227-0689-00L System Identification **03 October 2022**

Exercise 2: Least Squares

Background reading

The background material for this exercise is Sections 10.1, 13.2 and Appendix II of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1:

Consider the least-squares (LS) estimation problem:

$$Y = \Phi\theta + \epsilon,$$

where Φ is the regressor matrix and θ is the parameter vector to be estimated

$$Y := \begin{bmatrix} y(0) \\ \vdots \\ y(N-1)) \end{bmatrix}, \qquad \Phi := \begin{bmatrix} \varphi^\top(0) \\ \vdots \\ \varphi^\top(N-1) \end{bmatrix}, \qquad \theta := \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_p \end{bmatrix}.$$

Assume that the noise, ϵ, is zero-mean Gaussian and correlated with $E\left\{\epsilon\epsilon^\top\right\} = R$. In this exercise we look for a linear estimator $\hat{\theta}$ of the form,

$$\hat{\theta} = Z^\top Y, \tag{2.1}$$

which is unbiased and minimizes its variance. For a given Φ show the following:

1. For a linear estimator of the form (2.1) to be unbiased we require that $Z^\top \Phi = I$.
2. The covariance matrix of any linear unbiased estimator of the form (2.1) is $\operatorname{cov}\left\{\hat{\theta}\right\} = Z^\top R Z$.
3. The covariance matrix of the best linear unbiased estimator (BLUE) $\hat{\theta}_Z$ with $\hat{\theta}_Z = \left(\Phi^\top R^{-1}\Phi\right)^{-1}\Phi^\top R^{-1} Y$ is $\operatorname{cov}\left\{\hat{\theta}_Z\right\} = \left(\Phi^\top R^{-1}\Phi\right)^{-1}$.
4. The best linear unbiased estimator $\hat{\theta}_Z$ exhibits the smallest variance in the class of all unbiased estimators, i.e. $\operatorname{cov}\left\{\hat{\theta}_Z\right\} \leq \operatorname{cov}\left\{\hat{\theta}\right\}$.

Hint: All covariance matrices are positive semi-definite and in our case we can assume that R is positive definite. The inverse of a positive definite matrix is also positive definite.

Problem 2:

Show that the following problems can be solved using linear least squares, with the estimate given by

$$\hat{\theta} = (X^T X)^{-1} X^T y. \tag{2.2}$$

For each write down $\hat{\theta}$, X, and y.

1. fitting the coeficients of a polynomial of degree k, $p(x) = \sum_{i=0}^{k} a_i x^i$, to n i.i.d. observations.
2. fitting the equation of a circle, $(x - x_c)^2 + (y - y_c)^2 = r^2$, to n i.i.d. observations.

Hint 1: Write the expression in matrix form, with the coefiecients of the polynomial isolated in a vector.

Hint 2: The general equation for a conic section is $F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$, with a, b and c not all zero. A circle is a conic section.

Problem 3: (Matlab)

Data was collected on the position of two comets traveling in the solar system. You are provided with the files C-2017-K2.mat and P-2011-NO1.mat, containing xy observations in the orbit plane, in light-years.[1]

The general equation for a conic section is $F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$, with a, b and c not all zero. An ellipse satisfies $b^2 - 4ac < 0$, a parabola $b^2 - 4ac = 0$, and an hyperbole $b^2 - 4ac > 0$.

1. Write the linear least squares estimate for the general equation of a conic section.
2. Using Matlab estimate the coeficients of the equation for each of the comets.
3. Are any of these comets periodic?

Hint 1: The equation is satified for any scaling of the parameters, you can set $f = 1$ to avoid a trivial solution.

[1] This data was generated from the published orbit elements of the comets, and are not direct observations.