CSE 102
Introduction to Analysis of Algorithms
Proof of Stirling's Formula

Stirling's formula gives an asymtotic approximation to the factorial function $n!$. It has several different versions and multiple proofs. We prove the following version.

Theorem 1
If n is a positive integer, then

(1) $$n\,! \sim \sqrt{2\pi n}\cdot\left(\tfrac{n}{e}\right)^n.$$

A stronger version of the theorem is

Theorem 2
If n is a positive integer, then

(2) $$n\,! \;=\; \sqrt{2\pi n}\cdot\left(\tfrac{n}{e}\right)^n\cdot\left(1+\Theta\left(\tfrac{1}{n}\right)\right).$$

Observe that (1) is an immediate consequence of (2), as the following limit shows.

$$\lim_{n\to\infty}\frac{n\,!}{\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n} \;=\; \lim_{n\to\infty}\left(1+\Theta\left(\frac{1}{n}\right)\right)=1$$

We will not prove Theorem 2, but see *Concrete Mathematics* by Grahan, Knuth & Patashnik for details.

We proceed by proving three preliminary lemmas, then formula (1) will follow. First define the *double factorial* function $n!!$ (also called the *semi-factorial*) by

$$n!! = \begin{cases} n(n-2)(n-4)\cdots 6\cdot 4\cdot 2 & \text{if } n \text{ is even} \\ n(n-1)(n-4)\cdots 5\cdot 3\cdot 1 & \text{if } n \text{ is odd,} \end{cases}$$

or equivalently,

$$(2n)!! = (2n)(2n-2)(2n-4)\cdots 6\cdot 4\cdot 2,$$

and

$$(2n+1)!! = (2n+1)(2n-1)(2n-2)\cdots 5\cdot 3\cdot 1.$$

The following useful facts are easily verified.

$$(2n)!! = 2^n\cdot n!$$

$$(2n+1)!!\cdot(2n)!! = (2n+1)!$$

$$(2n)!!\cdot(2n-1)!! = (2n)!$$

Lemma 1
For any positive integer n,

$$\frac{(2n)!!}{(2n+1)!!} < \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!} < \frac{(2n-2)!!}{(2n-1)!!}$$

Proof:
Define $I_k = \int_0^{\pi/2} (\cos x)^k dx$, for $k = 0, 1, 2, \dots$. Then $I_0 = \pi/2$ and $I_1 = 1$. Integration by parts gives, for $k \geq 2$:

$$\begin{aligned}
I_k &= \int_0^{\pi/2} (\cos x)^{k-1} \cdot \cos x \, dx \\
&= \left[(\cos x)^{k-1} \cdot \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} (-1)(k-1)(\cos x)^{k-2} (\sin x)^2 \, dx \\
&= 0 + \int_0^{\pi/2} (k-1)(\cos x)^{k-2}(1 - (\cos x)^2) \, dx \\
&= (k-1) \int_0^{\pi/2} (\cos x)^{k-2} \, dx - (k-1) \int_0^{\pi/2} (\cos x)^k \, dx \\
&= (k-1) I_{k-2} - (k-1) I_k
\end{aligned}$$

Therefore $I_k + (k-1)I_k = (k-1)I_{k-2}$, and hence $kI_k = (k-1)I_{k-2}$, yielding the recurrence

$$I_k = \left(\frac{k-1}{k}\right) I_{k-2}.$$

Iterating this formula gives us

$$I_{2n} = \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right)\left(\frac{2n-5}{2n-4}\right) \cdots \left(\frac{1}{2}\right) I_0 = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$$

$$I_{2n+1} = \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right)\left(\frac{2n-4}{2n-3}\right) \cdots \left(\frac{2}{3}\right) I_1 = \frac{(2n)!!}{(2n+1)!!}$$

and

$$I_{2n-1} = \left(\frac{2n-2}{2n-1}\right)\left(\frac{2n-4}{2n-3}\right)\left(\frac{2n-6}{2n-5}\right) \cdots \left(\frac{2}{3}\right) I_1 = \frac{(2n-2)!!}{(2n-1)!!}$$

Since $(\cos x)^k$ is a decreasing function of k for $x \in [0, \pi/2]$, its integral I_k also decreases as k increases. Therefore $I_{2n+1} < I_{2n} < I_{2n-1}$, proving that

$$\frac{(2n)!!}{(2n+1)!!} < \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!} < \frac{(2n-2)!!}{(2n-1)!!}$$

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Lemma 2
Define the sequence A_n, for n a positive integer, by

$$A_n = \binom{2n}{n} \cdot 2^{-2n} = \frac{(2n)!}{n! \cdot n! \cdot 2^{2n}}.$$

Then $\lim_{n\to\infty} \left(A_n \cdot \sqrt{n\pi}\right) = 1$.

Proof:
Observe that

$$A_n = \frac{(2n)!}{(n! \cdot 2^n)^2} = \frac{(2n)!}{\left((2n)!!\right)^2} = \frac{(2n)!/(2n)!!}{(2n)!!} = \frac{(2n-1)!!}{(2n)!!}.$$

Therefore, by Lemma 1, we have

$$\frac{(2n)!!}{(2n+1)!!} < \frac{\pi}{2} \cdot A_n < \frac{(2n-2)!!}{(2n-1)!!}.$$

Multiply all terms in the above inequality by

$$\frac{(2n-1)!!}{(2n-2)!!} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{(2n)!!}{(2n-2)!!} = A_n \cdot 2n$$

to obtain

$$\frac{2n}{2n+1} < A_n^2 \cdot n\pi < 1$$

Since $\lim_{n\to\infty} 2n/(2n+1) = 1$, the last inequality implies $\lim_{n\to\infty} (A_n^2 \cdot n\pi) = 1$, and hence

$$\lim_{n\to\infty} \left(A_n \cdot \sqrt{n\pi}\right) = 1$$

as claimed. ∎

Lemma 3
For any positive integer n, we have

$$\ln\left(1 + \frac{1}{n}\right)^{(n+1/2)} = 1 + \frac{1}{12n^2} - \Theta\left(\frac{1}{n^3}\right)$$

Proof:
We integrate both sides of the following identity

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 - \cdots\cdots$$

(for $-1 < t < 1$) to obtain

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$$

$$= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \cdots\cdots\right]_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\cdots$$

(for $-1 < x < 1$). Upon setting $x = 1/n$, we get

$$\ln\left(1+\frac{1}{n}\right)^{(n+1/2)} = \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right)$$

$$= \left(n+\frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\cdots\right)$$

$$= 1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots\cdots$$
$$+\frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{6n^3} - \frac{1}{8n^4} + \cdots\cdots$$

$$= 1 + \frac{1}{12n^2} - \frac{1}{12n^3} + \cdots\cdots$$

$$= 1 + \frac{1}{12n^2} - \Theta\left(\frac{1}{n^3}\right)$$

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Proof of Theorem 1:
Define the sequence B_n by $B_0 = 1$, and

$$B_n = \frac{n!}{n^n e^{-n}\sqrt{2\pi n}}$$

for $n \geq 1$. We will show that $B_n \to 1$ as $n \to \infty$, proving (1). We have $n! = B_n \cdot n^n e^{-n}\sqrt{2\pi n}$, and hence for $n \geq 1$:

$$n + 1 = \frac{(n+1)!}{n!}$$

$$= \frac{B_{n+1} \cdot (n+1)^{(n+1)} e^{-(n+1)}\sqrt{2\pi(n+1)}}{B_n \cdot n^n e^{-n}\sqrt{2\pi n}}$$

$$= \left(\frac{B_{n+1}}{B_n}\right) \cdot e^{-1} \cdot \left(\frac{n+1}{n}\right)^n \cdot \frac{(n+1)^{3/2}}{n^{1/2}}$$

Therefore, for $n \geq 1$, we have

$$\left(\frac{B_{n+1}}{B_n}\right) = (n+1) \cdot e \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{n^{1/2}}{(n+1)^{3/2}}$$

$$= e \cdot \left(\frac{n}{n+1}\right)^{n+1/2}$$

$$= e \cdot \left(1 + \frac{1}{n}\right)^{-(n+1/2)}$$

Lemma 3 gives us $\ln\left(1 + \frac{1}{n}\right)^{(n+1/2)} = 1 + \frac{1}{12n^2} - \Theta\left(\frac{1}{n^3}\right)$, which implies

$$\ln\left(1 + \frac{1}{n}\right)^{(n+1/2)} > 1 \implies \left(1 + \frac{1}{n}\right)^{(n+1/2)} > e$$

$$\implies \frac{B_{n+1}}{B_n} = e \cdot \left(1 + \frac{1}{n}\right)^{-(n+1/2)} < 1$$

$$\implies B_{n+1} < B_n$$

hence the sequence B_n is monotone decreasing. The sequence is also bounded below (by 0), and therefore has a limit. Let $B = \lim_{n\to\infty} B_n$. We claim that $B \neq 0$. To see this, note

$$B_{n+1} = \frac{B_1}{B_0} \cdot \frac{B_2}{B_1} \cdot \frac{B_3}{B_2} \cdots \cdots \frac{B_{n+1}}{B_n}.$$

By the above calculation and Lemma 3 we see that

$$\ln(B_{n+1}) = \sum_{k=0}^{n} \ln\left(\frac{B_{k+1}}{B_k}\right)$$

$$> \sum_{k=1}^{n} \ln\left[e \cdot \left(1 + \frac{1}{k}\right)^{-(k+1/2)}\right]$$

$$= \sum_{k=1}^{n} \left[1 - \ln\left(1 + \frac{1}{k}\right)^{(k+1/2)}\right]$$

$$= \sum_{k=1}^{n} \left[1 - \left(1 + \frac{1}{12k^2} - \Theta\left(\frac{1}{k^3}\right)\right)\right]$$

$$= \sum_{k=1}^{n} \left[-\frac{1}{12k^2} + \Theta\left(\frac{1}{k^3}\right)\right]$$

Observe that since $\sum_{k=1}^{\infty}(1/k^2)$ converges (by the integral test), the above series also converges. This shows that $\ln(B_{n+1}) \nrightarrow -\infty$, so that $B_{n+1} \nrightarrow 0$, and hence $B \neq 0$, as claimed. Finally,

$$\frac{B_{2n}}{B_n^2} = \frac{B_{2n}\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi \cdot 2n}}{\left(B_n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^2} \cdot \frac{\sqrt{n\pi}}{2^{2n}}$$

$$= \frac{(2n)!}{(n!)^2} \cdot \frac{\sqrt{n\pi}}{2^{2n}}$$

$$= \binom{2n}{n} \cdot 2^{-2n}\sqrt{n\pi},$$

and so by Lemma 2 we have $B_{2n}/B_n^2 = A_n \cdot \sqrt{n\pi} \ \rightarrow \ 1$. But also both $B_n \rightarrow B$ and $B_{2n} \rightarrow B$, hence

$$\frac{B}{B^2} = 1$$

(Note this last step requires $B \neq 0$, which was proved above.) Therefore $B = B^2$, and upon canceling we obtain $B = 1$. This completes the proof of Theorem 1.

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