CSE 102
Homework Assignment 2

1. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta\big(f(n)\big)$. Prove that $\sum_{i=1}^{n} f(i) = \Theta\big(nf(n)\big)$. Hint: emulate the example on page 4 of the handout on asymptotic growth rates in which it is shown that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$.

2. Use the result of the preceding problem to prove that $\log(n!) = \Theta(n\log(n))$, without using Stirling's formula.

3. Use Stirling's formula to determine a constant $a > 0$ such that $\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$.

4. Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

   Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg(n))$.

5. Let $T(n)$ be defined by the recurrence

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & \text{if } n \geq 2 \end{cases}$$

   Show that $\forall n \geq 1$: $T(n) \leq (4/3)n^2$, and hence $T(n) = O(n^2)$.

6. Prove that the First Principle of Mathematical Induction implies the Second Principle of Mathematical Induction. (This is Exercise 4 at the end of the handout on Induction Proofs.)