

Monolithic solver for Computational Fluid-Structure Interaction

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The front page depicts a section of the root system of the exceptional Lie group E_8 , projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

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Chapter 1

Introduction to Fluid-Structure Interaction

The interaction between fluids and structures can be observed all around us in nature. From a flag waving in the wind to a large windmill at sea, when we breathe in air and our lungs expand, when our hearts fill up with blood and muscles contract to push blood into our arteries, are all examples of fluids and structures interacting. A flag waving in the wind is mainly air(fluid) exerting force on the flag(structure), making the flag flutter. However, the structure can also move the fluid. When we breathe air into our lungs, our diaphragm contracts and moves downward, increasing the space in our chest cavity making the lungs expand. As the lungs expand air is sucked through the mouth and nose, filling up the lungs. In this manner both fluid and structures can exert force, and consequently interact with each other.

A bridge is a rigid structure interacting with wind(fluid), this interaction is an example of Fluid-Structure Interaction(FSI). A profound example of wind interacting with a bridge, is the collapse of the Tacoma Narrows Bridge in 1940 [1]. The bridge collapsed only two months after being opened. The collapse was due to strong winds(64 km/h) interacting with the structure, making the bridge flutter and ultimately collapse. No human lives were lost in the collapse, but a Cocker Spaniel named Tubby left behind in a car was not that lucky and lost its life in the bridge collapse.

Modeling FSI is used today when constructing for instance windmills. Windmills are normally considered stiff relative to the flowing air, hence giving a big difference in density between fluid and structure. The structure will in

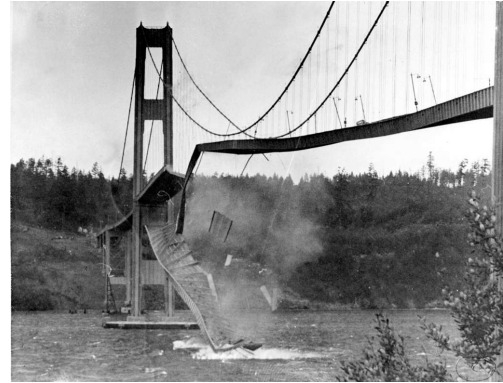


Figure 1.1: Tacoma Narrows bridge still standing with large deformations

Figure 1.2: Tacoma Narrows bridge collapsed

the case of a windmill only give rise to small deformations. However, modeling FSI in hemodynamics(dynamics of blood flow) deems more challenging. In the latter cases the densities in fluid and structure are more similar and blood flow usually induces large deformations in the veins and arteries. In both cases the fluid flow tends to transitions towards turbulence. All of these cases have challenges which gives the need to have rigid stabile FSI solvers.

A branch of fluid dynamics is called Computational Fluid Dynamics, where computers and numerical analyses are used to solve fluid problems. I would argue that a similar name is used for FSI, namely *Computational Fluid Structure Interaction* (CFSI). However in the literature FSI is used to describe CFSI and the name FSI will be used in this thesis. Also, even though FSI is Fluid-*Structure* Interaction, the word solid will also be used to describe a structure.

The main goal of this master thesis is to build a framework to solve FSI problems, investigating different approaches and schemes. The framework will be validated and verified using the *Method of Manufactured Solutions*, accompanying a classic, well known benchmark, known to test the robustness of a FSI solver.

Chapter 2

Continuum mechanics in different frames of reference

All matter is made up of small building blocks called atoms, and the mathematics of the fundamental laws at this tiny scale is called quantum mechanics. The characteristic scales at which quantum mechanics operate is in the nanometer range, which is far smaller than most length scales found in nature. If we zoom out to the *macroscale*, the scale on which phenomena are large enough to be visible for the human eye, matter like solids and fluids behave statistically uniform, and it is reasonable to assume the matter can be mathematically described, and modeled, as a continuum. The continuum hypothesis states that there exist no space inside the matter and the matter completely fills up the space it occupies. We can use mathematics with basic Newtonian physical laws to model fluids and solids when they are assumed continuous. These newtonian laws are generally expressed in two frames of reference, Lagrangian and Eulerian, depending on the the scientific branch.

To exemplify these frameworks we can imagine a river running down a mountain. In the Eulerian framework we are the observer standing besides the river looking at the flow. We are not interested in each fluid particle but only how the fluid acts as a whole flowing down the river. The Eulerian description of fluid mechanics is advantageous as we can imagine the fluid continuously deforming along the river side.

The Lagrangian description is particularly beneficial for describing solid mechanical problems, as one is generally.

In the Lagrangian description we have to imagine ourselves on a leaf going down the river with the flow. The Lagrangian description is particularly beneficial for describing solid mechanical problems, as one is generally interested in where the solid particles are in relation to each other. A solid mechanical example is modeling a deflecting beam attached to a wall. To model deflection one needs to know where all the particles are compared to each other. The more force that is applied the more the particles move in relation to each other. The Lagrangian description helps us to easier model a problem where need to know the particles spatial-relation to each other

This chapter introduces briefly the fluid and solid equations with their respective boundary conditions. A detailed description of the Lagrangian framework and the stress and strain relations are covered in the appendix.

2.1 Conservation of linear momentum for a solid

With matter assumed continuous, fundamental physical laws like conservation of mass and conservation of momentum can be applied to derive a differential equation describing the motions of a solid. Information about the particular material of a solid is added through constitutive relations. The differential solid equation will be stated in the Lagrangian reference system [15], in the solid time domain \mathcal{S} as:

$$\rho_s \frac{\partial \mathbf{d}^2}{\partial t^2} = \nabla \cdot (\mathbf{P}) + \rho_s \mathbf{f} \quad \text{in } \mathcal{S} \quad (2.1)$$

written in terms of the deformation \mathbf{d} , of the solid.. \mathbf{P} is the first Piola-Kirchhoff stress tensor. The derivation of Piola-Kirchhoff is included in the Appendix for the sake of completeness. Body forces are denoted as \mathbf{f} , and are forces that originate outside the body and act on the mass of the body.

Body forces are forces originating from sources outside of the body[12] that act on the volume (or mass) of the body. Saying that body forces are due to outside sources implies that the interaction between different parts of the body (internal forces) are manifested through the contact forces alone

2.2 Conservation of linear momentum for a fluid

The differential equations describing the velocity and pressure in a fluid are called the Navier-Stokes (N-S) equations. The N-S equations are derived, like the solid equation, from principles of mass and momentum conservation, assuming fluid to act as a continuum. The fluid equations are stated in an Eulerian framework. In the Eulerian framework the domain has fixed spatial points where the fluid passes through. The N-S equations are written in the fluid time domain \mathcal{F} as an incompressible fluid:

$$\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot \sigma_f + \rho_f \mathbf{f} \quad \text{in } \mathcal{F} \quad (2.2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{F} \quad (2.3)$$

where \mathbf{u} is the fluid velocity, p is the fluid pressure, ρ stands for density. \mathbf{f} is body force and σ_f is the Cauchy stress tensor, $\sigma_f = \mu_f(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - pI$ denoting a newtonian fluid. I denotes the Identity matrix.

There does not yet exist an analytical solutions to the N-S equations for every fluid problem. Only simplified fluid problems can be solved [24] analytically using the N-S equations. Actually there is a prize set out by the Clay Mathematics Institute of 1 million dollars to whomever can show the existence and smoothness of Navier-Stokes solutions [5], as apart of their millennium problems. Nonetheless this does not stop us from discretizing and solving N-S numerically.

One difficulty in the N-S equations is the nonlinearity appearing in the convection term on the left hand side. Non-linearity is most often handled using a lagging velocity function, Newtons method, or Picard iterations. Another difficulty is finding a suitable equation to solve for the pressure field [2]. The difficulty with pressure has been tackled using for instance the incremental pressure correction scheme (IPCS).

2.3 Fluid and Structure Boundary conditions

To complete the fluid and solid equations, boundary conditions need to be imposed. The fluid flows and the solid moves within the boundaries noted

as $\partial\mathcal{F}$ and $\partial\mathcal{S}$ respectively.

On the Dirichlet boundary, $\partial\mathcal{F}_D$ and $\partial\mathcal{S}_D$, we impose a given value. Dirichlet conditions can be initial conditions or set values, such as zero on the fluid boundary for a "no slip" condition. The Dirichlet conditions are defined for \mathbf{u} and p :

$$\mathbf{u} = u_0 \text{ on } \partial\mathcal{F}_D \quad (2.4)$$

$$p = p_0 \text{ on } \partial\mathcal{F}_D \quad (2.5)$$

$$\mathbf{d} = d_0 \text{ on } \partial\mathcal{S}_D \quad (2.6)$$

$$\mathbf{w}(\mathbf{X}, t)_0 = \frac{\partial \mathbf{d}(t=0)}{\partial t} \text{ on } \partial\mathcal{S}_D \quad (2.7)$$

The forces on the boundaries equal a possible external force \mathbf{f} . These are enforced on the Neumann boundaries $\partial\mathcal{F}_N$ and $\partial\mathcal{S}_N$:

$$\sigma \cdot \mathbf{n} = f \text{ on } \partial\mathcal{F}_N \quad (2.8)$$

$$P \cdot \mathbf{n} = f \text{ on } \partial\mathcal{S}_N \quad (2.9)$$

Chapter 3

Fluid-Structure Interaction Problem

The following chapter will be devoted to introducing the full FSI problem, with all the equations, conditions, and discretizations needed to build a Fluid-Structure Interaction solver.

When computing FSI problems the computing domain is split into three parts, fluid, structure, and interface. Fluid and structure domains are separated, and different constitutive equations are solved in each domain. The spatial points in which fluid and structure join is called the interface. The treatment of the interface gives the two main methods for solving FSI problems [9]. The first method is called fully Eulerian. In a fully Eulerian framework, both the fluid and structure equations are defined and solved in a purely Eulerian description. The interface in a fully Eulerian framework is tracked across a fixed domain [21]. The fully Eulerian description is suited for fluid problems but is problematic for structure problem. And certainly for FSI problems where tracking of the interface across the domain is a difficult task.

The second approach is the *Arbitrary Lagrangian Eulerian*(ALE). The ALE method entails formulating the fluid equations in a type of Eulerian framework and the solid in a Lagrangian framework. The entire domain itself moves with the structural displacements and the fluid moves through these points. In the ALE framework we get the best of both worlds, in that fluid and solid are described in their natural states. The structure equation will

remain as previously stated (2.1), and we will need to change the fluid equations to take into account the moving domain changing the fluid velocity.

Dealing with the movement of the domain is done in two ways. One way is to move the domain itself in relation to the structural displacements, and use this new domain to calculate the equations for every iteration. Moving the domain gives advantages as we can explicitly represent the fluid-structure interface, and the equations are stated in a more familiar manner. But problems arise when there are large deformations in the solid giving large deformations into the fluid domain. Moving the mesh with large deformation can be a challenge in that the domain can overlap causing singularities.

The second approach to ALE, which will be used in this thesis, is to calculate from **reference frame**. When solving equations from a reference frame we solve the equations on an initial, stress free domain, and use a series of mappings to account for the movements of the current time domain. It is the displacements in the in the domain that determines the value of the mappings between frames of reference.

From a technical point of view, moving both the mesh and using a reference frame are equivalent [15]. Since the reference frame method does not need a function to move the mesh between each time iteration, it can be less time consuming. The interface is also located in the same position, making the interface easy to track.

There are generally two types approaches when discretizing an FSI scheme. The first is the partitioned approach where fluid and structure are solved sequentially. The partitioned approach is appealing in that we have a wealth of knowledge and techniques on how to solve each of these kinds of problems in an efficient manner. The difficulty however is dealing with the interface. There are kinematic and dynamic conditions needed in FSI, and the coupling of these conditions is where problems arise. Explicit coupling schemes are known to be unconditionally unstable for standard Dirichlet-Neumann strategies when there is a large amount of added-mass in the system [7, 22]. There are however, schemes which offer added-mass free stability with explicit coupling, where the interface is treated through a Robin-Neumann coupling. The first scheme was made for coupling of a thin walled structure by Fernandez et al., 2013 [6]. Later a scheme was made with an extension coupling with a thick walled structure by Fernandez et.al,2015 [7]. The scheme for coupling

thick walled structures is rather complex and uses a number of techniques that are out of the scope of this thesis. (This may be in more detail in a later chapter (discussion and further work.))

In this thesis the other approach is used, named monolithic. In the monolithic approach all of the equations are solved at once. The monolithic approach has the advantage of offering numerical stability for problems with strong added-mass effects [9], and are fully coupled. The disadvantage over the partitioned approach is that we loose flexibility when solving many equations simultaneously, and the computing matrices can quickly become large and computationally costly. But the overall the ease of implementation and numerical stability makes monolithic the preferred choice in this thesis.

The following chapter starts by introducing the mappings needed to change between current and reference domain. Lastly, the equations will be discretized following the notation and ideas from [15]:

3.1 Mapping between different frames of reference

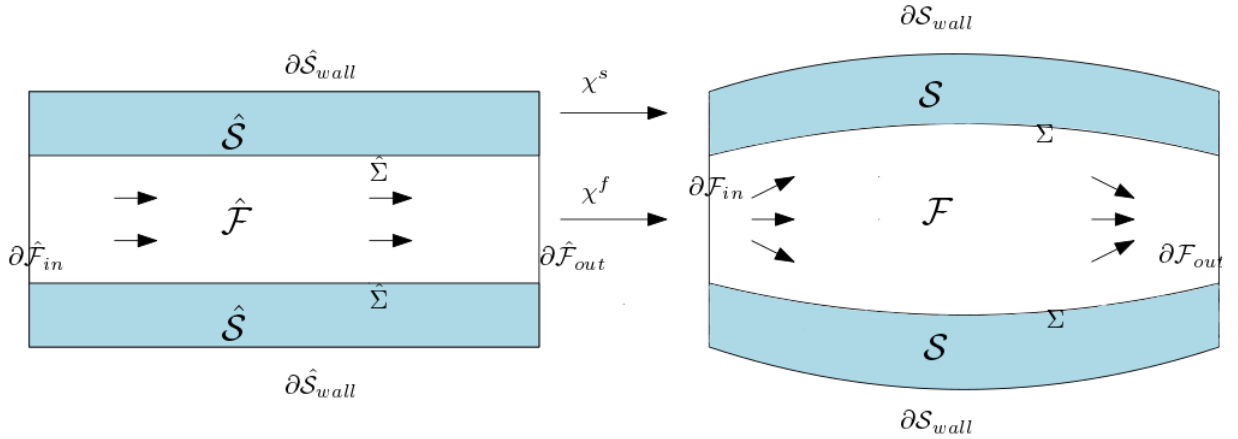


Figure 3.1: Mapping of domain from reference to current

Let $\hat{\mathcal{V}}$ be a reference volume and $\mathcal{V}(t)$ be the current time volume. Then using (A.5) and (A.6) to define a mapping between the volumes from the current to the reference configurations:

$$\int_{\mathcal{V}(t)} 1 dx = \int_{\hat{\mathcal{V}}} J dx \quad (3.1)$$

The gradients acting on a vector \mathbf{u} will also be mapped between current and reference configurations:

$$\int_{\mathcal{V}(t)} \nabla \mathbf{u} dx = \int_{\hat{\mathcal{V}}} J \nabla \mathbf{u} F^{-1} dx \quad (3.2)$$

The divergence of a vector \mathbf{u} will be mapped between configurations as:

$$\int_{\mathcal{V}(t)} \nabla \cdot \mathbf{u} dx = \int_{\hat{\mathcal{V}}} \nabla \cdot (J F^{-1} \mathbf{u}) dx \quad (3.3)$$

3.2 Governing equations for Fluid-Structure Interaction

The following section formulates the equations in the Eulerian, Lagrangian, and the ALE description. I start by Briefly talking about time derivatives in the different configurations, to understand the need to change the fluid equation in the ALE description [26]:

3.2.1 Derivatives in different frameworks

In the Lagrangian framework the total and partial derivatives have the following relation:

$$D_t f(x, t) = \partial_t f(x, t) \quad (3.4)$$

The total and partial derivatives have the following relation in the Eulerian framework:

$$D_t f(x, t) = \mathbf{u} \cdot \nabla f + \partial_t f(x, t) \quad (3.5)$$

whilst in the ALE framework this concept is extended to take into account the motion of the domain:

$$D_t f(x, t) = \mathbf{w} \cdot \nabla f + \partial_t f(x, t) \quad (3.6)$$

Equation 3.6 shows that in the Lagrangian framework \mathbf{w} is zero and for the Eulerian framework $\mathbf{w} = \mathbf{u}$.

3.2.2 Solid equation

Let $\Omega \in \hat{\mathcal{S}} \cup \hat{\mathcal{F}}$ be a global domain that is made up of the fluid, structure, and the interface. The interface is stated as: $\hat{\Sigma} \in \hat{\mathcal{S}} \cap \hat{\mathcal{F}}$. Let \mathbf{u} be a global velocity function that describes the fluid velocity in the fluid domain and the structure velocity in the structure domain, $\mathbf{u} = \mathbf{u}_s \cup \mathbf{u}_f \in \Omega$. The deformation function is also defined as a global function $\mathbf{d} = \mathbf{d}_s \cup \mathbf{d}_f \in \Omega$. Using global velocity and displacement functions makes the velocity and displacement continuous across the entire domain. This is a nice feature which will be apparent once the interface conditions are stated. .

The solid equation will be written in terms of the solid velocity \mathbf{u}_s , in contrast to 2.1 which was written in terms of the deformation. Expressing the solid balance laws in the Lagrangian formulation from the initial configuration:

$$\rho_s \frac{\partial \mathbf{u}}{\partial t} = \nabla \cdot (P) + \rho_s f \quad \text{in } \hat{\mathcal{S}} \quad (3.7)$$

3.2.3 Fluid equations

In the ALE description the fluid domain is moving, giving the need to redefine the velocity in the convective term in (2.2) to account for the moving domain:

$$\mathbf{u} \cdot \nabla \mathbf{u} \rightarrow \left(\mathbf{u} - \frac{\partial \mathbf{d}}{\partial t} \right) \cdot \nabla \mathbf{u} \quad (3.8)$$

Where \mathbf{d} is the deformation in the fluid domain. The domain velocity $\mathbf{w} = \frac{\partial \mathbf{d}}{\partial t}$ is written in terms of the deformation as the partial time derivative.

The change in the fluid equations denoted from the current configuration using the aforementioned mappings, maps the fluid equations to the reference configuration:

$$\int_{\mathcal{V}(t)} \rho_f \frac{\partial \mathbf{u}}{\partial t} = \int_{\hat{\mathcal{V}}} \rho_f J \frac{\partial \mathbf{u}}{\partial t} dx \quad (3.9)$$

$$\int_{\mathcal{V}(t)} \nabla \mathbf{u} \left(\mathbf{u} - \frac{\partial d}{\partial t} \right) dx = \int_{J\hat{\mathcal{V}}} ((\nabla \mathbf{u}) F^{-1} \left(\mathbf{u} - \frac{\partial d}{\partial t} \right)) dx \quad (3.10)$$

$$\int_{\mathcal{V}(t)} \nabla \cdot \mathbf{u} dx = \int_{J\hat{\mathcal{V}}} \nabla \cdot (J F^{-1} \mathbf{u}) dx \quad (3.11)$$

$$\int_{\mathcal{V}(t)} \nabla \cdot \sigma_f dx = \int_{J\hat{\mathcal{V}}} \nabla \cdot (J F^{-1} \hat{\sigma}_f) dx \quad (3.12)$$

$$\hat{\sigma}_f = -pI + \mu(\nabla \mathbf{u} F^{-1} + F^{-T} \mathbf{u}^T) \quad (3.13)$$

Assembling all these terms together with (2.2) gives the fluid equations from a reference frame:

$$\rho_f J \left(\frac{\partial \mathbf{u}}{\partial t} dx + (\nabla \mathbf{u}) F^{-1} \left(\mathbf{u} - \frac{\partial d}{\partial t} \right) \right) = \nabla \cdot (J F^{-1} \hat{\sigma}_f) + J \rho_f f \quad (3.14)$$

$$\nabla \cdot (J F^{-1} \mathbf{u}) = 0 \quad (3.15)$$

3.3 Interface conditions at the boundary

Figure 3.1 shows a typical Fluid-Structure Interaction domain. The fluid is surrounded by elastic walls, like for instance a blood vessel. Inflow of fluid at $\partial \hat{\mathcal{F}}_{in}$, a change in pressure, or a motion of the solid determines the flow of the fluid. The fluid's stress on the walls causes deformation in the solid domain and vice versa. The interface is where these energies are transferred and we therefore need conditions on the interface.

The three interface conditions comes from simple physical properties and consist of [15]:

- *Kinematic condition:* $\mathbf{u}_f = \mathbf{u}_s$ on $\hat{\Sigma}$. The fluid and structure velocities are equal on the interface. Meaning the fluid moves with the interface at all times.

Since we use a global function for \mathbf{u} in both fluid and structure domains, this condition is upheld.

The fluid and solid velocities are usually in different coordinate systems,

the solid velocity is then not available in Eulerian Coordinates. We instead link fluid velocity at the interface by using the fact that $\mathbf{u}_s = \frac{\partial \mathbf{d}}{\partial t}$. Setting $\mathbf{u}_f = \frac{\partial \mathbf{d}}{\partial t}$ at the interface.

- *Dynamic condition:* $\sigma_f n_f = \sigma_s n_s$ on $\hat{\Sigma}$.

The dynamic interface condition relates to Newtons third law of action and reaction. The forces on the interface area, here written as the normal forces are balanced on the interface. These forces will be written in a Lagrangian formulation:

$$J\sigma_f F^{-T} n_f = P n_s \text{ on } \hat{\Sigma}.$$

The dynamic condition is a Neumann condition that belongs to both subproblems.

- *Geometrical condition:* This condition says that the fluid and structure domains do not overlap, but rather that elements connect so the functions needing to transfer force are continuos across the entire domain.

3.4 Lifting operators for domain representation and mesh quality preservation

The kinematic interface conditions states that the fluid moves with the solid, and therefore the fluid domain needs to move in accordance with the solid deformations. The deformations from the structure are extrapolated through the interface into the fluid domain using different lifting operators. The lifting operators redistributes the interior node locations to uphold the mesh quality in the fluid domain. The choice of lifting operator is important for the overall FSI problem to be calculated [27]. When large deformations occur, we need a good lifting operator to uphold the integrity of the computing domain. A poor choice may make the mesh overlap and singularities may occur. When extrapolating deformation from the solid to the fluid domain, the fluid domain itself acts as a structure, deforming according to the deformations from the structure domain.

I will in this section present different lifting operators, that act differently on the computational domain. In 5.2.2 the techniques will be tested and investigated.

3.4.1 Harmonic extension

For small to moderate deformations an harmonic extension can be used. The harmonic extension uses the Laplace equation, transporting the deformations from the solid into the fluid domain. A variable $\alpha_u > 0$ can be multiplied to the Laplace equation, to control the amount of lifting of deformations to the fluid domain.

$$-\alpha_u \nabla^2 \mathbf{d} = 0 \quad \text{in } \hat{\mathcal{F}} \quad (3.16)$$

$$\mathbf{d} = 0 \quad \text{on } \partial \hat{\mathcal{F}} / \hat{\Sigma} \quad (3.17)$$

$$\mathbf{d}_f = \mathbf{d}_s \quad \text{on } \hat{\Sigma} \quad (3.18)$$

When using the harmonic lifting operator the variable α_u is very important when calculating moderate deformations. For small deformations a constant can be used for α_u . But for larger deformations we need to be a bit more clever. A good strategy for choosing α_u is proposed by Wick in [27], and further discussed in [19] and [12]. This alpha gets bigger when closer to the interface:

$$\alpha_u = \frac{1}{x^q} \quad (3.19)$$

where x^q is the distance from the interface. If $q = 0$ the laplacian is recovered. When the distance becomes larger, α_u gets smaller, and vice versa. Defining α_u in this manner is a smart choice since it upholds the cell structure closer to the interface where most of the cell distortion appears.

3.4.2 Biharmonic extension

The last extension is the biharmonic extension. The biharmonic extension provides more freedom than the harmonic and linear elastic in choosing boundary conditions and choice of parameter $\alpha_u > 0$. This is because the biharmonic extension, extends the deformation in a manner that upholds the integrity of the cells even in large deformations, even with α_u as a constant. In its simplest form it is written as:

$$-\alpha_u \nabla^4 \mathbf{d} = 0 \quad \text{in } \hat{\mathcal{F}} \quad (3.20)$$

The biharmonic extension is calculated with a mixed formulation where we introduce a new function ω (not to be confused with the domain velocity), the function is added to the system so that we solve for 4 functions:

$$\omega = \alpha_u \nabla^2 \mathbf{d} \quad \text{and} \quad -\alpha_u \nabla^2 \omega = 0 \quad \text{in } \hat{\mathcal{F}} \quad (3.21)$$

with the two types of boundary conditions. The first boundary conditions being:

$$\mathbf{d} = \partial_n \mathbf{d} = 0 \quad \text{on } \partial \hat{\mathcal{F}} \setminus \hat{\Sigma} \quad (3.22)$$

$$\mathbf{d}_f = \mathbf{d}_s \quad \text{on } \hat{\Sigma} \quad (3.23)$$

The second type of boundary condition imposes conditions on \mathbf{d} and ω , and are written in terms of single component functions $d^{(1)}, d^{(2)}$ and $\omega^{(1)}, \omega^{(2)}$

$$\mathbf{d}^{(1)} = \partial_n \mathbf{d}^{(1)} = 0, \text{ and } \omega^{(1)} = \partial_n \omega^{(1)} = 0 \quad \text{on } \partial \hat{\mathcal{F}}_{in,out} \quad (3.24)$$

$$\mathbf{d}^{(2)} = \partial_n \mathbf{d}^{(2)} = 0, \text{ and } \omega^{(2)} = \partial_n \omega^{(2)} = 0 \quad \text{on } \partial \hat{\mathcal{F}}_{walls} \quad (3.25)$$

$$(3.26)$$

Since the biharmonic extension is of fourth order character it will have a higher computational cost [15] than the harmonic or linear elastic.

3.5 Discretization of monolithic FSI equations

After introducing all the equations and boundary conditions needed to solve a FSI problem. We are ready to discretize the equations into one monolithic scheme. The equations will be discretized and solved using finite difference and finite element methods. I will introduce a so called θ -scheme which will make it easy to implement different schemes, by choosing a value for θ . I will briefly introduce the spaces needed to discretize, following the ideas and notations of [26]:

Spaces

Let $X \in \mathbb{R}^d, d \in \{1, 2\}$ be a time dependent domain we define:

$$\hat{V}_X := H^1(X), \quad \hat{V}_X^1 := H_0^1(X) \quad (3.27)$$

H^1 indicating a Hilbert space and

$$\hat{L}_X := L^2(X), \quad \hat{L}_X^0 := L^2(X)/\mathbb{R} \quad (3.28)$$

L^2 indicating a standard Lebesgue space.

The trial and test spaces for the velocity variable in the fluid domain are defined as:

$$\hat{V}_{f,u}^0 := \{\hat{u} \in H_0^1(\mathcal{F}) : \hat{u}_f = \hat{u}_s \text{ on } \hat{\Sigma}\} \quad (3.29)$$

The artificial displacement spaces in the moving fluid domain are defined:

$$\hat{V}_{f,d}^0 := \{\hat{d} \in H_0^1(\mathcal{F}) : \hat{d}_f = \hat{d}_s \text{ on } \hat{\Sigma}\} \quad (3.30)$$

$$\hat{V}_{f,d}^0 := \{\hat{d} \in H_0^1(\mathcal{F}) : \hat{\psi}_f = \hat{\psi}_s \text{ on } \hat{\Sigma}\} \quad (3.31)$$

Discretization

The temporal discretization is done using finite difference schemes and the spatial is treated with the finite element method. Employing a so called θ -scheme that will enables us to easily switch between schemes.

In the domain Ω and time interval $[0, T]$:

Find $U = \{\mathbf{u}, \mathbf{d}, p\} \in \hat{X}_D^0$ where $\hat{X}_D^0 := \{\mathbf{u}_f^D + \hat{V}_{f,\mathbf{u}}^0\} \times \hat{L}_f \times \{\mathbf{d}_f^D + \hat{V}_{f,\hat{f}}^0\} \times \{\mathbf{d}_f^D + \hat{V}_{f,\hat{f}}^0\} \times \hat{L}_f \times \hat{L}_s^0$ such that:

$$\int_0^T A(U)(\Psi) dt = \int_0^T \hat{F}(\Psi) dt \quad \forall \Psi \in \hat{X} \quad (3.32)$$

where $\Psi = \{\phi, \psi, \gamma\}$

$$\hat{X} = \hat{V}_{f,\mathbf{u}}^0 \times \hat{L}_f \times \hat{V}_{f,\mathbf{d},\hat{\Sigma}}^0 \times \hat{V}_s^0 \times \hat{L}_f^0 \times \hat{L}_s^0$$

I first introduce the scheme using, for simplicity, the harmonic mesh motion. Let \mathbf{u} be a global function in the entire domain instead of \mathbf{u}_f in the fluid and \mathbf{u}_s in the solid. Same for the test functions. This is done for ease of reading and also for the ease of implementation later.

The full monolithic FSI variational form reads:

$$A(U) = (J\rho_f\partial_t\mathbf{u}, \phi) - (J(\nabla\mathbf{u})F^{-1}(\mathbf{u} - \partial_t d), \phi)_{\hat{\mathcal{F}}} \quad (3.33)$$

$$+ (J\sigma_f F^{-T}, \nabla\phi)_{\hat{\mathcal{F}}} \quad (3.34)$$

$$+ (\rho_s\partial_t\mathbf{u}, \phi)_{\hat{\mathcal{S}}} + (P, \nabla\phi)_{\hat{\mathcal{S}}} \quad (3.35)$$

$$+ (\alpha_u\nabla\mathbf{u}, \nabla\psi)_{\hat{\mathcal{F}}} + (\nabla \cdot (JF^{-1}\mathbf{u}), \gamma)_{\hat{\mathcal{F}}} \quad (3.36)$$

$$+ \delta((\partial_t\mathbf{d}, \psi)_{\hat{\mathcal{S}}} - (\mathbf{u}, \psi)_{\hat{\mathcal{S}}}) \quad (3.37)$$

$$+ (J\sigma_{f,p}F^{-T}, \nabla\phi) \quad (3.38)$$

The condition (3.37), is weighted with a δ value. This is a critical detail for the program (detailed later) to run. The weighting says in a weak manner that this condition is important for the overall program.

Introducing the *One step- θ scheme* from [26], which has the advantage of easily being changed from a backward (implicit), forward(explicit), or a Crank-Nicholson (implicit) scheme, by changing the value of θ . The Crank-Nicholson scheme is of second order, but suffers from instabilities in this monolithic scheme for certain time step values [26]. A remedy for these instabilities is to chose a Crank-Nicholson scheme that is shifted towards the implicit side. How this is done will become evident once the scheme is defined.

The variational form is defined by dividing into four categories. This may seem strenuous at first, but the need for it will become evident when implementing the θ -scheme. The four divided categories consists of: a time group A_T (with time derivatives), implicit A_I , pressure A_P and the rest A_E (stress terms and convection):

$$A_T(U) = (J\rho_f\partial_t\mathbf{u}, \phi) - (J(\nabla\mathbf{u})F^{-1}(\partial_t\mathbf{d}), \phi)_{\hat{\mathcal{F}}} \quad (3.39)$$

$$+ (\rho_s\partial_t\mathbf{u}, \phi)_{\hat{\mathcal{S}}} + (\partial_t\mathbf{d}, \psi)_{\hat{\mathcal{S}}} \quad (3.40)$$

$$A_I(U) = (\alpha_u\nabla\mathbf{u}, \nabla\psi)_{\hat{\mathcal{F}}} + (\nabla \cdot (JF^{-1}\mathbf{u}), \gamma)_{\hat{\mathcal{F}}} \quad (3.41)$$

$$A_E(U) = (J(\nabla u)F^{-1}\mathbf{u}, \phi)_{\hat{\mathcal{F}}} + (J\sigma_{f,u}F^{-T}, \nabla\phi)_{\hat{\mathcal{F}}} \quad (3.42)$$

$$+ (P, \nabla\phi)_{\hat{\mathcal{S}}} - (\mathbf{u}, \psi)_{\hat{\mathcal{S}}} \quad (3.43)$$

$$A_P(U) = (J\sigma_{f,p}F^{-T}, \nabla\phi) \quad (3.44)$$

Notice that the fluid stress tensors have been split into a velocity and pressure

part.

$$\sigma_{f,u} = \mu(\nabla u F^{-1} + F^{-T} \nabla u) \quad (3.45)$$

$$\sigma_{f,p} = -pI \quad (3.46)$$

For the time group, discretization is done in the following way:

$$A_T(U^{n,k}) \approx \frac{1}{k}(\rho_f J^{n,\theta}(\mathbf{u}^n - \mathbf{u}^{n-1}), \phi)_{\hat{\mathcal{F}}} - \frac{1}{k}(\rho_f(\nabla u)(\mathbf{d}^n - \mathbf{d}^{n-1}), \phi)_{\hat{\mathcal{F}}} \quad (3.47)$$

$$+ \frac{1}{k}(\rho_s(\mathbf{u}^n - \mathbf{u}^{n-1}), \phi)_{\hat{\mathcal{S}}} + \frac{1}{k}((\mathbf{d}^n - \mathbf{d}^{n-1}), \psi)_{\hat{\mathcal{S}}} \quad (3.48)$$

The Jacobian is written with superscript θ as:

$$J^{n,\theta} = \theta J^n + (1 - \theta) J^{n-1} \quad (3.49)$$

Finally we can introduce the *One step- θ scheme*: Find $U^n = \{\mathbf{u}^n, \mathbf{d}^n, p^n\}$

$$A_T(U^{n,k}) + \theta A_E(U^n) + A_P(U^n) + A_I(U^n) = \quad (3.50)$$

$$- (1 - \theta) A_E(U^{n-1}) + \theta \hat{f}^n + (1 - \theta) \hat{f}^{n-1} \quad (3.51)$$

Notice that the scheme is selected by the choice of θ . By choosing $\theta = 1$ we get the back Euler scheme, for $\theta = \frac{1}{2}$ we get the Crank-Nicholson scheme and for the shifted Crank-Nicholson we set $\theta = \frac{1}{2} + k$, effectively shifting the scheme towards the implicit side. \hat{f} is the body forces. The shifting of the scheme towards the implicit side is important for long term stability for certain time step values [26]. The shifting will be investigated in chapter 5.

Chapter 4

Implementation of Fluid Structure Interaction in FEniCS

This Chapter shows the implementation of the FSI code in FEniCS [10]. FEniCS is platform used to solve partial differential equations. Code is written in python and FEniCS makes it easy to run efficient finite element code.

The complete code consist of many hundreds of lines of code, and therefore only the most essential parts are covered. The code that has been added is added so that a reader with a minimal skill set in scientific computing and basic knowledge of the finite element method could implement a version of the code.

4.1 Mappings

The deformation gradient and the Jacobian is made using python functions. The deformation gradient python function takes in the deformation and returns the deformation gradient. Identity in FEniCS is a function that makes the identity matrix, with the size determined by a input value. The identity takes the length of the deformation function which for 2D cases is 2. grad is a built in FEniCS function that takes a vectorfunction and take the gradient.

```
1 def F_(d): """inputs deformation """
```

```

2         return (Identity(len(d)) + grad(d))  """Returns the deformation
3         gradient"""
4 def J_(d):
5     return det(F_(d))

```

4.2 Variational form

The variational form can be written directly into FEniCS. A big advantage in FEniCS is that one can write all the forms and add them together to make the full variational form. The vector functions and functions such as the displacement, velocity, and pressure, are written with a script n meaning at which time the function is valued.

```

1 d_["n"]  """deformation in the current timestep """
2 u_["n-1"] """velocity in the last timestep """
3 p_["n-2"] """pressure in the second last timestep """

```

θ is the value θ that determines the scheme. k is the timestep Δt . F_{fluid_linear} is the linear part of the variational form, and $F_{fluid_nonlinear}$ is the nonlinear part. Each part is assembled later when using newtons-method.

Constant is a function in FEniCS that represents a constant value unknown at compile-time. Its value can be changed without the need to re-compilation of C++ code, which FEniCS utilizes. σ_f and σ_f is the velocity and pressure parts of the fluid stress tensor. grad and div are built-in functions in FEniCS that computes the gradient and divergence and given vector. dx_f denotes the fluid domain. $\text{inv}()$ is a built-in function in FEniCS outputs the inverse of the given matrix. ψ , ϕ and γ are the test functions.

```

1 J_theta = theta*J_(d_["n"]) + (1 - theta)*J_(d_["n-1"])
2
3 F_fluid_linear = rho_f/k*inner(J_theta*(v_["n"] - v_["n-1"]), psi)*dx_f
4
5 F_fluid_nonlinear = Constant(theta)*rho_f*\
6 inner(J_(d_["n"])*grad(v_["n"])*inv(F_(d_["n"]))*v_["n"], psi)*dx_f
7
8 F_fluid_nonlinear += inner(J_(d_["n"])*sigma_f_p(p_["n"], d_["n"])*\
9 inv(F_(d_["n"])).T, grad(psi))*dx_f

```



```

10
11 F_fluid_nonlinear += Constant(theta)*inner(J_(d_["n"])\
12 *sigma_f_u(v_["n"], d_["n"], mu_f)*inv(F_(d_["n"])).T, grad(psi))*dx_f
13
14 F_fluid_nonlinear += Constant(1 - theta)*inner(J_(d_["n-1"])*\
15 sigma_f_u(v_["n-1"], d_["n-1"], mu_f)*inv(F_(d_["n-1"])).T, grad(psi))*
    dx_f
16
17 F_fluid_nonlinear += \
18 inner(div(J_(d_["n"])*inv(F_(d_["n"]))*v_["n"]), gamma)*dx_f
19
20 F_fluid_nonlinear += Constant(1 - theta)*rho_f*\
21 inner(J_(d_["n-1"])*grad(v_["n-1"])*inv(F_(d_["n-1"]))*v_["n-1"], psi)*
    dx_f
22
23 F_fluid_nonlinear -= rho_f*inner(J_(d_["n"])*\
24 grad(v_["n"])*inv(F_(d_["n"]))*((d_["n"]-d_["n-1"])/k), psi)*dx_f

```

Fsolidlinear is the linear part of the variational form, and Fsolidnonlinear is the nonlinear part. Piola1 is the first Piola-Kirchhoff stress tensor.

```

1 delta = 1E10
2 F_solid_linear = rho_s/k*inner(v_["n"] - v_["n-1"], psi)*dx_s +\
3 delta*(1/k)*inner(d_["n"] - d_["n-1"], phi)*dx_s -\
4 delta*inner(Constant(theta)*v_["n"] + Constant(1-theta)*v_["n-1"], phi)*
    dx_s
5
6 F_solid_nonlinear = inner(Piola1(Constant(theta)*d_["n"] +\
7 Constant(1 - theta)*d_["n-1"], lamda_s, mu_s), grad(psi))*dx_s

```

The weighed delta is talked about in section 3.5

4.3 Newtons method implementation for solving Fluid structure interaction

To solve a non-linear FSI problem, the newtons method has been implemented, ideas and code taken from from [25].

The derivative of the nonlinear variational parts is take with respect to dvp,

which is the mixed function of displacement, velocity and pressure. $-F$ is assembly of full variational form.

There is also an if test which only assembles the Jacobian the first and tenth time. This reuses the Jacobian to improve runtime, which is discussed in chapter 6. Lastly the the mpi line is when the code is running in parallel.

```

1 def newtonsolver(F, J_nonlinear, A_pre, A, b, bcs, \
2                 dvp_, up_sol, dvp_res, rtol, atol, max_it, T, t, **
3                 monolithic):
4     Iter      = 0  """ Setting initial values """
5     residual   = 1
6     rel_res    = residual
7     lambda    = 1
8
9     while rel_res > rtol and residual > atol and Iter < max_it:
10        if Iter % 10 == 0:
11            A = assemble(J_nonlinear, tensor=A, form_compiler_parameters
12            = {"quadrature_degree": 4})
13            A.axpy(1.0, A_pre, True)
14            A.ident_zeros()
15
16            b = assemble(-F, tensor=b)
17
18            [bc.apply(A, b, dvp_["n"].vector()) for bc in bcs]
19            up_sol.solve(A, dvp_res.vector(), b)
20            dvp_["n"].vector().axpy(lambda, dvp_res.vector())
21            [bc.apply(dvp_["n"].vector()) for bc in bcs]
22            rel_res = norm(dvp_res, 'l2')
23            residual = b.norm('l2')
24            if isnan(rel_res) or isnan(residual):
25                print "type rel_res: ", type(rel_res)
26                t = T*T
27
28            if MPI.rank(mpi_comm_world()) == 0:
29                print "Newton iteration %d: r (atol) = %.3e (tol = %.3e), r
30                (rel) = %.3e (tol = %.3e) " \
31                % (Iter, residual, atol, rel_res, rtol)
32            Iter += 1
33
34    return dict(t=t)

```

Chapter 5

Verification and Validation of the Fluid Structure Interaction Implementation.

The general approach when solving a real world problem with numerical computing, starts by defining the mathematics, implementing the equations numerically and solve the equations on a computer.

The produced solutions are used to extract data of interest. A question then immediately arises, is the solution and the extracted data trustworthy.

To answer this question we need to answer another question, are the equations solved correct numerically, if so is the problem defined correct mathematically. Without answering these questions, being confident that your solutions are correct is difficult [18]. This process of generating evidence that computed solutions meets certain requirements to fulfill an intended purpose, in the context of scientific computing, is known as Verification and Validation. The goal of this section will hence be to verify and validate the different numerical schemes outlined in the two previous chapters.

The chapter starts with the process of Verification where the fluid and structure parts of the code will be verified. Following will be Validation of the code where I implement at a well known benchmark testing the fluid and structure parts individually and as a full FSI problem. After that I investigate the impact of using different time order schemes, and different lifting operator.

5.1 Verification

Verification, in the context of scientific computing, is the process of determining whether or not the implementation of numerical algorithms in computer code, is done correctly. [13]. In verification, we get evidence that the numerical model derived from mathematics is solved correctly by the computer. The strategy is to identify, quantify and reduce errors caused by mapping a mathematical model to a computational model. Verification does not address whether or not the computed solutions are in alignment with physics in the real world. It only tells us that our model is computed correctly or not. To verify that we are computing correctly we can compare our computed solution to an exact solution. But the problem is that there are no known exact solutions to, for instance, the Navier-Stokes equations, other than for very simplified problems.

In Verification there are multiple classes of test that can be performed, and the most rigorous is the *Method of Manufactured Solution* (MMS) [13]. Rather than looking for an exact solution, we manufacture one. The idea is to make a solution *a priori*, and use this solution to generate an analytical source term for the governing PDEs and then run the PDE with the source term to get a solution hopefully matching the manufactured one. The manufactured solution does not need to have a physically realistic relation, since the solution deals only with the mathematics. The overall idea is to make solutions in a way that all terms of the given PDE are tested. The procedure of MMS is as follows [13]:

- We define a mathematical model on the form $L(\mathbf{u}) = 0$ where $L(\mathbf{u})$ is a differential operator and u is a dependent variable.
- Define the analytical form of the manufactured solution $\hat{\mathbf{u}}$
- Use the model $L(u)$ with $\hat{\mathbf{u}}$ inserted to obtain an analytical source term $f = L(\hat{\mathbf{u}})$
- Initial and boundary conditions are enforced from $\hat{\mathbf{u}}$
- Then use this source term to calculate the solution \mathbf{u} , $L(\mathbf{u}) = f$

After the solution has been computed we perform systematic convergence tests [16]. The idea behind order of convergence tests is based on the behavior of the error between the manufactured exact solution and the computed solution. If we let \mathbf{u} be the numerical solution and $\hat{\mathbf{u}}$ be the exact solution,

$||.||$ be the L^2 norm, we define the error as:

$$E = ||\mathbf{u} - \hat{\mathbf{u}}|| \quad (5.1)$$

When we decrease the node spacing ($\Delta x, \Delta y$ or Δz) or decrease timestep size(Δt), we expect the solution to convergence towards a given solution and hence the error to get smaller. It is the rate of this error that lets us know wether the solution is converging correctly. If we assume that the number of spatial points are equal in all directions the error is expressed as

$$E = C_1 \Delta x^k + C_2 \Delta t^l \quad (5.2)$$

where $k = m + 1$ and m is the polynomial degree of the spatial elements. The error is hence dependent on the number of spatial points and the timestep. If instance Δt is reduced significantly, Δx will dominate, and Δt will be negligible. If we then look at two error where E_{n+1} has finer mesh than E_n , using (5.2):

$$\frac{E_{n+1}}{E_n} = \left(\frac{\Delta x_{n+1}}{\Delta x_n} \right)^k \quad (5.3)$$

$$k = \frac{\log(\frac{E_{n+1}}{E_n})}{\log(\frac{\Delta x_{n+1}}{\Delta x_n})} \quad (5.4)$$

k can is used to find the observed order of convergence and match with the theoretical order of convergence for each given problem.

The manufactured solutions should be chosen to be non-trivial and analytic [13, 16]. The solutions should be manufactured so that no derivatives vanish. For this reason trigonometric and exponential functions can be a smart choice, since they are smooth and infinitely differentiable. In short, a good manufactured solution is one that is complex enough so that it rigorously tests each part of the equations.

Starting with the verification of the solid part of the code with given displacement and velocity. Then verifying the fluid part with a given displacement, also testing the mappings between configurations. To do a full verification of the entire FSI problem, one needs to take into account the condition of continuity of velocity on the interface [4], the stresses need to equal on the interface and the flow needs to be divergence free. Manufacturing such a

solution is very difficult [3]. The author has yet to find a paper that manufactures a solution fulfilling all the condition in a rigorous manner. MMS of full FSI is therefore out the scope of this thesis.

5.1.1 Method of Manufactured Solution on the implementation of the Solid equation

The first MMS is with the solid problem alone. Testing the solid equation 3.7 with the condition $\mathbf{u} = \frac{\partial \mathbf{d}}{\partial t}$.

Solutions $\hat{\mathbf{d}}$ and $\hat{\mathbf{u}}$ are manufactured to meet the requirements of MMS such as smoothness and complexity, choosing functions with sine and cosine. The derivatives does not become zero and we have time and space dependencies.

$$\begin{aligned}\hat{\mathbf{d}}_{\mathbf{e}} &= (\cos(y)\sin(t), \cos(x)\sin(t)) \\ \hat{\mathbf{u}}_{\mathbf{e}} &= (\cos(y)\cos(t), \cos(x)\cos(t))\end{aligned}$$

The manufactured solutions are used to make a source term f_s :

$$\rho_s \frac{\partial \hat{\mathbf{u}}}{\partial t} - \nabla \cdot (P(\hat{\mathbf{d}})) = f_s$$

The solid variational formulation is written in weak form with test functions ϕ and ψ as:

$$\left(\rho_s \frac{\partial \mathbf{u}}{\partial t}, \phi\right)_{\hat{\mathcal{S}}} + (P, \nabla \phi)_{\hat{\mathcal{S}}} = f_s \quad (5.5)$$

$$\left(\mathbf{u} - \frac{\partial \mathbf{d}}{\partial t}, \psi\right)_{\hat{\mathcal{S}}} = 0 \quad (5.6)$$

These equations are solved together and we solve for d and u on a unit square domain. The number N denotes the number of spatial points in x and y direction. The functions u and d will be computed to match the source term. The computations ran with 10 timesteps and the error was calculated for each time step and then the mean of all the errors were used to calculate the convergence rates.

Even though there are two equations, we do not make a source for the second equation, since the solutions are made to uphold the criteria $\mathbf{u} = \frac{\partial \mathbf{d}}{\partial t}$.

In the tables below we investigate convergence in space and time. Starting with checking order of convergence in space, setting $m = 1$, the expected order of convergence will be 2.

N	Δt	m	E_u	\mathbf{k}_u	E_d	\mathbf{k}_d
4	1×10^{-7}	1	0.0068828	-	3.7855×10^{-9}	-
8	1×10^{-7}	1	0.0017204	2.0002577	9.4622×10^{-10}	2.0002577
16	1×10^{-7}	1	0.0004300	2.0000622	2.3654×10^{-10}	2.0000622
32	1×10^{-7}	1	0.0001075	2.0000154	5.9136×10^{-11}	2.0000154
64	1×10^{-7}	1	0.0000268	2.0000038	1.4783×10^{-11}	2.0000038

Table 5.1: Method of Manufactured Solution on the implementation of the Solid equation in space with $m = 1$

Lastly I check convergence in time. The scheme tested is of first order, with a value of $\theta = 1$, hence a order of convergence of 1 is expected. Here i set $N = 64$ and the timestep is halved for each computation.

N	Δt	$E_u [\times 10^{-6}]$	\mathbf{k}_u	$E_u [\times 10^{-8}]$	\mathbf{k}_d
64	0.1	0.027663		0.0034221	
64	0.05	0.013390	1.0467	0.0018093	0.9194
64	0.025	0.007016	0.9324	0.0009246	0.9685
64	0.0125	0.003645	0.9444	0.0004688	0.9798
64	0.00625	0.001828	0.9957	0.0002414	0.9571

Table 5.2: Method of Manufactured Solution on the implementation of the Solid equation in time

Comments on the Solid MMS

The table 5.1 shows a decrease in error with increased number of spatial points. The order of convergence tends toward the expected value of 2, hence increasing our confidence in the implementation of the solid equation.

Table 5.2 shows a order of convergence tending towards 1, which was expected. With order of convergence tests passed for both time and space for

the solid equation and the condition $\mathbf{u} = \frac{\partial \mathbf{d}}{\partial t}$, the implementation of the solid equation is satisfactory.

5.1.2 MMS of Fluid equations with prescribed motion

Starting by prescribing a motion to d and w and manufacturing solutions $\hat{\mathbf{u}}$ and \hat{p} . Setting $\hat{\mathbf{u}} = \mathbf{w}$:

$$\begin{aligned}\hat{\mathbf{d}} &= (\cos(y)\sin(t), \cos(x)\sin(t)) \\ \hat{\mathbf{u}} = \hat{\mathbf{w}} &= (\cos(y)\cos(t), \cos(x)\cos(t)) \\ \hat{p} &= \cos(x)\cos(t)\end{aligned}$$

Solutions are made to uphold the criterias :

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{d}}{\partial t} = \mathbf{w}$$

Whilst testing the implementations of the fluid equations, the opportunity arises to also test the mappings between current and reference configurations. The source term f_f is hence made without mappings:

$$\rho_f \frac{\partial \hat{\mathbf{u}}}{\partial t} + \nabla \hat{\mathbf{u}} \left(\hat{\mathbf{u}} - \frac{\partial \hat{\mathbf{d}}}{\partial t} \right) - \nabla \cdot \sigma(\hat{\mathbf{u}}, \hat{p})_f = f_f$$

Then f_f is used and mapped to the reference configuration and computed:

$$\rho_f J \frac{\partial \mathbf{u}}{\partial t} + (\nabla \mathbf{u}) F^{-1} \left(\mathbf{u} - \frac{\partial \mathbf{d}}{\partial t} \right) + \nabla \cdot (J \hat{\sigma}_f F^{-T}) = J f_f$$

The computations are done on a unit square domain and the computations ran with 10 timesteps and the error was calculated for each time step and then the mean of all the errors was taken and used to calculate the convergence rates.

The scheme tested for convergence time is of first order, with a value of $\theta = 1$, hence a order of convergence of 1 is expected. Here i set $N = 64$ and the timestep is halved for each computation.

N	Δt	E_u	$\mathbf{k_u}$	E_p	$\mathbf{k_p}$
64	0.1	5.1548×10^{-5}	-	0.008724	-
64	0.05	2.5369×10^{-5}	1.0228	0.004290	1.0240
64	0.025	1.2200×10^{-5}	1.0561	0.002058	1.0596
64	0.0125	0.56344×10^{-5}	1.1145	0.0009556	1.1068

Table 5.3: MMS ALE FSI u=w

In testing the spatial convergence I set $m = 2$, hence giving an expected order of convergence in the velocity of 3 and pressure of 2.

N	Δt	m	E_u	k_u	E_p	k_p
2	1×10^{-6}	2	8.6955×10^{-4}	-	0.01943	-
4	1×10^{-6}	2	1.0844×10^{-4}	3.0032	0.00481	2.0140
8	1×10^{-6}	2	0.1354×10^{-4}	3.0007	0.00119	2.0120
16	1×10^{-6}	2	0.0169×10^{-4}	3.0001	0.00029	2.0074

Table 5.4: MMS ALE FSI u=w

Comments on the Solid MMS

In table ?? the order of convergence converges as expected towards 1. In 5.4 the MMS testing convergence in space converges toward 3 for velocity and 2 for pressure as was expected when setting $m = 2$. Both results helps in building confidence that the implementation of the mapped fluid equation is correct.

5.2 Validation

After the code has been verified, we move on to Validation which is the process of determining if a model gives an accurate representation of real world physics within the bounds of the intended use [18]. A model is made for a specific purpose, its only valid with respect to that purpose [11]. If the purpose is complex and trying to answer multiple questions, then the validity needs to be determined for each question. The idea is to validate the solver, *brick by brick*, Starting with simple testing of each part of the model and build more complexity and eventually test the whole model.

Three issues have been identified in this process [18]: Quantifying the accuracy of the model by comparing responses with experimental responses, interpolation of the model to conditions corresponding to the intended use and determining the accuracy of the model for the conditions under which its meant to be used. Well known benchmarks will be used as reference points and these tests supply us with a problem setup, initial and boundary conditions, and lastly results that we can compare with.

The process of Validation is also, as I have experienced, a way to figure out at what size timestep and number of spatial points the model can handle. As we will see in the chapter all the benchmarks are run with different timesteps and number of cells to see how the model reacts. The problem with using benchmarks with known data for comparison is that we do not test the model blindly. It is easier to mold the model to the data we already have. As Oberkampf and Trucano in [18] puts it "Knowing the "correct" answer beforehand is extremely seductive, even to a saint.". Knowing the limitations of our tests will therefore strengthen our confidence in the model. It really can be an endless process of verifying and validating if one does not clearly know the bounds of sufficient accuracy. [18]

The following tests are done with the solid and fluid alone. Testing both steady and unsteady cases. Lastly the full FSI model is tested with steady and unsteady cases.

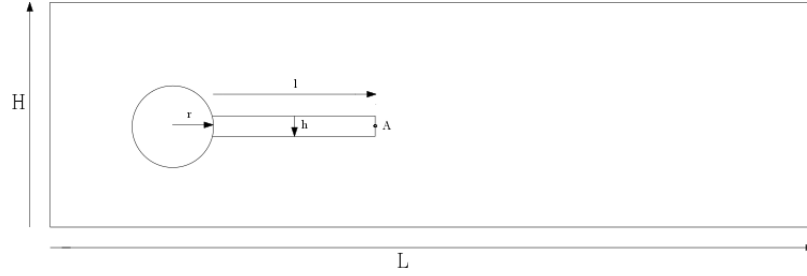
5.2.1 Fluid-Structure Interaction between an elastic object and laminar incompressible flow

The goal of this benchmark is to verify the fluid and solid solver first separately and then together as a full FSI problem [8]. This benchmark is based on the older benchmark "flow around cylinder" with fluid considered incompressible and in the laminar regime where the structure deformations are large, in one case the deformations are about 2.5 larger than the height of the elastic object. The problem is setup with the solid submerged in the fluid, so that oscillations in the fluid can deform the structure. Measuring the drag and lift around the circle and bar, and measure structural displacement at a given point. This benchmark will be used to test and verify different numerical methods and code implementations. Testing the robustness and efficiency of the FSI implementation.

A very important thing to notice about this paper [8] is that it is a proposal for a benchmark, as it is called "Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow". So the point of the paper is give a specific problem setup which others can contribute result-data. In the summary of the paper it is stated: "The next step will be the specification of how to submit and to collect the results, and the publication of the test configurations in an international journal. Moreover, it is planned to prepare a webpage for collecting and pre-senting the FSI results. As we have learned from [3], a very important aspect will be the submission of the results on (at least) 3 different meshes and time steps." In sight of this, a paper was published in 2010 by J. Hron, Turek, et al, [20] that compared results of different discretizations and solution approaches. This paper [20] gives 7 different methods and results for two of the FSI test cases. They state in the numerical results that "However, also clear differences between the different approaches with regard to accuracy are visible. Particularly for the drag and lift values, which lead to differences of up to order 50%, and also for the displacement values which are in the range of 10% errors.". With this in mind it is important to know that the referential values used are only those reported from the original paper, which only looked at one implementation. While the paper which compares results, only 2 of the 7 contributions were schemes of monolithic nature, which are the closest one should refer to in this thesis. In the Appendix A.2 is a copy of the results from the paper comparing schemes, showing different results for different discretization, with different time steps and unknowns.

Problem Defintion

Domain



The computational domain consists of a circle with an elastic bar behind the circle. The circle is positioned at (0.2, 0.2) making it 0.05 of center from bottom to top, **this shift in the domain is done to induce oscillations to an otherwise steady flow.** The fluid oscillations gives a force to the elastic bar. The parameters of the domain are:

$L = 2.5$, $H = 0.41$, $l = 0.35$, $h = 0.02$, $A = (0.2, 0.6)$

Boundary conditions

A parabolic profile has been prescribed to the velocity that increases over time until a time = 2 has been reached.

$$u(0, y) = 1.5u_0 \frac{y(H - y)}{(\frac{H}{2})^2}$$

$$u(0, y, t) = u(0, y) \frac{1 - \cos(\frac{\pi}{2}t)}{2} \text{ for } t < 2.0$$

$$u(0, y, t) = u(0, y) \text{ for } t \leq 2.0$$

Setting no slip on the "floor" and "ceiling" so to speak.

$$u(x, y, t) = 0 \text{ on } \partial\mathcal{F}_{floorandceiling}$$

$$p(x, y, t) = 0 \text{ on } \partial\mathcal{F}_{outlet}$$

Quantities for comparison

When the fluid moves around the circle and bar it exerts a force. These are split into drag and lift and calculated as follows:

$$(F_d, F_L) = \int_S \sigma_f n dS$$

where S is the part of the circle and bar in contact with the fluid.

We set a point $A = (0.2, 0.6)$ on the right side of the bar. This point is used to track the deformation.

In the unsteady time dependent problems, the values are represented by mean and amplitude values:

$$mean = \frac{1}{2}(max + min) \quad (5.7)$$

$$amplitude = \frac{1}{2}(max - min) \quad (5.8)$$

$$(5.9)$$

In each test the numbers with ref are the values taken from the original benchmark paper [8]

Results

CFD test

The first two CFD tests are run with Reynolds numbers 20 and 100 converging to steady solutions of drag and lift around the circle. The CFD 3 problem has a Reynolds number 200 which will induce oscillations behind the circle, giving fluctuations in the drag and lift, hence giving unsteady solutions. The CFD tests can be run in two ways. The first way assumes the bar to be rigid object, meaning that the computing domain is just the fluid domain. The other way, which is implemented here, is running the full FSI code and setting $\rho_s = 10^6$ and $\mu_s = 10^{12}$, so that the bar is almost completely rigid, only giving rise to very small deformation (in the $10^{-9} - 10^{-10}$ range).

Figure 5.1: Picture of entire FSI computing domain with 11556 cells

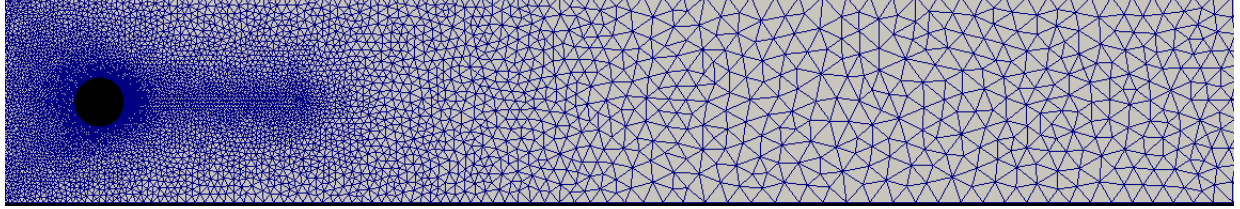
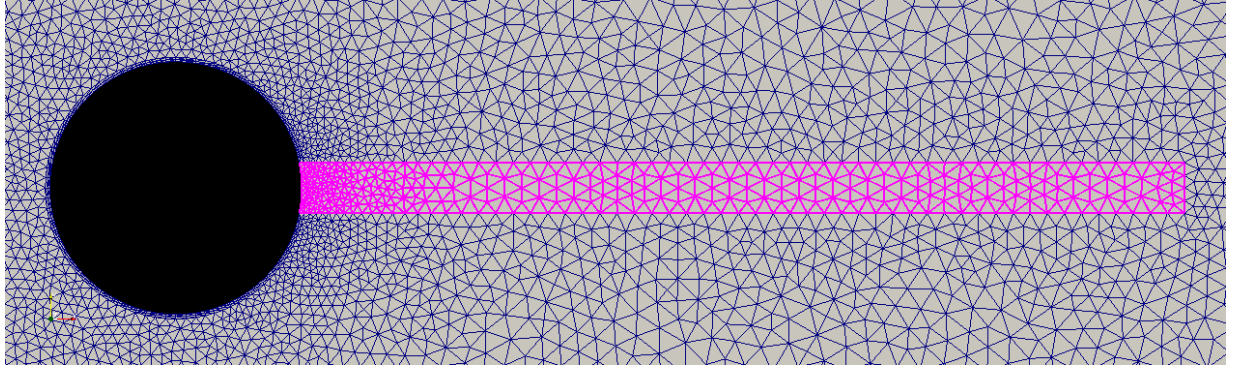


Figure 5.2: Picture of FSI computing domain with 11556 cells, zoomed in with the solid domain marked in pink



Parameters	CFD1	CFD2	CFD3
$\rho_f [10^3 \frac{kg}{m^3}]$	1	1	1
$\nu_f [10^{-3} \frac{m^2}{s}]$	1	1	1
$U [\frac{m}{s}]$	0.2	1	2
$Re = \frac{Ud}{\nu_f}$	20	100	200

Table 5.5: Summary of all the parameters in CFD tests

elements	dofs	Drag	Lift
2474	21749	14.059	1.100
7307	63365	14.110	1.080
11556	99810	14.200	1.1093
ref		14.29	1.119

Table 5.6: Results of CFD1 case run as full FSI, with almost rigid bar

elements	dofs	Drag	Lift
2474	21749	134.9	10.38
7307	63365	135.4	10.0
11556	99810	136.1	10.41
ref		136.7	10.53

Table 5.7: Results of CFD2 case run as full FSI with almost rigid bar

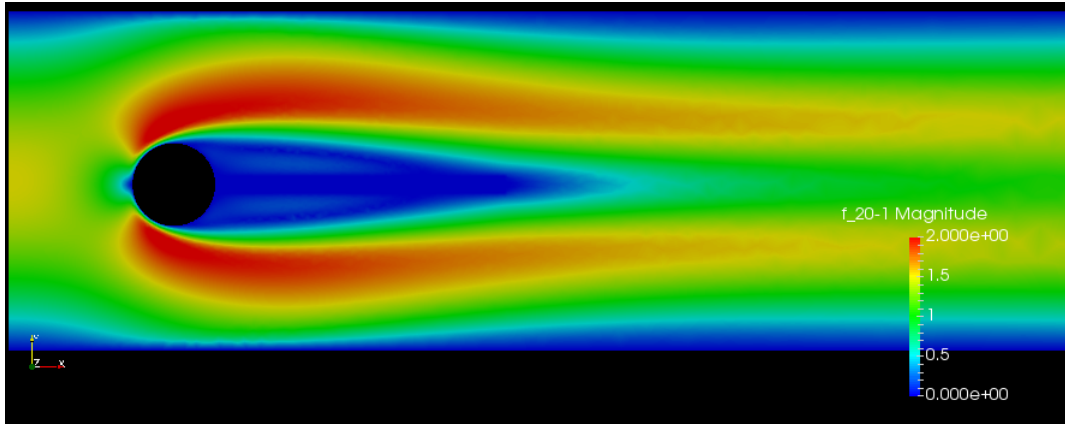


Figure 5.3: CFD2 steady state case fluid flow

Discussion of CFD results

In the CFD2 case notice, in figure 5.2.1, that since the circle and bar are positioned non symmetric in the y direction there is more fluid flowing closer to the upper boundary. This non symmetry is what induces oscillations when the fluid inlet velocity is higher. The CFD2 case has an inlet velocity just below the point of inducing oscillations.

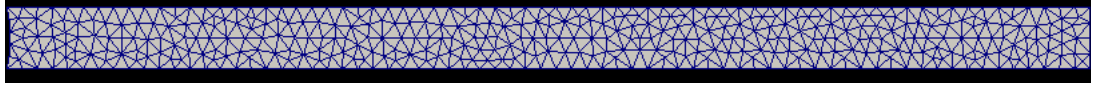
CSM test

The CSM test are calculated using only the bar as computing domain. A gravity term g has been added, which has the same value throughout the CSM tests but changing the parameters of the solid. The tests CSM1 and CSM2 are steady state solutions. The difference between them is a more slender bar. The CSM 3 test is unsteady and even more slender causing the bar to move up and down. Since there is no resistance from any fluid, the

unsteady test should ,if energy is preserved, make the bar move up to the initial state and down to the same point.

Our quantity for comparing there will be the deformation at the point A .

Figure 5.4: Picture of the coarsest solid mesh used in the MMS test



Parameters	CSM1	CSM2	CSM3
$\rho_f [10^3 \frac{kg}{m^3}]$	1	1	1
$\nu_f [10^{-3} \frac{m^2}{s}]$	1	1	1
u_0	0	0	0
$\rho_s [10^3 \frac{kg}{m^3}]$	1	1	1
ν_s	0.4	0.4	0.4
$\mu_s [10^6 \frac{m^2}{s}]$	0.5	2.0	0.5
g	2	2	2

Table 5.8: Summary table of the parameters used in the CSM tests

elements	dofs	$d_x(A)[\times 10^{-3}]$	$d_y(A)[\times 10^{-3}]$
725	1756	-5.809	-59.47
2900	6408	-6.779	-64.21
11600	24412	-7.085	-65.63
46400	95220	-7.116	-65.74
ref	ref	-7.187	-66.10

Table 5.9: Results of the steady CSM1 case from coarse to fine mesh.

Elements	Dofs	$d_x(A)[\times 10^{-3}]$	$d_y(A)[\times 10^{-3}]$
725	1756	-0.375	-15.19
2900	6408	-0.441	-16.46
11600	24412	-0.462	-16.84
46400	95220	-0.464	-16.87
ref	ref	-0.469	-16.97

Table 5.10: Results of the steady CSM2 case from coarse to fine mesh.

elements	dofs	$d_x(A)[\times 10^{-3}]$	$d_y(A)[\times 10^{-3}]$
725	1756	-11.743 ± 11.744	-57.952 ± 58.940
2900	6408	-13.558 ± 13.559	-61.968 ± 63.440
11600	24412	-14.128 ± 14.127	-63.216 ± 64.744
46400	95220	-14.182 ± 14.181	-63.305 ± 64.843
ref		-14.305 ± 14.305	-63.607 ± 65.160

Table 5.11: Results of the unsteady CSM3 case with mesh from coarse to fine.

Figure 5.5: Results for CSM3 showing Displacement of point A

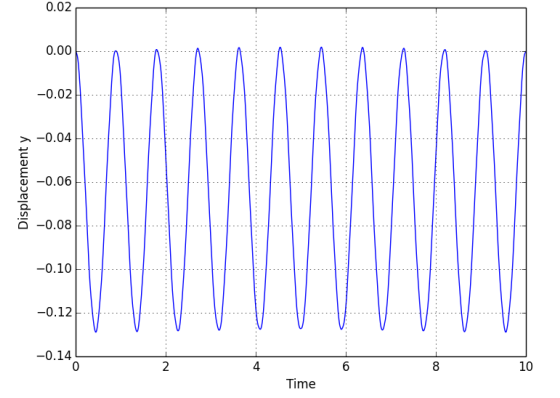
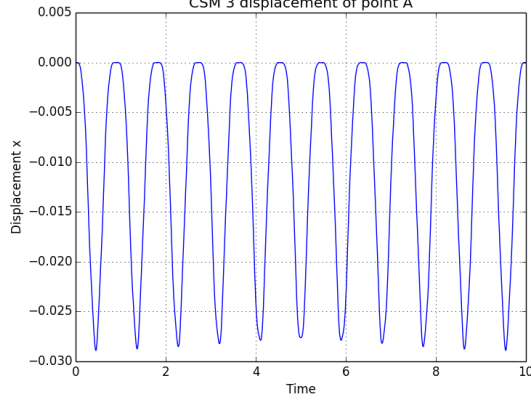


Figure 5.6: Displacement in x direction, timeinterval (0,10)

Figure 5.7: Displacement in y direction, timeinterval (0,10)

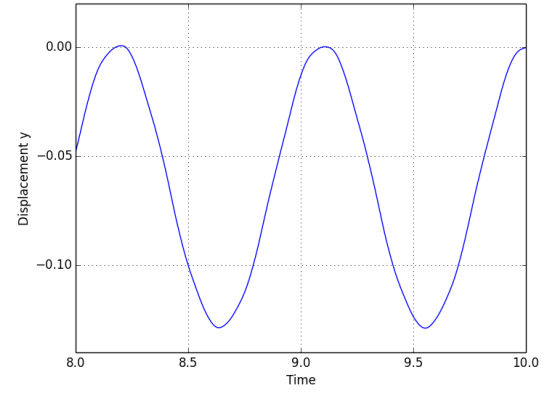
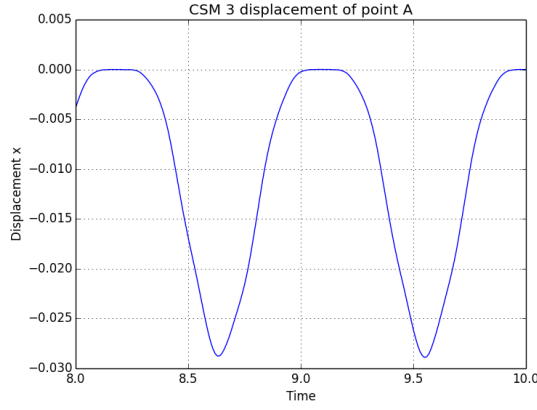


Figure 5.8: Displacement in x direction, timeinterval (8,10)

Figure 5.9: Displacement in y direction, timeinterval (8,10)

Discussion of the CSM results

The tables 5.9, 5.10 shows the results of the two steady state cases with different bar-stiffness. As the number of spatial points in the domain increases the solutions converge towards the referential value. The CSM3 case shown in table 5.11 shows the same trend of converging toward referential value when the number of spatial points are increased.

The plot 5.2.1 is of displacement at the point A in x and y direction of the CSM3 test. The CSM3 test was run with Crank-Nicholson, $\theta = 0.5$, and it can be seen that in the y displacement the bar returns to it initial state, that is with zero displacement. This results indicates that the energy in the system has been preserved.

FSI tests

Lastly the full FSI problem ran. Here we can see in 5.2.2 that now both fluid and structure has a mesh. The tests are run with 2 different inflows conditions. FSI1 gives a steady state solution while the others are unsteady. FSI-2 gives the largest deformation is therefore considered the most difficult of the three [14], giving deformations of 2.5 times greater than the flag height. Vinje 2016 [23] reported not being able to compute FSI-2 model due to shortcomings in the linear elasticity solid model. The FSI-3 test has the highest inflow speed giving medium deformations but more rapid oscillations.

Parameters	FSI1	FSI2	FSI3
$\rho_f [10^3 \frac{kg}{m^3}]$	1	1	1
$\nu_f [10^{-3} \frac{m^2}{s}]$	1	1	1
u_0	0.2	1	2
$Re = \frac{Ud}{\nu_f}$	20	100	200
$\rho_s [10^3 \frac{kg}{m^3}]$	1	10	1
ν_s	0.4	0.4	0.4
$\mu_s [10^6 \frac{m^2}{s}]$	0.5	0.5	2

Table 5.12: FSI Parameters

FSI1

Cells	Dofs	$d_x(A) [\times 10^{-3}]$	$d_y(A) [\times 10^{-3}]$	Drag	Lift
2474	21749	0.0229	0.8265	14.0581	0.7546
7307	63365	0.02309	0.7797	14.1077	0.7518
11556	99810	0.02295	0.8249	14.2046	0.7613
ref	ref	0.0227	0.8209	14.295	0.7638

Table 5.13: Results of FSI 1 test case

FSI2

FSI3 results

Cells	Dofs	$d_x(A)[\times 10^3]$	$d_y(A)[\times 10^3]$	Drag	Lift
2474	21249	-1.79 ± 1.80	3.29 ± 2.64	439.36 ± 12.04	1.96 ± 142.31
7307	63365	-2.48 ± 2.48	1.64 ± 3.28	449.77 ± 18.02	3.41 ± 153.47
11556	99810	-2.47 ± 2.45	1.27 ± 3.28	456.60 ± 18.73	-1.55 ± 153.46
ref	ref	-2.69 ± 2.56	1.48 ± 34.38	457.3 ± 22.66	2.22 ± 149.78

Table 5.14: FSI3 unsteady test case results with $\Delta t = 0.01$, with harmonic lifting operator

Comment on FSI tests

The FSI1 test gives a low fluid velocity and gives very low displacements. FSI1 is therefore not a rigid test for FSI. In fact I experienced in the beginning of making the FSI solver, that even with a wrong implementation i got good FSI1 results. However it is a good test for early checks, because if FSI1 is wrong the rest will definitely not work.

5.2.2 Comparing different lifting operators

This section is devoted to comparing different lifting operators from 3.4. The comparisons will be run using a version of the CSM test discussed earlier, with the same computing domain as the Hron Turek benchmark. Also the FSI2 case from the previous section will be investigated. The tests will compare the different techniques by investigating how the deformation from the solid domain is lifted into the fluid domain. This is done by investigating a plot of the mesh after deformation to see how much cells distort and where the cells distort. This is visualized using Paraview.

In the version of the CSM1 case the fluid is initially at rest and with no inflow on the fluid. A gravitational force is applied to the structure much like the previous CSM test. The only difference is that we now use the full domain from the 5.2.1 . The test is run as time-dependent with a the backward Euler scheme, leading to a steady state solution. In the first test case the parameters from CSM1 are used.

A plot of the FSI2 test case with different lifting operators has been added at the end

Boundary conditions

The computing domain is the same as used in the previous section. The upper, lower and left boundary is set as no slip, that is no velocity in the fluid. On the right boundary there is a do nothing, and zero pressure.

Quantities for comparison

The different techniques will be plotted with the minimal value of the Jacobian. The Jacobian determinant of the deformation rate, and **if the Jacobian is zero anywhere in the domain it means that the cells overlap and can cause singularity in the matrices during assembly.**

A plot of the deformation in the domain has been added, to visualize how the different mesh motion techniques work. It is possible to see that if get thin triangles in the computational domain then the lifting operator is not good enough, and we might get singularities in the computing matrix.

Different lifting operators with testcase CSM1

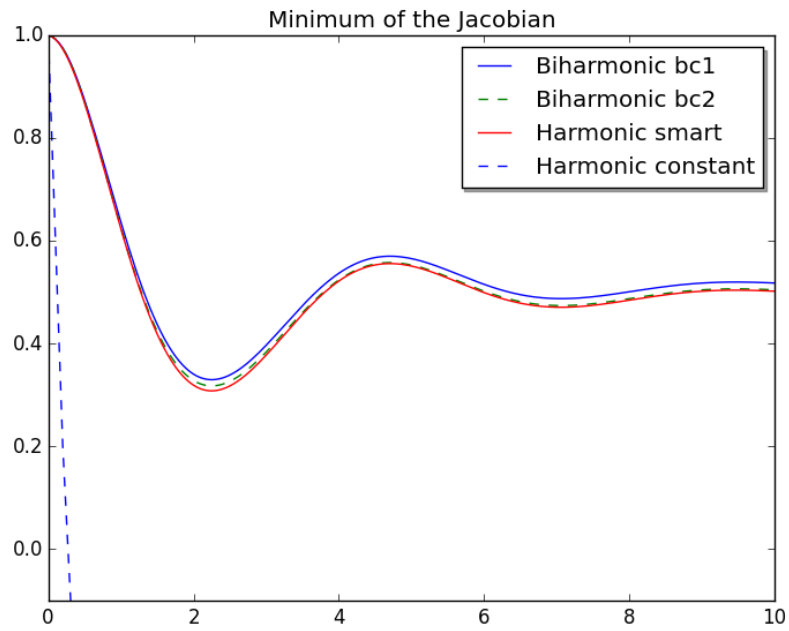


Figure 5.10: plot of the minimum of J in entire domain, using CSM1 test.
 $\Delta t = 0.05$

Figure 5.11: Results of testing different lifting operator using the CSM1 testcase computing full FSI

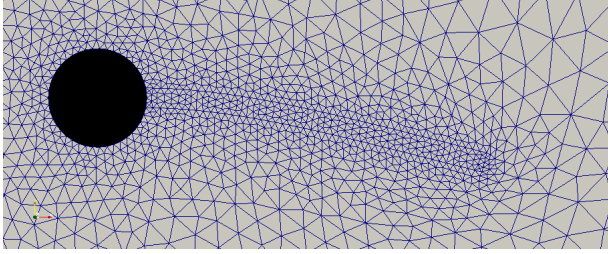


Figure 5.12: Harmonic smart

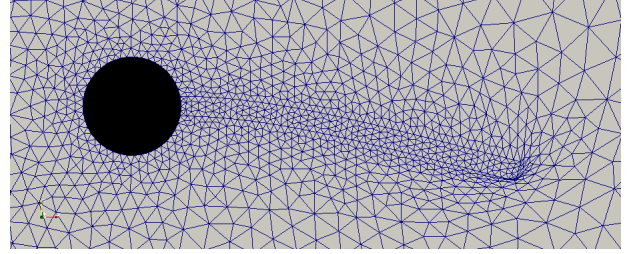


Figure 5.13: Harmonic constant

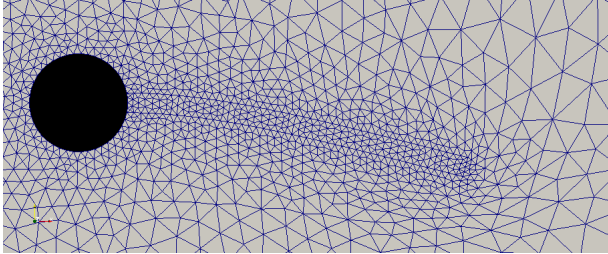


Figure 5.14: Biharmonic bc1

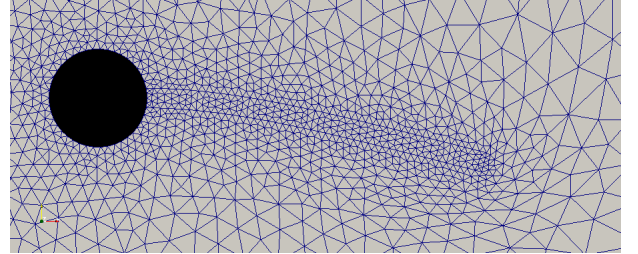


Figure 5.15: Biharmonic bc2

Technique	$d_y(A)[\times 10^{-3}]$	$d_x(A)[\times 10^{-3}]$
Harmonic	65.406	7.036
Constant	43.033	2.999
Bibc1	65.404	7.036
Bibc2	65.405	7.036

Table 5.15: Displacements results of different lifting operators of CSM1 test

Looking at figure 5.2.2 which shows the minimum of the Jacobian of the entire domain. The Harmonic with a constant α_u parameter. Gives overlapping cells quickly and computations stop. While the harmonic named smart meaning an α_u that is greater closer to the interface, and both the biharmonic techniques gives good and similar results. All three uphold, as we can see in 5.2.2, the integrity of the cells.

FSI2 with Lifting operator

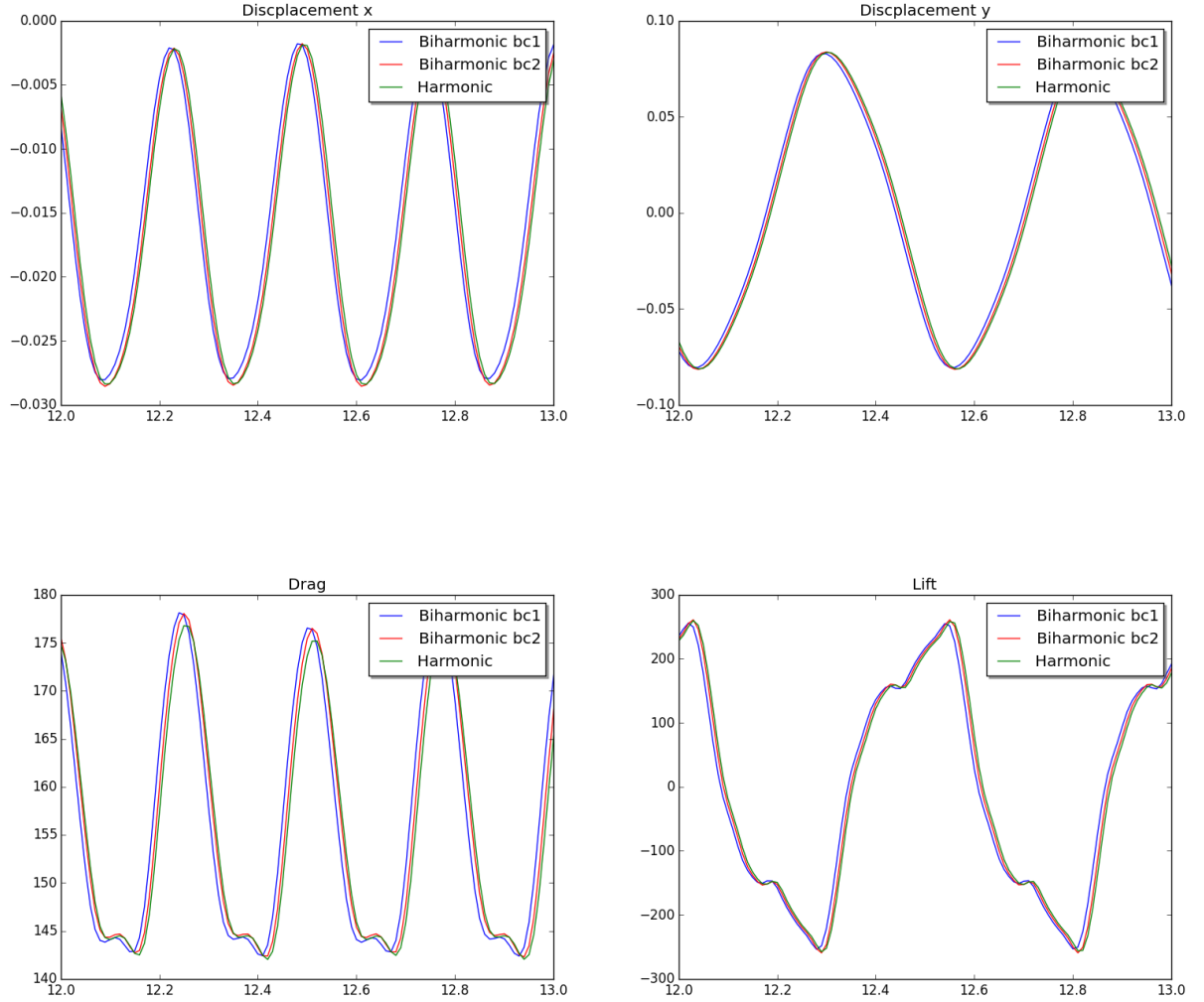


Figure 5.16: FSI2 with different lifting operators: Harmonic, Biharmonic bc1 and bc2. $\Delta t = 0.01$

Figure 5.2.2 shows the harmonic and the two biharmonic mesh motion techniques for the FSI2 case. All three are similar and only a slight change in the period can be noticed. This indicates that with a clever α_u the harmonic technique can be chosen. This is an advantage since the harmonic techniques

is the least computationally costly.

5.2.3 Temporal stability

The following section will give a brief insight in to the effects of choosing different θ values in the θ -scheme. The benchmark test FSI-2 has been used to since it is known to blow up with certain values of θ and Δt . Only the effects of Drag as been studied as the three other quantities shows the same behavior.

Figure 5.17: Drag for FSI2 with $\Delta t = 0.01$ with different values for θ

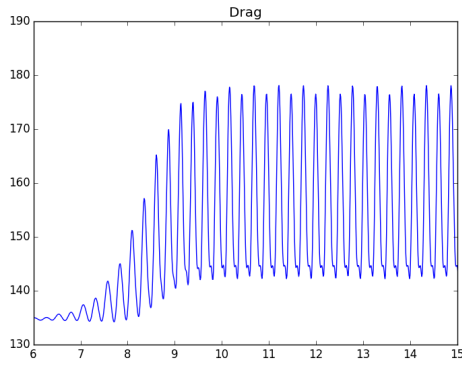


Figure 5.18: $\theta = 0.51$

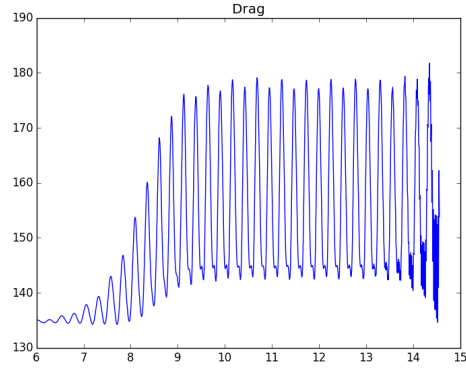


Figure 5.19: $\theta = 0.50$

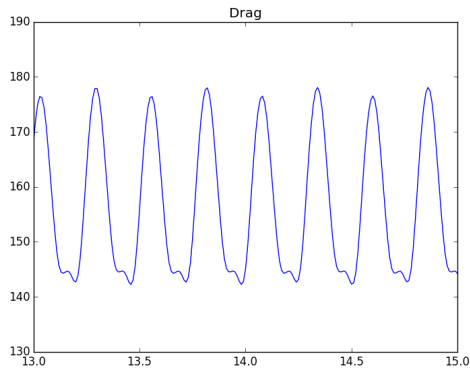


Figure 5.20: $\theta = 0.51$

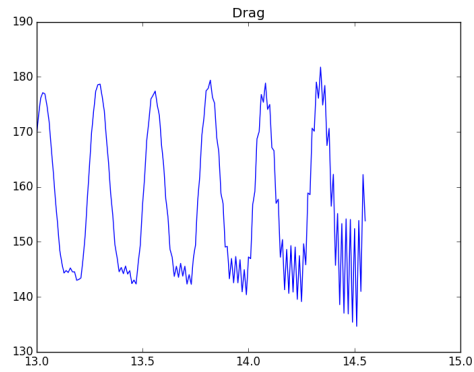


Figure 5.21: $\theta = 0.50$

5.2.3 show the plots of Drag with $\Delta t = 0.01$. It shows the instability when choosing $\theta = 0.5$. The Crank-Nicholson scheme is stable until about 13 seconds where we see that it blows up and the solver diverges. While the shifted Crank-Nicholson, $\theta = 0.5 + \Delta t$, is stable throughout the computing time.

It has been reported in [26] that the Crank-Nicholson, $\theta = 0.5$, scheme is stable throughout the computing time by setting $\Delta t = 0.001$.

Put in plot of $\theta = 0.5$ and $\Delta t = 0.001$ to see if its stable longer? ++++++

Chapter 6

CPU time improvement techniques

Modeling Fluid Structure Interaction accurately has been difficult not only because of the lack of stable schemes. It is also difficult because of the increase in runtime when modeling both fluid and structure, especially with monolithic schemes. When accuracy is needed in CFSI the number of cells can quickly become large. The strain on computers increases giving increased runtime.

To solve non-linear FSI problems I use in this these a Newton solver. In the Newton solver we have to assemble the Jacobian of the matrix, assemble the residual and solve for each iteration, until convergence is met. Working on this thesis the issue of runtime was certainly a problem, taking days and weeks for a simulation to finish.

I have in this chapter added the runtime improvements I have made to my newton solver. The point of this chapter is not to rigorously experiment with time improvement techniques. But merely to show what helped me make the simulation time bearable. And hopefully help others when employing a newton solver. The ideas covered are not new and has been implemented in Gabriel Balaban?s master thesis in 2012 [17].

6.1 Newton runtime profile

In table 6.1 I ran the FSI1 problem with no optimizations, checking the amount of time spent on: Jacobian Assembly, residual assembly and time to

solve.

Method	Runtime [s]	Runtime [%]	Calls
Assembly of Jacobian	60.7	94.4	5
Assembly or residual	0.6	0.9	5
Solve	2.8	4.4	5
Fulltime	64.3	100%	-

Table 6.1: Newton solver timed with no optimizations, $\Delta t = 0.5$

In table 6.1 it shows that the majority of time spent is in the assembly of the Jacobian. Therefore energy will be spent on reducing the time spent on assembling the Jacobian. In the next sections I will introduce two ways of improving time spent on assembling greatly. The first is reusing of the Jacobian and the second is employing a function in FENiCS called quadrature reduce. I will compare the timesaving techniques to table 6.1.

6.2 Reusing the Jacobian

The majority of the time spent on in the newton solver is assembling the Jacobian. In most cases a small $\delta t = 10^{-2}, 10^{-3}$ is employed, which in turn means that the Jacobian only differs by a small amount. The trick therefore is to reuse the Jacobian. This is done by telling the solver that we only assemble the Jacobian for a given number of iterations. The rest of the newton solver stays the same and keeps iterating until convergence, but the Jacobian matrix stays the same for a set number of iterations. When employed it was noticed that we needed a larger number of iterations to reach convergence, but the overall time of the Newton solver was much faster.

Method	Runtime [s]	Runtime [%]	Calls
Assembly of Jacobian	11.5 (-80%)	68.7	1 (-20%)
Assembly or residual	0.9 (+50%)	5.8	9 (+46%)
Solve	4.2 (+50%)	25.0	9 (+46%)
Fulltime	16.8 (-74 %)	-	-

Table 6.2: Parts of the Newtonsolver timed with reuse of the jacobian run with $\Delta t = 0.5$

In table 6.2 the same FSI1 test is done with $\Delta t = 0.5$. Even with a timestep that is fairly large, we get great improvements in runtime (-74%). The Jacobian assembles once meaning that at this timestep the Jacobian was calculated once and used again 9 times. What happens when we reuse the Jacobian, we have to iterate more times (9 times in this case) and as we can see the runtime in assembling the residual has increased by 50 %. This is a much less costly process, so even when iterating more times we get a reduce in runtime.

6.3 Quadrature reduce

Assembling of the Jacobian matrix with non-linear functions, requires a high number of quadrature points. When the accuracy of the Jacobian can be reduced, we can reduce the number of quadrature degree. This improves the runtime but reduces the accuracy leading to more iterations per time step. Reducing the quadrature degree can lead to blow up of the system in some cases. But is of great help in many other cases.

The FSI1 test is run with $\Delta t = 0.5$ like the section above.

Method	Runtime [s]	Runtime [%]	Calls
Assembly of Jacobian	4.9 (-92%)	60.3	5 (0%)
Assembly or residual	0.5 (-17%)	6.9	5 (0%)
Solve	2.6 (-7%)	31.9	5 (0%)
Fulltime	8.2 (-87%)	-	-

Table 6.3: Parts of the Newtonsolver timed with quadrature reduce run with $\Delta t = 0.5$

Reducing the quadrature degree as we can see gives a 92 % decrease in time spent assembling the Jacobian even with the same number of calls. The full time spent went down by 87 %.

6.4 Summary of runtime improvement techniques

Method	Runtime [s]	Runtime [%]	Calls
Assembly of Jacobian	1.2 (-98%)	18.1	1 (-20%)
Assembly or residual	0.9 (+50%)	14.7	9 (+46%)
Solve	4.4 (+57%)	66.2	9 (+46%)
Fulltime	6.6 (-89%)	-	-

Table 6.4: Newton solver timed with jacobian reuse and quadrature reduce run with $\Delta t = 0.5$

Finally I used both the reuse of Jacobian and the reduction of the quadrature. As we can see in table 6.4 the total runtime went down 89%. In my work on this thesis I used these two techniques as often as i could. Both techniques worked with great success in the FSI2 and FSI2 case.

Chapter 7

Insights, conclusions, and future work

7.1 Insights

When I started working on my thesis the plan was to build a Fluid-Structure Interaction solver meant to be applied to a hemodynamic problem. The plan was to first build a monolithic solver to get an introduction to FSI modeling. Then move on to a partitioned approach, specifically the scheme implemented by Fernandez 2013 [6]. Starting from scratch I thought building the monolithic solver would be easy, so I work on the monolithic scheme and the partitioned scheme in parallel, this process took a long time. I eventually figured out how difficult it is to build a FSI solver from scratch that can handle large deformations and high fluid velocity, and that building the monolithic solver and validating the solver in a rigorous manner would be enough. When starting from scratch I had to make all the mistakes needed to understand what it takes to build a solver that could handle large deformations, understand the importance good lifting operator has on FSI and the need for a long time stable scheme. An added difficulty is that the field of FSI is still very young so there are not many scientist researching the topic, meaning good information is hard to come by.

7.2 Conclusions

This thesis has developed a framework for solving Fluid-Structure Interaction problem using a monolithic scheme. I have introduced the mathematics and physics that governs fluids and structures, and the conditions needed to model FSI problems. I have discussed the challenges one meets when trying to model and compute FSI problems. I have introduced the θ -scheme with a shifted version of the Crank-Nicholson scheme, meant to uphold stability for long time FSI problems. The code has been verified in parts using MMS.

Using the proposal benchmark from Hron and Turek 2010, I have validated the FSI solver first in fluid and structure separately and finally for a full FSI problem. I have compared data to contributions made by others imposing similar schemes. I have shown how crucial the choice of lifting operator is for computing FSI problems with moderate to large deformations, and the need for a stable scheme for long run times.

7.3 Future work

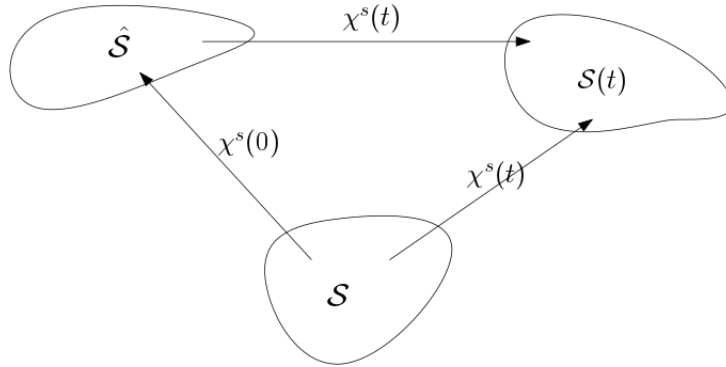
Computing FSI problems is difficult for a many reasons but the main reason FSI computing is not used in a commercial sense yet is the long run time for simulations. For future work the problem of long run time should be tackled by looking at partitioned schemes. When a FSI scheme is partitioned we can use existing solvers for the structure and fluid parts, increasing run times significantly. The difficulty is still to transfer the energies between the fluid and structure when each problem is solved separately.

Appendices

Appendix A

Appendix

A.1 Lagrangian description of solid mechanics



Let $\hat{\mathcal{S}}$, \mathcal{S} , $\mathcal{S}(t)$ be the initial stress free configuration of a given body, the reference and the current configuration respectively. I define a smooth mapping from the reference configuration to the current configuration:

$$\chi^s(\mathbf{X}, t) : \hat{\mathcal{S}} \rightarrow \mathcal{S}(t) \quad (\text{A.1})$$

Where \mathbf{X} denotes a material point in the reference domain and χ^s denotes the mapping from the reference configuration to the current configuration. Let $d^s(\mathbf{X}, t)$ denote the displacement field which describes deformation on a body. The mapping χ^s can then be specified from a current position plus the displacement from that position:

$$\chi^s(\mathbf{X}, t) = \mathbf{X} + d^s(\mathbf{X}, t) \quad (\text{A.2})$$

which can be written in terms of the displacement field:

$$d^s(\mathbf{X}, t) = \chi^s(\mathbf{X}, t) - \mathbf{X} \quad (\text{A.3})$$

Let $w(\mathbf{X}, t)$ be the domain velocity which is the partial time derivative of the displacement:

$$w(\mathbf{X}, t) = \frac{\partial \chi^s(\mathbf{X}, t)}{\partial t} \quad (\text{A.4})$$

A.1.1 Deformation gradient

The deformation gradient describes the rate at which a body undergoes deformation. Let $d(\mathbf{X}, t)$ be a differentiable deformation field in a given body, the deformation gradient is then:

$$F = \frac{\partial \chi^s(\mathbf{X}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{X} + d^s(\mathbf{X}, t)}{\partial \mathbf{X}} = I + \nabla d(\mathbf{X}, t) \quad (\text{A.5})$$

which denotes relative change of position under deformation in a Lagrangian frame of reference. We can observe that when there is no deformation. The deformation gradient F is simply the identity matrix.

Let the Jacobian determinant, which is the determinant of the of the deformation gradient F , be defined as:

$$J = \det(F) \quad (\text{A.6})$$

The Jacobian determinant is used to change between volumes, assuming infinitesimal line and area elements in the current ds, dx and reference dV, dX configurations. The Jacobian determinant is therefore known as a volume ratio.

A.1.2 Strain

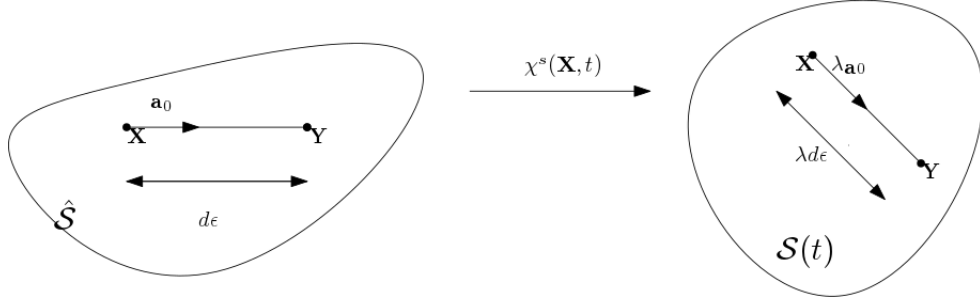


Figure A.1: Deformation of a line element with length $d\epsilon$ into a line element with length $\lambda d\epsilon$

Strain is the relative change of location between two particles. Strain, strain rate and deformation is used to describe the relative motion of particles in a continuum. This is the fundamental quality that causes stress [15].

Observing two neighboring points \mathbf{X} and \mathbf{Y} . Let \mathbf{Y} be described by adding and subtracting the point \mathbf{X} and rewriting \mathbf{Y} from the point \mathbf{X} plus a distance $d\mathbf{X}$:

$$\mathbf{Y} = \mathbf{Y} + \mathbf{X} - \mathbf{X} = \mathbf{X} + |\mathbf{Y} - \mathbf{X}| \frac{\mathbf{Y} - \mathbf{X}}{|\mathbf{Y} - \mathbf{X}|} = \mathbf{X} + d\mathbf{X} \quad (\text{A.7})$$

Let $d\mathbf{X}$ be denoted by:

$$d\mathbf{X} = d\epsilon \mathbf{a}_0 \quad (\text{A.8})$$

$$d\epsilon = |\mathbf{Y} - \mathbf{X}| \quad (\text{A.9})$$

$$\mathbf{a}_0 = \frac{\mathbf{Y} - \mathbf{X}}{|\mathbf{Y} - \mathbf{X}|} \quad (\text{A.10})$$

where $d\epsilon$ is the distance between the two points and \mathbf{a}_0 is a unit vector

We see now that $d\mathbf{X}$ is the distance between the two points times the unit vector or direction from \mathbf{X} to \mathbf{Y} .

A certain motion transform the points \mathbf{Y} and \mathbf{X} into the displaced positions $\mathbf{x} = \chi^s(\mathbf{X}, t)$ and $\mathbf{y} = \chi^s(\mathbf{Y}, t)$. Using Taylor's expansion \mathbf{y} can be expressed in terms of the deformation gradient:

$$\mathbf{y} = \chi^s(\mathbf{Y}, t) = \chi^s(\mathbf{X} + d\epsilon \mathbf{a}_0, t) \quad (\text{A.11})$$

$$= \chi^s(\mathbf{X}, t) + d\epsilon F \mathbf{a}_0 + \mathcal{O}(\mathbf{Y} - \mathbf{X}) \quad (\text{A.12})$$

where $\mathcal{O}(\mathbf{Y} - \mathbf{X})$ refers to the small error that tends to zero faster than $(\mathbf{X} - \mathbf{Y}) \rightarrow \mathcal{O}$.

Setting $\mathbf{x} = \chi^s(\mathbf{X}, t)$ It follows that:

$$\mathbf{y} - \mathbf{x} = d\epsilon F \mathbf{a}_0 + \mathcal{O}(\mathbf{Y} - \mathbf{X}) \quad (\text{A.13})$$

$$= F(\mathbf{Y} - \mathbf{X}) + \mathcal{O}(\mathbf{Y} - \mathbf{X}) \quad (\text{A.14})$$

Let the **stretch vector** be $\lambda_{\mathbf{a}_0}$, which goes in the direction of \mathbf{a}_0 :

$$\lambda_{\mathbf{a}_0}(\mathbf{X}, t) = F(\mathbf{X}, t) \mathbf{a}_0 \quad (\text{A.15})$$

Looking at the square of λ :

$$\lambda^2 = \lambda_{\mathbf{a}_0} \lambda_{\mathbf{a}_0} = F(\mathbf{X}, t) \mathbf{a}_0 F(\mathbf{X}, t) \mathbf{a}_0 \quad (\text{A.16})$$

$$= \mathbf{a}_0 F^T F \mathbf{a}_0 = \mathbf{a}_0 C \mathbf{a}_0 \quad (\text{A.17})$$

We have not introduced the important right Cauchy-Green tensor:

$$C = F^T F \quad (\text{A.18})$$

Since \mathbf{a}_0 is just a unit vector, we see that C measures the squared length of change under deformation. We see that in order to determine the stretch one needs only the direction of \mathbf{a}_0 and the tensor C. C is also symmetric and positive definite $C = C^T$. I also introduce the Green-Lagrangian strain tensor E:

$$E = \frac{1}{2}(F^T F - I) \quad (\text{A.19})$$

which is also symmetric since C and I are symmetric. The Green-Lagrangian strain tensor E has the advantage of having no contributions when there is no deformations. Where the Cauchy-Green tensor gives the identity matrix for zero deformation.

A.1.3 Stress

While strain, deformation and strain rate only describe the relative motion of particles in a given volume, stress give us the internal forces between neighboring particles. Stress is responsible for deformation and is therefore crucial in continuum mechanics. The unit of stress is force per area.

that completely define the state of stress at a point inside a material in the deformed state, placement, or configuration. The tensor relates a unit-length direction vector \mathbf{n} to the stress vector $\mathbf{T}(\mathbf{n})$ across an imaginary surface perpendicular to \mathbf{n} :

Introducing the Cauchy stress tensor σ_s , which define the state of stress inside a material. The version of Cauchy stress tensor is defined by the material model used. If we use this tensor on an area, taking the stress tensor times the normal vector $\sigma_s \mathbf{n}$ we get the forces acting on that area.

Stress tensor defined from the Cauchy by the constitutive law of St. Venant-Kirchhoff hyperelastic material model:

$$\sigma_s = \frac{1}{J} F (\lambda_s (tr E) I + 2\mu_s E) F^T \quad (\text{A.20})$$

Using the deformation gradient and the Jacobian determinant., I get the first Piola-Kirchhoff stress tensor \mathbf{P} :

$$\mathbf{P} = J \sigma F^{-T} \quad (\text{A.21})$$

This is known as the *Piola Transformation* and maps the tensor into a Lagrangian formulation which will be used when stating the solid equation.

Introducing the second Piola-Kirchhoff stress tensor \mathbf{S} :

$$\mathbf{S} = J F^{-1} \sigma F^{-T} = F^{-1} \mathbf{P} = \mathbf{S}^T \quad (\text{A.22})$$

from this relation the first Piola-Kirchhoff tensor can be expressed by the second:

$$\mathbf{P} = \mathbf{F} \mathbf{S} \quad (\text{A.23})$$

A.2 Results

Table 3 Results for unsteady benchmark FSI3.

	Unknowns	Δt	$u_1(A) [\times 10^{-3}]$	$u_2(A) [\times 10^{-3}]$	F_D	F_L
1	61318	1.0e-3	-2.54 ± 2.41	1.45 ± 32.80	450.3 ± 23.51	-0.10 ± 143.0
	237286	2.0e-3	-2.88 ± 2.73	1.53 ± 34.94	458.6 ± 27.18	2.08 ± 153.1
	237286	1.0e-3	-2.87 ± 2.73	1.54 ± 34.94	458.6 ± 27.31	2.00 ± 153.3
	237286	5.0e-4	-2.86 ± 2.72	1.53 ± 34.90	458.6 ± 27.27	2.01 ± 153.4
	941158	1.0e-3	-2.91 ± 2.77	1.47 ± 35.26	459.9 ± 27.92	1.84 ± 157.7
2a	11250	5.0e-3	-2.48 ± 2.24	1.27 ± 36.50	–	–
2b	7176	5.0e-3	-2.44 ± 2.32	1.02 ± 31.82	473.5 ± 56.97	8.08 ± 283.8
	7176	2.0e-3	-2.48 ± 2.39	0.92 ± 32.81	471.3 ± 62.28	6.11 ± 298.6
	7176	1.0e-3	-2.58 ± 2.49	0.94 ± 33.19	470.4 ± 64.02	4.65 ± 300.3
	27744	5.0e-3	-2.43 ± 2.27	1.41 ± 31.73	483.7 ± 22.31	2.21 ± 149.0
	27744	2.0e-3	-2.63 ± 2.61	1.46 ± 33.46	483.3 ± 24.48	2.08 ± 161.2
	27744	1.0e-3	-2.80 ± 2.64	1.45 ± 34.12	483.0 ± 25.67	2.21 ± 165.3
	42024	2.5e-3	-2.40 ± 2.26	1.39 ± 31.71	448.7 ± 21.16	1.84 ± 141.3
	42024	1.0e-3	-2.53 ± 2.38	1.40 ± 32.49	449.7 ± 22.24	1.61 ± 142.8
	42024	5.0e-4	-2.57 ± 2.42	1.42 ± 32.81	450.1 ± 22.49	1.49 ± 143.7
	72696	2.5e-3	-2.64 ± 2.48	1.38 ± 33.25	451.1 ± 24.57	2.04 ± 150.6
	72696	1.0e-3	-2.79 ± 2.62	1.28 ± 34.61	452.0 ± 25.78	1.91 ± 152.7
	72696	5.0e-4	-2.84 ± 2.67	1.28 ± 34.61	452.4 ± 26.19	2.36 ± 152.7
3	19488	1.0e-3	-3.02 ± 2.83	1.41 ± 35.47	458.2 ± 28.32	2.41 ± 145.6
	19488	5.0e-4	-3.02 ± 2.85	1.42 ± 35.63	458.7 ± 28.78	2.23 ± 146.0
	19488	2.5e-4	-3.02 ± 2.85	1.32 ± 35.73	458.7 ± 28.80	2.23 ± 146.0
	76672	1.0e-3	-2.78 ± 2.62	1.44 ± 34.36	459.1 ± 26.63	2.41 ± 151.3
	76672	5.0e-4	-2.78 ± 2.62	1.44 ± 34.35	459.1 ± 26.62	2.39 ± 150.7
	76672	2.5e-4	-2.77 ± 2.61	1.43 ± 34.43	459.1 ± 26.50	2.36 ± 149.9
	304128	1.0e-3	-2.86 ± 2.70	1.45 ± 34.93	460.2 ± 27.65	2.47 ± 154.9
	304128	5.0e-4	-2.86 ± 2.70	1.45 ± 34.90	460.2 ± 27.47	2.37 ± 153.8
	304128	2.5e-4	-2.88 ± 2.72	1.47 ± 34.99	460.5 ± 27.74	2.50 ± 153.9
4	81120	9.0e-5	-5.18 ± 5.04	1.12 ± 45.10	477.0 ± 48.00	7.00 ± 223.0
	324480	2.0e-5	-4.54 ± 4.34	1.50 ± 42.50	467.5 ± 39.50	16.20 ± 188.7
5	2480814	5.1e-5	-2.88 ± 2.71	1.48 ± 35.10	463.0 ± 31.30	1.81 ± 154.0
6	7059	5.0e-4	-1.60 ± 1.60	1.50 ± 25.90	525.0 ± 22.50	-0.55 ± 106.0
	27147	5.0e-4	-2.00 ± 1.89	1.45 ± 29.00	434.0 ± 17.50	2.53 ± 88.6
7	271740	5.0e-4	-3.04 ± 2.87	1.55 ± 36.63	474.9 ± 28.12	3.86 ± 165.9

Figure A.2: Results from different contributions in from the paper Turek et.al 2010 [20]

[20]

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