cse 102 4-18-24

Theorem

If T is a tree on n vertices, then T has n-1 edges.

I. if T has $n=1$ vertex, then it can have no edges

T $\bullet$

$\therefore$ base case is satisfied

II d. $\forall n > 1 : (P(1) \wedge \dots \wedge P(n-1)) \rightarrow P(n)$

ind. hyp.

Let $n > 1$. Assume that for all k in the range $1 \leq k < n$ that: any tree on k vertices has $k-1$ edges. We must show that if T is a tree on n vertices, then T has $n-1$ edges.

ind. conc.

Assume T is a tree on n vertices. Let e be any edge in T, and remove it.

This results in two trees, each with fewer than n vertices, call them T_1, T_2 with k_1 and k_2 vertices, respectively. We have $k_1 < n$ & $k_2 < n$. By the induction hypothesis, T_1 has $k_1 - 1$ edges and T_2 has $k_2 - 1$ edges, resp. Since no vertices were removed,

$$n = k_1 + k_2 .$$

Therefore,

$$(\#\text{edges in } T) = (k_1 - 1) + (k_2 - 1) + 1$$

$$= (k_1 + k_2) - 1 - \cancel{1} + \cancel{1}$$

$$= n - 1 .$$

Result follows for all n by 2^{nd} PMI.

Justification of PMI (1 & 2)

- Well ordering Property (WOP) of $\mathbb{Z}^+$.

  Any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Theorem 1 (WOP $\Rightarrow$ 1st PMI (form #b))

$$\left[P(1) \wedge \left(\forall n>1: P(n-1) \to P(n)\right)\right] \to \forall n \geq 1: P(n)$$

Proof:

Assume both $P(1)$ and $\forall n>1: P(n-1) \to P(n)$ are true.

let

$$S = \{ n \in \mathbb{Z}^+ \mid P(n) \text{ is false} \}$$

It is sufficient to show $S = \phi$, since then $P(n)$ is true for all $n \geq 1$.

Assume, to get a $\times$ that $S \neq \phi$.

By the W.O.P of $\mathbb{Z}^+$, S contains a least element, call it m.

so $\boxed{m \in S}$

and any $k < m$ is not in S.

Since $P(1)$ is true, we have $1 \notin S$. $\therefore$ $m \neq 1$ $\therefore$ $m > 1$ $\therefore$ $m-1 \geq 1$, i.e. $m-1$ is a positive integer. We know $m-1 \notin S$ so $\boxed{P(m-1)}$ is true.

Also since $P(n-1) \rightarrow P(n)$ for all $n > 1$. In particular for $m = n$, we have

$$\boxed{P(m-1) \rightarrow P(m)}$$

is true. Therefore $P(m)$ must be true.

Thus $\boxed{m \notin S}$. This contradicts the very definition of m as smallest element in S. This $\times$ shows our assumption was false.

$\therefore S = \emptyset$

$\therefore \forall n \geq 1 : P(n)$ is true

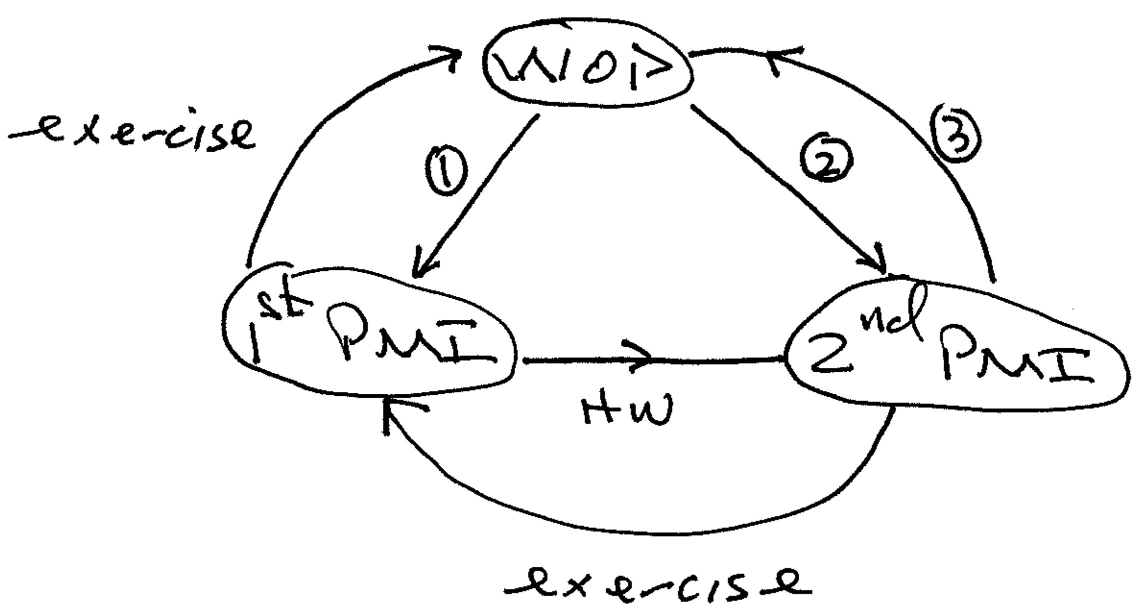
$\blacksquare$

Thm 2: WOP $\Rightarrow$ 2^{nd} PMI.

Thm 3: 2^{nd} PMI $\Rightarrow$ WOP.

Homework: 1^{st} PMI $\Rightarrow$ 2^{nd} PMI

WOP $\Rightarrow$ 1^{st} PMI $\Rightarrow$ 2^{nd} PMI $\Rightarrow$ WOP



Handout: Recurrence Relations

Ex

$$T(n) = \begin{cases} c & n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + dn & n \geq 2 \end{cases}$$

Goal: find a (asymptotic) solution.

3 methods:

- substitution
- recursion tree (iteration)
- master theorem

Substitution

Ex. $T(n) = \begin{cases} 2 & 1 \le n < 3 \\ 3T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$

Guess: $T(n) = O(n \log n)$

must show $\exists$ pos. c, n_0 such that

$$\forall n \ge n_0 : T(n) \le c \cdot n \log n$$

need at least 2 base cases

$n=1$: $T(1) \le c \cdot 1 \cdot \log(1)$ ✗

$n=2$: $T(2) \le c \cdot 2 \cdot \log(2)$

lowest base case n_0 is at least 2

call $\underline{n_1}$ <u>highest</u> base case

mimic induction step.

II. $\forall\ n > n_1 : \left(P(n_0) \wedge \cdots \wedge P(n-1)\right) \rightarrow P(n)$