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FSI solver

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

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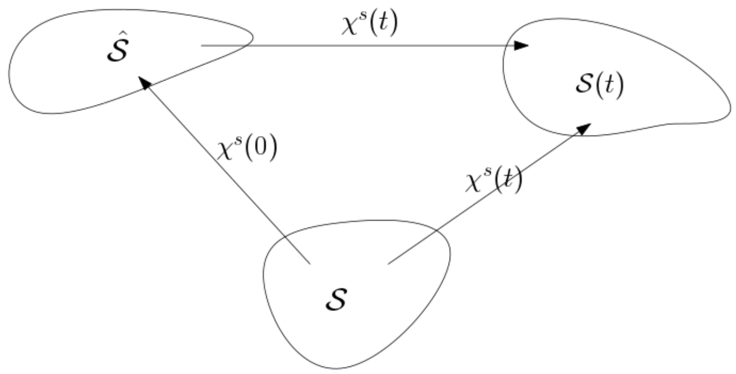
Chapter 1

Continuum mechanics

Materials like fluid and solids are made up of atoms, and between atoms there is space. However solids and fluids can be modeled if we assume them to exist as a continuum. This means that there exist no space inside the materials and they fill completely up the space they occupy. Using mathematics we can model fluids and solids using basic physical laws. These laws are generally defined in two frameworks, Lagrangian and Eulerian. A way of thinking about these frameworks is if we imagine a river running down a mountain. In the Eulerian framework we are the observer standing still besides the river looking at the flow. We are not interested in each fluid particle but only how the fluid acts as a whole flowing down the river. This approach fits the fluid problem nicely as we can imagine the fluid continuously deforming along the river side.
In the Lagrangian description we have to imagine ourselves on a leaf going down the river with the flow. Looking out as the mountain moves and we stand still compared to the fluid particles. This description fits a solid problem nicely since we are generally interested in where the solid particles are in relation to each other. In this chapter I will introduce both of these frameworks and the equations which are needed to model Fluid Structure Interaction. I start with the Lagrangian description and the solid equation.

1.1 Lagrangian physics

I will start by providing a short introduction to Lagrangian physics for the sake of completeness. For a more detailed look see [3].



I define $\hat{\mathcal{S}}$ as the initial stress free configuration of a given body, $\mathcal{S}$ as the reference and $\mathcal{S}(t)$ as the current configuration respectively. I need to define a smooth mapping from the reference configuration to the current configuration:

$$\chi^s(\mathbf{X}, t) : \hat{\mathcal{S}} \rightarrow \mathcal{S}(t) \tag{1.1}$$

Where $\mathbf{X}$ denotes a material point in the reference domain and χ^s denotes the mapping from the reference configuration to the current configuration. Let $d^s(\mathbf{X}, t)$ denote the displacement field which describes deformation on a body. I then set the mapping χ^s to be specified from the its current position plus the displacement from that position:

$$\chi^s(\mathbf{X}, t) = \mathbf{X} + d^s(\mathbf{X}, t) \tag{1.2}$$

which can be written in terms of the displacement field:

$$d^s(\mathbf{X}, t) = \chi^s(\mathbf{X}, t) - \mathbf{X} \tag{1.3}$$

I can then define w($\mathbf{X}$,t) as the domain velocity which is the partial time derivative:

$$w(\mathbf{X}, t) = \frac{\partial \chi^s(\mathbf{X}, t)}{\partial t} \tag{1.4}$$

1.1.1 Deformation gradient

To describe the rate at which a body undergoes deformation I will need to define a deformation gradient. If $d(\mathbf{X}, t)$ is a differentiable deformation field in a given body. I define the deformation gradient as:

$$F = \frac{\partial \chi^s(\mathbf{x}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{X} + d^s(\mathbf{X}, t)}{\partial \mathbf{X}} = I + \nabla d(\mathbf{X}, t) \tag{1.5}$$

which denotes relative change of position under deformation in a Lagrangian frame of reference. We see that when there is no motion, hence no deformation. The deformation gradient F is simply the identity matrix.
We also need a way to change between volumes, from the reference (dv) to current (dV) configuration. This is defined with the Jacobian, which is the determinant of the of the deformation gradient F:

$$J = \det(F) \tag{1.6}$$

The Jacobian is used to change between volumes ,if I assume infinitesimal line and area elements in the current ds, dx and reference dV, dX configurations. The volume elements dv, dV can be expressed by the dot product:

$$dv = ds \cdot dx = JdSdX \tag{1.7}$$

This is used to get the Nansons formula:

$$ds = JF^{-T}dS \tag{1.8}$$

which holds for an arbitrary line element in different configurations. This will be useful later on when describing equations in different configurations.

1.1.2 Strain

The relative change of location between two particles is called strain. This is the fundamental quality the causes stress. [9]. If we look at two neighboring points $\mathbf{X}$ and $\mathbf{Y}$. I can describe $\mathbf{Y}$ with:

$$\mathbf{Y} = \mathbf{Y} + \mathbf{X} - \mathbf{X} = \mathbf{X} + |\mathbf{Y} - \mathbf{X}|\frac{\mathbf{Y} - \mathbf{X}}{|\mathbf{Y} - \mathbf{X}|} = \mathbf{X} + d\mathbf{X} \tag{1.9}$$

I write $d\mathbf{X} = d\epsilon\mathbf{a}_0$, where $d\epsilon = |\mathbf{Y} - \mathbf{X}|$ is the distance between the two points and $\mathbf{a}_0 = \frac{\mathbf{Y}-\mathbf{X}}{|\mathbf{Y}-\mathbf{X}|}$ A certain motion transform the points $\mathbf{Y}$ and $\mathbf{X}$ into the displaced positions $\mathbf{x} = \chi^s(\mathbf{X}, t)$ and $\mathbf{y} = \chi^s(\mathbf{Y}, t)$. Using Taylor?s expansion $\mathbf{y}$ can be expressed in terms of deformation gradient:

$$\begin{aligned} \mathbf{y} =& \chi^s(\mathbf{Y}, t) = \chi^s(\mathbf{X} + d\epsilon\mathbf{a}_0, t) && (1.10) \\ =& \chi^s(\mathbf{X}, t) + d\epsilon F\mathbf{a}_0 + o(\mathbf{Y} - \mathbf{X}) && (1.11) \\ & && (1.12) \end{aligned}$$

where $o(\mathbf{Y}-\mathbf{X})$ refers to the small error that tends to zero faster than $\mathbf{X}-\mathbf{Y}$
Next I define the **stretch vector** $\lambda_{\mathbf{a}0}$:

$$\lambda_{\mathbf{a}0}(\mathbf{X}, t) = F(\mathbf{X}, t)\mathbf{a}_0 \tag{1.13}$$

If we look at the square of λ

$$\lambda^2 = \lambda_{\mathbf{a}0}\lambda_{\mathbf{a}0} = F(\mathbf{X},t)\mathbf{a}_0 F(\mathbf{X},t)\mathbf{a}_0 \tag{1.14}$$
$$= \mathbf{a}_0 F^T F \mathbf{a}_0 = \mathbf{a}_0 C \mathbf{a}_0 \tag{1.15}$$

We have not introduced the important right Cauchy-Green tensor:

$$C = F^T F \tag{1.16}$$

which is symmetric and positive definite $C = C^T$. I also introduce the Green-Lagrangian strain tensor E:

$$E = \frac{1}{2}(F^T F - I) \tag{1.17}$$

which is also symmetric since C and I are symmetric. This measures the squared length change under deformation.

1.1.3 Stress

Stress is the internal forces between neighboring particles. I introduce the Cauchy stress tensor:

$$\sigma_s = \frac{1}{J} F(\lambda_s(trE)I + 2\mu_s E)F^T \tag{1.18}$$

Using (1.8) I get the first Piola-Kirchhoff stress tensor P:

$$P = J\sigma F^{-T} \tag{1.19}$$

I also introduce the second Piola-Kirchhoff stress tensor S:

$$S = JF^{-1}\sigma F^{-T} = F^{-1}P = S^T \tag{1.20}$$

from this relation I can write the first Piola-Kirchhoff tensor by the second:

$$P = FS \tag{1.21}$$

1.2 Solid equation

From the principles of conservation of mass and momentum, I get the solid equation stated in the Lagrangian reference system (Following the notation from [9]):

$$\rho_s \frac{\partial d^2}{\partial t^2} = \nabla \cdot (P) + \rho_s f \tag{1.22}$$

where I used the first Piola-Kirchhoff stress tensor.

Locking

The problem og shear locking can happen FEM computations with certain elements. [mek4250 Kent] - Locking occurs if $\lambda >> \nu$ that is, the material is nearly incompressible. The reason is that all the elements discussed in this course are poor at approximating the divergence. Locking refers to the case where the displacement is to small because the divergence term essentially lock the displacement. It is a numerical artifact not a physical feature. [Verbatum]

1.3 Fluid equations

The fluid equation will be stated in an Eulerian framework. In this framework the domain has fixed points where the fluid passes through. The Navier-Stokes equations are derived using principles of mass and momentum conservation. These equations describes the velocity and pressure in a given fluid continuum. They are here written in the time domain $\mathcal{F}$:

$$\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = \nabla \cdot \sigma_f + f \tag{1.23}$$

$$\nabla \cdot u = 0 \tag{1.24}$$

where u is the fluid velocity, p is the fluid pressure, ρ stands for constant density, f is body force and $\sigma_f = \mu_f(\nabla u + \nabla u^T) - pI$
We will only compute incompressible fluids.
There does not yet exist an analytical solutions to the N-S equations, only simplified problems can be solved [13]. But this does not stop us from discretizing and solving them numerically.
Before these equations can be solved we need to impose boundary conditions.

1.3.1 Boundary conditions

On the Dirichlet boundary $\partial\mathcal{F}_D$ we impose a given value. This can be initial conditions or set to zero as on walls with "no slip" condition. These conditions needs to be defined for both u and p

$$u = u_0 \text{ on } \partial\mathcal{F}_D$$

$$p = p_0 \text{ on } \partial\mathcal{F}_D$$

The forces on the boundaries need to equal an eventual external force $\mathbf{f}$

$$\sigma \cdot \mathbf{n} = f \text{ on } \partial\mathcal{F}_N$$

Bibliography

[1] Miguel A. Fernández, Jimmy Mullaert, and Marina Vidrascu. Explicit robin-neumann schemes for the coupling of incompressible fluids with thin-walled structures. *Computer Methods in Applied Mechanics and Engineering*, 267:566–593, 2013.

[2] Miguel A. Fernández, Jimmy Mullaert, and Marina Vidrascu. Generalized Robin-Neumann explicit coupling schemes for incompressible fluid-structure interaction: Stability analysis and numerics. *International Journal for Numerical Methods in Engineering*, 101(3):199–229, 2015.

[3] G Holzapfel. Nonlinear solid mechanics: A continuum approach for engineering, 2000.

[4] Jaroslav Hron and Stefan Turek. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. *Fluid-Structure Interaction*, 53:371–385, 2006.

[5] Jie Liu, Rajeev K. Jaiman, and Pardha S. Gurugubelli. A stable second-order scheme for fluid-structure interaction with strong added-mass effects. *Journal of Computational Physics*, 270:687–710, 2014.

[6] Cm Macal. Proceedings of the 2005 Winter Simulation Conference ME Kuhl, NM Steiger, FB Armstrong, and JA Joines, eds. *Simulation*, pages 1643–1649, 2005.

[7] William L. Oberkampf and Christopher J. Roy. *Verification and Validation in Scientific Computing.* Cambridge University Press, Cambridge, 2010.

[8] T Richter and T Wick. On time discretizations of Fluid-structure interactions. *Multiple Shooting and Time Domain Decomposition MEthods*, pages 377–400, 2013.

[9] Thomas Richter. Fluid Structure Interactions. 2016.

[10] Noelle Selin. Verification and Validation. (February), 2014.

[11] K. Stein, T. Tezduyar, and R. Benney. Mesh Moving Techniques for Fluid-Structure Interactions With Large Displacements. *Journal of Applied Mechanics*, 70(1):58, 2003.

[12] E. H. van Brummelen. Added Mass Effects of Compressible and Incompressible Flows in Fluid-Structure Interaction. *Journal of Applied Mechanics*, 76(2):021206, 2009.

[13] Frank M White. Viscous Fluid Flow Viscous. *New York*, Second:413, 2000.

[14] Thomas Wick. Adaptive Finite Element Simulation of Fluid-Structure Interaction with Application to Heart-Valve Dynamics. *Institute of Applied Mathematics, University of Heidelber*, page 157, 2011.

[15] Thomas Wick. Fluid-structure interactions using different mesh motion techniques. *Computers and Structures*, 89(13-14):1456–1467, 2011.