Worksheet 3 – Random Numbers and Monte Carlo

Note: *If you want to make things easier for yourself for questions 1-2, either write some code to do the sums, or make a worksheet in excel. It can all be done with a calculator, but it will be a bit more laborious.*

1. Estimate the integral
$$I = \int_0^1 \frac{1}{1+x^2}dx$$
(true answer $I = \frac{\pi}{4}$), using Monte Carlo methods. Use the following sequence of random numbers in the interval $(0, 1)$:

   0.983 0.264 0.141 0.939 0.189 0.891 0.158 0.559 0.994 0.569

   (*If you prefer, you can use your own sequency of 10 random numbers instead*).

   Compare the answers using the above sequence directly, then using the weight function $w_1(x) = \frac{1}{3}(4-2x)$. Use a graph of the integrant function $f(x) = (1+x^2)^{-1}$ and $w_1(x)$ to explain your answers. Add to your graph $w_2(x) = \frac{6}{5}(1 - \frac{1}{2}x^2)$. Would this be a better weight function in principle? What are its drawbacks in practice?

2. Use the sequence of random numbers in Q1 to estimate the integral
$$I = \int_0^{\frac{\pi}{2}} \sin(x)dx$$
using MC integration. Use an appropriate linear weight function $w(x)$ to re-evaluate the integral using importance sampling. Compare the estimated standard error of your answers with the real error (which you can calculate as the integral can be done exactly, and is equal to 1).

3. A set of uniform deviates $\{x_i\}$ between 0 and 1 is transformed via the equation $y_i = \tan[\pi(x_i - \frac{1}{2})]$. Find the probability density function $p(y)$ generated by this procedure. What is the domain of y in this case?

4. The "fundamental transformation law of probabilities" for 2 variables is
$$p(y_1, y_2)dy_1dy_2 = p(x_1, x_2)\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} dy_1dy_2,$$
where the term in $|\quad|$ is the "Jacobian determinant" (it should be taken by modulus to give positive probability distributions). Show that for uniform deviates x_1 and x_2, and transformations
$$y_1 = \sqrt{-2\ln x_1}\cos(2\pi x_2); \qquad y_2 = \sqrt{-2\ln x_1}\sin(2\pi x_2),$$
each of the y_i $(i = 1, 2)$ is independently distributed according to a Gaussian distribution, $p(y_i) = \frac{1}{\sqrt{2\pi}}\exp(-y_i^2/2)$.

   [Hint: To re-arrange the transformations to be in a form $x(y)$ instead of $y(x)$, consider combinations $y_1^2 + y_2^2$ and y_2/y_1. Once you obtain the formulas for $x(y)$, proceed to differentiating and calculating the Jacobian.]