Work Sheet 1 - De-dimensionalisation and Discretisation

1. The equation for a falling stone is

$$m\frac{dv}{dt} + Bv^2 = mg.$$

What are the dimensions of each term? How many variables are there, and how many parameters? How many parameters would you expect to reduce this to by de-dimensionalisation? What sign should B have, and why?

De-dimensionalise the falling stone equation, keeping as many dimensionless products as possible $= 1$. Explain how you would calculate the true value of distance moved by the stone, its momentum and its kinetic energy in terms of your dimensionless variables.

2. An "inverted pendulum" has a mass m at the top of a rod of length ℓ free to pivot at its base. If a trolley (mass M) is attached to the pivot, it may be moved horizontally (along the x axis) to keep the pendulum upright (think of someone trying to balance a ruler on the end of her finger). The equations of motion for the pendulum are

$$\begin{cases} (M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = 0\ , \\ \quad m\ell(-g\sin\theta + \ddot{x}\cos\theta + \ell\ddot{\theta}) = 0\ , \end{cases}$$

where the dots denote differentiation with respect to time t, g is acceleration due to gravity, and θ is the angle of the rod to the vertical.

There are 3 variables in this problem. Identify them, and state their dimensions. Then de-dimensionalise the coupled equations, keeping as many terms as possible of order 1, and deriving expressions for any dimensionless products.

3. Use the derived on lectures "centred difference" approximations for derivatives to express the quantities $\nabla\Phi(\mathbf{r})$, $\nabla\cdot\mathbf{F}(\mathbf{r})$ and $\nabla\times\mathbf{F}(\mathbf{r})$ on a 3-dimensional cubic grid, with a grid spacing of a in x, y and z. $\Phi(\mathbf{r})$ is a scalar field, $\mathbf{F}(\mathbf{r})$ a vector field.

4. To find even-symmetry eigen-states of a Gaussian potential well, you need to solve the following dimensionless Schrödinger equation

$$\frac{-d^2\psi}{d\xi^2} - U_0\exp(-\xi^2/w^2)\psi = E\psi\ ,$$

where U_0 and w are positive constants which define the depth and width of the potential well, respectively. You need to solve this equation on the interval $(0 \le \xi \le L)$ with the following boundary conditions:

$$\frac{d\psi}{d\xi}(0) = 0\ , \qquad \psi(L) = 0\ .$$

Discretise this equation on a regular grid in ξ of spacing a; ensure the discretisation error is $O(a^2)$ at worst. What is the value of the step size a? What coordinates the first ($j = 1$) and the last ($j = N$) grid points correspond to? Apply the boundary conditions. Write down the corresponding matrix eigen-value problem, and specify the matrix. How the boundary conditions should be changed to find odd-symmetry eigen-states?

5. You need to solve the following equation:

$$\frac{d^2\Phi}{dx^2} - 3\exp(-\pi x^2)\Phi = 0 \tag{1}$$

on a symmetric interval $-L/2 \leq x \leq L/2$ with the periodic boundary condition $\Phi(x+L) = \Phi(x)$.

By expanding the solution in complex Fourier series,

$$\Phi = \sum_n \phi_n \exp(ik_n x)\ , \qquad k_n = \frac{2\pi n}{L}$$

show that the above differential equation may be re-written as

$$m^2 \frac{4\pi^2}{L^2}\phi_m + \sum_n V_{m-n}\phi_n = 0. \tag{2}$$

where V_{m-n} is an integral which you should define, but do not attempt to solve.

Write down equation (2) in the matrix form, assuming that m and n take the values 0, ± 1, and ± 2.

Show that for $L \gg 1/\sqrt{\pi}$ the coefficients V_{m-n} can be approximated as:

$$V_{m-n} \approx \frac{3}{L}\exp[-\pi(m-n)^2/L^2]$$

[Hint: use the known Fourier Transform pairs from the table in the formula book]