Propositional Logic: An Introduction

Type Theory and Mechanized Reasoning

Lecture 5

`CAS CS 400`

Introduction

Administrivia

- Assignment 1 is out. Another assignment will be out on Thursday.
- The course has been approved for the semester.
- Details about the final project will be available next week.

Objectives

1. Introduce propositional logic, a simple form of logic for reasoning about Boolean connectives.

2. Use Agda to implement propositional logic.

3. Use propositional logic as a setting for learning important concepts and terms in logic.

Unicode cheatsheet

`→` is `\->`

`ℕ` is `\bN`

`×` is `\times`

`∧` is `\and`

`∨` is `\or`

`¬` is `\neg`

`_ᵖ` is `\^p`

Practice Problem

*Write a function **down** which, given a natural number **n**, returns a vector with the values **n** through **1**.*

*Write a function **up**, which return a vector with the number from **1** to **n**.*

Agda Tutorial: Interaction

Interaction: At a High Level

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Agda is a functional language, which means a lot of pattern matching.

Agda is pure, there are no print statements, so we need to know how to compute values.

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modulo : ℕ → ℕ → ℕ
modulo = ?
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Within a hole, we can determine the types of everything in the environment.

We can even try to fill in a value (which can have it's own holes).

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4. (C-c C-SPACE) write a definition and try to fill it in.
5. Rinse and repeat.

Interaction Cheat Sheet

`C-c C-l`	`load file`
`C-c C-,`	`check type in hole`
`C-c C-c`	`pattern match within hole`
`C-c C-SPACE`	`try to fill in hole`
`C-c C-n`	`compute value of term`

demo

Propositional Logic: Motivation

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(this is also what you need for a programming language)

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sequence of characters → sequence of lexemes → sequence of tokens → parse tree

We're not going to focus on *syntax or parsing* in this course.

We will almost always presume we already have a parse tree.

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[sym '(', var 'X', con 'and', var 'Y', sym ')']
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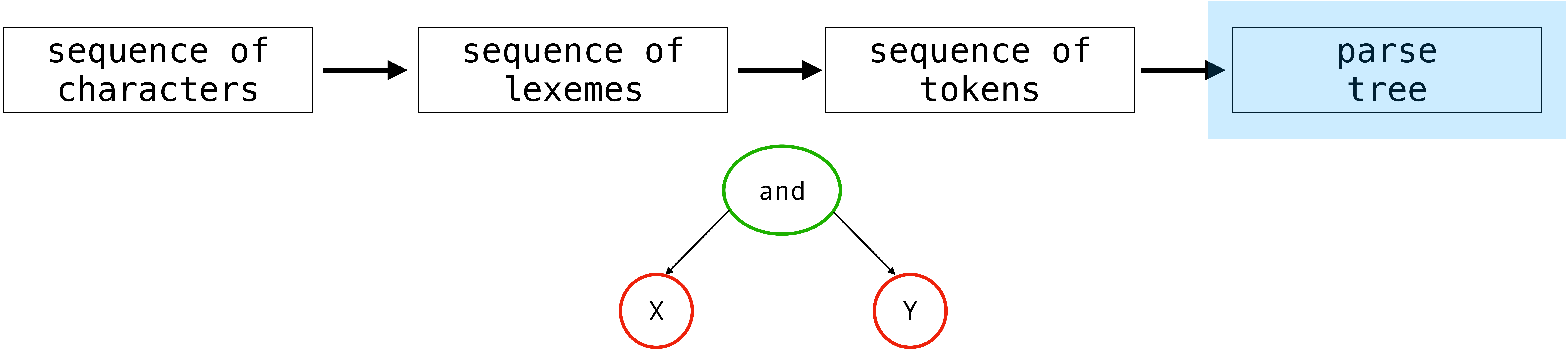

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In classical logic (what we will consider today) meaning will mean ***truth***.

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Very interesting question from a philosophical perspective.

In classical logic (what we will consider today) meaning will mean ***truth***.

In intuitionistic logic, meaning will be something different.

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» It is cold.

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- $\underbrace{\texttt{If}}_{\text{connective}}\ \overbrace{\texttt{it is raining}}^{\text{proposition}}\ \underbrace{\texttt{and}}_{\text{connective}}\ \overbrace{\texttt{it is cold}}^{\text{proposition}}\ \underbrace{\texttt{then}}_{\text{connective}}\ \overbrace{\texttt{it is raining}}^{\text{proposition}}.$

Why is this?

Boolean Connectives

$$x \boxed{\land} y$$

conjunction (and)

Propositional logic is the study of logical (Boolean) connectives.

- *What do connectives mean? How do they affect truth?*
- *What connectives exists? How many do we need?*
- *How do connectives interact?*

Propositional Logic and Programming Conditionals

```
if is_raining and not is_warm:
    # some code
```

Propositional logic is the study of **Bools.**

- *When can I replace one conditional with another?*
- *Why is there only* ***and****,* ***or****, and* ***not****?*
- *Why can conditionals short-circuit?*

Propositional Logic: Syntax

Propositional Variables

$$\boxed{x} \land \boxed{y}$$

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*We will assume a countable number of variable symbols like in algebra

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- If P is a formula then so it $\neg P$.*
- If P and Q are formulas then so are $P \wedge Q$ and $P \vee Q$ and $P \rightarrow Q$.

*P is a *meta-variable*. It refers to an arbitrary formula. It is not the same as a propositional variable.

Examples

$$P \wedge (Q \wedge \neg Z)$$

$$A \to (B \to (\neg C \vee B))$$

$$A \wedge (A \wedge (A \wedge A))$$

Note. Parentheses are *meta-syntactical*. Remember that these formulas are shorthand for trees.

Example (More formally)

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P is a formula and $Q \wedge \neg Z$ is a formula so $P \wedge (Q \wedge \neg Z)$ is a formula.

Informal Meaning

$\neg \equiv$ "not"

$\wedge \equiv$ "and"

$\vee \equiv$ "or"

$\rightarrow \equiv$ "implies"

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$\neg \equiv$ "not"

$\land \equiv$ "and"

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R It's raining

C It's cold

$(R \land C) \rightarrow R$ If it's raining and cold then it's raining

Understanding Check:
English to Formula

Syntax: In Agda

```agda
data Formula : Set where
  _ᵖ : String → Formula
  ¬ᵖ_ : Formula → Formula
  _∧ᵖ_ : Formula → Formula → Formula
  _∨ᵖ_ : Formula → Formula → Formula
  _→ᵖ_ : Formula → Formula → Formula
```

The tree structure of formulas is implicit in it being an ADT.

Syntax: What's Next?

Remember. We haven't actually given formulas *meaning*. That is, we haven't given a semantics.

But we can ask about:

» Function on formulas
» Transformations of formulas

Example: Depth (In Mathematical English)

Definition. The **depth** of a formula is defined as the depth of its corresponding tree:

$$d(x) = 0$$
$$d(\neg P) = 1 + d(P)$$
$$d(P \square Q) = \max(d(P), d(Q)) + 1$$

where $\square$ is any one of '$\wedge$' or '$\vee$' or '$\rightarrow$'.

Example: Depth (In Agda)

```
Let's do a demo.
```

Propositional Logic: Semantics

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Given an expression $3x + (2y \times 6h^2)$ and values for x, y, and h, we can compute the value of the entire expression.

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We think of a propositional variable as having the values of either **true** or **false**.

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A valuation v is like a state of affairs.

The idea. *If we know the state of affairs, we can determine the truth or falsity of any statement.*

Partial Valuations

$$v(x) = \text{true}$$
$$v(y) = \text{false}$$
$$v(z) = \text{false}$$
$$v(\_) = \text{false}$$

$$(x \wedge y) \vee z$$

Note. We will typically only care about a small collection of variables.

We can assume all unspecified variables are assigned to be false.

Valuation (In Agda)

```
Let's do a demo.
```

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This is where we give a formula meaning with respect to a state of the affairs.

We have to say what we want the truth of a statement to be based on its constituent parts.

Evaluation: In Agda

```
Let's do a demo.
```

Evaluation: In English

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I leave it as an exercise to write out a "mathy" definition, but the shape will be similar, e.g.,

$$\overline{v}(\neg P) = \begin{cases} \mathsf{false} & \overline{v}(P) = \mathsf{true} \\ \mathsf{true} & \texttt{otherwise} \end{cases}$$

Understanding Check: *Evaluation*

Propositional Logic:
Semantic Notions

Validity and Satisfiability

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Definition. A formula ϕ is **satisfiable** if there is *some* valuation which makes ϕ true, e.g.

$$x \vee y$$

Understanding Check: Validity

Understanding Check:
Validity

Entailment

Definition. A set of formulas Γ **entails** ϕ (written $\Gamma \vDash \phi$) if *every* valuation which make *every* formula in Γ true, also makes ϕ true.

Example. $\{x \to y, y \to z\} \vDash x \to z$

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*This does not account for short-circuiting.

Understanding Check: Logical Equivalence

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Propositional Logic: Functional Completeness

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Theorem. For any propositions P and Q, $P \otimes Q$ is logically equivalent to $(P \wedge Q) \vee (\neg P \vee \neg Q)$.

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Consider the exclusive–or operations $P \otimes Q$.

Why didn't we include this in our presentation of propositional logic?

Theorem. For any propositions P and Q, $P \otimes Q$ is logically equivalent to $(P \wedge Q) \vee (\neg P \vee \neg Q)$.

A connective is a boolean function.

Boolean Functions

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f is **represented** by a formula ϕ on with propositional variables $x_1, \ldots, x_n$ if for any valuation v,

$$f(v(x_1), \ldots, v(x_n)) = \overline{v}(\phi)$$

Functional Completeness

Theorem. Every n–variate boolean function is represented by a formula.

Let's try to prove it...