Problem 8

(a) Prove, using the Pumping Lemma for Regular Languages, that there is no Universal Regular Expression

Statement:

A Universal Regular Expression, URE would be a single regular expression that could describe every any regular expression R and its input x such that U matches R$x. Assume that there exists a URE, to demonstrate that such a regular expression does not exist, we will use the Pumping Lemma for Regular Languages which provides necessary properties of all regular languages.

Proof:

1. According to the Pumping Lemma for Regular Languages, for any regular language L, there exists a constant p such that any string s in L with $|s| \geq p$ can be split into three parts to be $s = xyz$ that satisfies the following conditions:

    1. $|xy| \leq p$
    2. $|y| > 0$
    3. The string $xy^i z$ in L for all $i \geq 0$

2. Consider a regular expression R over the alphabet {a, b}. For example, R can be $R = a^*b$. This regular expression matches strings such as "aab" but not "aabb". Based on the assumption, U is able to match both R and the strings in L(R).

3. We take the strings s = “$a^p b$” where p is the pumping length from the Pumping Lemma. Then, we decompose s into three parts to be $s = xyz$, where $|xy| \leq p$, $|y| > 0$ and the string xy^iz in L for all $i \geq 0$. However, if we pump y, as in, repeat y, we generate strings like $a^{(p+1)} b$ which are not in the language of R because R only allows exactly one "b". This contradicts the assumption that L(R) is regular and follows the Pumping Lemma. Thus, it is proven that no such Universal Regular Expression exists.

Conclusion:

There is no Universal Regular Expression because it contradicts the Pumping Lemma for Regular Languages.

(b) Prove, using the Pumping Lemma for Context-Free Languages, that there is no Universal Context-Free Grammar

Statement:

A Universal Context-Free Grammar, UCFG would be a single context-free grammar U which can generate any context-free grammar G and a string x such that U generates G$x. Assume that there exists a UCFG, to show that such a grammar cannot exist, we will use the Pumping Lemma for Context-Free Languages which provides necessary conditions for a language to be context-free.

Proof:

1. According to the Pumping Lemma for Context-Free Languages for any context-free language L , there exists a constant p such that any strings in L with $|s| \geq p$ can be split into five parts to be $s = uvwxy$ that satisfies the following conditions:

   1. $|vwx| \leq p$

   2. $|vx| > 0$

   3. The string $uv^i wx^i y$ in L for all $i \geq 0$

2. Consider a context-free grammar G over the alphabet {a, b}, such as G = {S -> aSb | Îµ}. This CFG generates palindromes, for example, "aba". Based on the assumption, U is able to generate both G and strings in L(G).

3. We take the string s = "a^p b^p" where p is the pumping length from the Pumping Lemma. Then, we decompose s into five parts to be $s = uvwxy$ where $|vwx| \leq p$, and v and x are non-empty. If we pump or repeat v and x, we would have strings such as "a^(p+1) b^(p+1)" which are not in L(G) because L(G) only contains palindromes. This contradicts the assumption that L(G) is context-free and follows the Pumping Lemma and proves that no such Universal Context-Free Grammar exists. Thus, it is proven that no such Universal Context-Free Grammar exists.

Conclusion:

There is no Universal Context-Free Grammar because it contradicts the Pumping Lemma for Context-Free Languages.