UiO : Department of Mathematics
University of Oslo

FSI solver

Sebastian Gjertsen

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

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Chapter 1

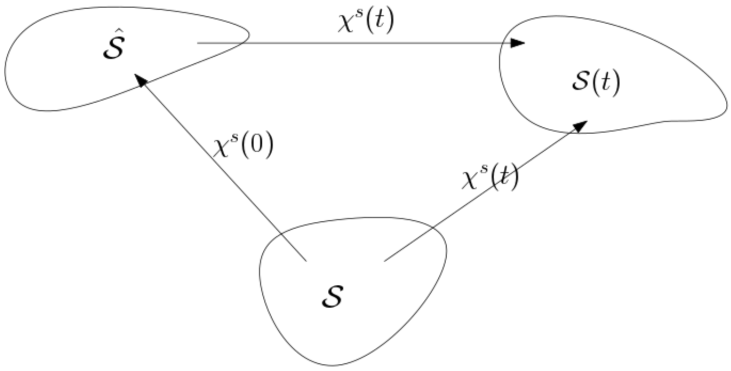
Continuum mechanics

All materials are made up of atoms, and between atoms there is space. The laws that govern these atoms are complex and are very difficult to model. However materials like solids and fluids can be modeled if we assume them to exist as a continuum. This means that we assume that there exist no space inside the materials and they fill completely up the space they occupy. We can use mathematics with basic physical laws to model fluids and solids when they are assumed continuous. These laws are generally expressed in two frames of reference, Lagrangian and Eulerian. To exemplify these frameworks we can imagine a river running down a mountain. In the Eulerian framework we are the observer standing still besides the river looking at the flow. We are not interested in each fluid particle but only how the fluid acts as a whole flowing down the river. This approach fits the fluid problem as we can imagine the fluid continuously deforming along the river side.

In the Lagrangian description we have to imagine ourselves on a leaf going down the river with the flow. Looking out as the mountain moves and we stand still compared to the fluid particles. This description fits a solid problem nicely since we are generally interested in where the solid particles are in relation to each other. Modeling for instance a beam attached to a wall at one end and a weight at the other end. We can imagine the beam bending and to model this bending we need to where all the particles are compared to each other. The more the particles move in relation to each other the more stress there is in the beam.

In this chapter I will introduce both of these frameworks and the equations which are needed to model Fluid and Structure separately. I start with the Lagrangian description, by providing a short introduction to Lagrangian physics for the sake of completeness. Then introduce the solid and fluid equations. For a more detailed look on Lagrangian physics and the solid equation see [**?**].

1.1 Lagrangian physics



Let $\hat{\mathcal{S}}$, $\mathcal{S}$, $\mathcal{S}(t)$ be the initial stress free configuration of a given body, the reference and as the current configuration respectively. I define a smooth mapping from the reference configuration to the current configuration:

$$\chi^s(\mathbf{X}, t) : \hat{\mathcal{S}} \to \mathcal{S}(t) \tag{1.1}$$

Where $\mathbf{X}$ denotes a material point in the reference domain and χ^s denotes the mapping from the reference configuration to the current configuration. Let $d^s(\mathbf{X}, t)$ denote the displacement field which describes deformation on a body. The mapping χ^s can then be specified from its current position plus the displacement from that position:

$$\chi^s(\mathbf{X}, t) = \mathbf{X} + d^s(\mathbf{X}, t) \tag{1.2}$$

which can be written in terms of the displacement field:

$$d^s(\mathbf{X}, t) = \chi^s(\mathbf{X}, t) - \mathbf{X} \tag{1.3}$$

Let w($\mathbf{X}$,t) be the domain velocity which is the partial time derivative of the displacement:

$$w(\mathbf{X}, t) = \frac{\partial \chi^s(\mathbf{X}, t)}{\partial t} \tag{1.4}$$

1.1.1 Deformation gradient

To describe the rate at which a body undergoes deformation I will need to define a deformation gradient. Let $d(\mathbf{X}, t)$ be a differentiable deformation field in a given body, the deformation gradient is then:

$$F = \frac{\partial \chi^s(\mathbf{x}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{X} + d^s(\mathbf{X}, t)}{\partial \mathbf{X}} = I + \nabla d(\mathbf{X}, t) \tag{1.5}$$

which denotes relative change of position under deformation in a Lagrangian frame of reference. We can observe that when there is no deformation. The deformation gradient F is simply the identity matrix.

We also need a way to change between volumes, from the reference ($\int dv$) to current ($\int dV$) configuration. This is defined with the Jacobian, which is the determinant of the of the deformation gradient F:

$$J = \det(F) \tag{1.6}$$

The Jacobian is used to change between volumes, assuming infinitesimal line and area elements in the current ds, dx and reference dV, dX configurations.

1.1.2 Strain

The relative change of location between two particles is called strain. Strain, strain rate and deformation is used to describe the relative motion of particles in a continuum. This is the fundamental quality that causes stress [5].

If we observe two neighboring points $\mathbf{X}$ and $\mathbf{Y}$. I can describe $\mathbf{Y}$ with:

$$\mathbf{Y} = \mathbf{Y} + \mathbf{X} - \mathbf{X} = \mathbf{X} + |\mathbf{Y} - \mathbf{X}| \frac{\mathbf{Y} - \mathbf{X}}{|\mathbf{Y} - \mathbf{X}|} = \mathbf{X} + d\mathbf{X} \tag{1.7}$$

Let $d\mathbf{X}$ be denoted by:

$$d\mathbf{X} = d\epsilon \mathbf{a}_0 \tag{1.8}$$

$$d\epsilon = |\mathbf{Y} - \mathbf{X}| \tag{1.9}$$

$$\mathbf{a}_0 = \frac{\mathbf{Y} - \mathbf{X}}{|\mathbf{Y} - \mathbf{X}|} \tag{1.10}$$

where $d\epsilon$ is the distance between the two points and $\mathbf{a}_0$ is a unit vector

We see now that $d\mathbf{X}$ is the distance between the two points times the unit vector or direction from $\mathbf{X}$ to $\mathbf{Y}$.

A certain motion transform the points $\mathbf{Y}$ and $\mathbf{X}$ into the displaced positions $\mathbf{x} = \chi^s(\mathbf{X}, t)$ and $\mathbf{y} = \chi^s(\mathbf{Y}, t)$. Using Taylor?s expansion $\mathbf{y}$ can be expressed in terms of the deformation gradient:

$$\mathbf{y} = \chi^s(\mathbf{Y}, t) = \chi^s(\mathbf{X} + d\epsilon\mathbf{a}_0, t) \tag{1.11}$$

$$= \chi^s(\mathbf{X}, t) + d\epsilon F\mathbf{a}_0 + \mathcal{O}(\mathbf{Y} - \mathbf{X}) \tag{1.12}$$

where $\mathcal{O}(\mathbf{Y} - \mathbf{X})$ refers to the small error that tends to zero faster than $(\mathbf{X} - \mathbf{Y}) \rightarrow \mathcal{O}$.

If I set $\mathbf{x} = \chi^s(\mathbf{X}, t)$ It follows that:

$$\mathbf{y} - \mathbf{x} = d\epsilon F\mathbf{a}_0 + \mathcal{O}(\mathbf{Y} - \mathbf{X}) \tag{1.13}$$

$$= F(\mathbf{Y} - \mathbf{X}) + \mathcal{O}(\mathbf{Y} - \mathbf{X}) \tag{1.14}$$

Let the **stretch vector** be $\lambda_{\mathbf{a}0}$, which goes in the direction of $\mathbf{a}_0$:

$$\lambda_{\mathbf{a}0}(\mathbf{X}, t) = F(\mathbf{X}, t)\mathbf{a}_0 \tag{1.15}$$

If we look at the square of λ:

$$\lambda^2 = \lambda_{\mathbf{a}0}\lambda_{\mathbf{a}0} = F(\mathbf{X}, t)\mathbf{a}_0 F(\mathbf{X}, t)\mathbf{a}_0 \tag{1.16}$$

$$= \mathbf{a}_0 F^T F\mathbf{a}_0 = \mathbf{a}_0 C\mathbf{a}_0 \tag{1.17}$$

We have not introduced the important right Cauchy-Green tensor:

$$C = F^T F \tag{1.18}$$

Since $\mathbf{a}_0$ is just a unit vector, we see that this measures the squared length of change under deformation. We see that in order to determine the stretch one needs only the direction of $\mathbf{a}_0$ and the tensor C. C is also symmetric and positive definite $C = C^T$. I also introduce the Green-Lagrangian strain tensor E:

$$E = \frac{1}{2}(F^T F - I) \tag{1.19}$$

which is also symmetric since C and I are symmetric.

1.1.3 Stress

While strain, deformation and strain rate only describe the relative motion of particles in a given volume. Stress give us the internal forces between neighboring particles. Stress is responsible for deformation and is therefore crucial in continuum mechanics. The unit of stress is force per area.

I introduce the Cauchy stress tensor:

$$\sigma_s = \frac{1}{J} F(\lambda_s (trE) I + 2\mu_s E) F^T \tag{1.20}$$

If we use this tensor on an area that is taking the stress tensor times the normal vector $\sigma_s \mathbf{n}$ we get the forces acting on that area.

Using the deformation gradient and the Jacobian, I get the first Piola-Kirchhoff stress tensor P:

$$P = J\sigma F^{-T} \tag{1.21}$$

This is known as the *Piola Transformation* and maps the tensor into a Lagrangian formulation which will be used when stating the solid equation.

I also introduce the second Piola-Kirchhoff stress tensor S:

$$S = JF^{-1}\sigma F^{-T} = F^{-1}P = S^T \tag{1.22}$$

from this relation the first Piola-Kirchhoff tensor can be expressed by the second:

$$P = FS \tag{1.23}$$

1.2 Solid equation

The solid equation describes the motion of a solid. It is derived from the principles of conservation of mass and momentum. Stated in the Lagrangian reference system [5]:

$$\rho_s \frac{\partial d^2}{\partial t^2} = \nabla \cdot (P) + \rho_s f \tag{1.24}$$

written in terms of the deformation d, where I used the first Piola-Kirchhoff stress tensor. f is a force acting on the solid body. Considering the material law from the previous section. The stresses depend on the displacements d which again depends on the strain. The solid equation will be finalized by stating the different boundary conditions needed to solve a solid problem.

1.2.1 Solid Boundary Conditions

The solid moves within the boundary of $\partial\mathcal{S}$. On the Dirichlet boundary $\partial\mathcal{S}_D$ we impose a given value. This can be initial conditions or set to zero as on walls with "no slip" condition. These conditions are defined for d and w:

$$d = d_0 \text{ on } \partial\mathcal{S}_D \tag{1.25}$$

$$w(\mathbf{X}, t)_0 = \frac{\partial d(t=0)}{\partial t} \tag{1.26}$$

The forces on the boundaries need to equal an eventual external force $\mathbf{f}$. These are are enforced on the Neumann boundaries:

$$P \cdot \mathbf{n} = f \text{ on } \partial\mathcal{S}_N$$

1.3 Fluid equations

The fluid equation will be stated in an Eulerian framework. In this framework the domain has fixed points where the fluid passes through. The Navier-Stokes equations are like the solid equation derived using principles of mass and momentum conservation. These equations describes the velocity and pressure in a given fluid continuum. They are here written in the fluid time domain $\mathcal{F}$ as an incompressible fluid:

$$\rho_f\big(\frac{\partial u}{\partial t} + u \cdot \nabla u\big) = \nabla \cdot \sigma_f + f \tag{1.27}$$

$$\nabla \cdot u = 0 \tag{1.28}$$

where u is the fluid velocity, p is the fluid pressure, ρ stands for constant density, f is body force and $\sigma_f = \mu_f(\nabla u + \nabla u^T) - pI$
There does not yet exist an analytical solutions to the N-S equations, only simplified problems can be solved [**?**]. But this does not stop us from discretizing and solving them numerically.
Before these equations can be solved we need to impose boundary conditions.

1.4 Boundary conditions

Lastly I need to impose boundary conditions for both the solid and fluid equation. The fluid flows within the boundary noted as $\partial\mathcal{F}$ and the solid moves within $\partial\mathcal{S}$. The place in which these two meet will be covered in the next chapter. On the Dirichlet boundary $\partial\mathcal{F}_D$ we impose a given value. This can be initial conditions or set to zero as on walls with "no slip" condition. These conditions are defined for u p and d:

$$u = u_0 \text{ on } \partial\mathcal{F}_D$$

$$p = p_0 \text{ on } \partial\mathcal{F}_D$$

$$d = d_0 \text{ on } \partial\mathcal{S}_D$$

The forces on the boundaries need to equal an eventual external force $\mathbf{f}$. These are are enforced on the Neumann boundaries:

$$\sigma \cdot \mathbf{n} = f \text{ on } \partial\mathcal{F}_N$$

$$P \cdot \mathbf{n} = f \text{ on } \partial\mathcal{S}_N$$

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