CSE 102 4-4-24

<u>Defn</u>

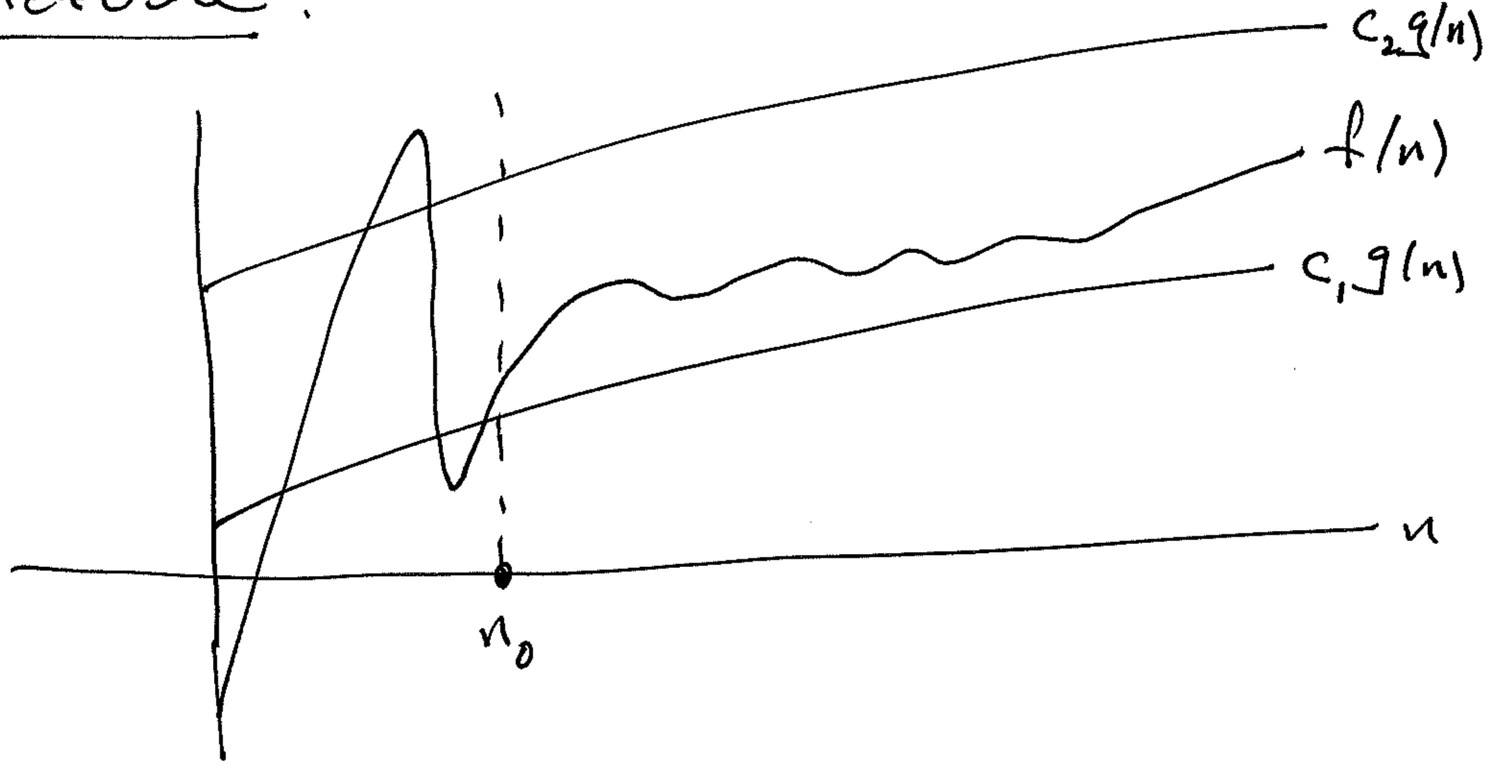
$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

equivalently:

$$\Theta(g(n)) = \{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}$$

we say: $g(n)$ is a tight asymptotic bound on $f(n)$

Picture:



Fact: if $P(n)$, $q(n)$ are polynomials, then

- $P(n) = O(q(n))$ iff $\deg(P(n)) \leq \deg(q(n))$
- " Ω " " " $\geq$ "
- " Θ " " " $=$ "

Exercise

Prove $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

Exercise

let $c > 0$. Prove

- $c \cdot g(n) = O(g(n))$
- $c \cdot g(n) = \Omega(g(n))$
- $c \cdot g(n) = \Theta(g(n))$

Proof

we must show $\exists$ pos. c_1, n_1 s.t.

$\forall n \geq n_1$: $0 \leq c_1 g(n) \leq c \cdot g(n)$. $(*)$

~~let~~ Define $c_1 = c$, $n_1 = 1$. Then

~~$\forall n \geq n_1$~~ $(*)$ holds. ∎

Ex. Prove $\sqrt{n+10} = \Theta(\sqrt{n})$

Proof:

we must find pos. c_1, c_2, n_0

s.t. $\forall\ n \geq n_0$:

$$0 \leq c_1 \cdot \sqrt{n} \leq \sqrt{n+10} \leq c_2 \cdot \sqrt{n}.$$

Let $c_1 = 1$, $c_2 = \sqrt{2}$, $n_0 = 10$.

Then if $n \geq n_0$, we have

$$-10 \leq 0 \quad \text{and} \quad 10 \leq n$$

$$\therefore\ -10 \leq (1-1)\ n \quad \text{and} \quad 10 \leq (2-1)\ n$$

$$\therefore\ -10 \leq (1-c_1^2)\ n \quad \text{and} \quad 10 \leq (c_2^2-1)\ n$$

$$\therefore\ c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n$$

$$\therefore \quad 0 \le c_1^2 n \le n+10 \le c_2^2 n$$

$$\therefore \quad 0 \le c_1 \sqrt{n} \le \sqrt{n+10} \le c_2 \sqrt{n}.$$ $\blacksquare$

<u>Exercise</u>: Prove.

Let $a, b \in \mathbb{R}$ with $b > 0$. Then

$$(n+a)^b = \Theta(n^b)$$

<u>Theorem</u>

If (1) <u>$f(n) \le h(n)$ for all sufficiently large n</u>

and (2) <u>$h(n) = O(g(n))$</u>, then

$$f(n) = O(g(n)).$$

Proof

(1) says that there exist pos. n_2 s.t. $\forall n \geq n_2$: $f(n) \leq h(n)$.

(2) says that there exist pos. c_1, n_1 s.t. $\forall n \geq n_1$: $0 \leq h(n) \leq c_1 \cdot g(n)$

We must show that there exist positive c and n_0 s.t. $\forall n \geq n_0$

$$0 \leq f(n) \leq c \cdot g(n).$$

Let $c = c_1$, and $n_0 = \max(n_1, n_2)$.

Then if $n \geq n_0$ we have both $n \geq n_1$ and $n \geq n_2$. Hence

$$\underset{\substack{\uparrow \\ \text{blanket assumption}}}{0 \leq} f(n) \leq h(n) \leq c \cdot g(n)$$

$\blacksquare$.

<u>Exercise.</u>

If (1) $f(n) \geq h(n)$ for all suff. large n, and (2) $h(n) = \Omega(g(n))$, then

$$f(n) = \Omega(g(n)).$$

<u>Exercise</u>

If $h_1(n) \leq f(n) \leq h_2(n)$ for all suff. large n, and $h_1(n) = \Omega(g(n))$ and $h_2(n) = O(g(n))$, then

$$f(n) = \Theta(g(n)).$$

Ex. Let $K \geq 1$ be an integer. Then

$$\sum_{i=1}^{n} i^K = \Theta(n^{K+1})$$

Proof

observe that

$$\sum_{i=1}^{n} i^K \leq \sum_{i=1}^{n} n^K = n \cdot n^K = n^{K+1} = O(n^{K+1}).$$

Also

$$\sum_{i=1}^{n} i^K \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^K$$

$$\geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lceil \tfrac{n}{2} \rceil^K$$

$$= \left(n - \lceil \tfrac{n}{2} \rceil + 1\right) \lceil \tfrac{n}{2} \rceil^K$$

$$= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \left\lceil \frac{n}{2} \right\rceil^{k} \quad \left\{ \begin{array}{l} \text{since} \\ \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n \end{array} \right.$$

$$> \left(\frac{n}{2} - \cancel{1} + \cancel{1}\right) \left(\frac{n}{2}\right)^{k} \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor > x - 1 \\ \lceil x \rceil \geq x \end{array} \right.$$

$$= \left(\frac{n}{2}\right)^{k+1}$$

$$= \left(\frac{1}{2}\right)^{k+1} \cdot n^{k+1} = \Omega\left(n^{k+1}\right)$$

Hence

$$\sum_{i=1}^{n} i^{k} = \Theta\left(n^{k+1}\right).$$

■

Recall

$$k=1: \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

$$k=2: \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$$

$$k=3: \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \Theta(n^4)$$

$$\vdots$$

<u>note</u>: what do we mean by

$$f(n) = 3n^2 + \underbrace{\Theta(n)}_{\substack{\text{stands for some} \\ \text{anonymous function} \\ h(n) = \Theta(n)}} \quad ?$$

<u>Exercise</u> Prove

$$\sum_{i=1}^{n} \underbrace{\Theta(i)}_{\uparrow} = \Theta(n^2)$$

means some $f(i) = \Theta(i)$

i.e. $\boxed{\text{if } f(i) = \Theta(i), \text{ then } \sum_{i=1}^{n} f(i) = \Theta(n^2)}$

Defn

$$o(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < c \cdot g(n)\}$$

Recall

$$O(g(n)) = \{f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

observe: $o(g(n)) \subseteq O(g(n))$.

If $f(n) = o(g(n))$ we say $g(n)$ is strict asymptotic upper bound for $f(n)$.

<u>lemma</u>

$$f(n) = o(g(n)) \text{ iff } \lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$$