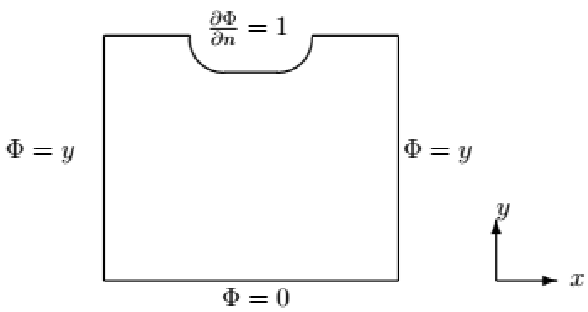
Worksheet 2 – Methods for solving PDEs

1. You need to solve numerically the Laplace's equation

$$\nabla^2\Phi = 0$$

in two dimensions (x, y) in the region shown in the figure. Boundary conditions for the top, bottom, left and right sides of the region are indicated in the figure. $\partial\Phi/\partial n = \hat{n}\cdot\nabla\Phi$ denotes the normal derivative of Φ.



(a) Discretise the Laplace's equation using a regular grid with step size h in each coordinate. Write down the resulting equaion for a generic point away from any boundaries.

(b) Choose a point close to the right boundary of the region to illustrate how the Dirichlet boundary condition is implemented.

(c) Choose a point close to a straight section of the upper boundary of the region to illustrate how the Neumann boundary condition is implemented.

(d) Why is the Neumann boundary condition along the top of the region difficult to deal with in the FDM? Briefly outline an approach (within the FDM) that might be used to deal with this Neumann boundary condition. Explain why FEM is much better at dealing with suh boundary conditions.

(e) Explain briefly how your FDM discretisation will lead to a matrix equation. Show how the symmetry of this problem might be used to reduce the number of grid points needed to compute a solution of a given accurace, and estimate the time saving involved.

2. This question is about solving the following $1D$ Boundary Value Problem:

$$\frac{d^2 f}{dx^2} + f = \epsilon \cos\left(\frac{\pi}{2}x\right) , \qquad \frac{df}{dx}(0) = 0 , \qquad f(1) = F_R$$

with FEM method.

(a) Show that the weak formulation of this problem is:

$$\int_0^1 \left(-\frac{df}{dx}\frac{d\eta}{dx} + f\eta\right) dx = \int_0^1 \epsilon \cos\left(\frac{\pi}{2}x\right) \eta dx ,$$

What are the conditions (if any) imposed on the test function $\eta(x)$?

(b) Using the grid with three nodal points at $x = 0$, $x = 0.5$, and $x = 1$ and the "tent" basis functions v_1, v_2, v_3 located at these points, write down the general expression for the FEM solution of this problem. Sketch the functions v_1, v_2 and v_3. On a separate plot make a sketch of the FEM solution.

(c) Using the expression for $f(x)$ you obtained in part (b), and using v_1 and v_2 as the two test functions, show that the discretised weak formulation can be written in the matrix form as:

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

where ϕ_1 and ϕ_2 are the values of the function f at the nodal points $x = 0$ and $x = 0.5$, respectively. Define all elements of the matrix L_{ij} and of the right-hand side vector R_i.

(d) Consider a particular case with $\epsilon = 0$ and $F_R = \cos(1)$. Using the analytical experessions for the linear interpolant "tent" functions (see lecture slides), perform the integrations and obtain the numerical values for all the matrix and rhs vector coefficients. Solve the resulting matrix problem. Compare your approximate solution with the analytical solution known for this case: $f_{\text{true}} = \cos(x)$.

3. Consider the following PDE

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + \exp\left[-x^2\right]\psi$$

on the interval $0 \leq x \leq L$, with the auxiliarly conditions:

$$\psi(x, t = 0) = 5\exp(-2x^2) , \qquad \frac{\partial \psi}{\partial x}(x = 0, t) = 0, \qquad \psi(x = L, t) = 0 .$$

You need to solve it using the "Forward time centred space (FTCS)" approach. Discretize the equation using a regular N-point grid in x, and a fixed time step Δt.

Write down the resulting **iteration** equations for:

(a) a generic grid point away from the boundaries in x-coordinate;

(b) the two points at each of the boundaries, implementing the boundary conditions as specified above.

Explain briefly how you implement the initial condition and progress with the iterations to obtain $\psi(x, t)$ at all the subsequent values of time.