A Course In Business Statistics

4th Edition

Chapter 8

Introduction to Hypothesis Testing

Chapter Goals

After completing this chapter, you should be able to:

- Formulate null and alternative hypotheses for applications involving a single population mean or proportion
- Formulate a decision rule for testing a hypothesis
- Know how to use the test statistic, critical value, and p-value approaches to test the null hypothesis
- Know what Type I and Type II errors are
- Compute the probability of a Type II error

What is a Hypothesis?

- A hypothesis is a claim (assumption) about a population parameter:

  - population mean

Example: The mean monthly cell phone bill of this city is μ = $42

  - population proportion

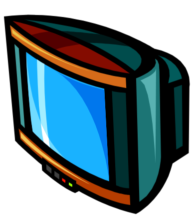
Example: The proportion of adults in this city with cell phones is p = .68

The Null Hypothesis, H_0

- States the assumption (numerical) to be tested

> **Example:** The average number of TV sets in U.S. Homes is at least three ($H_0 : \mu \geq 3$)

- Is always about a population parameter, not about a sample statistic

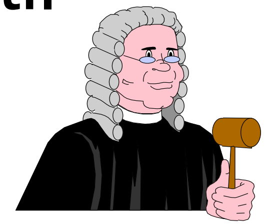


$$H_0 : \mu \geq 3 \qquad\qquad \cancel{H_0 : \bar{x} \geq 3}$$

The Null Hypothesis, H_0

(continued)

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty
- Refers to the status quo
- Always contains "=" , "≤" or "≥" sign
- May or may not be rejected



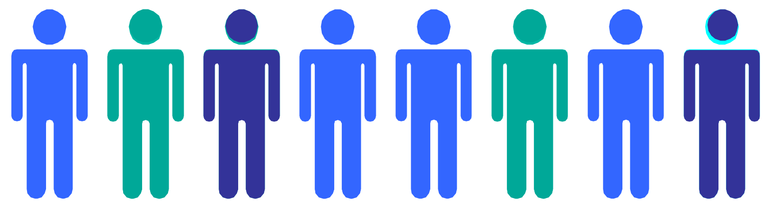
The Alternative Hypothesis, H_A

- Is the opposite of the null hypothesis
  - e.g.: The average number of TV sets in U.S. homes is less than 3 ($H_A: \mu < 3$)
- Challenges the status quo
- Never contains the "=" , "≤" or "≥" sign
- May or may not be accepted
- Is generally the hypothesis that is believed (or needs to be supported) by the researcher

Hypothesis Testing Process

Claim: **the population mean age is 50. (Null Hypothesis:**

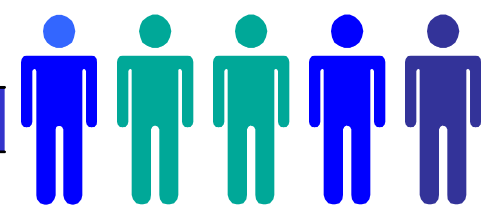
H_0: μ = 50)



Population

Now select a random sample

Is $\bar{x} = 20$ likely if $\mu = 50$?



Sample

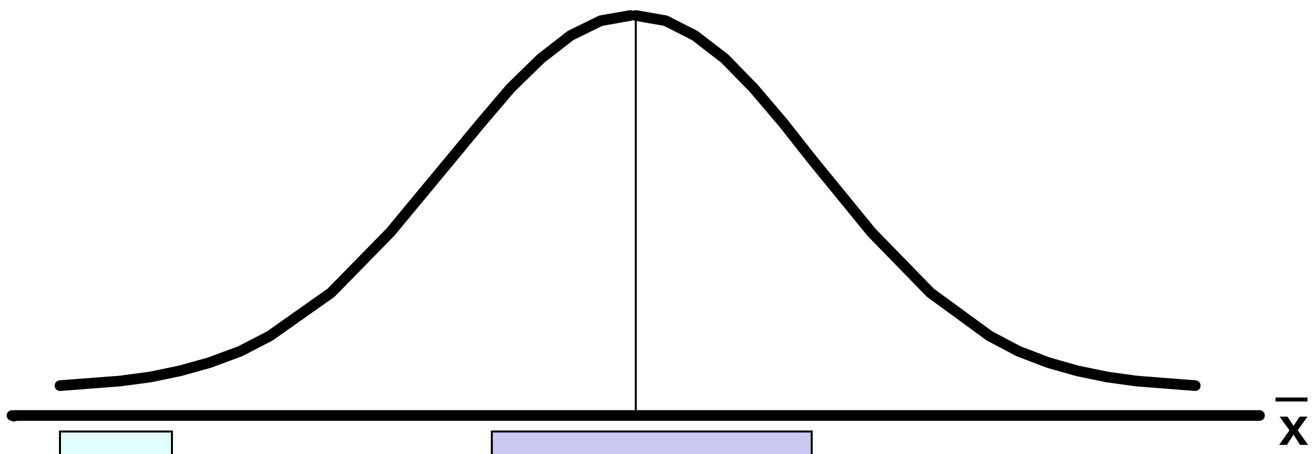
Suppose the sample mean age is 20: $\bar{x} = 20$

If not likely,

REJECT Null Hypothesis

Reason for Rejecting H_0

Sampling Distribution of $\bar{x}$



20

μ = 50
If H_0 is true

If it is unlikely that we would get a sample mean of this value ...

... if in fact this were the population mean…

... then we reject the null hypothesis that $\mu = 50$.

Level of Significance, α

- **Defines unlikely values of sample statistic if null hypothesis is true**
  - Defines rejection region of the sampling distribution
- Is designated by α , (level of significance)
  - Typical values are .01, .05, or .10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test

Level of Significance and the Rejection Region

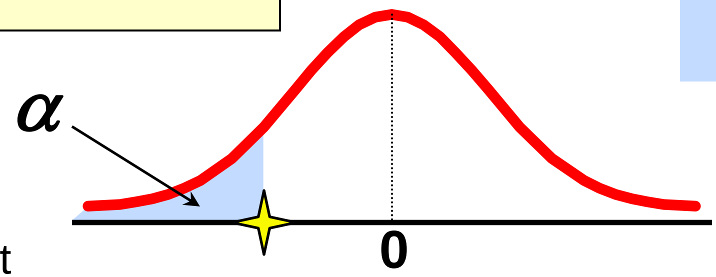
Level of significance = α

Represents critical value

Rejection region is shaded

H_0: $\mu \geq 3$

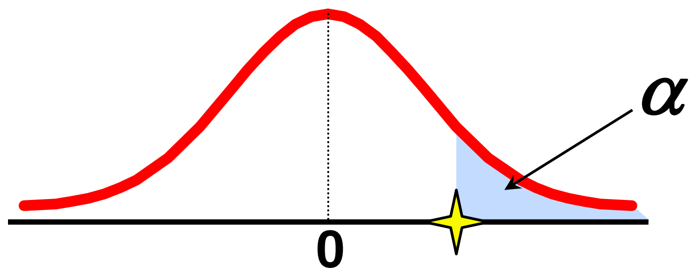
H_A: $\mu < 3$



Lower tail test

H_0: $\mu \leq 3$

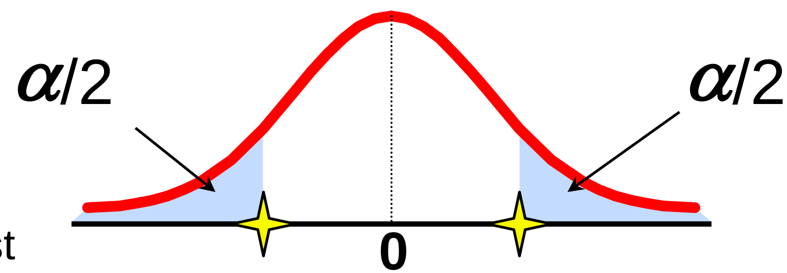
H_A: $\mu > 3$



Upper tail test

H_0: $\mu = 3$

H_A: $\mu \neq 3$



Two tailed test

Errors in Making Decisions

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error

The probability of Type I Error is α

- Called level of significance of the test
- Set by researcher in advance

Errors in Making Decisions

(continued)

- **Type II Error**
  - Fail to reject a false null hypothesis

The probability of Type II Error is β

Outcomes and Probabilities

Possible Hypothesis Test Outcomes

	State of Nature	
Decision	H_0 True	H_0 False
Do Not Reject $\mathbf{H_0}$	**No error** **$(1 - \alpha)$**	**Type II Error** **(β)**
Reject $\mathbf{H_0}$	**Type I Error** **(α)**	**No Error** **$(1 - \beta)$**

Key:
Outcome
(Probability)

Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

> If Type I error probability (α) ↑, then
> Type II error probability (β) ↓

Factors Affecting Type II Error

- All else equal,
  - $\beta \uparrow$ when the difference between hypothesized parameter and its true value $\downarrow$
  - $\beta \uparrow$ when $\alpha \downarrow$
  - $\beta \uparrow$ when $\sigma \uparrow$
  - $\beta \uparrow$ when $n \downarrow$

Critical Value Approach to Testing

- Convert sample statistic (e.g.: $\bar{X}$) to test statistic (*Z or t statistic*)
- Determine the critical value(s) for a specified level of significance α from a table or computer
- If the test statistic falls in the rejection region, reject H_0 ; otherwise do not reject H_0

Lower Tail Tests

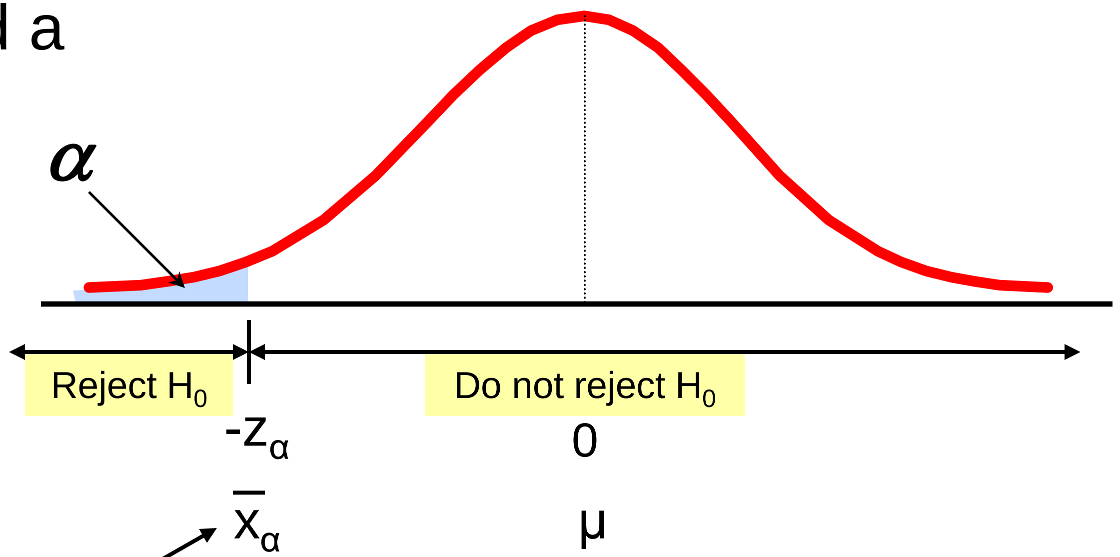
- The cutoff value,

  $-z_\alpha$ or $\bar{x}_\alpha$, is called a

  critical value

H_0: $\mu \geq 3$

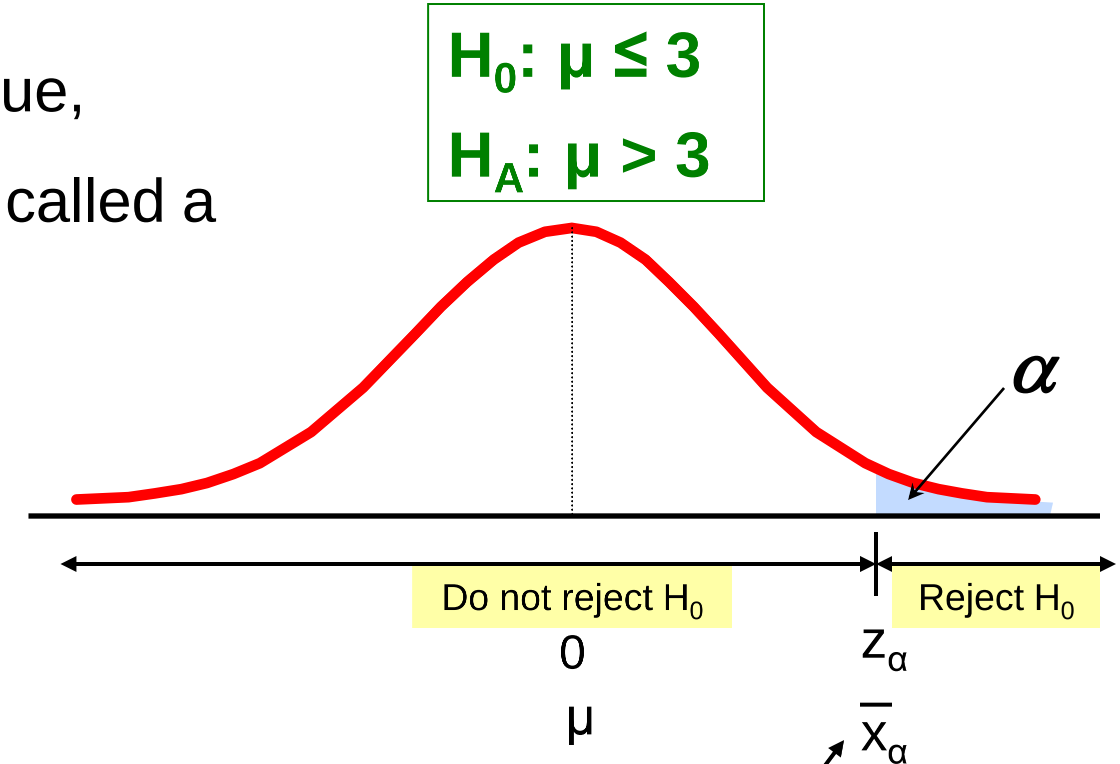
H_A: $\mu < 3$



$$\bar{x}_\alpha = \mu - z_\alpha \frac{\sigma}{\sqrt{n}}$$

Upper Tail Tests

- The cutoff value, z_α or $\bar{x}_\alpha$, is called a critical value



$$\bar{x}_\alpha = \mu + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Two Tailed Tests

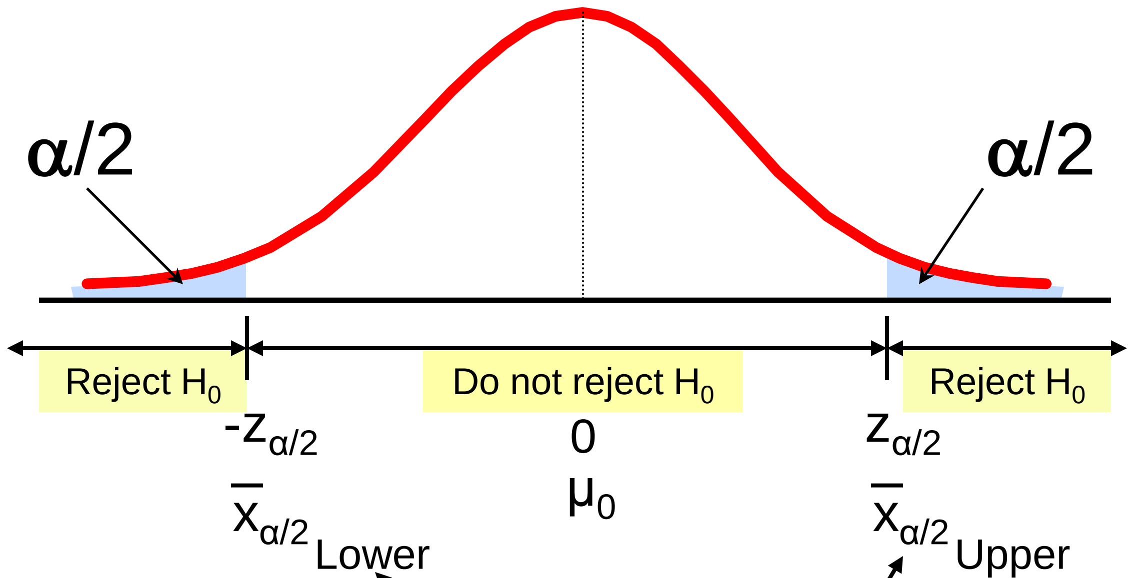
- There are two cutoff values (critical values):

$\boxed{\pm z_{\alpha/2}}$

or

$$\bar{X}_{\alpha/2\ \text{Lower}} \qquad \bar{X}_{\alpha/2\ \text{Upper}}$$

H_0: μ = 3
H_A: μ ≠ 3



$$\bar{x}_{\alpha/2} = \mu \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

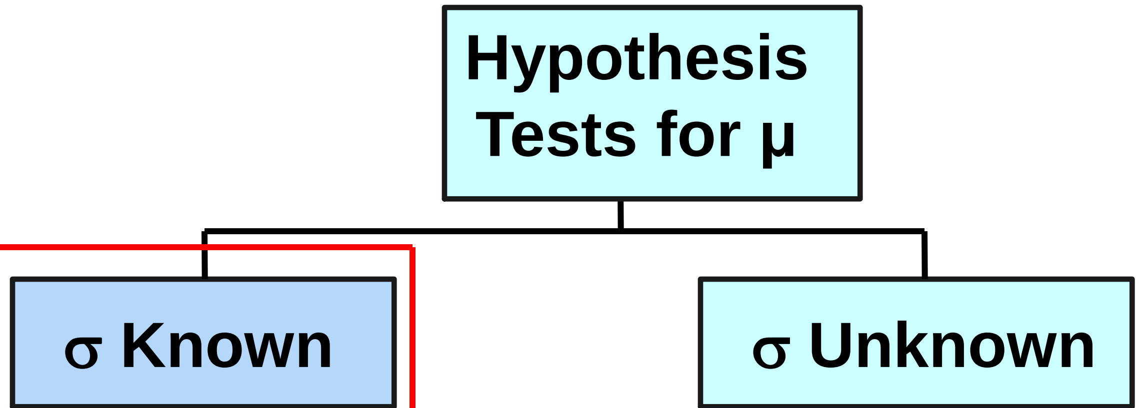
Critical Value Approach to Testing

- Convert sample statistic ($\bar{x}$) to a test statistic (*Z or t* statistic)

Hypothesis Tests for μ

- **σ Known**
- **σ Unknown**

Calculating the Test Statistic



The test statistic is:

$$z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Calculating the Test Statistic

(continued)

Hypothesis Tests for μ

σ Known

σ Unknown

Working With Large Samples

The test statistic is:

$$t_{n-1} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

But is sometimes approximated using a z:

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Calculating the Test Statistic

(continued)

Hypothesis Tests for μ

σ Known

σ Unknown

Using Small Samples

The test statistic is:

$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

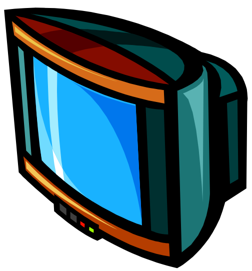
(The population must be approximately normal)

Review: Steps in Hypothesis Testing

- 1. Specify the population value of interest
- 2. Formulate the appropriate null and alternative hypotheses
- 3. Specify the desired level of significance
- 4. Determine the rejection region
- 5. Obtain sample evidence and compute the test statistic
- 6. Reach a decision and interpret the result

Hypothesis Testing Example

Test the claim that the true mean # of TV sets in US homes is at least 3.
(Assume $\sigma = 0.8$)

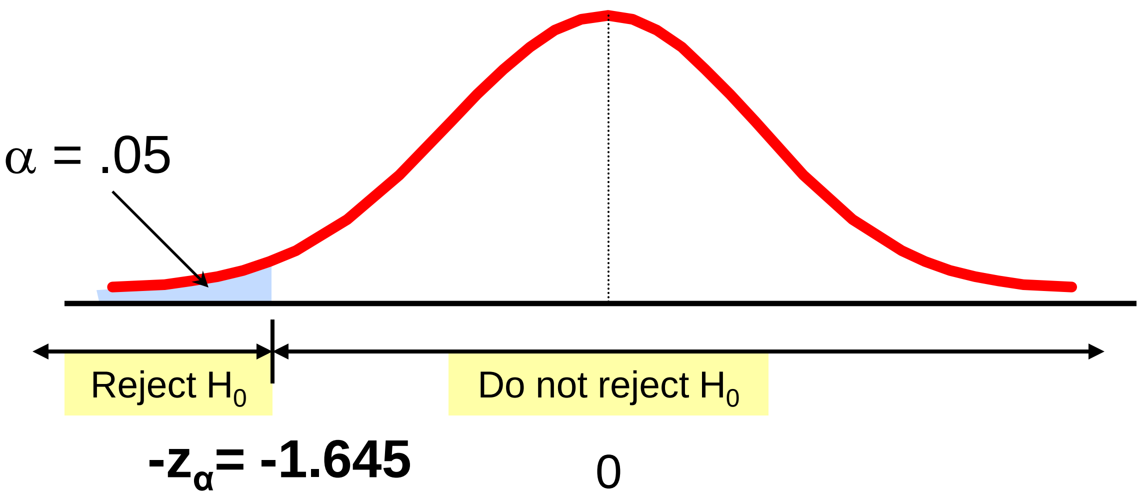


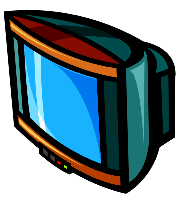
- 1. Specify the population value of interest
  - The mean number of TVs in US homes
- 2. Formulate the appropriate null and alternative hypotheses
  - H_0: $\mu \geq 3$ H_A: $\mu < 3$ (This is a lower tail test)
- 3. Specify the desired level of significance
  - Suppose that $\alpha = .05$ is chosen for this test

Hypothesis Testing Example

(continued)

- 4. Determine the rejection region





This is a one-tailed test with α = .05.
Since **σ is known**, the cutoff value is a z value:

Reject H_0 if $z < z_\alpha = -1.645$; otherwise do not reject H_0

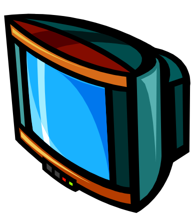
Hypothesis Testing Example

- 5. Obtain sample evidence and compute the test statistic

Suppose a sample is taken with the following results: $n = 100$, $\bar{x} = 2.84$ ($\sigma = 0.8$ is assumed known)

- Then the test statistic is:

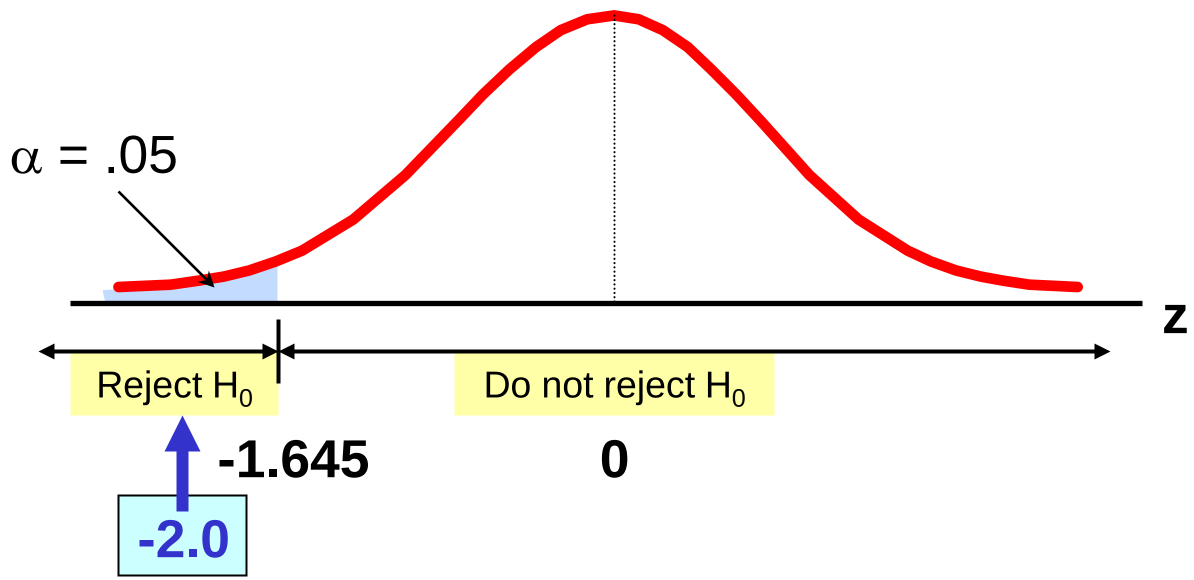
$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$



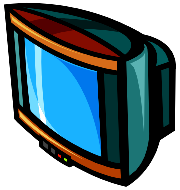
Hypothesis Testing Example

(continued)

- 6. Reach a decision and interpret the result



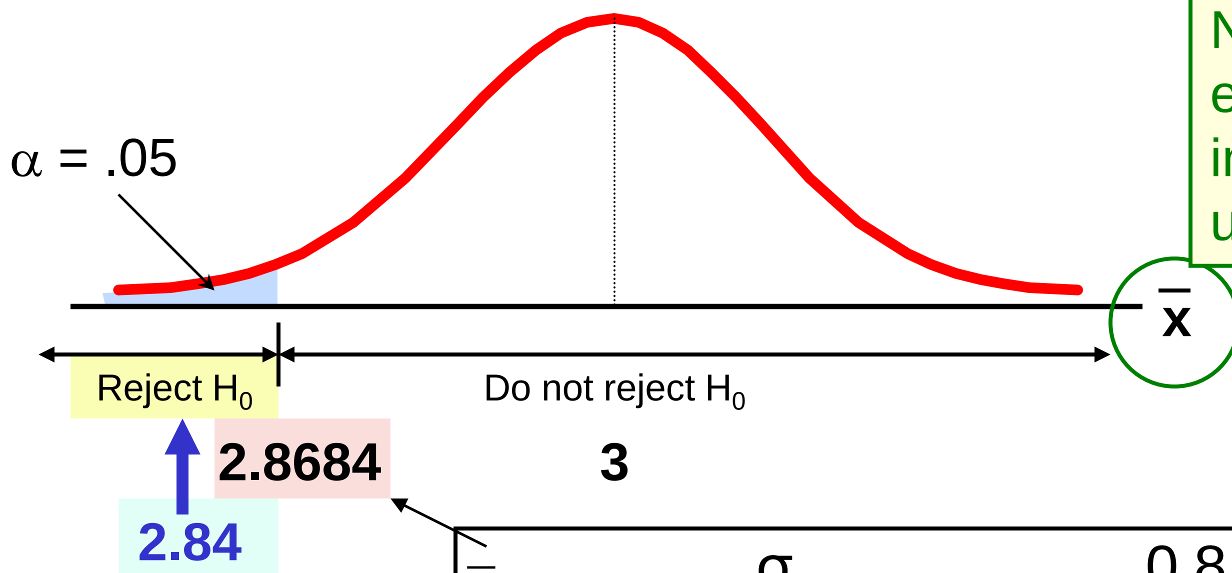
Since $z = -2.0 < -1.645$, we reject the null hypothesis that the mean number of TVs in US homes is at least 3



Hypothesis Testing Example

(continued)

- An alternate way of constructing rejection region:



Now expressed in $\bar{x}$, not z units

$$\bar{x}_\alpha = \mu - z_\alpha \frac{\sigma}{\sqrt{n}} = 3 - 1.645 \frac{0.8}{\sqrt{100}} = 2.8684$$

Since $\bar{x}$ = 2.84 < 2.8684, we reject the null hypothesis

p-Value Approach to Testing

- Convert Sample Statistic (e.g. $\bar{x}$) to Test Statistic (*z or t* statistic)
- Obtain the p-value from a table or computer
- Compare the p-value with α
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0

p-Value Approach to Testing

(continued)

- p-value: Probability of obtaining a test statistic more extreme (≤ or ≥) than the observed sample value given H_0 is true
  - Also called observed level of significance
  - Smallest value of α for which H_0 can be rejected

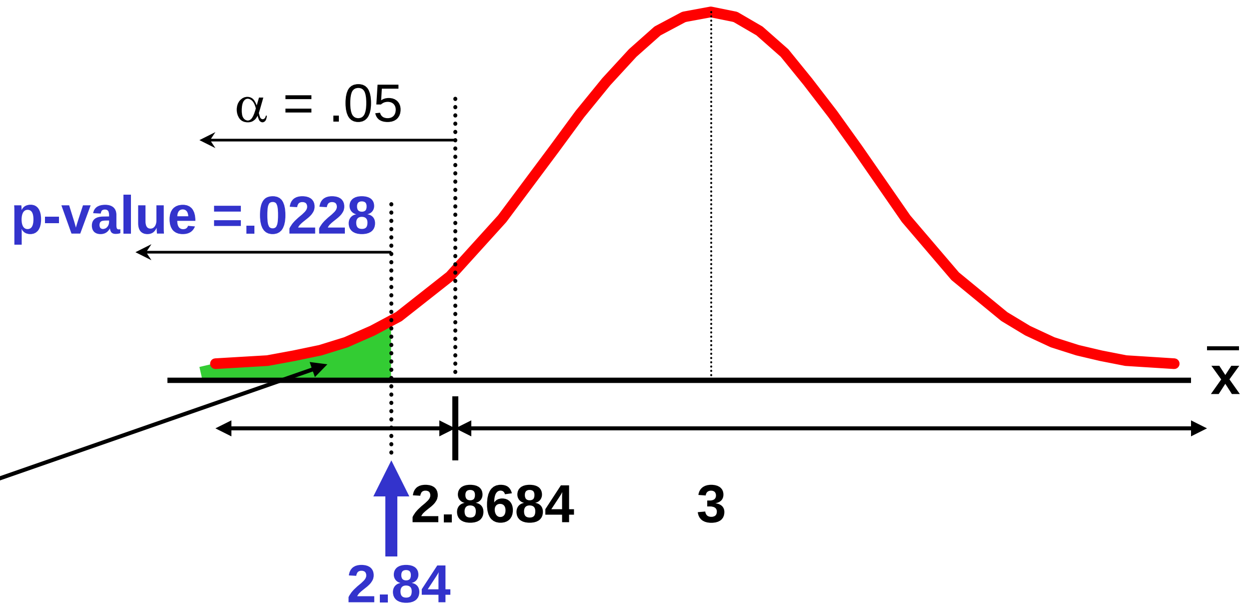
p-value example

- Example: How likely is it to see a sample mean of 2.84 (or something further below the mean) if the true mean is μ = 3.0?

$$P(\bar{x} < 2.84 \mid \mu = 3.0)$$

$$= P\left(z < \frac{2.84 - 3.0}{0.8 / \sqrt{100}} \right)$$

$$= P(z < -2.0) = .0228$$



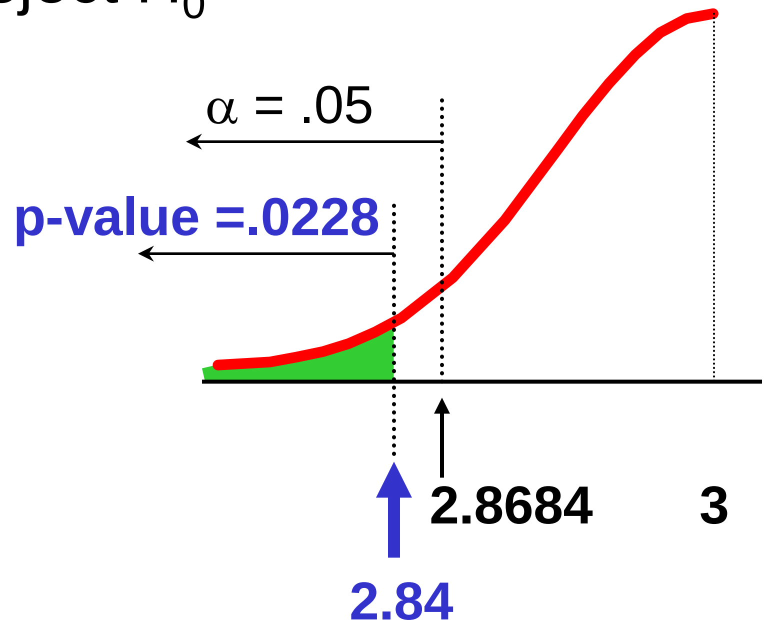
p-value example

(continued)

- Compare the p-value with α
  - If p-value $<$ α , reject H_0
  - If p-value $\geq$ α , do not reject H_0

Here: p-value = .0228
α = .05

Since .0228 < .05, we reject the null hypothesis



Example: Upper Tail z Test for Mean (σ Known)

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over $52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)

Form hypothesis test:

H_0: $\mu \leq 52$ the average is not over $52 per month

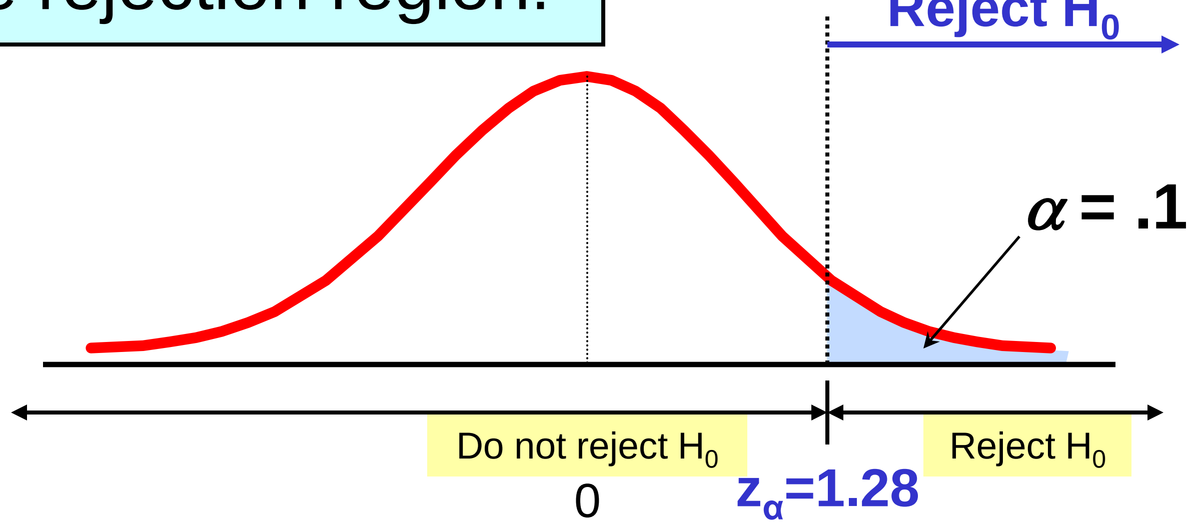
H_A: $\mu > 52$ the average **is** greater than $52 per month (i.e., sufficient evidence exists to support the manager's claim)

Example: Find Rejection Region

(continued)

- Suppose that α = .10 is chosen for this test

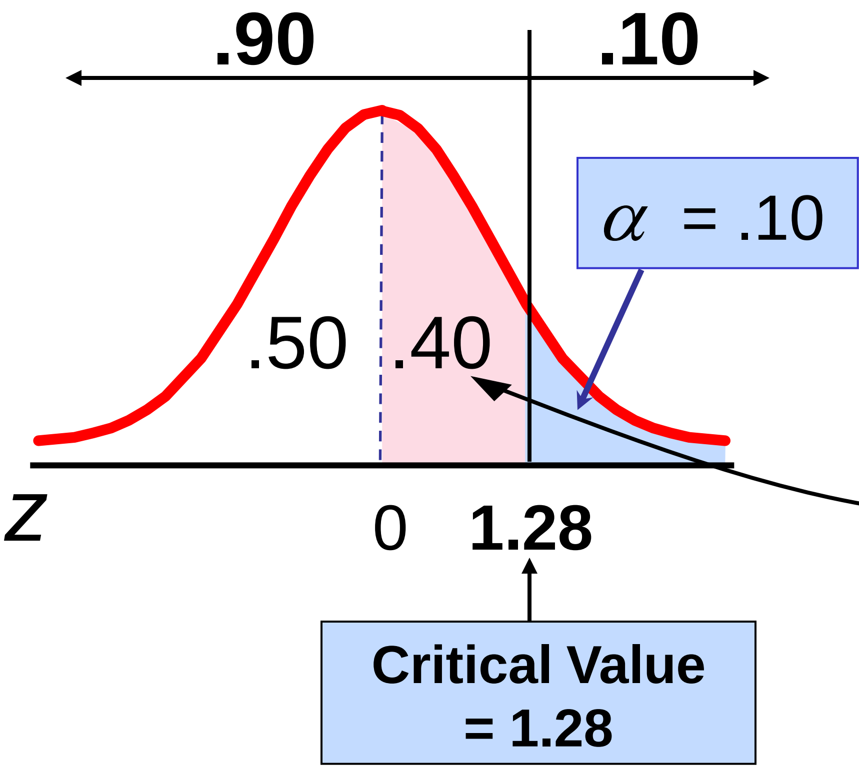
Find the rejection region:



Reject H_0 if $z > 1.28$

Review: Finding Critical Value - One Tail

What is z given α = 0.10?



Standard Normal Distribution Table (Portion)

Z	.07	.08	.09
1.1	.3790	.3810	.3830
1.2	.3980	.3997	.4015
1.3	.4147	.4162	.4177

Example: Test Statistic

(continued)

Obtain sample evidence and compute the test statistic

Suppose a sample is taken with the following results: $n = 64$, $\bar{x} = 53.1$ (σ=10 was assumed known)

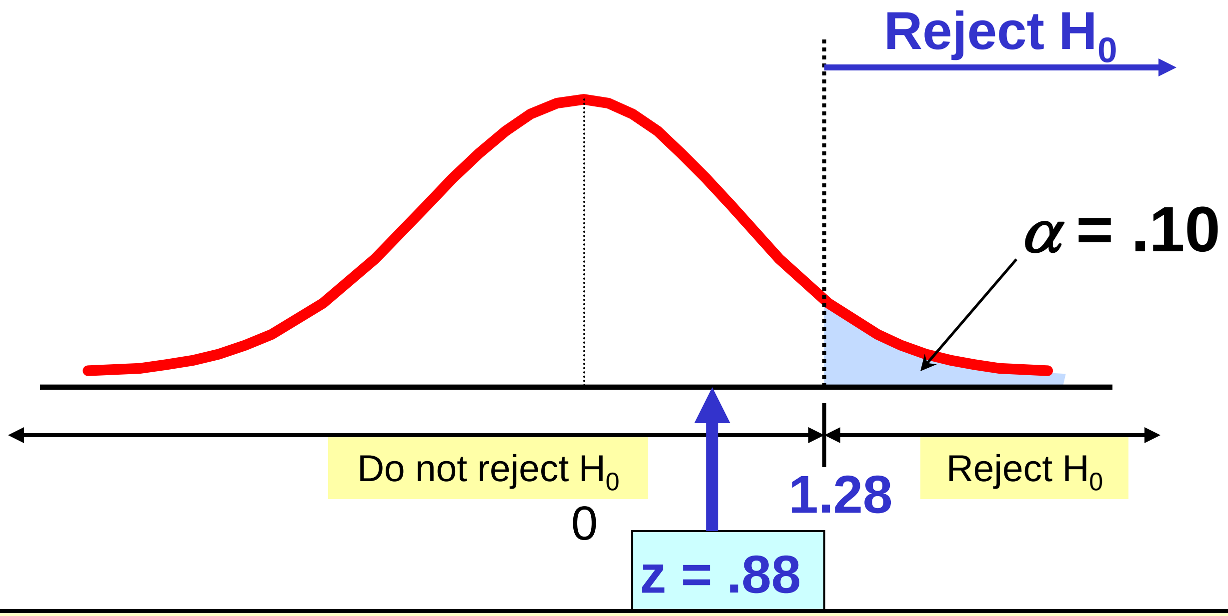
- Then the test statistic is:

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Example: Decision

(continued)

Reach a decision and interpret the result:



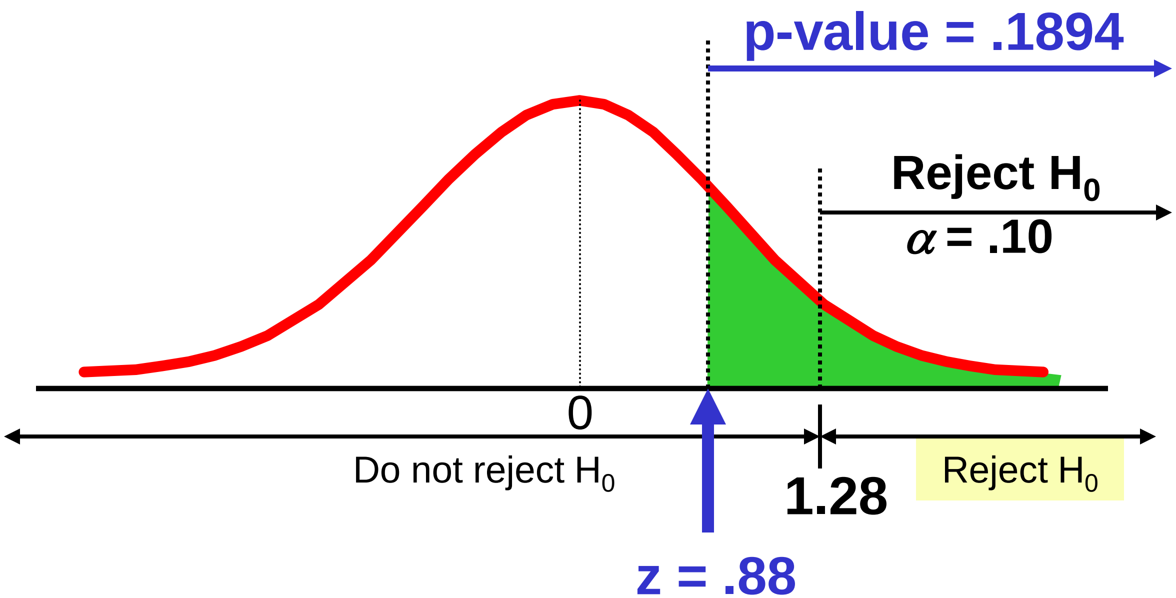
Do not reject H_0 since z = 0.88 ≤ 1.28

i.e.: there is not sufficient evidence that the mean bill is over $52

p -Value Solution

(continued)

Calculate the p-value and compare to α



$$P(\bar{x} \geq 53.1 \mid \mu = 52.0)$$

$$= P\left(z < \frac{53.1 - 52.0}{10 / \sqrt{64}} \right)$$

$$= P(z \geq 0.88) = .5 - .3106$$

$$= .1894$$

Do not reject H_0 since p-value = .1894 > α = .10

Example: Two-Tail Test (σ Unknown)

The average cost of a hotel room in New York is said to be \$168 per night. A random sample of 25 hotels resulted in $\bar{x}$ = \$172.50 and s = \$15.40. Test at the α = 0.05 level.

(Assume the population distribution is normal)

H_0: $\mu = 168$

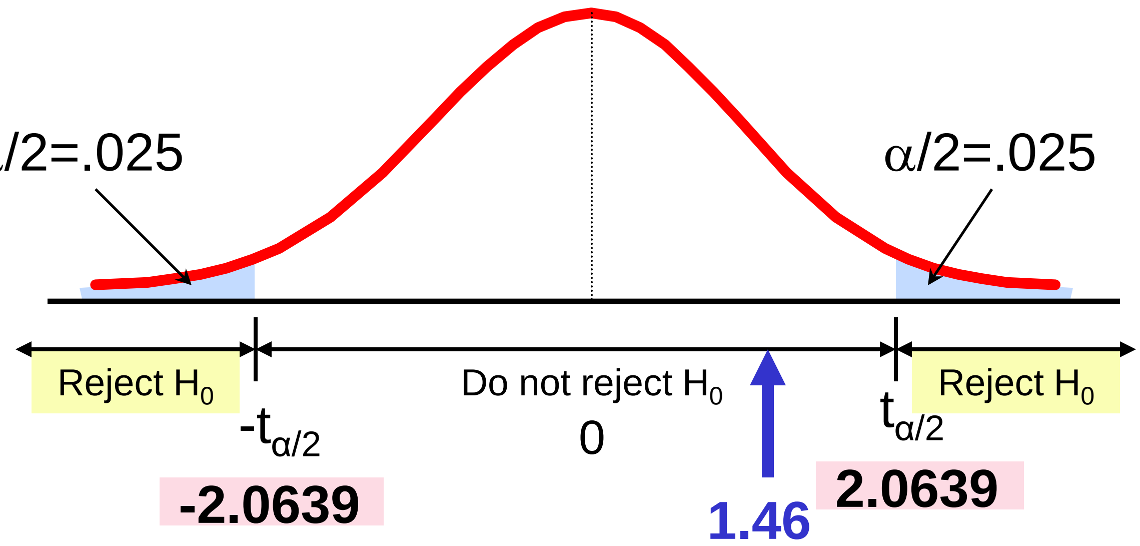
H_A: $\mu \neq 168$

Example Solution: Two-Tail Test

H_0: $\mu = 168$
H_A: $\mu \neq 168$

- $\alpha = 0.05$
- n = 25
- σ is unknown, so use a **t statistic**
- Critical Value:

$t_{24} = \pm\, 2.0639$



$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

Do not reject H_0: not sufficient evidence that true mean cost is different than $168

Hypothesis Tests for Proportions

- Involves categorical values
- Two possible outcomes
  - "Success" (possesses a certain characteristic)
  - "Failure" (does not possesses that characteristic)
- Fraction or proportion of population in the "success" category is denoted by p

Proportions

(continued)

- Sample proportion in the success category is denoted by $\bar{p}$

- $$\bar{p} = \frac{x}{n} = \frac{\text{number of successes in sample}}{\text{sample size}}$$

- When both np and n(1-p) are at least 5, $\bar{p}$ can be approximated by a normal distribution with mean and standard deviation

- $$\mu_{\bar{p}} = p \qquad \sigma_{\bar{p}} = \sqrt{\frac{p(1-p)}{n}}$$

Hypothesis Tests for Proportions

- The sampling distribution of $\bar{p}$ is normal, so the test statistic is a z value:

$$z = \frac{\bar{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

Hypothesis Tests for p

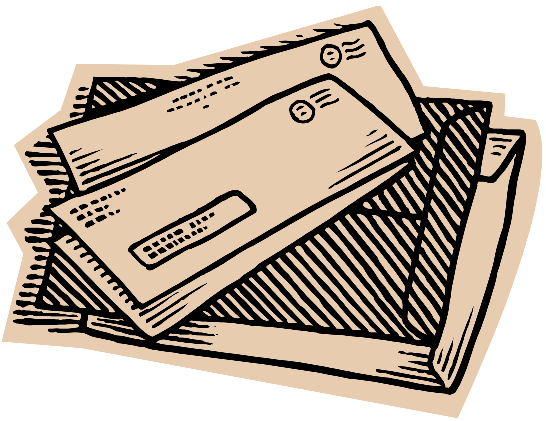
$np \geq 5$ and $n(1-p) \geq 5$

$np < 5$ or $n(1-p) < 5$

Not discussed in this chapter

Example: z Test for Proportion

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the $\alpha = .05$ significance level.



Check:

$$n\,p = (500)(.08) = 40$$

$$n(1-p) = (500)(.92) = 460$$

✓

Z Test for Proportion: Solution

H_0: p = .08

H_A: p ≠ .08

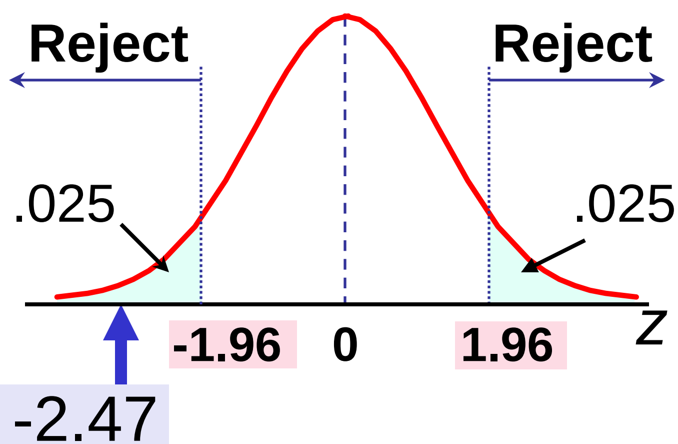
α = .05

n = 500, $\bar{p}$ = .05

Test Statistic:

$$z = \frac{\bar{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Critical Values: ± 1.96



Decision:

Reject H_0 at α = .05

Conclusion:

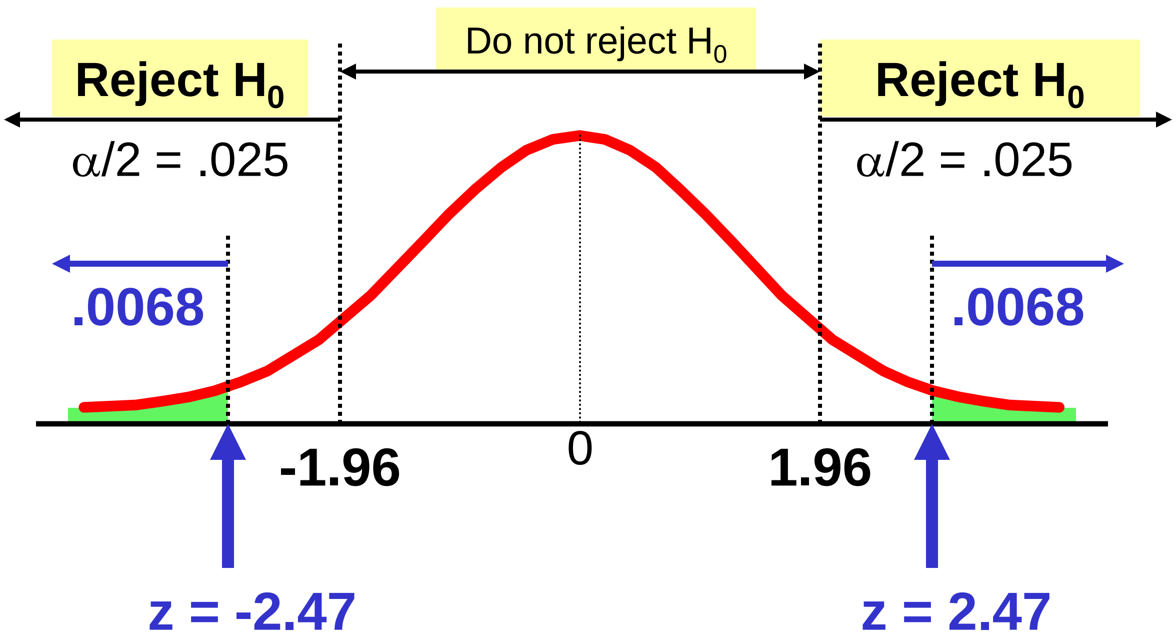
There is sufficient evidence to reject the company's claim of 8% response rate.

p -Value Solution

(continued)

Calculate the p-value and compare to α

(For a two sided test the p-value is always two sided)



p-value = .0136:

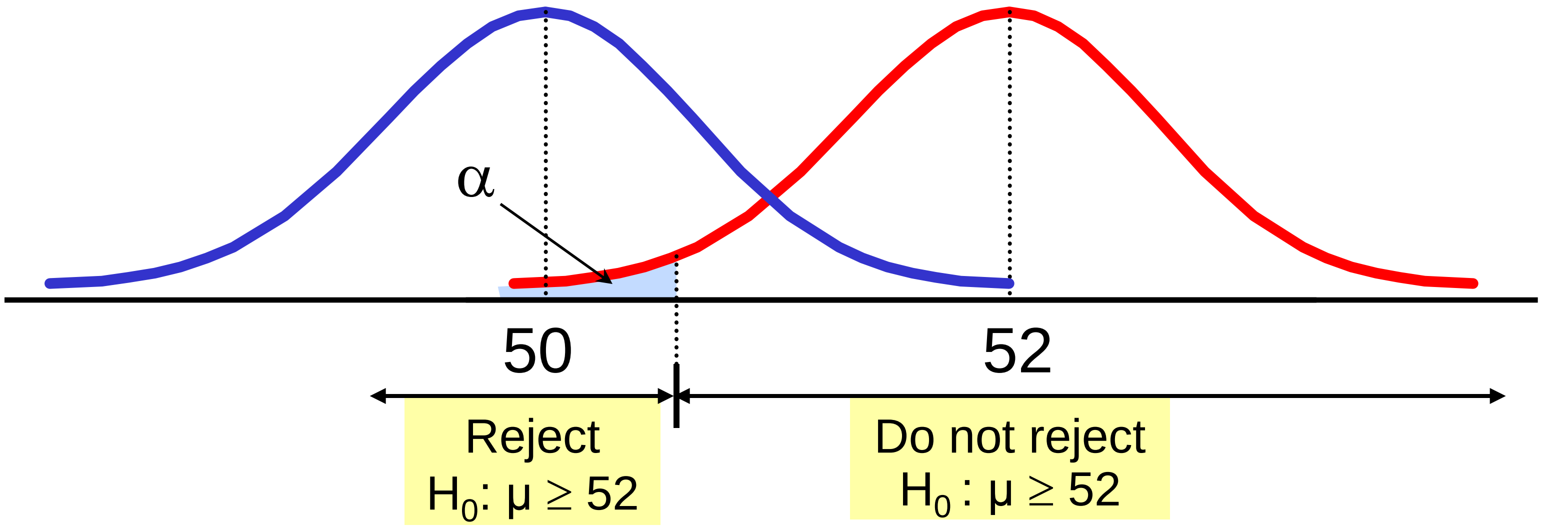
$$P(z \le -2.47) + P(x \ge 2.47)$$
$$= 2(.5 - .4932)$$
$$= 2(.0068) = 0.0136$$

Reject H_0 since p-value = .0136 < α = .05

Type II Error

- **Type II error is the probability of failing to reject a false H_0**

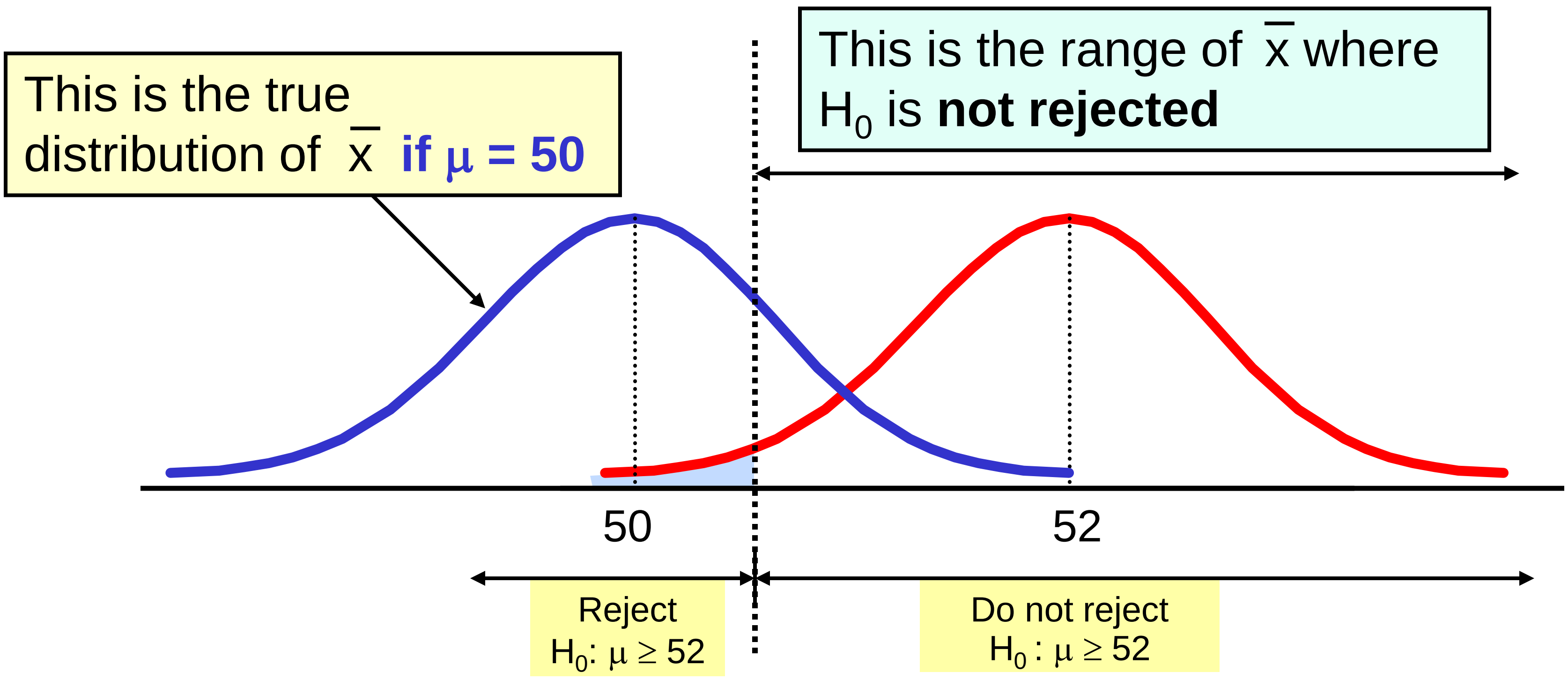
Suppose we fail to reject **H_0: $\mu \geq 52$**
when in fact the true mean is **$\mu = 50$**



Type II Error

(continued)

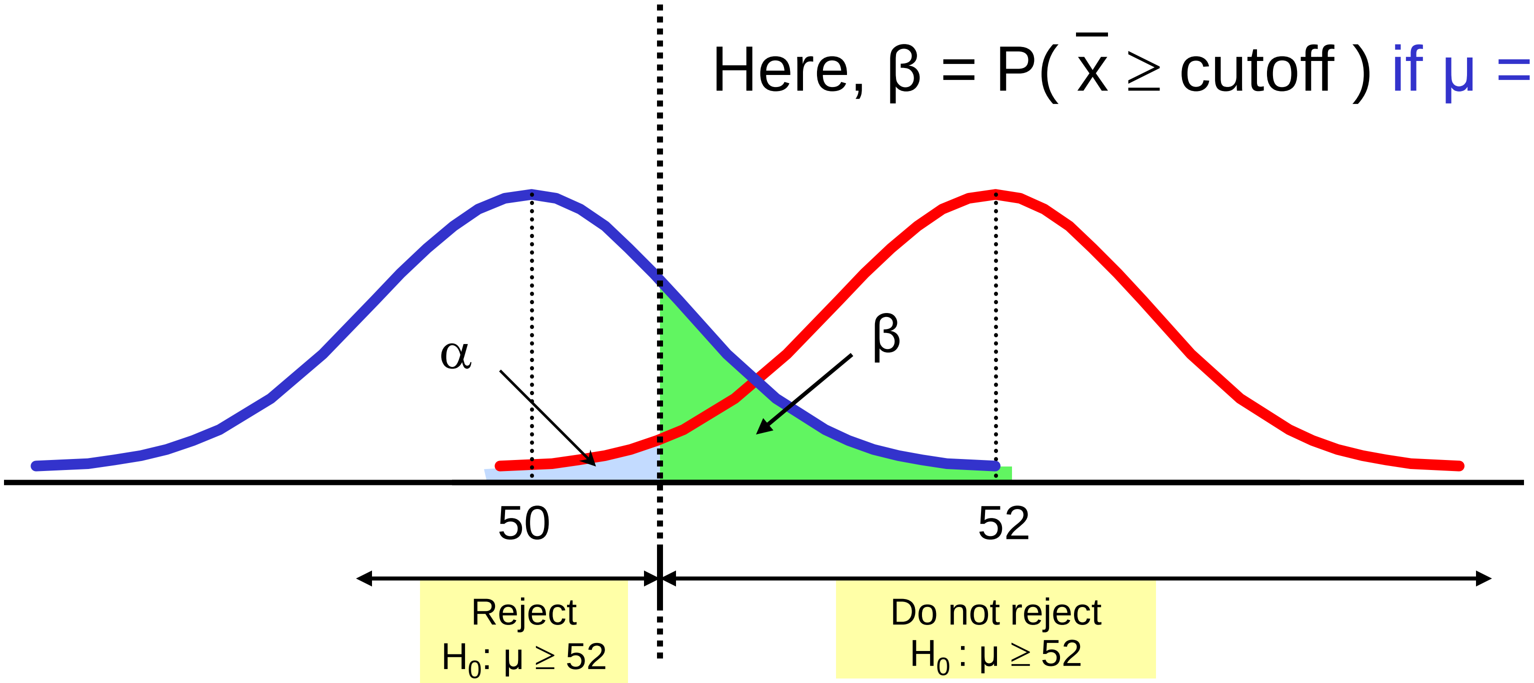
- Suppose we do not reject H_0: $\mu \geq 52$ when in fact the true mean is $\mu = 50$



Type II Error

(continued)

- Suppose we do not reject H_0: $\mu \geq 52$ when in fact the true mean is $\mu = 50$



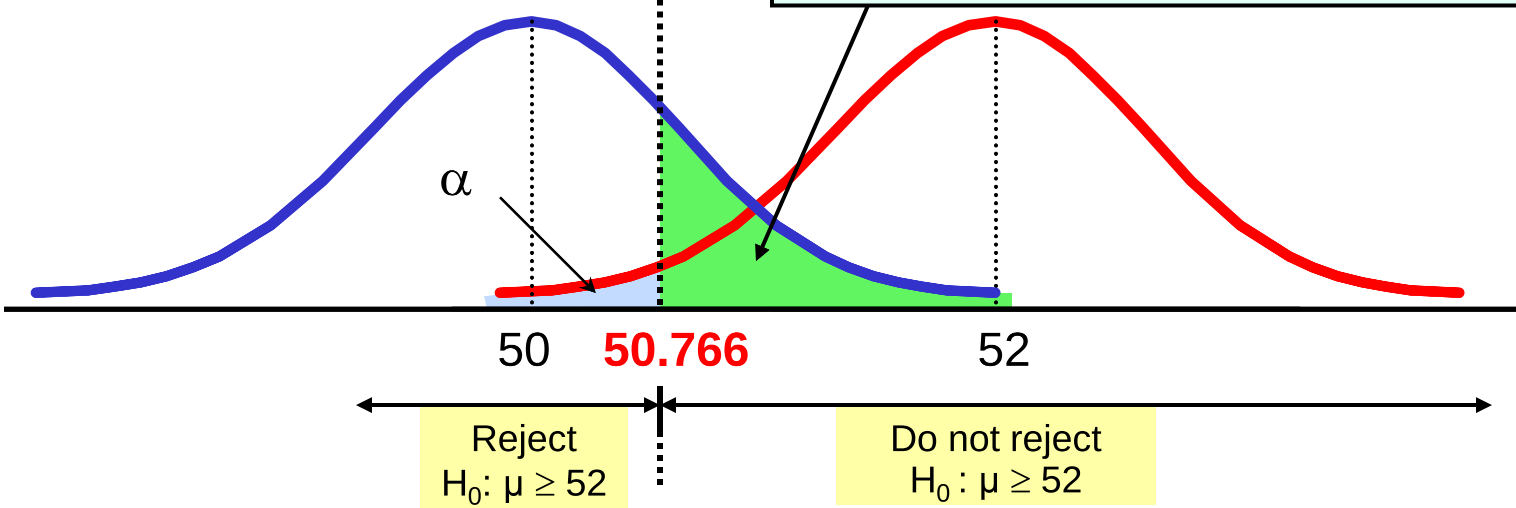
Calculating β

- Suppose n = 64 , σ = 6 , and α = .05

$$\text{cutoff} = \bar{x}_\alpha = \mu - z_\alpha \frac{\sigma}{\sqrt{n}} = 52 - 1.645 \frac{6}{\sqrt{64}} = \boxed{50.766}$$

(for $H_0 : \mu \geq 52$)

So $\beta = P(\bar{x} \geq 50.766)$ if $\mu = 50$

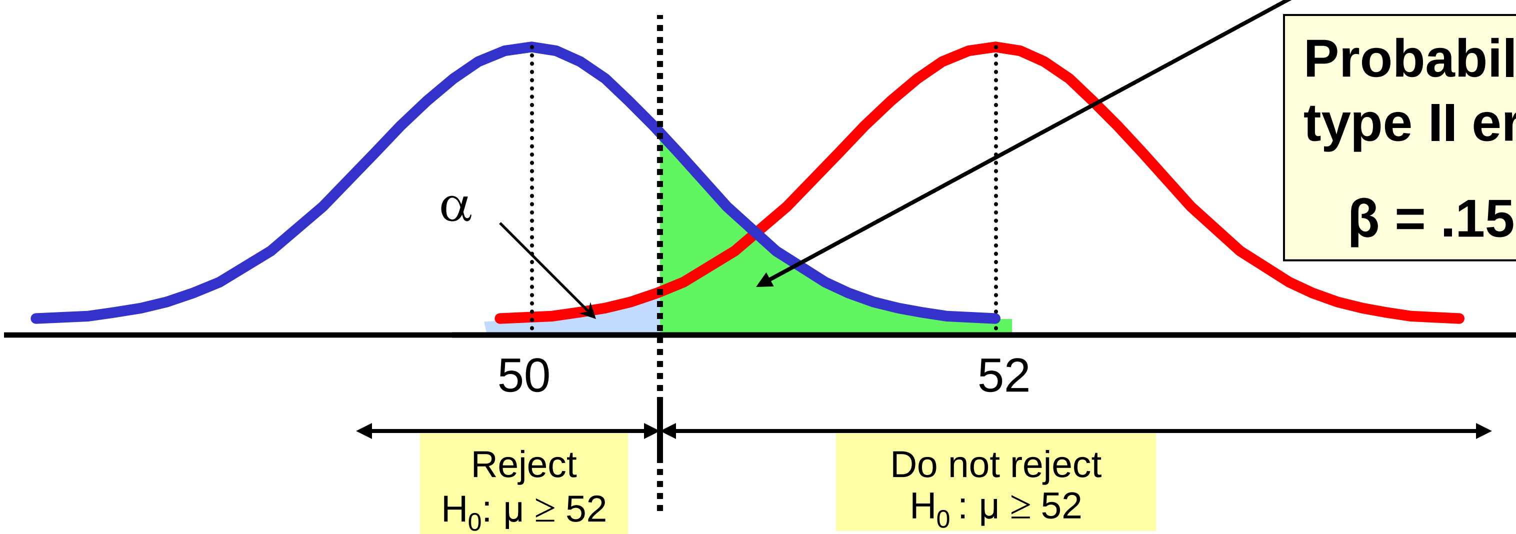


Calculating β

(continued)

- Suppose n = 64 , σ = 6 , and α = .05

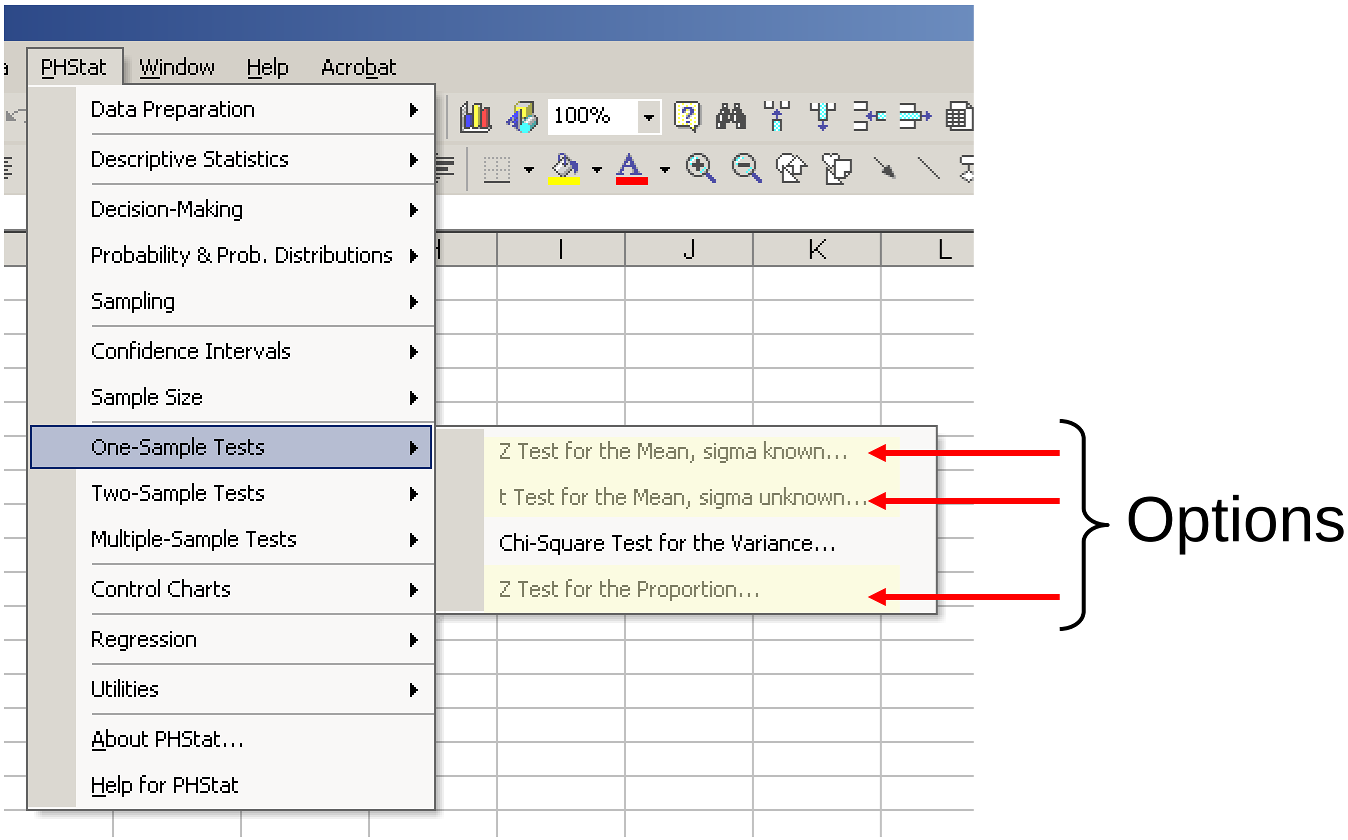
$$P(\bar{x} \geq 50.766 \mid \mu = 50) = P\left(z \geq \frac{50.766 - 50}{6/\sqrt{64}} \right) = P(z \geq 1.02) = .5 - .3461 = \boxed{.1539}$$



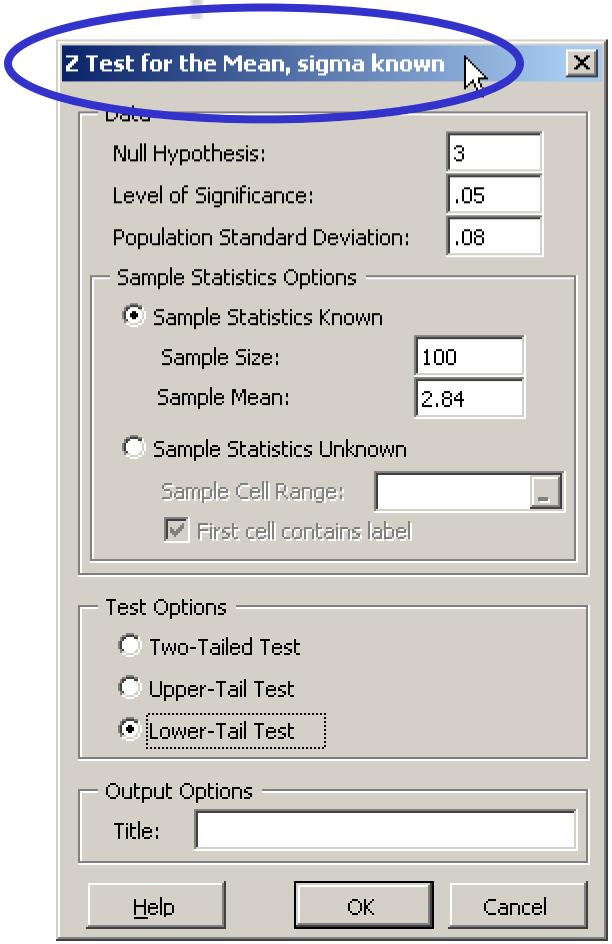
Probability of type II error:

β = .1539

Using PHStat



Sample PHStat Output



	A	B
1	**Z Test of Hypothesis for the Mean**	
2		
3	**Data**	
4	**Null Hypothesis**	3
5	**Level of Significance**	0.05
6	**Population Standard Deviation**	0.8
7	**Sample Size**	100
8	**Sample Mean**	2.84
9		
10	Intermediate Calculations	
11	Standard Error of the Mean	0.08
12	Z Test Statistic	-2
13		
14	**Lower-Tail Test**	
15	**Lower Critical Value**	-1.644853476
16	***p*-Value**	0.022750062
17	**Reject the null hypothesis**	
18		

Input

Output

Chapter Summary

- Addressed hypothesis testing methodology
- Performed z Test for the mean (σ known)
- Discussed p–value approach to hypothesis testing
- Performed one-tail and two-tail tests . . .

Chapter Summary

(continued)

- Performed t test for the mean (σ unknown)
- Performed z test for the proportion
- Discussed type II error and computed its probability