CSE 102 4-9-24

Lemma $f(n) = o(g(n))$ iff $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0$

Proof

note: $f(n) = o(g(n))$ iff

$$\forall c>0, \exists n_0>0, \forall n \geq n_0: 0 \leq f(n) < c\cdot g(n)$$

iff

$$\forall c>0, \exists n_0>0, \forall n \geq n_0: 0 \leq \frac{f(n)}{g(n)} < c$$

iff

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0$$

Ex. $\ln(n) = o(n)$

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 .$$

Ex. $n^k = o(e^n)$ since

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{kn^{k-1}}{e^n}\right)$$

$$= k(k-1)\lim_{n\to\infty}\left(\frac{n^{k-2}}{e^n}\right)$$

$\lceil k \rceil$ applications of l'Hop. $\Bigg\{ \quad \vdots$

$$= \text{const} \cdot \lim_{n\to\infty}\left(\frac{1}{e^n}\right) = 0$$

Exercise

Prove $o(g(n)) \cap \Omega(g(n)) = \phi$.

Defn

$$\omega(g(n)) = \{f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c g(n) < f(n)\}$$

Recall

$$\Omega(g(n)) = \{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c g(n) \leq f(n)\}$$

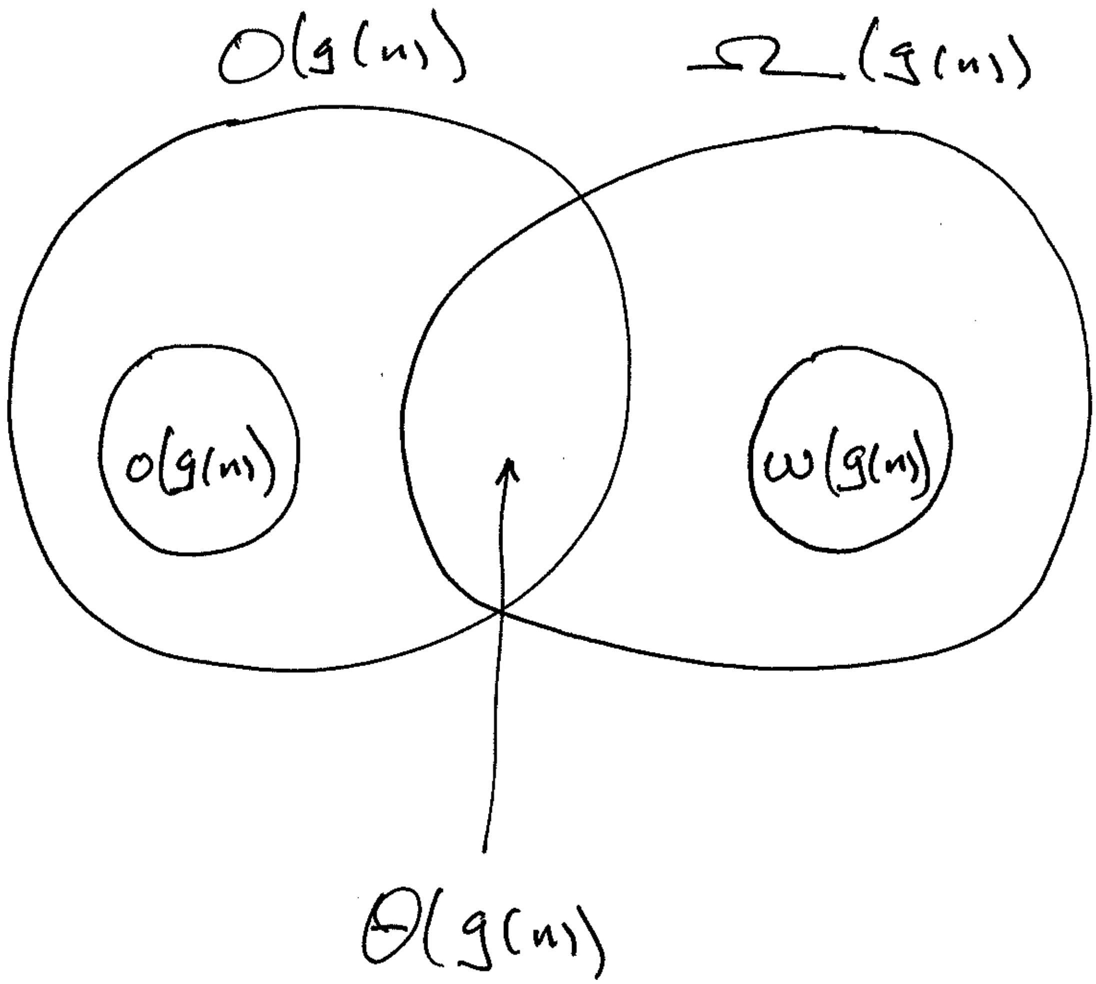
Thus: $\omega(g(n)) \subseteq \Omega(g(n))$

Exercise Prove

- $f(n) = \omega(g(n))$ iff $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \infty$

- $\omega(g(n)) \cap O(g(n)) = \phi$
- $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$

<u>Picture</u>:



Theorem

If $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = L$ where $0 \le L < \infty$

then $f(n) = O(g(n))$.

Warning: converse is false

Proof

The defn of the limit is

$$\forall \varepsilon > 0, \exists n_0 > 0, \forall n \ge n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon$$

Since this holds for all ε, we can set $\varepsilon = 1$. Then

$$\exists n_0 > 0, \forall n \ge n_0 : \left| \frac{f(n)}{g(n)} - L \right| < 1$$

$\therefore\ \exists n_0 > 0, \forall n \geq n_0 :\ -1 < \frac{f(n)}{g(n)} - L < 1$

$\therefore\ \quad '' \quad\quad '' \quad :\ \frac{f(n)}{g(n)} < (1+L)$

$\therefore\ \quad '' \quad\quad '' \quad :\ f(n) < (1+L)g(n)$

$\therefore\ \quad '' \quad\quad '' \quad :\ 0 \leq f(n) \leq \underbrace{(1+L)}_{c = L+1 > 0} g(n)$

blanket assumption

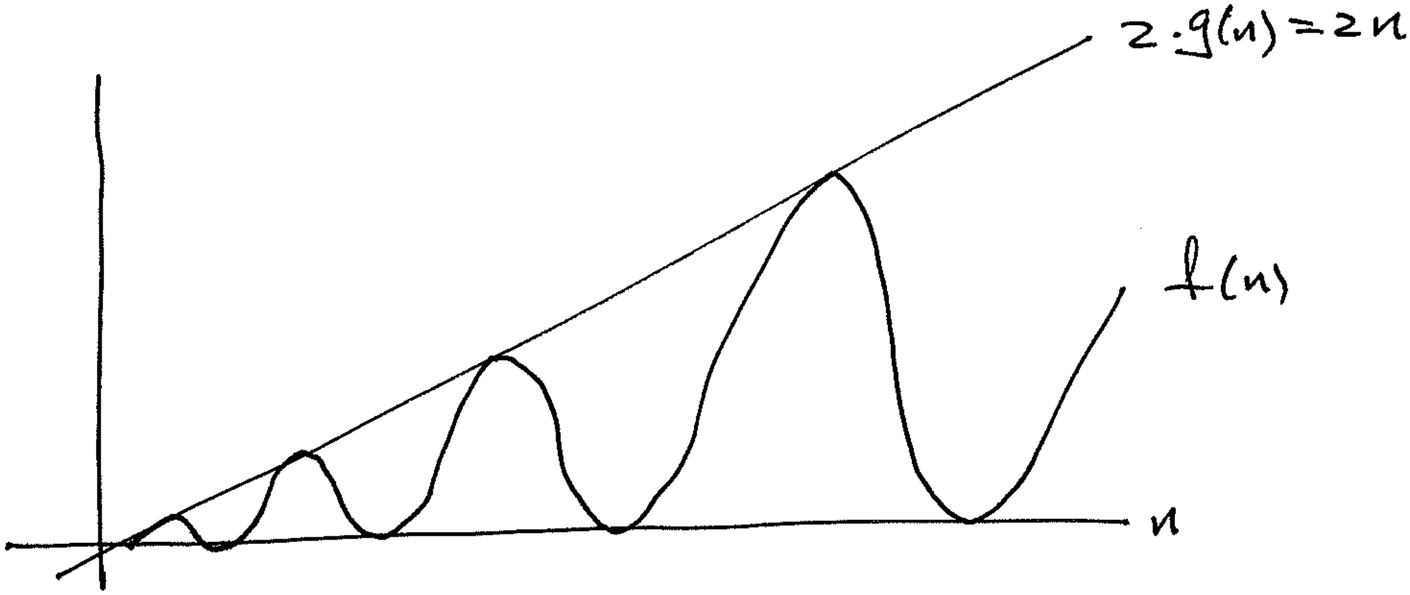
$\therefore\ f(n) = O(g(n))$. $\blacksquare$

<u>Exercise</u> if $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = L$

- if $0 < L \leq \infty$, then $f(n) = \Omega(g(n))$
- if $0 < L < \infty$, then $f(n) = \Theta(g(n))$

<u>note</u>: converses are false

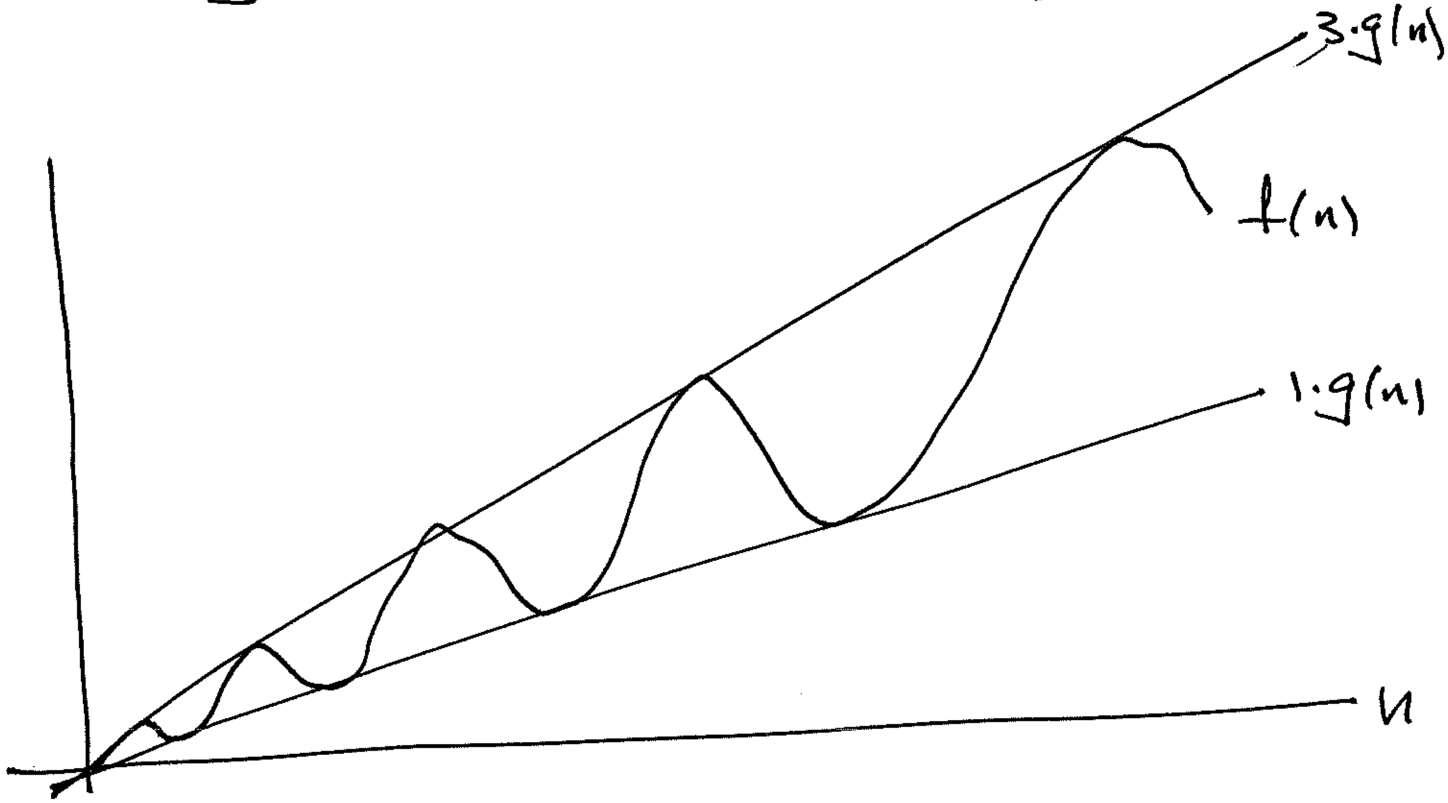
<u>Ex.</u> A: $g(n) = n$, $f(n) = (1+\sin(n))\cdot n$



Thus $f(n) = O(g(n))$. But

$\frac{f(n)}{g(n)} = 1+\sin(n)$ has no limit.

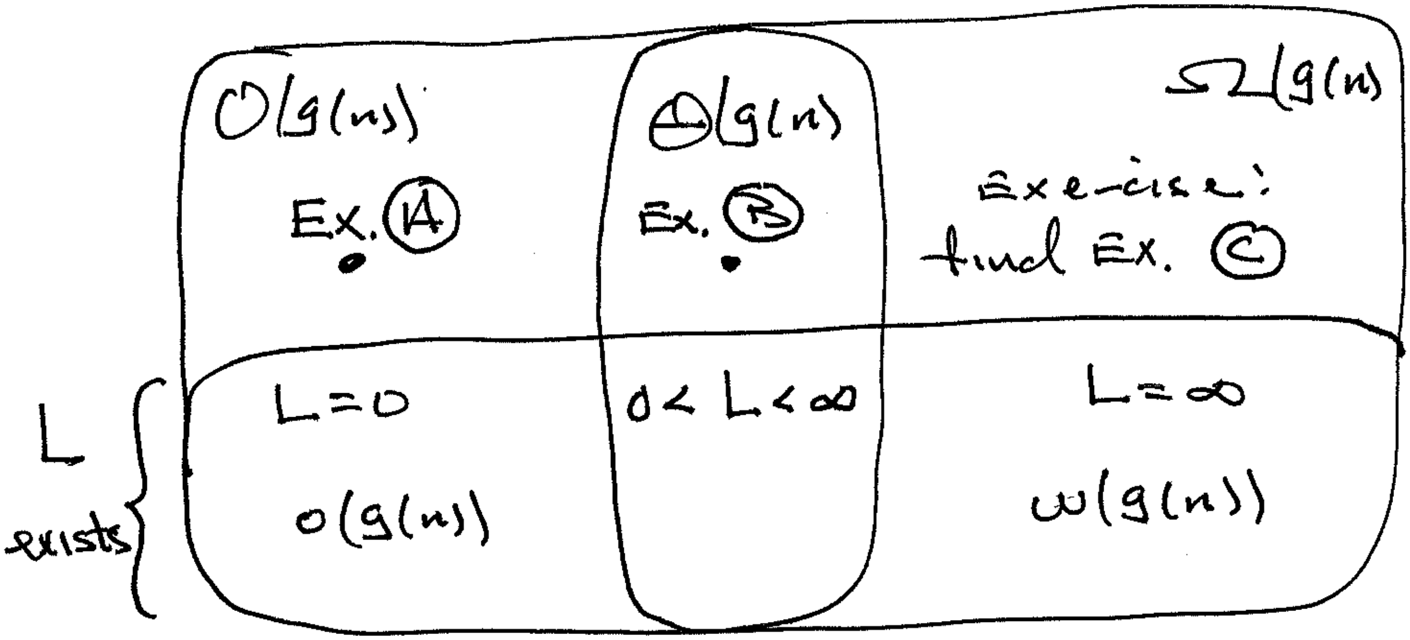
Ex (B) $g(n) = n$, $f(n) = (2 + \sin(n))n$



so $f(n) = \Theta(g(n))$, but

$\frac{f(n)}{g(n)} = 2 + \sin(n)$ has no limit

Picture



let $L = \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)$, if it exists

Exercises

- if $P(n)$ is a Polynomial of deg k, then $P(n) = \Theta(n^k)$. ✓

Proof:

we have

$$P(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$$

where $a_k > 0$, $a_{k-1}, \dots a_0 \in \mathbb{R}$. Thus

$$\frac{P(n)}{n^k} = a_k + \underbrace{\frac{a_{k-1}}{n}}_{\downarrow \, 0} + \dots + \underbrace{\frac{a_1}{n^{k-1}}}_{\downarrow \, 0} + \underbrace{\frac{a_0}{n^k}}_{\downarrow \, 0}$$

$$\therefore \lim_{n \to \infty} \left(\frac{P(n)}{n^k} \right) = a_k > 0$$

- let $\alpha, \beta > 0$ then

$$n^{\alpha} = \begin{cases} o(n^{\beta}) & \text{if } \alpha < \beta \; \checkmark \\ \Theta(n^{\beta}) & \text{if } \alpha = \beta \; \checkmark \\ \omega(n^{\beta}) & \text{if } \alpha > \beta \; \checkmark \end{cases}$$

<u>Proof</u>:

$$\frac{n^{\alpha}}{n^{\beta}} = n^{\alpha - \beta} \longrightarrow \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta \\ \infty & \text{if } \alpha > \beta \end{cases} \qquad \blacksquare$$

- Let $a, b > 1$. Then

$$a^n = \begin{cases} o(b^n) & \text{if } a < b \\ \Theta(b^n) & \text{if } a = b \\ \omega(b^n) & \text{if } a > b \end{cases}$$

Proof

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \longrightarrow \begin{cases} 0 & \text{if } a < b \\ 1 & \text{if } a = b \\ \infty & \text{if } a > b \end{cases}$$

$\blacksquare$

B

• $f(n) + \underbrace{o(f(n))} = \Theta(f(n))$

some function $h(n)$

satisfying $h(n) = o(f(n))$.

we show $f(n) + h(n) = \Theta(f(n))$

$$\frac{f(n)+h(n)}{f(n)} = 1 + \underset{\downarrow \atop 0}{\frac{h(n)}{f(n)}} \longrightarrow 1 \in (0,\infty)$$

$\therefore$ $f(n) + h(n) = \Theta(f(n))$. ∎

Analogy

analogous to

$$f(n) = O(g(n)) \equiv x \leq y$$

$$f(n) = \Theta(g(n)) \equiv x = y$$

$$f(n) = \Omega(g(n)) \equiv x \geq y$$

$$f(n) = o(g(n)) \equiv x < y$$

$$f(n) = \omega(g(n)) \equiv x > y$$

Handout: some common functions

Stirlings Formula

let $n \in \mathbb{Z}^+$. Then

$$n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$