PH32055: Nonlinear Physics

Schedule for 2025/2026 academic year
Problem classes: Fri @16:15 in weeks 3,5
Week 1. 29 Sep:
-Mon 17:15 8W3.14;
-Thu 16:15 8W3.13
Week 2. 6 Oct:
-Mon 17:15 8W3.14;
-Thu 16:15 8W3.13
Week 3. 13 Oct:
-Mon 17:15 8W3.14;
-Thu 16:15 8W3.13;
-Fri 16:15 8W 3.13
Week 4. 20 Oct:
-Mon 17:15 8W3.14;
-Thu 16:15 8W3.13
Week 5. 27 Oct:
-Mon 17:15 8W3.14;
-Thu 16:15 8W3.13;
-Fri 16:15 8W 3.13

Contents

Notes below closely follow and contain many extracts "Strogatz: Nonlinear Dynamics and Chaos". An electronic version of the book is available from the library.

This course teaches how to understand properties of solutions of nonlinear equations without fully solving them.

1 NONLINEAR DIFFERENTIAL EQUATIONS AND PHASE SPACE.

1.1 Introduction

Nonlinear systems arise throughout physics and beyond. Examples include climate dynamics, planetary motion, water waves, superconductivity and superfluidity, photon–atom interactions, and lasers.

In this context, we use the word system in a broad sense: not simply as a system of equations, but as a physical object or process whose behaviour in time is described by one or more governing equations. These equations may be algebraic, ordinary differential, partial differential, or some combination thereof.

By dynamics we mean the time evolution of such systems.

A system is said to be nonlinear when its overall behaviour cannot be obtained by simply adding together the behaviours of its elementary components. In other words, the principle of linear superposition no longer applies. A familiar consequence is that the output is not proportional to the input.

For most nonlinear differential equations of physical relevance, exact solutions are unavailable. This difficulty has led to the development of qualitative methods that allow us to analyze nonlinear systems in general terms, often independent of their specific physical setting. This body of ideas now forms a distinct branch of mathematical physics.

Nonlinear systems are often grouped into two broad families: conservative and dissipative. Conservative systems conserve energy, while dissipative systems do not, owing to interactions or exchanges with the environment.

This course begins with the study of nonlinear ordinary differential equations, and then extends to nonlinear partial differential equations.

1.2 Linear and nonlinear ordinary differential equations

Let's consider a harmonic oscillator equation describing small oscillations of a pendulum

$$\ddot{\theta} + \gamma\dot{\theta} + \omega^2\theta = 0 \tag{1.1}$$

This is an example of a linear system. If $\theta = y_1$ and $\theta = y_2$ are solutions, then $\theta = a_1 y_1(t) + a_2 y_2(t)$ is also a solution, where $a_{1,2}$ are arbitrary constants.

Let's define an angle θ and angular velocity as two unknowns

$$\theta = x_1 \tag{1.2}$$

$$\dot{\theta} = x_2, \tag{1.3}$$

then the oscillator equation can be written as

$$\dot{x}_1 = x_2 \tag{1.4}$$
$$\dot{x}_2 = -\gamma x_2 - \omega^2 x_1 \tag{1.5}$$

$$\dot{\vec{x}} = \hat{L}\vec{x},\ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \hat{L} = \begin{pmatrix} 0 & 1 \\ -\gamma & -\omega^2 \end{pmatrix} \tag{1.6}$$

Contrary, pendulum equation for arbitrary large angles is a nonlinear equation

$$\ddot{\theta} + \gamma\dot{\theta} + \omega^2 \sin\theta = 0,\ \ \sin\theta = \theta - \frac{1}{3!}\theta^3 + \dots \tag{1.7}$$

Rearranging we find

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \quad f_1(x_1, x_2) = x_2 \\ \dot{x}_2 &= f_2(x_1, x_2), \quad f_2(x_1, x_2) = -\gamma x_2 - \omega^2 \sin x_1, \end{aligned} \tag{1.8}$$

where f_2 is a nonlinear function.

We can rewrite the above equation in a vector form

$$\dot{\vec{x}} = \vec{F}(\vec{x}),\ \vec{F} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}. \tag{1.9}$$

1.3 Phase space

A general nonlinear system described by ordinary differential equations is

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_N) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_N) \\ &\dots \\ \dot{x}_N &= f_N(x_1, x_2, \dots, x_N) \end{aligned} \tag{1.10}$$

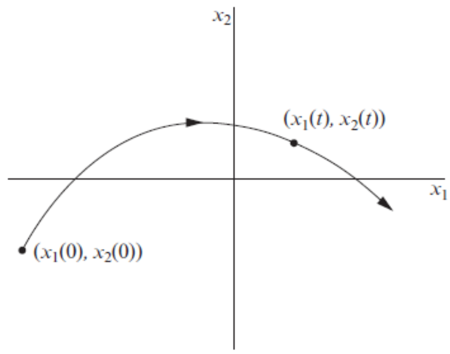


Figure 1: Phase points and phase trajectory.

or, in its vector form,

$$\dot{\vec{x}} = \vec{F}(\vec{x}) \tag{1.11}$$

The space spanned by $x_1, \ldots x_N$ is called PHASE SPACE.

A point in phase space $x_1(t_0), \ldots x_N(t_0)$ is called PHASE POINT.

A curve traced by a phase point $x_1(t), \ldots x_N(t)$ when time progresses from $t = 0$ to $t = t_{end}$ is called PHASE TRAJECTORY.

$\vec{F}(\vec{x})$ defines the velocity of a phase point. A set of velocity vectors defines a flow of an imaginary phase fluid. The term flow is often used interchangeably with dynamical system. Velocity vectors are tangential to phase trajectories.

If one considers a one dimensional Newton equation, $m\ddot{x} = F(x)$, then its phase space is two dimensional, $\dot{x}_1 = x_2, \dot{x}_2 = \frac{1}{m}F(x_1)$.

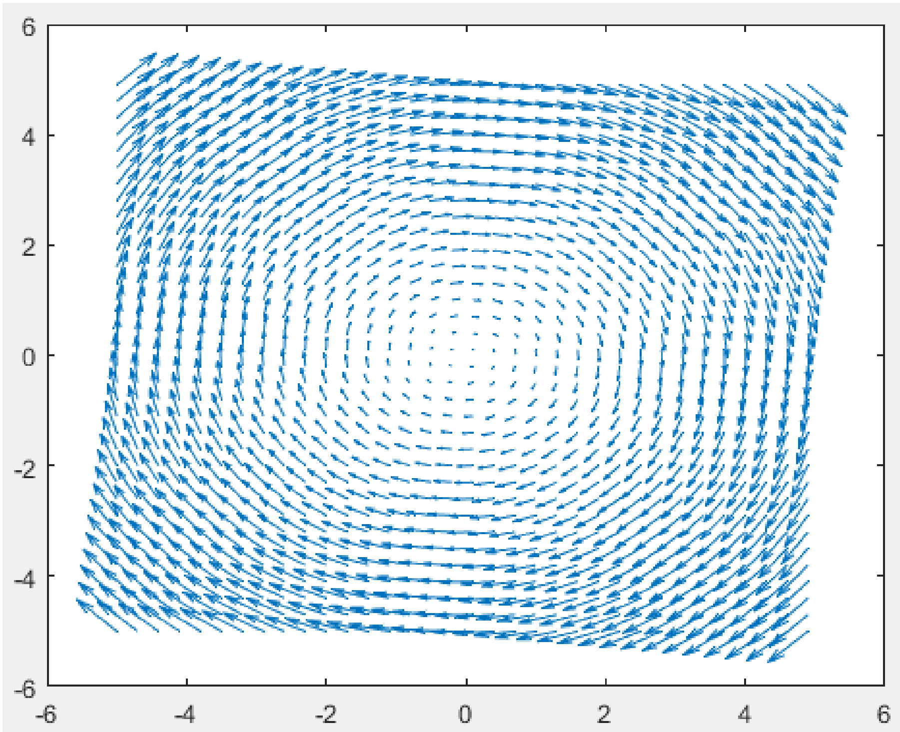


Figure 2: Here is a velocity plot generated using a matlab code for
$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$. Matlab commands are
$\mathrm{x0} = 5; \mathrm{xend} = \mathrm{x0}; \mathrm{step} = 0.3; \mathrm{scale} = 1.2;$
$[\mathrm{x1}, \mathrm{x2}] = \mathrm{meshgrid}(-\mathrm{x0} : \mathrm{step} : \mathrm{xend}, -\mathrm{x0} : \mathrm{step} : \mathrm{xend});$
$\mathrm{f1} = \mathrm{x2};$
$\mathrm{f2} = -\mathrm{x1};$
$\mathrm{quiver}(\mathrm{x1}, \mathrm{x2}, \mathrm{f1}, \mathrm{f2}, \mathrm{scale})$
What physical system does it describe and what phase trajectories correspond to this velocity plot?

2 DYNAMICS IN 1D PHASE SPACE

2.1 Fixed points and stability

We now spend considerable time with the simplest case of one dimensional phase space, i.e.,

$$\dot{x} = f(x), \tag{2.1}$$

which properties can be then used to understand several common scenarios of nonlinear dynamics which happen independently from dimensionality of the phase space. This does not mean that extra dimensions can not introduce qualitatively distinct dynamics. So that, some effects that happen in 2D do not happen in 1D, and some effects that happen in 3D do not happen in 2D.

We start building our general theoretical framework by considering an example

$$f(x) = \sin x,\ \ x(t=0) = x_0. \tag{2.2}$$

We separate variables

$$dt = \frac{dx}{\sin x} \tag{2.3}$$

and integrate

$$t = -\ln\left|\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right| + C \tag{2.4}$$

$$t = \ln\left|\frac{\frac{1}{\sin x_0} + \frac{\cos x_0}{\sin x_0}}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}\right| \tag{2.5}$$

This is an awkward equation, since one wants to write a solution as $x(t)$ and not $t(x)$.

Let's approach the problem from a geometric point of view by plotting velocity as a function of coordinate. This is just a sinusoid in our case.

Points where sinusoid crosses zero, $\sin x = 0$, $x = \pi j$, $j = 0, \pm 1, \pm 2, \ldots$, have zero velocity. They are called **fixed points**.

They are also called **equilibria, stationary points, steady states, or constant solutions.**

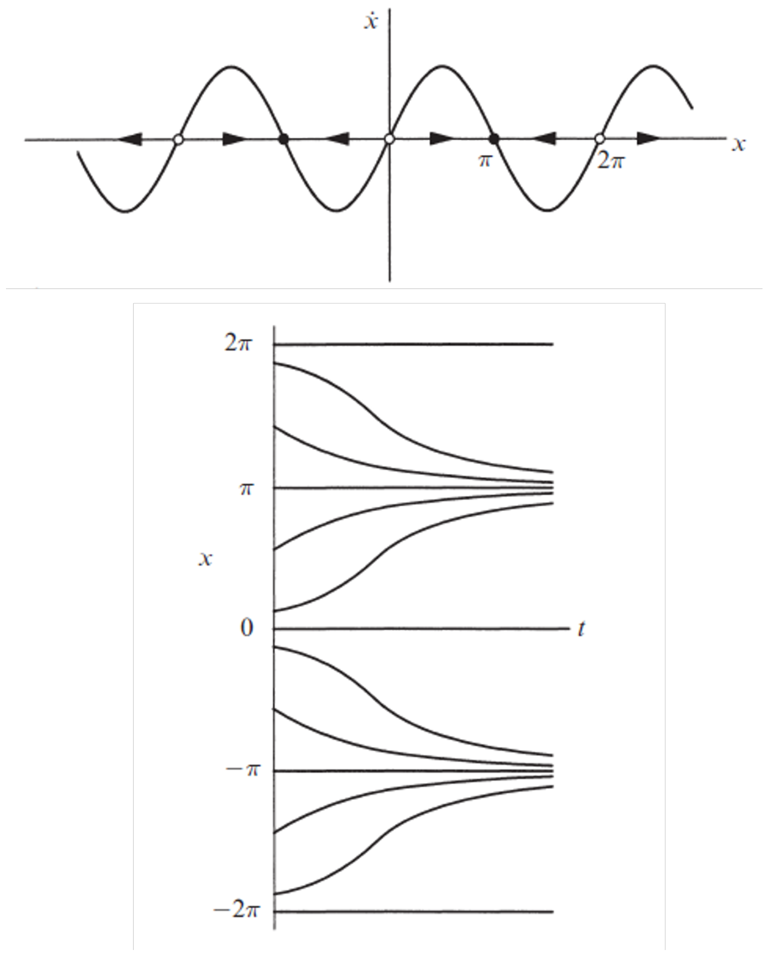


Figure 3:
Top: Velocity plot, fixed points and flow (velocity) directions for $\dot{x} = \sin x$. Directions of two flows around a fixed point determine if the point stable or unstable.
Bottom: Corresponding time evolution of $x(t)$ for a range of initial conditions, which further illustrates difference between stable and unstable fixed points.

In a general 1D case, such points are solutions of an algebraic equation

$$f(x) = 0 \tag{2.6}$$

In the N-D phase space, fixed points are solutions of a system of N algebraic (non-differential) equations

$$\vec{F}(\vec{x}) = 0 \tag{2.7}$$

Let's now consider different initial conditions, $x(t = 0) = x_0$, and observe that depending on a choice of x_0, velocity can be either positive or negative, hence some equilibria appear as repelling points and others are attractors. The latter are called stable equilibria and the former are unstable equilibria.

To visualise stability and instability of equilibria one can imagine fluid flowing along x-axis with velocity (value and direction) varying from point to point.

Stable fixed points (sinks or attractors) are the ones where any small deviation yields a velocity vector returning the flow back to the same point. **Unstable fixed points** (sources or repellers) have the imaginary fluid flowing away from them on both sides.

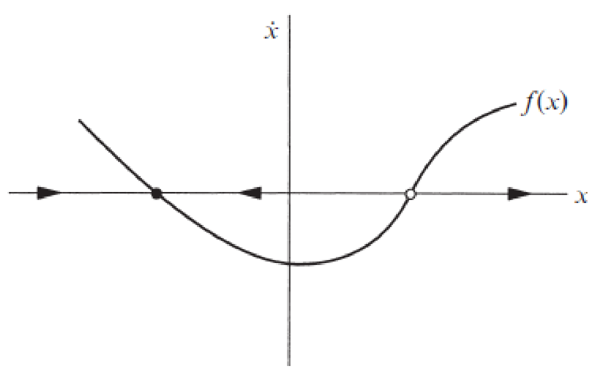


Figure 4: Stable and unstable fixed points for a general flow in 1D phase space.

2.2 Population growth

The simplest model for population growth is

$$\dot{N} = rN,\ r > 0 \tag{2.8}$$

r is growth rate. Solution

$$N(t) = N_0 e^{rt},\ r > 0 \tag{2.9}$$

implies exponential growth of population.

In practice, limited resources (food) available will start limit the growth rate. Obviously, this effect must depend on N, i.e., we need to generalise the model by assuming that the growth is itself a function of N, $r(N)$, e.g.,

$$r = R(1 - N/k) \tag{2.10}$$

This gives a so-called logistic model

$$\dot{N} = RN(1 - N/k) \tag{2.11}$$

Analyse logistic model using its velocity plot

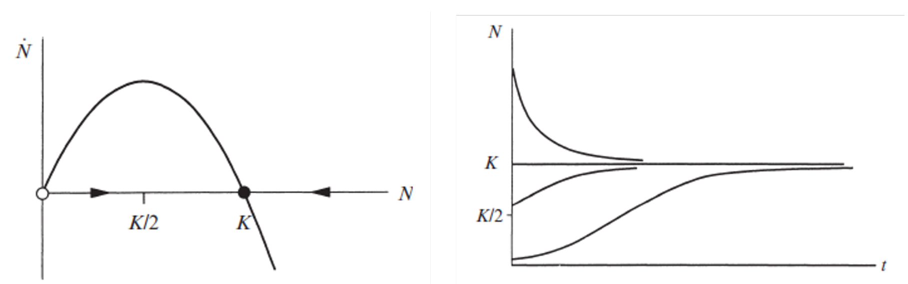


Figure 5: Logistic map plots.
Top: Population change/velocity plot and fixed points.
Bottom: Time dependencies.

2.3 Linear Stability Analysis in 1D phase-space.

Velocity plots are in fact a very general way of analysing stability.

Let's now consider a less general definition of stability, which is called linear stability. This is the most practical and easy to apply tool used in daily research. Linear stability analysis can also be readily applied for N-dimensional phase space. This is unlike the method of velocity plots, which is a good tool for N=1,2, and 3.

General equations for velocity, $\dot{x}$, and for equilibria, $x = a$, in 1D phase-space are

$$\dot{x} = f(x), \quad f(a) = 0. \tag{2.12}$$

Well posed nature inspired problems usually comply with an assumption that $f(x)$ and df/dx are continuous functions around $x = a$.

Linear stability of $x = a$ is found by considering a small time-dependent deviation $\varepsilon(t)$,

$$x(t) = a + \varepsilon(t) \tag{2.13}$$

and applying a so called linearisation procedure

$$\dot{\varepsilon} = f(a+\varepsilon) = f(a) + f'\varepsilon + \cdots = f'\varepsilon + \ldots, \quad f' = \left.\frac{df}{dx}\right|_{x=a}. \tag{2.14}$$

Thus we reduced a nonlinear problem for x to an approximate but linear, and therefore always solvable, problem for ε:

$$\varepsilon(t) = \varepsilon(0)e^{f't} \tag{2.15}$$

Hence, $x = a$ is stable if $f' < 0$ and unstable if $f' > 0$.

The timescale $\tau \sim 1/f'$ characterizes the interval over which the dynamics remain approximately linear (quasi-linear), before nonlinear effects become significant.

Class exercises:

Apply linear stability analysis for $N = k$ *fixed point in the logistic model.*

Apply linear stability analysis for all fixed points of $\dot{x} = x^2 - 1$.

Linear stability fails to explain stability if $f' = 0$

Class exercises:

Apply velocity plots to work out stability of $x = 0$ *in equations:* $\dot{x} = x^2$, $\dot{x} = x^3$, $\dot{x} = -x^3$

2.4 Impossibility of oscillations in 1D phase space

In 1D phase space, velocity can not change its direction between the fixed points and the phase point can not pass through a fixed point, therefore, oscillations are not possible.

This changes in 2D and higher-dimensional phase spaces.

2.5 Potentials

There is another powerful approach to visualise dynamics and stability in 1D phase spaces, which is not however often generalisable beyond 1D.

$$\dot{x} = f(x), f(x) = -\frac{dU}{dx}. \tag{2.16}$$

Thus, fixed points are given by maxima and minima of $U(x)$

$$\frac{dU}{dx} = 0, x = a. \tag{2.17}$$

Phase points always move down potential, $dV/dt < 0$, therefore minima are stable and maxima are unstable equilibria. Indeed,

$$\frac{dU}{dt} = \frac{dU}{dx}\frac{dx}{dt} = -\left(\frac{dU}{dx}\right)^2 \tag{2.18}$$

Note, that U is not energy. U is not conserved, $dU/dt \neq 0$. **Class exercises**:

Bistability: Use linear stability analysis and potential plot to analyse stability of fixed points in $\dot{x} = x - x^3$.

2.6 Solving equations numerically

h is time step.

First order method:

Euler's method

$$\frac{x_{n+1} - x_n}{h} = f(x_n) \tag{2.19}$$

$$x_n = x(t = nh) \tag{2.20}$$

$$\boxed{x_{n+1} = hf(x_n) + x_n} \tag{2.21}$$

Second order methods:

Mid-point method.

Improvement of the derivative value in the Euler's formula -

$$f(x_n) \to f(x(t = nh + \tfrac{1}{2}h)) \tag{2.22}$$

$$\boxed{x_{n+1} = hf(x_{n+1/2}) + x_n} \tag{2.23}$$

$$x_{n+1/2} = \tfrac{1}{2}hf(x_n) + x_n \tag{2.24}$$

Average derivative method (Heun's method)

Improvement of the derivative value in the Euler's formula -

$$f(x_n) \to \tfrac{1}{2}(f(x_n) + f(\tilde{x}_{n+1})) \tag{2.25}$$

$$\boxed{x_{n+1} = h\tfrac{1}{2}(f(x_n) + f(\tilde{x}_{n+1})) + x_n} \tag{2.26}$$

$$\tilde{x}_{n+1} = hf(x_n) + x_n \tag{2.27}$$

3 BIFURCATIONS IN 1D PHASE SPACE

Given the triviality of the dynamics, what's important about one-dimensional systems? Answer: Dependence on parameters. The qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occur are called bifurcation points.

There is only a small number of bifurcation scenarios with fixed points. Majority of them are common between 1d and Nd phase spaces.

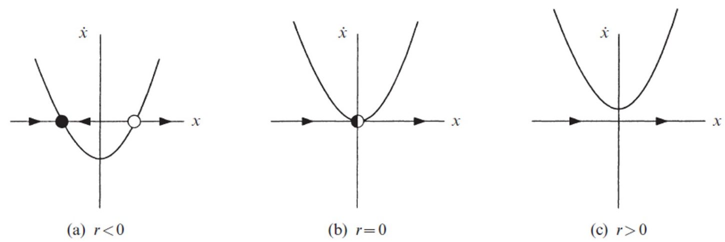
Throughout the discussions that follow, the letters r, h and p will be often used to denote bifurcation parameters in the equations. Letters a, b, c will be sometimes used for fixed points in phase space.

In the popular media bifurcation theory is sometimes called theory of catastrophes.

3.1 Creation and disappearance of fixed points: Saddle-node or blue-sky bifurcation

The saddle-node bifurcation, also known as a fold bifurcation, is the basic mechanism by which fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate. The prototypical example of a saddle-node bifurcation is given by the first-order system

$$\dot{x} = r + x^2 \tag{3.1}$$



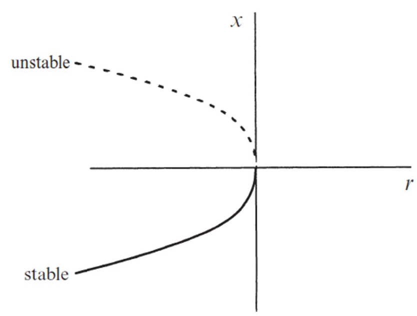


Figure 6: Saddle-node or fold bifurcation: Velocity plots and parameter dependence plot, called **bifurcation diagram**

where r is a parameter, which may be positive, negative, or zero. When r is negative, there are two fixed points, one stable and one unstable. When r is zero two fixed points coalesce to one, when $r < 0$ there no fixed points.

Class exercises:

Show that the first-order system $\dot{x} = r - x - e^{-x}$ undergoes a saddle-node bifurcation as r is varied, and find the value of r at the bifurcation point. Use the method described in the Strogatz book and then use an alternative approach based on the linear stability analysis.

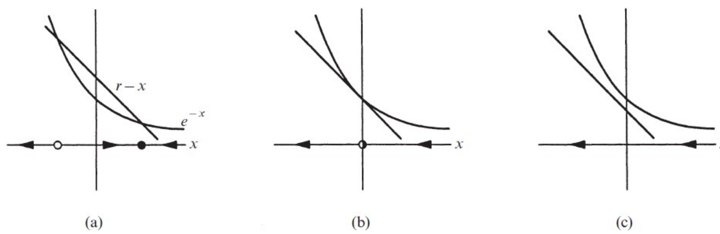


Figure 7: Graphical solution for the saddle-node bifurcation exercise

3.2 Normal forms

The examples $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$ are representative of all saddle-node bifurcations; that's why we called them "prototypical." The idea is that, close to a saddle-node bifurcation in Nd phase space, the dynamics typically look like is given by simple $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$ models.

Let's revisit equation $\dot{x} = r - x - e^{-x}$. Two solutions coalesce to $x = 0$ at $r = 1$. Let's assume that $r - 1 = R$ and x are small and apply Taylor expansion

$$\dot{x} = r - x - e^{-x} = R + 1 - x - (1 - x + \tfrac{1}{2}x^2 + \dots) = R - \tfrac{1}{2}x^2 + \dots \tag{3.2}$$

This has the same algebraic form as $\dot{x} = r - x^2$, and can be made to agree exactly by appropriate rescaling.

It's easy to understand why saddle-node bifurcations typically have this algebraic form. We just ask ourselves: how can two fixed points, i.e., two real-valued roots of $f(x, r)$, collide and disappear as a parameter r is varied? Graphically, fixed points occur where the graph of $f(x)$ intersects the x-axis. For a saddle-node bifurcation to be possible, we need two nearby roots of $f(x)$; this means $f(x)$ must look locally parabolic.

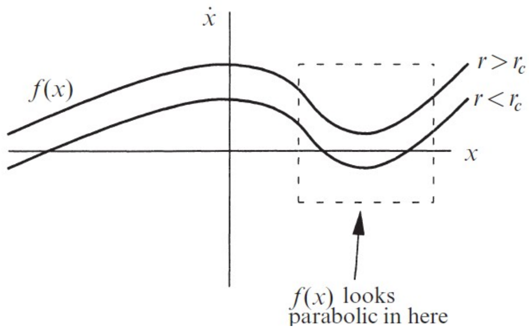


Figure 8: Plot for the normal form chapter. $r_c = R$. Saddle-node bifurcation.

Here is the more algebraic version of this arguments. Let's assume that two solution coalesce to $x = a$ at $r = R$

$$\begin{aligned}
\dot{x} &= f(x, r) \qquad (3.3)\\
&= f(a, R) + \frac{\partial f}{\partial x}\Big|_{x=a,r=R}(x-a) + \frac{\partial f}{\partial r}\Big|_{x=a,r=R}(r-R)\\
&+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x=a,r=R}(x-a)^2 + \ldots\\
&= \frac{\partial f}{\partial r}\Big|_{x=a,r=R}(r-R) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x=a,r=R}(x-a)^2 + \ldots
\end{aligned}$$

The above equation agrees with the form of our prototypical examples. We are assuming that the values of derivatives involved are not zeros, which is the typical case. If, for instance, 2nd derivative in x also happened to vanish at the bifurcation point, then one needs to proceed to the higher order terms in the Taylor expansion.

What we called "prototypical" examples is known as "normal forms", i.e., $\dot{x} = r + x^2$ is a normal forms for saddle-node bifurcations in all systems. Equivalently one can write $\dot{y} = \varepsilon - y^2$ ($x = -y$, $r = -\varepsilon$). This bifurcation is also called a blue-sky bifurcation, bcs two solutions appear out of nowhere (out of the clear blue sky)

3.3 Transcritical bifurcation

There are many physical systems where a fixed point exists for all values of a control parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, regardless of the value of the growth rate. However, such a fixed point may coalesce and exchange stability with another fixed point as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability. The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2 \tag{3.4}$$

Here, there are two fixed points $x = 0$ and $x = r$, which co-exist for all r and coalesce for $r = 0$.

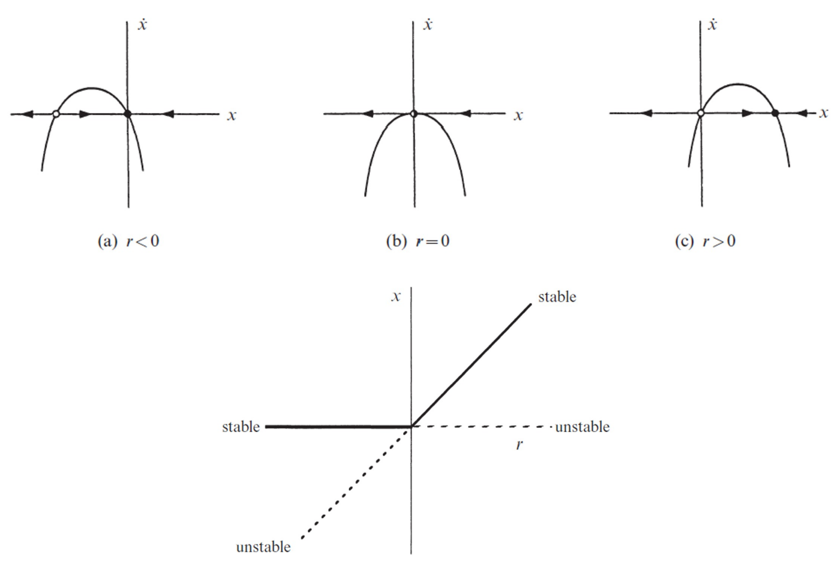


Figure 9: Transcritical bifurcation: Velocity plots and parameter dependence plot, called **bifurcation diagram**

Class exercises: *Transcritical bifurcation with two control parameters:*

Show that the first-order system

$$\dot{x} = x(1-x^2) - r(1-e^{-px}) \tag{3.5}$$

undergoes a transcritical bifurcation when the parameters p, r satisfy a certain equation, to be determined. This equation defines a bifurcation curve in the (p, r) parameter space. Then find an approximate formula for the fixed point that bifurcates from the trivial solution $x = 0$, assuming that the parameters are close to the bifurcation curve and determine parameter space ranges of stability of both solutions.

For small x, i.e., close to the $x = 0$ solution, we find

$$\begin{aligned} \dot{x} &= x(1-x^2) - r(1-e^{-px}) \\ &= x(1-x^2) - r(1-(1-px+\tfrac{1}{2}p^2x^2+\dots)) \\ &= x(1-pr) + \tfrac{1}{2}p^2x^2 + \dots \end{aligned} \tag{3.6}$$

Hence, 2nd fixed point is

$$x \approx \frac{pr-1}{p^2/2}, \tag{3.7}$$

transcritical bifurcation condition (two solutions coincide) is

$$p = \frac{1}{r} \tag{3.8}$$

and $x = 0$ solution is stable for $pr > 1$.

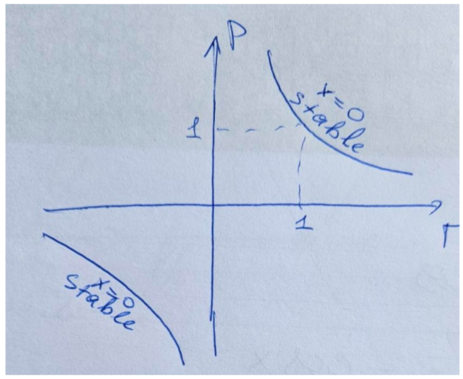


Figure 10: 2D parameter space with transcritical bifurcation lines for the exercise.

3.4 Pitchfork bifurcation

We turn now to a third kind of bifurcation, the so-called pitchfork bifurcation. This bifurcation is common in physical problems that have a symmetry. For example, many problems have a spatial symmetry between left and right, e.g., buckling problem. In such cases, fixed points tend to appear and disappear in symmetrical pairs. In the buckling example, the beam is stable in the vertical position if the load is small. In this case there is a stable fixed point corresponding to zero deflection. But if the load exceeds the buckling threshold, the beam may buckle to either the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born.

There are two very different types of pitchfork bifurcations-supercritical pitchfork bifurcation and subcritical pitchfork bifurcation.

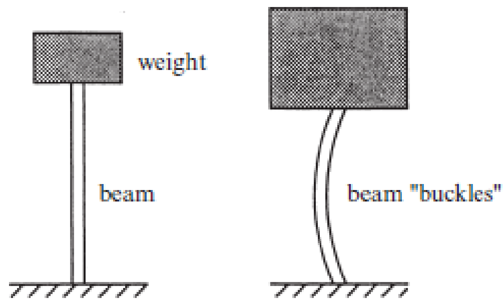


Figure 11: Buckling bifurcation has left-right symmetry

3.5 Supercritical pitchfork bifurcation

The normal form of the supercritical pitchfork is

$$\dot{x} = rx - x^3 \tag{3.9}$$

Note that this equation is invariant under the change of variables $x \to -x$. That is, if we replace x by $-x$ and then cancel the resulting minus signs on both sides of the equation, we get the same equation. This invariance is the mathematical expression of the left-right symmetry mentioned earlier.

When $r < 0$ the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since now solutions no longer decay exponentially in time — instead the decay is a much slower algebraic function of time. This lethargic decay is called critical slowing down in the physics literature. Finally, when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x = \pm\sqrt{r}$.

While analysing a practical problem with left-right symmetry and doing Taylor expansions for small x, one

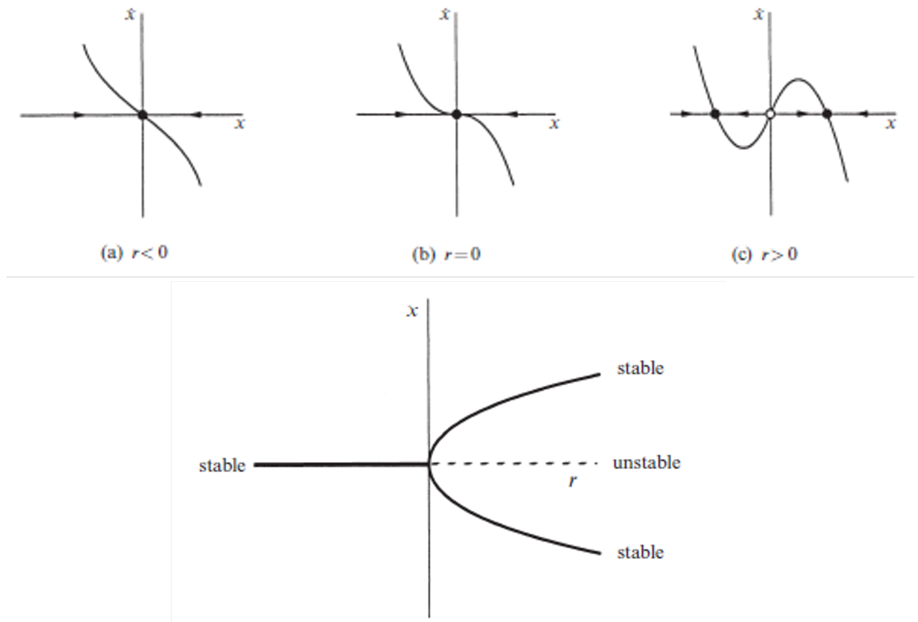


Figure 12: Supercritical pitchfork bifurcation: Velocity vs coordinate plots and solution dependence vs control parameter.

would usually derive an equation

$$\dot{x} = rx - \alpha x^3 \tag{3.10}$$

which is reducible to the normal form on scaling α away. But what if α can change its sign and become exactly zero? This is an indication that in the proximity of such a point a sought equation becomes

$$\dot{x} = rx - \alpha x^3 + \beta x^5, \tag{3.11}$$

which still complies with the $x \to -x$ symmetry. Keeping this observation in mind, we proceed by considering the subcritical case.

3.6 Laser threshold

We consider a laser, which consists of a collection of special "laser-active" atoms bounded by partially reflecting mirrors at either end. An external energy source is used to excite or "pump" the atoms out of their ground state with wave function ψ_g and energy E_g to an excited state with ψ_e and energy E_e. Transition frequency is then $\omega_t = (E_e - E_g)/\hbar$. Population inversion is defined as $D \sim |\psi_e|^2 - |\psi_g|^2$ and coherence (also called polarization) is defined as $P \sim \psi_e \psi_g^*$.

Each atom can be thought of as a little antenna radiating energy. When the pumping is relatively weak, the laser acts just like an ordinary lamp: the excited atoms oscillate independently of one another and emit randomly phased light waves. Now suppose we increase the strength of the pumping. At first nothing different happens, but then suddenly, when the pump strength exceeds a certain threshold, the atoms begin to oscillate in phase—the lamp has turned into a laser. Now the trillions of little antennas act like one giant antenna and produce a beam of radiation that is much more coherent and intense than that produced below the laser threshold.

Assuming that laser operates at the atomic transition frequency coinciding (which is not always the case) and using some simplifications of Maxwell and Schroedinger equations one can derive a system of equations for the electric field amplitude E generated by the laser, population inversion, D, and atomic polarization, P, (Strogatz, problems 3.3.2. and 9.1.4. Note that our course applies

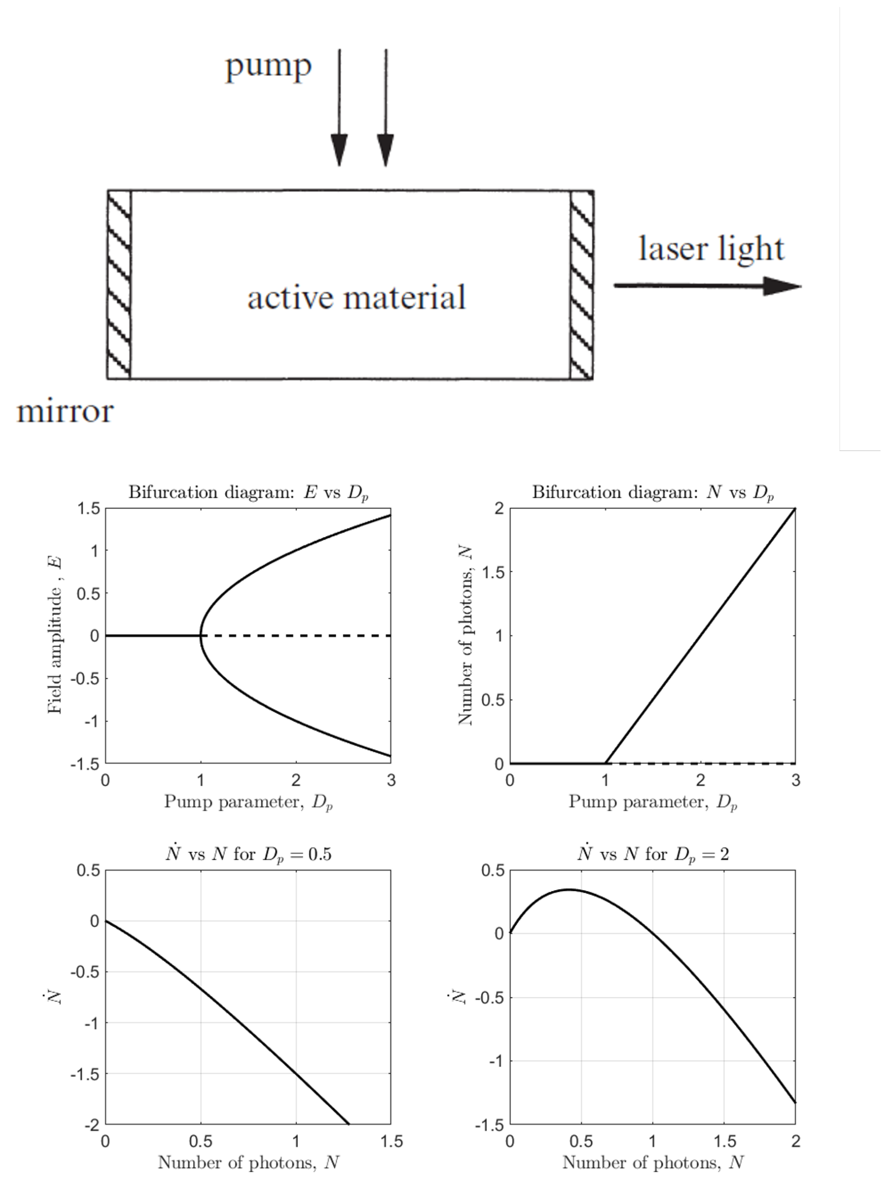


Figure 13: Laser threshold

slightly different arrangement of control parameters in the equation for D.)

$$\frac{1}{\kappa}\dot{E} = P - E \tag{3.12}$$

$$\frac{1}{\gamma_d}\dot{D} = D_p - D - EP \tag{3.13}$$

$$\frac{1}{\gamma_p}\dot{P} = ED - P \tag{3.14}$$

where κ is the decay rate in the laser cavity due to beam transmission through the mirrors, γ_p and γ_d are decay rates of the atomic polarization and population inversion, respectively, and D_p is the population inversion created by pumping without the laser action. All parameters here are positive.

The laser model is invariant relative to the following symmetry transformation,

$$(E, D, P) \to (-E, D, -P). \tag{3.15}$$

In the simplest case $\gamma_{p,d} \gg \kappa$; then P and D relax rapidly to steady values, and hence we can assume $\dot{P} = 0$ and $\dot{D} = 0$.

Then one can solve equations for D and P,

$$D = \frac{D_p}{1 + E^2} \tag{3.16}$$

$$P = ED = \frac{ED_p}{1 + E^2} \tag{3.17}$$

and substitute the above to the field equation,

$$\dot{E} = \kappa E \left(\frac{D_p}{1 + E^2} - 1 \right), \tag{3.18}$$

where $E \to -E$ symmetry is preserved, therefore, we anticipate pitchfork bifurcation.

There are three fixed points:
No lasing regime, $E = 0$, is stable for $D_p < 1$;
Two lasing regimes $E = \pm\sqrt{D_p - 1}$ are stable for $D_p > 1$.

It is a clear from the structure of solutions that we deal here with a supercritical pitchfork bifurcation.

Note 1: for plotting lasing solutions on the bifurcation digram it is convenient to write D_p vs E,

$$D_p = 1 + E^2, \tag{3.19}$$

rather than dealing with square roots.

Note 2: The electric field is given by $E \cos \omega_t t$, where E is real, but can be either positive or negative, corresponding to the phase flip by π. If resonator frequency, ω_r, is different from ω_t, model needs to be generalised by adding the detuning parameter, $\delta = \omega_r - \omega_t$, which makes E and P complex.

Note 3: Instead of using the field amplitude E, one can transform the reduced laser model to use the number of photons as an unknown,

$$N = E^2. \tag{3.20}$$

Hence

$$\dot{N} = 2\kappa N \left(\frac{D_p}{1 + N} - 1 \right), \tag{3.21}$$

Here, there are two fixed points $N = 0$, $N = D_p - 1$ and pitchfork bifurcation has transformed to transcritical one.

3.7 Subcritical pitchfork bifurcation

In the supercritical case, the cubic term is stabilizing $\dot{x} = rx - x^3$, which is seen from the fact that $x = 0$ is a stable point for $r = 0$. If instead the cubic term is destabilizing,

$$\dot{x} = rx + x^3 \tag{3.22}$$

then we have a subcritical pitchfork bifurcation.

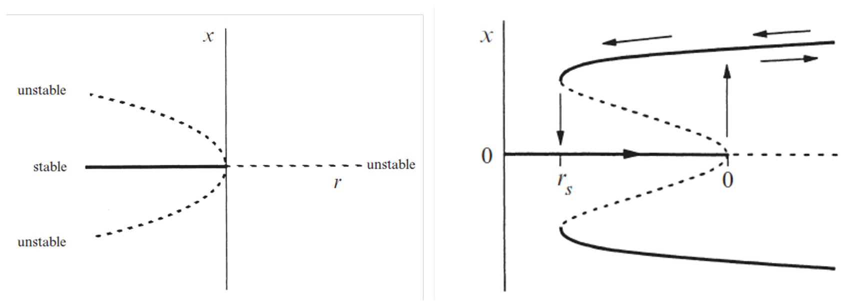


Figure 14: Subcritical pitchfork bifurcation: Solution dependence vs control parameter without and with stabilising quintic term.

The pitchfork is now inverted. The non-zero fixed points $x = \pm\sqrt{-r}$ are unstable, and exist only for $r < 0$, which motivates the term "subcritical." More importantly, the origin is stable for $r < 0$ and unstable for $r > 0$, as in the supercritical case, but now there is no stable solution for $r > 0$. In real physical systems, such an instability development above threshold is usually opposed by the stabilizing influence of higher-order terms. Assuming that the system is still symmetric under $x \to -x$, the first stabilizing term must be x^5.

Hence, the normal for subcritical pitchfork bifurcation is

$$\dot{x} = rx + x^3 - x^5, \tag{3.23}$$

To plot bifurcation diagram for non-trivial solutions, it is convenient to use $r(x)$,

$$r = x^4 - x^2. \tag{3.24}$$

Class Exercise: Find all fixed points of the normal form of subcritical pitchfork bifurcation. Workout an expression for $r = r_s$ where saddle-node bifurcation is taking place (see figure 12). Solution: $dr/dx = 4x^3 - 2x = 2x(2x^2 - 1) = 0$, $x_s = \pm 1/\sqrt{2}$, $r_s = -1/4$. Apply velocity method to determine stability of fixed points for $r < r_s$, $r = r_s$, $r_s < r < 0$, $r = 0$, $r > 0$.

The coexistence of different stable states allows for the possibility of jumps and hysteresis as r is varied. Suppose we start the system in the state $x = 0$, and then slowly increase r. Then the state remains at the origin until $r = 0$, when the origin loses stability. Now the slightest nudge will cause the state to jump to one of the large-amplitude branches. With further increases of r, the state moves out along the large-amplitude branch. If r is now decreased, the state remains on the large-amplitude branch, even when r is decreased below 0! We have to lower r even further (down past r_s) to get the state to jump back to the origin. This lack of reversibility as a parameter is varied is called hysteresis.

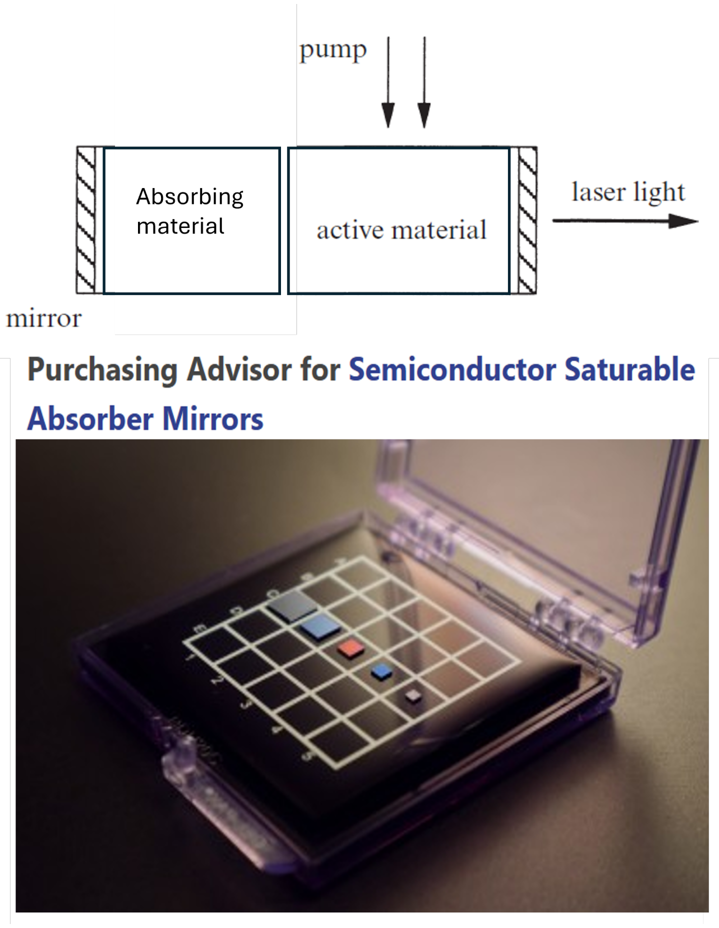


Figure 15: Laser with saturable absorber.

3.8 Laser with saturable (nonlinear) absorber

$$\frac{1}{\kappa}\dot{E} = P_1 + P_2 - E \tag{3.25}$$

$$\frac{1}{\gamma_{da}}\dot{D}_a = A - D_a - EP_a \tag{3.26}$$

$$\frac{1}{\gamma_{pa}}\dot{P}_a = ED_a - P_a \tag{3.27}$$

$$\frac{1}{\gamma_{db}}\dot{D}_b = -B - D_b - \alpha EP_b \tag{3.28}$$

$$\frac{1}{\gamma_{pb}}\dot{P}_b = ED_b - P_b \tag{3.29}$$

where nonlinear absorber is described by an additional pair of equations. Absorber is not pumped and therefore its population inversion is negative, see minus in front of B and plus in front of A. Proceeding as before with solving equations for $P_{a,b}$ and $D_{a,b}$, we find

$$P_a = ED_a = \frac{EA}{1+E^2} \tag{3.30}$$

$$P_b = ED_b = \frac{-EB}{1+\alpha E^2} \tag{3.31}$$

and substitute the above to the field equation,

$$\dot{E} = \kappa E \left(\frac{A}{1+E^2} - \frac{B}{1+\alpha E^2} - 1 \right). \tag{3.32}$$

One can see that of $\alpha = 1$ (saturation is the same for gain and absorption), then the bifurcation diagram is like we derived for a laser, assuming $D_p = A - B$. However, if $\alpha \neq 1$, then the nature of transition towards lasing regime can change from super to sub-critical.

To observe this transition, let's fix $\alpha > 1$ and plot E vs A for several gradually increasing values of B. This is conveniently achieved by noting that

$$A = (1+E^2) \left(\frac{B}{1+\alpha E^2} + 1 \right) \tag{3.33}$$

is valid for non-trivial fixed points.

To analyse this transition in more details, let's analyse bifurcation boundaries in the (A, B) parameter space. Let's switch again to the number of photons ($N > 0$) representation

$$\dot{N} = 2\kappa N \left(\frac{A}{1+N} - \frac{B}{1+\alpha N} - 1 \right). \tag{3.34}$$

It is clear that we have trivial fixed point:
No lasing regime, $N = 0$, which is stable for $A - B < 1$. Adding fractions inside the bracket, we find that non-trivial fixed points satisfy

$$N^2 + \frac{1}{\alpha}[B - \{\alpha(A-1) - 1\}]N + \frac{1}{\alpha}[B - (A-1)] = 0 \quad (3.35)$$

Only positive solutions for N have physical meaning. The above equation can have either one or two positive roots.

For a general equation $x^2 + px + q = 0$ to have two positive roots (subcritical bifurcation), three conditions must be satisfied: $p^2 > 4q$, $q > 0$, and $-p > 0$.

In our context,
1) $p = 0$ corresponds to pitchfork bifurcation (either super- or sub-critical)
2) $p^2 = 4q$ corresponds to saddle-node bifurcation.
3) $p = q = 0$ gives condition when super-critical bifurcation becomes sub-critical.

Let's use (A, B)-coordinate system as parameter space.

Conditions $q > 0$ and $-p > 0$ are expressed as
$B > B_1 = A - 1$ (pitchfork bifurcation) and
$B < B_2 = \alpha(A - 1) - 1$
and condition $p^2 > 4q$ (saddle-node bifurcation) is
$B > B_+$ and $B < B_-$, $B_\pm = \alpha(A+1) - 1 \pm 2\sqrt{\alpha A(\alpha - 1)}$

Extra questions:
(1) Sketch or plot using any available computational environment bifurcation diagrams when B is used as a bifurcation parameter for several fixed values of A and $\alpha = 3$?
(2) Find an expression A_{cr} vs α signalling transition

from supercritical to subcritical bifurcation diagram for B used as a control parameter?

(3) Find an expression B_{cr} vs α signalling transition from supercritical to subcritical bifurcation diagram for A used as a control parameter?

(4) Sketch $\dot{E}$ vs E for A, B and α within the hysteresis range and use your plot to illustrate stability of all fixed points.

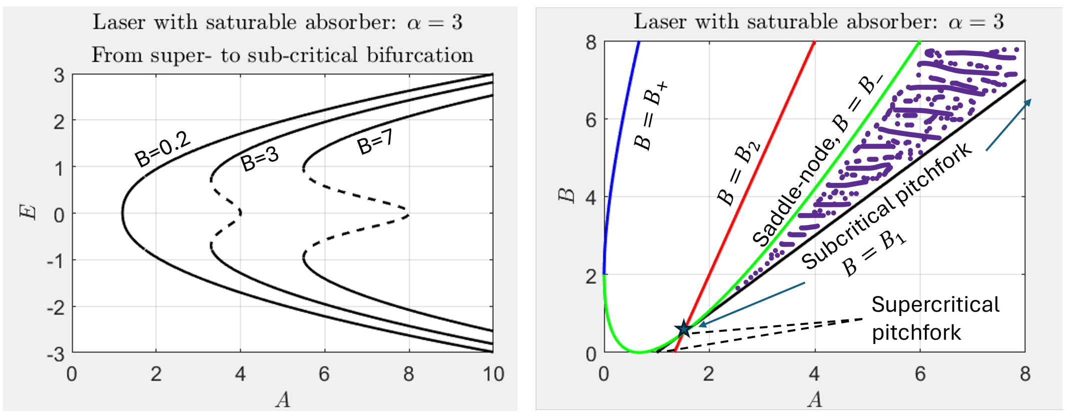


Figure 16: Bifurcations in a laser with saturable absorber. Dashed area in the right plot marks parameter range where four non-trivial lasing solutions coexist, corresponding to the hysteresis (bistability) range. Four solutions for E correspond to two solutions for $N = E^2$.

3.9 Imperfect bifurcations

In some real-world circumstances, the symmetry is only approximate— an imperfection leads to a slight difference between left and right. We now want to see what happens when such imperfections are present.

We consider a supercritical pitchfork bifurcation,

$$\dot{x} = h + rx - x^3, \tag{3.36}$$

where a small **imperfection parameter** h breaks $x \to -x$ symmetry.

We plot the graphs of $y = rx - x^3$ and $y = h$ on the same axes, and look for intersections, which occur at the fixed points. When $r < 0$, the cubic is monotonically decreasing, and so it intersects the horizontal line $y = -h$ in exactly one point. The more interesting case is $r > 0$; then one, two, or three intersections are possible, depending on the value of h. Thus we have upto 3 fixed points, which are slightly shifted three fixed points known explicitly for $h = 0$.

The critical case occurs when the horizontal line is just tangent to either the local minimum or maximum of the cubic; then we have a saddle-node bifurcation. Demonstrate that the saddle node bifurcations happen for

$$h = h_c = \frac{2r}{3}\sqrt{\frac{r}{3}} \tag{3.37}$$

Thus, if $r > 0$, then we have one fixed point $|h| > h_c$ and three fixed points for $|h| < h_c$, If $r < 0$, we always have one fixed point. Two saddle-node bifurcation curves meet tangentially and terminate at a point $h = r = 0$, which is called a cusp point.

Sweeping parameter h back and forward across the bistability range makes the system to follow the hysteresis loop. Note: I am not associating this with some sort of catastrophic scenario as is done in Strogatz book, but feel free to explore this further and shape your own opinion.

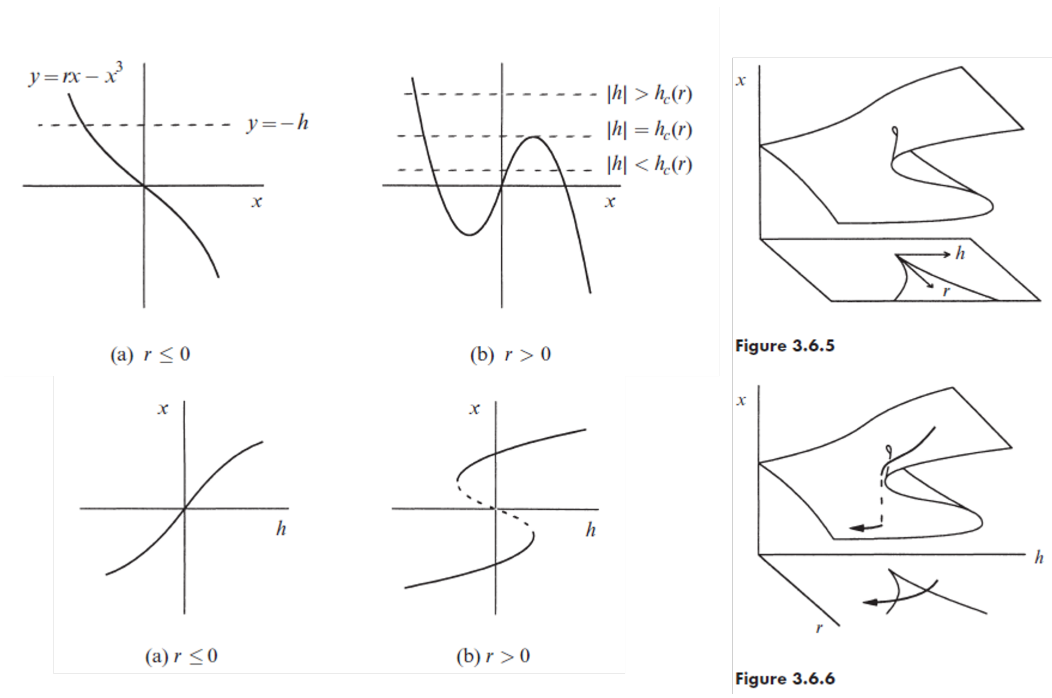


Figure 17: Imperfect pitchfork. Two plots in top left: Solving for fixed points. Two plots in bottom left: Solutions vs h. Two plots in right column show solutions vs h and r.

3.10 Insect outbreak

For a biological example of bistability and hysteresis, we turn now to a model for the sudden outbreak of an insect called the spruce budworm. This insect is a serious pest in eastern Canada, where it attacks the leaves of the

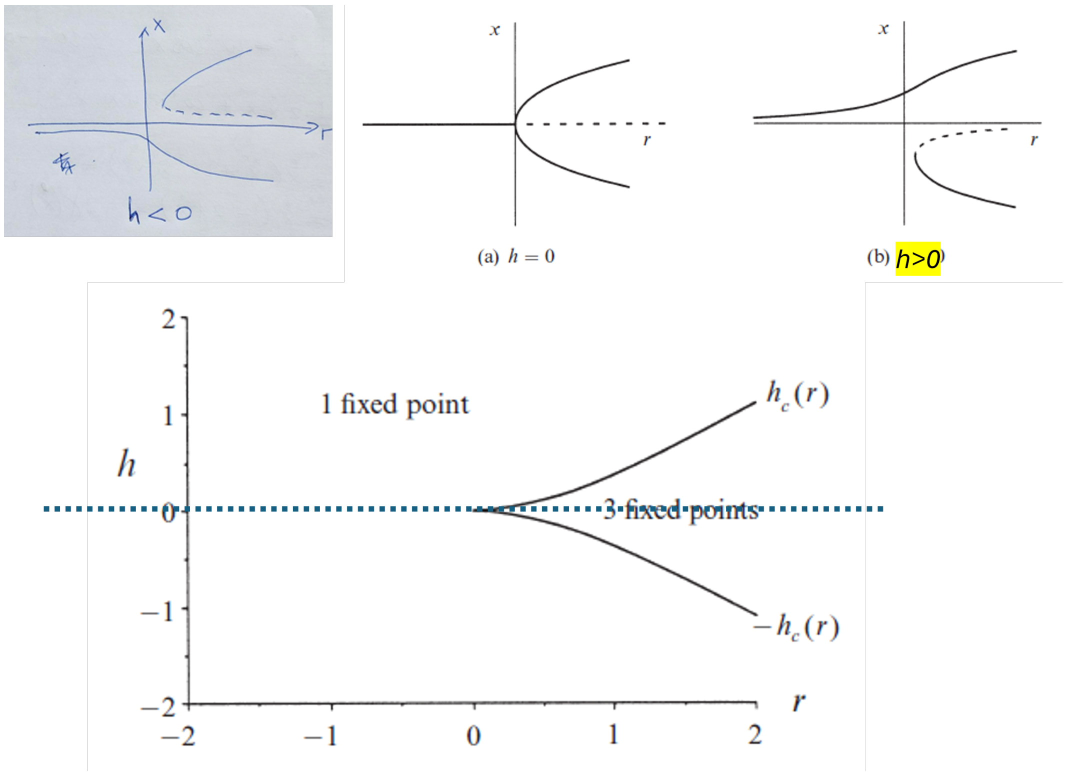


Figure 18: Imperfect pitchfork. Top: Solutions as a function of r, explaining why it is called imperfect pitchfork. Bottom: Bifurcation diagram in (h, r)-parameter space.

balsam fir tree. When an outbreak occurs, the budworms can defoliate and kill most of the fir trees in the forest in about four years.

In the absence of predators, the budworm population $x(t) > 0$ is assumed to grow logistically with growth rate r and carrying capacity k. The carrying capacity depends on the amount of foliage left on the trees, and so it is a slowly drifting parameter; at this stage we treat it as fixed. The term $p(x)$ represents the death rate due to predation, chiefly by birds.

$$\dot{x} = rx(1-x/k)-p(x),\ p(x) = \frac{x^2}{1+x^2}, x > 0, r > 0, k > 0 \tag{3.38}$$

There is almost no predation, $p \to 0$, when budworms are scarce, $x \to 0$; the birds seek food elsewhere. However, once the budworms population exceeds a certain critical value, $x = 1$ (we use scaled model), the predation p turns on sharply and then saturates (the birds are eating as fast as they can), $p \to 1$ for $x \to \infty$. Practically, this means that the fur forest dies.

$$\dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2} \tag{3.39}$$

Trivial fixed point $x = 0$ exists unconditionally, i.e., for any values of parameters.

Non-trivial fixed points are solutions of

$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2} \tag{3.40}$$

We solve it graphically by looking for intersections of

$$y_1 = r\left(1 - \frac{x}{k}\right), \quad y_2 = \frac{x}{1+x^2} \tag{3.41}$$

As one can see there could be 1 or three non-trivial fixed points

Condition of saddle-node bifurcation (appearance of fixed points b and c) is that y_1 and y_2 intersect tangentially, i.e.,

$$y_1 = y_2; \quad r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2} \tag{3.42}$$

$$y_1' = y_2'; \quad -\frac{r}{k} = \frac{(1+x^2) - 2x^2}{(1+x^2)^2} \tag{3.43}$$

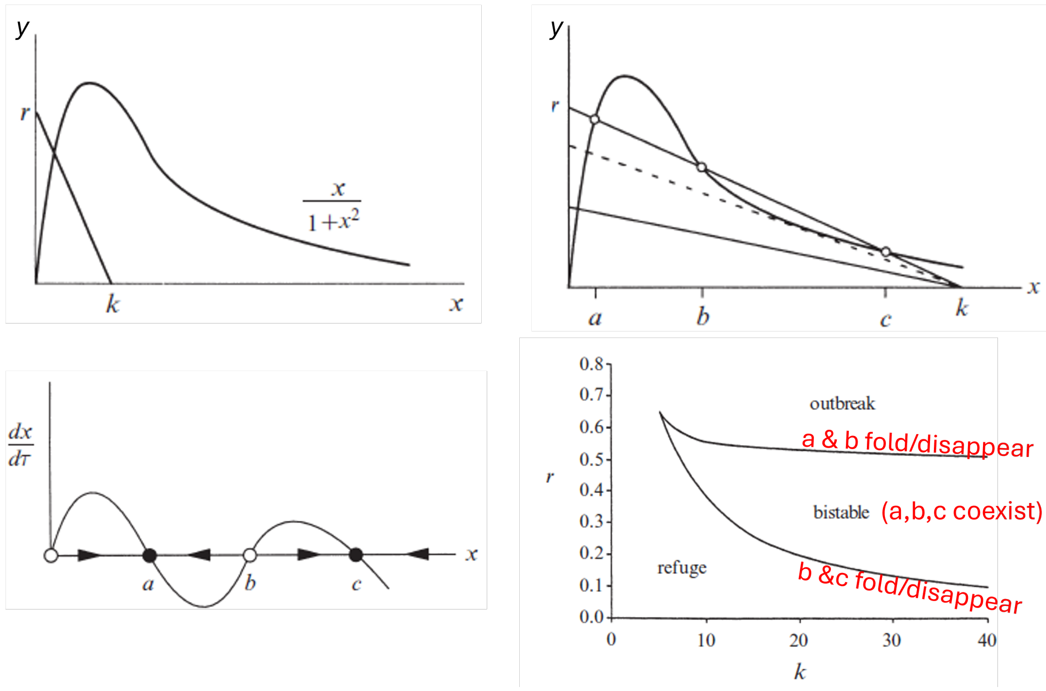


Figure 19: Insect outbreak: fixed points and stability.

Combining 2 conditions (taking r/k from the latter and substituting to the former) yields

$$r = \frac{2x^3}{(1+x^2)^2}, \quad k = \frac{2x^3}{x^2-1}, \quad x > 1 \tag{3.44}$$

Building a 3 column matrix of x, r, k values for $x > 1$, we can plot r vs k corresponding to the saddle-node bifurcations

Hence a is stable, b is unstable, and c is stable, for r and k in the range corresponding to three positive fixed points. The smaller stable fixed point a is called the refuge level of the budworm population, while the larger stable point c is the outbreak level. From the point of view of pest control, one would like to keep the population at a and away from c. The fate of the system is determined by the initial condition x_0; an outbreak occurs if and only if $x_0 > b$. In this sense the unstable equilibrium b plays the role of a threshold.

An outbreak can also be triggered by a saddle-node bifurcation. If the parameters r and k drift in such a way that the fixed point a disappears, then the population will jump suddenly to the outbreak level c. The situation is made worse by the hysteresis effect—even if the parameters are restored to their values before the outbreak, the population will not drop back to the refuge level.

3.11 Summary of normal forms

Saddle-node (blue-sky):

$$\dot{x} = r - x^2 \tag{3.45}$$

Transcritical:

$$\dot{x} = rx - x^2 \tag{3.46}$$

Supercritical pitchfork:

$$\dot{x} = rx - x^3 \tag{3.47}$$

Subcritical pitchfork:

$$\dot{x} = rx + x^3 - x^5 \tag{3.48}$$

4 DYNAMICS ON A CIRCLE

So far we've concentrated on the equation $\dot{x} = f(x)$, which we visualized as a vector field on the line. Now it's time to consider a new kind of differential equation and its corresponding phase space. This equation,

$$\dot{\theta} = f(\theta), \quad f(\theta) = f(\theta + 2\pi) \tag{4.1}$$

corresponds to a vector field on the circle. Here θ is a point on the circle and $\dot{\theta}$ is the angular velocity. Like the line, the circle is one-dimensional, but it has an important new property: by flowing in one direction, a particle can eventually return to its starting place. Thus periodic solutions become possible for the first time in this book! To put it another way, vector fields on the circle provide the most basic model of systems that can oscillate.

4.1 Uniform oscillator

A point on a circle can be described by its *phase* (or equivalently, an angle). For example, in the case of a pendulum, the motion may be expressed in terms of an amplitude a and a phase θ:

$$x = a\cos\theta, \qquad y = a\sin\theta.$$

In this section, we will assume that the dependence of a on time is negligible, while θ varies significantly with time, but all its variations can be approximately captured by a first order differential equation.

Then the simplest (uniform) oscillator of all is one in which the rate of change of θ is a time-independent constant, i.e., angular velocity is a constant:

$$\dot{\theta} = \omega \tag{4.2}$$

The solution is

$$\theta = \omega t + \theta_0 \tag{4.3}$$

, which corresponds to uniform motion around the circle at an angular frequency ω and initial phase θ_0. Period of this solution (time corresponding to the phase change from 0 to 2π) is

$$T = \int_0^T dt = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \frac{2\pi}{\omega} \tag{4.4}$$

Notice that we have said nothing about the amplitude of the oscillation. There really is no amplitude variable in our system. If we had an amplitude as well as a phase variable, we'd be in a two-dimensional phase space; this situation is more complicated and will be discussed later in the course.

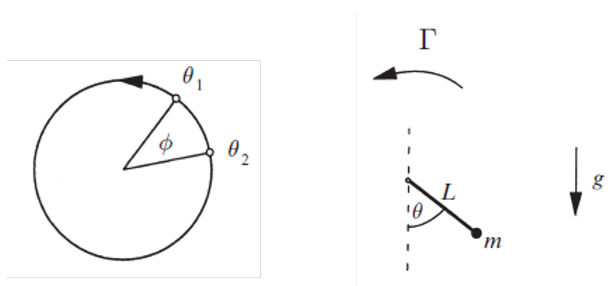


Figure 20: Left: Relative phase=difference between two phases of two independent oscillations. Right: Driven overdamped pendulum

Exercise: Two independent uniform oscillators: Beat phenomenon

Two joggers, Speedy and Pokey, are running at a steady pace around a circular track. It takes Speedy T_1 seconds to run once around the track, whereas it takes Pokey $T_2 > T_1$ seconds. Of course, Speedy will periodically overtake Pokey. Find this period, T. Periodic coincidence of two phases/angles in this examples is called - beat phenomenon. Introduce a relative phase, i.e., angle difference, $\phi = \theta_1 - \theta_2$, and show that

$$T = \frac{1}{T_1^{-1} - T_2^{-1}} \tag{4.5}$$

4.2 Non-uniform oscillator 1: Driven overdamped pendulum

2nd law of Newton for a pendulum driven by an applied torque Γ is

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma \tag{4.6}$$

$$\frac{L}{g}\ddot{\theta} + \frac{b}{mgL}\dot{\theta} + \sin\theta = \frac{\Gamma}{mgL} \tag{4.7}$$

$$\frac{1}{\omega_0^2}\ddot{\theta} + \frac{1}{\kappa}\dot{\theta} + \sin\theta = \mu \tag{4.8}$$

Here $\frac{1}{\kappa} = b/(mL^2)$ is the characteristic damping time, ω_0 is natural frequency of small oscillations ($\sin\theta \approx \theta$) and μ is the dimensionless driving parameter.

If damping rate κ is relatively small compared to frequency, then

$$\boxed{\frac{1}{\kappa}\dot{\theta} = \mu - \sin\theta}, 0 \leq \theta \leq 2\pi \tag{4.9}$$

$\boldsymbol{\mu > 1}$ - there are no fixed points. Driving is sufficiently strong to overcome friction, so that pendulum makes full 2π loops. Its motion is however, non-uniform, i.e., angular velocity is changing as it loops.

Period of oscillations for $\mu > 1$ is

$$T = \int_0^T dt = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \frac{1}{\kappa}\int_0^{2\pi} \frac{d\theta}{\mu - \sin\theta} = \frac{2\pi}{\kappa\sqrt{\mu^2 - 1}} \tag{4.10}$$

$\boldsymbol{0 < \mu < 1}$ then there are two fixed points.

$$\sin\theta = \mu, \cos\theta = \pm\sqrt{1 - \mu^2} \tag{4.11}$$

Intuitively it is clear that then one with $\theta > \pi/2$ (cos is negative) is unstable and the one with $\theta < \pi/2$ (cos is positive) is stable.

$\boldsymbol{\mu = 0}$ (no driving) still there are two fixed points: $\theta = 0$ (stable) and $\theta = \pi$ (unstable).

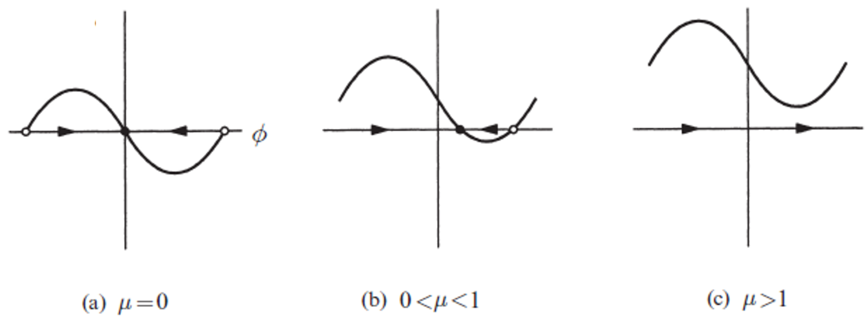


Figure 21: $\dot{\theta}$ vs θ for overdamped pendulum

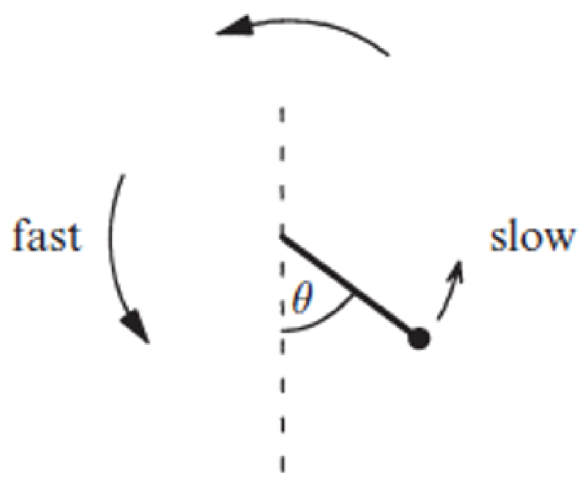


Figure 22: Bottleneck effect

$\boldsymbol{\mu = 1}$ there is one unstable (semi-stable) fixed point, $\theta = \pi/2$. Driving exactly balances friction and pendulum tends to freeze horizontally when approaching equilibrium from below. It takes long time for a pendulum to approach such a semistable situation. This phenomenon also persists when saddle-node bifurcation still has not happened but μ is close to 1 (bottleneck effect), so that velocity of the pendulum becomes near zero when it approaches and slowly passes $\theta = \pi/2$.

4.3 Non-uniform oscillator 2: Fireflies and entrainment

In some parts of southeast Asia, thousands of male fireflies gather in trees at night and flash on and off in unison. How does the synchrony occur? Certainly the fireflies don't start out synchronized; they arrive in the trees at dusk, and the synchrony builds up gradually as the night goes on. The key is that the fireflies influence each other: When one firefly sees the flash of another, it slows down or speeds up so as to flash more nearly in phase on the next cycle.

Hanson (1978) studied this effect experimentally, by periodically flashing a light at a firefly and watching it try to synchronize. For a range of periods close to the firefly's natural period (about 0.9 sec), the firefly was able to match its frequency to the periodic stimulus. In this case, one says that the firefly had been entrained by the stimulus. However, if the stimulus was too fast or too slow, the firefly could not keep up and entrainment was lost—then a kind of beat phenomenon occurred.

We assume that the stimulus lamp has phase Θ and flashes with frequency Ω and that the firefly phase is θ and its natural frequency is ω, so that when two do not interact the system is

$$\dot{\Theta} = \Omega \tag{4.12}$$

$$\dot{\theta} = \omega \tag{4.13}$$

When a firefly phase (moment of flash) is behind the lamp flash, i.e., $\Theta > \theta$, firefly likes to speed up and bring the next flash forward. When a firefly phase (moment of flash) is ahead the lamp flash, i.e., $\Theta < \theta$, firefly likes to slow down.

One could think to modify the firefly equation as

$$\dot{\theta} = \omega + A(\Theta - \theta), \quad A > 0 \tag{4.14}$$

but one should also account for periodicity, so that the model becomes

$$\dot{\Theta} = \Omega \tag{4.15}$$

$$\dot{\theta} = \omega + A \sin(\Theta - \theta) \tag{4.16}$$

To solve it we introduce the relative phase $\phi = \Theta - \theta$

$$\dot{\phi} = (\Omega - \omega) - A \sin \phi \tag{4.17}$$

Dividing by A we find nearly same model as used in the previous chapter, so that the stable fixed points corresponding to the constant time-independent delay between the stimulus and firefly flashes exist for

$$-A < \Omega - \omega < A. \tag{4.18}$$

Since ω and A are the intrinsic parameter of fireflies, it is convenient to rewrite the above condition in terms of our only control parameter Ω,

$$\Omega_- < \Omega < \Omega_+, \quad \Omega_\pm = \omega \pm A. \tag{4.19}$$

We note that solution for the flashing lamp equation remains

$$\Theta = \Omega t \tag{4.20}$$

Hence a stationary value of ϕ_* means that

$$\theta = \Omega t - \phi_* \tag{4.21}$$

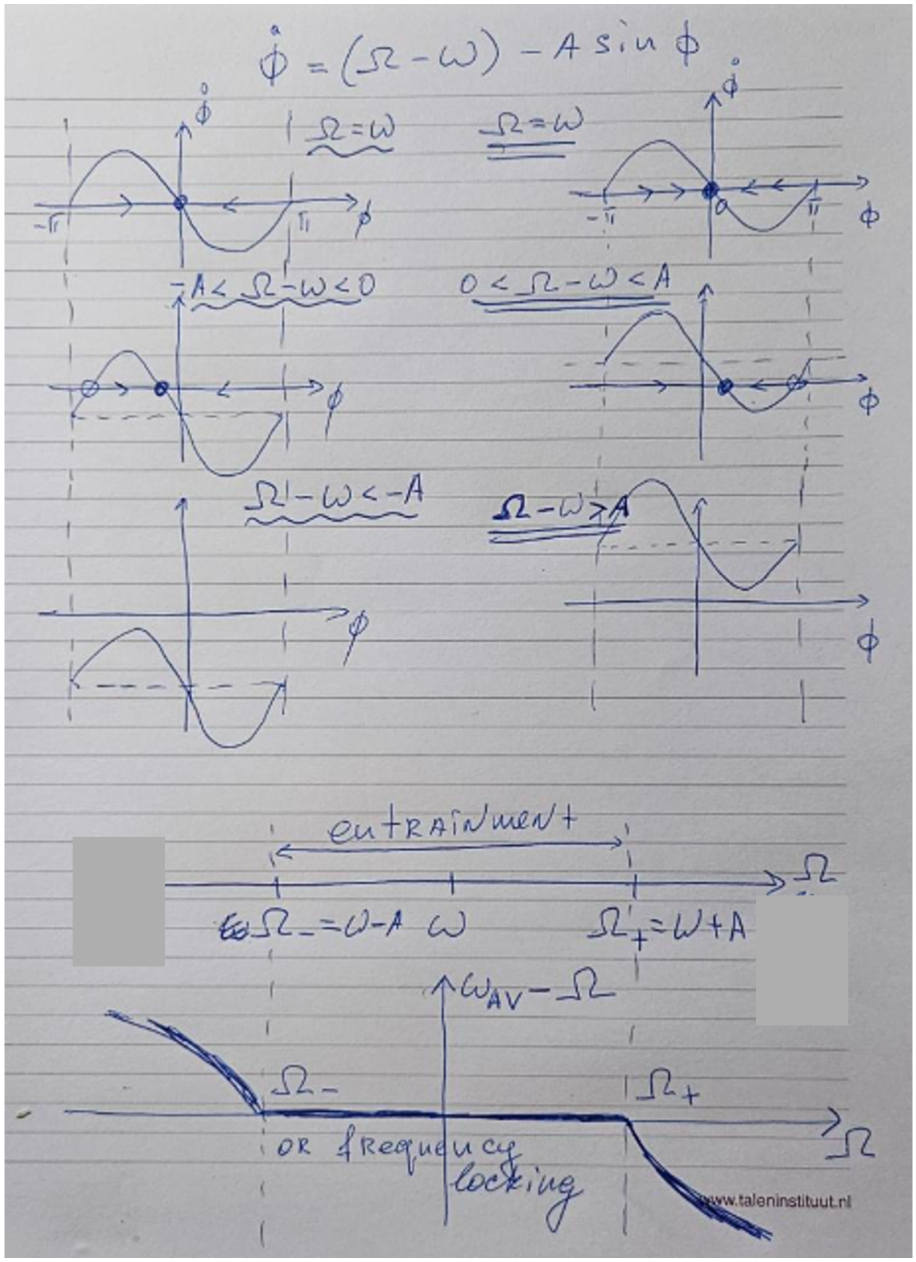


Figure 23: Inside the entrainment (frequency locking) range there is a stable fixed point for ϕ and hence fireflies oscillate with the same frequency as the driving lamp.

If Ω goes outside the entrainment range, then phase difference is growing ($\Omega > \Omega_+$) or decreasing ($\Omega < \Omega_-$) continuously. In these regimes the firefly phase is

$$\theta = \Omega t - \phi(t) \tag{4.22}$$

Average frequency of flashing is

$$\mathbf{\Omega > \Omega_+ :} \quad \omega_{av} = \Omega - \frac{1}{T}\int_0^T \dot{\phi} dt = \Omega - \frac{1}{T}\int_0^{2\pi} d\phi = \Omega - \frac{2\pi}{T} \tag{4.23}$$

(if $\dot{\phi} > 0$ for all ϕ, it means that ϕ is increasing for t going from 0 to T)

$$\mathbf{\Omega < \Omega_- :} \quad \omega_{av} = \Omega - \frac{1}{T}\int_0^T \dot{\phi} dt = \Omega - \frac{1}{T}\int_{2\pi}^{0} d\phi = \Omega + \frac{2\pi}{T} \tag{4.24}$$

(if $\dot{\phi} < 0$ for all ϕ, it means that ϕ is decreasing for t going from 0 to T)

Here, the period is found as before

$$\begin{aligned} T &= \int_0^{2\pi} \frac{d\phi}{\Omega - \omega - A\sin\phi} = \frac{2\pi}{\sqrt{(\Omega-\omega)^2 - A^2}} \\ &= \frac{2\pi}{\sqrt{(\Omega-\Omega_+)(\Omega-\Omega_-)}}, \ \ \Omega_\pm = \omega \pm A \end{aligned} \tag{4.25}$$

Hence average frequency of firefly flashing is

$$\mathbf{\Omega > \Omega_+ :} \quad \omega_{av} = \Omega - \sqrt{(\Omega-\Omega_+)(\Omega-\Omega_-)} \tag{4.26}$$

$$\mathbf{\Omega < \Omega_- :} \quad \omega_{av} = \Omega + \sqrt{(\Omega-\Omega_+)(\Omega-\Omega_-)} \tag{4.27}$$

Thus in the former case, the lamp is too fast for fireflies to catch with. While in the former case, the fireflies can not go slow enough. In either case, the mismatch is due to limitations of the firefly's internal bio-clock.