Discretized model of the ionosphere potential

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1 Equation Simplicifaction

The initial equation given by the goodman paper is as follows:

$$
\begin{aligned}
j_R = \frac{1}{R_e^2}\Bigg[& \left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\theta^2} \\
& + \frac{1}{\sin\theta}\left(\Sigma_{\theta\phi} + \Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\phi\partial\theta} \\
& + \frac{\Sigma_{\phi\phi}}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \Bigg\{\frac{\partial}{\partial\theta}\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
& + \cot\theta\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
& + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\Bigg\}\frac{\partial\psi}{\partial\theta} \\
& + \left\{\frac{\partial}{\partial\theta}\left(\frac{\Sigma_{\theta\phi}}{\sin\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \frac{\Sigma_{\theta\phi}\cot\theta}{\sin\theta}\right\}\frac{\partial\psi}{\partial\phi}\Bigg]_{R=R_e}
\end{aligned}
$$

By moving the R_E^2 term to the left side and then multiplying both sides by $\sin^2\theta$, we arrive at:

$$
\begin{aligned}
\sin^2\theta\cdot R_e^2 j_R = \Bigg[& \sin^2\theta\cdot\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\theta^2} \\
& + \sin\theta\left(\Sigma_{\theta\phi} + \Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\frac{\partial^2\psi}{\partial\phi\partial\theta} \\
& + \Sigma_{\phi\phi}\frac{\partial^2\psi}{\partial\phi^2} + \Bigg\{\sin^2\theta\cdot\frac{\partial}{\partial\theta}\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
& + \sin\theta\cos\theta\left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2}\right) \\
& + \sin\theta\cdot\frac{\partial}{\partial\phi}\left(\Sigma_{\phi\theta} - \frac{\Sigma_{\phi R}\tan\theta}{2}\right)\Bigg\}\frac{\partial\psi}{\partial\theta} \\
& + \left\{\sin^2\theta\cdot\frac{\partial}{\partial\theta}\left(\frac{\Sigma_{\theta\phi}}{\sin\theta}\right) + \frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \Sigma_{\theta\phi}\cdot\cos\theta\right\}\frac{\partial\psi}{\partial\phi}\Bigg]_{R=R_e}
\end{aligned}
$$

As given by the comment on the paper by O.Amm, the conductance tensor is:

$$\Sigma_{(R,\theta,\phi)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Sigma_0 \Sigma_P}{C} & \frac{\Sigma_0 \Sigma_H(-\cos\varepsilon)}{C} \\ 0 & \frac{\Sigma_0 \Sigma_H \cos\varepsilon}{C} & \Sigma_P + \frac{\Sigma_H^2 \sin^2\varepsilon}{C} \end{pmatrix}$$

From this, it is clear that $\Sigma_{\theta\phi} = -\Sigma_{\phi\theta}$ and $\Sigma_{\theta R} = \Sigma_{\phi R} = 0$. We can therefore, make subsitutions $\Sigma_{\psi\theta} = -\Sigma_{\theta\psi}$ on lines 2 and 6:

$$\begin{aligned}
\sin^2\theta \cdot R_e^2 j_R = \Bigg[& \sin^2\theta \cdot \left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2} \right) \frac{\partial^2\psi}{\partial\theta^2} \\
& + \sin\theta \left(\frac{\Sigma_{\phi R}\tan\theta}{2} \right) \frac{\partial^2\psi}{\partial\phi\partial\theta} \\
& + \Sigma_{\phi\phi} \frac{\partial^2\psi}{\partial\phi^2} + \Bigg\{ \sin^2\theta \cdot \frac{\partial}{\partial\theta} \left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2} \right) \\
& + \sin\theta\cos\theta \left(\Sigma_{\theta\theta} - \frac{\Sigma_{\theta R}\tan\theta}{2} \right) \\
& + \sin\theta \cdot \frac{\partial}{\partial\phi} \left(-\Sigma_{\theta\phi} - \frac{\Sigma_{\phi R}\tan\theta}{2} \right) \Bigg\} \frac{\partial\psi}{\partial\theta} \\
& + \left\{ \sin^2\theta \cdot \frac{\partial}{\partial\theta} \left(\frac{\Sigma_{\theta\phi}}{\sin\theta} \right) + \frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \Sigma_{\theta\phi} \cdot \cos\theta \right\} \frac{\partial\psi}{\partial\phi} \Bigg]_{R=R_e}
\end{aligned}$$

Additionally, we can remove all terms with a $\Sigma_{\psi R}$ or $\Sigma_{\theta R}$:

$$\begin{aligned}
\sin^2\theta \cdot R_e^2 j_R = \Bigg[& \sin^2\theta \cdot \Sigma_{\theta\theta} \frac{\partial^2\psi}{\partial\theta^2} \\
& + \Sigma_{\phi\phi} \frac{\partial^2\psi}{\partial\phi^2} \\
& + \left\{ \sin^2\theta \cdot \frac{\partial\Sigma_{\theta\theta}}{\partial\theta} + \sin\theta\cos\theta \cdot \Sigma_{\theta\theta} - \sin\theta \cdot \frac{\partial\Sigma_{\theta\phi}}{\partial\phi} \right\} \frac{\partial\psi}{\partial\theta} \\
& + \left\{ \sin^2\theta \cdot \frac{\partial}{\partial\theta} \left(\frac{\Sigma_{\theta\phi}}{\sin\theta} \right) + \frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \Sigma_{\theta\phi} \cdot \cos\theta \right\} \frac{\partial\psi}{\partial\phi} \Bigg]_{R=R_e}
\end{aligned}$$

The $\sin^2\theta \cdot \frac{\partial}{\partial\theta} \left(\frac{\Sigma_{\theta\phi}}{\sin\theta} \right)$ term in the first ϕ derivative coefficient can be expanded with the quotient rule, giving:

$$\begin{aligned}\sin^2\theta \cdot \frac{\partial}{\partial\theta}\left(\frac{\Sigma_{\theta\phi}}{\sin\theta}\right) &= \sin^2\theta \cdot \frac{\sin\theta\partial_\theta\Sigma_{\theta\phi} - \Sigma_{\theta\phi}\cos\theta}{\sin^2\theta} \\ &= \sin\theta \cdot \frac{\partial\Sigma_{\theta\phi}}{\partial\theta} - \Sigma_{\theta\phi} \cdot \cos\theta\end{aligned}$$

This cancels out the other term of $\Sigma_{\theta\phi} \cdot \cos\theta$ in the same coefficient. Separating out the coefficients, we arrive at the following expression for the original equation:

$$\sin^2\theta R_E^2 j_r = \kappa_{\theta\theta}\frac{\partial^2\psi}{\partial\theta^2} + \kappa_{\psi\psi}\frac{\partial^2\psi}{\partial\psi^2} + \kappa_\theta\frac{\partial\psi}{\partial\theta} + \kappa_\phi\frac{\partial\psi}{\partial\phi} \tag{1a}$$

$$\kappa_{\theta\theta} = \sin^2\theta \cdot \Sigma_{\theta\theta} \tag{1b}$$

$$\kappa_{\psi\psi} = \Sigma_{\phi\phi} \tag{1c}$$

$$\kappa_\theta = \sin^2\theta \cdot \frac{\partial\Sigma_{\theta\theta}}{\partial\theta} + \sin\theta\cos\theta \cdot \Sigma_{\theta\theta} - \sin\theta \cdot \frac{\partial\Sigma_{\theta\phi}}{\partial\phi} \tag{1d}$$

$$\kappa_\phi = \sin\theta \cdot \frac{\partial\Sigma_{\theta\phi}}{\partial\theta} + \frac{\partial\Sigma_{\phi\phi}}{\partial\phi} \tag{1e}$$

At this point, there are no longer any cross terms remaining in the equation, only first and second order derivatives of ψ.

2 Discretization