CSE 102 5-21-24

Recorded Lecture

- hw 6 : Due tonight 10 PM
- Midterm 2 : Thursday 5/23
- Mid 2 review solutions now Posted.

Greedy solution to coin change Problem: See notes from 5-18-24

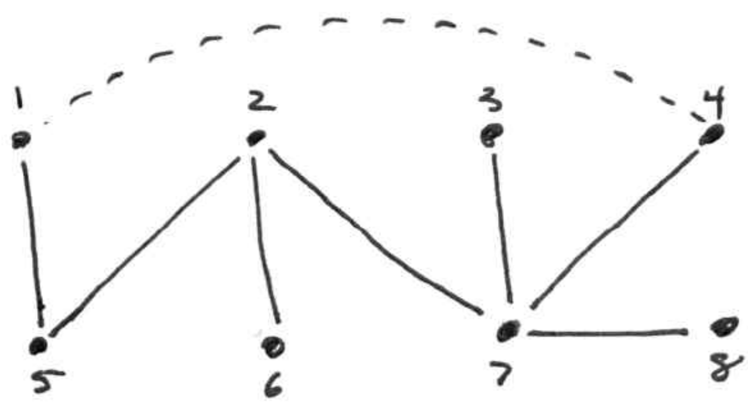
<u>note</u>: if $a \equiv b \pmod{m}$, then $a + km \equiv b \pmod{m}$, and converseley.

<u>Problem</u> : Minimum weight spanning trees in a weighted graph (MWST)

see Graph Theory handout

- connected ✓
- cycle ✓
- Acyclic ✓
- Tree : connected & acyclic. ✓

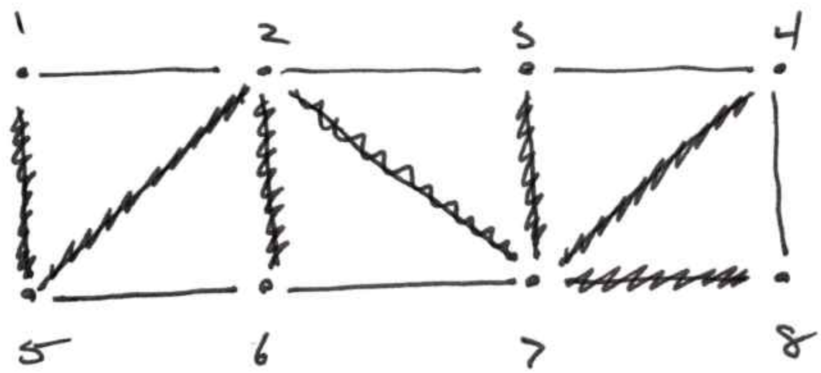
<u>Ex</u> Tree



$n = |V| = 8$, $m = |E| = 7$

- A subgraph of G is called a spanning tree if it is a tree, and it includes all vertices of G

<u>Ex.</u> spanning tree



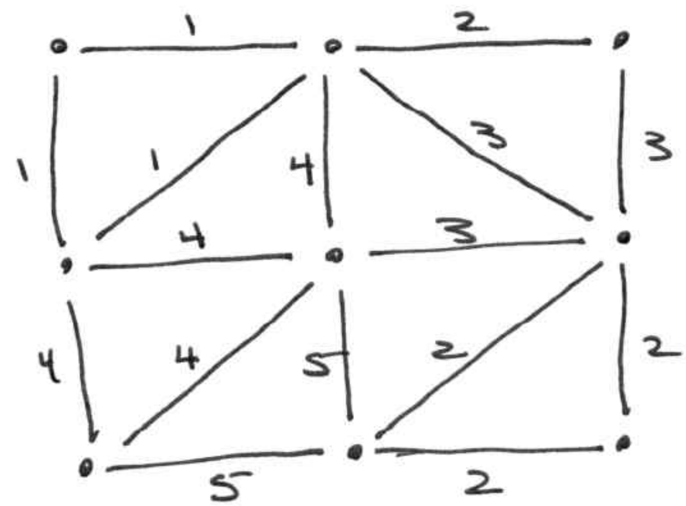
<u>note</u>

G contains a spanning tree iff G is connected.

Suppose $G = (V, E)$ has a weight function on edges

$$w : E \to \mathbb{R}_+$$

Ex



- The weight of a spanning tree is the sum of the weights of its edges.

Problem MWST

Given a connected graph G, find a spanning tree in G of minimum weight.

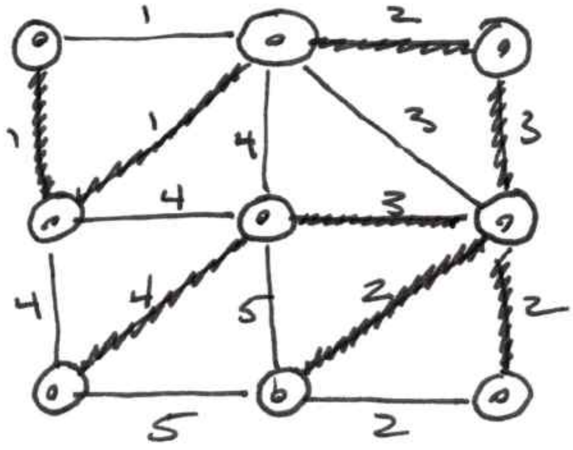
Two Algorithms.

Prim's Algorithm (23.2)

- choose an initial vertex (tree)
- Amongst all edges incident with the current tree, whose addition would not create a cycle, choose one of minimum weight.
- stop when $n-1$ edges have been chosen.

<u>Thm</u> The tree constructed by Prim is a MWST.

<u>Ex</u>

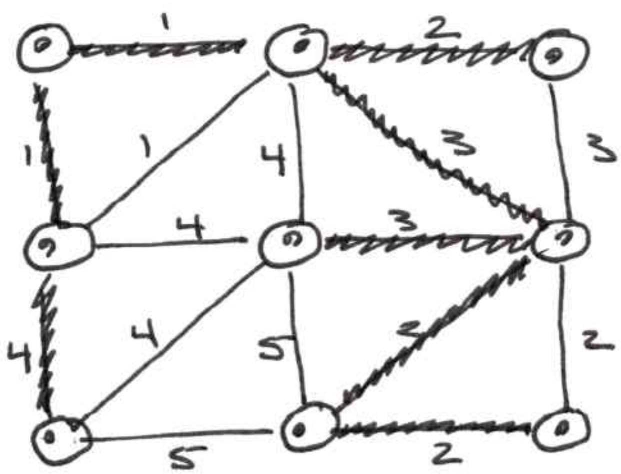


$w(T) = 18$

Kruskal's Algorithm (23.2)

- Choose an edge of minimum weight
- Amongst all edges which do not create a cycle with previously selected edges, choose one of minimum weight.
- stop when $n-1$ edges have been selected.

Ex.



$W(T) = 18$

Theorem

This spanning tree has minimum weight among all spanning trees in G.

<u>Proof</u>

Let T be the sp. tree created by Kruskal, let S be any <u>other</u> sp. tree. We must show

$$w(T) \leq w(S)$$

Let $e_1, e_2, e_3, \ldots, e_{n-1}$ be the edges of T in the order selected by Kruskal. Since $S \neq T$, there is a first edge e_k not in S. i.e.

$$\{e_1, e_2, \ldots, e_{k-1}\} \subseteq E(S)$$

and $$e_k \notin E(S)$$

Let H be the subgraph

$$H = S + e_k$$

By the treeness theorem, H contains

A unique cycle which includes e_K, call it C. Note: C must contain an edge $e \in E(S)$ which is not in T, $e \notin E(T)$. why? otherwise C would be contained in the acyclic T.

Now remove e from H to obtain a subgraph R.

$$R = H - e = S + e_K - e$$

Since R is connected and has $n-1$ edges, its another spanning tree.

The nature of Kruskal guarantees that $w(e_K) \le w(e)$.

Pf. e does not form a cycle with $\{e_1, \dots, e_{k-1}\}$ since

$$\{e_1, \dots, e_{k-1}, e\} \subseteq E(S)$$

Thus if $w(e) < w(e_k)$, then Kruskal would have chosen e on the k^{th} step, and not chosen e_k. Therefore $w(e_k) \leq w(e)$. ///

Thus R is a sp. tree in G with one more edge in common with T than S does, and

$$w(R) \leq w(S)$$

If $R = T$ we're done. Otherwise we do it again with R in place of S.

i.e. we construct another sp. tree R_2 with one more edge in common with T than R, and with $w(R_2) \leq w(R)$. Continuing we construct a sequence

$$w(T) \leq \ldots\ldots \leq w(R_2) \leq \underset{R_1}{\overset{\shortparallel}{w(R)}} \leq \underset{R_0}{\overset{\shortparallel}{w(S)}}$$

which must eventually reach T.

so $w(T) \leq w(S)$, as required.

$\blacksquare$