Estimating the elementary charge with the Millikan experiment.

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1 Introduction

The objective of this experiment was to determine the elementary charge using a Millikan apparatus. Oil droplets are electrically charged due to the friction they experience in the capillary of the apparatus' atomizer and then enter a chamber that consists of two charged parallel plates which produce a constant electric field. Three forces then act on the oil droplets; the downward gravitational force $F_g = m_o g$ where m_o is the mass of the oil droplet and g is the gravitational acceleration, the upward buoyant force $F_g = -m_a g$ where m_a is the mass of the air displaced by the oil droplet, the electric force $F_E = QE$ where Q is the charge of the oil droplet and E is the electric field and Stokes' resistive force $F_d = 6\pi r \eta \nu$ where r is the radius of the spherical oil droplet, η is the viscosity and ν is the velocity under laminar flow conditions. Depending on the set voltage, the motion of the oil droplet changes as follows;

i) If the oil droplet is floating in the chamber, the net force is zero which can be described by:

$$g(m_o - m_a) - \frac{QV_{stop}}{d} = 0$$

Equation 1: Static particle equation

Where V_{stop} is the voltage which allows the droplet to remain suspended and d is the distance between the charged parallel plates.

ii) If the droplet is moving downward with terminal velocity ν_t, the net force is zero and can be described by:

$$g(m_o - m_a) - 6\pi r \eta \nu_t = 0$$

Equation 2: Terminal falling velocity equation

iii) If the oil droplet is accelerating upwards the equation of motion is described by

$$ma = -g(m_o - m_a) + QE - 6\pi r \eta \nu$$

Equation 3: Upwards accelerating droplet equation

The elementary charge can then be determined using two methods. If the experimenter determines the stopping voltage and terminal velocity of an oil droplet, the following equation can be derived from **Equation 1** and **Equation 2**.

$$Q = C \frac{v_t^{\frac{3}{2}}}{v_{stop}}$$

Equation 4: Method one charge calculation model

Where C is a constant defined by:

$$C = \frac{18\pi d\eta^{\frac{3}{2}}}{\sqrt{2}\sqrt{\rho_o - \rho_a}\sqrt{g}}$$

Constant 1

Where the densities $\rho_o = \frac{3m_o}{4\pi r^3}$ and $\rho_a = \frac{3m_a}{4\pi r^3}$

Furthermore, if the experimenter determines the terminal velocity of the oil drop, the voltage V_{up} at which the oil droplet is moving upward with a constant speed v_2, the following equation can be derived from **Equations 1,2 and 3**;

$$Q = C(v_1 + v_2) \frac{v_t^{\frac{1}{2}}}{v_{up}}$$

Equation 5: Method two charge calculation model

Where the constant C is the same one determined using the first method. Using either method, the greatest common divisor of the individual experimental oil droplet charges can be found, which should then correspond to the value of the elementary charge as the experimental charges should be integer multiples of the elementary charge.

2 Methodology and Procedure

This experiment was a replication of the Millikan oil drop experiment. The experimental apparatus consisted of a variable voltage power supply connected to two plates making up the top and bottom of a transparent observation chamber. A Hewlett-Packard 3476A Digital Multimeter set to a range of 1100 V was connected to the terminals on the chamber to measure the voltage across the plates. This created an approximately uniform electric field within the chamber. An atomizer was connected to the chamber that allowed for charged oil droplets to be sprayed into the chamber. A camera that was attached to a microscopic lens was set up to point inside chamber and provided the ability to track individual droplets' vertical position using a computer program. This experimental apparatus is shown in **Figure 1**.
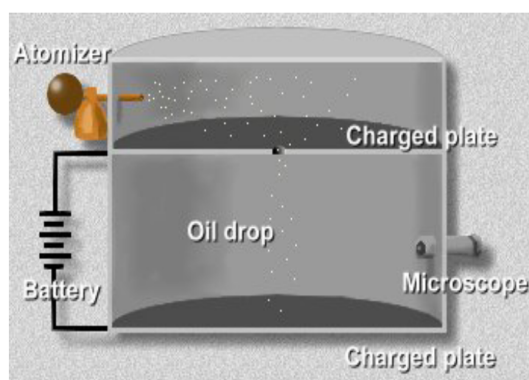

Figure 1: Experimental apparatus diagram

For each measurement, the following procedure was followed. The power supply was set to a value that was approximately in the middle of its output range (generally $\sim 300V$). The atomizer was used to spray a mist of droplets into the chamber. Once an appropriate droplet was located, the camera was moved and focused to get an ideal image for tracking, allowing the computer program to track the droplet's vertical position. The voltage was then adjusted until the droplet was stationary, and the value was recorded as the stopping voltage and the computer program was set up to track the vertical position of the droplet.

Adjusting the voltage on the power supply to $0V$ eliminated the electric field within the chamber. The droplet was allowed to accelerate and then, once it had reached a constant velocity, the position values were saved to a spreadsheet for a small interval of time. These position values in units of pixels were used to calculate the terminal velocity of the droplet.

The power supply was then returned to the previously determined stopping voltage to stop the movement. The power supply was then increased to a value larger than the stopping voltage. This voltage value was recorded with the corresponding stopping voltage. The same tracking technique was used to get a set of positions as the particle traveled upwards and used to calculate the upwards velocity.

3   Results

Note, all referenced functions, equations and constants can be found in the appendix.

To determine the uncertainty of the stopping and upward voltages, **Function 1** was defined based off of the Hewlett-Packard 3476A Digital Multimeter specifications.

To find the charge of the oil droplet for each trial, the terminal velocity and constant upward velocity had to be determined from the experimental data. To convert the position data from pixels to units of meters, the position data was divided by a factor of 540000 since the calibration factor of the left camera was found to be $(540 \pm 1) px/mm$. Since the camera had a frame rate of $10 Hz$, for each individual trial and its respective position data set, an array of times was generated where the time elapsed per frame was $0.1s$. The two sets of position and time data were then fitted to a linear model as implemented by **Function 2** by passing them through scipy's curve_fit function. The uncertainty was determined using curve_fit's covariance values since the diagonals of the covariance matrix represent the variability of each parameter and the square root of this was taken to calculate the standard error. From this an array of terminal and constant upward velocities was determined.

Note **Constant 1** was derived using **Equation 1**, **Equation 2** and **Equation 3** and its numerical value, $2.0 \times 10^{-10} \frac{kg}{s^{1/2}m^{1/2}}$ was determined using known physical constants which can be found in the appendix.

Method 1

Equation 4 as implemented by **Function 3** was used to generate an array of charges for each individual trial by passing in the stopping voltage and terminal velocity arrays. The uncertainties of the charges was determined using **Equation 6** as implemented by **Function 5**.

Method 2

Equation 5 as implemented by **Function 4** was used to generate an array of charges for each individual trial by passing in the terminal velocity, constant upward velocity and upward voltage arrays. The uncertainties of the charges was determined using **Equation 7** as implemented by **Function 6**.

Histograms for both methods were then generated as depicted in **Figure 2** and **Figure 3**. **Function 7** was used to determine the average uncertainty of the charges for both methods.
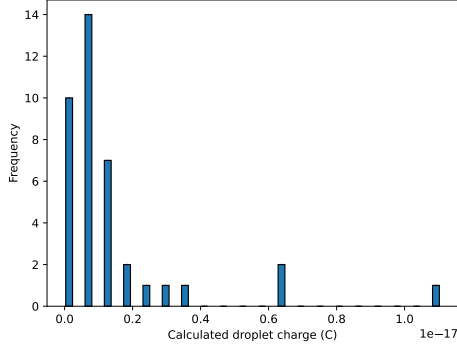
Figure 2: Histogram showing the frequency of droplet charges as calculated with method one
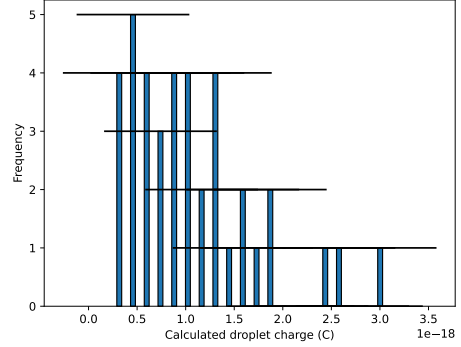

Figure 3: Histogram showing the frequency of droplet charges as calculated with method two

The values that had been determined with **Functions 4, 5, 6 and 7** were all not in line with the measurement precision which required 3 significant figures. Additionally, a requirement of greatest common divisor calculation in numpy is that the values must all be of the int type. **Function 8** was defined to resolve both of these issues. The function is supplied with a list and the required number of significant figures. The function will then return a list that represents the values rounded to three significant figures and scaled up to integers, as well as the scaling factor that the values were scaled by. For example 1.234×10^{-19} and 5.678×10^{-20} will be scaled to 1230 and 568 respectively and the scaling factor will be 20.

This function was applied both to the results for method one and method two and additionally was applied to the uncertainty of each measurement for the purpose of error propagation. This satisfied the requirements of the values being integers and ensured that the values had the correct precision. The numpy gcd function was then used to calculate the greatest common divisor of all four data sets and then the values were returned to their original order of magnitude by dividing by ten to the power of the scaling factor returned from **Function 8**.

To determine the average radius of an oil droplet, the average terminal velocity was determined using numpy's average function and **Equation 8**.

4   Analysis and Discussion

The value of the elementary charge determined using methods 1 and 2 were $1.0 \times 10^{-21} \pm 1.0 \times 10^{-23} C$ and $1.0 \times 10^{-21} \pm 1.0 \times 10 - 21 C$ respectively, however the accepted value of 1.6×10^{-19} does not fall within the range of uncertainty for either measurement. To determine the uncertainties of the experimental elementary charges, the greatest common divisors of the individual charge uncertainties were determined, as they similarly would have been scalar multiples of their respective experimental elementary charge's uncertainty. Obtaining experimental charge values that are perfect integer multiples of the accepted value is difficult as measurements such as the voltage can fluctuate during data collection. Furthermore, rounding may have also been a source of error. When the gcd function was applied to the output of **Function 8**, none of the data sets had a common factor greater than one. This meant that when the calculated value was scaled down to the appropriate order of magnitude, the value was just one divided by the scaling factor from **Function 8**. It follows that using the method of determining the greatest common divisor did not yield accurate results.

Despite the inaccuracy of the determined values, examining **Figure 2** and **Figure 3**, the smallest experimental charge values of the droplets fell within the same order of the accepted elementary charge value, 10^{-19}. Furthermore, almost all charge values for both methods, except one which can be considered as an outlier, were greater than the accepted value of the elementary charge. Using method 1, the minimum value found disregarding the outlier was $2.6 \times 10^{-19} \pm 1.6 \times 10^{-19} C$. Using method 2, the minimum value found was $3.2 \times 10^{-19} \pm 8.7 \times 10^{-19} C$. The accepted value falls within the range of uncertainty for both values and so the minimum experimental droplet charges ultimately act as better estimates of the true elementary charge value.

The average droplet radius found was $1.1 \times 10^{-6} m$. Droplets with larger radii were easier to track using the tracking software since droplets with smaller radii would often dissipate during data collection and the trial would have to be redone. In regards to the buoyant force, the mass of the displaced air can be determined by $m_a = \frac{4}{3} r^3 \rho_a$. Furthermore, the mass of the oil droplet can be determined by $m_o = \frac{4}{3} r^3 \rho_o$ and since $\rho_o >> \rho_a$, it follows that $m_o >> m_a$. In **Equation 1**, **Equation 2** and **Equation 3**, the difference between the masses only matters. Thus, the buoyant force is not significant in this experiment.

5   Conclusion

Although the experimental elementary charges using both methods and finding their greatest common divisors did not yield accurate values, examining **Figure 2** and **Figure 3** confirms that the charge of the droplets cannot fall below the accepted value. It follows that the elementary charge can be quantified using the Millikan apparatus and corresponding equations, however, it is difficult to do so as various sources of error as discussed in the previous section may heavily skew the experimental results.

6 Appendix

Constants

$$C = \frac{18\pi d\eta^{\frac{3}{2}}}{\sqrt{2}\sqrt{\rho_o - \rho_a}\sqrt{g}}$$

Constant 1

Equations

$$g(m_o - m_a) - \frac{Qv_{stop}}{d} = 0$$

Equation 1: Static particle equation

$$g(m_o - m_a) - 6\pi r\eta v_t = 0$$

Equation 2: Terminal falling velocity equation

$$ma = -g(m_o - m_a) + QE - 6\pi r\eta v$$

Equation 3: Upwards accelerating droplet equation

$$Q = C\frac{v_t^{\frac{3}{2}}}{V_{stop}}$$

Equation 4: Method one charge calculation model

$$Q = C(v_1 + v_2)\frac{v_t^{\frac{1}{2}}}{v_{up}}$$

Equation 5: Method two charge calculation model

$$u(Q(x,y)) = \sqrt{\left(\frac{3c}{2}\frac{x^{\frac{1}{2}}}{y}\right)^2 (u(x))^2 + \left(\frac{-Cx^{\frac{3}{2}}}{y^2}\right)(u(y))^2}$$

Equation 6: Method one charge calculation error propagation (derived from general uncertainty propagation equation)

$$u(Q(x,y,z)) = \sqrt{\left(\frac{3}{2}\frac{C}{z}x^{\frac{1}{2}} + \frac{1}{2}C\frac{y}{zx^{\frac{1}{2}}}\right)^2 (u(x))^2 + \left(\frac{Cx^{\frac{1}{2}}}{z}\right)^2 (u(y))^2 + \left(\frac{-Cx^{\frac{3}{2}}}{z^2} - \frac{Cyx^{\frac{1}{2}}}{z^2}\right)^2 (u(z))^2}$$

Equation 7: Method two charge calculation error propagation (derived from general uncertainty propagation equation)

$$r = \sqrt{\frac{\frac{9}{2}\eta v_t}{g\rho_o - \rho_a}} = \frac{3\sqrt{\eta}\sqrt{v_t}}{\sqrt{2}\sqrt{g}\sqrt{\rho_o - \rho_a}}$$

Equation 8: Equation used to calculate average terminal velocity

Functions

```python
def uncertaintyVoltage(reading, ran):
    return 0.006 * reading + 0.001 * ran
```

Function 1: Function used to calculate voltage uncertainty

```python
def linear(time, velocity, intercept):
    return time * velocity + intercept
```

Function 2: General linear model

```python
def chargeMethodOne(velocityOne, voltageOne):
    return constant * velocityOne ** (3 / 2) / voltageOne
```

Function 3: Method one charge model (implements Equation 4)

```python
def chargeMethodTwo(velocityOne, velocityTwo, voltageTwo):
    return (
        constant
        * (abs(velocityOne) + velocityTwo)
        * abs(velocityOne) ** (1 / 2)
        / voltageTwo
    )
```

Function 4: Method two charge model (implements Equation 5)

```python
def uncertaintyChargeMethodOne(
    velocityOne, voltageOne, uncertaintyOne, uncertaintyTwo
):
    return np.sqrt(
        (3 * constant / 2 * np.sqrt(abs(velocityOne)) / voltageOne) ** 2
        * uncertaintyOne ** 2
        + (-1 * constant * abs(velocityOne) ** (3 / 2) / voltageOne ** 2) ** 2
        * uncertaintyTwo ** 2
    )
```

Function 5: Method one uncertainty propagation (implements Equation 6)

```python
    def uncertaintyChargeMethodTwo(
        velocityOne,
        velocityTwo,
        voltageTwo,
        uncertaintyOne,
        uncertaintyTwo,
        uncertaintyThree,
    ):
        return np.sqrt(
            (
                (3 * constant / 2 * np.sqrt(abs(velocityOne)))
                + (
                    constant
                    / 2
                    * velocityTwo
                    / voltageTwo
                    / np.sqrt(abs(velocityOne))
                )
            )
            ** 2
            * uncertaintyOne ** 2
            + (constant * np.sqrt(abs(velocityOne)) / voltageTwo) ** 2
            * uncertaintyTwo ** 2
            + (
                (-1 * constant * abs(velocityOne) ** (3 / 2) / voltageTwo ** 2)
                - (
                    constant
                    * velocityTwo
                    * np.sqrt(abs(velocityOne))
                    / voltageTwo ** 2
                )
            )
            ** 2
            * uncertaintyThree ** 2
        )
```

Function 6: Method two uncertainty propagation (implements Equation 7)

```python
def averageUncertainty(uncertainty):
    s = 0
    for value in uncertainty:
        s += value ** 2
    return np.sqrt(s) / len(uncertainty)
```

Function 7: Function used to calculate average uncertainty for histogram

```python
def scaleAndRound(data, sigFigs):
    """
    Returns a length two tuple. The first element is the values in <data> scaled
    by the order of magnitude of the smalest element in <data> and rounds them
    to the number of significant figures defined in <sigFigs>. The second
    element is the factor that all of the values are scaled by.
    """
    scaleFactor = -(math.floor(math.log(min(data), 10)) - sigFigs + 1)
    scaled = np.rint((data * 10 ** scaleFactor).astype(float)).astype(int)
    for i in range(len(scaled)):
        nonSigDigits = len(str(scaled[i])) - sigFigs
        if nonSigDigits != 0:
            scaled[i] = round(scaled[i], -nonSigDigits)
    return (scaled, scaleFactor)
```

Function 8: Function used to prepare data sets for greatest common divisor calculation

Data

Sample	Charge as calculated with method one (C)
1	$5.14 \times 10^{-19} \pm 6.1 \times 10^{-21}$
2	$3.28 \times 10^{-19} \pm 3.49 \times 10^{-21}$
3	$1.02 \times 10^{-18} \pm 6.83 \times 10^{-20}$
4	$1.2 \times 10^{-18} \pm 1.4 \times 10^{-20}$
5	$1.02 \times 10^{-18} \pm 1.6 \times 10^{-20}$
6	$4.27 \times 10^{-19} \pm 5.23 \times 10^{-21}$
7	$1.98 \times 10^{-18} \pm 2.74 \times 10^{-20}$
8	$2.61 \times 10^{-19} \pm 4.61 \times 10^{-21}$
9	$6.73 \times 10^{-18} \pm 1.57 \times 10^{-19}$
10	$1.82 \times 10^{-18} \pm 2.66 \times 10^{-20}$
11	$8.07 \times 10^{-19} \pm 1.89 \times 10^{-20}$
12	$3.58 \times 10^{-19} \pm 4.84 \times 10^{-21}$
13	$1.58 \times 10^{-18} \pm 1.98 \times 10^{-20}$
14	$1.4 \times 10^{-18} \pm 1.48 \times 10^{-20}$
15	$2.36 \times 10^{-18} \pm 3.41 \times 10^{-20}$
16	$1.65 \times 10^{-18} \pm 1.63 \times 10^{-20}$
17	$1.39 \times 10^{-18} \pm 1.38 \times 10^{-20}$
18	$1.15 \times 10^{-17} \pm 2.48 \times 10^{-19}$
19	$1.15 \times 10^{-18} \pm 1.24 \times 10^{-20}$
20	$4.67 \times 10^{-19} \pm 4.27 \times 10^{-21}$
21	$7.33 \times 10^{-19} \pm 8.32 \times 10^{-21}$
22	$7.44 \times 10^{-19} \pm 8.09 \times 10^{-21}$
23	$3.27 \times 10^{-19} \pm 3.16 \times 10^{-21}$
24	$1.03 \times 10^{-18} \pm 9.63 \times 10^{-21}$
25	$1.29 \times 10^{-18} \pm 1.36 \times 10^{-20}$
26	$1.34 \times 10^{-19} \pm 1.53 \times 10^{-21}$
27	$9.53 \times 10^{-19} \pm 1.06 \times 10^{-20}$
28	$6.64 \times 10^{-18} \pm 9.25 \times 10^{-20}$
29	$1.1 \times 10^{-18} \pm 1.34 \times 10^{-20}$
30	$1.03 \times 10^{-18} \pm 1.32 \times 10^{-20}$
31	$8.77 \times 10^{-19} \pm 8.96 \times 10^{-21}$
32	$5.26 \times 10^{-19} \pm 5.73 \times 10^{-21}$
33	$1.44 \times 10^{-18} \pm 1.9 \times 10^{-20}$
34	$2.52 \times 10^{-18} \pm 2.79 \times 10^{-20}$
35	$3.08 \times 10^{-19} \pm 2.86 \times 10^{-21}$
36	$9.01 \times 10^{-19} \pm 1.35 \times 10^{-20}$
37	$1.26 \times 10^{-18} \pm 1.59 \times 10^{-20}$
38	$3.97 \times 10^{-18} \pm 6.05 \times 10^{-20}$
39	$3.32 \times 10^{-18} \pm 4.62 \times 10^{-20}$

Table 1: Calculated charge values for each sample using method one

Sample	Charge as calculated with method one (C)
1	$5.14 \times 10^{-19} \pm 6.1 \times 10^{-21}$
2	$3.28 \times 10^{-19} \pm 3.49 \times 10^{-21}$
3	$1.02 \times 10^{-18} \pm 6.83 \times 10^{-20}$
4	$1.2 \times 10^{-18} \pm 1.4 \times 10^{-20}$
5	$1.02 \times 10^{-18} \pm 1.6 \times 10^{-20}$
6	$4.27 \times 10^{-19} \pm 5.23 \times 10^{-21}$
7	$1.98 \times 10^{-18} \pm 2.74 \times 10^{-20}$
8	$2.61 \times 10^{-19} \pm 4.61 \times 10^{-21}$
9	$6.73 \times 10^{-18} \pm 1.57 \times 10^{-19}$
10	$1.82 \times 10^{-18} \pm 2.66 \times 10^{-20}$
11	$8.07 \times 10^{-19} \pm 1.89 \times 10^{-20}$
12	$3.58 \times 10^{-19} \pm 4.84 \times 10^{-21}$
13	$1.58 \times 10^{-18} \pm 1.98 \times 10^{-20}$
14	$1.4 \times 10^{-18} \pm 1.48 \times 10^{-20}$
15	$2.36 \times 10^{-18} \pm 3.41 \times 10^{-20}$
16	$1.65 \times 10^{-18} \pm 1.63 \times 10^{-20}$
17	$1.39 \times 10^{-18} \pm 1.38 \times 10^{-20}$
18	$1.15 \times 10^{-17} \pm 2.48 \times 10^{-19}$
19	$1.15 \times 10^{-18} \pm 1.24 \times 10^{-20}$
20	$4.67 \times 10^{-19} \pm 4.27 \times 10^{-21}$
21	$7.33 \times 10^{-19} \pm 8.32 \times 10^{-21}$
22	$7.44 \times 10^{-19} \pm 8.09 \times 10^{-21}$
23	$3.27 \times 10^{-19} \pm 3.16 \times 10^{-21}$
24	$1.03 \times 10^{-18} \pm 9.63 \times 10^{-21}$
25	$1.29 \times 10^{-18} \pm 1.36 \times 10^{-20}$
26	$1.34 \times 10^{-19} \pm 1.53 \times 10^{-21}$
27	$9.53 \times 10^{-19} \pm 1.06 \times 10^{-20}$
28	$6.64 \times 10^{-18} \pm 9.25 \times 10^{-20}$
29	$1.1 \times 10^{-18} \pm 1.34 \times 10^{-20}$
30	$1.03 \times 10^{-18} \pm 1.32 \times 10^{-20}$
31	$8.77 \times 10^{-19} \pm 8.96 \times 10^{-21}$
32	$5.26 \times 10^{-19} \pm 5.73 \times 10^{-21}$
33	$1.44 \times 10^{-18} \pm 1.9 \times 10^{-20}$
34	$2.52 \times 10^{-18} \pm 2.79 \times 10^{-20}$
35	$3.08 \times 10^{-19} \pm 2.86 \times 10^{-21}$
36	$9.01 \times 10^{-19} \pm 1.35 \times 10^{-20}$
37	$1.26 \times 10^{-18} \pm 1.59 \times 10^{-20}$
38	$3.97 \times 10^{-18} \pm 6.05 \times 10^{-20}$
39	$3.32 \times 10^{-18} \pm 4.62 \times 10^{-20}$

Table 2: Calculated charge values for each sample using method one