Assignment Optimization methods and algorithms

Group 21, course A-LZ

1 Linear model

Objective function:

$$\min \sum_{i=1}^{|E|} \sum_{j=1}^{|E|} n_{i,j} \sum_{\Delta=0}^{5} \frac{2^{5-\Delta}}{|S|} Y_{i,j,\Delta} \tag{1}$$

Constraints:

$$Y_{i,j,\Delta} \geq X_{i,k} + X_{j,k+\Delta} - 1 \quad \forall i,j \in E, \forall \Delta \in \{0,...,5\}, \forall k = 1,...,|T| - \Delta \tag{2}$$

$$n_{i,j} Y_{i,j,0} = 0 \quad \forall i,j \in E \tag{3}$$

$$\sum_{j=1}^{|E|} X_{i,j} = 1 \quad \forall i \in E \tag{4}$$

$$X_{i,j} \in \{0,1\} \quad \forall i \in E, \forall j \in T \tag{5}$$

$$Y_{i,j,\Delta} \in \{0,1\} \quad \forall i,j \in E, \forall \Delta \in \{0,1,2,3,4,5\} \tag{6}$$

2 Description of the model

2.1 Problem data

- E is the set of all exams.
- T is the set of all time slots.
- $N = (n_{i,j})_{i,j}$ is a matrix whose entry $n_{i,j}$ represents the number of students enrolled in both exams i and j ($i, j \in E$). It is assumed that $n_{i,i} = 0$ for all i.

2.2 Variables

- $X = (x_{i,j})_{i,j}$ is a matrix whose entry $x_{i,j} = 1$ if the exam i is located in time slot j, otherwise $x_{i,j} = 0$ ($i \in E, j \in T$).

- $Y_{i,j,\Delta}$ ($i, j \in E, \Delta \in \{0, 1, \ldots, 5\}$) is a boolean variable describing the distance between the time slots in which exams i and j are located: in particular $Y_{i,j,\Delta} = 1$ if the exam j is located Δ time slots after exam i, otherwise $Y_{i,j,\Delta} = 0$.

2.3 Objective function

- First of all, we observe that if $n_{i,j} = 0$ the penalty is not evaluated for the exams i,j. Otherwise, the penalty is computed if and only if the exam j is located Δ time slots after the exam i ($Y_{i,j,\Delta} = 1$). Notice that for each couple of exams i,j, it can exists at most one $\Delta \in \{0, 1, 2, 3, 4, 5\}$ such that $Y_{i,j,\Delta} = 1$. Then, the objective function is obtained by summing the penalties for all possible couples of exams i,j.

2.4 Constraints

- (2): For all i,j in E, for all k in $\{1, ..., |T| - \Delta\}$ and for every Δ in $\{0, ..., 5\}$, this constraint forces $Y_{i,j,\Delta}$ to be equal to 1 if both $X_{i,k}$ and $X_{j,k+\Delta}$ are equal to 1 (i.e. exam j is located Δ time slots after exam i). This constraint doesn't ensure us that $Y_{i,j,\Delta} = 0$ if at least one of $X_{i,k}$ and $X_{j,k+\Delta}$ is equal to 0, but this is not a problem because of the objective function, since we need to minimize it (without constraints, the optimum would be $Y_{i,j,\Delta} = 0$).

- (3): If $n_{i,j} > 0$ this constraint forces $Y_{i,j,0}$ to be equal to 0, because two exams which have common students can't be placed in the same time slot. So $Y_{i,j,0}$ never contributes to the computation of the objective function.

- (4): This constraint ensures that every exam is located in exactly one and only one time slot.

3 Examples in MPL

3.1 Example 1

$$N = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}, \ |E| = 3, \ |T| = 5$$

We have ignored the total number of students because it doesn't affect the optimal solution since it is a constant. So the value "MIN Cost" should be divided by the number of students to obtain the value of the objective function that we have defined in the model formulation.

```
SOLUTION RESULT


  Optimal integer solution found


    MIN Cost              =        38.0000


DECISION VARIABLES
VARIABLE x[exam,timeslot] :

    exam  timeslot          Activity     Reduced Cost
  ----------------------------------------------------
      0         0             1.0000           0.0000
      0         1             0.0000           0.0000
      0         2             0.0000           0.0000
      0         3             0.0000           0.0000
      0         4             0.0000           0.0000
      1         0             0.0000           0.0000
      1         1             0.0000           0.0000
      1         2             1.0000           0.0000
      1         3             0.0000           0.0000
      1         4             0.0000           0.0000
      2         0             0.0000           0.0000
      2         1             0.0000           0.0000
      2         2             0.0000           0.0000
      2         3             0.0000           0.0000
      2         4             1.0000           0.0000
  ----------------------------------------------------
```

The solution is:

	t_1	t_2	t_3	t_4	t_5
Exams	e_1		e_2		e_3

3.2 Example 2

$$N = \begin{pmatrix} 0 & 20 & 30 \\ 20 & 0 & 2 \\ 30 & 2 & 0 \end{pmatrix},\ |E| = 3,\ |T| = 5$$

```
SOLUTION RESULT


  Optimal integer solution found


    MIN Cost              =         172.0000


DECISION VARIABLES
VARIABLE x[exam,timeslot] :

    exam  timeslot          Activity     Reduced Cost
  ----------------------------------------------------
      0         0             1.0000           0.0000
      0         1             0.0000           0.0000
      0         2             0.0000           0.0000
      0         3             0.0000           0.0000
      0         4             0.0000           0.0000
      1         0             0.0000           0.0000
      1         1             0.0000           0.0000
      1         2             0.0000           0.0000
      1         3             1.0000           0.0000
      1         4             0.0000           0.0000
      2         0             0.0000           0.0000
      2         1             0.0000           0.0000
      2         2             0.0000           0.0000
      2         3             0.0000           0.0000
      2         4             1.0000           0.0000
  ----------------------------------------------------
```

The solution is:

	t_1	t_2	t_3	t_4	t_5
Exams	e_1			e_2	e_3

3.3 Example 3 - No feasible solutions

$$N = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix},\ |E| = 3,\ |T| = 2$$

```
    MODEL STATISTICS

Problem name:       Unfeasible_model

Filename:           Unfeasible_model.mpl
Date:               November 11, 2017
Time:               12:51
Parsing time:       0.03 sec
MPL version:        5.0.6.114  (64-bit)

Solver name:        CPLEX  (11.2.1)
Objective value:    0.000000000000
Integer nodes:      0
Improving nodes:    0
Iterations:         0
Solution time:      0.02 sec
Solver result:      Integer infeasible solution
Result code:        103

Model result:       Solution not available (4)
Constraints:        27
Variables:          42
Integers:           60
Nonzeros:           66
Density:            6 %

END
```