Chapter 5 Optical coefficients

Reflection at a surface is described by the "coefficient of reflection" or "reflectivity", R, defined as the ratio of *reflected* power to *incident* power.

The coefficient of transmission or transmissivity, T, is defined as the ratio of *transmitted* to *incident* power.

If there is no absorption or scattering, then conservation of energy implies $R + T = 1$.

R and T are determined by the refractive index n in the Fresnel equations (PH22006) 5.1).

The Fresnel coefficients for P- and S-polarized light are:

$$r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos\theta_t - Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} \text{ and } t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i}$$

$$r_{\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} \text{ and } t_{\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_t}$$

Figure 5.1: Fresnel equations for reflection and transmission *amplitudes* and arbitrary incident angle written in terms of optical impedances, from the notes for PH22006. ●

Here, we are focusing on material properties so will use only normal incidence, and can assume that one medium is air, as shown in Fig.5.2. Note that at normal incidence, there is no difference between s and p-polarised reflection (check Fig.5.1). The Fresnel equation for reflected intensity in terms of n is then:

$$R = \left| \frac{n-1}{n+1} \right|^2. \tag{5.1}$$

Refraction and phase velocity v are also controlled by refractive index ($v = c/n$). From Maxwell's equations,

$$n = \sqrt{\mu_r \varepsilon_r}. \tag{5.2}$$

For magnetic materials at frequencies above microwaves, the magnetic properties cannot change quickly enough to follow the applied field, so that $\mu_r \sim 1$. Then, n is only determined by the relative permittivity ε_r, giving $n = \sqrt{\varepsilon_r}$.

The *absorption* of light is quantified by its absorption coefficient, α, which is is defined (5.3) as the fractional change in intensity (power/area) per unit length of the medium. It has units of m^{-1}.

$$\frac{\delta I}{I} = -\alpha \delta x \tag{5.3}$$

By integration of (5.3) we obtain the law of exponential attenuation, often called *Beer's Law*, (5.4):

$$I(x) = I_0 e^{-\alpha x} \tag{5.4}$$

Reflection, absorption and transmission in a slab of optical material are shown in Fig.5.2:
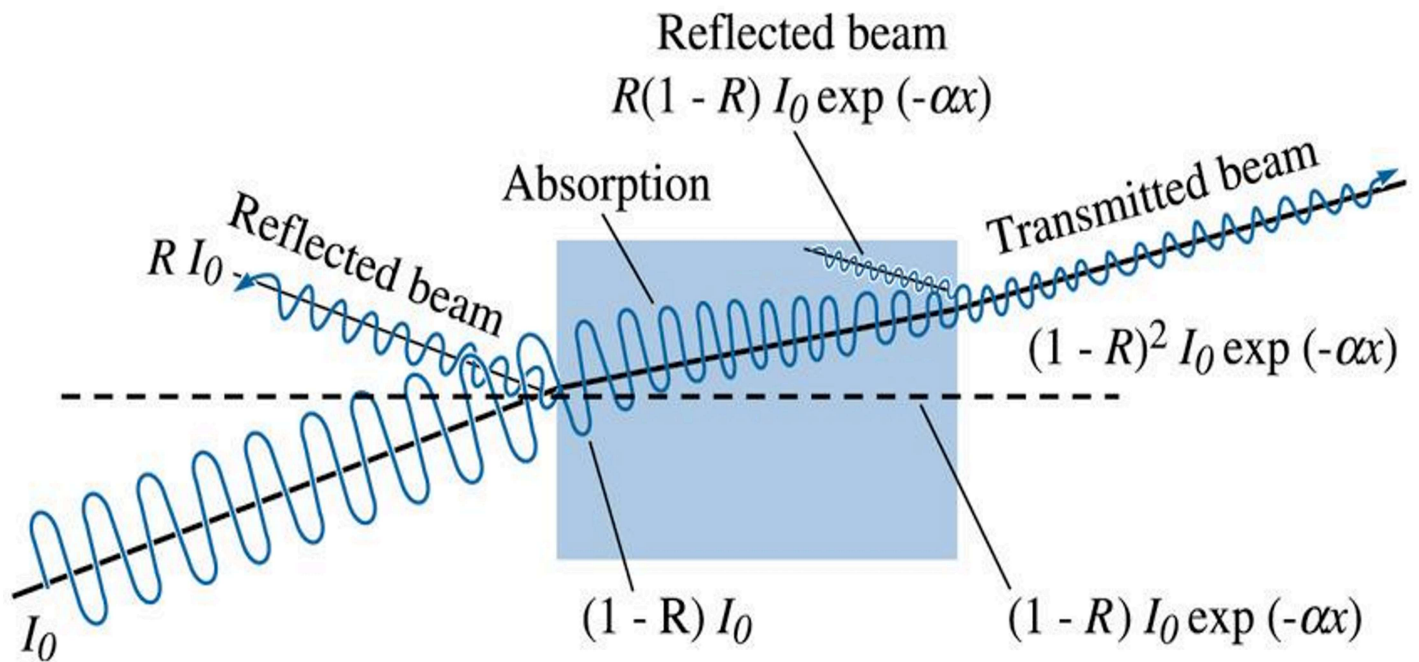

Figure 5.2: Refraction, absorption and transmission through a slab of material. In this course, we will consider only normal incidence. ●

Since scattering removes light from the direction of the transmitted beam, its effect on the transmitted intensity is similar to that of absorption, and Beer's Law (5.4) can be adapted (5.5):

$$I(z) = I_0 \exp(-N \sigma_s z) \tag{5.5}$$

where N is the number of scattering centres per unit volume and σ_s is the *scattering cross section* of one scattering centre (σ_s has units of area). The scattering is called Rayleigh scattering if the scattering is elastic and the object diameter is much smaller than the wavelength of light; for

Rayleigh scattering, (5.6) will be proved later:

$$\sigma_s\left(\lambda\right) \propto \frac{1}{\lambda^4} \tag{5.6}$$

If a material scatters light strongly, so that one photon undergoes many scattering events, (5.5) breaks down and light propagation is better described as a *diffusion* process with a *mean free path* l: for a thickness L of material, the total transmission T is (5.7)

$$T \propto \frac{l}{L}, L \gg l \tag{5.7}$$

A famous example of this is the diffusion of photons emitted at the centre of the Sun, which have a mean free path of order 1 mm and undergo around 10^{25} collisions before escaping; this takes a photon around 170,000 years to achieve; see Mitalas and Sills (1992).

5.1 Complex refractive index

Here, we continue setting up the terminology and mathematical machinery needed to link the fundamental properties of a material to its optical behaviour.

By generalising n to be complex, $\tilde{n}$, we can represent the attenuation of light as it travels through a medium.The complex refractive index is written (5.8):

$$\tilde{n} = n + i\kappa = \sqrt{\tilde{\mu}_r \tilde{\varepsilon}_r} \tag{5.8}$$

and κ is called the *extinction coefficient*. If the optical field of a plane wave travelling in the z direction is written (5.9):

$$E(z,t) = E_0 e^{i(kz - \omega t)} \tag{5.9}$$

then the complex refractive index appears in the definition of k:

$$k = \tilde{n}\frac{\omega}{c} = \left(n + i\kappa\right)\frac{\omega}{c}.$$

Substituting $(n + i\kappa)\omega/c$ for k in Eq. (5.9), the wave is a product of the usual plane wave and a real decaying exponential:

$$E(z,t) = E_0 e^{i\omega(\tilde{n}z/c - t)} = E_0 e^{-(\kappa\omega/c)z} e^{i\omega(nz/c - t)}.$$

If we calculate the intensity of this wave, as defined in (5.10), we obtain the initial intensity multiplied by a decaying exponential, which is exactly Beer's Law (5.4):

$$I = \frac{EE^*}{Z_0} = I_0 e^{-\alpha z} \tag{5.10}$$

and the absorption coefficient α is related to κ:

$$\alpha = \frac{2\kappa\omega}{c} = \frac{4\pi\kappa}{\lambda}. \tag{5.11}$$

We have not *proved* Beer's Law; really, we have just shown that it tells us a complex refractive index is necessary to describe attenuation. From Eq. (5.2), this means that the frequency-dependent dielectric constant (a.k.a. relative permittivity) is also complex:

$$\tilde{\varepsilon}_r = (n + i\kappa)^2. \tag{5.12}$$

We are getting closer to fundamental material properties. The dielectric constant can be related to the polarisation $\vec{P}$ induced in the material by an applied electric field $\vec{E}$:

$$\vec{D} = \varepsilon_0 \tilde{\varepsilon}_r \vec{E} = \varepsilon_0 \vec{E} + \vec{P} \tag{5.13}$$

$$\vec{P} = \varepsilon_o \tilde{\chi} \vec{E} \tag{5.14}$$

$$\tilde{\chi}(\omega) = \tilde{\varepsilon}_r(\omega) - 1. \tag{5.15}$$

Here, $\tilde{\chi}$ is the *electric susceptibility* (dimensionless) and, again, this is in general a complex quantity. $\tilde{n}$ is an experimental quantity whereas $\tilde{\chi}$ is generally what is calculated theoretically; this is why both closely-connected quantities persist in the scientific literature.

It is common also to split the complex dielectric constant into real and imaginary parts and these combine n, κ as in (5.17):

$$\tilde{\varepsilon}_r = (n + i\kappa)^2 = \varepsilon_1 + i\varepsilon_2 \tag{5.16}$$

$$\begin{aligned} \varepsilon_1 &= n^2 - \kappa^2 \\ \varepsilon_2 &= 2n\kappa \end{aligned} \tag{5.17}$$

The inversion of (5.17) to obtain n and κ in terms of ε_1 and ε_2 (5.18) is straightforward (but left as an exercise for the student!):

$$n = \frac{1}{\sqrt{2}} \left(\varepsilon_1 + \left(\varepsilon_1^2 + \varepsilon_2^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\kappa = \frac{1}{\sqrt{2}} \left(-\varepsilon_1 + \left(\varepsilon_1^2 + \varepsilon_2^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

(5.18)

We can obtain simpler expressions if absorption is either very strong or very weak, as follows.

5.2   Weakly absorbing materials, $\varepsilon_1 \gg \varepsilon_2$

One might be tempted to set κ to zero; this makes the refractive index real and switches off absorption completely (5.20):

$$n \sim \frac{1}{\sqrt{2}} \left(\varepsilon_1 + \varepsilon_1 \right)^{\frac{1}{2}} \sim \sqrt{\varepsilon_1}$$

(5.19)

$$\kappa \sim \frac{1}{\sqrt{2}} \left(-\varepsilon_1 + \varepsilon_1 \right)^{\frac{1}{2}} \sim 0$$

(5.20)

However, this is going too far. Using the binomial theorem, a few steps lead to a more useful approximation (5.22):

$$\kappa = \frac{1}{\sqrt{2}} \left(-\varepsilon_1 + \varepsilon_1 \left(1 + \frac{\varepsilon_2^2}{\varepsilon_1^2} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\kappa \sim \frac{1}{\sqrt{2}} \left(-\varepsilon_1 + \varepsilon_1 \left(1 + \frac{1}{2} \frac{\varepsilon_2^2}{\varepsilon_1^2} \right) \right)^{\frac{1}{2}}$$

$$\kappa \sim \frac{\varepsilon_2}{2\sqrt{\varepsilon_1}}$$

(5.21)

$$n = \sqrt{\varepsilon_1}$$
$$\kappa = \frac{\varepsilon_2}{2n}$$

(5.22)

5.3   Strongly absorbing materials, $\varepsilon_2 \gg \varepsilon_1$

By neglecting ε_1 in comparison to ε_2, we see that n and κ tend to become equal:

$$n \sim \frac{1}{\sqrt{2}} \left(\varepsilon_1 + \varepsilon_2 \right)^{\frac{1}{2}} \sim \sqrt{\frac{\varepsilon_2}{2}} \tag{5.23}$$

$$\kappa \sim \frac{1}{\sqrt{2}} \left(-\varepsilon_1 + \varepsilon_2 \right)^{\frac{1}{2}} \sim \sqrt{\frac{\varepsilon_2}{2}} \tag{5.24}$$

5.4   Fresnel coefficient at normal incidence

Substituting the complex refractive index into the expression (5.1) for the Fresnel coefficient for light incident normally on a flat surface, we obtain:

$$R = \left| \frac{\tilde{n} - 1}{\tilde{n} + 1} \right|^2 \quad = \frac{(n-1)^2 + \kappa^2}{(n+1)^2 + \kappa^2}$$

$$T = 1 - R \quad\quad = \frac{4n}{(n+1)^2 + \kappa^2} \tag{5.25}$$

We are now in a position to substitute n and κ for weakly- or strongly-absorbing materials into these expressions, to predict how R and T are influenced by absorption in the material. This is a major step forward from the understanding reached in PH22006.

References

Mitalas, R., and K. R. Sills. 1992. "On the Photon Diffusion Time Scale for the Sun" 401 (December): 759. https://doi.org/10.1086/172103.