

Processing Signal and **Machine Learning**

1. Statistical Learning

1.1. Definition Statistical Model

Statistical Model: $\{X, \mathbb{F}, P_{\theta}; \theta \in \Theta\}$

Sample Space: Observation Space: \mathbb{X} Sigma Algebra Probability:

 $T: \mathbb{X} \mapsto \{\theta_0, \theta_1\}, x \mapsto T(x)$

Null Hypothesis: $H_0: \theta \in \Theta_0$ Alternative Hypothesis: $H_1: \theta \in \Theta_1$

Cost Criterion G_T :

 $G_T: \{\theta_0,\theta_1\} \overset{\text{\tiny -}}{\mapsto} [0,1], \theta \mapsto P(\{T(X)=1\}|\theta)$ $= E[T(X); \theta] = \int T(x) f_X(x|\theta) dx$

Error Level α : $G_T(\theta_0) \le \alpha$ Two Error Types:

False Alarm: $\theta = \theta_0, T(x) = 1$ $G_T(\theta_0) = P(\{T(X) = 1\} | \theta_0)$

Detection Error: $\theta = \theta_1, T(x) = 0$ $1 - G_T(\theta_1) = P(\{T(X) = 0\} | \theta_1)$

1.2. Maximum Likelihood Test

ML Ratio Test Statistic:

$$R(x) = \begin{cases} \frac{f_X(x|\theta_1)}{f_X(x|\theta_0)} \; ; & f_X(x|\theta_0) > 0 \\ \infty & ; & f_X(x|\theta_0) = 0 \text{ and } f_X(x|\theta_1) > 0 \end{cases}$$

$$T_{\mathsf{ML}}: \mathbb{X} \mapsto \{0, 1\}, x \mapsto \begin{cases} 1 & ; & R(X) > c \\ 0 & ; & \mathsf{otherwise} \end{cases}$$

if $c \neq 1$ False Alarm Error Probability can be adjusted o Neyman Pear-

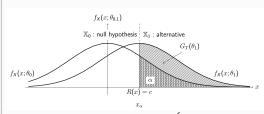
1.3. Neyman-Pearson-Test

The best test of P_0 against P_1 is

$$T_{\text{NP}}(x) = \begin{cases} 1 & R(x) > c \\ \gamma & R(x) = c \\ 0 & R(x) < c \end{cases} \qquad \begin{array}{ll} \text{Likelihood-Ratio:} \\ R(x) = \frac{f_X(x|\theta_1)}{f_X(x|\theta_0)} \end{cases}$$

 $\gamma = \frac{\alpha - P_0(\{R > c\})}{P_0(\{R = c\})}$ Errorlevel α

Steps: For α calculate x_{α} , then $c = R(x_{\alpha})$



ROC Graphs: plot $G_T(\theta_1)$ as a function of $G_T(\theta_0)$

1.4. Bayes Test (MAP Test)

Prior knowledge about possible hypotheses:

Prior knowledge about possible hypotheses:

$$P(\{\theta \in \Theta_0\}) + P(\{\theta \in \Theta_1\}) = 1$$

$$T_{\mathsf{Bayes}} = \underset{T}{\operatorname{argmin}} \{P_{\epsilon}\} = \begin{cases} 1 & ; & \frac{f_{X}(x|\theta_{1})}{f_{X}(x|\theta_{0})} > c \\ 0 & ; & \mathsf{otherwise} \end{cases}$$

$$= \begin{cases} 1 & ; & \mathsf{P}(\theta_1|x) > \mathsf{P}(\theta_0|x) \\ 0 & ; & \mathsf{otherwise} \end{cases}$$

with :
$$P_{\epsilon} = P(\theta_0)G_T(\theta_0) + P(\theta_1)(1 - G_T(\theta_1)), \quad c = \frac{P(\theta_0)}{P(\theta_1)}$$

if
$$P(\theta_0) = P(\theta_1) \rightarrow T_{\mathsf{Baves}} = T_{\mathsf{ML}}$$

$$\begin{array}{l} \text{Multiple Hypothesis } \{\theta_0,...,\theta_k\}; \mathbb{X}_0,...,\mathbb{X}_k \in \mathbb{X} \\ T_{\text{Bayes}} = \mathop{\mathrm{argmin}}_{k \in 1,...,K} \{P(\theta_k|x)\} \end{array}$$

Loss Function

$$L(T(x),\theta) = \begin{cases} L_0 & ; \quad T(x) = 1, \text{ but } \theta = \theta_0 \quad \text{(FALSE ALARM)} \\ L_1 & ; \quad T(x) = 0, \text{ but } \theta = \theta_1 \quad \text{(DETEC. ERROR)} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

 L_i denotes the Loss Value in cases where the correct decision parameter θ is missed

$$\operatorname{Risk}(T) = \mathsf{E}[L(T(X), \theta)] = \mathsf{E}[\mathsf{E}[L(T(x), \theta)|x = X]]$$

1.5. Linear Alternative Tests

Estimate normal vector \boldsymbol{w}^{\top} and w_0 , which separate \mathbb{X} into \mathbb{X}_0 and \mathbb{X}_1 $\log R(\underline{\boldsymbol{x}}) = -\frac{1}{2} \ln(\frac{\det(\underline{\boldsymbol{C}}_1)}{\det(\underline{\boldsymbol{C}}_0)}) - \frac{1}{2} (\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_1)^{\top} \underline{\boldsymbol{C}}_1^{-1} (\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_1) +$ $+\frac{1}{2}(\underline{x}-\underline{\mu}_0)^{\top}\underline{C}_0^{-1}(\underline{x}-\underline{\mu}_0) = \ln(\frac{P(\theta\in\Theta_0)}{P(\theta\in\Theta_1)})$ (seperating surface)

For Gaussian $f_X(x;\mu_k,C_k)$ with θ_0 and θ_1 corresponding to $\{\mu_0,C_0\}$ and $\{\mu_1,C_1\}$, it follows that

- if $C_0 \neq C_1$, log R(x) = 0 is non-linear and the separating surfaces are surfaces of second order: parabolic, hyperbolic, or elliptic
- if $C_0 = C_1$, log R(x) = 0 is affine and thus defines a hyperplane in \mathbb{X} which decomposes \mathbb{X} into \mathbb{X}_0 and \mathbb{X}_1 , i.e.,

$$T: \mathbb{X} \to \mathbb{R}, \underline{\boldsymbol{x}} \mapsto \begin{cases} 1 & \underline{\boldsymbol{w}}^\top \underline{\boldsymbol{x}} > w_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{split} &-\text{ case 1: } \underline{C}_0 = \underline{C}_1 = \sigma^2 \underline{I}_N \\ &\underline{\underline{w}}^\top = (\underline{\mu}_1 - \underline{\mu}_0)^\top, \\ &w_0 = \frac{1}{2}(\underline{\mu}_1^\top \underline{\mu}_1 - \underline{\mu}_0^\top \underline{\mu}_0) - \sigma^2 \ln(\frac{P(\theta \in \Theta_1)}{P(\theta \in \Theta_0)}) \end{split}$$

 $\begin{array}{cccc} \underline{w} & \text{colinear with } (\underline{\mu}_1 - \underline{\mu}_0) \\ \rightarrow & \text{hyperplane orthogonal to } (\underline{\mu}_1 - \underline{\mu}_0) \end{array}$

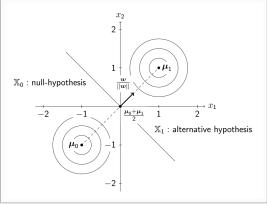
- case 2:
$$\underline{C}_0 = \underline{C}_1 = \underline{C}$$

$$\underline{w}^\top = (\underline{\mu}_1 - \underline{\mu}_0)^\top \underline{C}^{-1},$$

$$w_0 = \frac{1}{2}(\underline{\mu}_1 - \underline{\mu}_0)^\top \underline{C}^{-1}(\underline{\mu}_1 + \underline{\mu}_0) - \ln(\frac{P(\theta \in \Theta_1)}{P(\theta \in \Theta_0)})$$
in general \underline{w} not colinear with $(\underline{\mu}_1 - \underline{\mu}_0)$
 \rightarrow hyperplane not orthogonal to $(\underline{\mu}_1 - \underline{\mu}_0)$

• if $C_0=C_1$ and $\mu_0=-\mu_1$, log R(x)=0 is linear and defines a separating hyperplane in $\mathbb X$ which contains the origin, i.e.,

$$T: \mathbb{X} \to \mathbb{R}, \underline{\boldsymbol{x}} \mapsto \begin{cases} 1 & \underline{\boldsymbol{w}}^{\top}\underline{\boldsymbol{x}} > 0 \\ 0 & \text{otherwise} \end{cases}$$



2. Learning and Generalization

2.1. Empirical Risk Function and Generalization Error

ML scenarios (unknown Stochastical Model) base learning on: $Risk_{emp}(T; \mathbb{S}) = \frac{1}{M} \sum_{i=1}^{M} L(T(\underline{\mathbf{x}}_i), y_i), \quad (\underline{\mathbf{x}}_i, y_i) \in \mathbb{S}$ $\underline{\mathbf{x}} \mapsto T(\underline{\mathbf{x}}; \mathbb{S}) \quad T = \operatorname{argmin} \{ Risk_{emp}(T'; \mathbb{S}) \}$

good Generalization: $Risk_{emp}(T; \mathbb{S}_{test})$ similar to $Risk_{emp}(T; \mathbb{S})$ bad Generalization:

- ullet small $\mathbb T$ that does not cover $T_{opt} o$ cannot be selected by ML ⇒ strong mismatch between the desired and derived Test and refers to a sort of Bias Error Term
- too rich $\mathbb{T} \to \text{fluctuating of the available data (measurement noise)}$ is interpreted as meaningful information
- ⇒ Overfitting: leads to an increased Variance Error Term

2.2. Bias-Variance Decomposition

$$\begin{array}{lll} Risk &=& E_{S,X,Y}[L(T(X;S),Y)] &=& E_X[1-P_{Y\mid X}(Y=T_B(X))] \\ T_B(X)) &+& \overline{(1-P_{S\mid X}(T(X;S)=T_B(X)))} \\ T_B(X)) &-& 1)], & T_B(X) \text{ is the unknown } Bayes \ \textit{Test} \end{array}$$

If the potential set $\mathbb S$ would be selected from a distribution such that the derived Test $T(\mathbf{x}; \mathbb{S})$ and the corresponding Bayes Test $T_B(\mathbf{x})$ are identical almost surely, then the Risk Function achieves its minimum value which is equal to the Irreducible Error $E_X[1-P_{Y|X}(Y=T_B(X))]$ (denotes the probability that for a given input x the Bayes Test $T_{R}(X)$ decides for the false label u).

3. Classification Trees and Random Forests

3.1. CART Algorithms

Generate Binary Trees by splitting X at each (internal/root) node: $\mathbb{X}_{i,left} = \{\underline{\mathbf{x}} \in \mathbb{X}_i | x_{j_i} < \tau_i\} \quad \mathbb{X}_{i,right} = \mathbb{X}_i \setminus \mathbb{X}_{i,left}$

Root/Internal node: Binary decision based on chosen threshold $\tau_i \in \mathbb{R}$, feature $x_{j_i} = [\underline{\mathbf{x}}]_{j_i}$ with $j_i \in \mathbb{J} = \{1, ..., dim[\mathbb{X}]\}$ aims at minimiz $ing\ Risk_{emp}(T_{CART})$

Terminal node: n_i corresponds to subset $X_i \in X \to has$ no more children; outputs a decision

 $\Rightarrow x \mapsto n_i(x)$

Empirical Impurity Measure: choose j_i and τ_i at n_i by: $I_{CART}(\mathbb{S}_i) = \sum_{k=1}^{K} (1 - \hat{P}_{Y|X}(Y) = \theta_k | \{\underline{\mathbf{x}} \in \mathbb{S}_i\} \}$ X_i ; S_i)) $\hat{P}_{Y|X}(Y = \theta_k | \{\underline{\mathbf{x}} \in X_i\}; S_i)$

 $\hat{P}_{Y|X}(Y = \theta_k | \{\underline{\mathbf{x}} \in \mathbb{X}_i\}; \mathbb{S}_i) = \frac{M_k(\mathbb{S}_i)}{M(\mathbb{S}_i)} = \frac{|\{(\underline{\mathbf{x}}, y) \in \mathbb{S}_i | y = \theta_k\}|}{|\mathbb{S}_i|}$

 $\Rightarrow \qquad \{j_i, \tau_i\} \qquad = \underset{j \in \mathbb{J}, \tau \in \mathbb{R}}{\operatorname{argmin}} \Big\{ \sum_{k=1}^K \Big(1 - \frac{M_k(\mathbb{S}_i, left)}{M(\mathbb{S}_i, left)} \Big) \frac{M_k(\mathbb{S}_i, left)}{M(\mathbb{S}_i)} + \Big(1 - \frac{M_k(\mathbb{S}_i, right)}{M(\mathbb{S}_i, right)} \Big) \frac{M_k(\mathbb{S}_i, right)}{M(\mathbb{S}_i)} \Big)$ **Overfitting**(comes with high purity) can be controlled by a *Test Set*

Decision Rule: At terminal node n_i , input $\underline{\mathbf{x}}$ is assigned to $T_{CART}(\underline{\mathbf{x}}; \mathbb{S}) : \mathbb{X} \mapsto \{1, ..., K\}, \underline{\mathbf{x}} \mapsto \operatorname{argmax}\{\overline{M}_k(\mathbb{S}_i)\}$

Gini Impurity Index: I_{CART}

$$\sum_{k=1}^K (1 - P_Y|_X(\underline{\boldsymbol{y}} = \theta_k | \{\underline{\mathbf{x}} \in \mathbb{X}\})) P_Y|_X(\underline{\boldsymbol{y}} = \theta_k | \{\underline{\mathbf{x}} \in \mathbb{X}\}) \right| :$$

$$\sum_{k=1}^K \sum_{j=1, j \neq k}^K P_{Y|X}(\underline{\boldsymbol{y}} = \boldsymbol{\theta}_j | \{\underline{\mathbf{x}} \in \mathbb{X}\}) P_{Y|X}(\underline{\boldsymbol{y}} = \boldsymbol{\theta}_k | \{\underline{\mathbf{x}} \in \mathbb{X}\})$$

3.2. Random Forests

Avoid Overfitting (here: CART) \Rightarrow combine independent Hypothesis Tests: e.g. by Majority Vote

 $T_{maj}(\underline{\mathbf{x}}) = majority\{T_{CART}(\underline{\mathbf{x}}; \mathbb{S}^{(t)}, \nu^{(t)})\}_{t=1}^{t_{max}}$

Randomization Parameter ν_t controls an additionally introduced Randomness between the individual Tests.

 \Rightarrow Variance of $T_{ava}(\mathbf{x})$ is reduced by $1/t_{max}$ with respect to the Vari ance of the individual test.

Random Forest Method:

- $T_{RF}(\underline{\mathbf{x}}) = majority\{T_{CART}(\underline{\mathbf{x}}; \mathbb{S}^{(t)}, \mathbb{J}^{(t)})\}_{t=1}^{t_{max}}$
- Stochastic Independence by Bootstrapping of training samples (random sampling from $\mathbb S$ with replacement) \Rightarrow large t_{max} guarantees excellent performance (yet Tests are still correlated)
- Overfitting not considered (maximum purity) ⇒ small bias of RF

4. Hypothesis Testing

making a decision based on the observations

4.1. Definition

Null hypothesis $H_0: \theta \in \Theta_0$ (Assumed first to be true) Alternate hypothesis $H_1:\theta\in\Theta_1$ (The one to proof)

Descision rule $\varphi: \mathbb{X} \to [0,1]$ with

 $\varphi(x)=1$: decide for H_1 , $\varphi(x)=0$: decide for H_0 Error level α with $\mathsf{E}[d(\mathsf{X})|\theta] \le \alpha, \forall \theta \in \Theta_0$

Error Type	Decision Reality	H_1 false (H_0 true)	H_1 true (H_0 false
1 (FA) False	H_1 rejected	True Negative	False Negativ (Type 2)
Alarm	$(H_0 \ {\sf accepted})$	$P = 1 - \alpha$	$P = \beta$

2 (DE) H_1 accepted False Positive (Type 1) True Positive Detection (H_0 rejected)

Power: Sensitivity/Recall/Hit Rate: $\frac{TP}{TP+FN}=1-\beta$ Specificity/True negative rate: $\frac{TN}{FP \perp TN} = 1 - \alpha$ Precision/Positive Prediciton rate: TP Accuracy: $\frac{TP+TN}{P+N} = \frac{2-\alpha-\beta}{2}$

4.1.1. Design of a test

Cost criterion $G_{\varphi}:\Theta\to [0,1], \theta\mapsto \mathsf{E}[d(X)|\theta]$

False Positive lower than α : $G_d(\theta)|_{\theta \in \Theta_0} \leq \alpha, \forall \theta \in \Theta_0$ False Negative small as possible: $\max\{G_d(\theta)|_{\theta\in\Theta_1}\}, \forall \theta\in\Theta_1$

4.2. Sufficient Statistics

Sufficiency for a test T(X) means that no other test statistic, i.e., function of the observations \underline{x} , contains additional information about the parame-

 $f_{X|T}(x|T(x) = t, \theta) = f_{X|T}(x|T(x) = t)$

5. Math

 $\pi \approx 3.14159$ $e \approx 2.71828$ $\sqrt{2} \approx 1.414$ $\sqrt{3} \approx 1.732$ Binome, Trinome Binome, Trinome $(a\pm b)^2=a^2\pm 2ab+b^2 \qquad a^2-b^2=(a-b)(a+b)\\ (a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm b^3\\ (a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$

Folgen und Reihen

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \qquad \sum_{n=0}^{\infty} \frac{\mathbf{z}^n}{n!} = e^{\mathbf{z}}$$
 Aritmetrische Summenformel Geometrische Summenformel Exponentialreihe

Mittelwerte $(\sum \text{von } i \text{ bis } N)$ (Median: Mitte einer geordneten Liste) $\overline{x}_{ar} = \frac{1}{N} \sum x_i \geq \overline{x}_{geo} = \sqrt[N]{\prod x_i} \geq \overline{x}_{hm} = \frac{N}{\sum \frac{1}{x_i}}$ Arithmetisches Geometrisches Mittel Harmonisches $\sum \frac{1}{x_i}$

Bernoulli-Ungleichung: $(1+x)^n \ge 1 + nx$ Ungleichungen: $\left|\underline{\boldsymbol{x}}^{\top}\cdot\boldsymbol{y}\right|\leq\left\|\underline{\boldsymbol{x}}\right\|\cdot\left\|\boldsymbol{y}\right\|$ $||x| - |y|| \le |x \pm y| \le |x| + |y|$ Dreiecksungleichung **Mengen:** De Morgan: $\overline{A \cap B} = \overline{A} \uplus \overline{B}$ $\overline{A \uplus B} = \overline{A} \cap \overline{B}$

5.1. Exp. und Log. $e^x := \lim_{n \to \infty} (1 + \frac{x}{n})^n$ $e \approx 2,71828$ $a^{x} = e^{x \ln a} \qquad \log_{a} x = \frac{\ln x}{\ln a}$ $\ln(x^{a}) = a \ln(x) \qquad \ln(\frac{x}{a}) = \ln x - \ln a$ $\ln x \le x - 1$ log(1) = 0

5.2. Matrizen $oldsymbol{A} \in \mathbb{K}^{m imes n}$

 $m{A} = (a_{i\,i}) \in \mathbb{K}^{m imes n}$ hat m Zeilen (Index i) und n Spalten (Index j) $(\mathbf{A} + \mathbf{B})^{\top} = \mathbf{A}^{\top} + \mathbf{B}^{\top} \qquad (\mathbf{A} \cdot \mathbf{B})^{\top} = \mathbf{B}^{\top} \cdot \mathbf{A}^{\top}$ $(\mathbf{A}^{\top})^{-1} = (\mathbf{A}^{-1})^{\top}$ $(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$ $\dim \mathbb{K} = n = \operatorname{rang} \mathbf{A} + \dim \ker \mathbf{A} \quad \operatorname{rang} \mathbf{A} = \operatorname{rang} \mathbf{A}^{\top}$

5.2.1. Quadratische Matrizen $A \in \mathbb{K}^{n \times n}$ regulär/invertierbar/nicht-singulär $\Leftrightarrow \det(\mathbf{A}) \neq 0 \Leftrightarrow \operatorname{rang} \mathbf{A} = n$ singulär/nicht-invertierbar $\Leftrightarrow \det(\mathbf{A}) = 0 \Leftrightarrow \operatorname{rang} \mathbf{A} \neq n$ orthogonal $\Leftrightarrow \mathbf{A}^{\top} = \mathbf{A}^{-1} \Rightarrow \det(\mathbf{A}) = \pm 1$ symmetrisch: $\mathbf{A} = \mathbf{A}^{\top}$ schiefsymmetrisch: $\mathbf{A} = -\mathbf{A}^{\top}$

5.2.2. Determinante von $\widetilde{A} \in \mathbb{K}^{n \times n} \colon \det(\widetilde{A}) = |\widetilde{A}|$ $\det \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = \det \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} = \det (\underline{\boldsymbol{A}}) \det (\underline{\boldsymbol{D}})$ $det(\mathbf{A}) = det(\mathbf{A}^T)$ $\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{B})\det(\mathbf{A}) = \det(\mathbf{B}\mathbf{A})$

Hat \widetilde{A} $\widetilde{2}$ linear abhang. Zeilen/Spalten $\Rightarrow |A| = 0$ 5.2.3. Eigenwerte (EW) λ und Eigenvektoren (EV) v

 $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ det $\mathbf{A} = \prod \lambda_i$ Sp $\mathbf{A} = \sum a_{ii} = \sum \lambda_i$

Eigenwerte: $det(\mathbf{A} - \lambda \mathbf{1}) = 0$ Eigenvektoren: $ker(\mathbf{A} - \lambda_i \mathbf{1}) = \mathbf{v}_i$ EW von Dreieck/Diagonal Matrizen sind die Elem. der Hauptdiagonale. 5.2.4. Spezialfall 2×2 Matrix A

$$\begin{aligned} &\det(\underline{A}) = ad - bc \\ &\operatorname{Sp}(\underline{\underline{A}}) = a + d \end{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \underline{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &\lambda_{1/2} = \frac{\operatorname{Sp} \underline{A}}{2} \pm \sqrt{\left(\frac{\operatorname{sp} \underline{A}}{2}\right)^{2} - \det \underline{A}} \end{aligned}$$

$$\frac{\partial \underline{\underline{x}}^{\top}\underline{y}}{\partial \underline{\underline{x}}} = \frac{\partial \underline{y}^{\top}\underline{\underline{x}}}{\partial \underline{\underline{x}}} = \underline{y} \qquad \frac{\partial \underline{\underline{x}}^{\top}\underline{A}\underline{\underline{x}}}{\partial \underline{\underline{x}}} = (\underline{\underline{A}} + \underline{\underline{A}}^{\top})\underline{\underline{x}}$$
$$\frac{\partial \underline{\underline{x}}^{\top}\underline{A}\underline{y}}{\partial \underline{\underline{A}}} = \underline{\underline{x}}\underline{y}^{\top} \qquad \frac{\partial \det(\underline{\underline{B}}\underline{A}\underline{C})}{\partial \underline{\underline{A}}} = \det(\underline{\underline{B}}\underline{A}\underline{C}) \left(\underline{\underline{A}}^{-1}\right)^{\top}$$

5.2.6. Ableitungsregeln ($\forall \lambda, \mu \in \mathbb{R}$)

 $(\lambda f + \mu g)'(x) = \lambda f'(x) + \mu g'(x_0)$ Linearität: Produkt: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$ $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \quad \left(\frac{\text{NAZ-ZAN}}{\text{N}^2}\right)$ Quotient: Kettenregel (f(g(x)))' = f'(g(x))g'(x)

5.3. Integrale $\int e^x dx = e^x = (e^x)'$

Partielle Integration: $\int uw' = uw - \int u'w$ $\int f(g(x))g'(x) dx = \int f(t) dt$

F(x) - C	f(x)	f'(x)
$\frac{1}{q+1}x^{q+1}$	x^q	qx^{q-1}
$\frac{2\sqrt{ax^3}}{3}$	\sqrt{ax}	$\frac{\frac{a}{2\sqrt{ax}}}{\frac{1}{x}}$
$x \ln(ax) - x$	ln(ax)	$\frac{1}{x}$
$\frac{1}{a^2}e^{ax}(ax-1)$	$x \cdot e^{ax}$	$e^{ax}(ax+1)$
$\frac{a^x}{\ln(a)}$	a^x	$a^x \ln(a)$
$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\cosh(x)$	sinh(x)	$\cosh(x)$
$-\ln \cos(x) $	$\tan(x)$	$\frac{1}{\cos^2(x)}$

$$\int e^{at} \sin(bt) dt = e^{at} \frac{a \sin(bt) + b \cos(bt)}{a^2 + b^2}$$

$$\int \frac{dt}{\sqrt{at+b}} = \frac{2\sqrt{at+b}}{a} \qquad \int t^2 e^{at} dt = \frac{(ax-1)^2 + 1}{a^3} e^{at}$$

$$\int te^{at} dt = \frac{at-1}{a^2} e^{at} \qquad \int x e^{ax^2} dx = \frac{1}{2a} e^{ax^2}$$

5.3.1. Volumen und Oberfläche von Rotationskörpern um x-Achse $O = 2\pi \int_{a}^{b} f(x) \sqrt{1 + f'(x)^2} dx$ $V = \pi \int_a^b f(x)^2 dx$

6. Support Vector Machines

Motivation and Background

6.1. Kernel Methods

Kernel Methods is non-parametic estimation, these make no assumption on statistical model → purely Data-Based.

Test Statistic
$$\mathbb{X} \to \mathbb{R}, \mathbf{x} \mapsto S(\mathbf{x}) = \sum_{k=1}^M \lambda_k g(\mathbf{x}, \mu_\mathbf{k})$$

linear combination of Kernel Function $g(., \mu_k)$, g() generally non-linear pos. definite

 μ_k : representative for Sample Set $S = \{x_1, ..., x_M\}$

 λ_k : weight coefficient determined by learning

Sample Set S is Empirical Characterization of Unknown Statistical Model Infernce of λ_k based on Sample Set or Training Set is called **Learning**

6.2. Kernel Tests

Statistical Hypothesis Test, where a Sufficient Test Statistic is compared to threshold(i.e.R(x) \geq c) decomposes sample space \mathbb{X} into two disjoint $subsets(\mathbb{X} = \mathbb{X}_0 \cup \mathbb{X}_1)$

Seperating surface between X₀ and X₁ given by:

 $\{\mathbf{x}|R(\mathbf{x})=c\}$ The relative postion of a sample x_i to the separating surface determines choice of hypothesis

$$\mathbb{S} = \{(x_1, y_1), ..., (x_M, y_M)\}$$

 $x_i \in \mathbb{R}^N, \ y_i \in \{\Theta_0, \Theta_1\}$ Inference of Hypothesis Test based on a Sample Set that includes Labeling y_i of the elements x_i is called Supervised Learning

Size M of samples has to statisfy: M > dim(X)

Because underlying statistical model is unknown, true θ_0 and θ_1 irrelevant \rightarrow replace them by e.g. -1,+1 for decision between hypotheses

6.3. Linear Kernels

Test Statistic for linear test

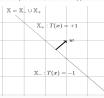
$$S(x) = \sum_{i=1}^{M} \lambda_i \mathbf{x_i}^T \mathbf{x} + wo = \mathbf{w}^T \mathbf{x} + wo \quad \mathbf{w} = \sum_{i=1}^{M} \lambda_i x_i$$

Hyperplane defined by w(normal vector or weight vector) and w_0 approximates seperating surface between X_ and X_+ \rightarrow Decistion rule T(x):

$$T(\mathbf{x}) = sign(S(\mathbf{x})) = \begin{cases} +1 & ; & \mathbf{w}^T \mathbf{x} + wo \ge 0 \\ -1 & ; & otherwise \end{cases}$$

Linear Kernel Test in sample space X:

(Orientation of w chosen such that w points into direction of θ_1 ("+1" hypothesis))



To determine w and w_0 formulate problem as constrained optimalization problem with the constraints:

 $\forall k \in \{1, ...M\} : T(\mathbf{x}_k) = y_k$

 \Rightarrow Support Vector Methods: $y_k(\mathbf{w}^T\mathbf{x}_k + wo) \ge \epsilon, \forall k$

Robust solution: maximize margin ϵ for constant norm of \mathbf{w}

Application

6.4. Support Vector Methods

only feasible for normalized weight vectors

$$\begin{aligned} \max_{w} & \epsilon \text{ s.t. } & y_k \frac{\mathbf{w}^T}{\|\mathbf{w}\|_2} \mathbf{x}_k \geq \epsilon, \forall k \text{ , } w_0 = 0 \\ & \Leftrightarrow \min_{w} \frac{1}{2} \|\mathbf{w}\|_2^2 \text{ s.t. } & y_k \mathbf{w}^T \mathbf{x}_k \geq 1, \forall k \\ \text{Optimization Problem convex} & \rightarrow \mathbf{Langragian Method} \end{aligned}$$

Dual Problem: maxmin $\Phi(\mathbf{w}, \mathbf{u})$ s.t. $\mathbf{u} \geq 0$

Langragian Multiplier: $u_k > 0$ Langragian Fct: $\Phi(\mathbf{w}, \mathbf{u}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + \sum_{k=1}^{M} u_k (1 - y_k \mathbf{w}^T \mathbf{x_k})$ $\frac{\partial \Phi(\mathbf{w}, \mathbf{u})}{\partial \mathbf{w}}|_{\mathbf{w} = \mathbf{w}(\mathbf{u})} \cdot = 0 \leftrightarrow \mathbf{w}(\mathbf{u}) = \sum_{k=1}^{M} \underbrace{u_k y_k}_{\mathbf{k}} \mathbf{x_k}$

Evaluate dual function:

Evaluate that function:
$$\begin{split} & \Phi(\mathbf{w}(\mathbf{u}), \mathbf{u}) = \Phi(\sum_{k=1}^{M} u_k y_k \mathbf{x}_k, u_1..., u_M) \\ & = -\frac{1}{2} \sum_{k=1}^{M} \sum_{l=1}^{M} u_k u_l y_k y_l \mathbf{x}_k^T \mathbf{x}_l + \sum_{k=1}^{M} u_k \\ & = -\frac{1}{2} \mathbf{u}_1^T \mathbf{Y} \mathbf{X} \mathbf{X}^T \mathbf{Y} \mathbf{u} + \mathbf{1}^T \mathbf{u} \end{split}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^{\mathbf{T}} \\ \vdots \\ \mathbf{x}_M^{\mathbf{T}} \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} y_1 \\ & \ddots \\ & & y_M \end{bmatrix}, \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Alternativ to approach above

Iterative Solution:

Choose one element \mathbf{x}_k out of sample set $\mathbb{S} = \{\mathbf{x_1}, ..., \mathbf{x_M}\}$ and

$$u_k \leftarrow u_k + \max_{\{\eta \frac{\partial \phi(\mathbf{u})}{\partial u_k}, -u_k\}, \forall k\}$$

 $u_k \leftarrow u_k + \max\{ \eta \frac{\partial \phi(\mathbf{u})}{\partial u_k}, -u_k \}, \forall k$ Necessary and sufficient condition for existence of solution given by: $1 \in \mathsf{conce}[\mathbf{Y}\mathbf{X}\mathbf{X}^{\mathbf{T}}\mathbf{Y}]$

6.5. Suport Vectors

Dual OP.: $\max_{\mathbf{x}} \sum_{k=1}^{M} (-\frac{1}{2} \sum_{l=1}^{M} u_k u_l y_k y_l \mathbf{x_k^T} \mathbf{x_l} + u_k)$ s.t. $u_k \geq 0$

Optimal Dual Variables $u_1^*, ..., u_M^*$ either active $u_k > 0$ or inactive $u_k = 0$

Elements of S with active dual variables = Support Vectors $\mathbb{S}_{SV} = \{ \mathbf{x}_k \in \mathbb{S} | u_k^* > 0 \}$

Elements with inactive dual variables dont contribute to Kernel Test **Optimal Weight Vektor** $\mathbf{w}^* = \mathbf{w}(\mathbf{u}^*)$ of Kernel Test constructed by

Support Vectors only: $\mathbf{w}^* = \sum u_k^* y_k \mathbf{x}_k$

Number of Support Vectors approx. size of dim $[X] \rightarrow$ selection of Support Vectors reduces computational complexity of Kernel Test

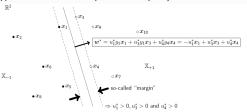


Fig. 2.2: The elements $x_h \in \mathbb{S}$ with ACTIVE DUAL VARIABLES $u_h^* > 0$ are called SUPPORT VECTORS

- Exists only if S Linearly Separable
- $w_0 \neq 0$ no (straightforward) iterative solution available
- if Linearly Inseperable method generalized by slack variables for controlled violation of constraints

 $\begin{array}{l} \rightarrow \text{ instead of } \min \frac{1}{2}\mathbf{w^Tw} \text{ s.t. } y_k\mathbf{w^Tx}_k \geq 1 \text{ we get} \\ \min \frac{1}{2}\mathbf{w^Tw} + \rho \sum_{k=1}^{M} \epsilon_k \text{ s.t.} y_k\mathbf{w^Tx}_k \geq 1 - \epsilon_k, \forall k, \underline{\epsilon}, \rho \geq 0 \end{array}$

6.6. Kernel Trick

Linear Hypothesis Test often not sufficient → Kernel Trick: Generalize linear methods to non-linear approximation of seperating surfaces $(\{x | \log R(\mathbf{x}) = c\})$

Basic Idea: Transfer problem statement into higher-dimensional space(without introducing additional degrees of freedom) by Feature Map $\varphi: \mathbb{S} \to \mathbb{S}_{\varphi}$

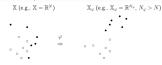


Fig. 2.3: Transfer the problem statement into a higher-dimensional (inner product) space without intro ducing additional degrees of freedom by means of a so-called Feature Map $\varphi: \mathbb{S} \to \mathbb{S}_{\omega}$.

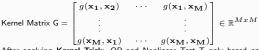
Construction of Linear Test in \mathbb{R}^3 correspondes to Non-Linear Test in \mathbb{R}^2

$$T: \mathbb{R}^3 \to \{-1, +1\}, \varphi(\mathbf{x}) \mapsto \begin{cases} +1; & \mathbf{w}_{\varphi}^T \varphi(\mathbf{x}) \ge 0 \\ -1; & otherwise \end{cases}$$

Linear kernel in \mathbb{X}_{φ} represents nonlinear kernel in \mathbb{X} \to choose g(...)directly instead of finding transformation φ

$$\langle \varphi(\mathbf{x}), \varphi(\mathbf{y}) \rangle =: g(\mathbf{x}, \mathbf{y})$$

In Optimization Problem and resulting Dual Function and Variables replace \mathbf{x} by $\varphi(\mathbf{x}_k) \to \mathsf{Dual}$ OP: $\max_{\mathbf{u} > 0} \{ -\mathbf{u^T Y G Y u} + \mathbf{1^T u} \}$



After applying Kernel Trick: OP and Nonlinear Test T only based on Kernel Function g. transformation φ becomes obsolete

 $\text{Hypothesis Test(nonlinear):} \quad T: \mathbf{x} \mapsto sign(\sum_{k=1}^{M} u_k^* y_k g(\mathbf{x_k}, \mathbf{x}))$

Possible Kernels for Kernel Trick

Linear Kernel: $g_{lin}(\mathbf{x}, \mathbf{x}_k) = \mathbf{x}_k^T \mathbf{x}$

Polynomial Kernel: $g_{nolu}(\mathbf{x}, \mathbf{x}_k) = (\mathbf{x}_k^T \mathbf{x} + 1)^d$

Sigmoid Kernel: $g_{sigm}(\mathbf{x}, \mathbf{x}_k) = \tanh(\beta(\mathbf{x}_k^T \mathbf{x}) + w_0)$ Radial Kernel: $g_{rbf}(\mathbf{x}, \mathbf{x}_k) = \exp(-\frac{1}{2\pi^2} \|\mathbf{x} - \mathbf{x}_k\|_2^2)$

Support Vector Machine Representation.

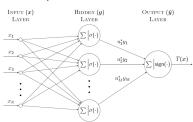


Fig. 2.4: The interpretation of a SUPPORT VECTOR MACHINE as a NEURAL NETWORK with three layers and a non-linear function σ . For POLYNOMIAL KERNELS each SINGLE HIDDEN LAYER UNIT is described by $g_{\text{poly}}(x, x_k) = \sigma(z_k)$, with $\sigma(z_k) = z_k^d$ and $z_k = x_k^T x + 1$

7. Probability Theory Basics

7.1. Kombinatorik

Mögliche Variationen/Kombinationen um k Elemente von maximal n Elementen zu wählen bzw. k Elemente auf n Felder zu verteilen:

	Mit Reihenfolge	Reihenfolge egal
Mit Wiederholung Ohne Wiederholung	$\frac{n^k}{\frac{n!}{(n-k)!}}$	$\binom{n+k-1}{k} \binom{n}{k}$

Permutation von n mit jeweils k gleichen Elementen: $\frac{n!}{k_1! \cdot k_2! \cdot \ldots}$ Binomialkoeffizient $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k! \cdot (n-k)!}$

Binomialkoeffizient
$$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k! \cdot (n-k)!}$$

 $\binom{n}{0} = 1$ $\binom{n}{1} = n$ $\binom{4}{2} = 6$ $\binom{5}{2} = 10$ $\binom{6}{2} = 15$

7.2. Der Wahrscheinlichkeitsraum (Ω, \mathbb{F}, P)

Ergebnismenge $\Omega = \{\omega_1, \omega_2, \ldots\}$ Ergebnis $\omega_i \in \Omega$ Ereignis $A_i \subseteq \Omega$ $\mathrm{P}(A) = \frac{|A|}{|\Omega|}$ Ereignisalgebra $\mathbb{F} = \{A_1, A_2, \dots\}$ $P: \mathbb{F} \to [0, 1]$ Wahrscheinlichkeitsmaß

7.3. Wahrscheinlichkeitsmaß P

$$P(A) = \frac{|A|}{|\Omega|}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

7.3.1. Axiome von Kolmogorow

 $\text{Nichtnegativit\"at:} \qquad \mathsf{P}(A) \geq 0 \Rightarrow \mathsf{P}: \mathbb{F} \mapsto [0,1]$

$$\begin{split} & \mathbf{P} \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mathbf{P}(A_i), \\ & \text{wenn } A_i \cap A_j = \emptyset, \, \forall i \neq j \end{split}$$

7.4. Bedingte Wahrscheinlichkeit

Bedingte Wahrscheinlichkeit für A falls B bereits eingetreten ist: $P_B(A) = P(A|B) = \frac{P(A \cap B)}{P(B)}$

7.4.1. Totale Wahrscheinlichkeit und Satz von Bayes Es muss gelten: $\bigcup_i B_i = \Omega$ für $B_i \cap B_j = \emptyset$, $\forall i \neq j$

 $\begin{array}{ll} & i \in I \\ & \text{Totale Wahrscheinlichkeit:} & & \mathsf{P}(A) = \sum\limits_{i \in I} \mathsf{P}(A|B_i) \, \mathsf{P}(B_i) \\ & \text{Satz von Bayes:} & & \mathsf{P}(B_k|A) = \frac{\mathsf{P}(A|B_k) \, \mathsf{P}(B_k)}{\sum\limits_{i \in I} \mathsf{P}(A|B_i) \, \mathsf{P}(B_i)} \\ & & \\ & & \\ \end{array}$

Multiplikationssatz: $P(A \cap B) = P(A|B) P(B) = P(B|A) P(A)$

7.5. Zufallsvariable

 $X: \Omega \mapsto \Omega'$ ist Zufallsvariable, wenn für jedes Ereignis $A' \in \mathbb{F}'$ im Bildraum ein Ereignis A im Urbildraum \mathbb{F} existiert, sodass $\{\omega \in \Omega | X(\omega) \in A'\} \in \mathbb{F}$

7.6. Distribution

Bezeichnung	Abk.	Zusammenhang
Wahrscheinlichkeitsdichte	pdf	$f_{X}(x) = \frac{\mathrm{d}F_{X}(x)}{\mathrm{d}x}$
Kumulative Verteilungsfkt.	cdf	$F_X(x) = \int_{-\infty}^x f_X(\xi) d\xi$

Joint CDF: $F_{X,Y}(x,y) = P(\{X \le x, Y \le y\})$

7.7. Relations between $f_{\mathbf{X}}(x), f_{\mathbf{X},\mathbf{Y}}(x,y), f_{\mathbf{X}\mid\mathbf{Y}}(x|y)$

$$f_{X,Y}(x,y) = f_{X\mid Y}(x,y) f_{Y}(y) = f_{Y\mid X}(y,x) f_{X}(x)$$

$$\int_{\text{Joint PDF}}^{\infty} f_{X,Y}(x,\xi) \, \mathrm{d}\xi = \int_{-\infty}^{\infty} f_{X\mid Y}(x,\xi) f_{Y}(\xi) \, \mathrm{d}\xi = f_{X}(x)$$

$$\underbrace{\int_{\text{Marginalization}}^{\infty} f_{X\mid Y}(x,\xi) f_{Y}(\xi) \, \mathrm{d}\xi}_{\text{Total Probability}} = f_{X}(x)$$

7.8. Bedingte Zufallsvariablen

Ereignis A gegeben: $F_{X|A}(x|A) = P(\{X \le x\}|A)$ ZV Y gegeben: $F_{X|Y}(x|y) = P(\{X \le x\} | \{Y = y\})$ $p_{X\mid Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_{Y}(y)}$ $f_{X\mid Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_{Y}(y)} = \frac{\mathrm{d}F_{X\mid Y}(x|y)}{\mathrm{d}x}$

7.9. Unabhängigkeit von Zufallsvariablen

 X_1, \dots, X_n sind stochastisch unabhängig, wenn für jedes $x \in \mathbb{R}^n$ gilt:

$$\begin{split} F_{X_1, \cdots, X_n}(x_1, \cdots, x_n) &= \prod_{i=1}^n F_{X_i}(x_i) \\ p_{X_1, \cdots, X_n}(x_1, \cdots, x_n) &= \prod_{i=1}^n p_{X_i}(x_i) \\ f_{X_1, \cdots, X_n}(x_1, \cdots, x_n) &= \prod_{i=1}^n f_{X_i}(x_i) \end{split}$$

8. Common Distributions

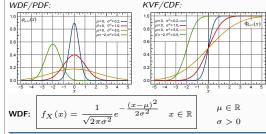
8.1. Binomialverteilung $\mathcal{B}(n,p)$ mit $p \in [0,1], n \in \mathbb{N}$ Folge von n Bernoulli-Experimenten

p: Wahrscheinlichkeit für Erfolg k: Anzahl der Erfolge

$$p_{\mathsf{X}}(k) = B_{n,p}(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k \in \{0,\dots,n\} \\ 0 & \mathsf{sonst} \end{cases}$$

 $\mathsf{E}[\mathsf{X}] = np \qquad \mathsf{Var}[\mathsf{X}] = np(1-p) \qquad G_X(z) = \left(pz + 1 - p\right)^n$ Frwartungswert

8.2. Normalverteilung



 $arphi_X(\omega) = e^{j\omega\mu - rac{\omega^2\sigma^2}{2}}$ Charakt. Funktion $Var(X) = \sigma^2$ $E(X) = \mu$ Frwartungswert

8.3. Sonstiges

Gammadistribution $\Gamma(\alpha, \beta)$: $E[X] = \frac{\alpha}{\beta}$

Exponential: $f(x, \lambda) = \lambda e^{-\lambda x}$ $E[X] = \lambda^{-1}$ $Var[X] = \lambda^{-2}$

9. Wichtige Parameter

9.1. Erwartungswert (1. zentrales Moment)

gibt den mittleren Wert einer Zufallsvariablen an

$$\begin{array}{cccc} \mu_X = \mathsf{E}[X] = \sum\limits_{x \in \Omega'} x \cdot \mathsf{P}_X(x) & \stackrel{\wedge}{=} & \int\limits_{\mathbb{R}} x \cdot f_X(x) \, \mathrm{d}x \\ & \text{diskrete } X : \Omega \! \to \! \Omega' & \text{stetige } X : \Omega \! \to \! \mathbb{R} \end{array}$$

 $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$ $X \leq Y \Rightarrow E[X] \leq E[Y]$ $\mathsf{E}[X^2] = \mathsf{Var}[X] + \mathsf{E}[X]^2$

E[X Y] = E[X] E[Y], falls X und Y stochastisch unabhängig

9.1.1. Für Funktionen von Zufallsvariablen q(x)

$$\mathsf{E}[g(\mathsf{X})] = \sum_{x \in \Omega'} g(x) \, \mathsf{P}_{\mathsf{X}}(x) \quad \stackrel{\wedge}{=} \quad \int\limits_{\mathbb{R}} g(x) f_{\mathsf{X}}(x) \, \mathrm{d}x$$

9.2. Varianz (2. zentrales Moment)

ist ein Maß für die Stärke der Abweichung vom Erwartungswert

$$\sigma_X^2 = \operatorname{Var}[X] = \operatorname{E}\left[(X - \operatorname{E}[X])^2\right] = \operatorname{E}[X^2] - \operatorname{E}[X]^2$$

$$\begin{aligned} & \overline{ \text{Var}[\alpha \, X + \beta] = \alpha^2 \, \text{Var}[X] } & \text{Var}[X] = \text{Cov}[X, X] \\ & \text{Var}\left[\sum_{i=1}^n X_i \right] = \sum_{i=1}^n \text{Var}[X_i] + \sum_{j \neq i} \text{Cov}[X_i, X_j] \end{aligned}$$

9.3. Kovarianz

Maß für den linearen Zusammenhang zweier Variablen

$$Cov[X, Y] = E[(X - E[X])(Y - E[Y])^{\top}] =$$

$$= E[X Y^{\top}] - E[X] E[Y]^{\top} = Cov[Y, X]$$

 $\begin{aligned} & \operatorname{Cov}[\alpha \, X + \beta, \, \gamma \, Y + \delta] = \alpha \gamma \, \operatorname{Cov}[X, \, Y] \\ & \operatorname{Cov}[X + U, \, Y + V] = \operatorname{Cov}[X, \, Y] + \operatorname{Cov}[X, \, V] + \operatorname{Cov}[U, \, Y] + \operatorname{Cov}[U, \, V] \end{aligned}$

9.3.1. Korrelation = standardisierte Kovarianz

$$\rho(\mathbf{X},\mathbf{Y}) = \frac{\mathsf{Cov}[\mathbf{X},\mathbf{Y}]}{\sqrt{\mathsf{Var}[\mathbf{X}]\cdot\mathsf{Var}[\mathbf{Y}]}} = \frac{C_{x,y}}{\sigma_{x}\cdot\sigma_{y}} \qquad \rho(\mathbf{X},\mathbf{Y}) \in [-1;1]$$

9.3.2. Kovarianzmatrix für
$$\underline{z} = (\underline{x}, \underline{y})^{\top}$$

$$\operatorname{Cov}[\underline{z}] = \underline{C}_{\underline{z}} = \begin{bmatrix} C_X & C_{XY} \\ C_{XY} & C_Y \end{bmatrix} = \begin{bmatrix} \operatorname{Cov}[X, X] & \operatorname{Cov}[X, Y] \\ \operatorname{Cov}[Y, X] & \operatorname{Cov}[Y, Y] \end{bmatrix}$$

$$\operatorname{Immer symmetrisch}: C_{xy} = C_{yx}! \text{ Für Matrizen}: \underline{C}_{\underline{x}\underline{y}} = \underline{C}_{y}^{\top}$$

10. Estimation

10.1. Estimation

Statistic Estimation treats the problem of inferring underlying characteristics of unknown random variables on the basis of observations of outputs of those random variables.

Sample Space Ω nonempty set of outputs of experiment Sigma Algebra $\mathbb{F} \subseteq 2^{\Omega}$ set of subsets of outputs (events) Probability $P : \mathbb{F} \mapsto [0, 1]$ Random Variable $X : \Omega \mapsto X$ mapped subsets of Ω

Observations: x_1, \ldots, x_N single values of Xpossible observations of XObservation Space X Unknown parameter $\theta \in \Theta$

parameter of propability function Estimator $O \longrightarrow : X \mapsto \Theta$ $\bigcirc - \bullet (X) = \hat{\theta}$, finds $\hat{\theta}$ from X

unknown parm θ estimation of param $\hat{\theta}$ R.V. of param. ⊖ estim. of R.V. of parm $T(X) = \hat{\Theta}$

10.2. Quality Properties of Estimators

$$\begin{array}{ll} \text{Consistent: If } \lim_{N \to \infty} \circ \longrightarrow (x_1, \dots, x_N) = \theta \\ \text{Bias Bias} (\circ \longrightarrow) := \mathsf{E} [\circ \longrightarrow (\mathsf{X}_1, \dots, \mathsf{X}_N)] - \theta \end{array}$$

unbiased if $Bias(\bigcirc \longrightarrow) = 0$ (biased estimators can provide better estimates than unbiased estimators.)

Variance $Var[\bigcirc \bullet] := E[(\bigcirc \bullet - E[\bigcirc \bullet])^2]$

10.3. Mean Square Error (MSE)

The MSE is an extension of the Variance
$$Var[o - \bullet] := E[(o - \bullet - E[o - \bullet])^2]:$$

$$\varepsilon[\circ - \bullet] = \mathsf{E} \left[(\circ - \bullet - \theta)^2 \right]^\mathsf{MSE:} \\ = \mathsf{E}[(\hat{\theta} - \theta)^2] = \mathsf{Var}(\circ - \bullet) + (\mathsf{Bias}[\circ - \bullet])^2$$

If
$$\Theta$$
 is also r.v. \Rightarrow mean over both (e.g. Bayes est.):

Mean MSE:
$$E[(\bigcirc - \bullet (X) - \Theta)^2]$$

 $E[E[(\bigcirc - \bullet (X) - \Theta)^2 | \Theta = \theta]]$

10.3.1. Minimum Mean Square Error (MMSE)

Minimizes mean square error: $\arg\min \mathsf{E}\left[(\hat{\theta}-\theta)^2\right]$

$$\mathsf{E}\left[(\hat{\theta} - \theta)^2\right] = \mathsf{E}[\theta^2] - 2\hat{\theta}\,\mathsf{E}[\theta] + \hat{\theta}^2$$

$$\text{Solution: } \frac{\mathrm{d}}{\mathrm{d}\hat{\theta}} \, \mathsf{E} \left[(\hat{\theta} - \theta)^2 \right] \stackrel{!}{=} 0 = -2 \, \mathsf{E}[\theta] + 2 \hat{\theta} \quad \Rightarrow \hat{\theta}_{\mathsf{MMSE}} = \mathsf{E}[\theta] \, \bigg|$$

10.4. Maximum Likelihood

Given model $\{\mathbb{X}, \mathbb{F}, \mathsf{P}_{\theta}; \theta \in \Theta\}$, assume $\mathsf{P}_{\theta}(\underline{\boldsymbol{x}})$ or $f_X(\underline{\boldsymbol{x}}, \theta)$ for observed data x. Estimate parameter θ so that the likelihood $L(x, \theta)$ or $L(\theta | X = \underline{x})$ to obtain \underline{x} is maximized

Likelihood Function: (Prob. for θ given x)

Discrete:
$$L(x_1, \dots, x_N; \theta) = P_{\theta}(x_1, \dots, x_N)$$

Continuous: $L(x_1, \ldots, x_N; \theta) = f_{X_1, \ldots, X_N}(x_1, \ldots, x_N, \theta)$ If N observations are Identically Independently Distributed (i.i.d.):

$$L(\underline{\boldsymbol{x}}, \theta) = \prod_{i=1}^{N} \mathsf{P}_{\theta}(x_i) = \prod_{i=1}^{N} f_{X_i}(x_i)$$

ML Estimator (Picks
$$\theta$$
): $O \longrightarrow {}_{ML} : X \mapsto \operatorname*{argmax} \{L(X, \theta)\} =$

$$= \operatorname*{argmax}\{\log L(\underline{\mathbf{X}}, \boldsymbol{\theta})\} \overset{\text{i.i.d.}}{=} \operatorname*{argmax}\big\{\sum \log L(x_i, \boldsymbol{\theta})\big\}$$

Find Maximum:
$$\frac{\partial L(\underline{x}, \theta)}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}\theta} \log L(x; \theta) \Big|_{\theta = \hat{\theta}} \stackrel{!}{=} 0$$

Solve for θ to obtain ML estimator function $\hat{\theta}_{\rm MI}$

Check quality of estimator with MSE

Maximum-Likelihood Estimator is Asymptotically Efficient. However, there might be not enough samples and the likelihood function is often not known

10.5. Uniformly Minimum Variance Unbiased (UMVU) Estimators (Best unbiased estimators)

Best unbiased estimator: Lowest Variance of all estimators. Fisher's Information Inequality: Estimate lower bound of variance if

- $L(x,\theta) > 0, \forall x, \theta$
- $L(x, \theta)$ is diffable for θ
- $\bullet \ \int_{\mathbb{X}} \frac{\partial}{\partial \theta} L(x,\theta) \, \mathrm{d}x = \frac{\partial}{\partial \theta} \int_{\mathbb{X}} L(x,\theta) \, \mathrm{d}x$ Score Function:

$$g(x,\theta) = \frac{\partial}{\partial \theta} \log L(x,\theta) = \frac{\frac{\partial}{\partial \theta} L(x,\theta)}{L(x,\theta)} \qquad \mathsf{E}[g(x,\theta)] = 0$$
 Fischer Information:

Fischer Information:
$$I_{\mathsf{F}}(\theta) := \mathsf{Var}[g(\mathsf{X},\theta)] = \mathsf{E}[g(x,\theta)^2] = -\,\mathsf{E}\left[\frac{\partial^2}{\partial \theta^2} \log L(\mathsf{X},\theta)\right]$$

Cramér-Rao Lower Bound (CRB): (if O- is unbiased

For N i.i.d. observations: $I_{\mathbf{r}}^{(N)}(x,\theta) = N \cdot I_{\mathbf{r}}^{(1)}(x,\theta)$

10.5.1. Exponential Models

If
$$f_X(x) = \frac{h(x)\exp\left(a(\theta)t(x)\right)}{\exp(b(\theta))}$$
 then $I_F(\theta) = \frac{\partial a(\theta)}{\partial \theta}\,\frac{\partial E[t(X)]}{\partial \theta}$

Some Derivations: (check in exam

Uniformly: Not diffable \Rightarrow no $I_F(\theta)$

Normal
$$\mathcal{N}(\theta, \sigma^2)$$
: $g(x, \theta) = \frac{(x - \theta)}{\sigma^2}$ $I_{\mathsf{F}}(\theta) = \frac{1}{\sigma^2}$ Binomial $\mathcal{B}(\theta, K)$: $g(x, \theta) = \frac{x}{\theta} - \frac{K - x}{1 - \theta}$ $I_{\mathsf{F}}(\theta) = \frac{K}{\theta(1 - \theta)}$

10.6. Bayes Estimation (Conditional Mean)

A Priori information about θ is known as probability $f_{\Theta}(\theta; \sigma)$ with random variable Θ and parameter σ . Now the conditional pdf $f_{X \mid \Theta}(x, \theta)$ is used to find θ by minimizing the mean MSE instead of uniformly MSE. Mean MSE for Θ : $\mathbb{E}\left[\mathbb{E}[(T(X) - \Theta)^2 | \Theta = \theta]\right]$

Conditional Mean Estimator:

Posterior
$$f_{\Theta|X}(\theta|\underline{x}) = \frac{f_{X|\Theta}(\underline{x})f_{\theta}(\theta)}{f_{\Theta}f_{X,\xi}(\underline{x},\xi)d\xi} = \frac{f_{X|\Theta}(\underline{x})f_{\theta}(\theta)}{f_{X}(x)}$$

Hint: to calculate $f_{\Theta|X}(\theta|\underline{x})$: Replace every factor not containing θ , such as $\frac{1}{f_{Y}(x)}$ with a factor γ and determine γ at the end such that $\int_{\Theta} f_{\Theta|X}(\theta|\underline{x}) d\theta = 1$

$$\mathsf{MMSE} : \mathsf{E}[\mathsf{Var}[X \mid \Theta = \theta]]$$

$$\begin{array}{ll} \text{Multivariate Gaussian: } X, \Theta \sim \mathcal{N} & \Rightarrow \sigma_X^2 = \sigma_X^2 \mid_{\Theta = \theta} + \sigma_\Theta \\ \bigcirc \bullet_{\text{CM}} : x \mapsto \text{E}[\Theta \mid X = x] = \underline{\mu}_\Theta + \underline{C}_\Theta, x \underline{C}_X^{-1} (\underline{x} - \underline{\mu}_X) \\ \text{MMSF}. \end{array}$$

$$\mathbb{E}\left[\left\| \circ - \bullet_{\mathsf{CM}} - \Theta\right\|_{2}^{2}\right] = \operatorname{tr}(\tilde{\boldsymbol{C}}_{\theta \mid X}) = \operatorname{tr}(\tilde{\boldsymbol{C}}_{\Theta} - \tilde{\boldsymbol{C}}_{\Theta}, \boldsymbol{X}\tilde{\boldsymbol{C}}_{X}^{-1}\tilde{\boldsymbol{C}}_{X}, \Theta)$$

Orthogonality Principle:

$$0 \longrightarrow \mathsf{CM}(\underline{X}) - \Theta \perp h(\underline{X}) \quad \Rightarrow \quad \mathsf{E}[(T_{\mathsf{CM}}(\underline{X}) - \Theta)h(\underline{X})] = 0$$

MMSE Estimator: $\hat{\theta}_{MMSE} = \arg \min MSE$

minimizes the MSE for all estimators

10.7. Example:

Estimate mean
$$\theta$$
 of X with prior knowledge $\theta \in \Theta \sim \mathcal{N}$: $X \sim \mathcal{N}(\theta, \sigma_{X\mid\Theta=\theta}^2)$ and $\Theta \sim \mathcal{N}(m, \sigma_{\Theta}^2)$

$$\hat{\theta}_{\mathsf{CM}} = \mathsf{E}[\Theta | \underline{X} = \underline{x}] = \frac{N \sigma_{\Theta}^2}{\sigma_X^2 | \Theta = \theta^{+N} \sigma_{\Theta}^2} \hat{\theta}_{\mathsf{ML}} + \frac{\sigma_X^2 | \Theta = \theta}{\sigma_X^2 | \Theta = \theta^{+N} \sigma_{\Theta}^2} m$$

For N independent observations x_i : $\hat{\theta}_{ML} = \frac{1}{N} \sum x_i$ Large $N \Rightarrow \mathsf{ML}$ better, small $N \Rightarrow \mathsf{CM}$ better

11. Linear Estimation

t is now the unknown parameter θ , we want to estimate u and \underline{x} is the input vector... review regression problem $y=A\underline{x}$ (we solve for \underline{x}), here we solve for \underline{t} , because \underline{x} is known (measured)! Confusing...

1. Training → 2. Estimation Training: We observe y and \underline{x} (knowing both) and then based on that we try to estimate y given x (only observe x) with a

linear model $\hat{u} = \mathbf{x}^{\top} \mathbf{t}$

Estimation:
$$\hat{y} = \underline{x}^{\top}\underline{t} + m$$
 or $\hat{y} = \underline{x}^{\top}\underline{t}$

Given: N observations (y_i, x_i) , unknown parameters t, noise m

$$\underline{\underline{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \underline{\underline{X}} = \begin{bmatrix} \underline{\underline{x}}_1^{ op} \\ \vdots \\ \underline{\underline{x}}_n^{ op} \end{bmatrix} \quad \text{Note: } \hat{y}
eq y!!$$

Problem: Estimate y based on given (known) observations \underline{x} and unknown parameter t with assumed linear Model: $\hat{y} = x^{\top} t$

Note
$$y = \underline{\underline{x}}^{\top}\underline{t} + m \rightarrow y = \underline{\underline{x}}'^{\top}\underline{t}'$$
 with $\underline{\underline{x}}' = \begin{pmatrix} \underline{\underline{x}} \\ 1 \end{pmatrix}$, $t' = \begin{pmatrix} \underline{\underline{t}} \\ m \end{pmatrix}$

Sometimes in Exams: $\hat{y} = x^{\top}t \Leftrightarrow \hat{x} = T^{\top}y$ estimate \underline{x} given y and unknown T

11.1. Least Square Estimation (LSE)

Tries to minimize the square error for linear Model: $\hat{y}_{1S} = x^{\top} t_{1S}$

Least Square Error: $\min \left| \sum_{i=1}^{N} (y_i - \underline{x}_i^{\top} \underline{t})^2 \right| = \min_{\underline{t}} \left\| \underline{y} - \underline{X}\underline{t} \right\|$

$$\underline{\boldsymbol{t}}_{\mathsf{LS}} = (\underline{\boldsymbol{X}}^{\top}\underline{\boldsymbol{X}})^{-1}\underline{\boldsymbol{X}}^{\top}\underline{\boldsymbol{y}}$$

$$\underline{\hat{y}}_{LS} = \underline{X}\underline{t}_{LS} \in span(X)$$

Orthogonality Principle: N observations $\boldsymbol{x}_i \in \mathbb{R}^d$ $Y - XT_{1S} \perp \operatorname{span}[X] \Leftrightarrow Y - XT_{1S} \in \operatorname{null}[X^{\top}]$, thus $X^{\top}(Y - XT_{1S}) = 0$ and if $N > d \wedge rang[X] = d$: $T_{\mathsf{LS}} = (\mathbf{X}^{\top} \mathbf{X})^{-1} \mathbf{X}^{\top} \mathbf{Y}$

11.2. Linear Minimum Mean Square Estimator (LMMSE)

Estimate y with linear estimator t, such that $\hat{y} = t^{\top}x + m$ Note: the Model does not need to be linear! The estimator is linear!

$$\hat{y}_{\mathsf{LMMSE}} = \mathop{\arg\min}_{t,m} \mathsf{E} \left[\left\| \underline{\boldsymbol{y}} - (\underline{\boldsymbol{t}}^{\top}\underline{\boldsymbol{x}} + m) \right\|_2^2 \right]$$

If Random joint variable
$$\underline{z} = \begin{pmatrix} \underline{x} \\ y \end{pmatrix}$$
 with

$$\underline{\mu}_{\underline{z}} = \begin{pmatrix} \underline{\mu}_{\underline{x}} \\ \underline{\mu}_{y} \end{pmatrix}$$
 and $\underline{C}_{\underline{z}} = \begin{bmatrix} C_{\underline{x}} & \underline{c}_{\underline{x}y} \\ c_{y\underline{x}} & c_{y} \end{bmatrix}$ then

LMMSE Estimation of
$$y$$
 given x is
$$\hat{y} = \mu_y + \underline{c}_{y\underline{w}} \underline{C}_{\underline{w}}^{-1} (\underline{x} - \underline{\mu}_{\underline{w}}) = \underline{\underline{c}_{y\underline{w}}} \underline{C}_{\underline{w}}^{-1} \underline{x} - \underline{\mu}_y + \underline{c}_{y\underline{w}} \underline{C}_{\underline{w}}^{-1} \underline{\mu}_{\underline{w}}$$

$$= t^{\top}$$

$$= m$$
Variance σ^2 for $\mathcal{N}(\mu, \sigma^2)$: $\hat{\sigma}_{\mathsf{ML}}^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2$

$$\text{Minimum MSE: E}\left[\left\|\underline{\boldsymbol{y}}-(\underline{\boldsymbol{x}}^{\top}\underline{\boldsymbol{t}}+m)\right\|_{2}^{2}\right]=c_{y}-c_{y\underline{\underline{x}}}C_{\underline{\underline{x}}}^{-1}\underline{c}_{\underline{\underline{x}}y}$$

Hint: First calculate \hat{y} in general and then set variables according to system equation.

Multivariate:
$$\hat{\underline{y}} = \tilde{\underline{x}}_{LMMSE}^{T} \underline{\underline{x}}$$
 $\tilde{\underline{x}}_{LMMSE}^{T} = \tilde{\underline{C}}_{y\underline{x}}\tilde{\underline{C}}_{x}^{-1}$

If
$$\underline{\mu}_z = \underline{0}$$
 then

Estimator
$$\hat{y} = \underline{c}_{y,\underline{x}} C_{\underline{x}}^{-1} \underline{x}$$

Minimum MSE:
$$\mathrm{E}[c_{y,\underline{x}}] = c_y - \underline{t}^{\top}\underline{c}_{\underline{x},y}$$

11.3. Matched Filter Estimator (MF)

For channel y = hx + v, Filtered: $t^{\top}y = t^{\top}hx + t^{\top}v$

Find Filter
$$\underline{t}^{\top}$$
 that maximizes SNR $=\frac{\|\underline{h}x\|}{\|\underline{v}\|}$

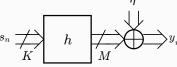
$$\underline{\boldsymbol{t}}_{\mathsf{MF}} = \max_{t} \left\{ \frac{\mathsf{E}\left[\left(\underline{\boldsymbol{t}}^{\top}\underline{\boldsymbol{h}}\boldsymbol{x}\right)^{2}\right]}{\mathsf{E}\left[\left(\underline{\boldsymbol{t}}^{\top}\underline{\boldsymbol{v}}\right)^{2}\right]} \right\}$$

In the lecture (estimate h

$$\underline{T}_{\mathsf{MF}} = \max_{T} \left\{ \frac{\left| \mathbf{E} \left[\underline{\hat{\boldsymbol{h}}}^H \underline{\boldsymbol{h}} \right] \right|^2}{\operatorname{tr} \left[\mathsf{Var} \left[\underline{T} \underline{\boldsymbol{n}} \right] \right]} \right\}$$

$$\hat{\underline{h}}_{\mathsf{MF}} = \underline{\underline{T}}_{\mathsf{MF}}\underline{\underline{y}}$$
 $\underline{\underline{T}}_{\mathsf{MF}} \propto \underline{\underline{C}}_{\underline{h}}\underline{\underline{S}}^H\underline{\underline{C}}_{\underline{n}}^{-1}$

11.4. Example



System Model: $oldsymbol{y}_{m} = H \underline{s}_{n} + \eta_{n}$

 $\begin{array}{ll} & \underbrace{\underline{-n}}_{n} & \underbrace{-n}_{n-k} \\ & \text{with } \underline{H} = (h_{m,k}) \in \mathbb{C}^{M \times K} & (m \in [1,M], k \in [1,K]) \\ \text{Linear Channel Model } \underline{y} = \underbrace{S\underline{h}}_{n} + \underline{n} \text{ with} \end{array}$ $h \sim \mathcal{N}(0, C_h)$ and $\overline{n} \sim \widetilde{\mathcal{N}}(0, \overline{C_n})$

Linear Estimator T estimates $\hat{m{h}} = T m{y} \in \mathbb{C}^{MK}$

$$\underline{\widetilde{T}}_{\mathrm{MMSE}} = \underline{C}_{\underline{h}\underline{y}}\underline{C}_{\underline{y}}^{-1} = \underline{C}_{\underline{h}}\underline{\widetilde{S}}^{\mathrm{H}}(\underline{\widetilde{S}}\underline{C}_{\underline{h}}\underline{\widetilde{S}}^{\mathrm{H}} + \underline{C}_{\underline{n}})^{-1}$$

$$\underline{T}_{\mathsf{ML}} = \underline{T}_{\mathsf{Cor}} = (\underline{\underline{S}}^{\mathsf{H}} \underline{C}_{\underline{\boldsymbol{n}}}^{-1} \underline{\underline{S}})^{-1} \underline{\underline{S}}^{\mathsf{H}} \underline{C}_{\underline{\boldsymbol{n}}}^{-1}$$

For Assumption $S^H S = N \sigma_s^2 \mathbf{1}_{K \times M}$ and $C_n = \sigma_n^2 \mathbf{1}_{N \times M}$

Estimator	Averaged Squared Bias	Variance
ML/Correlator	0	$KM \frac{\sigma_{\eta}^2}{N \sigma_s^2}$
Matched Filter	$\sum\limits_{i=1}^{KM} \lambda_i \left(rac{\lambda_i}{\lambda_1} - 1 ight)^2$	$\sum_{i=1}^{KM} \left(\frac{\lambda_i}{\lambda_1}\right)^2 \frac{\sigma_{\eta}^2}{N\sigma_s^2}$
MMSE	$\sum_{i=1}^{KM} \lambda_i \left(\frac{1}{1 + \frac{\sigma_{\eta}^2}{\lambda_i N \sigma_{\mathfrak{s}}^2}} - 1 \right)^2$	$\sum_{i=1}^{KM} \frac{1}{\left(1 + \frac{\sigma_{\eta}^2}{\lambda_i N \sigma_s^2}\right)^2} \frac{\sigma_{\eta}^2}{N \sigma_s^2}$

11.5. Estimators

Upper Bound: Uniform in $[0; \theta]$: $\hat{\theta}_{MI} = \frac{2}{N} \sum x_i$ Probability p for $\mathcal{B}(p, N)$: $\hat{p}_{\mathsf{ML}} = \frac{x}{N}$ $\hat{p}_{\mathsf{CM}} = \frac{x+1}{N+2}$

Mean μ for $\mathcal{N}(\mu,\sigma^2):\hat{\mu}_{\mathsf{ML}}^2=\frac{1}{N}\sum\limits_{}^{N}x_i$

12. Gaussian Stuff

12.1. Gaussian Channel

Channel:
$$Y = hs_i + N$$
 with $h \sim \mathcal{N}, N \sim \mathcal{N}$
$$L(y_1,...,y_N) = \prod_{i=1}^n f_{Y_i}(y_i,h)$$

$$f_{Y_{i}}(y_{i}, h) = \frac{11}{i=1} f_{Y_{i}}(y_{i}, h)$$
$$f_{Y_{i}}(y_{i}, h) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^{2}} (y_{i} - hs_{i})^{2}\right)$$

$$\hat{h}_{ML} = \underset{h}{\operatorname{argmin}} \{ \left\| \underline{\boldsymbol{y}} - h\underline{\boldsymbol{s}} \right\|^2 \} = \frac{\underline{\boldsymbol{s}}^{\top}\underline{\boldsymbol{y}}}{\underline{\boldsymbol{s}}^{\top}\underline{\boldsymbol{s}}}$$

If multidimensional channel: y = Sh + n:

$$L(\underline{\boldsymbol{y}},\underline{\boldsymbol{h}}) = \frac{1}{\sqrt{\det(2\pi\underline{\boldsymbol{C}})}} \exp\left(-\frac{1}{2}(\underline{\boldsymbol{y}} - \underline{\boldsymbol{S}}\underline{\boldsymbol{h}})^{\top}\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{y}} - \underline{\boldsymbol{S}}\underline{\boldsymbol{h}})\right)$$

$$l(\underline{\underline{y}},\underline{\underline{h}}) = \frac{1}{2} \left(\log(\det(2\pi \underline{\underline{C}}) - (\underline{\underline{y}} - \underline{\underline{S}}\underline{\underline{h}})^{\top} \underline{\underline{C}}^{-1} (\underline{\underline{y}} - \underline{\underline{S}}\underline{\underline{h}}) \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}h}(\underline{y} - \underline{S}\underline{h})^{\top}\underline{C}^{-1}(\underline{y} - \underline{S}\underline{h}) = -2\underline{S}^{\top}\underline{C}^{-1}(\underline{y} - \underline{S}\underline{h})$$
Gaussian Covariance: if $Y \sim \mathcal{N}(0, \sigma^2)$, $N \sim \mathcal{N}(0, \sigma^2)$:

Gaussian Covariance: if
$$Y \sim \mathcal{N}(0, \sigma^-)$$
, $N \sim \mathcal{N}(0, \sigma^-)$: $C_Y = \text{Cov}[Y, Y] = \text{E}[(Y - \mu)(Y - \mu)^\top] = \text{E}[YY^\top]$

For Channel Y = Sh + N: $E[YY^{\top}] = SE[hh^{\top}]S^{\top} + E[NN^{\top}]$

12.2. Multivariate Gaussian Distributions

A vector \mathbf{x} of n independent Gaussian random variables x_i is jointly Gaussian. If $\underline{\mathbf{x}} \sim \mathcal{N}(\underline{\boldsymbol{\mu}}_{\mathbf{x}}, \underline{\boldsymbol{C}}_{\underline{\mathbf{x}}})$:

$$\begin{split} f_{\underline{\mathbf{x}}}(\underline{\boldsymbol{x}}) &= f_{x_1, \dots, x_n} \left(\mathbf{x}_1, \dots, \mathbf{x}_n \right) = \\ &= \frac{1}{\sqrt{\det(2\pi \underline{\boldsymbol{C}}_{\underline{\mathbf{x}}})}} \exp\left(-\frac{1}{2} \left(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_{\underline{\mathbf{x}}} \right)^\top \underline{\boldsymbol{C}}_{\underline{\mathbf{x}}}^{-1} \left(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_{\underline{\mathbf{x}}} \right) \right) \end{split}$$

Affine transformations $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$ are jointly Gaussian with

$$\underline{\mathbf{y}} \sim \mathcal{N}(\underline{\underline{\mathbf{A}}}\underline{\underline{\boldsymbol{\mu}}}_{\mathbf{x}} + \underline{\mathbf{b}}, \underline{\underline{\mathbf{A}}}\bar{\underline{\boldsymbol{C}}}_{\mathbf{x}}\underline{\underline{\mathbf{A}}}^{\top})$$

All marginal PDFs are Gaussian as well

Ellipsoid with central point E[y] and main axis are the eigenvectors of

12.3. Conditional Gaussian

$$\begin{array}{l} \underline{A} \sim \mathcal{N}(\underline{\mu}_{\underline{A}}, \underline{C}_{\underline{A}}), \underline{B} \sim \mathcal{N}(\underline{\mu}_{\underline{B}}, \underline{C}_{\underline{B}}) \\ \Rightarrow (\underline{A} | \underline{B} = b) \sim \mathcal{N}(\underline{\mu}_{A \mid B}, \underline{C}_{\underline{A} \mid \underline{B}}) \end{array}$$

$$\begin{array}{l} \text{Conditional Mean:} \\ \mathsf{E}[\underline{A}|\underline{B}=\underline{b}] = \underline{\mu}_{\underline{A}|\underline{B}=\underline{b}} = \underline{\mu}_{\underline{A}} + \underbrace{\mathcal{C}}_{\underline{A}\underline{B}} \,\, \underbrace{\mathcal{C}}_{\underline{B}\underline{B}}^{-1} \, \left(\underline{b} - \underline{\mu}_{\underline{B}}\right) \end{array}$$

Conditional Variance:

$$C_{\underline{A}|\underline{B}} = C_{\underline{AA}} - C_{\underline{AB}} C_{\underline{BB}}^{-1} C_{\underline{BA}}$$

If CDF of gaussian distribution given $\Phi(z) \sim \mathcal{N}(0,1)$ then for $X \sim$ $\mathcal{N}(1,1)$ the CDF is given as $\Phi(x-\mu_x)$

13. Sequences

13.1. Random Sequences

Sequence of a random variable. Example: result of a dice is RV, roll a dice several times is a random sequence

13.2. Markov Sequence $X_n:\Omega \to X_n$

Sequence of memoryless state transitions with certain probabilities.

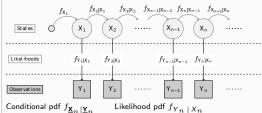
1. state:
$$f_{X_1}(x_1)$$

2. state: $f_{X_2 | X_1}(x_2 | x_1)$

n. state: $f_{X_n \mid X_{n-1}}(x_n \mid x_{n-1})$

13.3. Hidden Markov Chains

Problem: states X_i are not visible and can only be guessed indirectly as a random variable Y_i .



State-transision pdf $f_{X_n \mid X_{n-1}}$

$$\sum_{n} |\underline{\mathbf{Y}}_{n}| = \sum_{n} |\mathbf{Y}_{n}|$$

$$f_{\underline{\mathbf{X}}_n|\underline{\mathbf{Y}}_n} \propto f_{\underline{\mathbf{Y}}_n|\underline{\mathbf{X}}_n} \cdot \int_{\mathbb{X}} f_{\underline{\mathbf{X}}_n|\underline{\mathbf{X}}_{n-1}} \cdot f_{\underline{\mathbf{X}}_{n-1}|\underline{\mathbf{Y}}_{n-1}} \, \mathrm{d}\underline{\boldsymbol{x}}_{n-1}$$

14. Recursive Estimation

14.1. Kalman-Filter

recursively calculates the most likely state from previous state estimates and current observation. Shows optimum performance for Gauss-Markov Sequences.

$$\underline{\underline{x}}_n = \underline{G}_n \underline{\underline{x}}_{n-1} + \underline{B}\underline{\underline{u}}_n + \underline{\underline{v}}_n$$

$$\underline{\underline{y}}_n = \underbrace{H}_n \underline{\underline{x}}_n + \underline{\underline{w}}_n$$

With gaussian process/measurement noise $\underline{v}_n/\underline{w}_n$ Short notation: $\mathrm{E}[\underline{x}_n|\underline{y}_{n-1}] = \hat{\underline{x}}_{n|n-1}$ $\mathrm{E}[\underline{x}_n|\underline{y}_n] = \hat{\underline{x}}_{n|n}$ $\mathrm{E}[\underline{\boldsymbol{y}}_n|\underline{\boldsymbol{y}}_{n-1}] = \underline{\hat{\boldsymbol{y}}}_{n|n-1} \quad \mathrm{E}[\underline{\boldsymbol{y}}_n|\underline{\boldsymbol{y}}_n] = \underline{\hat{\boldsymbol{y}}}_{n|n}$

$$\begin{array}{l} \text{Mean: } \underline{\hat{x}}_{n|n-1} = \underline{G}_n \underline{\hat{x}}_{n-1|n-1} \\ \text{Covariance: } \underline{C}_{\underline{x}_n|n-1} = \underline{G}_n \underline{C}_{\underline{x}_{n-1|n-1}} \underline{G}_n^\top + \underline{C}_{\underline{v}} \end{array}$$

$$\begin{array}{l} \text{Mean: } \underline{\hat{x}}_{n|n} = \underline{\hat{x}}_{n|n-1} + \underbrace{K}_{n} \left(\underline{y}_{n} - \underbrace{H}_{n} \underline{\hat{x}}_{n|n-1} \right) \\ \text{Covariance: } \underline{C}_{\underline{x}_{n}|n} = \underline{C}_{\underline{x}_{n|n-1}} + \underbrace{K}_{n} \underbrace{H}_{n} \underline{C}_{\underline{x}_{n|n-1}} \end{array}$$

correction:
$$\mathsf{E}[\mathsf{X}_n \mid \Delta \mathsf{Y}_n = y_n]$$

$$\underline{\hat{x}}_{n|n} = \underbrace{\hat{\underline{x}}_{n|n-1}}_{\text{estimation E}[X_n \mid Y_{n-1} = y_{n-1}]} + \underbrace{\underbrace{K_n \left(\underline{y}_n - \underline{H}_n \hat{\underline{x}}_{n|n-1}\right)}_{\text{innovation:} \Delta y_n}}$$

With optimal Kalman-gain (prediction for $\underline{\boldsymbol{x}}_n$ based on Δy_n):

$$\underbrace{K_n = \mathcal{C}_{\underline{\boldsymbol{x}}_n|_{n-1}} \underbrace{H_n^\top}_{n} (\underbrace{\underbrace{H_n \mathcal{C}_{\underline{\boldsymbol{x}}_n|_{n-1}} \underbrace{H_n^\top}_{n} + \mathcal{C}_{\underline{\boldsymbol{w}}_n}}_{\mathcal{C}_{\delta y_n}})^{-1}}$$

Innovation: closeness of the estimated mean value to the real value $\Delta \underline{y}_n = \underline{y}_n - \underline{\hat{y}}_{n|n-1} = \underline{y}_n - \underline{\hat{H}}_n \underline{\hat{x}}_{n|n-1}$

Init:
$$\hat{\underline{x}}_{0|-1} = E[X_0]$$
 $\sigma_{0|-1}^2 = Var[X_0]$

MMSE Estimator: $\underline{\hat{x}} = \int \underline{x}_n f_{X_n \mid Y_{(n)}} (\underline{x}_n | \underline{y}_{(n)}) d\underline{x}_n$

For non linear problems: Suboptimum nonlinear Filters: Extended KF Unscented KF ParticleFilter

14.2. Extended Kalman (EKF)

Linear approximation of non-linear a, h $\underline{x}_n = g_n(\underline{x}_{n-1}, \underline{v}_n) \qquad \underline{v}_n \sim \mathcal{N}$ $y_n = h_n(\underline{x}_{n-1}, \underline{w}_n) \quad \underline{w}_n \sim \mathcal{N}$

14.3. Unscented Kalman (UKF)

Approximation of desired PDF $f_{X_n|Y_n}(x_n|y_n)$ by Gaussian PDF.

14.4. Particle-Filter

For non linear state space and non-gaussian noise

Non-linear State space:

$$\underline{\mathbf{x}}_n = g_n(\underline{\mathbf{x}}_{n-1}, \underline{\mathbf{v}}_n) \\
\underline{\mathbf{y}}_n = h_n(\underline{\mathbf{x}}_{n-1}, \underline{\mathbf{w}}_n)$$

Posterior Conditional PDF:
$$f_{X_n|Y_n}(x_n|y_n) \propto f_{Y_n|X_n}(y_n|x_n) \cdot \int_{\mathbb{X}} \underbrace{f_{X_n|X_{n-1}}(x_n|x_{n-1})}_{\text{the transition}} \underbrace{f_{X_n|X_{n-1}}(x_n|x_{n-1})}_{\text{lett conditional PDF}} dx_{n-1}$$

N random Particles with particle weight \boldsymbol{w}_{n}^{i} at time n

Monte-Carlo-Integration:
$$I = \mathsf{E}[g(\mathsf{X})] pprox I_N = \frac{1}{N} \sum\limits_{i=1}^N \tilde{g}(x^i)$$

Importance Sampling: Instead of $f_X(x)$ use Importance Density $q_X(x)$

$$I_N=\frac{1}{N}\sum\limits_{i=1}^N \tilde{w}^i g(x^i)$$
 with weights $\tilde{w}^i=\frac{f\chi(x^i)}{q\chi(x^i)}$

If $\int f_{X_n}(x) \, \mathrm{d}x \neq 1$ then $I_N = \sum_{i=1}^N \tilde{w}^i g(x^i)$

14.5. Conditional Stochastical Independence

$$\mathsf{P}(A\cap B|E)=\mathsf{P}(A|E)\cdot\mathsf{P}(B|E)$$

Given Y, X and Z are independent if

 $f_{Z | Y, X}(z|y, x) = f_{Z | Y}(z|y)$ or

 $f_{X,Z|Y}(x,z|y) = f_{Z|Y}(z|y) \cdot f_{X|Y}(x|y)$ $f_{Z|X,Y}(z|x,y) = f_{Z|Y}(z|y) \text{ or } f_{X|Z,Y}(x|z,y) = f_{X|Y}(x|y)$

15. Hypothesis Testing

making a decision based on the observations

15.1. Definition

Null hypothesis $H_0: \theta \in \Theta_0$ (Assumed first to be true) Alternate hypothesis $H_1: \theta \in \Theta_1$ (The one to proof)

Descision rule $\varphi: \mathbb{X} \to [0,1]$ with

 $\varphi(x)=1$: decide for H_1 , $\varphi(x)=0$: decide for H_0 Error level α with $E[d(X)|\theta] \le \alpha, \forall \theta \in \Theta_0$

Error Type	Decision	H_1 false (H_0 true)	H_1 true (H_0 false
1 (FA) False	H_1 rejected	True Negative	False Negative (Type 2)
Alarm	$(H_0 \; {\it accepted})$	$P = 1 - \alpha$	$P = \beta$
, ,	H_1 accepted or $(H_0$ rejected)	False Positive (Type 1) ${\sf P} = \alpha$	True Positive $P = 1 - \beta$

Power: Sensitivity/Recall/Hit Rate: $\frac{TP}{TP+FN} = 1 - \beta$

Specificity/True negative rate: $\frac{\text{TN}}{\text{FP+TN}} = 1 - \alpha$

Precision/Positive Prediciton rate: TP

Accuracy: $\frac{TP+TN}{P+N} = \frac{2-\alpha-\beta}{2}$

15.1.1. Design of a test

Cost criterion $G_{\varphi}:\Theta\to [0,1], \theta\mapsto \mathsf{E}[d(X)|\theta]$

False Positive lower than α : $G_d(\theta)|_{\theta \in \Theta_0} \leq \alpha, \forall \theta \in \Theta_0$

False Negative small as possible: $\max\{G_d(\theta)|_{\theta\in\Theta_1}\}, \forall \theta\in\Theta_1$

15.2. Sufficient Statistics

Sufficiency for a test T(X) means that no other test statistic, i.e., function of the observations \underline{x} , contains additional information about the parame-

$$f_{X|T}(x|T(x) = t, \theta) = f_{X|T}(x|T(x) = t)$$

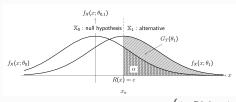
16. Tests

16.1. Nevman-Pearson-Test The best test of Po against P1 is

The best test of
$$r_0$$
 against r_1 is
$$d_{\mathrm{NP}}(x) = \begin{cases} 1 & R(x) > c \\ \gamma & R(x) = c \\ 0 & R(x) < c \end{cases} \qquad \text{Likelihood-Ratio:}$$

 $\gamma = \frac{\alpha - P_0(\{R > c\})}{P_0(\{R = c\})}$ Errorlevel α

Steps: For α calculate x_{α} , then $c = R(x_{\alpha})$



Maximum Likelihood Detector: $d_{ML}(x) = \begin{cases} 1 & R(x) > 1 \\ 0 & \text{otherwise} \end{cases}$ **ROC Graphs:** plot $G_d(\theta_1)$ as a function of $G_d(\theta_0)$

16.2. Bayes Test (MAP Detector)

Prior knowledge on possible hypotheses: $P(\{\theta \in \Theta_0\}) + P(\{\theta \in \Theta_0\})$ Θ_1) = 1, minimizes the probability of a wrong decision.

$$d_{\mathsf{Bayes}} = \begin{cases} 1 & \frac{f_{\mathsf{X}}(x|\theta_1)}{f_{\mathsf{X}}(x|\theta_0)} > \frac{c_0 \, \mathsf{P}(\theta_0|x)}{c_1 \, \mathsf{P}(\theta_1|x)} \\ 0 & \mathsf{otherwise} \end{cases} = \begin{cases} 1 & \mathsf{P}(\theta_1|x) > \mathsf{P}(\theta_0|x) \\ 0 & \mathsf{otherwise} \end{cases}$$

Multiple Hypothesis
$$d_{\mathsf{Bayes}} = \begin{cases} 0 & x \in \mathbb{X}_0 \\ 1 & x \in \mathbb{X}_1 \\ 2 & x \in \mathbb{X}_2 \end{cases}$$

16.3. Linear Alternative Tests

$$d: \mathbb{X} \to \mathbb{R}, \underline{\boldsymbol{x}} \mapsto \begin{cases} 1 & \underline{\boldsymbol{w}}^{\top}\underline{\boldsymbol{x}} - w_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Estimate normal vector $\underline{\boldsymbol{w}}^{\top}$, which separates \mathbb{X} into \mathbb{X}_0 and \mathbb{X}_1 $\log R(\underline{\boldsymbol{x}}) = \frac{\ln(\det(\underline{C}_0))}{\ln(\det(\underline{C}_1))} + \frac{1}{2}(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_0)^{\top}\underline{\boldsymbol{C}}_0^{-1}(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_0) -$

$$\log R(\underline{\boldsymbol{x}}) = \frac{1}{\ln(\det(\underline{\boldsymbol{C}}_1))} + \frac{1}{2}(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_0) + \underline{\boldsymbol{C}}_0^{-1}(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_0)$$

$$\begin{aligned} & -\frac{1}{2}(\underline{x} - \underline{\mu}_1)^\top \underline{C}_1^{-1}(\underline{x} - \underline{\mu}_1) = 0 \\ & \text{For 2 Gaussians, with } \underline{C}_0 = \underline{C}_1 = \underline{C} \colon \underline{w}^\top = (\underline{\mu}_1 - \underline{\mu}_0)^\top \underline{C} \\ & \text{and constant translation } w_0 = \frac{(\underline{\mu}_1 - \underline{\mu}_0)^\top \underline{C}(\underline{\mu}_1 - \underline{\mu}_0)}{2} \end{aligned}$$

