CSE 102 4-2-24

Handout: Asymptotic Growth

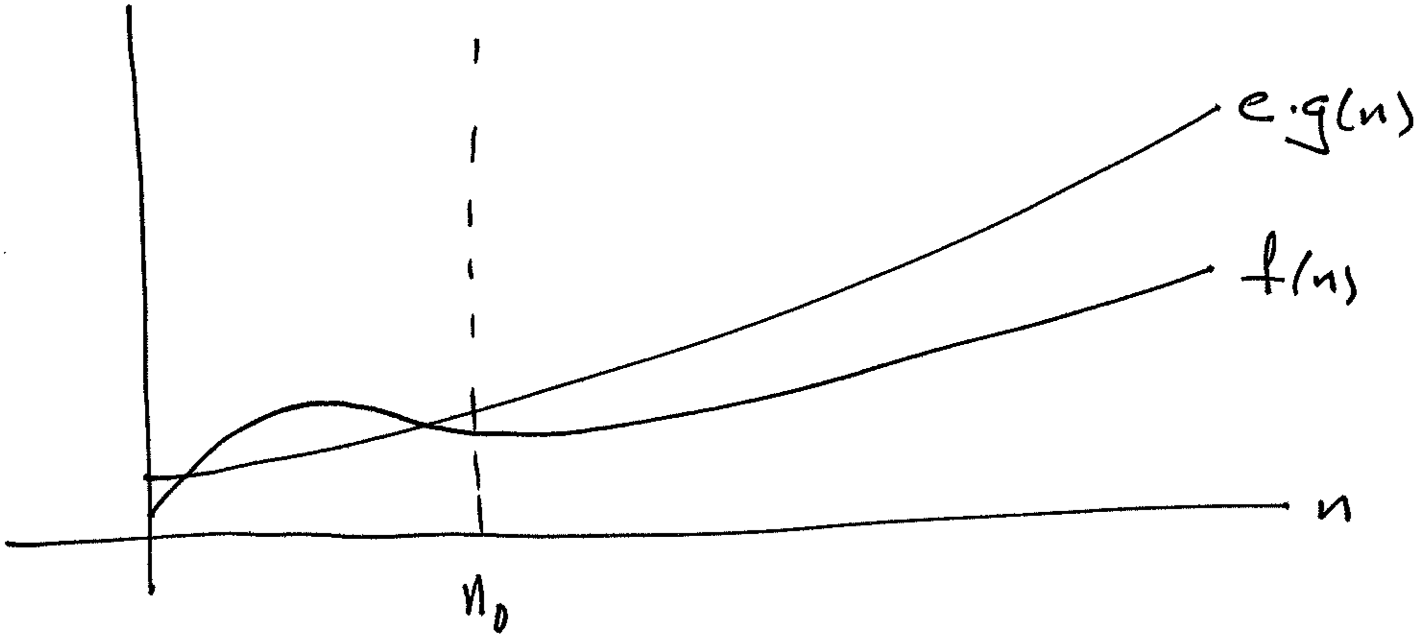
Defn let $g(n)$ be a function, The set $O(g(n))$ is

$$O(g(n)) = \{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

i.e. $f(n) \in O(g(n))$ iff

there exist pos. c, n_0 s.t. for all $n \geq n_0$:

$$0 \leq f(n) \leq c\, g(n)$$



<u>notation</u>: write $f(n) = O(g(n))$ to mean $f(n) \in O(g(n))$

<u>Defn</u> $f(n)$ is asymptotically non-negative iff $\exists\, n_0 > 0$ s.t. for all $n \geq n_0$: $f(n) \geq 0$.

Blanket assumption: all functions under discussion are asymptotically non-negative. (a.n.)

Defn $f(n)$ is asymptotically positive (a.p.) iff $\exists n_0 > 0$ s.t. for all $n \geq n_0$:

$f(n) > 0$.

Ex. $10n + 100 = O(n^2 - 40n + 500)$.

why? Observe that

$$0 \leq 10n + 100 \leq 1 \cdot (n^2 - 40n + 500)$$

holds for all $n \geq 40$. i.e.

$c = 1$ and $n_0 = 40$.

<u>note</u>: $an^1 + b = O(cn^2 + dn + e)$

for any constants a, b, c, d, e

with $a > 0$, $c > 0$.

more generally $P(n) = O(q(n))$

for any polynomials $P(n)$, $q(n)$

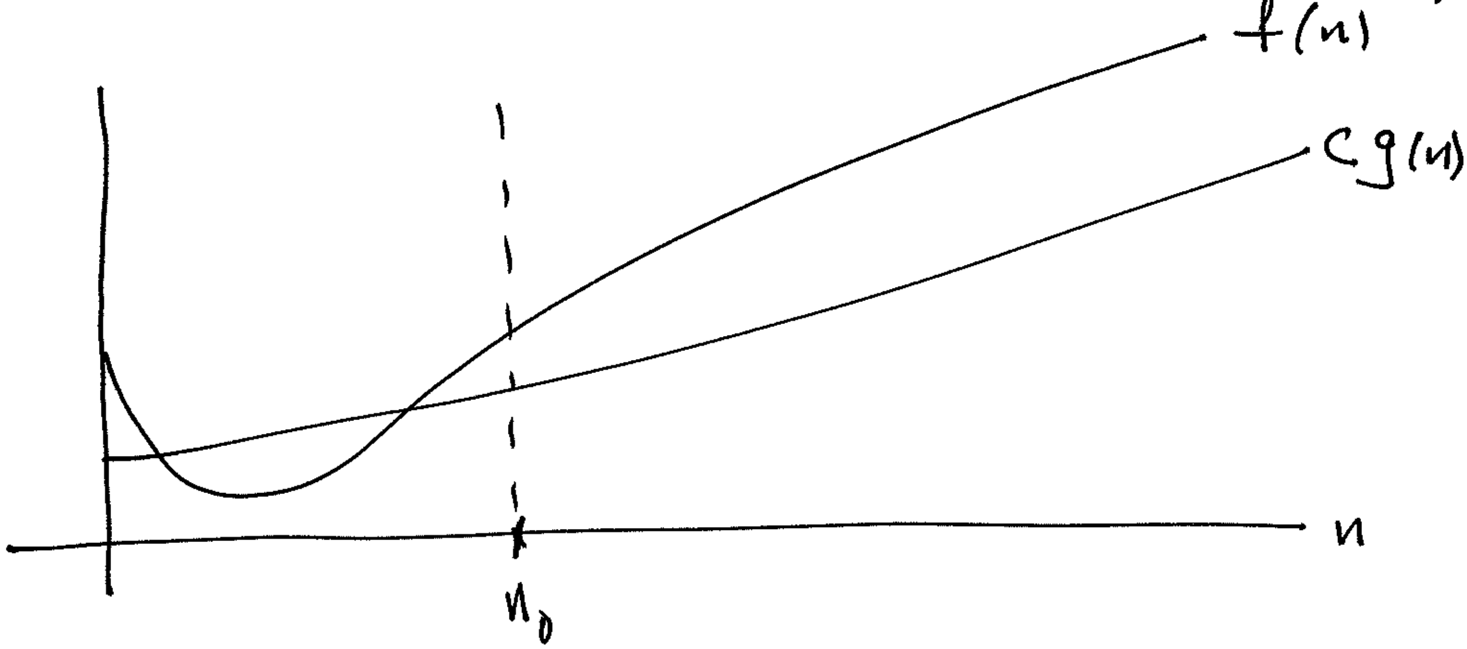
with

$$\deg(P(n)) \leq \deg(q(n))$$

Defn

let $g(n)$ be a function.

$$\Omega(g(n)) = \{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) \leq f(n) \}$$



Theorem

$$f(n) = O(g(n)) \;\; \underset{\Longleftarrow}{\overset{\Longrightarrow}{\text{iff}}} \;\; g(n) = \Omega(f(n))$$

Proof:

($\Rightarrow$) Let $f(n) = O(g(n))$. Then there exist positive c_1, n_1 s.t. $0 \le f(n) \le c_1 g(n)$ for all $n \ge n_1$. We must show there exist positive c_2, n_2 s.t. for all $n \ge n_2$:

$$0 \le c_2 f(n) \le g(n)$$

Let $c_2 = \frac{1}{c_1}$ and $n_2 = n_1$.

Since $n_2 = n_1$ we have $n \geq n_2$

$\Rightarrow n \geq n_1 \Rightarrow 0 \leq f(n) \leq c_1 g(n)$

$\Rightarrow 0 \leq \frac{1}{c_1} f(n) \leq g(n)$

$\Rightarrow 0 \leq c_2 f(n) \leq g(n)$.

This proves $(\Rightarrow)$.

$(\Leftarrow)$ is left as an exercise.

$\blacksquare$