CSE 102 4-23-24

- mid 1 on Thursday

11:40 – 12:45 exam

12:45 – 12:50 break

12:50 – 1:15 lecture

<u>note</u>

$$n > 5 \Rightarrow n \geq 6$$

$$\Rightarrow \frac{n}{3} \geq 2$$

$$\Rightarrow \left\lfloor \frac{n}{3} \right\rfloor \geq 2 \quad \checkmark$$

Master Theorem

Let $a \geq 1$, $b > 1$, $f(n)$ be asymptotically positive. Define $T(n)$ by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Then we have 3 cases:

① if $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ for some $\varepsilon > 0$, then

$$\boxed{T(n) = \Theta\left(n^{\log_b(a)}\right)}$$

② if $f(n) = \Theta(n^{\log_b a})$, then

$$\boxed{T(n) = \Theta(n^{\log_b a} \cdot \log(n)) = \Theta(f(n) \cdot \log n)}$$

③ if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$,

and if $a f(\frac{n}{b}) \le c f(n)$ for some $0 < c < 1$

for all suff. large n, then

$$\boxed{T(n) = \Theta(f(n))}$$

regularity condition

Ex $T(n) = 8T\left(\frac{n}{2}\right) + n^3$

$a = 8$, $b = 2$, $f(n) = n^3$

compare: n^3 to $n^{\log_2(8)} = n^3$

by case 2: $T(n) = \Theta(n^3 \cdot \log n)$

Ex. $T(n) = 5\,T\left(\frac{n}{4}\right) + n$

compare: n^1 to $n^{\log_4 5}$

let $\varepsilon = \log_4 5 - 1$. Then $\varepsilon > 0$

and $\log_4 5 - \varepsilon = 1$, so

$$n = n^{\log_4 5 - \varepsilon} = O(n^{\log_4 5 - \varepsilon})$$

By case 1: $T(n) = \Theta(n^{\log_4 5})$

EX $T(n) = 5T\left(\frac{n}{4}\right) + n^2$

compare: n^2 to $n^{\log_4 5}$

$5 < 16 \Rightarrow \log_4 5 < \log_4 16 = 2$

let $\varepsilon = 2 - \log_4 5$. Then $\varepsilon > 0$,

and $2 = \log_4 5 + \varepsilon$.

$\therefore \quad n^2 = \Omega(n^2) = \Omega\left(n^{\log_4 5 + \varepsilon}\right)$.

Also we need: $5\left(\frac{n}{4}\right)^2 \le c n^2$

i.e. $\frac{5}{16} n^2 \le c n^2$ i.e. $\frac{5}{16} \le c < 1$

Pick $c = \frac{5}{16}$. By case 3:

$$T(n) = \Theta(n^2)$$

Ex.

$$T(n) = 8T\left(\frac{n}{2}\right) + \underbrace{10n^3 + 15n^2 - n^{1.5} + n\log n + 1}_{\Theta(n^3)}$$

case 2: $T(n) = \Theta(n^3 \cdot \log n)$

sometimes we write recurrences in the form

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(f(n))$$

what if $f(n)$ is not a Polynomial?

Ex. $T(n) = T\left(\lfloor \frac{n}{2} \rfloor\right) + 2T\left(\lceil \frac{n}{2} \rceil\right) + \log(n!)$

Simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n \log n)$$

compare: $n \log n$ to $n^{\log_2 3}$ ← winner

let $\varepsilon = \frac{1}{2}(\log_2 3 - 1)$. Then $\varepsilon > 0$ and

$2\varepsilon = \log_2 3 - 1 \quad \therefore \; 1 + \varepsilon = \log_2 3 - \varepsilon$.

note $n \log n = o(n^{1+\varepsilon}) \subseteq O(n^{1+\varepsilon})$

why? $\frac{n \log n}{n^{1+\varepsilon}} = \frac{\cancel{n} \log n}{\cancel{n} \cdot n^{\varepsilon}} \longrightarrow 0$

so

$$n \log n = O\left(n^{\log_2 3 - \varepsilon}\right)$$

Case 1 : $T(n) = \Theta\left(n^{\log_2 3}\right)$.

<u>Note</u>: $\varepsilon = \log_2 3 - 1$ will not work

but any ε in $0 < \varepsilon < \log_2 3 - 1$ will.

<u>check</u>: $n \log n \neq O\left(n^{\log_2 3 - (\log_2 3 - 1)}\right)$

i.e. $n \log n \neq O(n^1)$

since $\frac{n \log n}{n} = \log n \longrightarrow \infty$, so

$n \log n = \omega(n)$, hence $n \log n \neq O(n)$.