CSE 102 4-11-24

Defn

We say $f(n)$ is asymptotically equivalent to $g(n)$, written

$$f(n) \sim g(n),$$

iff $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 1$

note: $f(n) \sim g(n) \Rightarrow f(n) = \Theta(g(n))$, but not converseley.

Exercise

Prove $f(n) \sim g(n)$ iff $f(n) = g(n) + o(g(n))$

Handout: Common functions (3.2)

- floors & ceilings : read this
- logs

Recall $\log_b(\cdot)$ is inverse of $\exp_b(\cdot) = b^{\cdot}$

so $a^{\log_a(x)} = x$

$\log_a(a^x) = x$

so

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)} = b^{\log_b(a)\cdot\log_a(x)}$$

$$\therefore \quad \log_b(x) = \underbrace{\log_b(a)}_{\text{const.}}\cdot\log_a(x) \qquad (*)$$

Thus $\log_b(n) = \text{const.} \cdot \log_a(n)$.

$$\therefore \quad \boxed{\log_b(n) = \Theta(\log_a(n))}$$

also

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Also

$$a^{\log_b(x)} = a^{\log_a(x)\cdot\log_b(a)}$$

$$= \left(a^{\log_a(x)}\right)^{\log_b(a)}$$

$$= x^{\log_b(a)}$$

$$\therefore \quad \boxed{a^{\log_b(x)} = x^{\log_b(a)}}$$

- Stirling's Formula

weak version: $$\boxed{n! \sim \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n}$$

strong version:

$$* \quad \boxed{n! = \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}$$

i.e.

$$\frac{n!}{\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n} - 1 = \Theta\left(\frac{1}{n}\right)$$

Corollary

(1) $n! = o(n^n)$ ✓

(2) $n! = \omega(b^n)$ ← exercise!

(3) $\log(n!) = \Theta(n \log(n))$

Proof of (1)

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n} \cdot \frac{\cancel{n^n}}{e^n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}{\cancel{n^n}}$$

$$= \frac{\sqrt{2\pi} \cdot n^{1/2}}{e^n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right) \longrightarrow 0$$

Proof of (3)

read handout

$\blacksquare$

Exercise:

Prove $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

Recall: $\binom{m}{k} = \dfrac{m!}{k!\,(m-k)!} \quad (0 \leq k \leq m)$

Proof

$$\binom{2n}{n} = \frac{(2n)!}{n!\,(2n-n)!}$$

$$= \frac{(2n)!}{(n!)^2}$$

$$= \frac{\sqrt{2\pi \cdot 2n} \cdot \left(\frac{2n}{e}\right)^{2n} \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^{n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)\right)^2}$$

$$= \frac{\cancel{2}\sqrt{\pi}\cdot n^{\frac{1}{2}} \cdot \frac{2^{2n}\cdot \cancel{n^{2n}}}{\cancel{e^{2n}}} \cdot \left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\cancel{2}\pi \cdot n \cdot \frac{\cancel{n^{2n}}}{\cancel{e^{2n}}} \cdot \left(1+\Theta\left(\frac{1}{n}\right)\right)^2}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} \cdot 4^n \cdot \left[\frac{\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}\right]$$

$$\therefore \frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}}\cdot\left[\cdots\right] \longrightarrow \frac{1}{\sqrt{\pi}}$$

$$\therefore \binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$$

$\blacksquare$