Xsuite
physics manual

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Contents

Chapter 1

Single-particle tracking

XTrack is a 6D single particle symplectic tracking code used to compute the trajectories of individual relativistic charged particles in circular accelerators. It has been developed based on SixTrack.
The physical models are collected from the main references [**?**, **?**, **?**, **?**, **?**, **?**, **?**], which contain more details of the derivation of the maps.

1.1 Notation and reference frame

The speed, momentum, energy, rest mass, charge of a particle are indicated by v, P, E, m and q, respectively. These quantities are related by the following equations:

$$v = \beta c \qquad E^2 - P^2c^2 = m^2c^4 \qquad E = \gamma mc^2 \qquad Pc = \beta E \tag{1.1}$$

where β and γ are the relativistic factors.
In a curvilinear reference frame defined by a constant curvature h_x in the $\hat{X}, \hat{Z}$ plane and parameterized by s, the position of the particle at a time t can be written as:

$$\mathbf{Q}(t) = \mathbf{r}(s) + x\,\hat{x}(s) + y\,\hat{y}(s), \tag{1.2}$$

and therefore identified by the coordinates s, x, y, t in the reference frame defined by $\hat{x}(s)$ and $\hat{y}(s)$. In particle tracking, s is normally used as independent parameter and t as a coordinate.
The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be derived in a curvilinear reference frame from the potentials $V(x, y, s, t)$ and $\mathbf{A}(x, y, s, t)$, where

$$\mathbf{A}(x, y, s, t) = A_x(x, y, s, t)\hat{x}(s) + A_y(x, y, s, t)\hat{y}(s) + A_s(x, y, s, t)\hat{z}(s) \tag{1.3}$$

and for which:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\partial_x V\hat{x} - \partial_y V\hat{y} - \frac{1}{1+hx}\partial_s V\hat{z} - \partial_t \mathbf{A} \tag{1.4}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \left(\partial_y A_s - \frac{\partial_s A_y}{1+hx}\right)\hat{x} + \left(\frac{\partial_s A_x - \partial_x(1+hx)A_s}{1+hx}\right)\hat{y} \tag{1.5}$$

$$+ \quad \left(\partial_x A_y - \partial_y A_x\right)\hat{z}. \tag{1.6}$$

In this reference frame the canonical momenta are:

$$P_x = m\gamma\dot{x} + qA_x, \quad P_y = m\gamma\dot{y} + qA_y, \quad P_s = m\gamma\dot{s}(1+hx)^2 + q(1+hx)A_s. \tag{1.7}$$

and the energy of a particle and the field is

$$E = qV + c\sqrt{(mc)^2 + \frac{(P_s - qA_s(1+hx))^2}{(1+hx)^2} + (P_x - qA_x)^2 + (P_x - qA_x)^2}. \tag{1.8}$$

1.2 Hamiltonian and particle coordinates

If $s(t)$ is monotonically increasing, it is possible to derive the equations of motion using s as the independent parameter, $(-t, E)$ as conjugate coordinates and $-P_s$ as Hamiltonian.

$$P_s = (1+hx)\left(\sqrt{\frac{(E-q\phi)^2}{c^2} - (mc)^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2} + qA_s\right) \tag{1.9}$$

Since in accelerators the orbits of the particles are often a perturbation of the reference trajectory followed by a particle with rest mass m_0, charge q_0, speed $\beta_0 c$ and momentum P_0, one could use the following derived quantities that usually assume small values:

$$p(x,y) = \frac{m_0}{m}\frac{P(x,y)}{P_0} \qquad \chi = \frac{q}{q_0}\frac{m_0}{m} \qquad a(x,y,s) = \frac{q_0}{P_0}A(x,y,s) \tag{1.10}$$

Note that here m is used to indicate the rest mass of particles of species different from the reference particle (which has mass m_0) and not the relativistic mass. Further rescaling the energy and charge density as

$$e(x,y,s) = \frac{m_0}{m}\frac{E(x,y,s)}{P_0} \qquad \varphi(x,y,s) = \frac{q_0}{P_0 c}\phi(x,y,s)\,, \tag{1.11}$$

and as all canonical momenta scale with the same factor, we can define a new Hamiltonian $\tilde{H}$ that still satisfies the same equations of motion:

$$\begin{aligned} \tilde{H}(x,y,-t,p_x,p_y,e) &= \frac{m_0}{m}\frac{1}{P_0}H(x,y,-t,P_x,P_y,E) \\ \tilde{H} &= -(1+hx)\left(\sqrt{\left(\frac{e}{c} - \chi\varphi\right)^2 - \frac{1}{\beta_0^2\gamma_0^2} - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right) \end{aligned} \tag{1.12}$$

1.2.1 Longitudinal coordinates

Different sets of longitudinal coordinates can be used:

$$\xi = s\frac{\beta}{\beta_0} - \beta ct \qquad \tau = \frac{s}{\beta_0} - ct \qquad \zeta = s - \beta_0 ct \tag{1.13}$$

$$\delta = \frac{P\frac{m_0}{m} - P_0}{P_0} \qquad p_\tau = \frac{1}{\beta_0}\frac{E\frac{m_0}{m} - E_0}{E_0} \qquad p_\zeta = \frac{1}{\beta_0^2}\frac{E\frac{m_0}{m} - E_0}{E_0} \tag{1.14}$$

where variables in the same columns are canonically conjugate.
The different longitudinal variables can be easily related to each other:

$$\xi = \beta\tau = \frac{\beta}{\beta_0}\zeta \tag{1.15}$$

$$p_\tau = \beta_0 p_\zeta \tag{1.16}$$

$$\delta = \sqrt{p_\tau^2 + 2\frac{p_\tau}{\beta_0} + 1} - 1 = \beta p_\tau + \frac{\beta - \beta_0}{\beta_0} \tag{1.17}$$

$$\delta = \sqrt{\beta_0^2 p_\zeta^2 + 2p_\zeta + 1} - 1 = \beta\beta_0 p_\zeta + \frac{\beta - \beta_0}{\beta_0} \tag{1.18}$$

$$\gamma = \gamma_0(1 + \beta_0 p_\tau) \tag{1.19}$$

$$\beta = \sqrt{1 - \frac{1 - \beta_0}{(1 + \beta_0 p_\tau)^2}} \tag{1.20}$$

For small energy deviations ($\delta \ll 1$, $p_\tau \ll 1$, $p_\zeta \ll 1$), we can neglect the terms of order δ^2, p_τ^2, p_ζ^2 and higher, hence the following approximations hold:

$$\delta \simeq \frac{p_\tau}{\beta_0} \tag{1.21}$$

$$\delta \simeq p_\zeta \tag{1.22}$$

$$\beta \simeq \beta_0 + (1 - \beta_0^2) p_\tau \tag{1.23}$$

1.2.2 Hamiltonian with different coordinate choices

The conjugate pairs can be generated by the following generating functions [1]

$$F_2 = xp_x + yp_y + \left(\frac{s}{\beta_0} - ct\right)\frac{1+\delta}{\beta} \tag{1.24}$$

$$F_2 = xp_x + yp_y + \left(\frac{s}{\beta_0} - ct\right)\left(p_\tau + \frac{1}{\beta_0}\right) \tag{1.25}$$

$$F_2 = xp_x + yp_y + \left(\frac{s}{\beta_0} - ct\right)\left(\beta_0 p_\zeta + \frac{1}{\beta_0}\right) \tag{1.26}$$

The Hamiltonians are then:

[1] $F_2(-t, p_{\mathrm{new}}, s)$, $e = \frac{\partial F_2}{\partial(-t)}$, $q_{\mathrm{new}} = \frac{\partial F_2}{\partial p_{\mathrm{new}}}$, $H_{\mathrm{new}} = H + \frac{\partial F_2}{\partial s}$

$$H_\delta = \frac{1+\delta}{\beta\beta_0} - (1+hx)\left(\sqrt{\left(\frac{1+\delta}{\beta} - \chi\varphi\right)^2 - \frac{1}{\beta_0^2\gamma_0^2} - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right)$$
$$H_\tau = \frac{p_\tau}{\beta_0} - (1+hx)\left(\sqrt{\left(p_\tau + \frac{1}{\beta_0} - \chi\varphi\right)^2 - \frac{1}{\beta_0^2\gamma_0^2} - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right)$$
$$H_\zeta = p_\zeta - (1+hx)\left(\sqrt{\left(\beta_0 p_\zeta + \frac{1}{\beta_0} - \chi\varphi\right)^2 - \frac{1}{\beta_0^2\gamma_0^2} - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right)$$

Note that things get complicated when using the pair (ξ, δ), as then the Hamiltonian contains terms in β, which in turn depends on the energy. In particular:

$$\frac{\partial\beta}{\partial\delta} = \beta\,\frac{1-\beta^2}{1+\delta} \tag{1.27}$$

For this reason we prefer using H_τ when deriving the equations of motion. Note that when $\varphi = 0$, the Hamiltonian simplifies into:

$$H_\tau = \frac{p_\tau}{\beta_0} - (1+hx)\left(\sqrt{(1+\delta)^2 - (p_x - \chi a_x)^2 - (p_y - \chi a_y)^2} + \chi a_s\right) \tag{1.28}$$

The following identities are useful to derive the equations of motion:

$$\frac{\partial\delta}{\partial p_\tau} = \frac{p_\tau + 1/\beta_0}{1+\delta} = \frac{1}{\beta} \tag{1.29}$$
$$\frac{\partial}{\partial\delta}\left(\frac{1+\delta}{\beta\beta_0}\right) = \frac{\beta}{\beta_0} \tag{1.30}$$

1.3 Symplectic integrators

Xsuite supports different Hamiltonian splittings for symplectic integration.

We call $\sqrt{\dots} = \sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$:

Map	**Parameters**	**Description**	
D	l	Drift exact	$H_D = \frac{p_\tau}{\beta_0} - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$
De	l	Drift expanded	$H_{De} = \frac{p_\tau}{\beta_0} - \frac{p_x^2+p_y^2}{2(1+\delta)}$
Dc	l	Correction drift exact	$H_{Dc} = H_D - H_{De}$
R	h, l	Curved drift	$H_R = \frac{p_\tau}{\beta_0} - (1+hx)\left(\sqrt{...}\right)$
Br	k0, l	Rectangular bend	$H_{Br} = \frac{p_\tau}{\beta_0} - \sqrt{...} - b_1 x$
B	k0, l, h	Exact Bend	$H_B = \frac{p_\tau}{\beta_0} - (1+hx)\left(\sqrt{...} - k_0\left(x - \frac{hx^2}{2(1+hx)}\right)\right)$
Kh	h, l	Thin curvature kick	$H_{Kh} = -hx$
K0h	k0,l	Weak focusing	$H_{K0h} = k_0 h \frac{x^2}{2}$
K0	k0, l	Dipole kick	$H_{K0} = k_0 x$
K1	k1, l	Quadrupole kick	$H_{K1} = k_1 \frac{x^2-y^2}{2}$
K1h	k1, h, l	Quad correction	$H_{K1h} = -k_1 h \frac{2x^3+3xy^2}{6}$
Kn	kn, ks, l	Multipole	$H_{Kn} = -\Re\left(\sum_{n=0}^{N}(k_n + i\hat{k}_n)\frac{(x+iy)^{n+1}}{(n+1)!}\right)$
Knh	kn, ks, h	Curved Multipoles	Not yet available
M	k0, k1, l, h	2nd order Hamiltonian	$H_{Kn} = \frac{p_\tau}{\beta_0} + \frac{1}{2}\frac{p_x^2+p_y^2}{(1+\delta)^2} + k_0\left(x - \frac{hx^2}{2(1+hx)}\right) + k_1\frac{x^2-y^2}{2}$
S	ks,l	Solenoid	$H_S = \frac{p_\tau}{\beta_0} - \sqrt{(1+\delta)^2 - \left(p_x - \frac{k_s}{2}y\right)^2 - \left(p_y + \frac{k_s}{2}\right)^2}$

Splitting schemes

Model	**Splitting**	$h = 0, ks = 0$	$h \neq 0, ks = 0$
3	rot-kick-rot	D(1/2) K D(1/2)	R(1/2) K Ch1 Ch2 Chn R(1/2)
2	bend-kick-bend	Br(1/2) K Br(1/2)	B(1/2) K Ch2 B(1/2)
4	matrix-kick-matrix	M(1/2) K Cd M(1/2)	M(1/2) K Ch2 Cd Chn M(1/2)
5	drift-kick-drift-exact	D(1/2) K D(1/2)	D(1/2) K Ch0 Ch1 Ch2 Chn D(1/2)
6	drift-kick-drift-expanded	De(1/2) K De(1/2)	De(1/2) K Ch0 Ch1 Ch2 Chn De(1/2)

Model	$h = 0, ks = 0$	$h \neq 0, ks = 0$	$h = 0, k_s \neq 0$
rot-kick-rot	D(1/2) K D(1/2)	R(1/2) K Ch1 Ch2 R(1/2)	S(1/2) K S(1/2
bend-kick-bend	Br(1/2) K Br(1/2)	B(1/2) K Ch2 B(1/2)	not available
matrix-kick-matrix	M(1/2) K M(1/2)	M(1/2) K Ch2 M(1/2)	not available
drift-kick-drift-exact	D(1/2) K D(1/2)	D(1/2) K Ch0 Ch1 Ch2 D(1/2)	not available
drift-kick-drift-expanded	De(1/2) K De(1/2)	De(1/2) K Ch0 Ch1 Ch2 De(1/2)	not available

Model	$h = 0, ks = 0$	$h \neq 0, ks = 0$
rot-kick-rot	D, K0 K1 Kn	R, K0 K0h K1 K1h Kn Knh
bend-kick-bend	Br, K1 Kn	B, K1 K1h Kn Knh
matrix-kick-matrix	M, Kn Dc	M, K1h Kn Knh Dc
drift-kick-drift-exact	D, K0 K1 Kn	D, Kh K0 K0h K1 K1h Kn Knh
drift-kick-drift-expanded	De, K0 K1 Kn	De, Kh K0 K0h K1 K1h Kn Knh

Hamiltonians

$$
\begin{aligned}
B &= \frac{p_\tau}{\beta_0} - (1+hx)\left(\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} - k_0\left(x - \frac{hx^2}{2(1+hx)}\right)\right) \\
Br &= \frac{p_\tau}{\beta_0} - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} - b_1 x \\
Kn &= -\Re\left(\hat{k}_0 y + i\hat{k}_1 \frac{(x+iy)^2}{2} + \sum_{n=2}^{N}(k_n + i\hat{k}_n)\frac{(x+iy)^{n+1}}{(n+1)!}\right) \\
R &= \frac{p_\tau}{\beta_0} - (1+hx)\left(\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}\right) \\
D &= \frac{p_\tau}{\beta_0} - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \\
De &= \frac{p_\tau}{\beta_0} - \frac{p_x^2 + p_y^2}{2(1+\delta)} \\
Cd &= D - De \\
Ch0 &= (h - k_0)x \\
Ch1 &= -k_0 h \frac{x^2}{2} \\
Ch2 &= -k_1 h \frac{2x^3 + 3xy^2}{6} \\
M &= \frac{p_\tau}{\beta_0} + \frac{1}{2}\frac{p_x^2 + p_y^2}{(1+\delta)^2} + k_0\left(x - \frac{hx^2}{2(1+hx)}\right) + k_1\frac{x^2 - y^2}{2} \\
S &= \frac{p_\tau}{\beta_0} - \sqrt{(1+\delta)^2 - \left(p_x - \frac{k_s}{2}y\right)^2 - \left(p_y + \frac{k_s}{2}\right)^2}
\end{aligned}
$$

Multipole definitions

$$
\begin{aligned}
k_n(s) &= \frac{q}{p_0}\frac{\partial^n}{\partial x^n} B_y(x,0,s) \\
\hat{k}_n(s) &= \frac{q}{p_0}\frac{\partial^n}{\partial x^n} B_x(x,0,s) \\
k_s(s) &= \frac{q}{p_0} B_s(0,0,s)
\end{aligned}
$$

TODO

Fringes

References

General Hamiltonian

$$H = \frac{p_\tau}{\beta_0} - (1 + hx)\left(\sqrt{(1+\delta)^2 - (p_x - a_x(x,y,h))^2 - (p_y - a_y(x,y,h))^2} + a_s(x,y,h)\right) \tag{1.31}$$