CSE 102
Homework Assignment 2
Solutions

1. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta\big(f(n)\big)$. Prove that $\sum_{i=1}^{n} f(i) = \Theta\big(nf(n)\big)$. Hint: emulate the example on page 4 of the handout on asymptotic growth rates in which it is shown that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$.

   Proof:
   Since $f(n)$ is increasing, we have $\sum_{i=1}^{n} f(i) \le \sum_{i=1}^{n} f(n) = nf(n) = O\big(nf(n)\big)$. Note also that

$$
\begin{aligned}
\sum_{i=1}^{n} f(i) &\ge \sum_{i=\lceil n/2\rceil}^{n} f(i) && \text{by discarding some positive terms} \\
&\ge \sum_{i=\lceil n/2\rceil}^{n} f(\lceil n/2\rceil) && \text{since } f(n) \text{ is increasing} \\
&= (n - \lceil n/2\rceil + 1) f(\lceil n/2\rceil) \\
&= (\lfloor n/2\rfloor + 1) f(\lceil n/2\rceil) \\
&> \big((n/2) - 1 + 1\big) f(n/2) && \text{since } \lfloor x\rfloor > x - 1,\ \lceil x\rceil \ge x \text{ and } f(n) \text{ is increasing} \\
&= (1/2)n\Omega\big(f(n)\big) && \text{since } f(n/2) = \Theta\big(f(n)\big) \subseteq \Omega\big(f(n)\big) \\
&= \Omega\big(nf(n)\big)
\end{aligned}
$$

   Thus $\sum_{i=1}^{n} f(i)$ is bounded above and below by functions in the classes $O\big(nf(n)\big)$ and $\Omega\big(nf(n)\big)$, respectively. By an exercise in the handout on asymptotic growth rates, we have $\sum_{i=1}^{n} f(i) = \Theta\big(nf(n)\big)$ as required. ■

2. Use the result of the preceding problem to prove that $\log(n!) = \Theta(n\log(n))$, without using Stirling's formula.

   Proof:
   Observe that the function $\log n$ is positive (for $n > 1$), increasing and $\log(n/2) = \log n - \log 2 = \Theta(\log n)$. Therefore we may take $f(n) = \log(n)$ in the preceding problem, giving

$$
\begin{aligned}
\log(n!) &= \log(1 \cdot 2 \cdot 3 \cdots n) \\
&= \sum_{i=1}^{n} \log(i) \\
&= \Theta(n\log n).
\end{aligned}
$$

■

3. Use Stirling's formula to determine a constant $a > 0$ such that $\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$.

   Proof:
   By Stirling's formula we have

$$
\binom{3n}{n} = \frac{(3n)!}{n! \cdot (2n)!}
$$

$$= \frac{\sqrt{2\pi \cdot 3n} \cdot \left(\frac{3n}{e}\right)^{3n} \cdot \left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^{n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)\right) \cdot \left(\sqrt{2\pi \cdot 2n} \cdot \left(\frac{2n}{e}\right)^{2n} \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)\right)}$$

$$= \frac{\sqrt{2\pi} \cdot \sqrt{3n} \cdot \frac{3^{3n} n^{3n}}{e^{3n}} \cdot \left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\sqrt{2\pi} \cdot \sqrt{n} \cdot \frac{n^n}{e^n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right) \cdot \sqrt{2\pi} \cdot \sqrt{2n} \cdot \frac{2^{2n} n^{2n}}{e^{2n}} \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)}$$

$$= \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{27^n/4^n}{\sqrt{n}} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right) \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)}.$$

Therefore

$$\lim_{n\to\infty} \frac{\binom{3n}{n}}{\frac{(27/4)^n}{\sqrt{n}}} = \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \lim_{n\to\infty} \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right) \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)} = \frac{\sqrt{3}}{2\sqrt{\pi}}$$

and since $0 < \frac{\sqrt{3}}{2\sqrt{\pi}} < \infty$, it follows that $\binom{3n}{n} = \Theta\left(\frac{(27/4)^n}{\sqrt{n}}\right)$. Hence $a = 27/4$. ■

4. Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg(n))$.

Proof:

I. When $n = 1$ we have $S(1) \geq \lg(1)$, which reduces to $0 \geq 0$, which is true.

II. Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. We must show that $S(n) \geq \lg(n)$. We have

$$\begin{aligned} S(n) &= S(\lceil n/2 \rceil) + 1 \\ &\geq \lg(\lceil n/2 \rceil) + 1 && \text{by the induction hypothesis with } k = \lceil n/2 \rceil \\ &\geq \lg(n/2) + 1 && \text{since } \lceil x \rceil \geq x \text{ for any } x \\ &= \lg(n) - \lg(2) + 1 \\ &= \lg(n). \end{aligned}$$

By the Second Principle of Mathematical Induction, $S(n) \geq \lg(n)$ for all $n \geq 1$. ■

5. Let $T(n)$ be defined by the recurrence

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & \text{if } n \geq 2 \end{cases}$$

Show that $\forall n \geq 1\colon\ T(n) \leq (4/3)n^2$, and hence $T(n) = O(n^2)$.

Proof:

I. For $n = 1$ we have $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$, so the base case is satisfied.

II. Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. In particular we have $T(\lfloor n/2 \rfloor) \leq (4/3)\lfloor n/2 \rfloor^2$. We must show that $T(n) \leq (4/3)n^2$. We have

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 \\ &\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis} \\ &\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\ &= (1/3)n^2 + n^2 \\ &= (4/3)n^2. \end{aligned}$$

By the Second Principle of Mathematical Induction, $T(n) \leq (4/3)n^2$ for all $n \geq 1$. ∎

6. Prove that the First Principle of Mathematical Induction implies the Second Principle of Mathematical Induction. (This is Exercise 4 at the end of the handout on Induction Proofs.)

Proof:

The 1st PMI asserts that for any propositional function $P(n)$, the following sentence holds.

$$P(1) \wedge [\forall n > 1\colon P(n-1) \to P(n)] \longrightarrow \forall n \geq 1\colon P(n)$$

The 2nd PMI says the following sentence is true for any propositional function $Q(n)$.

$$Q(1) \wedge [\forall n > 1\colon (Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1)) \to Q(n)] \longrightarrow \forall n \geq 1\colon Q(n)$$

We assume the 1st PMI, and show the 2nd PMI as a consequence. To that end, let $Q(n)$ be any propositional function, and define $P(n)$ by

$$P(n) = Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n).$$

In particular, we have

$$P(1) = Q(1),$$

$$P(n-1) = Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1),$$

and

$$P(n) = P(n-1) \wedge Q(n).$$

To prove the 2nd PMI, we assume both $Q(1)$ and $\forall n > 1\colon \big(Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1)\big) \to Q(n)$ are true. We must show that $\forall n \geq 1\colon Q(n)$ holds.

These assumptions give us that both $P(1)$ and $\forall n > 1: P(n-1) \rightarrow Q(n)$ are true. It is an elementary fact that $\forall n > 1: P(n-1) \rightarrow P(n-1)$, and therefore $\forall n > 1: P(n-1) \rightarrow P(n-1) \wedge Q(n)$ holds, which is equivalent to $\forall n > 1: P(n-1) \rightarrow P(n)$. We have shown that under our assumptions both $P(1)$ and $\forall n > 1: P(n-1) \rightarrow P(n)$ are true. The 1^{st} PMI now yields $\forall n \geq 1: P(n)$. Obviously $P(n) \rightarrow Q(n)$, so $\forall n \geq 1: Q(n)$ is also true.

We have shown that if $Q(1)$ and $\forall n > 1: \big(Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1)\big) \rightarrow Q(n)$ are both true, then $\forall n \geq 1: Q(n)$ must also be true, establishing the 2^{nd} PMI. ∎