CSE 102 10-6-22

Induction

let $P(n)$ be a propositional function, i.e.

$$P : \mathbb{Z}^+ \longrightarrow \{T, F\}$$

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

Suppose we wish to prove

$$\boxed{\forall n \geq 1 : P(n)}$$

To prove such a stmt by mathematical induction, we perform 2 steps.

I Base.

Prove $P(1)$ is true.

II. Induction.

Induction hypothesis

Prove $\forall n \geq 1: P(n) \rightarrow P(n+1)$

Let $n \geq 1$ be chosen arbitrarily.

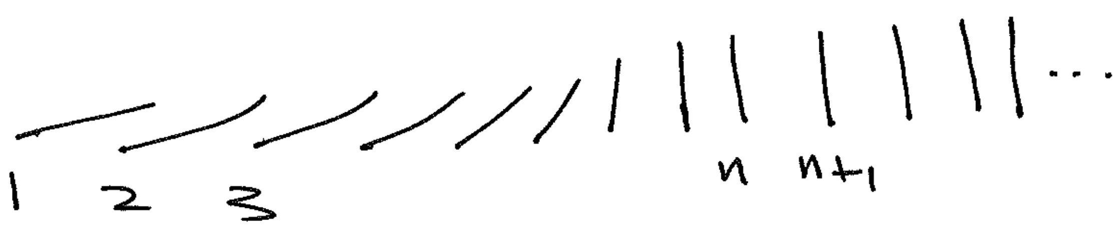
<u>Assume $P(n)$ is true.</u>

Show as a consequence $P(n+1)$ is also true.

When these are done, conclude

$$\forall n \geq 1 : P(n) .$$

Domino analogy:



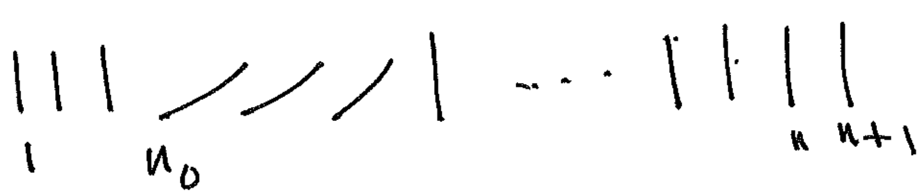
<u>minor variation</u>:

to show $\forall n \geq n_0 : P(n)$, do

I. Prove $P(n_0)$

$\boxed{\text{ind. hyp.}}$

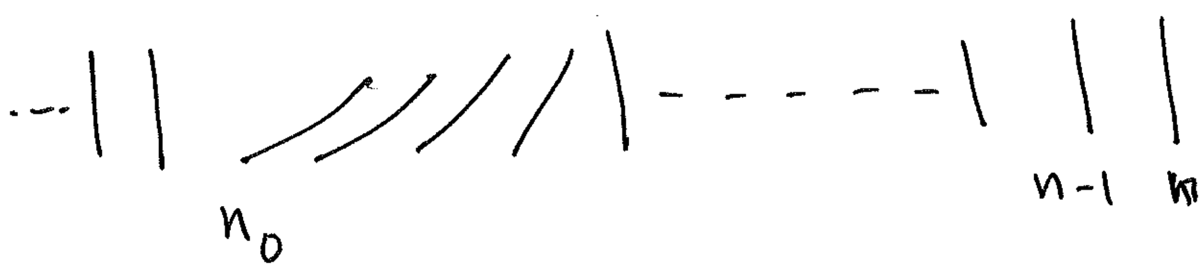
IIa. Prove $\forall n \geq n_0 : P(n) \rightarrow P(n+1)$



other variations on induction step.

$\boxed{\text{ind. hyp.}}$

II b. Prove $\forall n > n_0 : P(n-1) \rightarrow P(n)$



II a & II b are called <u>weak induction</u> or <u>1st principle of mathematical induction</u> (PMI).

II c. Prove

$$\forall n \geq n_0 : \overbrace{\left(P(n_0) \land P(n_0+1) \land \ldots \land P(n)\right)}^{\text{Induction hypothesis}} \rightarrow P(n+1)$$

Let $n \geq n_0$ be chosen arbitrarily

Assume for all k in range $n_0 \leq k \leq n$ that $P(k)$ is true. Show as a consequence that $P(n+1)$ is true.

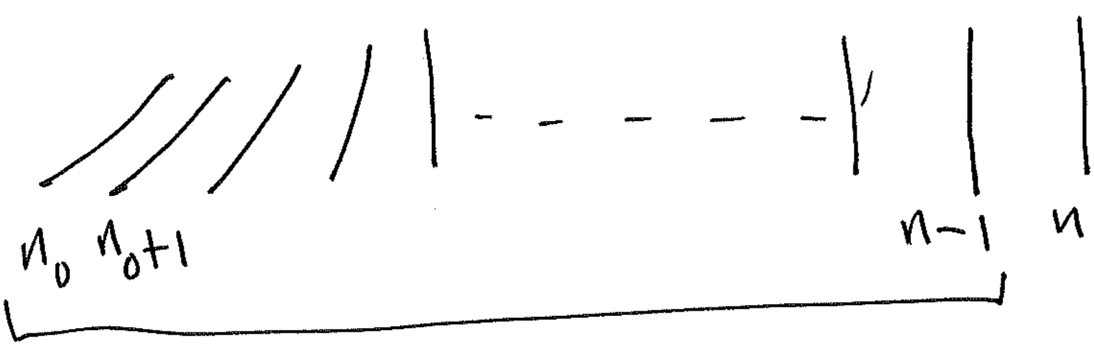
II d. Prove

$$\forall n > n_0 : \big(P(n_0) \wedge P(n_0+1) \wedge \dots \wedge P(n-1)\big) \to P(n)$$

let $n > n_0$ be arbitrary.

Assume for all k in range $n_0 \le k < n$ that $P(k)$ is true.

Show as a consequence that $P(n)$ is true.



We call IIe & IIIl strong induction or the 2nd PMI

Well ordering Property of $\mathbb{Z}^+$

any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Ex. show $\forall n \geq 1$:

$P(n)$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

Proof

I $P(1)$ says $\sum_{k=1}^{1} k^2 = \frac{1\cdot(1+1)(2\cdot 1+1)}{6}$

i.e. $1^2 = \frac{2\cdot 3}{6}$

i.e. $1 = 1$ ✓

ind. hyp

induction conclusion

II a. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$.

let $n \geq 1$ be arbitrary. Assume for this n that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

state ind. hyp.

we must show

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

state ind conc.

so

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \left\{\begin{array}{l}\text{by the}\\ \text{induction}\\ \text{hypothesis}\end{array}\right.$$

$$\vdots$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} .$$

[state where ind. hyp. is used

Result follows for all $n \geq 1$

by 1^{st} PMI. ∎

Ex. Let $x \in \mathbb{R}$, $x \neq 1$. Show

$$\forall n \geq 0: \boxed{\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}} \; P(n)$$

I. $P(0)$ says $\displaystyle\sum_{k=0}^{0} x^k = \frac{x^{0+1}-1}{x-1}$

i.e. $x^0 = \dfrac{x-1}{x-1}$, i.e. $1 = 1$.

II b. Show $\forall n > 0: P(n-1) \rightarrow P(n)$

Let $n > 0$ be arbitrary. Assume

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x-1}$$

We must show

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}.$$

So

$$\sum_{k=0}^{n} x^k = \left(\sum_{k=0}^{n-1} x^k\right) + x^n$$

$$= \left(\frac{x^n-1}{x-1}\right) + x^n \quad \Big\{ \text{by the Ind. hyp.}$$

$$= \frac{x^n-1 + x^n(x-1)}{x-1}$$

$$= \frac{\cancel{x^n}-1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1}-1}{x-1}.$$

Result follows by 1st PMI. ∎

Exercise

- $\forall n \geq 1 : \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

- $\forall n \geq 1 : \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

look this up

- $\forall n \geq 1 : \sum_{k=1}^{n} k^5 =$ ______________

Ex. define $T(n)$ for $n \in \mathbb{Z}^+$ by

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 & \text{if } n \geq 2 \end{cases}$$

$P(n)$

Prove: $\forall n \geq 1$: $\boxed{T(n) \leq \lg(n)}$

Hence $T(n) = O(\log n)$

<u>Proof.</u>

I. $P(1)$ says $T(1) \le \lg(1)$,

i.e. $0 \le 0$ ✓.

II d. $\forall n > 1 : \big(P(1) \wedge \ldots \wedge P(n-1)\big) \to P(n)$.

Let $n > 1$ be arbitrary. Assume for all k in the range $1 \le k < n$ that $T(k) \le \lg(k)$.

We must show

$$T(n) \le \lg(n).$$

so

$$T(n) = T\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + 1$$

$$\leq \lg\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + 1 \quad \left\{ \begin{array}{l} \text{by the ind} \\ \text{hyp. with} \\ k = \left\lfloor \tfrac{n}{2} \right\rfloor \end{array} \right.$$

$$\leq \lg\left(\tfrac{n}{2}\right) + 1 \quad \left\{ \begin{array}{l} \text{since } \lfloor x \rfloor \leq x \\ \text{and } \therefore \lg\lfloor x \rfloor \leq \lg(x) \end{array} \right.$$

$$= \lg(n) - \cancel{\lg 2} + \cancel{1}$$

$$= \lg(n).$$

$\therefore$ $T(n) \leq \lg(n)$. Result follows by 2^{nd} P.M.I. ∎

multiple base cases!

lowest basecase

highest base case

Base: $P(1), P(2), \ldots, P(n_0)$

Induction (IId): $\forall n > n_0 : \left(P(1) \wedge \cdots \wedge P(n-1)\right) \rightarrow P(n)$