CSE 102 4-16-24

Multiple base cases

I. Prove $P(1), P(2), \ldots, P(n_0)$

II (strong) Induction

show $\forall n > n_0 : (P(1) \wedge \ldots \wedge P(n-1)) \rightarrow P(n)$

Ex. Define $T : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$

$$T(n) = \begin{cases} 1 & \text{if } 1 \le n \le 2 \\ 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n & \text{if } n \ge 3 \end{cases}$$

show: $\forall n \ge 1 : \underbrace{\boxed{T(n) \le n^2}}_{P(n)}$ $\therefore T(n) = O(n^2)$

Proof

Two base cases $P(1), P(2)$.

I. $P(1)$ says: $T(1) \le 1^2$ i.e. $1 \le 1$ ✓

$P(2)$ says: $T(2) \le 2^2$ i.e. $1 \le 4$ ✓

II. Let $n > 2$. Assume for all k in the range $1 \le k < n$ that

$$T(k) \le k^2.$$

In Particular, for $k = \lfloor \frac{n}{3} \rfloor$, we have

$$T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) \le \left\lfloor \frac{n}{3} \right\rfloor^2.$$

We must show: $T(n) \le n^2$

Then

$$T(n) = 4T\left(\left\lfloor \tfrac{n}{3} \right\rfloor\right) + n \quad \left\{ \begin{array}{l} \text{by recurrence} \\ \text{for } T(n) \end{array} \right.$$

$$\leq 4 \cdot \left\lfloor \tfrac{n}{3} \right\rfloor^2 + n \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$\leq 4 \cdot \left(\tfrac{n}{3}\right)^2 + n \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor \leq x \end{array} \right.$$

$$= \tfrac{4}{9} n^2 + n$$

$$\vdots$$

$$\leq n^2$$

Since $n > 2 \Rightarrow \frac{9}{5} \leq n \Rightarrow n \leq \frac{5}{9} n^2 \Rightarrow \frac{4}{9} n^2 + n \leq n^2$.

$\therefore T(n) \leq n^2$.

$\therefore \forall n \geq 1 : T(n) \leq n^2$ by 2^{nd} PMI. ∎

why 2 base cases?

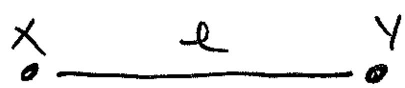
n	$\lfloor \frac{n}{3} \rfloor$	
1	0	
2	0	
3	1	
4	1	1
5	1	
6	2	
7	2	2
8	2	
9	3	
10	3	
11	3	
12	4	1
13	4	
14	4	
15	5	
16	5	
17	5	

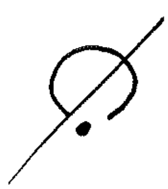
Graphs

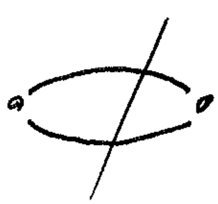
Defn A Graph is a Pair of sets

$$G = (\underset{\substack{\uparrow \\ \text{Vertex} \\ \text{set}}}{V}, \underset{\substack{\uparrow \\ \text{Edge} \\ \text{set}}}{E})$$

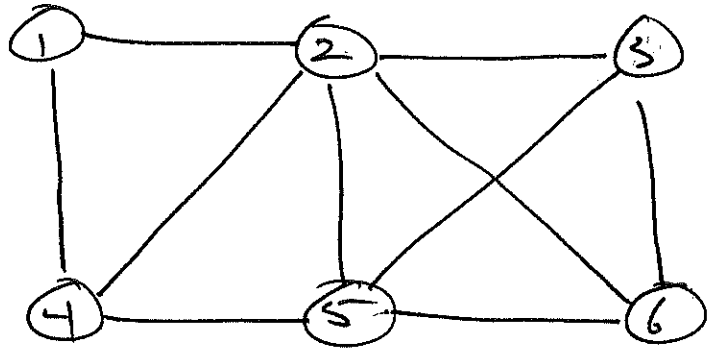
if $e \in E$, then $e = \{x, y\} = xy = yx$



note  since $x \neq y$

also  since E is a set

Ex.



$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{12, 14, 24, 25, 23, 26, 35, 36, 45, 56\}$

Defn an x-y Path in G is a seq of Vertices

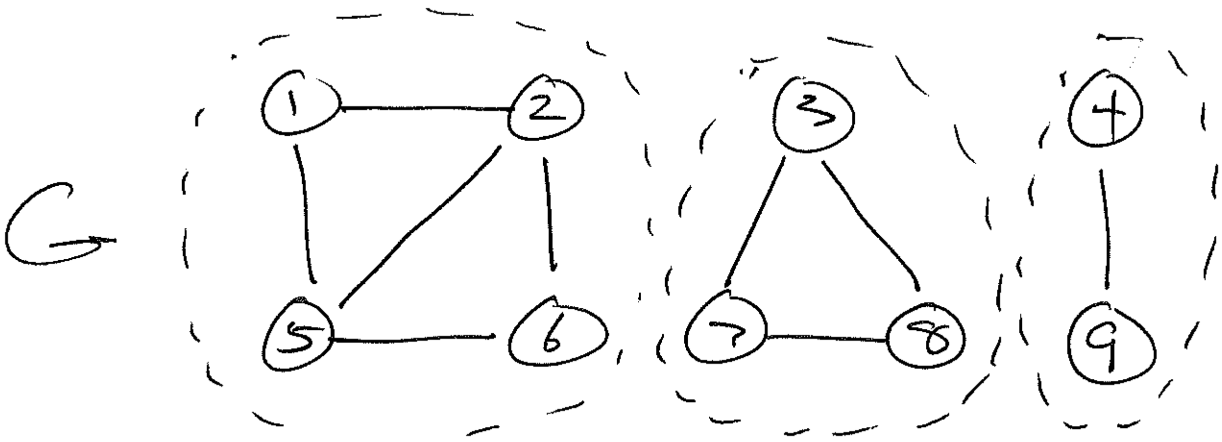
$\boxed{\text{length} = k}$

$x = v_0, v_1, \ldots, v_{k-1}, v_k = y$

- consecutive Pairs are adjacent
- no repetition of Vertices, unless $x = y$ (called a closed Path)

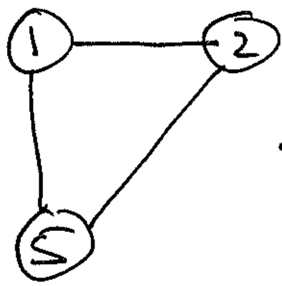
Defn G is connected iff for all $x, y \in V(G)$, G contains an x-y Path.

Ex



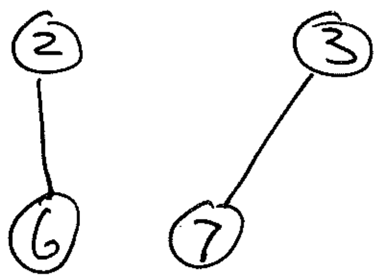
Defn A subgraph H of G is a Graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

Ex. $(\{1, 2, 5\}, \{12, 15, 25\})$



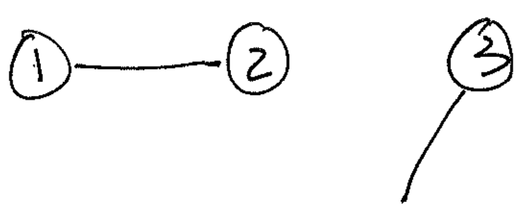
← connected

Ex. $(\{2, 3, 6, 7\}, \{26, 37\})$



← disconnected

Ex. $(\{1, 2, 3\}, \{12, 37\})$



← not a graph
so not a
subgraph

Defn A subgraph H of G is a connected component of G iff

① H is connected

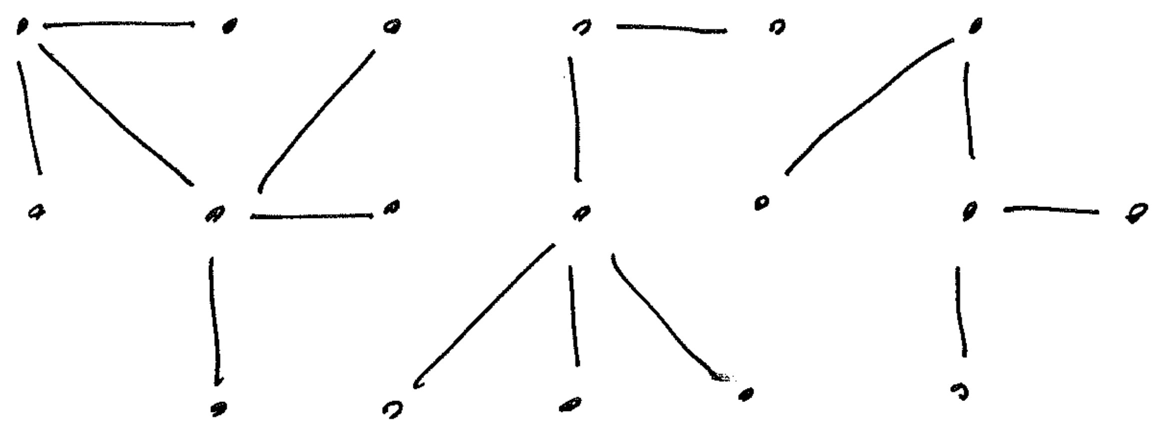
② H is maximal w.r.t. ①

Defn

A tree is a Graph which is both connected and acyclic

Defn a cycle is a closed path with at least one edge. acyclic means containing no cycles

Ex. Acyclic Graph (forest)



#vert:	7	6	5
#edges:	6	5	4

Theorem

$$\forall n \geq 1 : \overset{P(n)}{\boxed{\text{if } T \text{ is a tree on } n \text{ vertices, then } T \text{ has } (n-1) \text{ edges}}}$$

Proof

I. if T has 1 vertex

then necessarily T has 0 edges. so P(1) holds.

II d. $\forall n > 1: (P(1) \wedge \ldots \wedge P(n-1)) \to P(n)$

Let $n > 1$.