SAT Solvers:
In Practice

Type Theory and Mechanized Reasoning
Lecture 8

CAS CS 400

Introduction

Administrivia

Homework 2 is due on Thursday 11:59PM. Homework 3 will be released on Thursday (Tomorrow).

I will post about the **Final Project** on Friday. The rough outline:

» Groups of 1, 2, or 3
» Implementation, Proof, or Survey
» Weekly assignments will include progress reports
» By next week, you should have a group and a rough idea

Objectives

Finish our discussion on DPLL.

Talk about encoding with CNF formulas.

Build a sudoku solver with a SAT solver.

Agda Tutorial: Implicit Arguments

Recall: Polymorphism

```
k : (A : Set) -> (B : Set) -> A -> B -> A
k A B x y = x
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This is an example of parametric polymorphism which means that the definition of **k** is *completely agnostic to the types **A** and **B**.*

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So we can replace them with wildcards in the definition of **k**.

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The values of **A** and **B** can be solved based on the inputs given to **k**.

This solving process is call unification.

Unification and Implicit Arguments

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k : {A : Set} -> {B : Set} -> A -> B -> A
k x y = x

c : {A B C : Set} -> (A -> B) -> (B -> C) -> (A -> C)
c f g x = g (f x)

i : {A : Set} -> A -> A
i x = x

f : {A B : Set} -> A -> B -> A
f x = c (k x) i
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When a value can be solved via unification, then can made implicit without issues.

An Example with Dependent Types

```
head : {A : Set} -> {n : Nat} -> Vec A (suc n) -> A
head (x :: _) = x

foo : Nat
foo = head (1 :: 2 :: [])
```

After applying head to a value, the input **n** can be solved, so it can be made implicit.

Passing in Implicit Arguments

```
allVals : {n : Nat} -> List (Vec Bool n)
allVals {zero} = [] :: []
allVals {suc n} = go (allVals {n}) where
  go : List (Vec Bool n) -> List (Vec Bool (suc n))
  go [] = []
  go (x :: xs) = (true :: x) :: (false :: x) :: go xs
```

It is occasionally necessary to work directly with implicit arguments. To do this, we pass in values with {_}'s.

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The Takeaway

```
k : (A : Set) -> (B : Set) -> A -> B -> A
k _ _ x y = x
```

If you can pass a wildcard for argument in the definition of a function, it can be made implicit.

(Even if you can't it may still be useful).

Question

```
foo : Bool -> Set
foo true = Nat
foo false = Nat

bar : (b : Bool) -> foo b -> Nat
bar true x = x
bar false x = x
```

Can ***b*** *in* ***bar*** *be made implicit?*

Recap: SAT Solvers

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Example. $(x \vee y) \wedge (\neg x \vee y) \wedge (\neg x \vee \neg y)$ is satisfied by

$$v(y) = \mathsf{true} \quad v(x) = \mathsf{false} \quad v(\_) = \mathsf{false}$$

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$$\underbrace{(x \lor \neg y)}_{\text{clause}} \land \underbrace{\neg y}_{\text{clause}} \land \underbrace{(\neg y \lor \neg z \lor \neg w)}_{\text{clause}}$$

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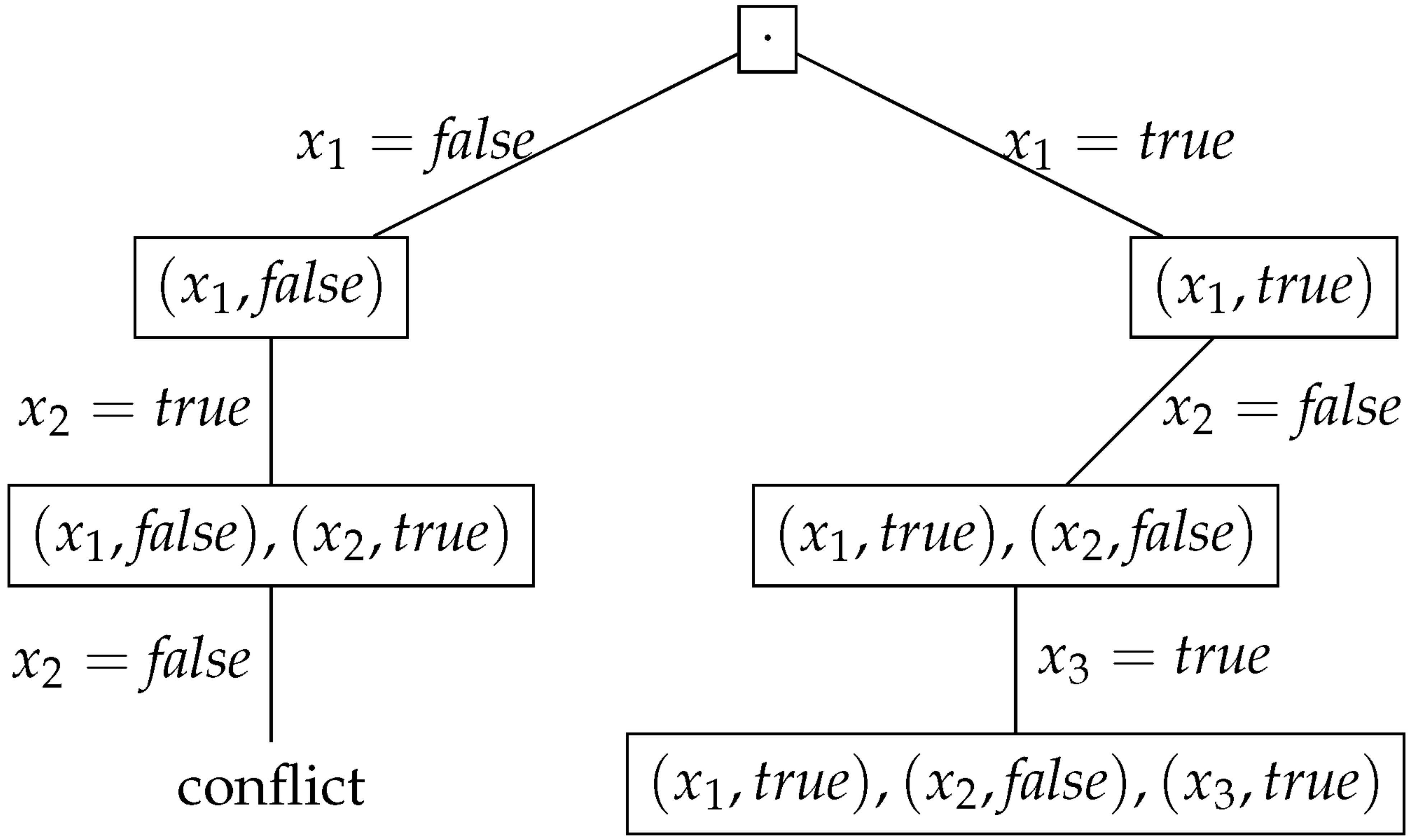
Given a CNF formula ϕ, determine a satisfying assignment for ϕ or determine that no such satisfying assignment exists.

Another view. Can we find an assignment which satisfies **at least one literal of every clause**?

Recall: DPLL



https://www.mdpi.com/2227-7390/10/13/2264

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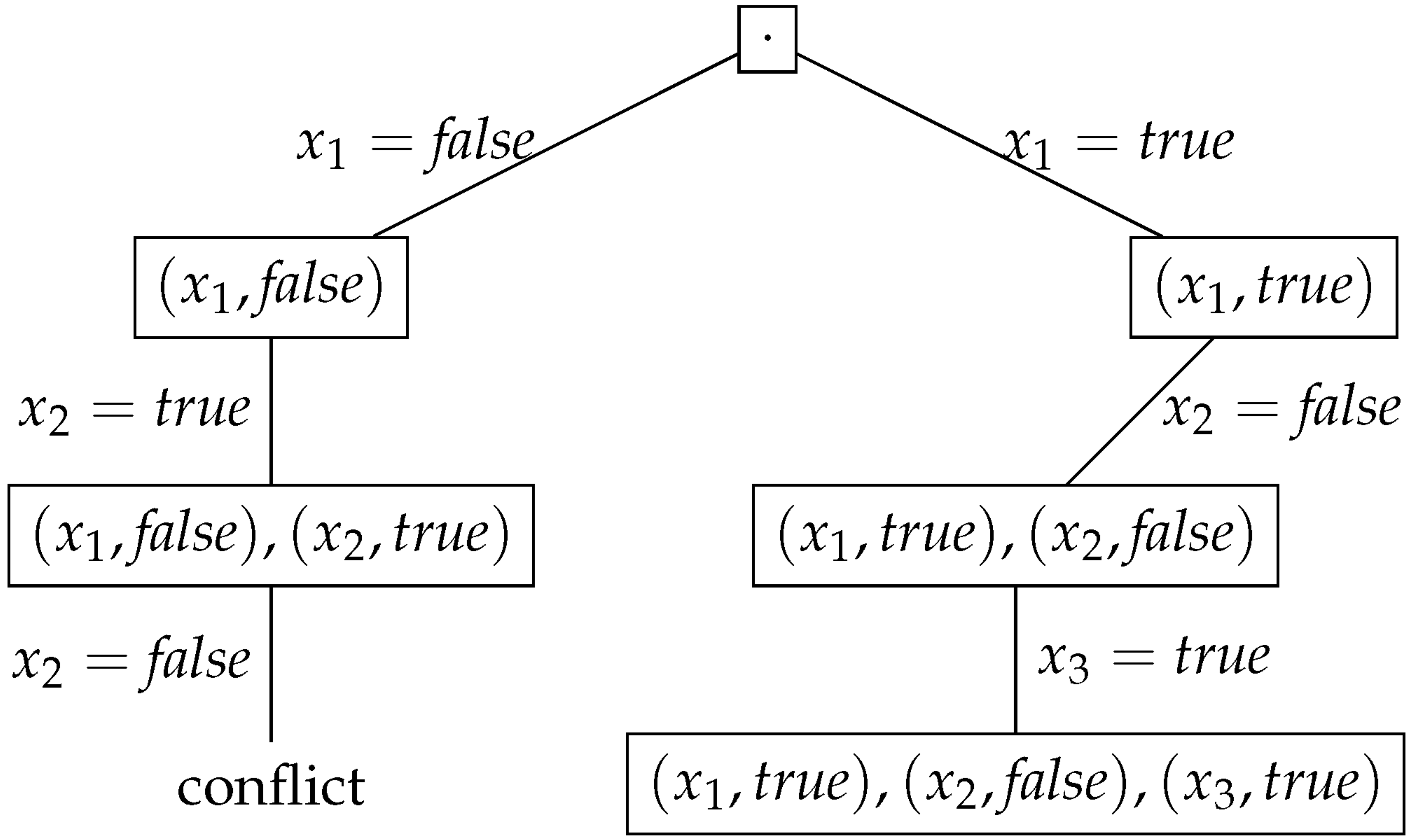

Idea. Build a satisfying assignment one variable at a time, updating the formula at each step.

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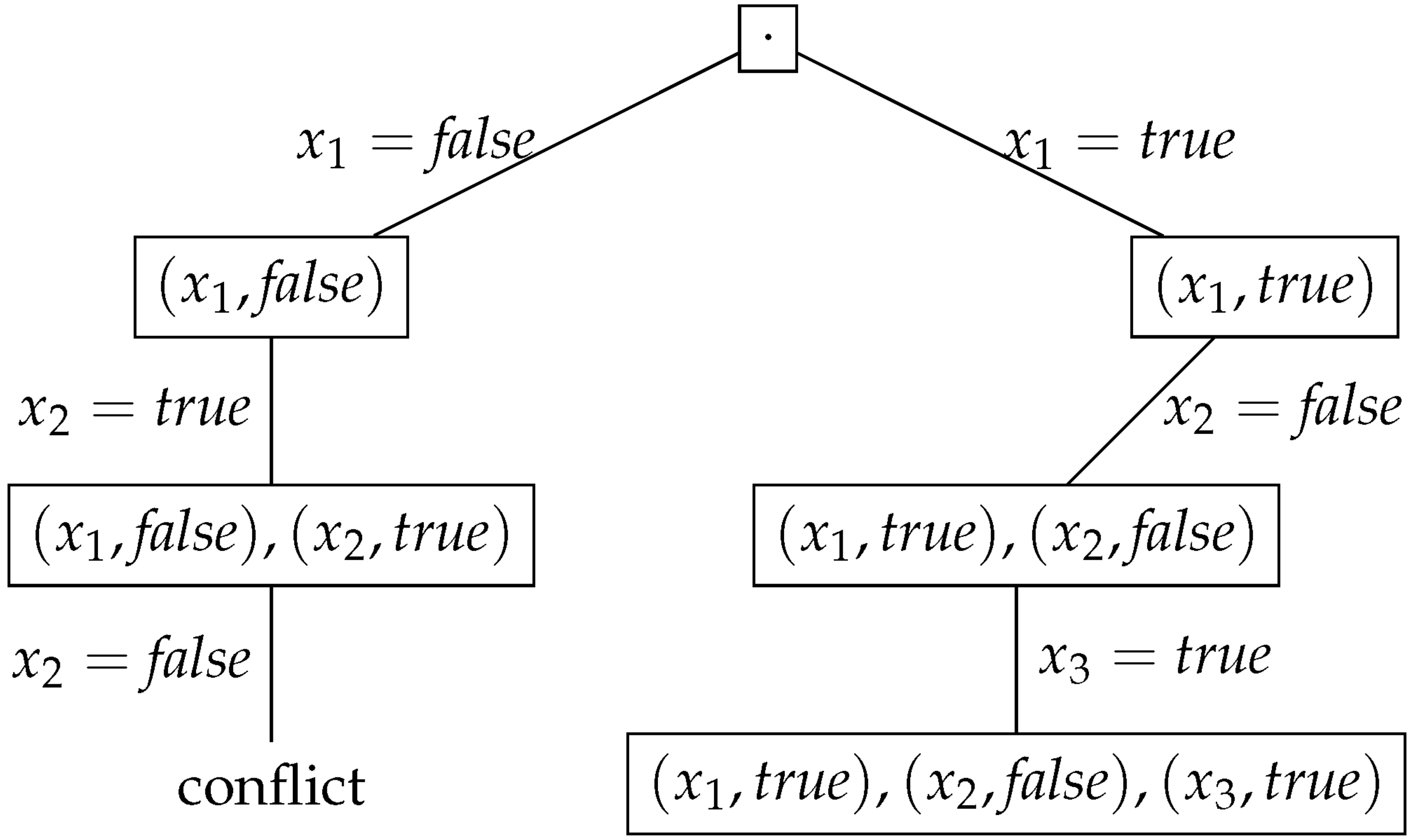

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This is a backtracking procedure, where a leaf is a satisfying assignment or a **conflict** (the assignment cannot be satisfying).

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Recall: Partial Assignments

Definition. A **partial assignment** is a set of literals.

Example. $\{x^1, y^0, z^1\}$

We think of this as a set of *assertions*, i.e., x is true and y is false and z is true.

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$$(C_1 \wedge C_2 \wedge \ldots \wedge C_k)|_l = (C_1|_l) \wedge (C_2|_l) \ldots \wedge (C_k|_l)^*$$

*This can be made more efficient.

General Restriction

Restriction by a partial assignment can be understood as repeated restriction by literals, e.g.*

$$\phi|_{\{l_1,l_2\}} = (\phi|_{l_1})|_{l_2}$$

*This elides questions about order of restrictions

Naive DPLL in Agda

```
{-# TERMINATING #-}
is-sat : CNF -> Bool
is-sat f with find-var f
is-sat f | Nothing = notb (has-empty f)
is-sat f | Just x with is-sat (restrict (x , true) f)
is-sat f | Just x | true = true
is-sat f | Just x | false = is-sat (restrict (x , false) f)
```

High Level: is-sat branches the choice of restricting by x or $\neg x$.

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Pure Literal Rule. If the formula has only appearance of l, then restrict the formula by l.

Example. $((x^0 \vee y^0) \wedge (x^0 \vee y^1) \wedge z^0)|_{x^0}$ becomes z^0

DPLL Pseudocode

```
Algorithm DPLL
    Input: A set of clauses Φ.
    Output: A truth value indicating whether Φ is satisfiable.
```

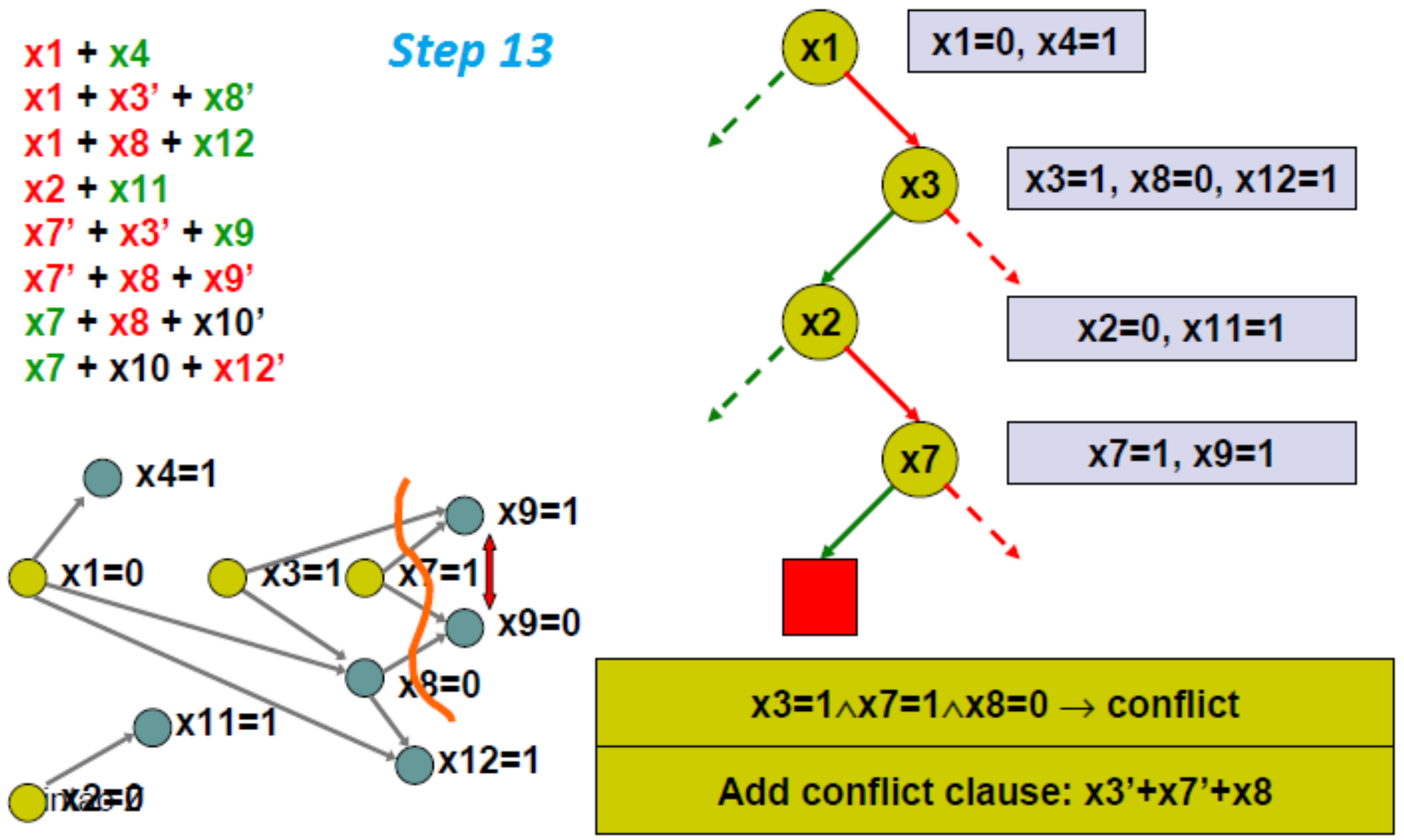
```
function DPLL(Φ)
    // unit propagation:
    while there is a unit clause {l} in Φ do
        Φ ← unit-propagate(l, Φ);
    // pure literal elimination:
    while there is a literal l that occurs pure in Φ do
        Φ ← pure-literal-assign(l, Φ);
    // stopping conditions:
    if Φ is empty then
        return true;
    if Φ contains an empty clause then
        return false;
    // DPLL procedure:
    l ← choose-literal(Φ);
    return DPLL(Φ ∧ {l}) or DPLL(Φ ∧ {¬l});
```

https://en.wikipedia.org/wiki/DPLL_algorithm

That's it. That's the
David–Putnam–Logemann–Loveland Procedure.

backtracking + unit propagation + pure literal rule

A More Complicated Algorithm: CDCL



Modern SAT solvers are built using a heuristic called **conflict driven clause learning (CDCL).**

Idea. When we find that a partial assignment creates a conflict, we can *add* clauses to our formula which might help the solver avoid making the same mistake again.

https://en.wikipedia.org/wiki/Conflict-driven_clause_learning#/media/File:Cdcl13.png

CNF Encodings

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Natural encodings in CNF may be very complex, or difficult to work with.

We need to build up a couple tricks for encoding.

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This only introduces one clause.

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$$\text{one}_{\leq}(x_1, \ldots, x_n) \quad = \bigwedge_{1 \leq i < j \leq n} \neg(x_i \wedge x_j) \quad \equiv \quad \bigwedge_{1 \leq i < j \leq n} \neg x_i \vee \neg x_j$$

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Question. *How many clauses does this create?*

"Exactly one" Encoding

Given $l_1, \ldots, l_n$, "exactly one of $l_1, \ldots, l_n$ is true" can be encoded as

$$\text{one}(l_1, \ldots, l_n) = \text{one}_{\leq}(l_1, \ldots, l_n) \wedge \text{one}_{\geq}(l_1, \ldots, l_n)$$

Note that this is still a CNF formula.

Understanding Check

Write a CNF encoding for $\text{two}_{\geq}(l_1, \ldots, l_n)$ *which expresses "at least two of* $l_1, \ldots, l_n$ *hold".*

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Some SAT solvers (e.g., CryptoMiniSat) have built-on XOR solvers.

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Question. *Why aren't the above formulas equivalent?*

A Trick using Equisatisfiability

$$\mathsf{one}_{\leq}(x_1, x_2, \ldots, x_6)$$

`and`

$$\mathsf{one}_{\leq}(x_1, x_2, y) \land \mathsf{one}_{\leq}(\neg y, x_3, x_4, z) \land \mathsf{one}_{\leq}(\neg z, x_5, x_6)$$

`are equisatisfiable.`

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are equisatisfiable.

This gives can be used to create a CNF formula with linearly many clauses.

Sudoku Solver

The Rules

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Every row has the numbers 1 through 9.

Every column has the numbers 1 through 9.

Every 3×3 box has the number 1 through 9.

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Question. *How many variables will the final formula have?*

The Rules as a CNF

We need clauses for:

1. Every square has a one number
2. The rows are well-formed
3. The columns are well-formed
4. The boxes are well-formed

5	3			7				
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$$\mathsf{one}_{\geq}(x_{i,1,v}, \ldots, x_{i,9,v}) = x_{i,1,v} \vee x_{i,2,v} \vee \ldots \vee x_{i,9,v} \quad (R_{i,v})$$

Example: Rows

Row i must have the number v :=

$$\mathsf{one}_{\geq}(x_{i,1,v}, \ldots, x_{i,9,v}) = x_{i,1,v} \lor x_{i,2,v} \lor \ldots \lor x_{i,9,v} \quad (R_{i,v})$$

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`Row` i `must have at most one` v `:=`

$$\text{one}_{\leq}(x_{i,1,v}, x_{i,2,v}, \ldots, x_{i,9,v}) \quad (S_{i,v})$$

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The rows are well-formed :=

$$\bigwedge_{1 \leq i,v \leq n} R_{i,v} \wedge S_{i,v} \equiv \bigwedge_{1 \leq i,v \leq n} \mathsf{one}(x_{i,1,v}, \ldots, x_{i,9,v})$$

Example: Rows

Columns and Boxes are similar.

Row i must have the number v :=

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Adding the Board

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Given a board, we need to add clauses which express that a given position already has a value.

<u>Example.</u> $x_{0,0,5} \wedge x_{0,1,3} \wedge x_{0,4,7} \wedge \ldots$

PySat

```
>>> from pysat.solvers import Glucose3
>>>
>>> g = Glucose3()
>>> g.add_clause([-1, 2])
>>> g.add_clause([-2, 3])
>>> print(g.solve())
>>> print(g.get_model())
...
True
[-1, -2, -3]
```

PySat is an interface for working with a number of different SAT solvers in Python.

Clauses are represented as lists of nonzero integers.

demo