Discretized model of the ionosphere potential

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1 Equation Simplicifaction

The initial equation given by the goodman paper is as follows:

$$\begin{aligned}
j_R = \frac{1}{R_e^2}\Bigg[&\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\theta^2} \\
&+ \frac{1}{\sin\theta}\left(\Sigma_{\theta\phi} + \Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\phi\partial\theta} \\
&+ \frac{\Sigma_{\phi\phi}}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \Bigg\{\frac{\partial}{\partial\theta}\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
&+ \cot\theta\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
&+ \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\Bigg\}\frac{\partial\psi}{\partial\theta} \\
&+ \left\{\frac{\partial}{\partial\theta}\left(\frac{\Sigma_{\theta\phi}}{\sin\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \frac{\Sigma_{\theta\phi}\cot\theta}{\sin\theta}\right\}\frac{\partial\psi}{\partial\phi}\Bigg]_{R=R_e}
\end{aligned} \tag{1}$$

By moving the R_E^2 term to the left side and then multiplying both sides by $\sin^2\theta$, we arrive at:

$$\begin{aligned}
\sin^2\theta \cdot R_e^2 j_R = \Bigg[&\sin^2\theta\cdot\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\theta^2} \\
&+ \sin\theta\left(\Sigma_{\theta\phi} + \Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\phi\partial\theta} \\
&+ \Sigma_{\phi\phi}\frac{\partial^2\psi}{\partial\phi^2} + \Bigg\{\sin^2\theta\cdot\frac{\partial}{\partial\theta}\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
&+ \sin\theta\cos\theta\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
&+ \sin\theta\cdot\frac{\partial}{\partial\phi}\left(\Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\Bigg\}\frac{\partial\psi}{\partial\theta} \\
&+ \left\{\sin^2\theta\cdot\frac{\partial}{\partial\theta}\left(\frac{\Sigma_{\theta\phi}}{\sin\theta}\right) + \frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \Sigma_{\theta\phi}\cdot\cos\theta\right\}\frac{\partial\psi}{\partial\phi}\Bigg]_{R=R_e}
\end{aligned}$$

By the comment on the paper by O.Amm, the conductance tensor (Equation 1 in the comment) is given as:

$$\Sigma_{(R,\theta,\phi)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Sigma_0 \Sigma_P}{C} & \frac{\Sigma_0 \Sigma_H (-\cos\varepsilon)}{C} \\ 0 & \frac{\Sigma_0 \Sigma_H \cos\varepsilon}{C} & \Sigma_P + \frac{\Sigma_H^2 \sin^2\varepsilon}{C} \end{pmatrix} \tag{2}$$

Where the values C, $\cos\varepsilon$ and $\sin\varepsilon$ are defined as follows:

$$C = \qquad \Sigma_0 \cos^2\theta + \Sigma_P \sin^2\theta \tag{3}$$

$$\cos\varepsilon \qquad = \frac{-2\cos\theta}{\sqrt{1+3\cos^2\theta}} \tag{4}$$

$$\sin\varepsilon \qquad = \tag{5}$$

Where Equation 3 is given in O.Amm's comment, and Equations 4 and 5 correspond to Equations 21 and 22 in Goodmans original paper respectively.
The conductance tensor (Equation 2) allows some terms to be eliminated. $\Sigma_{\theta\phi} = -\Sigma_{\phi\theta}$ and $\Sigma_{\theta R} = \Sigma_{\phi R} = 0$. We can therefore, make subsitutions $\Sigma_{\psi\theta} = -\Sigma_{\theta\psi}$ on lines 2 and 5:

$$\begin{aligned}
\sin^2\theta \cdot R_e^2 j_R = \Bigg[& \sin^2\theta \cdot \left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R} \tan\theta}{2} \right) \frac{\partial^2\psi}{\partial\theta^2} \\
& + \sin\theta \left(\frac{\Sigma_{\phi R} \tan\theta}{2} \right) \frac{\partial^2\psi}{\partial\phi\partial\theta} \\
& + \Sigma_{\phi\phi} \frac{\partial^2\psi}{\partial\phi^2} + \Bigg\{ \sin^2\theta \cdot \frac{\partial}{\partial\theta} \left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R} \tan\theta}{2} \right) \\
& + \sin\theta\cos\theta \left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R} \tan\theta}{2} \right) \\
& + \sin\theta \cdot \frac{\partial}{\partial\phi} \left(-\Sigma_{\theta\phi} - \frac{\Sigma_{\phi R} \tan\theta}{2} \right) \Bigg\} \frac{\partial\psi}{\partial\theta} \\
& + \Bigg\{ \sin^2\theta \cdot \frac{\partial}{\partial\theta} \left(\frac{\Sigma_{\theta\phi}}{\sin\theta} \right) + \frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \Sigma_{\theta\phi} \cdot \cos\theta \Bigg\} \frac{\partial\psi}{\partial\phi} \Bigg]_{R=R_e}
\end{aligned}$$

Additionally, we can remove all terms with a $\Sigma_{\psi R}$ or $\Sigma_{\theta R}$:

$$\sin^2\theta \cdot R_e^2 j_R = \Bigg[\sin^2\theta \cdot \Sigma_{\theta\theta} \frac{\partial^2 \psi}{\partial \theta^2}$$

$$+ \Sigma_{\phi\phi} \frac{\partial^2 \psi}{\partial \phi^2}$$

$$+ \left\{ \sin^2\theta \cdot \frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + \sin\theta\cos\theta \cdot \Sigma_{\theta\theta} - \sin\theta \cdot \frac{\partial \Sigma_{\theta\phi}}{\partial \phi} \right\} \frac{\partial \psi}{\partial \theta}$$

$$+ \left\{ \sin^2\theta \cdot \frac{\partial}{\partial \theta} \left(\frac{\Sigma_{\theta\phi}}{\sin\theta} \right) + \frac{\partial \Sigma_{\phi\phi}}{\partial \phi} + \Sigma_{\theta\phi} \cdot \cos\theta \right\} \frac{\partial \psi}{\partial \phi} \Bigg]_{R=R_e}$$

The $\sin^2\theta \cdot \frac{\partial}{\partial\theta}\left(\frac{\Sigma_{\theta\phi}}{\sin\theta}\right)$ term in the first ϕ derivative coefficient can be expanded with the quotient rule, giving:

$$\begin{aligned} \sin^2\theta \cdot \frac{\partial}{\partial \theta} \left(\frac{\Sigma_{\theta\phi}}{\sin\theta} \right) &= \sin^2\theta \cdot \frac{\sin\theta \partial_\theta \Sigma_{\theta\phi} - \Sigma_{\theta\phi}\cos\theta}{\sin^2\theta} \\ &= \sin\theta \cdot \frac{\partial \Sigma_{\theta\phi}}{\partial \theta} - \Sigma_{\theta\phi} \cdot \cos\theta \end{aligned}$$

This cancels out the other term of $\Sigma_{\theta\phi} \cdot \cos\theta$ in the same coefficient. Separating out the coefficients, we arrive at the following expression for Equation 1:

$$\begin{aligned} \sin^2\theta R_E^2 j_r &= \kappa_{\theta\theta} \frac{\partial^2 \psi}{\partial \theta^2} + \kappa_{\phi\phi} \frac{\partial^2 \psi}{\partial \phi^2} + \kappa_\theta \frac{\partial \psi}{\partial \theta} + \kappa_\phi \frac{\partial \psi}{\partial \phi} \\ \kappa_{\theta\theta} &= \sin^2\theta \cdot \Sigma_{\theta\theta} \\ \kappa_{\phi\phi} &= \Sigma_{\phi\phi} \\ \kappa_\theta &= \sin^2\theta \cdot \frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + \sin\theta\cos\theta \cdot \Sigma_{\theta\theta} - \sin\theta \cdot \frac{\partial \Sigma_{\theta\phi}}{\partial \phi} \\ \kappa_\phi &= \sin\theta \cdot \frac{\partial \Sigma_{\theta\phi}}{\partial \theta} + \frac{\partial \Sigma_{\phi\phi}}{\partial \phi} \end{aligned} \tag{6}$$

At this point, there are no longer any cross terms remaining in the equation, only first and second order derivatives of ψ. Using Equation 2, along with the expansions of C, $\cos\varepsilon$ and $\sin\varepsilon$ given in Equations 3, 4 and 5 the conductance tensor components remaining in the equation can be put in terms of θ, ϕ.

$$\begin{aligned}\Sigma_{\theta\theta} &= \frac{\Sigma_0 \Sigma_P}{C} = \frac{\Sigma_0 \Sigma_P}{\Sigma_0 \cos^2 \varepsilon + \Sigma_P \sin^2 \varepsilon} \\ &= \frac{\Sigma_0 \Sigma_P (1 + 3\cos^2 \theta)}{\Sigma_0 \cos^2 \theta + \Sigma_P \sin^2 \theta}\end{aligned}$$

$$\begin{aligned}\Sigma_{\theta\phi} &= \frac{-\Sigma_0 \Sigma_P \cos\varepsilon}{C} = \frac{-\Sigma_0 \Sigma_P \cos\varepsilon}{\Sigma_0 \cos^2 \varepsilon + \Sigma_P \sin^2 \varepsilon} \\ &= \frac{-\Sigma_0 \Sigma_P \cos\varepsilon (1 + 3\cos^2 \theta)}{4\Sigma_0 \cos^2 \theta + \Sigma_P \sin^2 \theta} = \frac{2\Sigma_0 \Sigma_P \cos\theta \sqrt{1 + 3\cos^2 \theta}}{4\Sigma_0 \cos^2 \theta + \Sigma_P \sin^2 \theta} \\ &= \frac{2\Sigma_0 \Sigma_P \sqrt{1 + 3\cos^2 \theta}}{4\Sigma_0 \cos\theta + \Sigma_P \sin\theta \tan\theta}\end{aligned}$$

$$\begin{aligned}\Sigma_{\phi\phi} &= \Sigma_P + \frac{\Sigma_H^2 \sin^2 \varepsilon}{\Sigma_0 \cos^2 \varepsilon + \Sigma_P \sin^2 \varepsilon} = \Sigma_P + \frac{\Sigma_H^2 \sin^2 \theta}{4\Sigma_0 \cos^2 \theta + \Sigma_P \sin^2 \theta} \cdot \frac{\sqrt{1 + 3\cos^2 \theta}}{\sqrt{1 + 3\cos^2 \theta}} \\ &= \Sigma_P + \frac{\Sigma^2 + H}{4\Sigma_0 \cot^2 \theta + \Sigma_P}\end{aligned}$$

This results in the following set of equations for the conductance tensor components:

$$\Sigma_{\theta\theta} = \frac{\Sigma_0 \Sigma_P (1 + 3\cos^2 \theta)}{\Sigma_0 \cos^2 \theta + \Sigma_P \sin^2 \theta} \tag{7}$$

$$\Sigma_{\theta\phi} = \frac{2\Sigma_0 \Sigma_P \sqrt{1 + 3\cos^2 \theta}}{4\Sigma_0 \cos\theta + \Sigma_P \sin\theta \tan\theta} \tag{8}$$

$$\Sigma_{\phi\phi} = \Sigma_P + \frac{\Sigma^2 + H}{4\Sigma_0 \cot^2 \theta + \Sigma_P} \tag{9}$$

2 Discretization

The resulting Equation 6 can be discretized using a central difference approximation across the ψ derivatives.

$$\left(\frac{\partial^2\psi}{\partial\theta^2}\right)_{i,j} = \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta\theta^2} \tag{10}$$

$$\left(\frac{\partial^2\psi}{\partial\phi^2}\right)_{i,j} = \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta\phi^2} \tag{11}$$

$$\left(\frac{\partial\psi}{\partial\theta}\right)_{i,j} = \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta\theta} \tag{12}$$

$$\left(\frac{\partial\psi}{\partial\phi}\right)_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta\phi} \tag{13}$$

Additionally, the same discretezation approximation can be applied to the Σ derivative terms.

$$\left(\frac{\partial\Sigma_{\theta\theta}}{\partial\theta}\right)_{i,j} = \frac{(\Sigma_{\theta\theta})_{i+1,j} - (\Sigma_{\theta\theta})_{i-1,j}}{2\Delta\theta} \tag{14}$$

$$\left(\frac{\partial\Sigma_{\theta\phi}}{\partial\phi}\right)_{i,j} = \frac{(\Sigma_{\theta\phi})_{i,j+1} - (\Sigma_{\theta\phi})_{i,j-1}}{2\Delta\phi} \tag{15}$$

$$\left(\frac{\partial\Sigma_{\theta\phi}}{\partial\theta}\right)_{i,j} = \frac{(\Sigma_{\theta\phi})_{i+1,j} - (\Sigma_{\theta\phi})_{i-1,j}}{2\Delta\theta} \tag{16}$$

$$\left(\frac{\partial\Sigma_{\phi\phi}}{\partial\phi}\right)_{i,j} = \frac{(\Sigma_{\phi\phi})_{i,j+1} - (\Sigma_{\phi\phi})_{i,j-1}}{2\Delta\phi} \tag{17}$$

It is possible to determine an analytical solution to all of these derivatives, however for the first iteration of this algorithm, a finite difference approach will be conducted.

3 Implementation

To implement the solver for Equation 6 will be implemented in C++ using a trilinus package that is yet to be determined. The steps for computation involve.

1. Computing conductance values Σ_0, Σ_H and Σ_P across the solution domain. This will be done using empirical data of the ionosphere. This data will then be used to calculate relevant values of the conductance tensor utilizing 7, 8 and 9.

2. Computing the finite difference approximation for the conductance derivatives shown in Equations 14, 15, 16 and 17. This will be done using trillinus. In a future iteration, this step may be replaced with an analytical solution to these equations.

3. Computing the finite difference approximation for Equation 6 using the values computed in step 1 and 2 also using trilinus.