Thesis Title

Institution Name

Author Name

Day Month Year

Kapittel 1

Equations and notation

Notation

Let $\Omega \in \mathbb{R}^d$ for $d \in \{2\}$, be a bounded domain with boundary $\partial\Omega$. The domain is made up of of two sub domains $\mathcal{F}$ for the fluid domain, and $\mathcal{S}$ for the solid. The interface between the domains are denoted by $\Sigma = \mathcal{F} \cap \mathcal{S}$. The reference or initial is denoted by $\hat{\mathbf{\Sigma}} = \hat{\mathcal{F}} \cap \hat{\mathcal{S}}$
u - Velocity in fluid and structure.
d - Displacement in the solid and in the fluid mesh. p - Pressure in the fluid.

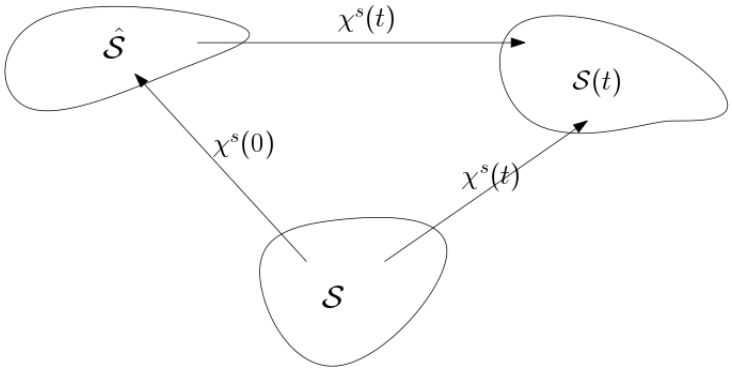
1.1 Solid Equations

In this chapter we will look briefly into the solid equation. The solid equation is most commonly described in a Lagrangian description. In a Lagrangian description the material particles are fixed with the grid points, this is a useful property when tracking the solid domain.

Lagrangian physics

Mapping and identites

We will start by providing a short introduction to Lagrangian physics for the sake of completeness.



We define $\hat{\mathcal{S}}$ as the initial stress free configuration of a given body, $\mathcal{S}$ as the reference and $\mathcal{S}(t)$ as the current configuration respectively. We need to define a smooth mapping from the reference configuration to the current configuration:

$$\chi^s(t) : \hat{\mathcal{S}} \to \mathcal{S}(t) \tag{1.1}$$

Following the notation of [4],where $\mathbf{X}$ denote a material point in the reference domain and χ^s denotes the mapping from the reference configuration. Let $d^s(\mathbf{X}, t)$ denote the displacement field

and w($\mathbf{X}$,t) the domain velocity. We then set the mapping

$$\chi^s(\mathbf{X}, t) = \mathbf{X} + d^s(\mathbf{X}, t) \tag{1.2}$$

where $d^s(\mathbf{X}, t)$ represents displacement field

$$d^s(\mathbf{X}, t) = \chi^s(\mathbf{X}, t) - \mathbf{X} \tag{1.3}$$

and the domain velocity is the partial time derivative:

$$w(\mathbf{X}, t) = \frac{\partial \chi^s(\mathbf{X}, t)}{\partial t} \tag{1.4}$$

Next we will need a function that describes the rate of deformation in the solid.

Deformation gradient

When a continuum body undergoes deformation and is moved from the reference configuration to some current configuration we need a deformation gradient that describes the rate of deformation in the body. If $d(\mathbf{X}, t)$ is a differentiable deformation field in a given body. We define the deformation gradient as:

$$F = \frac{\partial \chi}{\partial \mathbf{X}} = I + \nabla d(\mathbf{X}, t) \tag{1.5}$$

which denotes relative change of position under deformation in a Lagrangian frame of reference. A change in volume between reference (dv) and current (dV) configuration is defined as :

$$dv = JdV \tag{1.6}$$

$$J = \det(F) \tag{1.7}$$

Where J is the determinant of the deformation gradient F known as the Jacobian determinant or volume ratio. If there is no motion, that is $F = I$ and $J = 1$, there is no change in volume. But we can also have the constraint $J = 1$ with motion, but preserving the volume. If we assume infinitesimal line and area elements in the current ds, dx and reference dV, dX configurations. The volume elements dv, dV can be expressed by the dot product:

$$dv = ds \cdot dx = JdSdX \tag{1.8}$$

This is used to get the Nansons formula:

$$ds = JF^{-T}dS \tag{1.9}$$

which holds for an arbitrary line element in different configurations.

Strain

In continuum mechanics relative change of location of particles is called strain and this is the fundamental quality that causes stress in a material. [10]. An import strain measure is the right Cauchy-Green tensor

$$C = F^T F$$

which is symmetric and positive definite $C = C^T$. We also introduce the Green-Lagrangian strain tensor E:

$$E = \frac{1}{2}(F^T F - I)$$

which is also symmetric since C and I are symmetric. This measures the squared length change under deformation.

Stress

Stress is the internal forces between neighboring particles. We introduce the Cauchy stress tensor:

$$\sigma_s = \frac{1}{J} F(\lambda_s(trE)I + 2\mu_s E)F^T \tag{1.10}$$

Using (1.9) we get the first Piola-Kirchhoff stress tensor P:

$$P = J\sigma F^{-T} \tag{1.11}$$

We also introduce the second Piola-Kirchhoff stress tensor S:

$$S = JF^{-1}\sigma F^{-T} = F^{-1}P = S^T \tag{1.12}$$

from this relation we can write the first Piola-Kirchhoff tensor by the second:

$$P = FS \tag{1.13}$$

Solid equation

From the principles of conservation of mass and momentum, we get the solid equation stated in the Lagrangian reference system (Following the notation from [10]):

$$\rho_s \frac{\partial d^2}{\partial t^2} = \nabla \cdot (P) + \rho_s f \tag{1.14}$$

where we used the first Piola-Kirchhoff stress tensor.

Locking

The problem og shear locking can happen FEM computations with certain elements. [mek4250 Kent] - Locking occurs if $\lambda >> \nu$ that is, the material is nearly incompressible. The reason is that all the elements discussed in this course are poor at approximating the divergence. Locking refers to the case where the displacement is to small because the divergence term essentially lock the displacement. It is a numerical artifact not a physical feature. [Verbatum]

1.2 Fluid equations

The Navier-Stokes equations are derived using principles of mass and momentum conservation. These equations describes the velocity and pressure in a given fluid continuum. They are here written in the time domain $\mathcal{F}$:

$$\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = \nabla \cdot \sigma_f + f \tag{1.15}$$

$$\nabla \cdot u = 0 \tag{1.16}$$

where u is the fluid velocity, p is the fluid pressure, ρ stands for constant density, f is body force and $\sigma_f = \mu_f(\nabla u + \nabla u^T) - pI$
We will only compute incompressible fluids.
There does not yet exist an analytical solutions to the N-S equations, only simplified problems can be solved [14]. But this does not stop us from discretizing and solving them numerically.
Before these equations can be solved we need to impose boundary conditions.

Boundary conditions

On the Dirichlet boundary $\partial\mathcal{F}_D$ we impose a given value. This can be initial conditions or set to zero as on walls with no slip"condition. These conditions needs to be defined for both u and p

$$u = u_0 \text{ on } \partial\mathcal{F}_D$$

$$p = p_0 \text{ on } \partial\mathcal{F}_D$$

The forces on the boundaries need to equal an eventual external force $\mathbf{f}$

$$\sigma \cdot \mathbf{n} = f \text{ on } \partial\mathcal{F}_N$$

Kapittel 2

Fluid Structure Interaction Problem

In FSI the computing domain is split into three parts. Fluid, structure and interface. The Navier-Stokes equations are solved on the fluid domain and the structure equation on the structure. The interface is where the solid and fluid meet, and FSI is generally solved using two methods. These two main methods differ in their treatment of the interface [6]. The first uses a fixed mesh, known as Eulerian, where the interface is tracked across the mesh. This treatment is useful for a fluid problem since fluids are continuously deformable and tracking each fluid particle is not essential in computing fluid dynamics. For the solid this formulation is not very practical as the distance between particles are used to track the deformation, hence the particles need to be tracked. Tracking where the particles are, and consequentially the interface is, across meshpoints is a difficult task. One would need short spacing between meshpoints to accurately track the interface. In the second method (ALE) the interface moves with the mesh. The ALE formulation stands for Arbitrary Lagrangian Eulerian. This entails formulating the fluid equations in an Eulerian and the solid in a Lagrangian framework. The mesh itself moves with the structure and hence interface displacements, and the fluid moves through these points. In this way we get best of both world so to speak. The structure equation will remain as previously stated (1.14), and we will need to change the fluid equations to take into account the moving mesh changing the fluid velocity. This is done in two ways. One is to move the mesh itself in relation to the structural displacements, and calculate using this new mesh every time. This approach gives advantages as we can explicitly represent the fluid-structure interface, and the equations are stated in a more familiar manner. But problems arise when there are large deformations in the solid giving large deformations into the fluid domain. Moving the mesh with large deformation can be a challenge. In this thesis the ALE approach is used from a reference frame. Meaning we solve the equation on a initial, stress free domain, and use a series of mappings to account for the movements of the domain. The equations are mapped to the current domain, that is where the domain has moved to in the present time. It is the displacement in the solid and the displacement extrapolated from the solid to the fluid domain that determines the mappings. From a technical point of view, both formulations are equivalent.[10] But the ease of computing and time efficiency, in that we do not need an extra function to move the mesh between every timestep, contributes to the choice of using the mapped approached. Lastly in this chapter we look at how to solve the equations. This problem is tackled using either a monolithic or a partitioned approach. We will go into details later, but the overall idea is that a monolithic scheme involves computing all the equations together into one block. The partitioned splits up into parts, that is we solve the fluid problem and structure problem separately. The advantage of this is that we can use existing solvers and techniques for each of the problems, but the difficulty is the treatment of the interface. The monolithic approach however, offers more stability but is more costly as the size of the problem increases[6].

Mapping

Let $\hat{\mathcal{V}}$ be a reference volume and $\mathcal{V}(t)$ be the current time volume using [**?**]. Then using (1.5) and (1.6) we define a mapping between the volumes from the current to reference configurations:

$$\int_{\mathcal{V}(t)} 1dx = \int_{\hat{\mathcal{V}}} Jdx \tag{2.1}$$

The gradients acting on a vector $\mathbf{u}$ will also be mapped between current and reference configurations:

$$\int_{\mathcal{V}(t)} \nabla \mathbf{u} dx = \int_{\hat{\mathcal{V}}} J \nabla \mathbf{u} F^{-1} dx \tag{2.2}$$

Same for the divergence of a vector $\mathbf{u}$:

$$\int_{\mathcal{V}(t)} \nabla \cdot \mathbf{u} dx = \int_{\hat{\mathcal{V}}} \nabla \cdot (JF^{-1}\mathbf{u}) dx \tag{2.3}$$

Balance laws

We will formulate the equations in the Eulerian, Lagrangian and the ALE description.

Solid

We express the solid balance laws in the Lagrangian formulation from the initial configuration

$$\rho_s \frac{\partial^2 d}{\partial t^2} = \nabla \cdot (P) \quad in \quad \hat{\mathcal{S}} \tag{2.4}$$

Fluid

The fluid domain is moving, and therefore we need to redefine the velocity in the convective term in (1.15) to account for the moving domain

$$\mathbf{u} \cdot \nabla \mathbf{u} \rightarrow (\mathbf{u} - \frac{\partial d_f}{\partial t}) \cdot \nabla \mathbf{u} \tag{2.5}$$

where d_f is the deformation in the fluid domain. Now the actual fluid velocity will be $u - \frac{\partial d}{\partial t}$
The fluid equations are denoted from the initial configuration using the aforementioned mappings:

$$\int_{\mathcal{V}(t)} \rho_f \frac{\partial \mathbf{u}}{\partial t} dx = \int_{\hat{\mathcal{V}}} \rho_f J \frac{\partial \mathbf{u}}{\partial t} dx \tag{2.6}$$

$$\int_{\mathcal{V}(t)} \nabla \mathbf{u} (\mathbf{u} - \frac{\partial d}{\partial t}) dx = \int_{\hat{\mathcal{V}}} ((\nabla \mathbf{u}) F^{-1} (\mathbf{u} - \frac{\partial d}{\partial t}) dx \tag{2.7}$$

$$\int_{\mathcal{V}(t)} \nabla \cdot \mathbf{u} dx = \int_{\hat{\mathcal{V}}} \nabla \cdot (JF^{-1}\mathbf{u}) dx \tag{2.8}$$

$$\int_{\mathcal{V}(t)} \nabla \cdot \sigma_f dx = \int_{\hat{\mathcal{V}}} \nabla \cdot (JF^{-1}\hat{\sigma_{\mathbf{f}}}) dx \tag{2.9}$$

$$\hat{\sigma_{\mathbf{f}}} = -pI + \mu(\nabla \mathbf{u} F^{-1} + F^{-T}\mathbf{u}^T) \tag{2.10}$$

Putting this together (1.15) then turns into:

$$\rho_f J\big(\frac{\partial \mathbf{u}}{\partial t} dx + (\nabla \mathbf{u}) F^{-1} (\mathbf{u} - \frac{\partial d}{\partial t})\big) = \nabla \cdot (JF^{-1}\hat{\sigma_{\mathbf{f}}}) + J\rho_f f \tag{2.11}$$

$$\nabla \cdot (JF^{-1}\mathbf{u}) = 0 \tag{2.12}$$

Harmonic extension

To bind together the computation of fluid and structure domain, we need an harmonic extension to the boundary values. The solid deformation d is extended from the interface into the fluid domain and is done to help deal with big deformations in fluid domain. These big deformations can then cause several challenges to the ALE mapping. For this purpose define the following harmonic extension equation in the fluid domain:

$$-\alpha_u \nabla^2 d^f = 0 \quad in \ \hat{\mathbf{F}} \tag{2.13}$$

This equation is chosen for its good regularity and smoothing properties. A strategy for choosing α_u is proposed by Wick in [15], and further discussed in [12]. Where they set

$$\alpha_u = J^{-1} \tag{2.14}$$

This is a smart choice since J gets smaller closer to the interface, which again makes alpha large, this upholds the cell structure closer to the interface where most of the cell distortion appears. It is also possible to chose an harmonic extension with stiffening, which can give better control of the deformed meshes. This in practice behaves like a transport problem, transporting the deformation into the fluid domain. Another possibility is extension by pseudo-elasticity which defines the extension operator by means of the Navier-Lame equation. And lastly we can chose a biharmonic extension,

$$-\alpha_u \nabla^4 d^f = 0 \tag{2.15}$$

this gives the freedom of not requiring a careful choice of mesh dependent parameter. Implementation of the biharmonic extension is more tedious in that we have to implement an extra function and extra boundary conditions. It is also of fourth order character, and thus will have a high computational cost. [10]

Coupled Fluid Structure Interface conditions

We introduce a global domain $\Omega \in \mathcal{S} \cup \mathcal{F}$ that is made up of the fluid and the structure and the interface. We define a global velocity function u that is the fluid velocity in the fluid domain and the structure velocity in the structure domain. This can be done due to the interface condition making the velocity field continuous over the entire domain. Then the interface will be $\Gamma \in \mathcal{S} \cap \mathcal{F}$ We need to define conditions that couple that motions of the fluid and structure together. These consist of:

- Kinematic condition: $\mathbf{u}_f = \mathbf{u}_s \quad on\Gamma$. The fluid and structure velocities need to be equal on the boundary.
  We later realize this later by setting $\mathbf{u} - \frac{\partial d}{\partial t} = 0$ on the solid domain to strongly imply that the velocity on the solid is the derivative of the deformation.

- Dynamic condition: $\sigma_f n_f = \sigma_s n_s \quad on \ \Gamma$.
  This relates to Newtons third law of action and reaction. The forces on the interface area, here written as the normal stresses are balanced on the interface. These will be written in a Lagrangian formulation:
  $J\sigma_f F^{-T} n_f = F\Sigma n_s \quad on \ \Gamma$.
  This condition is often added to structure problem, since it is most often in FSI problem the fluid exerting force that cause deformation.

- Geometrical condition: This condition says that the fluid and structure domains do not overlap, but rather that elements connect so the functions needing to transfer force are continouos across the entire domain.

Monolithic FSI Problem

As stated in the introduction there are generally two types of schemes used when simulating FSI. The partitioned approach where fluid and structure are solved sequentially, is appealing in that we

have a wealth of knowledge and techniques on how to solve these kinds of problems in an efficient manner. The difficulty however is dealing with the interface. As we know there are kinematic and dynamic conditions needed in FSI, and the coupling of these conditions is where the problems arise. So called explicit coupled schemes are known to be unconditionally unstable for standard Dirichlet-Neumann strategies when there is a large amount of added-mass in the system [3], [13]. There are schemes which offer added-mass free stability with explicit coupling, where the interface is treated through a Robin-Neumann coupling. First for a coupling with a thin walled structure [2] and later an extension to thick wall [3]. These schemes are rather complex and uses a number of techniques that are out of the scope of this thesis. (This may be in more detail in a later chapter (discussion and further work.))
The other is approach is monolithic, where all of the equations are solved at once. This approach has the advantage of offering numerical stability for problems with strong added-mass effects [6], and are fully coupled. The disadvantage over the partitioned approach is that we loose flexibility when solving many equations simultaneously, and the problems can quickly become very large and computationally costly.
We start by stating the entire FSI ALE problem in a monolithic framework using the mapped approach:
Find $\mathbf{u} \in \hat{\mathcal{F}}, p \in \hat{\mathcal{F}}$ and $d \in \hat{\mathcal{S}}$ such that :

$$\rho_f J\big(\frac{\partial u}{\partial t} + (\nabla \mathbf{u})F^{-1}(\mathbf{u} - \frac{\partial d}{\partial t})\big) + \nabla \cdot (J\hat{\sigma_\mathbf{f}}F^{-T}) = 0 \qquad \text{on } \hat{\mathcal{F}} \tag{2.16}$$

$$\nabla \cdot (J\mathbf{u}F^{-T})\big) = 0 \qquad \text{on } \hat{\mathcal{F}} \tag{2.17}$$

$$\rho_s \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot FS_s, = 0 \qquad \text{on } \hat{\mathcal{S}} \tag{2.18}$$

$$\nabla^2 d = 0 \qquad \text{on } \hat{\mathcal{F}} \tag{2.19}$$

$$\mathbf{u} - \frac{\partial d}{\partial t} = 0 \qquad \text{on } \hat{\mathcal{S}} \tag{2.20}$$

$$J\sigma_f F^{-T} n_f = \sigma_s n_s \quad \text{on } \Gamma \tag{2.21}$$

2.1 Discretization

Now that we have stated the full FSI monolithic scheme we need to specify the ways in which the scheme is discretized. The temporal discretization is done using finite difference schemes and the spatial is treated with the finite element method. Following the ideas and notations of [15]. In the domain Ω and time interval [0,T]. We use for simplicity the harmonic mesh motion: Find $U = \{\mathbf{u}, d, p\} \in X_D$ bla bla function spaces.

$$A(U) = (J\rho_f \partial_t \mathbf{u}, \phi) - (J(\nabla u)F^{-1}(\mathbf{u} - \partial_t d), \phi)_{\hat{\mathbf{F}}} \tag{2.22}$$

$$+ (J\sigma_f F^{-T}, \nabla\phi)_{\hat{\mathbf{F}}} \tag{2.23}$$

$$+ (\rho_s \partial_t \mathbf{u}, \phi)_{\hat{\mathbf{S}}} + \big(FS_s, \nabla\phi\big)_{\hat{\mathbf{S}}} \tag{2.24}$$

$$+ (\alpha_u \nabla \mathbf{u}, \nabla\psi)_{\hat{\mathbf{F}}} + (\nabla \cdot (JF^{-1}\mathbf{u}), \gamma)_{\hat{\mathbf{F}}} \tag{2.25}$$

$$+ (\partial_t d, \psi)_{\hat{\mathbf{S}}} - (\mathbf{u}, \psi)_{\hat{\mathbf{S}}} \tag{2.26}$$

$$+ \big(J\sigma_{f,p}F^{-T}, \nabla\phi\big) \tag{2.27}$$

I will here formulate the *One step-θ scheme* from [15]. This θ scheme has the advantage of easily being changed from the backward (implicit), forward(excplicit) or Crank-Nicholson (implicit) scheme. The backward Euler scheme is of first order and is implicit in that it is using the newest time step appears on both sides of the equation. The Crank-Nicholson is of second order and both the current and previous time step is used. This scheme suffers from instabilities in its normal context, we will therefore look at a *shifted* Crank-Nicholson scheme.
We define the variational form by dividing into four categories: time term, implicit, pressure and the rest (stress, convection):

$$A_T(U) = (J\rho_f \partial_t \mathbf{u}, \phi) - (J(\nabla u)F^{-1}(\partial_t d), \phi)_{\hat{\mathbf{F}}} \tag{2.28}$$
$$+ (\rho_s \partial_t \mathbf{u}, \phi)_{\hat{\mathbf{S}}} + (\partial_t d, \psi)_{\hat{\mathbf{S}}} \tag{2.29}$$
$$A_I(U) = (\alpha_u \nabla \mathbf{u}, \nabla \psi)_{\hat{\mathbf{F}}} + (\nabla \cdot (JF^{-1}\mathbf{u}), \gamma)_{\hat{\mathbf{F}}} \tag{2.30}$$
$$A_E(U) = (J(\nabla u)F^{-1}\mathbf{u}, \phi)_{\hat{\mathbf{F}}} + (J\sigma_{f,u}F^{-T}, \nabla\phi)_{\hat{\mathbf{F}}} \tag{2.31}$$
$$+ \big(FS_s, \nabla\phi\big)_{\hat{\mathbf{S}}} - (\mathbf{u}, \psi)_{\hat{\mathbf{S}}} \tag{2.32}$$
$$A_P(U) = \big(J\sigma_{f,p}F^{-T}, \nabla\phi\big) \tag{2.33}$$

Here the stress tensors have been split into a velocity and pressure part.

$$\sigma_{f,u} = \mu(\nabla u F^{-1} + F^{-T}\nabla u) \tag{2.34}$$
$$\sigma_{f,p} = -pI \tag{2.35}$$

We also notice that the we have split up
For the time group we discretize in the following way:

$$A_T(U^{n,k}) \approx \frac{1}{k}\big(\rho_f J^{n,\theta}(u^n - u^{n-1}), \phi\big)_{\hat{\mathbf{F}}} - \frac{1}{k}\big(\rho_f(\nabla u)(d^n - d^{n-1}), \phi\big)_{\hat{\mathbf{F}}} \tag{2.36}$$
$$+ \frac{1}{k}\big(\rho_s J^{n,\theta}(u^n - u^{n-1}), \phi\big)_{\hat{\mathbf{S}}} + \frac{1}{k}\big(J^{n,\theta}(d^n - d^{n-1}), \psi\big)_{\hat{\mathbf{S}}} \tag{2.37}$$

And the Jacobian is written with superscript θ as:

$$J^{n,\theta} = \theta J^n + (1-\theta)J^{n-1} \tag{2.38}$$

We can now introduce the *One step-θ scheme*: Find $U^n = \{u^n, d^n, p^n\}$

$$A_T(U^{n,k}) + \theta A_E(U^n) + A_P(U^n) + A_I(U^n) = \tag{2.39}$$
$$- (1-\theta)A_E(U^{n-1}) + \theta F^n + (1-\theta)F^{n-1} \tag{2.40}$$

We see here that scheme is selected by the choice of θ. By choosing $\theta = 1$ we get the back Euler scheme, for $\theta = \frac{1}{2}$ we get the Crank-Nicholson scheme and shifted Crank-Nicholson we set $\theta = \frac{1}{2} + k$, effectively shifting the scheme towards the implicit side.

Finite Element method FSI in ALE

Variational formulation

Reference domain

We use 3 testfunctions, ϕ, ψ, γ. As mentioned before we use a global velocity function u for both the solid and fluid.

$$\rho_f J\big(\frac{\partial u}{\partial t} + (\nabla u)F^{-1}(u - \frac{\partial d}{\partial t}), \phi\big)_{\hat{\mathbf{F}}} + (J\sigma_f F^{-T}, \nabla\phi)_{\hat{\mathbf{F}}} = 0 \tag{2.41}$$
$$\big(\nabla \cdot (JuF^{-T}), \gamma\big)_{\hat{\mathbf{F}}} = 0 \tag{2.42}$$
$$\big(\rho_s \frac{\partial u}{\partial t}, \phi\big)_{\hat{\mathbf{S}}} + \big(FS_s, \nabla\phi\big)_{\hat{\mathbf{S}}} = 0 \tag{2.43}$$
$$\big(\nabla d, \nabla\psi\big)_{\hat{\mathbf{F}}} = 0 \tag{2.44}$$
$$\big(u - \frac{\partial d}{\partial t}, \psi\big)_{\hat{\mathbf{S}}} = 0 \tag{2.45}$$

Equation (5) has not been addressed and is added since we use a global function for velocity we need to force that the structure velocity is the time derivative of the deformation in the structure domain.

Spaces and Elements

The velocity and pressure copling in the fluid domain must satisfy the inf-sup condition. If not stabilization has to added. We here need to define some spaces that will have these desired properties. We denote $u_h \in V_h$ and $d_h \in W_h$, here the finite element pair og pressure and velocity mush satisfy the inf-sup condition given in ALE coordinates:

$$\inf_{p_h \in L_{h,f}} \sup_{v_h \in V_{h,f}} \frac{(p_h, div(J_f F_f^{-1} u_h))_{\mathcal{F}}}{\|\|J^{\frac{1}{2}} p_h\|\|_{\mathcal{F}} \|\|J_f^{\frac{1}{2}} \nabla u_h F_f^{-T}\|\|_{\mathcal{F}}} \geq \gamma$$

A good choice of spaces will be P2-P2-P1 for velocity, displacement and fluid pressure respectively.

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