A Course In Business Statistics

4th Edition

Chapter 9

Estimation and Hypothesis Testing for Two Population Parameters

Chapter Goals

After completing this chapter, you should be able to:

- Test hypotheses or form interval estimates for
  - two independent population means
    - Standard deviations known
    - Standard deviations unknown
  - two means from paired samples
  - the difference between two population proportions
  - Set up a contingency analysis table and perform a chi-square test of independence

Estimation for Two Populations

Estimating two population values

- Population means, independent samples
- Paired samples
- Population proportions

Examples:

- Group 1 vs. independent Group 2
- Same group before vs. after treatment
- Proportion 1 vs. Proportion 2

Difference Between Two Means

Population means, independent samples *

σ_1 and σ_2 known

Goal: Form a confidence interval for the difference between two population means, $\mu_1 - \mu_2$

The point estimate for the difference is

$$\bar{X}_1 - \bar{X}_2$$

Independent Samples

Population means, independent samples *

σ_1 and σ_2 known

- Different data sources
  - Unrelated
  - Independent
    - Sample selected from one population has no effect on the sample selected from the other population
- Use the difference between 2 sample means
- Use z test or pooled variance *t* test

σ_1 and σ_2 known

Population means, independent samples

σ_1 and σ_2 known *

Assumptions:

- Samples are randomly and independently drawn
- population distributions are normal or both sample sizes are ≥ 30
- Population standard deviations are known

σ_1 and σ_2 known

(continued)

Population means, independent samples

σ_1 and σ_2 known *

When σ_1 and σ_2 are known and both populations are normal or both sample sizes are at least 30, the test statistic is a z-value…

…and the standard error of $\bar{x}_1 - \bar{x}_2$ is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}}$$

σ_1 and σ_2 known

(continued)

Population means, independent samples

σ_1 and σ_2 known *

The confidence interval for $\mu_1 - \mu_2$ is:

$$\left(\bar{x}_1 - \bar{x}_2\right) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}}$$

σ_1 and σ_2 unknown

Population means, independent samples

- σ_1 and σ_2 known
- σ_1 and σ_2 unknown, *

Assumptions:

- populations are normally distributed
- the populations have equal variances
- samples are independent

σ_1 and σ_2 unknown

(continued)

Population means, independent samples

σ_1 and σ_2 known

σ_1 and σ_2 unknown, *

Forming interval estimates:

- The population variances are assumed equal, so use the two sample standard deviations and pool them to estimate σ
- the test statistic is a t value with $(n_1 + n_2 - 2)$ degrees of freedom

σ_1 and σ_2 unknown

(continued)

Population means, independent samples

- σ_1 and σ_2 known
- σ_1 and σ_2 unknown *

The pooled standard deviation is

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

σ_1 and σ_2 unknown

(continued)

Population means, independent samples

- σ_1 and σ_2 known
- σ_1 and σ_2 unknown *

The confidence interval for $\mu_1 - \mu_2$ is:

$$\left(\bar{x}_1 - \bar{x}_2\right) \pm t_{\alpha/2}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Where $t_{\alpha/2}$ has $(n_1 + n_2 - 2)$ d.f.,

and

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Paired Samples

Paired samples

Tests Means of 2 Related Populations

- Paired or matched samples
- Repeated measures (before/after)
- Use difference between paired values:

$$d = x_1 - x_2$$

- Eliminates Variation Among Subjects
- Assumptions:
  - Both Populations Are Normally Distributed
  - Or, if Not Normal, use large samples

Paired Differences

Paired samples

The i^{th} paired difference is d_i, where

$$d_i = x_{1i} - x_{2i}$$

The point estimate for the population mean paired difference is $\bar{d}$:

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n}$$

The sample standard deviation is

$$s_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n-1}}$$

n is the number of pairs in the paired sample

Paired Differences

(continued)

Paired samples

The confidence interval for $\bar{d}$ is

$$\bar{d} \pm t_{\alpha/2} \frac{s_d}{\sqrt{n}}$$

Where $t_{\alpha/2}$ has n - 1 d.f. and s_d is:

$$s_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}}$$

n is the number of pairs in the paired sample

Hypothesis Tests for the Difference Between Two Means

- Testing Hypotheses about $\mu_1 - \mu_2$

- Use the same situations discussed already:
  - Standard deviations known or unknown

Hypothesis Tests for Two Population Proportions

Two Population Means, Independent Samples

Lower tail test:	Upper tail test:	Two-tailed test:
$H_0: \mu_1 \geq \mu_2$ $H_A: \mu_1 < \mu_2$ i.e., $H_0: \mu_1 - \mu_2 \geq 0$ $H_A: \mu_1 - \mu_2 < 0$	$H_0: \mu_1 \leq \mu_2$ $H_A: \mu_1 > \mu_2$ i.e., $H_0: \mu_1 - \mu_2 \leq 0$ $H_A: \mu_1 - \mu_2 > 0$	$H_0: \mu_1 = \mu_2$ $H_A: \mu_1 \neq \mu_2$ i.e., $H_0: \mu_1 - \mu_2 = 0$ $H_A: \mu_1 - \mu_2 \neq 0$

Hypothesis tests for $\mu_1 - \mu_2$

Population means, independent samples

σ_1 and σ_2 known → Use a **z** test statistic

σ_1 and σ_2 unknown → Use s to estimate unknown σ, use a **t** test statistic and pooled standard deviation

σ_1 and σ_2 known

Population means, independent samples

- σ_1 and σ_2 known *
- σ_1 and σ_2 unknown

The test statistic for $\mu_1 - \mu_2$ is:

$$z = \frac{\left(\bar{x}_1 - \bar{x}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

σ_1 and σ_2 unknown

Population means, independent samples

σ_1 and σ_2 known

σ_1 and σ_2 unknown *

The test statistic for $\mu_1 - \mu_2$ is:

$$t = \frac{\left(\bar{x}_1 - \bar{x}_2\right) - \left(\mu_1 - \mu_2\right)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Where $t_{\alpha/2}$ has $(n_1 + n_2 - 2)$ d.f.,

and

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Hypothesis tests for $\mu_1 - \mu_2$

Two Population Means, Independent Samples

Lower tail test:

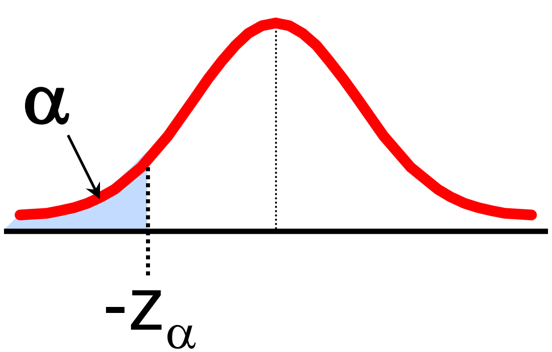
$H_0: \mu_1 - \mu_2 \geq 0$
$H_A: \mu_1 - \mu_2 < 0$

Upper tail test:

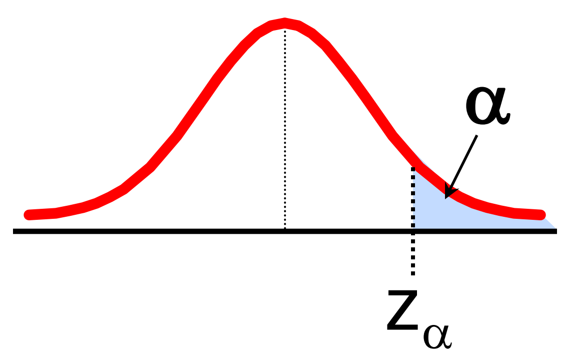
$H_0: \mu_1 - \mu_2 \leq 0$
$H_A: \mu_1 - \mu_2 > 0$

Two-tailed test:

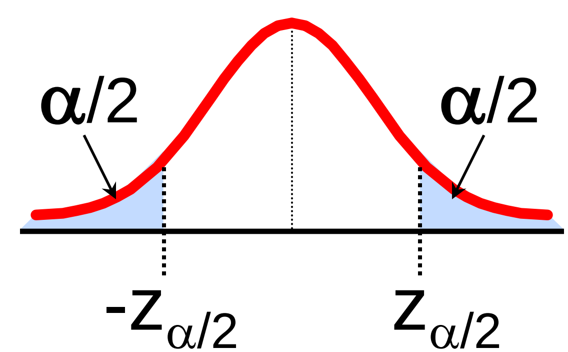
$H_0: \mu_1 - \mu_2 = 0$
$H_A: \mu_1 - \mu_2 \neq 0$



Reject H_0 if $z < -z_\alpha$



Reject H_0 if $z > z_\alpha$



Reject H_0 if $z < -z_{\alpha/2}$
or $z > z_{\alpha/2}$

Pooled t Test: Example

You're a financial analyst for a brokerage firm. Is there a difference in dividend yield between stocks listed on the NYSE & NASDAQ? You collect the following data:

	NYSE	NASDAQ
Number	**21**	**25**
Sample mean	**3.27**	**2.53**
Sample std dev	**1.30**	**1.16**

Assuming equal variances, is there a difference in average yield (α = **0.05**)?

Calculating the Test Statistic

The test statistic is:

$$t = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{\left(3.27 - 2.53\right) - 0}{1.2256\sqrt{\frac{1}{21} + \frac{1}{25}}} = \boxed{2.040}$$

$$s_p = \sqrt{\frac{\left(n_1 - 1\right)s_1^{\,2} + \left(n_2 - 1\right)s_2^{\,2}}{n_1 + n_2 - 2}} = \sqrt{\frac{\left(21 - 1\right)1.30^2 + \left(25 - 1\right)1.16^2}{21 + 25 - 2}} = 1.2256$$

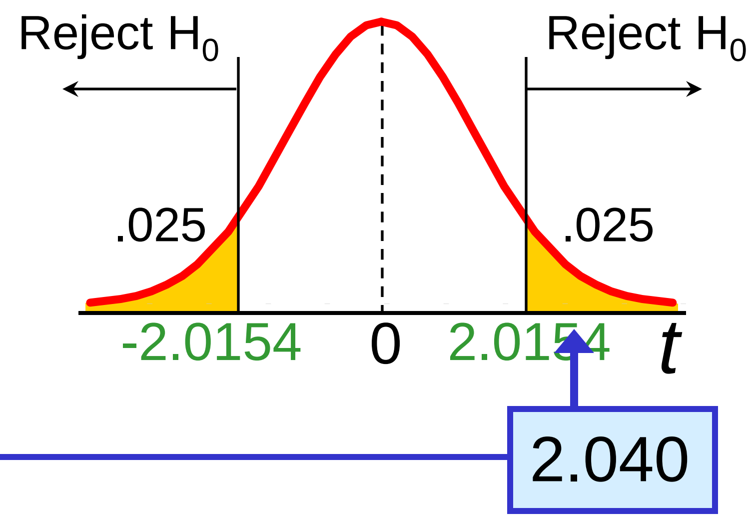
Solution

H_0: $\mu_1 - \mu_2 = 0$ i.e. ($\mu_1 = \mu_2$)
H_A: $\mu_1 - \mu_2 \neq 0$ i.e. ($\mu_1 \neq \mu_2$)

$\alpha = 0.05$

df = 21 + 25 - 2 = 44

Critical Values: t = ± 2.0154



Test Statistic:

$$t = \frac{3.27 - 2.53}{1.2256\sqrt{\frac{1}{21} + \frac{1}{25}}} = \boxed{2.040}$$

Decision:

Reject H_0 at $\alpha = 0.05$

Conclusion:

There is evidence of a difference in means.

Hypothesis Testing for Paired Samples

Paired samples

The test statistic for $\bar{d}$ is

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}$$

n is the number of pairs in the paired sample

Where $t_{\alpha/2}$ has n - 1 d.f. and s_d is:

$$s_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}}$$

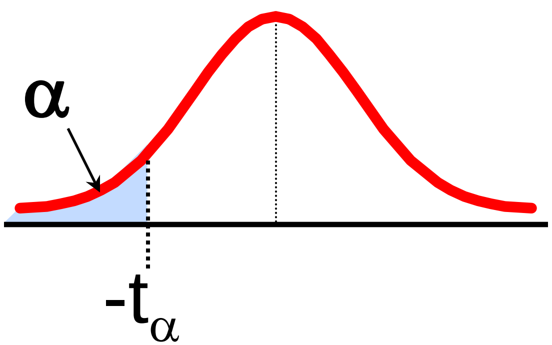
Hypothesis Testing for Paired Samples

(continued)

Paired Samples

Lower tail test:

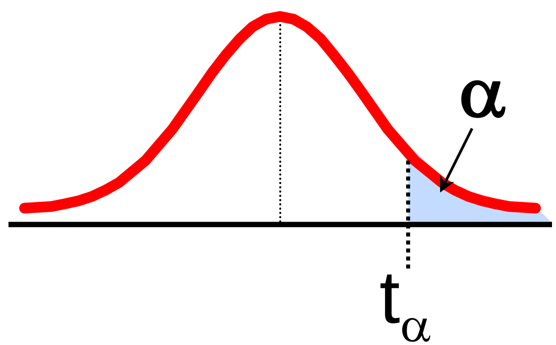
$H_0: \mu_d \geq 0$
$H_A: \mu_d < 0$



Reject H_0 if $t < -t_\alpha$

Upper tail test:

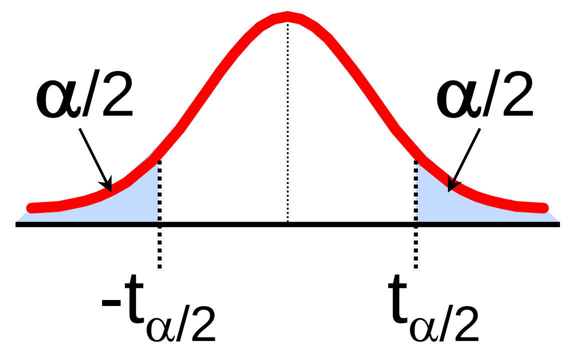
$H_0: \mu_d \leq 0$
$H_A: \mu_d > 0$



Reject H_0 if $t > t_\alpha$

Two-tailed test:

$H_0: \mu_d = 0$
$H_A: \mu_d \neq 0$



Reject H_0 if $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

Where t has $n - 1$ d.f.

Paired Samples Example

- Assume you send your salespeople to a "customer service" training workshop. Is the training effective? You collect the following data:

Salesperson	Number of Complaints: Before (1)	After (2)	(2) - (1) Difference, d_i
C.B.	6	4	- 2
T.F.	20	6	-14
M.H.	3	2	- 1
R.K.	0	0	0
M.O.	4	0	- 4
			-21

$$\bar{d} = \frac{\sum d_i}{n}$$

$$= -4.2$$

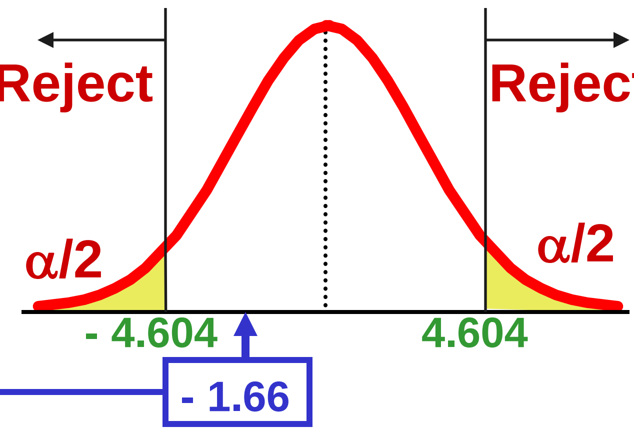
$$s_d = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n-1}}$$

$$= 5.67$$

Paired Samples: Solution

- Has the training made a difference in the number of complaints (at the 0.01 level)?

$$H_0:\ \mu_d = 0$$
$$H_A:\ \mu_d \neq 0$$



$\alpha = .01 \qquad \bar{d} = -4.2$

Critical Value = ± 4.604

d.f. = n - 1 = 4

Test Statistic:

$$t = \frac{\bar{d} - \mu_d}{s_d/\sqrt{n}} = \frac{-4.2 - 0}{5.67/\sqrt{5}} = \boxed{-1.66}$$

Decision: Do not reject *H*$_0$

(t stat is not in the reject region)

Conclusion: There is not a significant change in the number of complaints.

Two Population Proportions

Population proportions

Goal: Form a confidence interval for or test a hypothesis about the difference between two population proportions, $\boxed{p_1 - p_2}$

Assumptions:

$n_1 p_1 \geq 5$, $n_1(1-p_1) \geq 5$

$n_2 p_2 \geq 5$, $n_2(1-p_2) \geq 5$

The point estimate for the difference is $\bar{p}_1 - \bar{p}_2$

Confidence Interval for Two Population Proportions

Population proportions

The confidence interval for $p_1 - p_2$ is:

$$\left(\bar{p}_1 - \bar{p}_2\right) \pm z_{\alpha/2} \sqrt{\frac{\bar{p}_1(1-\bar{p}_1)}{n_1} + \frac{\bar{p}_2(1-\bar{p}_2)}{n_2}}$$

Hypothesis Tests for Two Population Proportions

Population proportions

Lower tail test:	Upper tail test:	Two-tailed test:
H_0: $p_1 \geq p_2$ H_A: $p_1 < p_2$	H_0: $p_1 \leq p_2$ H_A: $p_1 > p_2$	H_0: $p_1 = p_2$ H_A: $p_1 \neq p_2$
i.e.,	i.e.,	i.e.,
H_0: $p_1 - p_2 \geq 0$ H_A: $p_1 - p_2 < 0$	H_0: $p_1 - p_2 \leq 0$ H_A: $p_1 - p_2 > 0$	H_0: $p_1 - p_2 = 0$ H_A: $p_1 - p_2 \neq 0$

Two Population Proportions

Population proportions

Since we begin by assuming the null hypothesis is true, we assume $p_1 = p_2$ and pool the two $\bar{p}$ estimates

The pooled estimate for the overall proportion is:

$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$$

where x_1 and x_2 are the numbers from samples 1 and 2 with the characteristic of interest

Two Population Proportions

(continued)

Population proportions

The test statistic for

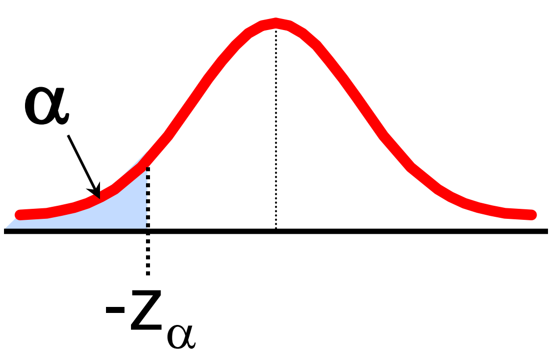
$p_1 - p_2$ is:

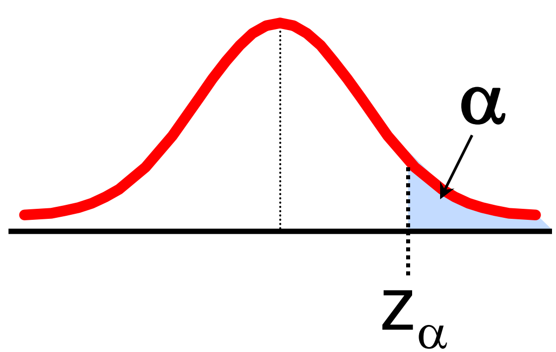
$$z = \frac{\left(\bar{p}_1 - \bar{p}_2\right) - \left(p_1 - p_2\right)}{\sqrt{\bar{p}\left(1 - \bar{p}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

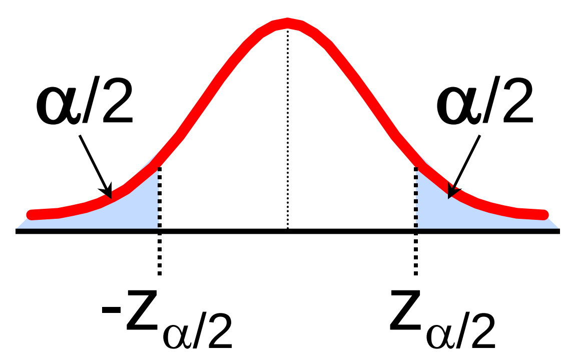
Hypothesis Tests for Two Population Proportions

Population proportions

Lower tail test:	Upper tail test:	Two-tailed test:
$H_0: p_1 - p_2 \geq 0$ $H_A: p_1 - p_2 < 0$	$H_0: p_1 - p_2 \leq 0$ $H_A: p_1 - p_2 > 0$	$H_0: p_1 - p_2 = 0$ $H_A: p_1 - p_2 \neq 0$







Reject H_0 if $z < -z_\alpha$

Reject H_0 if $z > z_\alpha$

Reject H_0 if $z < -z_{\alpha/2}$
or $z > z_{\alpha/2}$

Example: Two population Proportions

Is there a significant difference between the proportion of men and the proportion of women who will vote Yes on Proposition A?

- In a random sample, 36 of 72 men and 31 of 50 women indicated they would vote Yes
- Test at the .05 level of significance

Example: Two population Proportions

(continued)

- The hypothesis test is:

H_0: $p_1 – p_2 = 0$ (the two proportions are equal)
H_A: $p_1 – p_2 \neq 0$ (there is a significant difference between proportions)

- The sample proportions are:
  - Men: $\bar{p}_1 = 36/72 = .50$
  - Women: $\bar{p}_2 = 31/50 = .62$

- The pooled estimate for the overall proportion is:

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{36 + 31}{72 + 50} = \frac{67}{122} = .549$$

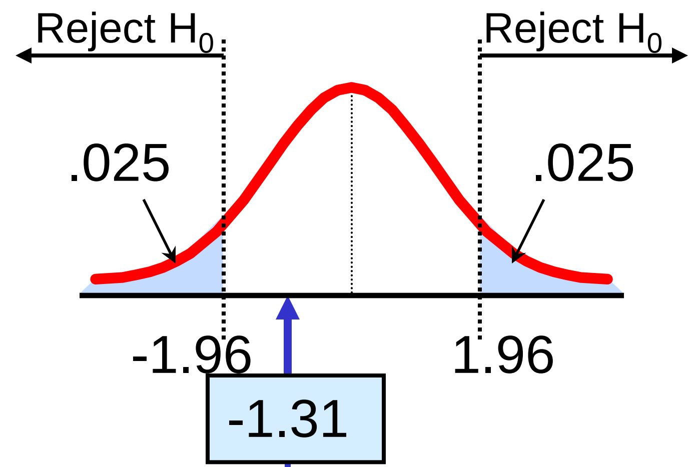
Example: Two population Proportions

(continued)

The test statistic for $p_1 - p_2$ is:

$$z = \frac{\left(\bar{p}_1 - \bar{p}_2\right) - \left(p_1 - p_2\right)}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{(.50 - .62) - (0)}{\sqrt{.549(1 - .549)\left(\frac{1}{72} + \frac{1}{50}\right)}} = \boxed{-1.31}$$



Critical Values = ±1.96
For α = .05

Decision: Do not reject H_0

Conclusion: There is not significant evidence of a difference in proportions who will vote yes between men and women.

Two Sample Tests in EXCEL

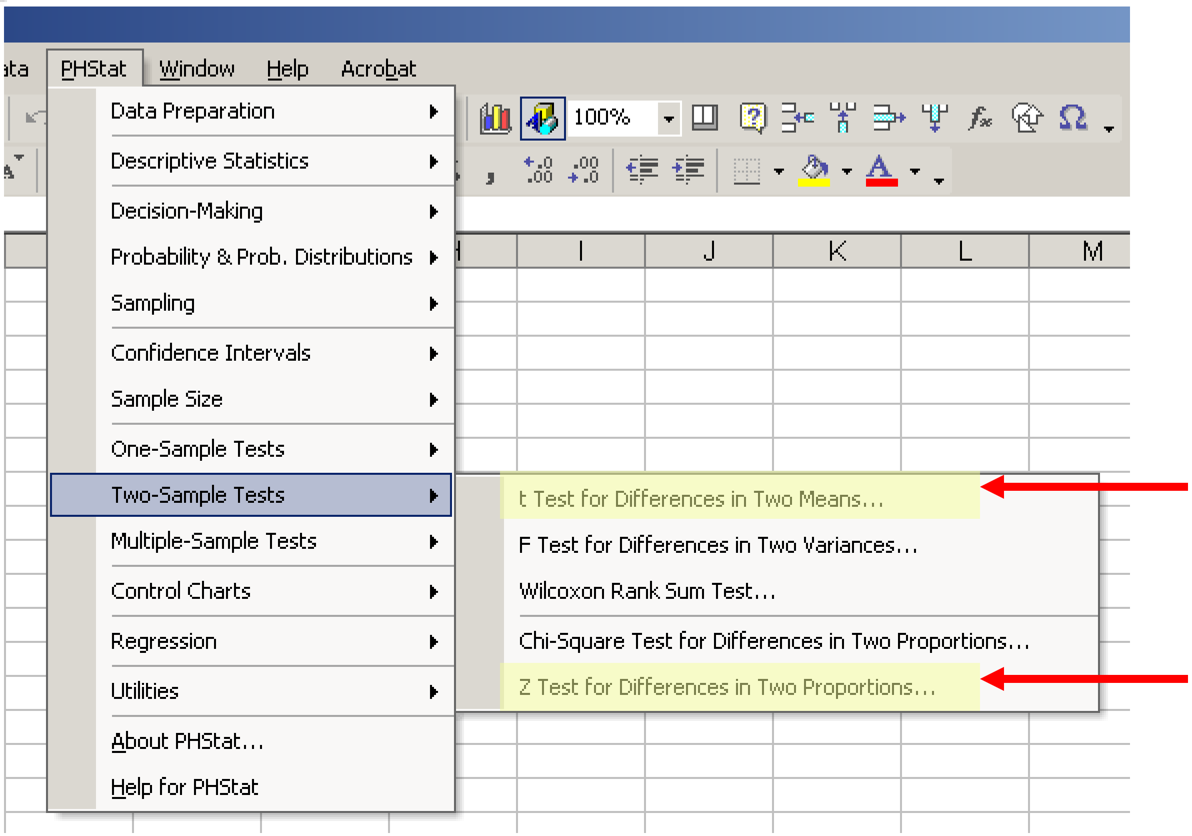
For independent samples:

- Independent sample Z test with variances known:
  - Tools | data analysis | z-test: two sample for means
- Independent sample Z test with large sample
  - Tools | data analysis | z-test: two sample for means
  - If the population variances are unknown, use sample variances

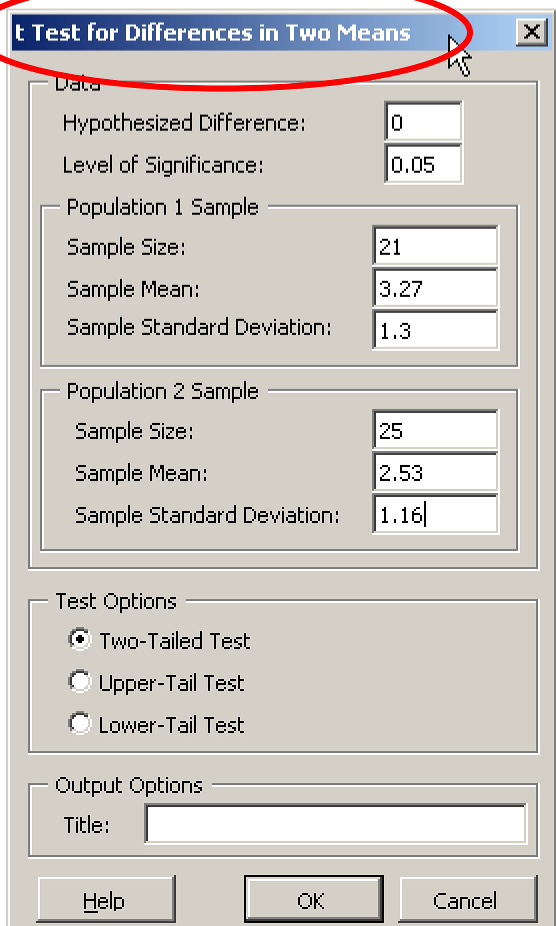
For paired samples (t test):

- Tools | data analysis… | t-test: paired two sample for means

Two Sample Tests in PHStat



Two Sample Tests in PHStat

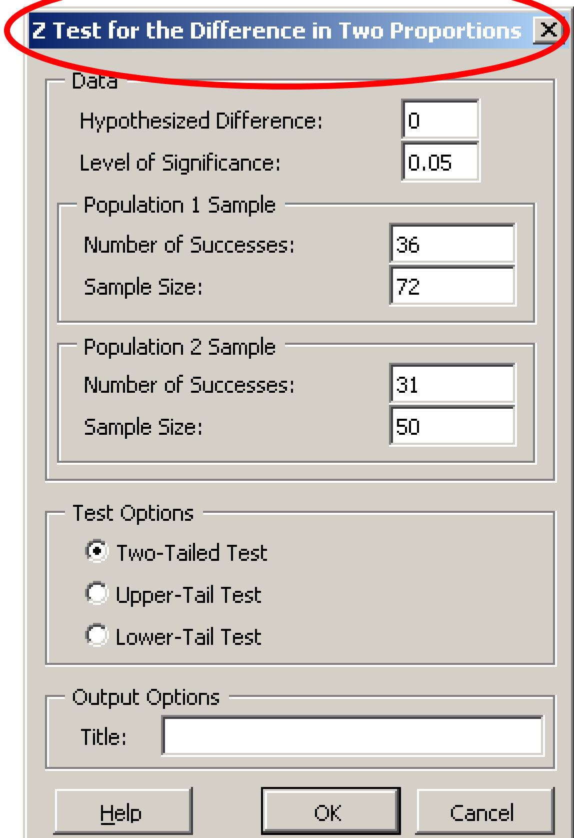


	A	B
1	**t Test for Differences in Two Means**	
2		
3	**Data**	
4	**Hypothesized Difference**	**0**
5	**Level of Significance**	**0.05**
6	**Population 1 Sample**	
7	**Sample Size**	**21**
8	**Sample Mean**	**3.27**
9	**Sample Standard Deviation**	**1.3**
10	**Population 2 Sample**	
11	**Sample Size**	**25**
12	**Sample Mean**	**2.53**
13	**Sample Standard Deviation**	**1.16**
14		
15	Intermediate Calculations	
16	Population 1 Sample Degrees of Freedom	20
17	Population 2 Sample Degrees of Freedom	24
18	Total Degrees of Freedom	44
19	Pooled Variance	1.502145
20	Difference in Sample Means	0.74
21	*t*-Test Statistic	2.039748
22		
23	**Two-Tailed Test**	
24	**Lower Critical Value**	**-2.01537**
25	**Upper Critical Value**	**2.015367**
26	***p*-Value**	**0.047407**
27	**Reject the null hypothesis**	

Input

Output

Two Sample Tests in PHStat



	A	B
1	Z Test for Differences in Two Proportions	
2		
3	**Data**	
4	**Hypothesized Difference**	**0**
5	**Level of Significance**	**0.05**
6	**Group 1**	
7	**Number of Successes**	**36**
8	**Sample Size**	**72**
9	**Group 2**	
10	**Number of Successes**	**31**
11	**Sample Size**	**50**
12		
13	Intermediate Calculations	
14	Group 1 Proportion	0.5
15	Group 2 Proportion	0.62
16	Difference in Two Proportions	-0.12
17	Average Proportion	0.549180328
18	Z Test Statistic	-1.310067478
19		
20	**Two-Tailed Test**	
21	**Lower Critical Value**	**-1.959962787**
22	**Upper Critical Value**	**1.959962787**
23	***p*-Value**	**0.190173138**
24	**Do not reject the null hypothesis**	

Input

Output

Contingency Tables

Contingency Tables

- Situations involving multiple population proportions
- Used to classify sample observations according to two or more characteristics
- Also called a crosstabulation table.

Contingency Table Example

Left-Handed vs. Gender

- Dominant Hand: Left vs. Right
- Gender: Male vs. Female

H_0: Hand preference is independent of gender

H_A: Hand preference is not independent of gender

Contingency Table Example

(continued)

Sample results organized in a contingency table:

sample size = n = 300:

120 Females, 12 were left handed

180 Males, 24 were left handed

Gender	Hand Preference		
	Left	Right	
Female	12	108	120
Male	24	156	180
	36	264	300

Logic of the Test

H_0: Hand preference is independent of gender

H_A: Hand preference is not independent of gender

- If H_0 is true, then the proportion of left-handed females should be the same as the proportion of left-handed males
- The two proportions above should be the same as the proportion of left-handed people overall

Finding Expected Frequencies

120 Females, 12 were left handed

180 Males, 24 were left handed

Overall:

P(Left Handed) = 36/300 = .12

If independent, then

P(Left Handed | Female) = P(Left Handed | Male) = .12

So we would expect 12% of the 120 females and 12% of the 180 males to be left handed…

i.e., we would expect (120)(.12) = 14.4 females to be left handed
(180)(.12) = 21.6 males to be left handed

Expected Cell Frequencies

(continued)

- Expected cell frequencies:

$$e_{ij} = \frac{(i^{th} \text{ Row total})(j^{th} \text{ Column total})}{\text{Total sample size}}$$

Example:

$$e_{11} = \frac{(120)(36)}{300} = 14.4$$

Observed v. Expected Frequencies

Observed frequencies vs. expected frequencies:

Gender	Hand Preference		
	Left	Right	
Female	Observed = 12 Expected = 14.4	Observed = 108 Expected = 105.6	120
Male	Observed = 24 Expected = 21.6	Observed = 156 Expected = 158.4	180
	36	264	300

The Chi-Square Test Statistic

The Chi-square contingency test statistic is:

$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(o_{ij} - e_{ij})^2}{e_{ij}}$$

with $\quad d.f. = (r-1)(c-1)$

- where:

  o_{ij} = observed frequency in cell (i, j)

  e_{ij} = expected frequency in cell (i, j)

  r = number of rows

  c = number of columns

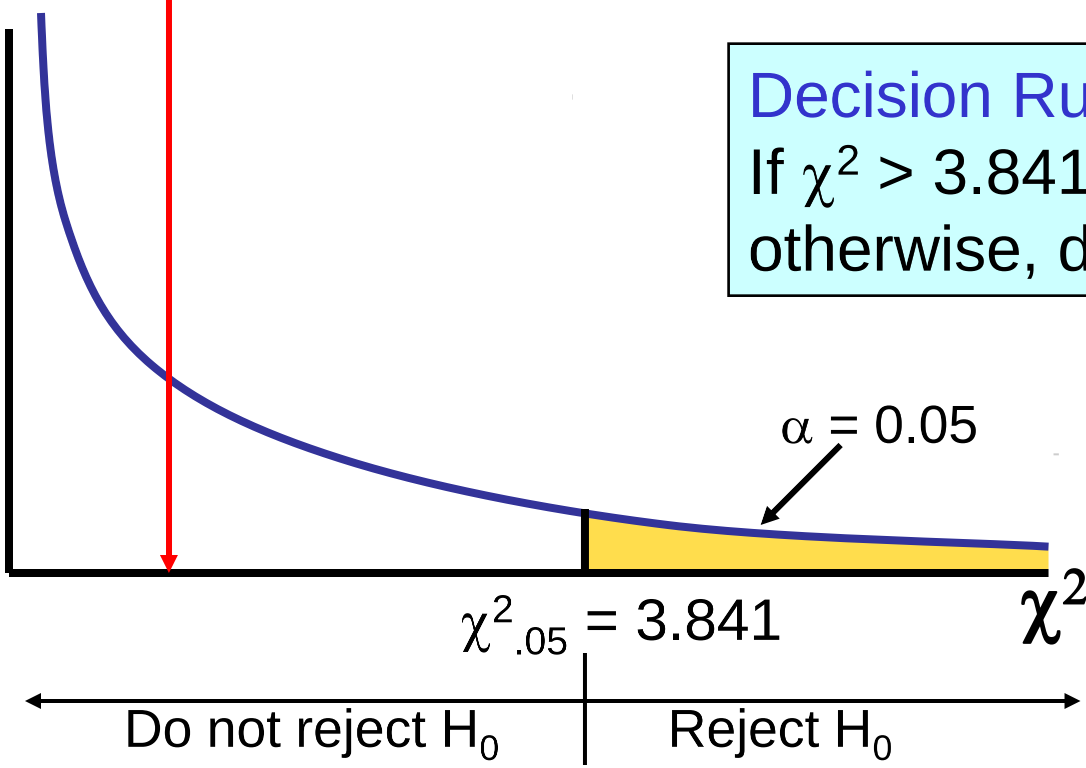
Observed v. Expected Frequencies

Gender	Hand Preference: Left	Hand Preference: Right	
Female	Observed = 12 Expected = 14.4	Observed = 108 Expected = 105.6	120
Male	Observed = 24 Expected = 21.6	Observed = 156 Expected = 158.4	180
	36	264	300

$$\chi^2 = \frac{(12-14.4)^2}{14.4} + \frac{(108-105.6)^2}{105.6} + \frac{(24-21.6)^2}{21.6} + \frac{(156-158.4)^2}{158.4} = 0.6848$$

Contingency Analysis

$$\chi^2 = \boxed{0.6848} \quad \text{with} \quad \text{d.f.} = (r-1)(c-1) = (1)(1) = 1$$



Decision Rule:
If $\chi^2 > 3.841$, reject H_0, otherwise, do not reject H_0

Here, $\chi^2 = 0.6848 < 3.841$, so we **do not reject H_0** and conclude that gender and hand preference are independent

Chapter Summary

- Used the chi-square goodness-of-fit test to determine whether data fits a specified distribution
  - Example of a discrete distribution (uniform)
  - Example of a continuous distribution (normal)
- Used contingency tables to perform a chi-square test of independence
  - Compared observed cell frequencies to expected cell frequencies

Chapter Summary

- Compared two independent samples
  - Formed confidence intervals for the differences between two means
  - Performed *Z* test for the differences in two means
  - Performed *t* test for the differences in two means
- Compared two related samples (paired samples)
  - Formed confidence intervals for the paired difference
  - Performed paired sample *t* tests for the mean difference
- Compared two population proportions
  - Formed confidence intervals for the difference between two population proportions
  - Performed Z-test for two population proportions
- Used contingency tables to perform a chi-square test of independence