CSE 102 4-27-24

Supplemental Lecture

Recall:

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

$\vdots$

$$T(n) = \sum_{i=0}^{k-1} 2^i \cdot \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k \quad \text{where } k = \lfloor \log_3(n) \rfloor \le \log_3(n)$$

$$\le \sum_{i=0}^{k-1} 2^i \left(\frac{n}{3^i}\right) + 2^{\log_3(n)}$$

$$= n \cdot \sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i + n^{\log_3(2)}$$

$$\le n \cdot \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i + n^{\log_3(2)}$$

$$= n \cdot \left(\frac{1}{1-\frac{2}{3}}\right) + n^{\log_3(2)}$$

$$= 3n + n^{\log_3(2)}$$

$$= O(n) \quad (\text{since } \log_3(2) < 1.)$$

$$\therefore\ T(n) = O(n)$$

Also by the recurrence

$$T(n) = 2T(\lfloor \tfrac{n}{3} \rfloor) + n \geq n = \Omega(n)$$

$$\therefore\ T(n) = \Omega(n)$$

$$\therefore\ T(n) = \Theta(n).$$

<u>Ex.</u>

$$T(n) = \begin{cases} 5 & 0 \leq n < 2 \\ T(n-2) + n & n \geq 2 \end{cases}$$

$$T(n) = n + T(n-2)$$

$$= n + (n-2) + T(n - 2\cdot 2)$$

$$= n + (n-2) + (n-2\cdot 2) + T(n - 2\cdot 3)$$

$$\vdots$$

$$= \sum_{i=0}^{k-1} (n-2i) + T(n-2k)$$

recursion terminates when

$$0 \le n - 2k < 2$$

$$2k \le n < 2k+2$$

$$k \le \frac{n}{2} < k+1$$

$$\therefore \boxed{k = \left\lfloor \frac{n}{2} \right\rfloor}$$

$$\therefore T(n) = n \cdot \sum_{i=0}^{k-1} 1 - 2 \sum_{i=\cancel{0}\,1}^{k-1} i + T(n-2k)$$

$$\therefore T(n) = n\cdot k - \cancel{2}\,\frac{k(k-1)}{\cancel{2}} + T(n-2k)$$

with $k = \lfloor \frac{n}{2} \rfloor$ we have

$$\boxed{T(n) = n\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 5}$$

$$\therefore T(n) = \Theta(n^2)$$

even, $T(n) \sim \frac{1}{4}n^2 + o(n^2)$

Ex.

$$T(n) = \begin{cases} 0 & n = 1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 & n \geq 1 \end{cases}$$

	Level
$T(n) = 1 + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$	1
$= 1 + 1 + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right)$	
$= 1 + 1 + T\left(\left\lfloor \frac{n}{2^2} \right\rfloor\right)$	2

$$= 1 + 1 + 1 + T\left(\left\lfloor \frac{\lfloor n/2^2 \rfloor}{2} \right\rfloor\right)$$

$$= 1 + 1 + 1 + T\left(\left\lfloor \frac{n}{2^3} \right\rfloor\right) \qquad 3$$

$$\vdots \qquad\qquad \vdots$$

$$= k + T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) \qquad k$$

recursion terminates when

$$\left\lfloor \frac{n}{2^k} \right\rfloor = 1$$

$$\Leftrightarrow \quad 1 \leq \frac{n}{2^k} < 2$$

$$\Leftrightarrow \quad 2^k \leq n < 2^{k+1}$$

$$\Leftrightarrow \quad k \leq \lg(n) < k+1$$

$$\therefore \quad k = \lfloor \lg(n) \rfloor$$

Thus

$$\boxed{T(n) = \lfloor \lg n \rfloor}$$

$$\therefore \; T(n) = \Theta(\log n)$$

Exercise check that $T(n) = \lfloor \lg n \rfloor$

really solves the recurrence

Exercise

use same technique to show

$$S(n) = \begin{cases} 0 & n = 1 \\ S(\lceil \frac{n}{2} \rceil) + 1 & n \geq 2 \end{cases}$$

has solution $\boxed{S(n) = \lceil \lg(n) \rceil}$

$\therefore \; S(n) = \Theta(\log n)$

Ex

$$T(n) = \begin{cases} c & 1 \leq n < n_0 \\ T(\lfloor \frac{n}{2} \rfloor) + d & n \geq n_0 \end{cases}$$

$$T(n) = d + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$

$$= d + d + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right)$$

$$= 2d + T\left(\left\lfloor \frac{n}{2^2} \right\rfloor\right)$$

$$\vdots$$

$$= kd + T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right)$$

To terminate recurssion, we seek the first (i.e. smallest) k such that

$$\left\lfloor \frac{n}{2^k} \right\rfloor < n_0$$

$$\iff \quad \frac{n}{2^k} < n_0$$

$$\iff \quad \frac{n}{n_0} < 2^k$$

$$\iff \quad k-1 \leq \lg\left(\frac{n}{n_0}\right) < k \quad \left\{ \text{since } k \text{ is the least such integer} \right.$$

$$\iff \quad k-1 = \left\lfloor \lg\left(\frac{n}{n_0}\right) \right\rfloor$$

$\therefore \quad k = \lfloor \lg(\frac{n}{n_0}) \rfloor + 1$

Thus

$$\boxed{T(n) = \left(\lfloor \lg(\tfrac{n}{n_0}) \rfloor + 1\right) \cdot d + c}$$

$\therefore \quad T(n) = \Theta(\log(n))$