CSE 102 6-6-24

Exercise

Show that at least $\binom{n}{2}$ 'adjacency' questions are necessary to determine if a graph on n vertices is acyclic.

Hint: adversary strategy:

Answer yes to any edge probe, unless it would prove the existence of a cycle.

Exercise

Let $b = x_1 x_2 x_3 x_4 x_5$ be a bit string of len. 5. Problem: determine whether b contains the substring '111'. Consider algorithms that solve the problem by peeking at individual bits.

(a) Write such an algorithm. Express as a decision tree. Try to do it in 4 peeks.

(b) Show 4 peeks are necessary with an adversary argument.

<u>Ex.</u>

Recall Problem of finding (min, max) of an array $A[1 \cdots n]$. We found an <u>algorithm</u> that does this in $\lceil \frac{3n}{2} \rceil - 2$ comparisons. Is this the best we can do?

- Decision tree lower bound:

$$\lceil \lg (n^2 - n + 1) \rceil$$ (<u>exercise</u>: Prove this)

<u>Theorem</u>

Any algorithm that performs only array comparisons must do at least $\lceil \frac{3n}{2} \rceil - 2$ comparisons to find (min, max) on an array of len. n.

<u>Proof</u>

Run any <u>algorithm</u> for this Problem against the following adversary.

store an array of len. n containing only marks: $+$, $-$

	1	2	3		n
A	$\pm$	$\pm$	$\pm$	. . . . .	$\pm$

initially array contains $2n$ marks

at all times $A[i]$ is either $\pm$, $+$, $-$, N (no mark). meanings are

- $\pm$ candidate for both max, min
- $+$ " " max only
- $-$ " " min only
- N " " neither

The adversary's answer to each probe $A_i < A_j$ depends on the current markings of A_i & A_j

- If A_i, A_j both contain $\pm$, then randomly select one element to become $+$ and other to become $-$ only, answer accordingly. This removes 2 marks.
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--- see table in handout ---

let

c_0 = # comp. that remove no marks

c_1 = " " " " 1 mark

c_2 = " " " " 2 marks

Assume algorithm halts after fewer than $\lceil \frac{3n}{2} \rceil - 2$ comparisons, and gives an answer! (min, max)

Exercise: show $\lceil \frac{3n}{2} \rceil - 2 = 2n - 2 - \lfloor \frac{n}{2} \rfloor$.

Therefore

$$c_0 + c_1 + c_2 < 2n - 2 - \lfloor \tfrac{n}{2} \rfloor$$

$$\therefore \quad c_1 + 2c_2 < (2n-2) + (c_2 - \lfloor \tfrac{n}{2} \rfloor) - c_0$$

note: $c_0 \geq 0$ and $c_2 \leq \lfloor \frac{n}{2} \rfloor$

Thus

$$c_1 + 2c_2 < 2n - 2$$

so

$$\#\text{marks removed} = 0 \cdot c_0 + 1 \cdot c_1 + 2 \cdot c_2$$

$$= c_1 + 2c_2$$

and

$$\#\text{marks removed} < 2n - 2$$

$\therefore$ $\#\text{marks remaining} > 2n - (2n-2) = 2$

$\therefore$ $\#\text{marks remaining} \geq 3$

Thus there are either 2 +'s or 2 −'s remaining. Say there are 2 +'s remaining. Then the adversary can contradict the algorithm's answer for the maximum. Similarly for 2-'s.

Therefore no correct algorithm can do fewer than $\lceil \frac{3n}{2} \rceil - 2$ comparisons. ∎

Ex. $n = 5$, $\lceil \frac{3n}{2} \rceil - 2 = 6$

	1	2	3	4	5
A	* −	+ *	* *	* *	* −

#marks = 10

	comparison	answer	# marks
1.	$A_1 < A_2$	yes	8
2.	$A_3 < A_4$	no	6
3.	$A_5 < A_3$	yes	5
4.	$A_1 < A_4$	yes	4
5.	$A_2 < A_3$	no	3