A Course In Business Statistics

4th Edition

Chapter 4

Using Probability and Probability Distributions

Important Terms

- Probability – the chance that an uncertain event will occur (always between 0 and 1)
- Experiment – a process of obtaining outcomes for uncertain events
- Elementary Event – the most basic outcome possible from a simple experiment
- Sample Space – the collection of all possible elementary outcomes

Examples

- Example 1:
  - Experiment: Tossing a coin (fair-coin) to observe what face turns up
  - Experimental outcomes: HEAD, TAIL
  - Probability of getting a HEAD is 0.5
- Example 2:
  - Experiment: Rolling a die (fair-die) to observe which number occurs
  - Experimental outcomes: 1,2,3,4,5,6
  - Probability of getting 3 is 1/6

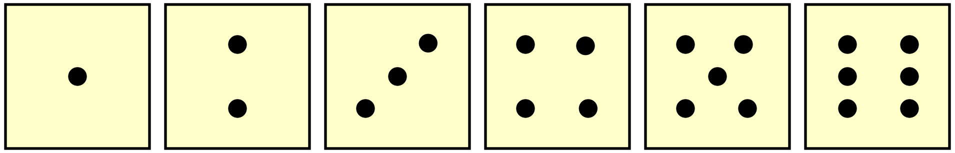
Examples

- Example 3:
  - Experiment: Drawing a card from a deck (well-shuffled French deck)
  - Experimental outcomes: ♣A, ♣ 2, ..., ♣ K, ♦A, ♦2,..., ♦K, ♥A, ♥2,..., ♥K, ♠A, ♠2,..., ♠K.
  - Probability of getting an ACE (could be a club ♣, diamond ♦, heart ♥ and a spade ♠, among 4 of the French suits) is 4/52.

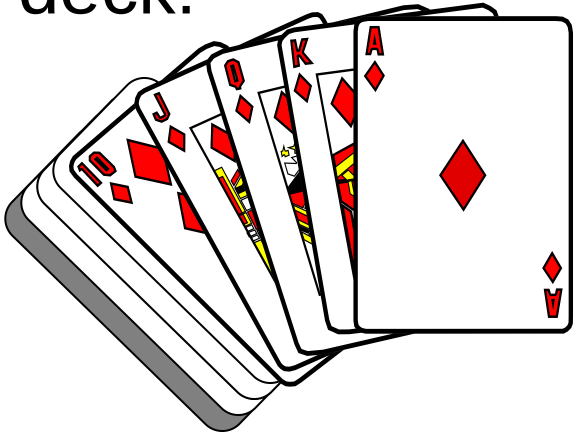
Sample Space

The Sample Space is the collection of all possible outcomes of an experiment

e.g. All 6 faces of a die:



e.g. All 52 cards of a bridge deck:



Events

- Elementary event – An outcome from a sample space with one characteristic
  - Example: A red card from a deck of cards
- Event – May involve two or more outcomes simultaneously
  - Example: An ace that is also red from a deck of cards

- Example 1:
  - Experiment: Toss a coin
  - Sample space: S={H, T}
  - Event: E={H} (looking for a head)
- Example 2:
  - Experiment: Roll a die
  - Sample space: S={1, 2, 3, 4, 5, 6}
  - Event: E={6} (Looking for a 6)
- Example 3:
  - Experiment: Draw a card from a 52 deck
  - Sample space: S={♣A, ♣ 2, ..., ♣ K, ♦A, ♦2,..., ♦K, ♥A, ♥2,..., ♥K, ♠A, ♠2,..., ♠K}
  - Event: E={♣A, ♦A, ♥A, ♠A} (Looking for an ACE)

Assigning Probability

- Classical Probability Assessment

$$P(E_i) = \frac{\text{Number of ways } E_i \text{ can occur}}{\text{Total number of elementary events}}$$

- Relative Frequency of Occurrence

$$\text{Relative Freq. of } E_i = \frac{\text{Number of times } E_i \text{ occurs}}{N}$$

- Subjective Probability Assessment

An opinion or judgment by a decision maker about the likelihood of an event

- For example, to estimate the probability that a randomly selected consumer prefers a smart phone to all other phones, we perform an experiment in which we ask a randomly selected consumer for his or her preference.
- There are two possible experimental outcomes: “prefers a smart phone” and “does not prefer a smart phone.”
- However, we have no reason to believe that these experimental outcomes are equally likely, so we cannot use the classical method. We might perform the experiment, say, 1,000 times by surveying 1,000 randomly selected consumers.
- Then, if 140 of those surveyed said that they prefer smart phones, we would estimate the probability that a randomly selected consumer prefers smart phones to all other soft drinks to be 140/1,000=0.14. This is called the ***relative frequency method*** for assigning probability.

- If we cannot perform the experiment many times, we might estimate the probability by using our previous experience with similar situations, intuition, or special expertise that we may possess.

- For example, a company president might estimate the probability of success for a onetime business venture to be 0.7. Here, on the basis of knowledge of the success of previous similar ventures, the opinions of company personnel, and other pertinent information, the president believes that there is a 70 percent chance the venture will be successful. When we use experience, intuitive judgement, or expertise to assess a probability, we call this a **subjective probability.** Such a probability may or may not have a relative frequency interpretation.

Rules of Probability

Rules for Possible Values and Sum

- **Individual Values**
  - $0 \le P(e_i) \le 1$
  - For any event e_i
- **Sum of All Values**
  - $$\sum_{i=1}^{k} P(e_i) = 1$$

where:

k = Number of elementary events in the sample space

e_i = i^{th} elementary event

- If an event never occurs, then the probability of this event equals 0.
  - The probability that I will be teaching in Yale University is 0.
- If an event is certain to occur, then the probability of this event equals 1.
  - The probability that you will graduate from IU School of Business is equal to 1, hopefully ☺

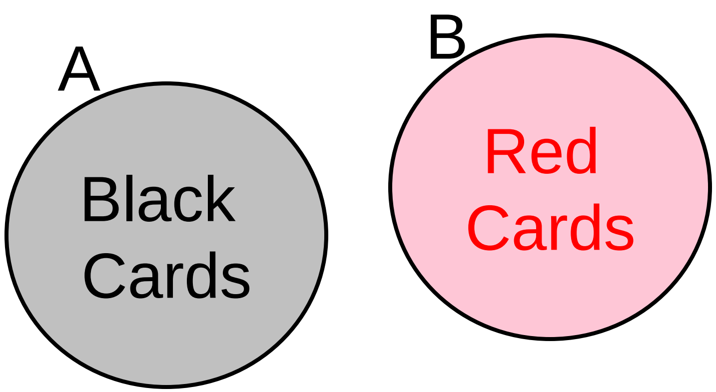
Probability Concepts

Events

- Mutually Exclusive Events
  - No commonality
- Independent Events
  - The occurance of one event **does not affect** the occurance of the other
- Dependent Events
  - The occurance of one event **affects** the occurance of the other

Probability Concepts

- **Mutually Exclusive Events**
  - If event A occurs, then event B cannot occur
  - A and B have no common elements



A card cannot be Black and Red at the same time.

Probability Concepts

- **Independent and Dependent Events**
  - Independent: Occurrence of one does not influence the probability of occurrence of the other
  - Dependent: Occurrence of one affects the probability of the other

Independent vs. Dependent Events

- **Independent Events**

A = heads on one flip of fair coin

B = heads on second flip of same coin

Result of second flip does not depend on the result of the first flip.

- **Dependent Events**

A = rain forecasted on the news

B = take umbrella to work

Probability of the second event is affected by the occurrence of the first event

Lottery Example

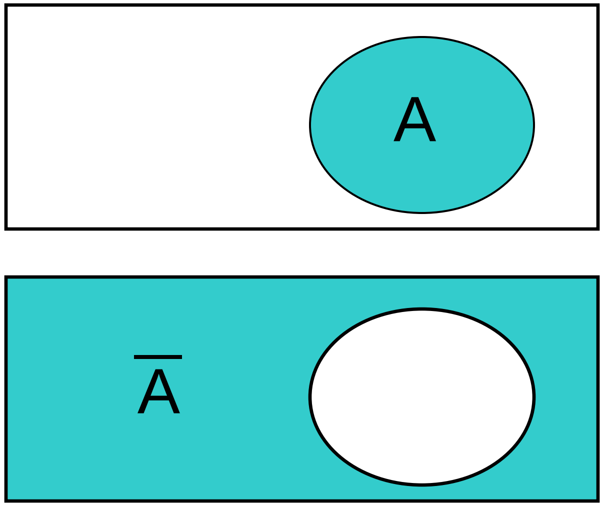
- Claims: Certain numbers appear very often and one after another!!!
- Is one week result of a lottery is dependent on the other week's result?
- Does the balls have a memory?

Probability Rules

1. Complement Rule
2. Addition Rule
3. Multiplication Rule

1. Complement Rule

- The complement of an event A is the collection of all possible elementary events **not** contained in event A. The complement of event A is represented by $\bar{A}$.



- Complement Rule:

$$P(A) = 1 - P(\bar{A})$$

Or, $$P(A) + P(\bar{A}) = 1$$

The Intersection and Union of 2 Events

- Given 2 events A and B,
  - The **intersection of *A* and *B*** is the event consisting of the sample space outcomes belonging to both *A* and *B*. The intersection is denoted by ∩. Furthermore, ∩ denotes **the probability that *both A and B will simultaneously occur.***
  - The **union of *A* and *B*** is the event consisting of the sample space outcomes belonging to *A* or *B* (or both). The union is denoted U. Furthermore, U denotes **the probability that *A or B (or both) will occur.***

2. Addition Rule for Elementary Events

- The probability of an event A_i is equal to the sum of the probabilities of the elementary events forming A_i.

- That is, if:

$$\boxed{A_i = \{a_1, a_2, a_3\}}$$

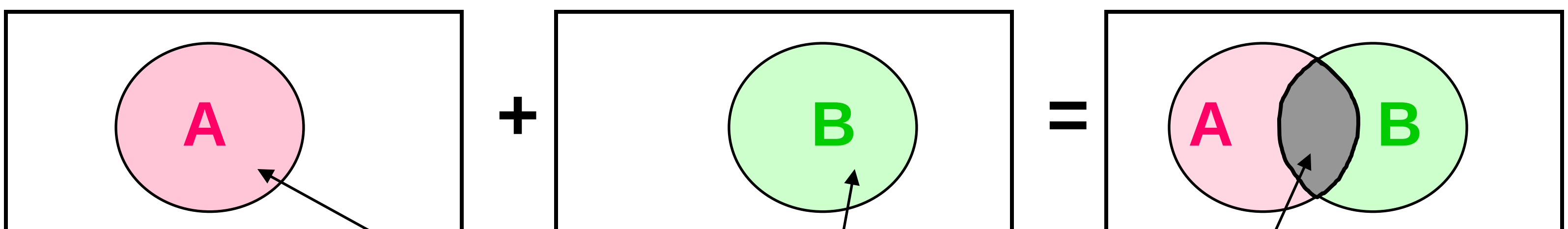
then:

$$\boxed{P(A_i) = P(a_1) + P(a_2) + P(a_3)}$$

Addition Rule for Two Events (one or the other outcome happens)

- **Addition Rule:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Don’t count common elements twice!

Addition Rule Example

P(Red or Ace) = P(Red) +P(Ace) - P(Red and Ace)

= 26/52 + 4/52 - $\boxed{2/52}$ = 28/52

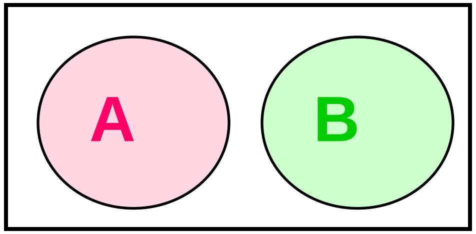
Don't count the two red aces twice!

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Addition Rule for Mutually Exclusive Events

- If A and B are mutually exclusive, then

$$P(A \cap B) = 0$$



So

$$\mathbf{P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{= 0 \text{ if mutually exclusive}}}$$

$$\mathbf{= P(A) + P(B)}$$

3. Conditional Probability

- The **probability of the event , given the condition that the event *B has occurred,*** is written as ***P*(*A*|*B*)**—pronounced "the probability of *A* given *B*." We often refer to such a probability as the **conditional probability of *A* given *B*.** Conditional probability for any two events A and B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ the probability of A given B

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$ the probability of B given A

Conditional Probability Example

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

- What is the probability that a car has a CD player, given that it has AC ?

i.e., we want to find $P(CD \mid AC)$

Conditional Probability Example

(continued)

- Of the cars on a used car lot, **70%** have air conditioning (AC) and **40%** have a CD player (CD). **20%** of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \text{ and } AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Conditional Probability Example

(continued)

- Given AC, we only consider the top row (70% of the cars). Of these, 20% have a CD player. 20% of 70% is about 28.57%.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \text{ and } AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

For Independent Events:

- Conditional probability for independent events A, B:

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = P(B)$$

4. Multiplication Rules (happening at the same time)

- Multiplication rule for two events A and B:

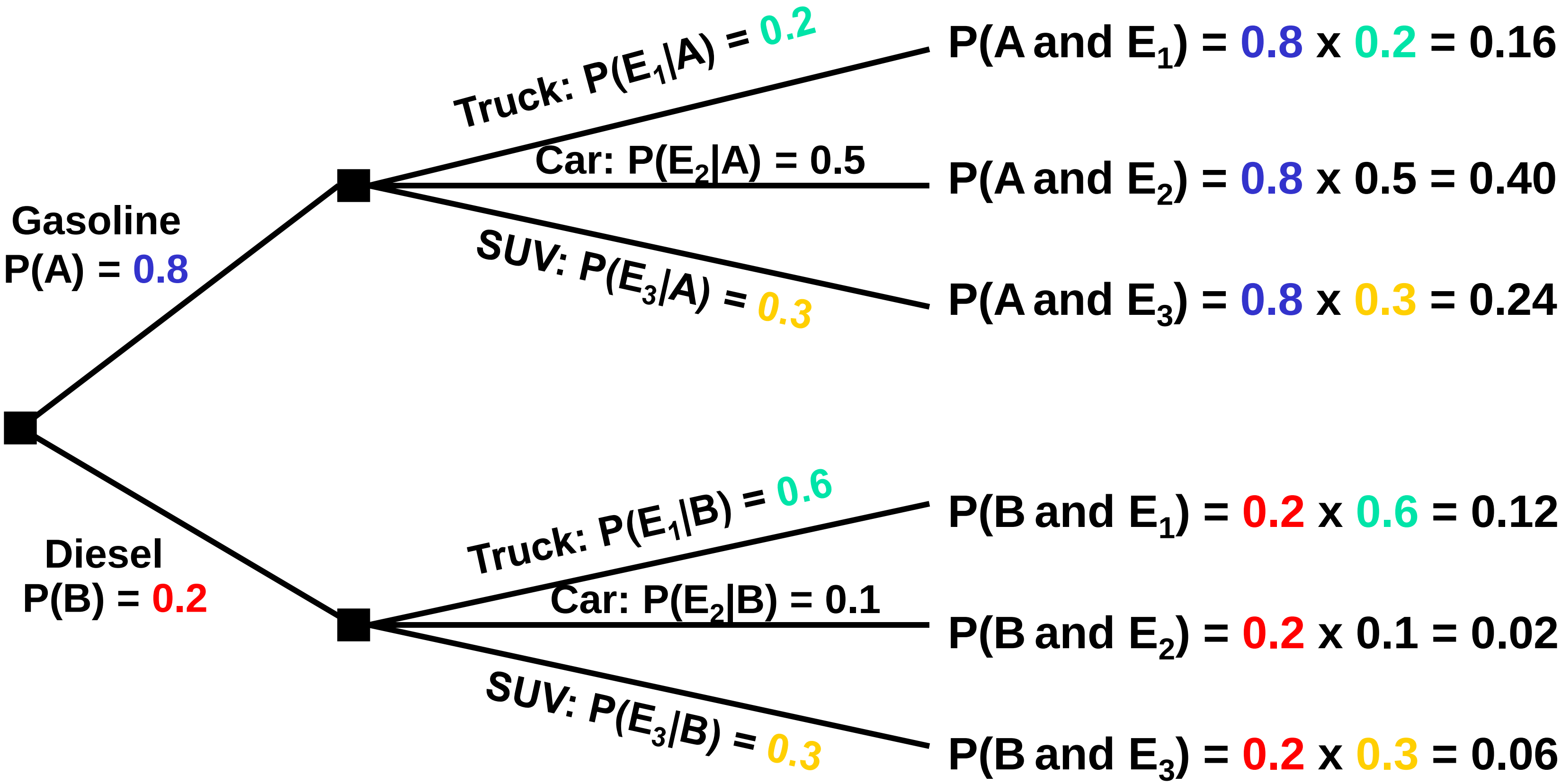
$$P(A \cap B) = P(A|B) \, . \, P(B)$$

$$= P(B|A) \, . \, P(A)$$

Note: If A and B are independent, then $P(A|B) = P(A)$
and the multiplication rule simplifies to $P(B|A) = P(B)$

$$P(A \cap B) = P(A) \, . \, P(B)$$

Tree Diagram Example



5. Bayes' Theorem

$$P(E_i|B) = \frac{P(E_i \cap B)}{P(B)} = \frac{P(B|E_i).P(E_i)}{P(B)}$$

$$P(B) = P(E_1 \cap B) + (E_2 \cap B) \cdots + P(E_k \cap B)$$

$$P(B) = P(B|E_1).P(E_1) + P(B|E_2).P(E_2) \cdots + P(B|E_k).P(E_k)$$

$$\boxed{P(E_i \mid B) = \frac{P(E_i)P(B \mid E_i)}{P(E_1)P(B \mid E_1) + P(E_2)P(B \mid E_2) + \ldots + P(E_k)P(B \mid E_k)}}$$

- where:

  E_i = i^{th} event of interest of the k possible events

  B = new event that might impact $P(E_i)$

  Events E_1 to E_k are mutually exclusive and collectively exhaustive

Example

- Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

- Event A_1. It rains on Marie's wedding.
- Event A_2. It does not rain on Marie's wedding.
- Event B. The weatherman predicts rain correctly.

Source: http://stattrek.com/probability/bayes-theorem.aspx

Solution

In terms of probabilities, we know the following:

- $P(A_1) = 5/365 = 0.0136985$ [It rains 5 days out of the year.]
- $P(A_2) = 360/365 = 0.9863014$ [It does not rain 360 days out of the year.]
- $P(B \mid A_1) = 0.9$ [When it rains, the weatherman predicts rain 90% of the time.]
- $P(B \mid A_2) = 0.1$ [When it does not rain, the weatherman predicts rain 10% of the time.]

Solution

We want to know $P(A_1 \mid B)$, the probability it will rain on the day of Marie's wedding, given a forecast for rain by the weatherman. The answer can be determined from Bayes' theorem, as shown below.

$$P(A_1 \mid B) = \quad P(A_1)\,P(B \mid A_1) \,/\, P(A_1)\,P(B \mid A_1) + P(A_2)\,P(B \mid A_2)$$

$$P(A_1 \mid B) = \quad (0.014)(0.9) \,/\, [\,(0.014)(0.9) + (0.986)(0.1)\,]$$

$$P(A_1 \mid B) = \quad 0.111$$

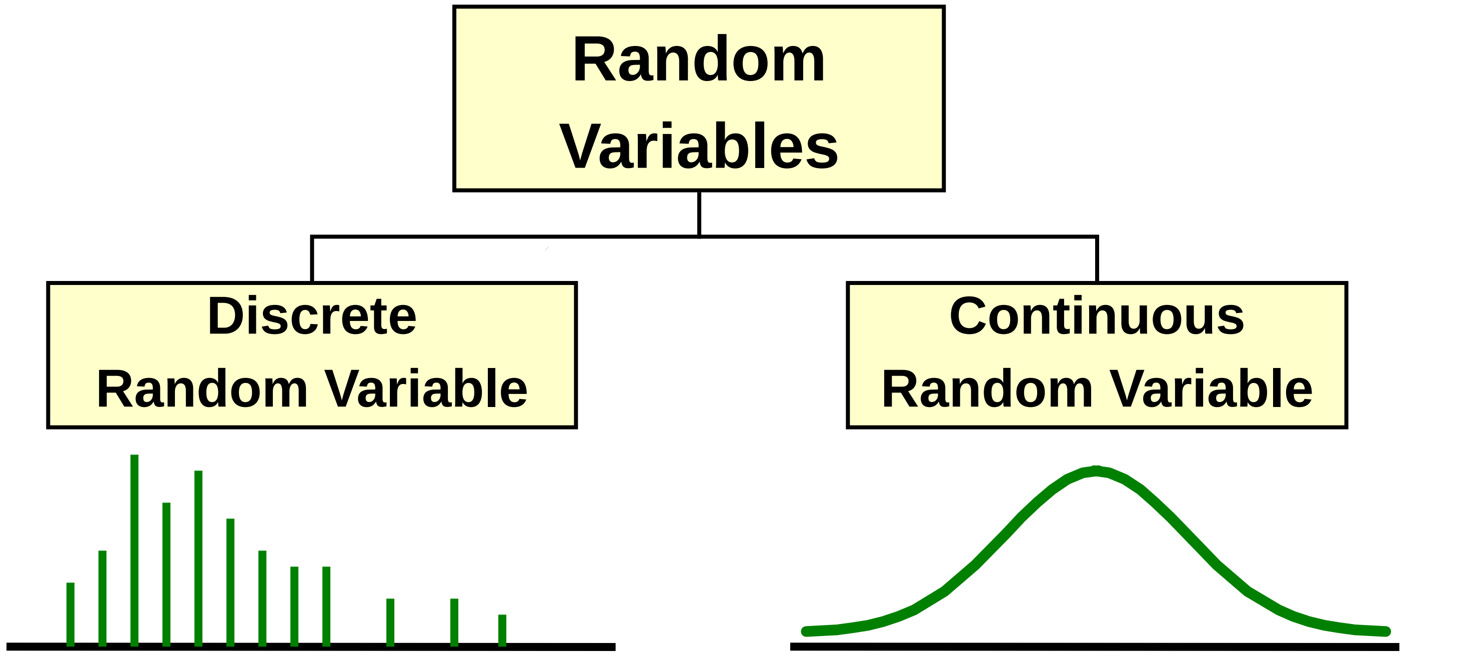
Note the somewhat unintuitive result. Even when the weatherman predicts rain, it only rains only about 11% of the time. Despite the weatherman's gloomy prediction, there is a good chance that Marie will not get rained on at her wedding.

Probability Distributions

- Probability distribution of
  - discrete random variables
  - continuous random variables

Introduction to Probability Distributions

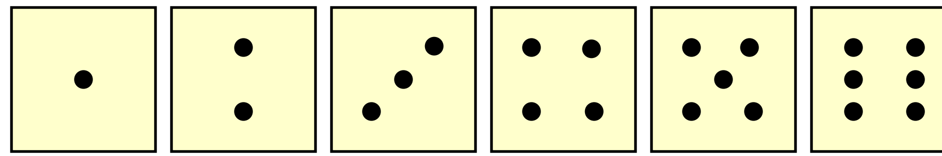
- **Random Variable**
  - Represents a possible numerical value from a random event



Discrete Random Variables

- Can only assume a countable number of values

Examples:



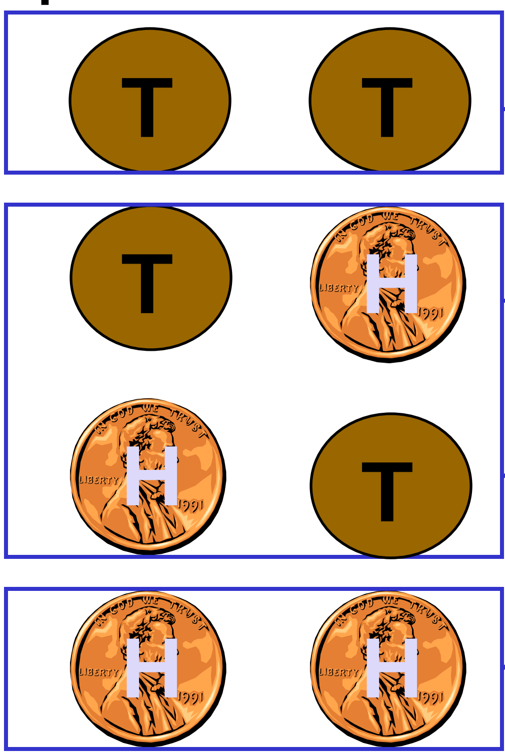
- **Roll a die twice**
  Let x be the number of times 4 comes up
  (then x could be 0, 1, or 2 times)

- **Toss a coin 5 times.**
  Let x be the number of heads
  (then x = 0, 1, 2, 3, 4, or 5)

Discrete Probability Distribution

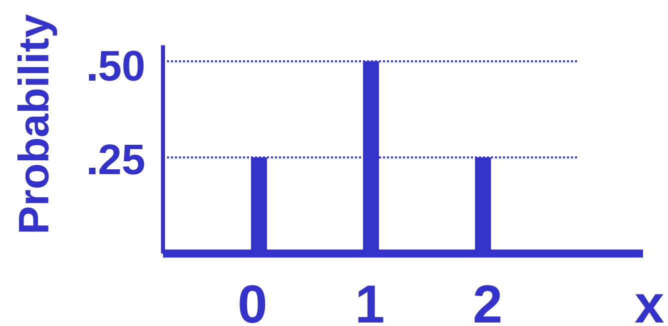
Experiment: Toss 2 Coins. Let x = # heads.

4 possible outcomes



Probability Distribution

x Value	Probability
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25



Discrete Probability Distribution

- A list of all possible [x_i , $P(x_i)$] pairs

  x_i = Value of Random Variable (Outcome)

  $P(x_i)$ = Probability Associated with Value

- x_i's are mutually exclusive
  (no overlap)
- x_i's are collectively exhaustive
  (nothing left out)
- $0 \leq P(x_i) \leq 1$ for each x_i
- $\Sigma P(x_i) = 1$

Discrete Random Variable Summary Measures

- Expected Value of a discrete distribution
  (Weighted Average)

$$E(x) = \Sigma x_i P(x_i)$$

- Example: Toss 2 coins,
  x = # of heads,
  compute expected value of x:

x	P(x)
0	.25
1	.50
2	.25

$E(x) = (0 \times .25) + (1 \times .50) + (2 \times .25)$
$= 1.0$

Discrete Random Variable Summary Measures

(continued)

- Standard Deviation of a discrete distribution

$$\sigma_x = \sqrt{\sum \{x - E(x)\}^2 P(x)}$$

where:

E(x) = Expected value of the random variable

x = Values of the random variable

P(x) = Probability of the random variable having the value of x