Propositional Proofs

Type Theory and Mechanized Reasoning
Lecture 9

CAS CS 400

Introduction

Administrivia

Homework 3 is due on Thursday 11:59PM. Homework 4 will be released on Thursday (Tomorrow).

I updated the syllabus for the course to include some information about the final project.

We'll take a bit of time today to put together groups for the final project.

Objectives

Introduce resolution as one way of verifying that a CNF formula is unsatisfiable, and connect this to DPLL.

Introduce Gentzen-style proofs as a way of verifying that an arbitrary formulas is valid.

Look at soundness and completeness.

Agda Tutorial: The Empty Type

Definition

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Fact. `There is no value we could write with the type` **`Empty`**`.`

`This fact is surprisingly useful.`

Inheriting Emptiness

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IsEmpty : Set -> Set
IsEmpty A = A -> Empty
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Fact. If there is a function from **A** to **Empty**, then **A** has no values.

We can "prove" that a type **A** has no values by construction a *function* from **A** to **Empty**.

Empty is Empty

```
empty-is-empty : IsEmpty Empty
empty-is-empty found-one = found-one
```

This is just the identity function.

But it indicates that the definition could be reasonable.

A More Interesting Example

```
data NotWell : Set where
  cannot : NotWell -> NotWell

emp : IsEmpty NotWell
emp (cannot but-I-did) = emp but-I-did
```

We can't create a value of type **NotWell.**

If we could we could use it to build an value of the empty type.

More Inheriting Emptiness

```
isEmpty-Prod :
  {A : Set} ->
  {B : Set} ->
  IsEmpty A ->
  IsEmpty B ->
  IsEmpty (Either A B)
isEmpty-Prod f g (left x) = f x
isEmpty-Prod f g (right x) = g x
```

***Either A B** is empty if **A** and **B** are.*

Verifying Truth

```
IsTrue : Bool -> Set
IsTrue true = Unit
IsTrue false = Empty
```

We can use dependent types and **Empty** to "lift" truth to the types.

Principle of Explosion

```
explode : {A : Set} -> Empty -> A
explode ()
```

If I *could* create a value of the empty type, I could create anything I want.

Note. If a type is empty, then Agda doesn't need to pattern match on it.

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Understanding Check

Write a function of type

```
{A : Set} -> A -> IsEmpty (IsEmpty A)
```

How do we "read" this type?

Proof Systems

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Question. *How do I convince someone that ϕ is satisfiable?*

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Question. *What about unsatisfiability?*

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For large formulas this is unreasonable.

Recall: Validity and Satisfiability

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Validity/Unsatisfiability Theorem. ϕ is a tautology if and only if $\neg\phi$ is unsatisfiable.

Determining satisfiability is at least as hard as determining validity.

There is a tight connection between satisfiability and proof.

Example

$$x_1 \land (\neg x_1 \lor x_2) \land \ldots \land (\neg x_{n-1} \lor x_n) \land \neg x_n$$

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Looking at all possible assignments seems unnecessary.

We want to formalize the intuition that we can demonstrate unsatisfiability by *reasoning* instead of brute force.

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There are many shapes of proof systems but generally they are:

- **Directed** We prove one thing *from* another thing
- **Checkable** We require that is is easy to check if a proof is correct.

Judgements (Sequents)

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Φ and Ψ are *statements*. Today that means **formulas**, later that will mean *typing statements*.

Inference Rules

$$\frac{J_1 \quad J_2 \quad \ldots \quad J_n}{J_{n+1}} \; (\texttt{condition})$$

In **inference rule** an way of describing an individual step in a proof.

It reads: "If the judgments $J_1, \ldots, J_n$ hold and the condition is met, then judgment J_{n+1} holds."

Example: Modus Ponens

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Note. P and Q are *meta-variables*, they stand for arbitrary formulas. Γ is also a meta-variable, but for a list of formulas.

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We say that $\Gamma \vdash \Psi$ holds if there is a derivation tree which has this judgment at the root.

Resolution

The Resolution Rule

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Fact. If v satisfies clauses $C \vee x$ and $D \vee \neg x$ then v satisfies $C \vee D$.

Resolution

axiom $$\frac{}{\phi \vdash C} \; (C \in \phi)$$

res. rule $$\frac{\phi \vdash C \vee x \quad \phi \vdash D \vee \neg x}{\phi \vdash C \vee D}$$

The resolution proof system $\mathscr{R}$ has the following two inference rules.

Note. The antecedent never changes, so we could drop it if we remember what ϕ is.

Example

$$\dfrac{\dfrac{x_1 \land (\neg x_1 \lor x_2) \land \neg x_2 \vdash x_1 \quad x_1 \land (\neg x_1 \lor x_2) \land \neg x_2 \vdash (\neg x_1 \lor x_2)}{x_1 \land (\neg x_1 \lor x_2) \land \neg x_2 \vdash x_2} \qquad x_1 \land (x_1 \lor x_2) \land \neg x_2 \vdash \neg x_2}{x_1 \land (\neg x_1 \lor x_2) \lor \neg x_2 \vdash \varnothing}$$

Example

$$\dfrac{\dfrac{x_1 \qquad \neg x_1 \vee x_2}{x_2} \qquad \neg x_2}{\varnothing}$$

The following is a resolution derivation of $x_1 \wedge (\neg x_1 \vee x_2) \wedge \neg x_2 \vdash \varnothing$.

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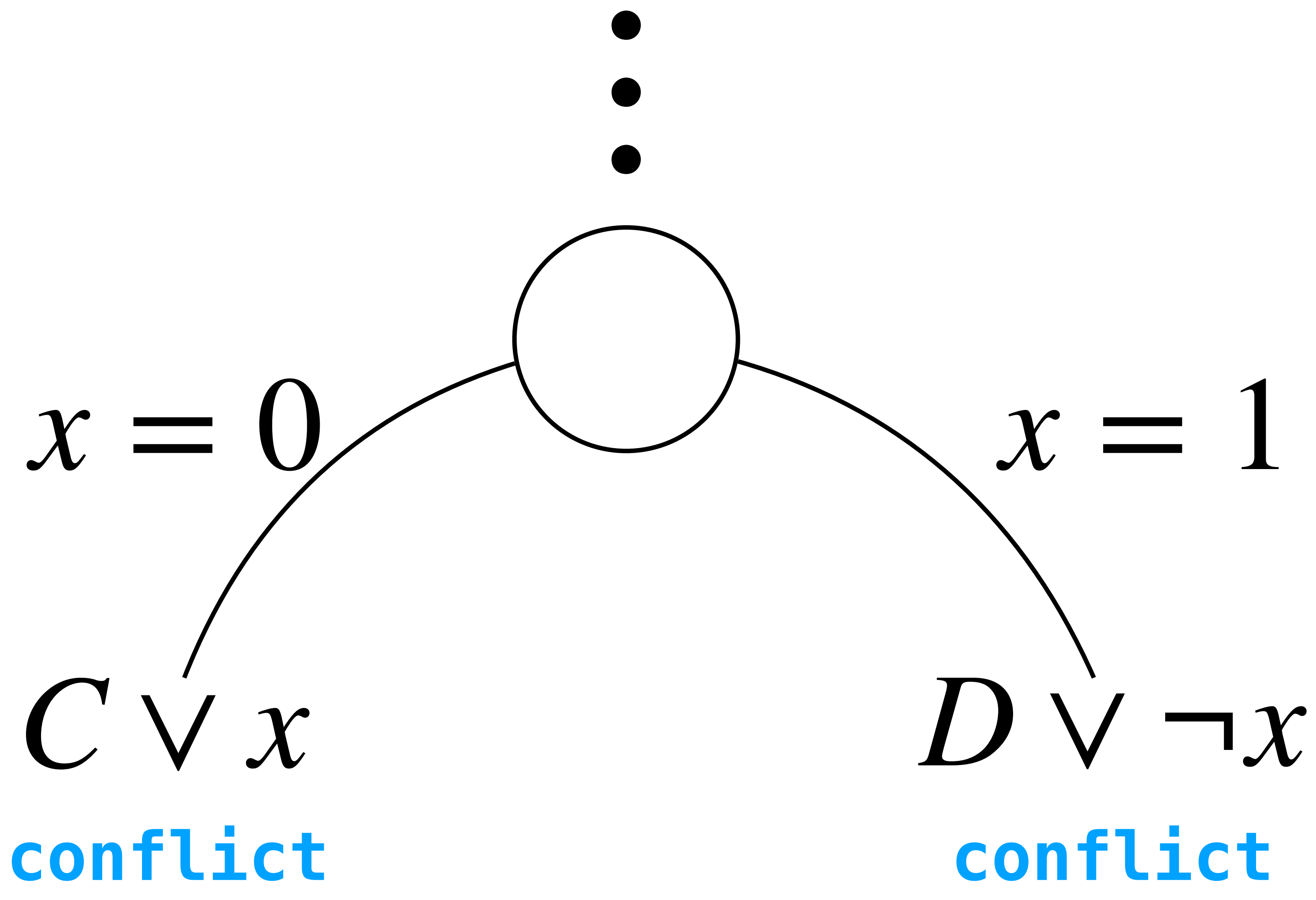
Soundness says that we cannot use a resolution proof to show that a satisfiable formula is unsatisfiable.

We call a derivation of $\phi \vdash \varnothing$ a **refutation** of ϕ.

DPLL and Resolution



$$\frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

`If` $C \vee x$ `is falsified and` $D \vee \neg x$ `is falsified, then` $C \vee D$ `is falsified by the assignment` *`without`* `assigning` x.

Completeness

Theorem. If ϕ is unsatisfiable, then $\phi \vdash_{\mathscr{R}} \varnothing$

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