CSE 102 5-16-24

- Project due Monday May 27
- Midterm 2: Thursday May 23

Greedy Algorithms

Ex. Continuous Knapsack.

Goal

(1) Maximize: $\sum_{i=1}^{n} \underbrace{x_i}_{\text{fraction stolen}} \underbrace{v_i}_{\text{value of } i^{th} \text{ object}}$

subject to:

(2) $\sum_{i=1}^{n} x_i \cdot \underbrace{w_i}_{\text{weight of } i^{th} \text{ object}} \le \underbrace{W}_{\text{capacity}}$

Here $0 \le x_i \le 1$ for $1 \le i \le n$.

A vector $x = (x_1, \ldots, x_n)$ is called feasible iff

$$\sum_{i=1}^{n} x_i w_i \le W$$

A <u>Greedy Strategy</u> consists of making locally optimal choices, then solving the subproblem arising from that choice, i.e. include the "best" object.

Knapsack (v, w, W)

```
n = len(v) = len(w)
(x_1, ..., x_n) = (0, ..., 0)
weight = 0
while weight < W
                                    unmarked
    i = the "best" remaining object
    if weight + w_i ≤ W
        x_i = 1
        weight = weight + w_i
        mark i as included
    else
        x_i = (W - weight) / w_i
        weight = W
return x
```

we define a selection function $f(i)$ that encodes the desirability of an object.

line (5) maximizes f over all unmarked objects.

Possible choices for f

- $f(i) = v_i$: sort by dec. values
- $f(i) = \frac{1}{w_i}$: sort by inc. weights
- $f(i) = \frac{v_i}{w_i}$: sort by dec. val to weight ratio

Ex. $W = 10$, $n = 5$

i	1	2	3	4	5
v_i	2	3	6.6	4	6
w_i	1	2	3	4	5
$\frac{v_i}{w_i}$	2	1.5	2.2	1	1.2

$f(i)$	X					total val
v_i	0	0	1	.5	1	14.6
$\frac{1}{w_i}$	1	1	1	1	0	15.6
$\frac{v_i}{w_i}$	1	1	1	0	.8	16.4 *

Theorem

If we maximize $\frac{v_i}{w_i}$ on line (5),

then Knapsack returns an optimal solution.

Proof

w.l.o.g. we may assume that the objects are already sorted by decreasing value/weight.

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \frac{v_3}{w_3} \geq \cdots \geq \frac{v_n}{w_n}$$

Let $x = (x_1, x_2, \ldots, x_n)$ be the solution returned by Knapsack.

If all $x_i = 1$, then x is obviously optimal. Otherwise let j be the first index such that

$$x_j < 1$$

We see from the algorithm that

$$x_i = 1 \quad \text{for} \quad 1 \le i < j$$

$$x_j < 1$$

$$x_i = 0 \quad \text{for} \quad j < i \le n$$

also: $$\sum_{i=1}^{n} x_i w_i = W$$

Let $y = (y_1, y_2, \dots, y_n)$ be an arbitrary feasible solution. Define

$$V(y) = \sum_{i=1}^{n} y_i v_i$$

we must show that

$$V(x) \geq V(y)$$

Since y is feasible, we know

$$\sum_{i=1}^{n} y_i w_i \leq W .$$

Thus

$$\sum_{i=1}^{n} (x_i - y_i) w_i = \sum_{i=1}^{n} x_i w_i - \sum_{i=1}^{n} y_i w_i$$

$$= W - \sum y_i w_i \geq 0$$

Now

$$V(x) - V(y) = \sum_{i=1}^{n} x_i v_i - \sum_{i=1}^{n} y_i v_i$$

$$= \sum_{i=1}^{n} (x_i - y_i) v_i$$

Observe that

- $i < j \Rightarrow x_i = 1 \Rightarrow x_i - y_i \geq 0$ and $\dfrac{v_i}{w_i} \geq \dfrac{v_j}{w_j}$

  $\therefore \quad (x_i - y_i)\left(\dfrac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\dfrac{v_j}{w_j}\right)$

- $i > j \Rightarrow x_i = 0 \Rightarrow x_i - y_i \leq 0$ and $\dfrac{v_i}{w_i} \leq \dfrac{v_j}{w_j}$

  $\therefore \quad (x_i - y_i)\left(\dfrac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\dfrac{v_j}{w_j}\right)$

- $i = j \Rightarrow (x_i - y_i)\left(\dfrac{v_i}{w_i}\right) = (x_i - y_i)\left(\dfrac{v_j}{w_j}\right)$

so $(x_i - y_i)\left(\frac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\frac{v_j}{w_j}\right)$ for all $1 \leq i \leq n$. Therefore

$$V(x) - V(y) = \sum_{i=1}^{n} (x_i - y_i) v_i$$

$$= \sum_{i=1}^{n} (x_i - y_i)\left(\frac{v_i}{w_i}\right) w_i$$

$$\geq \sum_{i=1}^{n} (x_i - y_i)\left(\frac{v_j}{w_j}\right) w_i$$

$$= \left(\frac{v_j}{w_j}\right) \cdot \sum_{i=1}^{n} (x_i - y_i) w_i$$

$$\geq 0$$

$\therefore V(x) \geq V(y)$. ∎

Runtime of Knapsack

- sort first: $\Theta(n \log n)$
- loop costs: $\Theta(n)$

total cost: $\Theta(n \log n)$.

<u>note</u>: This greedy strategy doesn't always work on the discrete Knapsack problem.