximation to value American options for GARCH processes with single lags in the variance dynamics. In contrast, this paper develops a closed-form solution for European option values (and hedge ratios) in a GARCH model. The model allows for multiple lags in the time series dynamics of the variance process and also allows for correlation between returns of the spot asset and variance. The single lag version of the model reconciles the discrete-time GARCH approach with the continuous-time stochastic volatility approach to option valuation by including Heston's (1993) closed-form stochastic volatility model as a continuous-time limit. This generalizes the Black–Scholes and Merton approach to option valuation because it is possible to value options by the absence of arbitrage only in the continuous-time limit, even though volatility is path dependent. In the BS model option values are functions of the current spot asset price, while in the GARCH model option values are functions of current and lagged spot prices. Except for this difference the models are operationally similar.

We test the empirical implications of our GARCH model in the S&P 500 index options market. As a benchmark model we choose the *ad hoc* BS model of Dumas, Fleming and Whaley (1998, henceforth DFW) that has the flexibility of fitting to the strike and term structure of observed implied volatilities by using a separate implied volatility for each option. It is found that the GARCH model has smaller valuation errors (out-of-sample) than the *ad hoc* BS model even though the *ad hoc* model is updated every period. In contrast, the parameters of the GARCH model are held constant over a sample period and variance is filtered from the history of asset prices. When we update the parameters of the GARCH model every period, the out-of-sample prediction errors decrease even further and substantially. Also the out-of-sample results remain essentially unchanged if we use the S&P 500 futures to filter the spot variance for our GARCH model instead of the S&P 500 cash index.

Our out-of-sample valuation results stand in contrast to previous empirical tests of the implied binomial tree/deterministic volatility models. In these tests DFW (1998) found that the *ad hoc* BS model dominated the deterministic volatility models in terms of out-of-sample options valuation errors in the S&P 500 index options market. Most of the options valuation improvements by the GARCH model are seen to result from its ability to simultaneously capture the path dependence in volatility and the negative correlation of volatility with index returns. This negative correlation allows the model to quickly adapt to changes in volatility associated with changes in the market levels. Also the negative correlation generates a negative skewness in the risk-neutral distribution of the S&P 500 index return. This is associated with the strike price and maturity specific biases in the index options market.

Section 1 describes the GARCH process and presents the option formula. Section 2 applies it to the S&P500 index option data, Section 3 reports the in-sample and out-of-sample results, while Section 4 concludes. Appendix A contains detailed calculations and derivations of the option formula while Appendix B contains the calculations regarding the convergence of the GARCH model to its continuous-time limit.

1. The Model

The model has two basic assumptions. The first assumption is that the log-spot price follows a particular GARCH process.

Assumption 1. *The spot asset price, $S(t)$ (including accumulated interest or dividends) follows the following process over time steps of length Δ,*

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t) \tag{1a}$$

$$h(t) = \omega + \sum_{i=1}^{p} \beta_i h(t-i\Delta) + \sum_{i=1}^{q} \alpha_i (z(t-i\Delta) - \gamma_i \sqrt{h(t-i\Delta)})^2, \tag{1b}$$

where r is the continuously compounded interest rate for the time interval Δ and $z(t)$ is a standard normal disturbance. $h(t)$ is the conditional variance of the log return between $t-\Delta$ and t and is known from the information set at time $t-\Delta$. The conditional variance in equation (1b), although distinct from the classic GARCH models of Bollerslev (1986) and Duan (1995), is quite similar to the NGARCH and VGARCH models of Engle and Ng (1993). The conditional variance $h(t)$ appears in the mean as a return premium. This allows the average spot return to depend on the level of risk.[1] Equation (1a) assumes that the expected spot return exceeds the riskless rate by an amount proportional to the variance $h(t)$. Since volatility equals the square root of $h(t)$, this implies the *return premium per unit of risk* is also proportional

[1] We assume that λ is constant, but option prices are very insensitive to this parameter. The functional form of this risk premium, $\lambda h(t)$, prevents arbitrage by ensuring that the spot asset earns the riskless interest rate when the variance equals zero.

to the square root of $h(t)$, exactly as in the Cox, Ingersoll and Ross (1985) model. In particular limiting cases the variance becomes constant. As the α_i and β_i parameters approach zero, it is equivalent to the Black–Scholes model observed at discrete intervals.

This paper will focus on the first-order case ($p = q = 1$) for pointing out some of the properties of the particular GARCH process. The first-order process remains stationary with finite mean and variance if $\beta_1 + \alpha_1\gamma_1^2 < 1$.[2] In this model one can directly observe $h(t + \Delta)$, at time t, as a function of the spot price as follows:

$$h(t+\Delta) = \omega + \beta_1 h(t) + \alpha_1 \frac{(\log(S(t)) - \log(S(t-\Delta)) - r - \lambda h(t) - \gamma_1 h(t))^2}{h(t)}. \quad (2)$$

α_1 determines the kurtosis of the distribution and α_1 being zero implies a deterministic time varying variance. The γ_1 parameter results in asymmetric influence of shocks; a large negative shock, $z(t)$ raises the variance more than a large positive $z(t)$. In general the variance process $h(t)$ and the spot return are correlated as follows,

$$\mathrm{Cov}_{t-\Delta}[h(t+\Delta), \log(S(t))] = -2\alpha_1\gamma_1 h(t). \quad (3)$$

Given positive α_1, positive value for γ_1 results in negative correlation between spot returns and variance. This is consistent with the postulate of Black (1976) and the leverage effect documented by Christie (1982) and others. Thus, γ_1 controls the skewness or the asymmetry of the distribution of the log-returns [see Heston, (1993) for an illustration in a continuous time framework]; the distribution is symmetric if γ_1 (and λ) is zero.

Although equations (1a) and (1b) refer to a stochastic process observed at a prescribed time interval Δ, they have an interesting continuous-time limit. It can be shown following Foster and Nelson (1994) that as the observation interval, Δ shrinks, the variance process $h(t)$ converges weakly to a variance process, $v(t)$ which is the square-root process of Feller (1951), Cox, Ingersoll and Ross (1985), and Heston (1993)

$$dv = \kappa(\theta - v)dt + \sigma\sqrt{v}dz, \quad (4)$$

where $z(t)$ is a Wiener process. The details are in Appendix B. Consequently the option valuation model (1) contains Heston's (1993) continuous-time

[2] In the multiple factor case one must add the additional condition that the polynomial roots of $x^p - \sum_{i=1}^{p}(\beta_i + \alpha_i\gamma_i^2)x^{p-i}$ lie inside the unit circle.

stochastic volatility model (that also admits a closed-form solution for option values) as a special case.[3]

At this point we cannot value options or other contingent claims because we do not know the risk-neutral distribution of the spot price. Motivated by previous lognormal option formulas, we rewrite equation (1) in the form

$$\log(S(t)) = \log(S(t-\Delta)) + r - \frac{1}{2}h(t) + \sqrt{h(t)}z^*(t) \tag{5a}$$

$$\begin{aligned} h(t) = \omega + \sum_{i=1}^{p} \beta_i h(t-i\Delta) + \sum_{i=2}^{q} \alpha_i \left(z(t-i\Delta) - \gamma_i \sqrt{h(t-i\Delta)}\right)^2 \\ + \alpha_1 \left(z^*(t-\Delta) - \gamma_1^* \sqrt{h(t-\Delta)}\right)^2, \end{aligned} \tag{5b}$$

where,

$$z^*(t) = z(t) + \left(\lambda + \frac{1}{2}\right)\sqrt{h(t)},$$

$$\gamma_1^* = \gamma_1 + \lambda + \frac{1}{2}.$$

Equations (5a) and (5b) appear to be "risk-neutral" versions of equations (1a) and (1b), though at this point, equations (5a) and (5b) are merely algebraic rearrangements of equation (1a) and (1b); there is no reason for the risk-neutral distribution of $z^*(t)$ to be normal. In order for $z^*(t)$ to have a standard normal risk-neutral distribution, we introduce Assumption 2.

Assumption 2. *The value of a call option with one period to expiration obeys the Black–Scholes-Rubinstein formula.*

This assumption is equivalent to Duan's (1995) valuation assumption. The Black–Scholes-Rubinstein formula is natural to use here because the spot price has a conditionally lognormal distribution over a single period. However, BS prices do not follow from absence of arbitrage with discrete-time trading. Instead, one must appeal to other arguments such as those of Rubinstein (1976) and Brennan (1979). If the BS formula holds for a single period, then the risk-neutral distribution of the asset price is lognormal with mean, $S(t-\Delta)e^r$. This implies that one can find a random variable, $z^*(t)$ which has a standard normal distribution under the risk-neutral probabilities.[4] We formalize this property as the following proposition.

[3] We can value options by the absence of arbitrage alone in the continuous-time model as the asset returns and variance are instantaneously perfectly correlated. Note however that returns and volatility are not perfectly correlated over any discrete interval of time.

[4] For details of the preference assumptions that give rise to risk neutralization in a discrete-time model with continuously distributed returns, one can refer to Duan (1995) for a GARCH model and Rubinstein (1976) and Brennan (1979) for the Black–Scholes model.

Proposition 1. *The risk-neutral process takes the same GARCH form as equations* (1a) *and* (1b) *with* λ *replaced by* $-1/2$ *and* γ_1 *replaced by* $\gamma_1^* = \gamma_1 + \lambda + 1/2$.

The proof of this proposition is trivial by noting that $z^*(t)$, γ_1^* and λ as defined above make the one period return from investing in the spot asset equal to the risk free rate in equation (5a). Assumptions 1 and 2 allow us to derive the values of all contingent claims that can be written as functions of the spot asset price. Since long-term options are functions of $S(t)$ and $h(t + \Delta)$, and $h(t + \Delta)$ can be written as a function of $S(t)$ in equation (2), this includes options of all maturities. Also note that unlike continuous-time stochastic volatility models, in the GARCH model, the parameter that governs the risk-neutral skewness, γ_1^* and the parameter that governs the actual skewness, γ_1 can differ due to the risk premium parameter, λ.

The risk premium parameter, λ has some useful interpretations. In practice, λ exerts a negligible influence on the current filtered values of $h(t)$. But it does change the risk-neutral distribution of future variance. The variance, $h(t+\Delta)$ does not carry an independent risk premium in this model. But since our empirical evidence shows that variance is highly negatively correlated with spot S&P 500 index returns, variance should behave like a "negative beta" asset.[5] If the spot asset has a positive return premium ($\lambda > -1/2$) and variance is negatively correlated with spot returns ($\gamma_1 > 0$) then Proposition 1 (and equation B1 of Appendix B) shows that the risk-neutral drift of variance will be higher than the true drift of variance. This means that implied volatility will typically be higher than expected future volatility. For example, the term structure of volatility inferred from option prices may be upward sloping even when the spot volatility, $h(t + \Delta)$ is not expected to increase. In this case a trader using the Black–Scholes formula would incorrectly perceive an arbitrage opportunity from selling and hedging volatility-sensitive options, e.g., long-term at-the-money straddles. However the Black–Scholes hedge ratio does not accommodate GARCH effects. Since volatility is very negatively correlated with S&P 500 returns, a short option position with a Black–Scholes hedge would actually be bullish on the market. While the option position would earn an expected return premium, the premium would be no higher than the return on a comparable levered market position. And in the event of a sharp market downturn, volatility would rise significantly, causing the short option position to lose considerable value. This is consistent with losses at "arbitrage" hedge funds, such as Long Term Capital Management, which held short index option positions in some markets in the third quarter of 1998 [Dunbar (1999)].

[5] Using equations (3) and (B2) of Appendix B, with the parameter estimates in Table 1 from spot S&P 500 returns, gives a correlation of -0.96 when $h(t)$ is at its long-run value. This is a reasonable approximation to the limiting diffusion case of Appendix B with perfect negative correlation.

We proceed to solve for the generating function of the GARCH process of (1a) and (1b) and use it to produce option values.

Let $f(\phi)$ denote the conditional generating function of the asset price

$$f(\phi) = E_t[S(T)^{\phi}]. \tag{6}$$

This is also the moment generating function of the logarithm of $S(T)$. The function $f(\phi)$ depends the parameters and state variables of the model, but these arguments are suppressed for notational convenience. We shall use the notation $f^*(\phi)$ to denote the generating function for the risk-neutral process in (5a) and (5b).

Proposition 2. *The generating function takes the log-linear form*

$$f(\phi) = S(t)^{\phi} \exp\Bigg(A(t; T, \phi) + \sum_{i=1}^{p} B_i(t; T, \phi) h(t + 2\Delta - i\Delta) + \sum_{i=1}^{q-1} C_i(t; T, \phi)(z(t + \Delta - i\Delta) - \gamma_i \sqrt{h(t + \Delta - i\Delta)})^2\Bigg), \tag{7}$$

where,

$$A(t; T, \phi) = A(t + \Delta; T, \phi) + \phi r + B_1(t + \Delta; T, \phi)\omega - \frac{1}{2}\ln(1 - 2\alpha_1 B_1(t + \Delta; T, \phi)) \tag{8a}$$

$$B_1(t; T, \phi) = \phi(\lambda + \gamma_1) - \frac{1}{2}\gamma_1^2 + \beta_1 B_1(t + \Delta; T, \phi) + \frac{1/2(\phi - \gamma_1)^2}{1 - 2\alpha_1 B_1(t + \Delta; T, \phi)}, \tag{8b}$$

for the single lag ($p = q = 1$) version and these coefficients can be calculated recursively from the terminal conditions:

$$A(T; T, \phi) = 0. \tag{9a}$$

$$B_1(T; T, \phi) = 0. \tag{9b}$$

The appendix derives the recursion formulas for the coefficients $A(t; T, \phi)$, $B_i(t; T, \phi)$, and $C_i(t; T, \phi)$ in the general case i.e., for any p and q.

Since the generating function of the spot price is the moment generating function of the logarithm of the spot price, $f(i\phi)$ is the characteristic function of the logarithm of the spot price. Note that to use the characteristic function, ϕ in equations (8a) and (8b) must be replaced by $i\phi$ everywhere. One can calculate probabilities and risk-neutral probabilities following Feller (1971) or Kendall and Stuart (1977) by inverting the characteristic function.

Proposition 3. *If the characteristic function of the log spot price is* $f(i\phi)$ *then*

$$E_t[\text{Max}(S(T)-K,0)] = f(1)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi f(1)}\right]d\phi\right)$$
$$-K\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right), \quad (10)$$

where Re[ ] *denotes the real part of a complex number.* Proposition 3 involves a somewhat new inversion formula, different from that of Heston (1993) and others. In particular, it enables us to calculate the expectation in (10) once we just have the characteristic function of the logarithm of the spot price, instead of calculating two separate integrals.[6]

An option value is simply the discounted expected value of the payoff, $\text{Max}(S(T)-K,0)$ calculated using the risk-neutral probabilities, i.e., using the characteristic function, $f^*(i\phi)$. In particular, a European option value is given by the following corollary.

Corollary. *At time t, a European call option with strike price K that expires at time T is worth*

$$C = e^{-r(T-t)}E_t^*[\text{Max}(S(T)-K,0)] = \frac{1}{2}S(t)$$
$$+\frac{e^{-r(T-t)}}{\pi}\int_0^\infty \text{Re}\left[\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}\right]d\phi$$
$$-\text{K}e^{-r(T-t)}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}\right]d\phi\right), \quad (11)$$

where $E_t^*[]$ *denotes the expectation under the risk-neutral distribution.* This completes the option valuation formula.[7] As in the Black–Scholes formula, (11) can be written as the asset price multiplied by a probability, $P_1()$ and the discounted strike price multiplied by a probability, $P_2()$. $P_2()$ is the risk neutral probability of the asset price being greater than K at maturity and the delta of the call value is simply $P_1()$. The other hedge ratios like the vega and the gamma can be calculated by straight differentiation in equation (11) and the expression for $P_1()$ respectively. Put option values can be calculated using the put-call parity. In contrast to the Black–Scholes formula, this formula is a function of the current asset price, $S(t)$, and the conditional variance,

[6] The new inversion formula exploits the inherent relationship between the two probabilities, $P_1()$ and $P_2()$ of a European option valuation model, thus requiring the calculation of only one integral, instead of two separate integrals as in Heston (1993).

[7] The integrands converge very rapidly and the integration can be very efficiently performed in fractions of a second using a numerical integration routine such as Romberg's method on an open interval [Press et al. (1992)] or quadrature based integration routines (end of Appendix A has some sample option values).

$h(t+\Delta)$. Since $h(t+\Delta)$ is a function of the observed path of the asset price, the option formula is effectively a function of current and lagged asset prices. In contrast to continuous-time models, volatility is a readily observable function of historical asset prices and need not be estimated with other procedures.

The next section describes the empirical performance of the single lag $(p = q = 1)$ version of the GARCH model in the S&P 500 index options market.

2. Empirical Analysis

The empirical analysis starts with a description of the options data. It proceeds to estimate the GARCH model with time series data on index returns and with options data.

2.1 Description of data

Intra-day data on S&P 500 index options traded on the Chicago Board Options Exchange (CBOE) are used to test the model. The raw data set is obtained directly from the exchange. The market for S&P 500 index options is the second most active index options market in the United States and, in terms of open interest in options, it is the largest. Unlike options on the S&P 100 index, there are no wild card features [see Fleming and Whaley (1994), French and Maberly (1992)] that can complicate valuation. Also it is easier to hedge S&P 500 index options because there is a very active market for the S&P 500 futures. In fact, according to Rubinstein (1994) it is one of the best markets for testing a European option valuation model.

As many of the stocks in the S&P 500 index pay dividends, one needs a time series of dividends for the index. We use the daily cash dividends for the S&P 500 index collected from the S&P 500 information bulletin.[8] We arrive at the present value of the dividends and subtract it from the current index level. For the risk free rate, the continuously compounded Treasury bill rates (from the average of the bid and ask discounts reported in the *Wall Street Journal*), interpolated to match the maturity of the option is used. Also the change in the expiration time of the SPX options on August 24, 1992 from close to open (see DFW) resulting in the reduction of the time to expiration by one day was taken into account.

The intra-day data set is sampled every Wednesday (or the next trading day if Wednesday is a holiday)[9] between 2:30 P.M. and 3:15 P.M. (central standard time, CST) for the years 1992, 1993 and 1994 to create the

[8] We thank the referee for suggesting the use of this dividend series and Jeff Fleming and the referee for making the dividend series available to us.

[9] In our options sample, all but one day are Wednesdays.

data that we work on. We follow DFW (1998) in filtering the intra-day data to create weekly data and use the mid-point of the bid-ask quote as the option price. As in DFW only options with absolute moneyness, $|K/F - 1|$, (K is the strike and F is the forward price) less than or equal to ten percent are included. In terms of maturity, options with time to maturity less than six days or greater than one hundred days are excluded.[10] However, unlike DFW, we do not infer the index level simultaneously with the other parameters in the estimation procedure. Instead, the level of the S&P 500 index reported for a particular record is used. The S&P 500 is a value-weighted index and the bigger stocks that trade more frequently constitute the bulk of the index level. Since intra-day data and not the end-of-the-day option prices is used, the problem with the index level being somewhat stale may not be severe enough to undermine an estimation procedure. In theory, one could possibly overcome this problem by using implied index levels from the put-call parity equation. However, this is conditional on put-call parity holding as an equality and in the presence of transactions costs (bid-ask spreads that are non-negligible), the equality becomes an inequality. Thus the implied index levels from the put-call parity equation may not equal the true index level. Also, even if one assumes away transactions costs, it is very difficult to create a sample of sufficient size by creating matched pairs of puts and calls because the level of the S&P 500 index changes quite frequently through the day. Another alternative is to use the implied index levels from S&P 500 futures prices. However, one must then assume that the futures and the options market are closely integrated.[11] The following criteria are also used as filters.

1) An option of a particular moneyness and maturity is represented only once in the sample on any particular day. In other words, although the same option may be quoted again in our time window (with same or different index levels) on a given day, only the first record of that option is included in our sample for that day.
2) A transaction must satisfy the no-arbitrage relationship (Merton, 1973) in that the call price has to be greater than or equal to the spot price minus the present value of the remaining dividends minus the discounted strike price. Similarly, the put price has to be greater than or equal to the present value of the remaining dividends plus the discounted strike price minus the spot price.

The data set consists of 10,100 records/observations. The average number of options per day is 65 with a minimum of 24 and a maximum of 106. The average bid-ask spread is \$0.481. The number of options of distinct maturities for various days were: 30 days—two maturities, 115 days—three maturities and 11 days—four maturities.

[10] See DFW for justification of the exclusionary criteria about moneyness and maturity.

[11] As will be discussed later, we do, however, use the S&P 500 futures data to address potential problems in filtering volatility from the history of asset prices in the GARCH model.

2.2 Estimation

The empirical analysis focuses mainly on the single lag version of the GARCH model. We set $\Delta = 1$ and use daily index returns to model the evolution of volatility. Unlike continuous-time stochastic volatility models in which the volatility process is unobservable, all the parameters in our valuation formula can be easily estimated directly from the history of asset prices. We do this with the maximum likelihood estimation (MLE) used by Bollerslev (1986) and many others.[12] To illustrate the importance of the skewness parameter, γ_1, we performed this estimation with an unrestricted model and with a restricted GARCH model in which γ_1 was constrained to equal zero (symmetric GARCH). Table 1 shows the maximum likelihood estimates of the GARCH model, both when γ_1 is non-zero and when it is restricted to zero, on the daily S&P 500 levels, closest to 2:30 P.M. (CST—Central Standard Time) from 01/08/92 to 12/30/94. The skewness parameter, γ_1 is substantially positive indicating that shocks to returns and volatility are strongly negatively correlated. Using a likelihood ratio test, the symmetric version is easily rejected implying that the negative correlation between returns and volatility is a significant feature of the S&P 500 time series. The daily volatility series (annualized) are shown in Figure 1A and 1B for the unrestricted and restricted/symmetric versions of the model.[13] These figures show that the skewness parameter γ_1 has an important effect on the qualitative behavior of the variance process. Including this parameter makes the filtered variance more volatile, and produces sudden drops in volatility in addition to sudden increases.

We also investigate how different our maximum likelihood estimation results would be if we use S&P 500 futures prices to imply out the S&P 500 index levels. Towards this purpose we use the closest to 2:30 P.M. (CST) lead/nearest maturity S&P 500 futures prices from 01/08/92 to 12/30/94 to get the implied S&P 500 index levels. These futures prices are created from tick-by-tick S&P 500 futures data sets that are obtained from the Futures Industry Institute.[14] Given a discrete dividend series, we use the following equation [see Hull (1998)] to get the implied spot price (i.e. S&P 500 index level)

$$F(t) = (S(t) - \text{PVDIV})e^{r(t)(T-t)} \tag{12}$$

where $F(t)$ denotes the futures price, PVDIV denotes the present value of dividends to be paid from time t until the maturity of the futures contract

[12] The procedure sets $h(0)$ equal to the sample variance of the changes in the logarithm of $S(t)$. Due to the strong mean reversion of volatility, all results were insensitive to the starting value of $h(0)$.

[13] Unreported results show that various symmetric and asymmetric GARCH specifications of Engle and Ng (1993) produce similar results to our symmetric and asymmetric models, respectively on the same data set. The values of the likelihood function are very similar and the time series graphs of filtered volatility (Figures 1A and 1B) lie virtually on the top of each other.

[14] We thank the referee for pointing us towards this source for obtaining tick-by-tick S&P 500 futures data.

Table 1

Maximum likelihood estimation

	α_1	β_1	γ_1	ω	λ	θ	$\beta_1+\alpha_1\gamma_1^2$	Log-Likelihood
GARCH (spot)	$1.32e{-}6$	0.589	421.39	$5.02e{-}6$	0.205	9.51%	0.823	3503.7
	$(0.03e{-}6)$	(0.007)	(11.01)	$(0.19e{-}6)$	(0.228)			
GARCH, $\gamma_1 = 0$ (spot)	$1.07e{-}6$	0.922		$1.63e{-}6$	0.732	9.33%	0.922	3492.4
	$(0.04e{-}6)$	(0.013)		$(0.43e{-}6)$	(0.22)			
GARCH (futures)	$1.33e{-}6$	0.586	424.69	$4.96e{-}6$	0.335	9.54%	0.826	3482.8
	$(0.03e{-}6)$	(0.006)	(9.2)	$(0.13e{-}6)$	(0.228)			
GARCH, $\gamma_1 = 0$ (futures)	$1.71e{-}6$	0.859		$3.57e{-}6$	0.671	9.71%	0.859	3467.6
	$(0.04e{-}6)$	(0.02)		$(0.43e{-}6)$	(0.22)			

Maximum Likelihood Estimates of the GARCH model with $p = q = 1$ and $\Delta = 1$ (day) using the spot/cash S&P 500 levels and the S&P 500 futures prices for the unrestricted ($\gamma_1 \neq 0$) and restricted ($\gamma_1 = 0$) model.

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t),$$

$$h(t) = \omega + \beta_1 h(t-\Delta) + \alpha_1\left(z(t-\Delta) - \gamma_1\sqrt{h(t-\Delta)}\right)^2$$

The log-likelihood function is $\sum_{t=1}^{T} -0.5\left(\log(h(t)) + z(t)^2\right)$, where T is the number of days in the sample. The daily cash index levels closest to (before) 2:30 P.M. (central standard time) from 01/08/92–12/30/94 are used. The futures prices are those of the shortest/lead maturity contracts closest to (before) 2:30 P.M. (central standard time). Number of Observations = 755. Asymptotic standard errors appear in parentheses. θ defined to be equal to $\sqrt{252(\omega+\alpha_1)/(1-\beta_1-\alpha_1\gamma_1^2)}$ is the annualized (252 days) long run volatility (standard deviation) implied by the parameter estimates. $\beta_1+\alpha_1\gamma_1^2$ measures the degree of mean reversion in that $\beta_1+\alpha_1\gamma_1^2 = 1$ implies that the variance process is integrated.

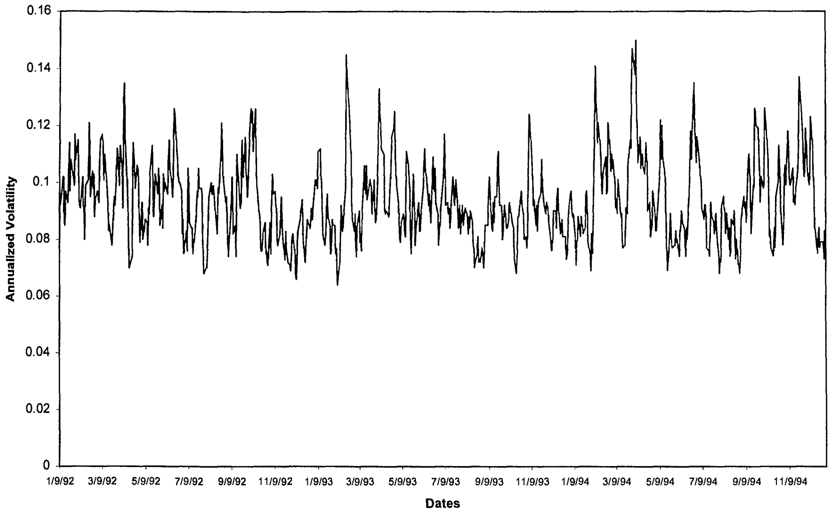

Figure 1A
This figure shows the daily annualized spot volatility from the unrestricted/asymmetric GARCH model from January 09, 1992 to December 30, 1994 using daily S&P 500 index levels (closest to and before 2:30 P.M., central standard time).

at time, T and $r(t)$ is the continuously compounded Treasury bill rate (from the average of the bid and ask discounts reported in the *Wall Street Journal*), interpolated to match the maturity of the futures contract.

Table 1 also reports the various parameter estimates and the value of the log-likelihood function using the futures data set. The parameters are very similar across the analysis of cash/spot and futures data for the unrestricted/asymmetric model. As with the cash/spot data, shocks to returns and volatility are negatively correlated (i.e. $\gamma_1 > 0$) using the S&P 500 futures data. Other features of the time series dynamics of volatility are quite similar across the cash/spot and futures data sets. For example, the parameter that measures the degree of mean reversion (as given by $\beta_1 + \alpha_1\gamma_1^2$) is 0.823 from the cash/spot data and 0.826 from the futures data. Similarly, the volatility of volatility, as measured by α_1, is $1.32e{-6}$ from the cash/spot data and $1.33e{-6}$ from the futures data. The annualized long-run mean of volatility/standard deviation as given by $\sqrt{252(\omega + \alpha_1)/(1 - \beta_1 - \alpha_1\gamma_1^2)}$ (assuming a year with 252 trading days) is 9.51% from the cash/spot data and 9.54% for the futures data. This is reflected in the actual filtered time series of $\sqrt{h(t+1)}$ (i.e. volatility, not variance). For example, the average (across our sample of 755 days) annualized level of $\sqrt{h(t+1)}$ is 9.34% from the cash/spot data and 9.46% from the futures data.[15] The mean

[15] For each day in our sample, the annualized level of volatility is $\sqrt{252\,h(t+1)}$.

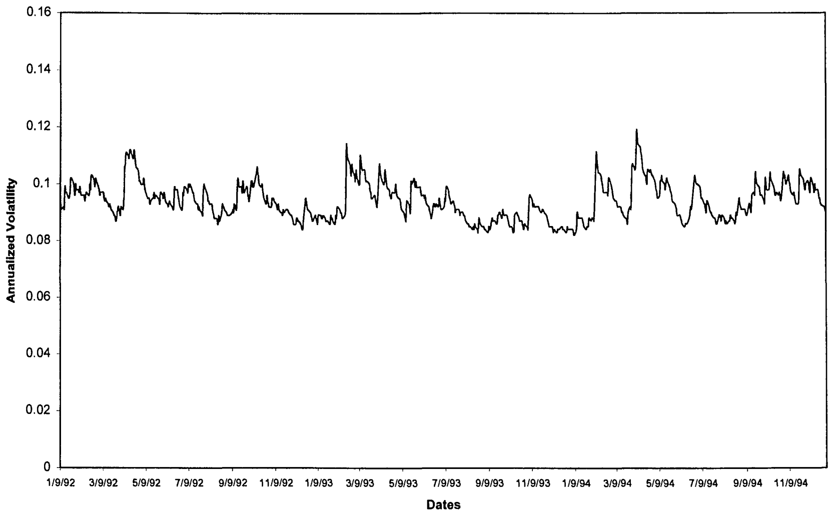

Figure 1B
This figure shows the daily annualized spot volatility from the restricted/symmetric GARCH model from January 09, 1992 to December 30, 1994 using daily S&P 500 index levels (closest to and before 2:30 P.M., central standard time).

difference (futures – spot) in the annualized volatility between the two time series is 0.12% with a standard deviation of 0.249% and a maximum difference of 1.4%.

In the restricted/symmetric model (i.e. $\gamma_1 = 0$), the differences between the parameter estimates from the futures and the cash/spot are somewhat greater than in the unrestricted model. For example, the annualized long-run mean of volatility is 9.33% from the cash/spot data and 9.71% from the futures data. Similarly, the average annualized level of $\sqrt{h(t+1)}$ is 9.39% from the cash/spot data and 9.72% from the futures data. The mean difference (futures – spot) between the two time series of annualized volatility is 0.3% with a standard deviation of 0.31% and a maximum difference of 2.3%. Note however that as with the spot/cash data, the futures data also rejects the restricted model in favor of the unrestricted model (using likelihood ratio test).

One could plug the parameter estimates obtained from the above MLEs (using historical asset prices) into the options valuation formula to compute option values. However, the information that one would be using is only historical and could be different from the expectations about the future evolution of the asset price that are embedded in option prices. In contrast, the information in option prices is forward looking. Since our model has a closed-form solution for option values, a natural candidate for parameter estimation is a non-linear least squares (NLS) procedure that tries to match model option values to observed option prices as closely as possible.

The option value at time t is not only a function of the current level of variance, $h(t+1)$ but also of the parameters that drive the variance process, namely, α_1, β_1, γ_1, λ and ω. One could do a cross-sectional fitting every week (i.e. each Wednesday) to imply out all these parameters including the variance, $h(t+1)$. But purely cross-sectional estimation from a single days's data suffers from two problems if the model has more parameters to be estimated than a single implied volatility as in the BS model. First of all, because of the limited sample size, there is a problem of overfitting as noted in DFW (1998). Furthermore, in the context of our model, this procedure does not use the information in the evolution of the index or equivalently the time series of historical volatilities. It is quite possible that the history of the index provides some information about the future over and above the information contained in the option prices. Our model can readily exploit the combined information in the history of asset prices (as variance is observable) and the cross-section of option prices (due to a closed-form solution). In order to take the implications of the model seriously, we hold the time invariant parameters constant over the first six-month period of each year. Later on we relax this restriction and allow the parameters of the GARCH model to be updated every week (in the process of computing out-of-sample valuation errors in the second half of each year). However, we always compute the variance, $h(t+1)$ from the history of asset prices.

As mentioned previously, for each year we choose the option prices in the interval, 2:30–3:15 P.M. (CST) for the first twenty six Wednesdays (or the next trading day) to create our sample. Let $e(i,t)$ denote the model error in valuing option i at time t, i.e., $e(i,t)$ is the difference between the model value of option and the market price of that option at time t. Then our criterion function is, $\sum_{t=1}^{T}\sum_{i=1}^{N_t} e(i,t)^2$, where T denotes the number of weeks in the sample and N_t is the number of options traded on the Wednesday (or the next trading day) of week t. The criterion function needs to be minimized over α_1, β_1, γ_1, ω and λ. Note that in order to minimize the above criterion function we need $h(t+1)$ for each t. However, at each t, $h(t+1)$ is a function of α_1, β_1, γ_1, ω, and λ and the history of asset prices [see equation (2)]. Therefore $h(t+1)$ is known at time t given these parameters.[16] This feature considerably simplifies our estimation procedure in contrast to continuous-time stochastic volatility models where the daily volatility is not known as a function of the history of asset prices. Consequently, one has to estimate the daily volatility separately. However, if the sample consists of a long time series of option prices, then the number of parameters increases proportionately with the number of days in the sample. In contrast, as noted previously, in our GARCH setting, only a few time-invariant parameters need to be estimated irrespective of the sample size.

[16] At each iteration of an optimization routine, $h(t+1)$ is needed to compute option prices. But given the parameters that are in use for that iteration, $h(t+1)$ is known from the history of asset prices.

For the BS model, a single implied volatility can be estimated for each day in the sample by minimizing the above criterion function. However, a BS model with a single implied volatility across all strikes and maturities, although consistent theoretically is perhaps too restrictive in practice. Since the GARCH model has four more parameters than the BS model, it may have an unfair advantage over the BS model. Therefore, we follow DFW and construct an *ad hoc* BS model in which each option has its own implied volatility depending on the strike price and time to maturity. Specifically, the spot volatility of the asset that enters the BS option formula is a function of the strike price and the time to maturity or combinations thereof (see DFW for details). For example, one functional form is,

$$\sigma = a_0 + a_1 K + a_2 K^2 + a_3 \tau + a_4 \tau^2 + a_5 K \tau, \tag{13}$$

where σ is the implied volatility (using BS) for an option of strike K and time to maturity τ. There are simpler parameterizations (subsets of the above, see DFW for details) that one can use when data are limited. As per DFW the particular functional form of σ we select on a given day depends on the number of distinct option maturities in the sample on that day. The coefficients of the *ad hoc* model are estimated every week via ordinary least squares, minimizing the sum of squared errors between the BS implied volatilities across different strikes (and maturities) and the model's functional form of the implied volatility.

The *ad hoc* BS model, although theoretically inconsistent is definitely a more challenging benchmark than the simple BS model for any competing option valuation model. Furthermore, DFW shows that the implied binomial tree or the deterministic volatility models of Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) underperform the *ad hoc* BS model in out-of-sample options valuation errors in the S&P 500 index options market. Thus comparing the GARCH to the *ad hoc* BS model should also yield insights on the relative efficacies of path-independent volatility models such as the implied binomial tree models and path-dependent volatility models such as the GARCH in terms of valuing options.

The NLS estimation procedure was carried out to estimate the parameters for the first six months of 1992, 1993 and 1994.[17] Although the options data are weekly, the conditional variance, $h(t+1)$, that is relevant for option values at time t is drawn from the daily evolution of index returns and not from the weekly evolution. Specifically we use the levels of the index closest to (before than) 2:30 P.M. (CST). The starting variance, $h(0)$, is kept fixed at the in-sample estimate of the variance i.e., the variance for the first six months of 1992, 1993 and 1994 respectively, computed from daily logarithmic returns.

[17] The NLS procedure that we use is the Levenberg-Marquardt method in Press et al. (1992).

3. Model Comparisons

This section describes the parameter estimates and in-sample and out-of-sample comparisons of the GARCH model with the BS model and the *ad hoc* BS model of DFW (1998).

3.1 In-sample model comparison

The parameter estimates from the NLS estimation for the three years, 1992, 1993, and 1994 and the average in-sample valuation errors appear in Table 2. The actual and risk-neutral skewness parameter, γ_1 and γ_1^*, respectively are always positive. This indicates that variance tends to rise when the index falls, and vice versa under both the actual and risk-neutral distribution. Because shocks to variance and index returns are negatively correlated, there is negative skewness in the distributions of multiperiod index returns. The long-run annualized volatility (standard deviation) implied by the options data for the three years, 1992, 1993 and 1994 are 13.1%, 10.6% and 10.6% respectively.[18]

The average option price in the sample is somewhat more than thirteen dollars, ranging from \$13.37 in 1993 to \$13.70 in 1994. The root mean squared error (RMSE) of the simple BS model is around one dollar in-sample, ranging from \$0.95 in 1992 to \$1.14 in 1994.[19] These are economically significant errors in matching market option prices. Nevertheless the BS model fits better than the symmetric GARCH ($\gamma_1^* = 0$) model. The root mean squared valuation errors (averaged across all options) for the symmetric GARCH model are \$1.06, \$0.986 and \$1.29 respectively, for the years 1992, 1993 and 1994. The symmetric GARCH model has worse in-sample fit than the BS model in every year. This version of the model allows conditional heteroskedasticity (as α_1 is non-zero) and it also allows a term structure of volatility. Since the BS model is a special case of the symmetric GARCH model, it may seem puzzling that the BS model fits better. However, the BS model has the advantage of using a different volatility to fit option prices each day. In contrast the symmetric GARCH model uses the actual time series of index returns to generate volatility. The symmetric GARCH model has more flexibility to fit the term structure of the option prices across maturity. But apparently this does not adequately compensate for the more accurate calibration of the simple BS model period by period.

The results improve dramatically with the full GARCH model that allows pronounced skewness/correlation effects with a nonzero γ_1^* parameter. This parameter allows better time-series tracking of volatility by accounting for changes in volatility that accompany upward or downward movements in the index. The skewness parameter also gives the model an important dimension of flexibility in valuing options across different strike prices. The root mean

[18] Since we only use options of up to 100 days in maturity, these numbers should be treated with caution.

[19] RMSE is the square root of the average squared valuation error.

Table 2

Non-linear least squares estimation

	α_1	β_1	γ_1	ω	λ	θ	$\beta_1+\alpha_1\gamma_1^2$	RMSE	Average price	Observations
1992										
BS								0.95	13.61	1744
GARCH-non-updated	1.16e−5	0.616	172.53	1.18e−5	−173.03	0.39	0.961	1.06	13.61	1744
($\gamma_1^*=0$)	(5.23e−6)	(0.128)	(151.1)	(5.9e−6)	(151.1)					
Ad hoc BS								0.526	13.61	1744
GARCH-non-updated	3.42e−6	0.286	426.99	2.76e−6	0.32	0.131	0.909	0.686	13.61	1744
	(0.52e−6)	(0.038)	(68.32)	(0.6e−6)	(0.49)					
1993										
BS								0.964	13.37	1750
GARCH-non-updated	1.08e−5	0.504	−196.57	9.09e−6	196.07	0.252	0.921	0.986	13.37	1750
($\gamma_1^*=0$)	(4.6e−7)	(0.08)	(138.6)	(1.21e−8)	(138.6)					
Ad hoc BS								0.586	13.37	1750
GARCH-non-updated	4.48e−6	0.105	412.31	1.48e−6	1.03	0.106	0.867	0.615	13.37	1750
	(0.12e−6)	(0.021)	(30.21)	(0.11e−6)	(2.05)					
1994										
BS								1.14	13.70	1662
GARCH-non-updated	9.39e−6	0.783	−122.95	8.0e−18	122.45	0.176	0.924	1.29	13.70	1662
($\gamma_1^*=0$)	(1.04e−7)	(0.068)	(108.7)	(3.2e−16)	(158.7)					
Ad hoc BS								0.644	13.70	1662
GARCH-non-updated	7.98e−7	0.214	969.24	8.22e−7	0.48	0.106	0.964	0.688	13.70	1662
	(0.45e−7)	(0.005)	(82.15)	(0.01e−6)	(1.06)					

Reports the parameter estimates and in-sample valuation errors from minimizing the sum of squared errors between model option values and market option prices in the first half of each year. Asymptotic standard errors appear in parentheses. Two versions of the GARCH model are used, one in which γ_1^* is unrestricted and another in which $\gamma_1^* = 0$. BS is the Black–Scholes model in which a single implied volatility is estimated across all strikes and maturities on a given day while *ad hoc* BS is an *ad hoc* version of the Black–Scholes model with strike and maturity specific implied volatilities. Both BS and the *ad hoc* BS are calibrated every week while the parameters of the GARCH model are held constant over the estimation period. θ is the annualized long run standard deviation under the GARCH (defined in Table 1) while $\beta_1+\alpha_1\gamma_1^2$ measures the degree of mean reversion in that $\beta_1+\alpha_1\gamma_1^2+1$ implies that the variance process is integrated. $\gamma_1^*=\gamma_1+\lambda+1/2$, measures the skewness of the risk neutral distribution. If $\gamma_1^*=0$, then $\gamma_1=-(\lambda+1/2)$. RMSE is the root mean squared pricing error (in \$). Average price is the average option price in the sample (in \$).

squared valuation errors for the GARCH model in 1992, 1993 and 1994 are 68.6 cents, 61.5 cents and 68.8 cents respectively. Recall that the corresponding errors in the BS model are 95 cents, 96.4 cents, and $1.14 respectively. Hence the GARCH model provides a substantially better in-sample fit. Thus although not updated and constrained to use the variance from the history of asset prices, the GARCH model fits better because it matches the shape of the option prices better. And the skewness parameter, γ_1^* is essential for this purpose. Bates (1999) and Nandi (1998) have previously emphasized the importance of skewness effects when applying continuous-time stochastic volatility models to explain index option prices.[20]

We should be able to get the best in-sample fit to options prices by using a weekly implied volatility along with a flexible fitting option formula. The *ad hoc* BS model has more flexibility than the GARCH model because it is designed to fit both the volatility smile in strike prices and the term structure of implied volatilities. Also it is updated every period. Table 2 shows that the *ad hoc* version of the BS model dominates the GARCH model in-sample in each of the three years. The in-sample RMSE over the first six months are 52.6 cents, 58.6 cents and 64.4 cents for 1992, 1993 and 1994 respectively. Thus a flexible but theoretically inconsistent model may appear to be better than our theoretically consistent GARCH model in terms of in-sample fit.

Of course the above comparison of the GARCH model with the *ad hoc* BS model is somewhat "unfair", because the implementation of the *ad hoc* BS model allows the model to be updated every week. It is not clear whether the improved in-sample fit of the *ad hoc* model stems from a more flexible functional form or from the instability of the functional form of the GARCH process over a long enough time period. In order to determine this we estimated an "updated" GARCH model by minimizing the sum of squared errors between model option values and market option prices, allowing the parameters to change every week. Although the parameters change each week, the variance, $h(t+1)$ is still drawn from the history of asset prices at time t. At each time we used the time-series of returns from the previous 252 days to filter the variance.[21] Since our ultimate goal is to compare out-of-sample valuation errors in the second half of each year, this updating is done only in the second half of each year. Table 3 presents a "fair" in-sample comparison of the *ad hoc* BS model with the updated GARCH model.[22] In every year the updated GARCH model fits options prices better than the *ad hoc* BS model in terms of root mean squared error, ranging from around 3 cents in 1994

[20] As mentioned earlier, the true time series skewness parameter is slightly different from the risk-neutral skewness parameter that affects option prices. However, in our model both of these skewness parameters are estimated to have the same sign.

[21] We have also used longer time intervals, such as two or three years for drawing the variance. However, the results are essentially the same due to the strong mean reversion in variance.

[22] To facilitate comparison with subsequent tables these results are reported for the second half of each year, but results are similar in the first half.

Table 3

In-sample comparison of the ad hoc BS model and the updated GARCH model

	RMSE	Average option price	Number of observations
1992			
Ad hoc BS	0.416	13.09	1550
GARCH (updated)	0.366	13.09	1550
1993			
Ad hoc BS	0.553	13.24	1511
GARCH (updated)	0.483	13.24	1511
1994			
Ad hoc BS	0.489	13.12	1881
GARCH-updated	0.459	13.12	1881

In-sample valuation errors (in $) from the weekly (every week) estimation using option prices in the second half of each year (1992, 1993 and 1994) for the updated GARCH and the *ad hoc* BS model. Note however that the last Wednesday of the first half of each year appears in this sample. Both the *ad hoc* BS and the GARCH model are estimated each week using ordinary least squares and non-linear least squares respectively. For the GARCH model, variance, $h(t+1)$ is drawn from the daily history (last 252 days) of S&P 500 levels (closest to and before 2:30 P.M., central standard time).

to around 5 cents in 1992 and 1993. Thus the flexibility of updating appears to make a difference in terms of its ability to fit options prices in-sample.

Table 4 reports the mean and standard deviations of the updated GARCH coefficients from this estimation. As we can see, the parameter that is least stable is ω, followed by β_1. The parameters, α_1 and γ_1 are relatively more stable. This should not be too surprising because option values are more sensitive to α_1 (that measures the volatility of volatility), and γ_1 (that controls the skewness of index returns) than they are to the other parameters. This stability is important for the GARCH model to fit the data reasonably well even with constant parameters.

Although both the *ad hoc* BS and the updated GARCH formulas appear quite "flexible", there is an important distinction between the two functional forms. The *ad hoc* BS formula is a function of the current index level. But the updated GARCH model is a function of the historical path of the index. Both models are consistent with the skew in implied volatilities and a term structure of volatility. However, the shape of the *ad hoc* model is constant by definition, whereas the shape of the GARCH option valuation function changes depending on the past sequence of spot returns. Of course flexibility is not the only criterion of an option valuation formula. DFW (1998)

Table 4

Mean estimates from the updated GARCH model using non-linear least squares

Parameter	Mean	Standard Deviation
α_1	$4.06e-6$	$7.52e-7$
β_1	0.159	0.084
γ_1	430.7	45.31
ω_1	$1.10e-6$	$1.24e-6$

This table reports the mean and standard deviation of the parameter estimates from the weekly (every week) estimation of the GARCH model (using option prices in the second half of each year, 1992, 1993 and 1994) using non-linear least squares. Note however that the last Wednesday of the first half of each year appears in this sample. Variance, $h(t+1)$ is drawn from the daily history (last 252 days) of S&P 500 levels (closest to and before 2:30 P.M., central standard time).

have shown that a more flexible model may dominate in-sample but have much less predictive power for out-of-sample options prices. This occurs when a misspecified model achieves good in-sample results by overfitting the data. We examine this issue in the next section by comparing all the models out-of-sample.

3.2 Out-of-sample model comparison

Having estimated the parameters in-sample from the first six months of each year, we turn to out-of-sample valuation performance of the unrestricted/asymmetric GARCH model for the next six months of each the three years under consideration. In computing out-of-sample option values for the second half of a particular year for the non-updated GARCH model, we keep the parameters fixed at their in-sample estimates for the particular year and obtain the conditional variance, $h(t+1)$, from the dynamics of daily asset returns. In updating the variance we use the same starting variance, $h(0)$ as in the in-sample estimation and for any given time, t, in the out-of-sample period, obtain $h(t+1)$ from the entire daily history of asset prices for that year[23] (given the in-sample estimates). Therefore the entire out-of-sample options calculations for the non-updated GARCH model are based on options prices from the first 6 months for that year. The BS, *ad hoc* BS, and updated GARCH models are also estimated every week in the second half of each year. For the BS, the estimated implied volatility from the current week is used to value options in the next week. For the *ad hoc* BS and updated GARCH models, the estimated parameters from the current week are used to value options in the next week. For the updated GARCH model, at each time, t, in the out-of-sample period, the variance, $h(t+1)$ is drawn from the daily history of S&P 500 levels including and up to the first day in the history of S&P 500 levels that was used for getting the in-sample estimates in the previous week.[24] The important distinction between the out-of-sample implementations is that the non-updated GARCH model predicts options values up to 26 weeks ahead, whereas the BS, *ad hoc* BS, and updated GARCH models only predict one week ahead.

Table 5 (Panel A) reports the out-of-sample valuation errors for the various models aggregated across all three out-of-sample periods. The aggregate root mean squared valuation errors are \$1.14, \$0.771, \$0.737 and \$0.58, respectively for the BS, *ad hoc* BS, non-updated GARCH and the updated GARCH respectively. Recall the BS formula fits worst in-sample and it is still the worst performer out-of-sample. The biggest deterioration occurs with

[23] Daily S&P 500 levels (closest to and before 2:30 P.M, CST). Note that the sample of option prices for each day start at 2:30 P.M (CST).

[24] For example, in computing out-of-sample option values on a given Wednesday, we would use the daily history of S&P 500 levels (closest to and before 2:30 P.M, CST) that go up to 252 days prior to the previous Wednesday.

Table 5
Out-of-sample valuation errors

Panel A: Aggregate valuation errors across all years

	RMSE	MAE	MOE	Average option price	Number of observations
BS	1.14	0.807	−0.125	13.1	4944
Ad hoc BS	0.771	0.47	0.011	13.1	4944
GARCH (non-updated)	0.737	0.492	0.057	13.1	4944
GARCH (updated)	0.58	0.373	0.031	13.1	4944

Panel B: Valuation errors by years

	RMSE	MAE	MOE	Average option price	Number of observations
1992					
BS	1.058	0.76	−0.09	13.01	1545
Ad hoc BS	0.689	0.418	0.022	13.01	1545
GARCH (non-updated)	0.833	0.571	0.366	13.01	1545
GARCH (updated)	0.479	0.303	−0.027	13.01	1545
1993					
BS	1.138	0.789	−0.161	13.20	1500
Ad hoc BS	0.899	0.544	0.008	13.20	1500
GARCH (non-updated)	0.689	0.454	0.029	13.20	1500
GARCH (updated)	0.624	0.401	−0.008	13.20	1500
1994					
BS	1.229	0.894	−0.135	13.09	1899
Ad hoc BS	0.725	0.448	0.002	13.09	1899
GARCH (non-updated)	0.691	0.452	−0.28	13.09	1899
GARCH (updated)	0.637	0.417	0.128	13.09	1899

Panel A reports the aggregate (across the three years, 1992, 1993 and 1994) out-of-sample valuation errors (in \$) for all options by various models. Panel B reports the out-of-sample valuation errors (in \$) by each year (1992, 1993 and 1994). Option values are computed every Wednesday (or the next trading day) in the second half of each year. For the GARCH (non-updated) model, option values are computed by holding the parameters at their in-sample estimates from Table 2 and updating the variance from the daily S&P 500 levels (closest to and before 2:30 P.M., central standard time). For the GARCH (updated) model, parameters are estimated in the previous week and variance is computed from the history of the daily S&P 500 levels (closest to and before 2:30 P.M., central standard time). BS is the Black–Scholes model in which a single implied volatility is estimated across all strikes and maturities on a given day while *ad hoc* BS is an *ad hoc* version of the Black–Scholes model with strike and maturity specific implied volatilities; both the BS and *ad hoc* BS are estimated every week and then used to value options in the following week. RMSE is the root mean squared out-of-sample/prediction valuation error in dollars. MOE (in \$) or the mean outside error measures the mean valuation error outside the bid-ask spread (difference between the model value and the ask price if the model value exceeds the ask price or the difference between the model value and the bid price if the bid price exceeds the model value). MAE (in \$) or the mean absolute error is the average absolute value of the valuation errors outside the bid-ask spread.

the *ad hoc* BS model. Although it has a competitive fit in-sample, it under-performs both the non-updated GARCH and the updated GARCH one week out-of-sample. Table 5 (Panel B) reports the out-of-sample RMSE for each out-of-sample period. We find that the *ad hoc* BS is superior to the non-updated GARCH in 1992, but is outperformed enough by the non-updated GARCH in 1993 and 1994 so that the aggregate errors are lower under the non-updated GARCH. Note however that the updated GARCH substantially outperforms the *ad hoc* BS in all three out-of-sample periods.

Although the *ad hoc* BS model is flexible, it achieves a tight in-sample fit only by overfitting the data. In contrast the GARCH model holds up surprisingly well. It continues to predict options values in the second half of the year using parameter estimates from only the first half. Furthermore, despite using a variance $h(t+1)$ filtered from the time series of index returns, it outperforms the BS and *ad hoc* BS formulas calibrated to the previous week's options. The updated GARCH model, however, provides the best predictive fit. While it is only slightly better than the *ad hoc* BS model in-sample, it does not suffer as much deterioration in out-of-sample fit. Consequently it is the best model both in-sample and out-of-sample.

Note that the updated GARCH model fits the dynamics of index returns jointly with the pattern of option prices across strike price and maturity on a given day. Thus it is somewhat restricted in that the spot variance, $h(t+1)$ is not a free parameter. However, this restriction improves the predictions. In unreported diagnostics we have also estimated the updated GARCH model with $h(t+1)$ as a fitted parameter, instead of filtering $h(t+1)$ from the historical time-series of index returns. This resembles Bakshi, Cao and Chen's (1997) estimation of continuous-time stochastic volatility models.[25] Naturally this improves the in-sample fit due to increased degrees of freedom. But it does not improve out-of-sample predictions (the variance is updated from the asset prices to compute out-of-sample values). Compared with the earlier updated GARCH, the refitting of all the parameters increases the out-of-sample root mean squared error by an average of 5 cents (across all years). Note that even with this increased error the updated GARCH model still produces better out-of-sample predictions than other models. But it produces the best predictions by simultaneously using the information in option prices and the history of the index.

Table 5 (Panel A and Panel B) also reports out-of-sample mean absolute error (MAE) that measures the absolute values of the valuation errors outside the bid-ask spread for all options.[26] The aggregate (i.e. across the three years) out-of-sample MAE's are \$0.807, \$0.47, \$0.492 and \$0.373 for the BS, *ad hoc* BS, the non-updated GARCH and the updated GARCH respectively. Table 6 and Table 7 report the valuation errors (both the RMSE in dollars and the percentage valuation error) by different option moneyness and maturity categories for puts and calls respectively. They also report the average error outside the bid-ask spread for each category. Looking at the valuation errors by moneyness, maturity and option type (i.e. a call or a put), we find that the GARCH model (both updated and non-updated) is able to value deep out-of-the-money options ($K/F < 0.95$ for puts and $K/F > 1.05$ for calls) better for all maturities than the *ad hoc* BS. For example, the RMSE for deep out-of-the-money puts ($K/F < 0.95$) that have less than forty days to mature is 28.2 cents for the non-updated GARCH and 22.7 cents for the updated GARCH versus 42.3 cents for the *ad hoc* BS and 55.4 cents for the BS

[25] The parameter ω turned out to be negative in three (out of 68 cases) using this kind of estimation. A negative ω does not guarantee the long-run variance to be positive, although the updated, one week ahead variance was positive due to the relatively higher value of the estimated $h(t+1)$ with respect to ω.

[26] Note that the MAE is not directly comparable to the RMSE because MAE is defined only when the valuation error does not fall within the bid-ask spread.

Table 6

Out-of-sample valuation errors for put options

		Days to Expiration								
		<40			[40–70]			>70		
Model	Moneyness	RMSE	% Error	MOE	RMSE	% Error	MOE	RMSE	% Error	MOE
BS										
	<0.95	0.554	92.79	−0.406	1.186	63.18	−1.011	1.645	54.52	−1.452
	[0.95–0.99)	0.572	31.22	−0.277	0.954	21.98	−0.566	1.322	21.90	−0.953
	[0.99,1.01)	0.873	17.34	0.522	0.965	12.00	0.513	0.965	9.44	0.340
	[1.01,1.05]	0.953	7.13	0.646	1.331	8.43	0.935	1.589	9.13	1.129
	>1.05	0.418	1.41	0.194	1.017	3.43	0.802	1.475	5.01	1.104
Ad hoc BS										
	<0.95	0.423	70.94	−0.127	0.545	29.03	−0.160	1.035	34.31	−0.427
	[0.95–0.99)	0.466	25.45	−0.086	0.697	16.06	0.192	0.709	11.76	0.224
	[0.99,1.01)	0.595	11.81	−0.254	0.762	9.47	0.206	0.795	7.78	0.059
	[1.01,1.05]	0.553	4.14	−0.241	0.802	5.08	0.002	1.087	6.25	−0.446
	>1.05	0.366	1.23	−0.016	0.421	1.42	−0.024	1.018	3.45	−0.593
GARCH (non-updated)										
	<0.95	0.282	47.14	−0.102	0.514	27.38	−0.176	0.612	20.31	−0.155
	[0.95–0.99)	0.581	31.70	0.142	0.787	18.13	0.181	0.847	14.03	0.233
	[0.99–1.01)	0.773	15.35	0.245	0.940	11.68	0.489	1.118	10.94	0.634
	[1.01–1.05]	0.522	3.91	0.018	0.844	5.35	0.223	1.149	6.60	0.619
	>1.05	0.311	1.05	−0.176	0.415	1.40	−0.029	0.718	2.44	0.297
GARCH (updated)										
	<0.95	0.227	38.08	0.00	0.361	19.22	0.022	0.507	16.80	−0.101
	[0.95,0.99)	0.539	29.43	0.291	0.627	14.44	0.342	0.698	11.57	0.164
	[0.99–1.01)	0.649	12.89	0.275	0.683	8.49	0.417	0.843	8.25	0.477
	[1.01–1.05]	0.459	3.44	0.005	0.526	3.33	0.079	0.763	4.39	0.395
	>1.05	0.315	1.06	0.220	0.404	1.36	−0.124	0.516	1.75	0.090

Reports out-of-sample valuation errors by moneyness and maturity for put options. Moneyness is defined to be K/F where K is the strike, F is the forward price. RMSE and MOE, both in dollars are as defined in Table 5. % error is the ratio of the RMSE to the average option price for that option category.

Table 7

Out-of-sample valuation errors for call options

		Days to Expiration								
		<40			[40–70]			>70		
Model	Moneyness	RMSE	% Error	MOE	RMSE	% Error	MOE	RMSE	% Error	MOE
BS										
	<0.95	0.937	2.78	−0.606	1.410	4.10	−0.946	2.157	6.03	−1.657
	[0.95, 0.99)	0.826	5.49	−0.470	1.217	6.78	−0.768	1.663	8.19	−1.226
	[0.99,1.01)	0.701	14.17	0.333	0.763	9.30	0.328	0.747	7.05	−0.096
	[1.01, 1.05]	0.785	68.63	0.588	1.151	38.80	0.916	1.155	25.32	0.817
	>1.05	0.337	191.82	0.232	0.919	157.19	0.732	1.189	98.46	1.005
Ad hoc BS										
	<0.95	0.628	1.86	0.199	0.687	2.00	0.039	1.186	3.31	−0.174
	[0.95, 0.99)	0.640	4.26	0.243	0.898	5.00	0.522	0.947	4.67	0.571
	[0.99, 1.01)	0.512	10.35	−0.064	0.826	10.08	0.393	0.941	8.88	0.357
	[1.01, 1.05]	0.405	35.42	−0.083	0.659	22.24	0.097	1.189	26.04	−0.168
	>1.05	0.163	92.32	−0.016	0.842	144.13	−0.235	1.581	130.96	−0.692
GARCH (non-updated)										
	<0.95	0.672	2.00	−0.441	0.730	2.12	−0.360	0.927	2.59	−0.523
	[0.95, 0.99)	0.687	4.57	−0.140	0.822	4.58	−0.059	0.899	4.43	−0.212
	[0.99, 1.01)	0.685	13.85	0.024	0.805	9.82	0.269	0.953	9.00	0.270
	[1.01, 1.05]	0.427	37.36	−0.141	0.595	20.08	0.073	0.826	18.09	0.269
	>1.05	0.161	91.44	−0.097	0.265	45.32	−0.156	0.391	32.34	0.020
GARCH (updated)										
	<0.95	0.607	1.80	−0.310	0.593	1.73	−0.160	0.860	2.40	−0.502
	[0.95, 0.99)	0.621	4.13	−0.015	0.609	3.39	0.050	0.720	3.55	−0.283
	[0.99, 1.01)	0.625	12.64	0.060	0.590	7.20	0.252	0.644	6.08	0.100
	[1.01, 1.05]	0.402	35.15	−0.192	0.455	15.35	−0.051	0.526	11.52	0.089
	>1.05	0.154	87.30	−0.093	0.295	50.51	−0.185	0.341	28.25	−0.135

Out-of-sample valuation errors by moneyness and maturity for call options. Moneyness is defined to be K/F where K is the strike and F is the forward price. RMSE and MOE, both in dollars are as defined in Table 5. % error is the ratio of the RMSE to the average option price for that option category.

(see Table 6). For near-the-money options, the results are mixed. We find that for short-term (<40 days to expiration) near-the-money ($0.99 <= K/F < 1.01$) call options, the *ad hoc* BS model has lower valuation errors than both versions of the GARCH. However, for medium-term (40–70 days) and long-term (> 90 days) near-the-money calls, the updated GARCH has lower valuation errors than the *ad hoc* BS while for medium-term near-the-money call options, the non-updated GARCH dominates the *ad hoc* BS. In contrast, for near-the-money put options, irrespective of maturity, the *ad hoc* BS is better than the non-updated GARCH, but not necessarily better than the updated GARCH. In terms of maturity only, the percentage valuation errors under the GARCH tend to decrease with an increase in maturity, especially for out-of-the-money options. Short-term (<40 days to expire) out-of-the-money options often tend to be the most difficult to value (in terms of percentage valuation error) under both GARCH and the *ad hoc* BS, although the magnitude of valuation errors under the GARCH is substantially lower. Figures 2A, 2B and 2C show the absolute percentage valuation errors for put options of the three different maturities by moneyness while figures 3A, 3B and 3C show the same for call options.

The superior out-of-sample valuation performance of the GARCH model is especially encouraging in the context of results reported by DFW (1998).

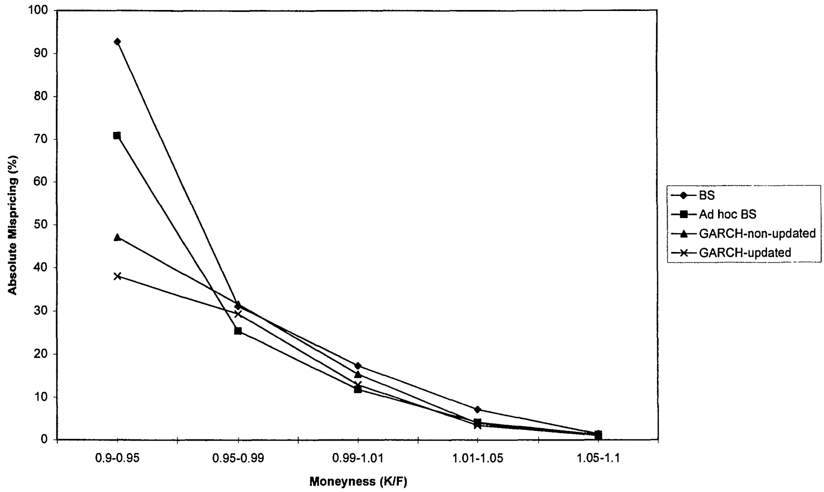

Figure 2A
This figure shows the percentage out-of-sample valuation errors (i.e. 100×RMSE/Option Price) for put options (less than 40 days to mature) by various models in the second half of each year. Out-of-sample option values are computed every Wednesday. BS is the Black–Scholes model and *ad hoc* BS is the *ad hoc* Black–Scholes model. BS and *ad hoc* BS are estimated from option prices every week. GARCH-non-updated is the GARCH model in which parameters have been estimated in the first half of each year, while GARCH-updated is the GARCH model in which parameters have been estimated in the previous week.

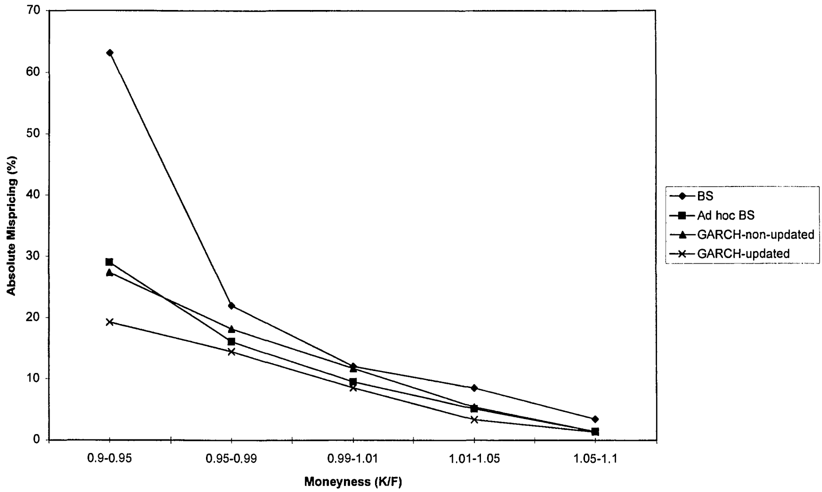

Figure 2B
This figure shows the percentage out-of-sample valuation errors (i.e. 100 × RMSE/Option Price) for put options (between 40 and 70 days to mature) by various models in the second half of each year. K is the strike and F is the forward price. Out-of-sample option values are computed every Wednesday. BS is the Black–Scholes model and *ad hoc* BS is the *ad hoc* Black–Scholes model. BS and *ad hoc* BS are estimated from option prices every week. GARCH-non-updated is the GARCH model in which parameters have been estimated in the first half of each year, while GARCH-updated is the GARCH model in which parameters have been estimated in the previous week.

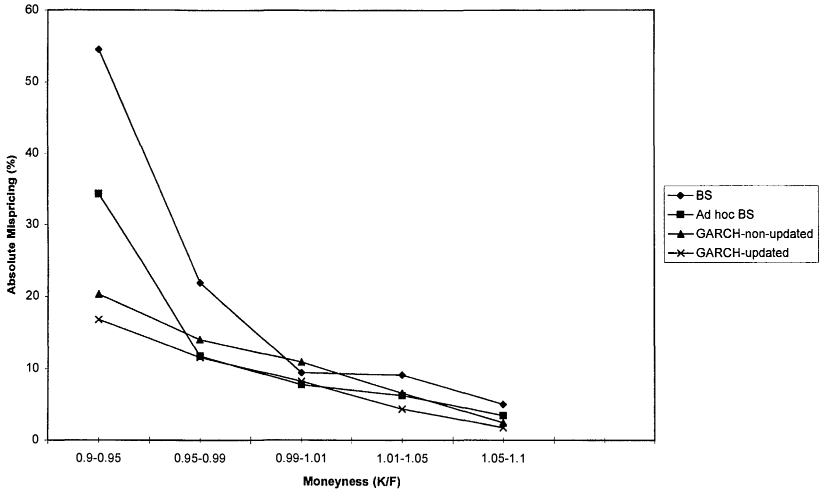

Figure 2C
This figure shows the percentage out-of-sample valuation errors (i.e. 100 × RMSE/Option Price) for put options (more than 70 days to mature) by various models in the second half of each year. K is the strike and F is the forward price. Out-of-sample option values are computed every Wednesday. BS is the Black–Scholes model and *ad hoc* BS is the *ad hoc* Black–Scholes model. BS and *ad hoc* BS are estimated from option prices every week. GARCH-non-updated is the GARCH model in which parameters have been estimated in the first half of each year, while GARCH-updated is the GARCH model in which parameters have been estimated in the previous week.

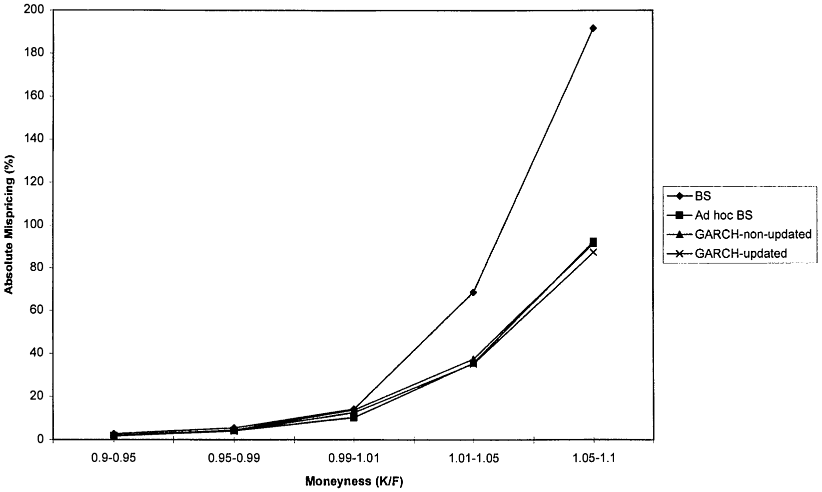

Figure 3A
This figure shows the percentage out-of-sample valuation errors (i.e. 100 × RMSE/Option Price) for call options (less than 40 days to mature) by various models in the second half of each year. K is the strike and F is the forward price. Out-of-sample option values are computed every Wednesday. BS is the Black–Scholes model and *ad hoc* BS is the *ad hoc* Black–Scholes model. BS and *ad hoc* BS are estimated from option prices every week. GARCH-non-updated is the GARCH model in which parameters have been estimated in the first half of each year, while GARCH-updated is the GARCH model in which parameters have been estimated in the previous week.

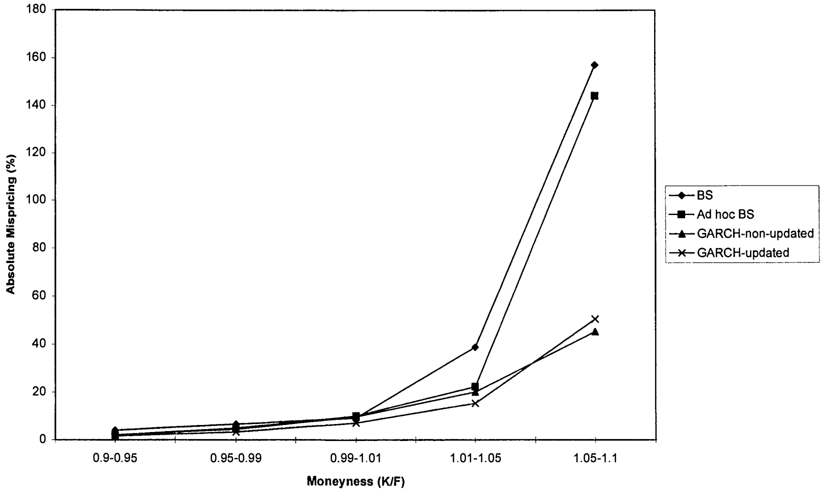

Figure 3B
This figure shows the percentage out-of-sample valuation errors (i.e. 100 × RMSE/Option Price) for call options (between 40 and 70 days to mature) by various models in the second half of each year. K is the strike and F is the forward price. Out-of-sample option values are computed every Wednesday. BS is the Black–Scholes model and *ad hoc* BS is the *ad hoc* Black–Scholes model. BS and *ad hoc* BS are estimated from option prices every week. GARCH-non-updated is the GARCH model in which parameters have been estimated in the first half of each year, while GARCH-updated is the GARCH model in which parameters have been estimated in the previous week.

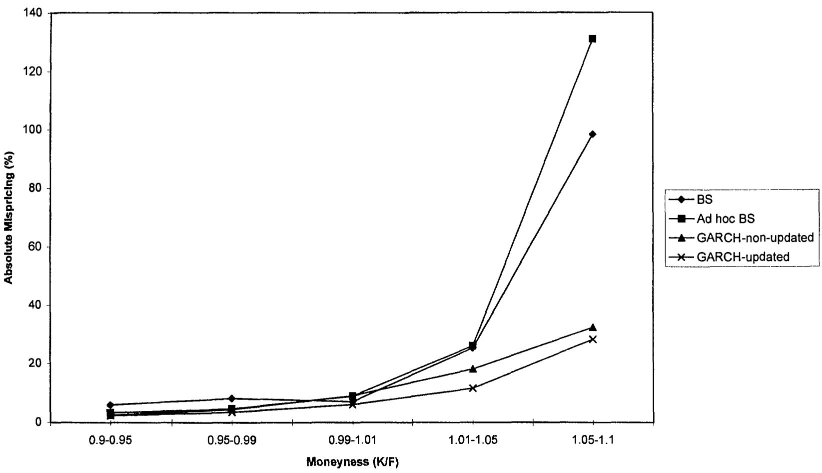

Figure 3C
This figure shows the percentage out-of-sample valuation errors (i.e. 100 × RMSE/Option Price) for call options (more than 70 days to mature) by various models in the second half of each year. K is the strike and F is the forward price. Out-of-sample option values are computed every Wednesday. BS is the Black–Scholes model and *ad hoc* BS is the *ad hoc* Black–Scholes model. BS and *ad hoc* BS are estimated from option prices every week. GARCH-non-updated is the GARCH model in which parameters have been estimated in the first half of each year, while GARCH-updated is the GARCH model in which parameters have been estimated in the previous week.

They found that the deterministic volatility models inspired by the implied binomial trees of Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) [see Brown and Toft (1999) for an extension of Rubinstein (1994)] underperform the *ad hoc* BS model in out-of-sample valuation tests in the S&P 500 index options market. While the deterministic volatility models can generate negative skewness in the distribution of asset returns, they are Markovian and assume that volatility is a function of the index level and time. The results of DFW show that this path-independence assumption is poorly specified for options prices. Path-independent volatility dynamics also contradict the extensively documented empirical literature on ARCH-GARCH effects or path dependence in volatility. The deterministic volatility models are quite capable of fitting the volatility smile of option prices in-sample. But they do not properly relate this shape to the path-dependent dynamics of volatility. Consequently these models do not properly update the volatility using subsequent index returns, leading to poor out-of-sample option predictions. In contrast the GARCH model is able to both fit the shape of options prices and use this information to predict volatility on the basis of subsequent index returns. The negative correlation between returns and volatility is an important element of this relationship.

3.3 Using S&P 500 futures to filter volatility

The GARCH model uses the history of asset prices (S&P 500 levels) to filter the variance, $h(t+1)$, that is needed to value options on day t, while the *ad hoc* BS and the simple BS model (with a single implied volatility) do not use the history of asset prices. The in-sample and out-of-sample valuation errors reported in the previous sections are computed using volatility filtered from the history of S&P 500 cash/spot levels. If stale price quotations affect the measured S&P 500 cash index levels then the volatilities are contaminated by the accumulated history of these quotation errors. Table 1 reports the slightly different volatility dynamics computed (using maximum likelihood estimation) from the history of S&P 500 cash/spot data and the history of S&P 500 futures data. A related question is how robust our options valuation results are if we use the S&P 500 futures to filter volatility. In other words, one needs to explore how minor changes in data affect our option valuation results. In order to address this issue, we first create a daily time series of the prices of lead/closest maturity S&P 500 futures contracts from the intra-day data on S&P 500 futures. Specifically as with the cash/spot, we use the closest to (before) 2:30 P.M. (CST) futures prices to filter volatility. However, we retain the same parameter estimates that were obtained from the previous in-sample optimizations (on the options data) using the S&P 500 cash/spot index. Table 8 reports the in-sample and out-of-sample RMSE in the various periods for the two versions of the GARCH model i.e. the non-updated and the updated GARCH model.

The out-of-sample RMSE for the non-updated GARCH model using S&P 500 futures (to filter variance) are 85 cents, 69 cents and 69.6 cents for 1992, 1993 and 1994 respectively. Recall that the corresponding out-of-sample RMSE using variance filtered from the S&P 500 cash index for the non-updated GARCH model are 83.3 cents, 68.9 cents and 69.1 cents respectively.

Table 8

In-sample and out-of-sample valuation errors using S&P 500 futures

	RMSE(in-sample)	RMSE (out-of-sample)
1992		
GARCH (non-updated)	0.711	0.85
GARCH (updated)	0.40	0.498
1993		
GARCH (non-updated)	0.723	0.69
GARCH (updated)	0.505	0.619
1994		
GARCH (non-updated)	0.689	0.696
GARCH (updated)	0.527	0.617

Reports the dollar root mean squared valuation errors (RMSE) for the in-sample and out-of-sample estimation (for 1992, 1993 and 1994) for the two versions of the GARCH model if the variance, $h(t+1)$ is filtered from the S&P 500 levels that are implied from the S&P 500 futures. The nearest maturity futures with prices closest to (before) 2:30 P.M. (central standard time) are used. Parameter estimates are fixed at the ones obtained from the in-sample estimations of the respective models using the S&P 500 cash/spot levels.

This shows that there is a maximum 1.7 cents difference for out-of-sample valuation errors if we change the data used to filter variance. For the updated GARCH model, the out-of-sample RMSE (using S&P 500 futures) are 49.8 cents, 61.9 cents and 61.7 cents for 1992, 1993 and 1994 respectively. Recall that the corresponding RMSE using variance filtered from the S&P 500 cash index for the updated GARCH model are 47.9 cents, 62.4 cents and 63.7 cents respectively. Thus at most, there is a 2 cent difference in terms of out-of-sample valuation errors for the updated GARCH model. Therefore using futures data to filter volatility does not change our primary result that the GARCH model outperforms the BS and the *ad hoc* BS out-of-sample and the quantitative implications of using index returns implied from the S&P 500 futures prices are quite trivial.

For in-sample valuation errors, the RMSE from the updated GARCH model using the S&P 500 futures are 40 cents, 50.5 cents and 52.7 cents for 1992, 1993 and 1994 respectively. Recall that the corresponding in-sample RMSE using the S&P 500 cash index are 36.6 cents, 48.3 cents and 45.9 cents respectively. Thus the differences (between futures and spot) in in-sample valuation errors are higher than the differences in out-of-sample valuation errors and the same is true of the differences in in-sample valuation errors if we use the non-updated GARCH model. These in-sample differences would undoubtedly be smaller if we re-estimated the parameters of the GARCH model from the options data based on the volatility filtered using the futures data. We do not pursue this because it would confound the effects of changing parameters with the effects of changing data. The empirical goal of our paper is to compare models in terms of out-of-sample performance, and the results demonstrate that the out-of-sample superiority of the GARCH model is insensitive to the data used to filter volatility.

4. Conclusion

This paper presents a closed-form solution for option values (and hedge ratios) when the variance of the spot asset follows a GARCH(p, q) process and is correlated with asset returns. The discrete-time GARCH model with single lag converges to Heston's (1993) continuous-time stochastic volatility model as the observation interval shrinks. In this limiting case the formula gives a unique option value that is based on the absence of arbitrage only (the option can be replicated by trading in the underlying asset and the risk free asset). This limit is a path-dependent continuous-time model of the type suggested by DFW (1998) to overcome the limitations of the path independent implied binomial tree models. In practice the numerical results of the discrete-time GARCH model with daily increments are quite close to those of the continuous-time model. But unlike continuous-time stochastic volatility models, the GARCH model can be easily estimated from observing only the history of asset prices. As the model has closed-form solutions for option values, one can easily combine the information in the cross-section of options with the time-series information in the history of asset prices to estimate model parameters.

Empirical analysis of the single lag version of the GARCH model on the S&P 500 index options data shows out-of-sample valuation improvements over a flexible *ad hoc* version of the BS model. The *ad hoc* BS model uses a separate implied volatility for each option (specific to its strike and time to maturity) extracted from market prices and is designed to produce a very close fit to the shape of the implied volatilities across strike prices and maturities; also it is updated every period. In contrast, the GARCH model filters the volatility from the history of asset prices and its parameters are held constant over the sample periods. If the parameters of the GARCH model are updated every period, the out-of-sample valuation improvements over the *ad hoc* model are more substantial. However, the GARCH model underperforms even the simple BS model when it does not allow correlation between index returns and volatility. DFW (1998) have shown that models which capture this correlation, but do not allow for path dependence in volatility are dominated by the *ad hoc* BS model in out-of-sample valuation errors in the S&P 500 index options market. We conclude that the out-of-sample valuation improvements of the GARCH model depends on the model's ability to simultaneously capture the path-dependence in volatility and the correlation of volatility with asset returns. Also our results are robust to using the history of S&P 500 futures to filter the variance instead of the S&P 500 cash index.

Although the empirical analysis in this paper has focused on the single lag version of the GARCH model, one could estimate the model with multiple lags. Since Assumption 2 states that the GARCH model is identical to the BS model for one-period options, additional lags will probably not improve valuation errors of short-term options. One might also want to incorporate a fat-tailed distribution of the one step ahead index returns. Also instead of filtering the variance from daily returns, one could potentially use intra-day levels of S&P 500 to update the variance intra-day. Attention could also be given to how the existence of huge open interest in deep-out-of-the-money puts could potentially impact the prices of these puts, a subject that does not lie within the realm of this paper, but may be important in itself. Although this paper has focused on equity/index options, the model can also be easily applied to currency options and quantos. One can also extend the model to bond options by assuming that (continuously compounded) interest rates follow an autoregressive moving-average process with the GARCH effects of equation (1b). This leads to a family of log-linear (affine) bond models that in continuous-time limit nest the diffusion models of Vasicek (1977), Heston (1990) and others.

Appendix A. Derivation of the Generating Function and Option Formula

Proof of Proposition 2.

Derivation of the Generating Function:

Let $x(t) = \log(S(t))$ and let $f(t; T, \phi)$ be the conditional generating function of $S(T)$, or equivalently the conditional moment generating function of $x(T)$ i.e.,

$$f(t; T, \phi) = E_t[\exp(\phi x(T))]. \tag{A1}$$

We shall guess that the moment generating function takes the log-linear form

$$\begin{aligned} f(t; T, \phi) = \exp\Bigg(&\phi x(t) + A(t; T, \phi) \\ &+ \sum_{i=1}^{p} B_i(t; T, \phi) h(t + 2\Delta - i\Delta) + \sum_{i=1}^{q-1} C_i(t; T, \phi) \\ &\times (z(t + \Delta - i\Delta) - \gamma_i \sqrt{h(t + \Delta - i\Delta)})^2\Bigg), \end{aligned} \tag{A2}$$

and solve for the coefficients $A()$, $B_i()$ and $C_i()$ as in Ingersoll (1987, p. 397), utilizing the fact that the conditional moment generating function is exponential affine in the state variables, $x(t)$ and the $h(t)s$. The fact that the conditional moment generating function is exponential affine can be easily verified by calculating the moment generating functions for $x(t+1)$, $x(t+2)$ and so on. Equation (A2) specifies the general form of this function for $x(T)$.

Since $x(T)$ is known at time T, equations (A1) and (A2) require the terminal condition

$$A(T; T, \phi) = B_i(T; T, \phi) = C_i(T; T, \phi) = 0. \tag{A3}$$

Applying the law of iterated expectations to $f(t; T, \phi)$, we get,

$$\begin{aligned} f(t; T, \phi) = E_t\big[f(t + \Delta; T, \phi)\big] = E_t\Bigg[\exp\Bigg(&\phi x(t + \Delta) + A(t + \Delta; T, \phi) \\ &+ \sum_{i=1}^{p} B_i(t + \Delta; T, \phi) h(t + 3\Delta - i\Delta) + \sum_{i=1}^{q-1} C_i(t + \Delta; T, \phi) \\ &\times \left(z(t + 2\Delta - i\Delta) - \gamma_i \sqrt{h(t + 2\Delta - i\Delta)}\right)^2\Bigg)\Bigg]. \end{aligned} \tag{A4}$$

Substituting the dynamics of $x(t)$ in equations (1a) and (1b) shows

$$\begin{aligned} f(t; T, \phi) = E_t\Bigg[\exp\Bigg(&\phi\Big(x(t) + r + \lambda h(t + \Delta) + \sqrt{h(t + \Delta)} z(t + \Delta)\Big) \\ &+ A(t + \Delta; T, \phi) + B_1(t + \Delta; T, \phi) \\ &\times \left(\beta_1 h(t + \Delta) + \alpha_1\big(z(t + \Delta) - \gamma_1 \sqrt{h(t + \Delta)}\big)^2\right) \\ &+ B_1(t + \Delta; T, \phi)\Bigg(\omega + \sum_{i=1}^{p-1} \beta_{i+1} h(t + 2\Delta - i\Delta) \\ &+ \sum_{i=1}^{q-1} \alpha_{i+1}\left(z(t + 2\Delta - i\Delta) - \gamma_{i+1}\sqrt{h(t + 2\Delta - i\Delta)}\right)^2\Bigg) \end{aligned}$$

$$+\sum_{i=1}^{p-1} B_{i+1}(t+\Delta;T,\phi)h(t+2\Delta-i\Delta)+C_1(t+\Delta;T,\phi)$$
$$\times\left(z(t+\Delta)-\gamma_1\sqrt{h(t+\Delta)}\right)^2+\sum_{i=1}^{q-2} C_{i+1}(t+\Delta;T,\phi)$$
$$\times\left(z(t+\Delta-i\Delta)-\gamma_{i+1}\sqrt{h(t+\Delta-i\Delta)}\right)^2\Bigg)\Bigg]. \quad \text{(A5)}$$

Rearranging terms through completing squares and some algebra shows

$$f(t;T,\phi)=E_t\Bigg[\exp\Bigg(\phi(x(t)+r)+A(t+\Delta;T,\phi)+B_1(t+\Delta;T,\phi)\omega$$
$$+\left(B_1(t+\Delta;T,\phi)\alpha_1+C_1(t+\Delta;T,\phi)\right)\Bigg(z(t+\Delta)$$
$$-\left(\gamma_1-\frac{\phi}{2(B_1(t+\Delta;T,\phi)\alpha_1+C_1(t+\Delta;T,\phi))}\right)$$
$$\times\sqrt{h(t+\Delta)}\Bigg)^2+\Bigg(\phi\lambda+B_1(t+\Delta;T,\phi)\beta_1+\phi\gamma_1$$
$$-\frac{\phi^2}{4(B_1(t+\Delta;T,\phi)\alpha_1+C_1(t+\Delta;T,\phi))}\Bigg)h(t+\Delta)$$
$$+B_1(t+\Delta;T,\phi)\sum_{i=1}^{p-1}\beta_{i+1}h(t+2\Delta-i\Delta)+\sum_{i=1}^{p-1}B_{i+1}$$
$$(t+\Delta;T,\phi)h(t+2\Delta-i\Delta)+B_1(t+\Delta;T,\phi)\Bigg(\sum_{i=1}^{q-1}\alpha_{i+1}$$
$$\times\left(z(t+2\Delta-i\Delta)-\gamma_{i+1}\sqrt{h(t+2\Delta-i\Delta)}\right)^2\Bigg)$$
$$+\sum_{i=1}^{q-2}C_{i+1}(t+\Delta;T,\phi)\Big(z(t+\Delta-i\Delta)$$
$$-\gamma_{i+1}\sqrt{h(t+\Delta-i\Delta)}\Big)^2\Bigg)\Bigg]. \quad \text{(A6)}$$

A useful result is that for a standard normal variable z

$$E\left[\exp(a(z+b)^2)\right]=\exp\left(-\frac{1}{2}\ln(1-2a)+\frac{ab^2}{1-2a}\right). \quad \text{(A7)}$$

Substituting this result in (A6) and subsequently equating terms in both sides of (A6) shows

$$A(t;T,\phi)=A(t+\Delta;T,\phi)+\phi r+B_1(t+\Delta;T,\phi)\omega$$
$$-\frac{1}{2}\ln\left(1-2\alpha_1B_1(t+\Delta;T,\phi)-2C_1(t+\Delta;T,\phi)\right). \quad \text{(A8)}$$

$$B_1(t;T,\phi)=\phi(\lambda+\gamma_1)-\frac{1}{2}\gamma_1^2+\beta_1B_1(t+\Delta;T,\phi)+B_2(t+\Delta;T,\phi)$$

$$+\frac{1/2(\phi-\gamma_1)^2}{1-2\alpha_1 B_1(t+\Delta;T,\phi)-2C_1(t+\Delta;T,\phi)} \tag{A9}$$

$$B_i(t;T,\phi)=\beta_i B_1(t+\Delta;T,\phi)+B_{i+1}(t+\Delta;T,\phi),\quad \text{for } 1<i\le p,$$
$$C_i(t;T,\phi)=\alpha_{i+1} B_1(t+\Delta;T,\phi)+C_{i+1}(t+\Delta;T,\phi),\quad \text{for } 1<i\le q-1.$$

One can use equations (A8) and (A9) to calculate the coefficients recursively starting with equation (A3).

Proof of Proposition 3.

Let $f(\phi)$ denote the moment generating function of the probability density, $p(x)$, where x is the logarithm of the terminal asset price.[27] Let $p^*(x)$ be an adjusted probability density defined by $p^*(x)=\exp(x)p(x)/f(1)$.[28] It is easy to see that it is a valid probability density because it is non-negative and $f(1)=E_t[\exp(x(T))]$ from equation (6). The moment generating function for $p^*(x)$ is

$$\begin{aligned}\int_{-\infty}^{\infty}\exp(\phi x)p^*(x)dx &= \frac{1}{f(1)}\int_{-\infty}^{\infty}\exp((\phi+1)x)p(x)dx\\ &= \frac{f(\phi+1)}{f(1)}.\end{aligned} \tag{A10}$$

Since the spot asset price is $\exp(x)$, the expectation (at time t) of a call option payoff separates into two terms with probability integrals.

$$\begin{aligned}E[\mathrm{Max}(e^x-K,0)] &= \int_{\ln(K)}^{\infty}\exp(x)p(x)dx - K\int_{\ln(K)}^{\infty}p(x)dx\\ &= f(1)\int_{\ln(K)}^{\infty}p^*(x)dx - K\int_{\ln(K)}^{\infty}p(x)dx.\end{aligned} \tag{A11}$$

Note that $f(i\phi)$ is the characteristic function corresponding to $p(x)$ and $f(i\phi+1)/f(1)$ is the characteristic function corresponding to $p^*(x)$. Feller [1971] and Kendall and Stuart [1977] show how to recover the "probabilities" from the characteristic functions

$$\int_{\ln(K)}^{\infty}p(x)dx=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{e^{-i\phi\ln(K)}f(i\phi)}{i\phi}\right]d\phi, \tag{A12}$$

and similarly the other integral of $p^*(x)$. Substituting equation (A12) into (A11) proves the proposition and noting that under the risk neutral distribution, $S(t)=e^{-r(T-t)}E[S(T)]$, demonstrates the corollary.

As mentioned before, the option values under our model can be gotten very easily with a numerical integration routine as the integrand converges very fast. For example, using the parameters from the MLE estimation (of the unrestricted/asymmetric GARCH model) on daily spot index returns that appear in Table 1, with $S=K=\$100$, $h(t+1)=(0.15*0.15)/252.0$ and setting the interest rate to zero, call options with 50 and 100 days to expiration have values of \$1.817 and \$2.481 respectively.[29]

[27] Note that $p(x)$ is basically the conditional (at time t) density function of $x(T)$.

[28] Note that $p^*(x)$ is not the density corresponding to $f^*(\phi)$.

[29] These option values were generated by using the integration routine "qromo()" of Press et al. (1992) as well as by using Gauss-Hermite quadrature (Press et al. (1992)) after normalizing the integrand by a factor proportional to the square-root of expected future variance (until the option expires).

Appendix B. Convergence to Continuous Time

In the single factor/single lag model the conditional mean and variance of $h(t)$ are

$$E_{t-\Delta}[h(t+\Delta)] = \omega + \alpha_1 + (\beta_1 + \alpha_1\gamma_1^2)h(t). \tag{B1}$$

$$\mathrm{Var}_{t-\Delta}[h(t+\Delta)] = \alpha_1^2(2 + 4\gamma_1^2 h(t)). \tag{B2}$$

There are various ways to approach a continuous-time limit as the time interval Δ shrinks. Since $h(t)$ is the variance of the spot return over time interval Δ, it should converge to zero. To measure the variance per unit of time we define $v(t) = h(t)/\Delta$ and $v(t)$ has a well defined continuous-time limit. The stochastic process, $v(t)$ follows the dynamics

$$v(t+\Delta) = \omega_v + \beta_v v(t) + \alpha_v(z(t) - \gamma_v\sqrt{v(t)})^2, \tag{B3}$$

where,

$$\omega_v = \frac{\omega}{\Delta},\ \beta_v = \beta_1,\ \alpha_v = \frac{\alpha_1}{\Delta},\ \gamma_v = \gamma_1\sqrt{\Delta}.$$

Let $\alpha_1(\Delta) = \frac{1}{4}\sigma^2\Delta^2$, $\beta_1(\Delta) = 0$, $\omega(\Delta) = (\kappa\theta - \frac{1}{4}\sigma^2)\Delta^2$, $\gamma_1(\Delta) = 2/(\sigma\Delta) - \kappa/\sigma$, and $\lambda(\Delta) = \lambda$. Noting that $v(t+\Delta)$ is observable at time t and taking conditional expectation at time $t-\Delta$,

$$E_{t-\Delta}[v(t+\Delta) - v(t)] = \kappa(\theta - v(t))\Delta + \frac{1}{4}\kappa^2 v(t)\Delta^2. \tag{B4}$$

$$\mathrm{Var}_{t-\Delta}[v(t+\Delta)] = \sigma^2 v(t)\Delta + \left(\frac{\sigma^4}{8} - \sigma^2\kappa v(t) + \frac{\sigma^2\kappa^2}{4}v(t)\Delta\right)\Delta^2. \tag{B5}$$

(Note that α_1, β_1, ω, γ_1 as defined above are not α_v, β_v, ω_v, and γ_v corresponding to the $v(t)$ process). The correlation between the variance process and the continuously compounded stock return is

$$\mathrm{Corr}_{t-\Delta}[v(t+\Delta), \log(S(t))] = \frac{-\mathrm{sign}(\gamma_v)\sqrt{2\gamma_v^2 v(t)}}{\sqrt{1 + 2\gamma_v^2 v(t)}}. \tag{B6}$$

As the time interval Δ shrinks the skewness parameter, $\gamma_v(\Delta)$ approaches positive or negative infinity. Consequently the correlation in equation (B6) approaches 1 (or negative 1) in the limit.

The variance process, $v(t)$ has a continuous-time diffusion limit following Foster and Nelson (1994). As the observation interval Δ shrinks, $v(t)$ converges weakly to the square-root process of Feller (1951), Cox, Ingersoll Ross (1985), and Heston (1993)

$$d\log(S) = (r + \lambda v)dt + \sqrt{v}dz \tag{B7a}$$

$$dv = \kappa(\theta - v)dt + \sigma\sqrt{v}dz, \tag{B7b}$$

where $z(t)$ is a Wiener process. Note that the same Wiener process drives both the spot asset and the variance. The limiting behavior of this GARCH process is very different from those of other GARCH processes such as GARCH 1-1 [Bollerslev (1986)] or many other asymmetric GARCH processes in which two different Wiener processes drive the spot assets and the variance. Also, while the above shows that the asset returns and variance processes under the data generating measure converge to well-defined continuous-time limits, one still needs to verify that the discrete risk-neutral processes converge to appropriate continuous-time limits if the discrete-time GARCH option values are to converge to their continuous-time limits.

As shown in Proposition 1, in the risk-neutral distribution, $\gamma_1{}^*$ equals $\gamma_1+\lambda+1/2$. Therefore, the risk-neutral parameter for the $v(t)$ process is,

$$\gamma_v^*(\Delta)=\gamma_1^*(\Delta)\sqrt{\Delta}=\frac{2}{\sigma\sqrt{\Delta}}-\left(\frac{\kappa}{\sigma}-\lambda-\frac{1}{2}\right)\sqrt{\Delta}. \tag{B8}$$

Consequently the risk-neutral process has a different mean,

$$\begin{aligned}E_{t-\Delta}^*[v(t+\Delta)-v(t)]&=\left[\kappa(\theta-v(t))+\sigma\left(\lambda+\frac{1}{2}\right)v(t)\right]\Delta\\&\quad+\frac{1}{4}\left(\kappa+\sigma\left(\lambda+\frac{1}{2}\right)\right)^2 v(t)\Delta^2.\end{aligned} \tag{B9}$$

Again following Foster and Nelson (1994), it follows that the continuous-time risk-neutral processes are

$$d\log(S)=\left(r-\frac{v}{2}\right)dt+\sqrt{v}dz^* \tag{B10a}$$

$$dv=\left(\kappa(\theta-v)+\sigma\left(\lambda+\frac{1}{2}\right)v\right)dt+\sigma\sqrt{v}dz^*, \tag{B10b}$$

where $z(t)^*$ is a Wiener process under the risk-neutral measure. As with the data generating measure, the same Wiener process drives both asset returns and variance under the risk-neutral measure. The above risk-neutral processes for the stock price and the variance are equivalent to the risk-neutral processes of Heston (1993) with the two Wiener processes therein being perfectly correlated. Consequently, the discrete-time GARCH option values converge to the continuous-time option values of Heston (1993) as Δ shrinks; such convergence has been verified numerically. Figure 4 shows how the discrete time GARCH model converges to the continuous-time model as Δ decreases (i.e. as the number of trading periods increases). The parameters

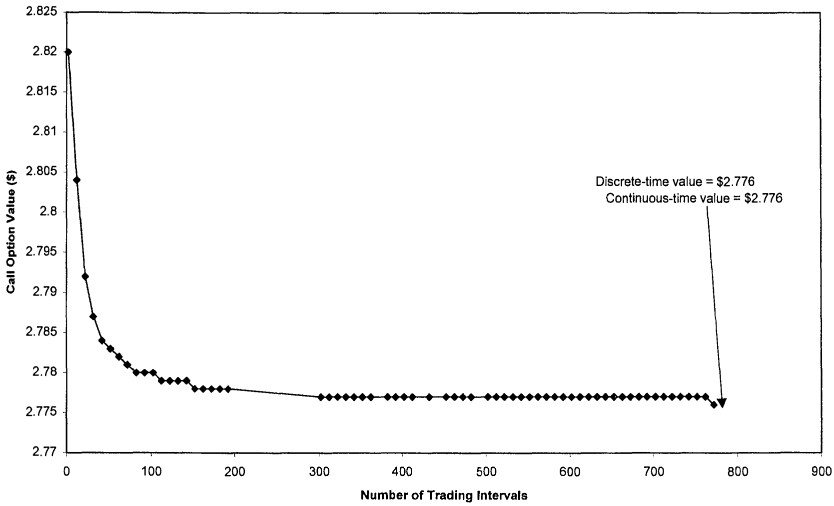

Figure 4
This figure shows how the discrete-time GARCH option values converge to the continuous-time option value with an increase in the number of trading intervals.

used for an at-the-money option with a spot asset price, S = \$100, strike price, K = \$100 with 0.5 years to maturity are, $\kappa - \sigma(\lambda + 1/2) = 2$, $\kappa\theta = 0.02$, $\rho = -1$, $\sigma = 0.1$, $v = 0.01$. Given these parameters, one can directly use Heston's (1993) formula to compute the value of an option.

As the two Wiener processes are perfectly correlated, one can value options solely by the absence of arbitrage only using the hedging arguments of Black and Scholes (1973) and Merton (1973) or equivalently by showing the existence of a unique risk-neutral distribution as per Cox and Ross (1976), Harrison and Kreps (1979), Harrison and Pliska (1981). In this case Assumption 2 is superfluous and merely states that options are properly priced at maturity. Note that although returns and volatility are perfectly correlated instantaneously in the continuous time model, they are imperfectly correlated over any discrete time interval. Also note that the parameter related to the asset risk premium, λ can influence option values unlike in the Black–Scholes-Merton setup due to the fact that the asset price process is non-Markovian. Since variance is a function of only the uncertainty emanating from changes in the asset prices and not driven by a separate Wiener process, the parameter related to the asset risk premium, λ appears in the drift of the risk-neutral variance instead of a volatility risk premium as in Heston (1993). This type of result has also been noted in Kallsen and Taqqu (1998).

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