Problem 5

(a) Using the same approach as you did in Problem 1, construct a Boolean expression ϕX using variables vi, where the expression is True if and only if the assignment of truth values to the variables represents a triangle-free vertex cover of X. The relationship between the number of clauses in your expression and the number of edges and triangles in X should match the expression that you gave in Problem 3 (a).

(v1 | v2) & (v1 | v3) & (v2 | v3) & (v4 | v5) & (v5 | v6) & (v4 | v7) & (v6 | v7) & (~v1 | ~v2 | ~v3)

(b) Recall the process described at the top of page 5 of taking any graph G and constructing a Boolean expression ϕG from that graph. In this question, we look at a restricted version of this problem where only 2-regular graphs are considered.

For any 2-regular graph G, let ϕG be a Boolean expression with n variables that is constructed from G, as described at the top of page 5.

Prove by induction that ϕG has at most 4n/3 clauses.

Base Case

We need to prove when ϕG has at most 4n/3 clauses when n = 3 for a 2-regular graph that has the minimum number of 3 vertices for a 2-regular graph to form.

When n = 3, ϕG = (v1 | v2) & (v1 | v3) & (v2 | v3) & (~v1 | ~v2 | ~v3)

ϕG has 4 clauses which is accepted by the requirement 4n/3 = 4(3)/3 = 4 clauses.

Hence, ϕG has at most 4n/3 clauses when n = 3 for a 2-regular graph that has the minimum number of vertices.

Inductive Hypothesis

Assume that ϕG has at most 4n/3 clauses when n = k for any 2-regular graph.

Inductive Step

We need to prove that ϕG has at most 4n/3 clauses when n = k+3.

Consider a 2-regular graph G with k+3 vertices, adding 3 vertices to a 2-regular graph would result in the addition of another triangle. This is because in a 2-regular graph, adding 3 vertices while maintaining degree 2 for each vertex inevitably forms a triangle. The new clauses introduced by the addition of 3 vertices include 3 clauses for the edges connecting the 3 new vertices and 1 clause to ensure the triangle formed by these 3 vertices is excluded from the vertex cover. Therefore, if the original graph ϕG has 4k/3 clause, 4 new clauses are added to that to become 4k/3+4 clauses for every 3 vertices added.

For n = k+3, 4n/3 = (4k+12)/3 = 4k/3+4

ϕG still adheres to the upper bound of 4n/3 after the addition of vertices. Hence, ϕG has at most 4n/3 clauses when n = k+3 for a 2-regular graph.

Conclusion

By the proof of mathematical induction, the Boolean expression ϕG for any 2-regular graph G with n vertices has at most 4n/3 clauses.