CSE 102 5-10-24

Supplemental Lecture (written only)

<u>Coin Changing Problem</u>: greedy solution

As before, given coins in denominations

$$d = (d_1, d_2, \ldots, d_n)$$

and an amount N to be paid, find $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_+^n$ such that

$$\text{Value} = d_1 x_1 + d_2 x_2 + \cdots + d_n x_n = N$$

and making $x_1 + x_2 + \cdots + x_n$ minimum.

The greedy strategy is:

- from among all denominations whose addition would not cause the value to exceed N, choose the largest.
- stop when value = N.

Exercise

show that for $d = (1, 5, 10, 25, 50)$ the greedy strategy yields an optimal solution for any N.

* Exercise

same for $d = (1, 5, 10, 25)$

Exercise

same for $d = (1, 5, 10)$

Exercise

show that for $d = (1, 10, 25)$, the greedy strategy does not yield an optimal solution for every N.

Exercise

Show that if $d = (1, 5, 10, 25)$, the greedy strategy yields an optimal solution for any N.

Proof

Let $x = (x_1, x_2, x_3, x_4)$ be an optimal solution for amount N, and let $g = (g_1, g_2, g_3, g_4)$ be the greedy result for the same N.

We have

$$(1) \quad x_1 + 5x_2 + 10x_3 + 25x_4 = N = g_1 + 5g_2 + 10g_3 + 25g_4$$

and

$$(2) \quad x_1 + x_2 + x_3 + x_4 \leq g_1 + g_2 + g_3 + g_4$$

We must show that inequality (2) is actually an equality, i.e.

$$x_1 + x_2 + x_3 + x_4 = g_1 + g_2 + g_3 + g_4 .$$

For this it is sufficient to show $x = g$, i.e. $x_1 = g_1$, $x_2 = g_2$, $x_3 = g_3$, $x_4 = g_4$.

This also shows that the optimal solution is unique, which is not always true in optimization problems.

We list a few facts:

(a) $0 \le x_1 < 5$ since if $x_1 \ge 5$ we can trade 5 pennies for 1 nickel, contradicting that x is optimal.

(b) $0 \le g_1 < 5$ since the greedy solution selects as many nickles as possible before any pennies

(c) $0 \le x_2 < 2$ since if $x_2 \ge 2$ we can trade 2 nickles for a dime, again showing that x is non-optimal, a contradiction

(d) $0 \le g_2 < 2$ since the greedy solution selects as many dimes as possible before any nickles.

(e) The value of all dimes, nickles and quarters in the optimal solution is less than 25 cents, i.e.

$$0 \le x_1 + 5x_2 + 10x_3 < 25$$

<u>Pf.</u>

If $x_1 + 5x_2 + 10x_3 \ge 25$, then we can replace $x_1 + x_2 + x_3$ coins in the optimal solution by 1 Quarter. It is impossible that $x_1 + x_2 + x_3 = 0$, and if $x_1 + x_2 + x_3 = 1$, we have either

$$(x_1, x_2, x_3) = (1, 0, 0) : \text{Value} = 1 < 25$$
$$\text{or } (x_1, x_2, x_3) = (0, 1, 0) : \text{Value} = 5 < 25$$
$$\text{or } (x_1, x_2, x_3) = (0, 0, 1) : \text{Value} = 10 < 25$$

Thus $x_1 + x_2 + x_3 \ge 2$. But again, this contradicts the optimality of x.

(f) $x_4 \le g_4$

Pf. the greedy solution makes $g_4 = \lfloor \frac{N}{25} \rfloor$. if $x_4 > g_3$, then

$25x_4 > N$, contradicting (1)

(g) $\boxed{x_4 = g_4}$

Pf. suppose $x_4 < g_4$. Then at least one quarter in the greedy solution must be accounted for by dimes nickles and quarters in the optimal solution. This contradicts (e).

using (g) equation (1) becomes

$$(3) \qquad x_1 + 5x_2 + 10x_3 = g_1 + 5g_2 + 10g_3$$

Reduce this equation modulo 5 to get

$$x_1 \equiv g_1 \pmod{5}$$

This together with (a) and (b) gives

$$\boxed{x_1 = g_1}$$

which yields $5x_2 + 10x_3 = 5g_2 + 10g_3$, and hence

$$x_2 + 2x_3 = g_2 + 2g_3 \quad . \tag{4}$$

Now reduce this equation modulo 2 to get

$$x_2 \equiv g_2 \pmod{2}$$

Together with (c) and (d), this gives

$$\boxed{x_2 = g_2}$$

Therefore from (4) we have $2x_3 = 2g_3$,

hence $\boxed{x_3 = g_3}$.

We've shown that $g = x$, proving that g is an optimal solution.

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