CSE 102
Homework Assignment 1
Solutions

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ asymptotically positive functions. Prove that $f(n) + g(n) = \Theta\big(\max\big(f(n),\ g(n)\big)\big)$.

   Proof:
   Since $f(n)$ and $g(n)$ are asymptotically positive, we know that there exists a positive constant n_0 such that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. For any such n we have

$$\begin{aligned} 0 &\leq \max\big(f(n), g(n)\big) \\ &\leq \min\big(f(n), g(n)\big) + \max\big(f(n), g(n)\big) \\ &\leq 2 \cdot \max\big(f(n), g(n)\big). \end{aligned}$$

   But $f(n) + g(n) = \min\big(f(n), g(n)\big) + \max\big(f(n), g(n)\big)$, and therefore

$$0 \leq 1 \cdot \max\big(f(n), g(n)\big) \leq f(n) + g(n) \leq 2 \cdot \max\big(f(n), g(n)\big)$$

   for all $n \geq n_0$, showing that $f(n) + g(n) = \Theta\big(\max\big(f(n), g(n)\big)\big)$. ∎

   Note: we really only used that f and g are asymptotically non-negative.

2. Prove or disprove: If $f(n) = \Theta\big(g(n)\big)$, then $f(n)^2 = \Theta(g(n)^2)$.

   Proof:
   Since $f(n) = \Theta\big(g(n)\big)$, there are positive constants c_1, c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

   for all $n \geq n_0$. Squaring this inequality gives $0 \leq c_1^2 g(n)^2 \leq f(n)^2 \leq c_2^2 g(n)^2$ for all $n \geq n_0$. Thus there exist positive constants d_1, d_2 and n_1 (namely $d_1 = c_1^2$, $d_2 = c_2^2$ and $n_1 = n_0$) such that

$$0 \leq d_1 g(n)^2 \leq f(n)^2 \leq d_2 g(n)^2$$

   for all $n \geq n_1$. This proves $f(n)^2 = \Theta(g(n)^2)$. ∎

3. Prove or disprove: If $f(n) = \Theta\big(g(n)\big)$, then $2^{f(n)} = \Theta\big(2^{g(n)}\big)$.

Solution:
The statement is false, as the following counter-example shows.

Let $f(n) = 2n$ and $g(n) = n$. Then $f(n) = \Theta\big(g(n)\big)$, but $2^{2n} = 4^n = \omega(2^n)$, and therefore $2^{f(n)} = \omega\big(2^{g(n)}\big)$, whence $2^{f(n)} \neq \Theta\big(2^{g(n)}\big)$. ∎

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions, and assume that $\lim_{n\to\infty} g(n) = \infty$. Prove that if $f(n) = \Theta\big(g(n)\big)$, then $\ln\big(f(n)\big) = \Theta\big(\ln\big(g(n)\big)\big)$.

Proof:
Assume $f(n) = \Theta\big(g(n)\big)$. Then there exist positive constants c_1, c_2 and n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$$

for all $n \ge n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the constant n_0 can be chosen large enough so that

$$1 \le c_1 g(n) \le f(n) \le c_2 g(n)$$

for all $n \ge n_0$. Take ln() of all terms in the preceding inequality to get

$$0 \le \ln(c_1) + \ln\big(g(n)\big) \le \ln\big(f(n)\big) \le \ln(c_2) + \ln\big(g(n)\big)$$

for all $n \ge n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the term $\ln\big(g(n)\big)$ dominates the constants $\ln(c_1)$ and $\ln(c_2)$, whence $\ln(c_1) + \ln\big(g(n)\big) = \Omega\big(\ln\big(g(n)\big)\big)$ and $\ln(c_2) + \ln\big(g(n)\big) = O\big(\ln\big(g(n)\big)\big)$. The desired result $\ln\big(f(n)\big) = \Theta\big(\ln\big(g(n)\big)\big)$ now follows by Exercise 4 on page 4 of the handout on asymptotic growth rates. ∎

5. (Problem 3.2-8) Show that if $f(n) \ln f(n) = \Theta(n)$, then $f(n) = \Theta(n/\ln n)$. Hint: use the result of the preceding problem.

Proof:
Assume $f(n) \ln f(n) = \Theta(n)$. Since $\lim_{n\to\infty} n = \infty$, we can, by the result of problem (4), take ln () of both sides of this equation to get

$$\ln(f(n) \ln f(n)) = \Theta(\ln(n))$$

so

$$\ln\big(f(n)\big) + \ln\Big(\ln\big(f(n)\big)\Big) = \Theta(\ln(n)).$$

Since $\ln\Big(\ln\big(f(n)\big)\Big) = o\Big(\ln\big(f(n)\big)\Big)$, we can, by Exercise 10g on page 7 of the handout on asymptotic growth rates, drop the lower order term and write this as $\ln\big(f(n)\big) = \Theta(\ln(n))$. Substituting this into our original assumption gives $f(n)\Theta(\ln n) = \Theta(n)$, and therefore

$$f(n) = \Theta(n) \cdot \frac{1}{\Theta(\ln n)} = \Theta(n) \cdot \Theta\left(\frac{1}{\ln n}\right) = \Theta\left(\frac{n}{\ln n}\right)$$

where we have used Exercise 12bc on page 8 of the handout on asymptotic growth rates. ■

6. Consider the statement: $f(cn) = \Theta\big(f(n)\big)$.

   a. Determine a function $f(n)$ and a constant $c > 0$ for which the statement is false.

      Example:
      Let $f(n) = 2^n$ and $c = 2$. Then the statement says $2^{2n} = \Theta(2^n)$, i.e. $4^n = \Theta(2^n)$. This is false since $4^n = \omega(2^n)$ by Exercise 10e on page 7 of the handout on Asymptotic Growth Rates, and since $\omega(2^n) \cap \Theta(2^n) = \emptyset$ by Exercise 7 on page 5 of the same handout. ■

   b. Determine a function $f(n)$ for which the statement is true for all $c > 0$.

      Example:
      Let $f(n) = n$. Then the statement says $cn = \Theta(n)$. This is true for any $c > 0$ by Exercise 2 on page 3 of the handout on Asymptotic Growth Rates. ■

7. Determine the asymptotic order of the expression $\sum_{i=1}^{n} a^i$ where $a > 0$ is a constant, i.e. find a simple function $g(n)$ such that the expression is in the class $\Theta(g(n))$. (Hint: consider the cases $a = 1$, $a > 1$, and $0 < a < 1$ separately.)

Solution: In the case $a = 1$ we have $\sum_{i=1}^{n} a^i = n = \Theta(n)$. Now assume $a \neq 1$ and use the formula for the sum of a geometric series:

$$\sum_{i=1}^{n} a^i = a\left(\sum_{i=0}^{n-1} a^i\right) = a\left(\frac{a^n - 1}{a - 1}\right) = \left(\frac{a}{a-1}\right) \cdot a^n - \left(\frac{a}{a-1}\right)$$

If $a > 1$ then $a^n \to \infty$ as $n \to \infty$, so the first term dominates the second, and $\sum_{i=1}^{n} a^i = \Theta(a^n)$. If $0 < a < 1$ then $a^n \to 0$, so the constant term (which is positive in this case) dominates, and therefore $\sum_{i=1}^{n} a^i = \Theta(1)$. Thus we have

$$\sum_{i=1}^{n} a^i = \begin{cases} \Theta(1) & \text{if } 0 < a < 1 \\ \Theta(n) & \text{if } a = 1 \\ \Theta(a^n) & \text{if } a > 1 \end{cases}$$

■

8. Use induction to prove that $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$ for all $n \geq 1$.

Proof:

I. $\sum_{k=1}^{1} k^4 = 1 = \frac{2\cdot 15}{30} = \frac{1\cdot(1+1)\cdot(6\cdot 1^3+9\cdot 1^2+1-1)}{30}$, and the base case is established.

II. Let $n \geq 1$. Assume for this *n* that

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

We must show that

$$\sum_{k=1}^{n+1} k^4 = \frac{(n+1)(n+2)(6(n+1)^3+9(n+1)^2+(n+1)-1)}{30}$$

We have

$$\sum_{k=1}^{n+1} k^4 = \left(\sum_{k=1}^{n} k^4\right) + (n+1)^4$$

$$= \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30} + (n+1)^4 \qquad \text{by the induction hypothesis}$$

$$= \frac{(n+1)[n(6n^3 + 9n^2 + n - 1) + 30(n+1)^3]}{30}$$

$$= \frac{(n+1)[6n^4 + 39n^3 + 91n^2 + 89n + 30]}{30}$$

$$= \frac{(n+1)(n+2)[6n^3 + 27n^2 + 37n + 15)]}{30}$$

$$= \frac{(n+1)((n+1)+1)[6(n+1)^3 + 9(n+1)^2 + (n+1) - 1)]}{30}$$

Thus the formula holds for $n + 1$ as well. The formula is valid for all n by induction. ■