Propositional Logic: An Introduction

Type Theory and Mechanized Reasoning
Lecture 5

```
CAS CS 400
```

Introduction

Administrivia

- Assignment 1 is out. Another assignment will be out on Thursday.
- The course has been approved for the semester.
- Details about the final project will be available next week.

Objectives

1. Introduce propositional logic, a simple form of logic for reasoning about Boolean connectives.

2. Use Agda to implement propositional logic.

3. Use propositional logic as a setting for learning important concepts and terms in logic.

Unicode cheatsheet

→ is `\->`

ℕ is `\bN`

× is `\times`

∧ is `\and`

∨ is `\or`

¬ is `\neg`

$\_^{p}$ is `\^p`

Practice Problem

Write a function ***down*** *which, given a natural number* ***n,*** *returns a vector with the values* ***n*** *through* ***1.***

Write a function ***up,*** *which return a vector with the number from* ***1*** *to* ***n.***

Agda Tutorial: Interaction

Interaction: At a High Level

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Agda is a functional language, which means a lot of pattern matching.

Agda is pure, there are no print statements, so we need to know how to compute values.

The Primary Tool: Holes

```
modulo : ℕ → ℕ → ℕ
modulo = ?
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They allow you to build up code piece-by-piece and check code frequently.

Within a hole, we can determine the types of everything in the environment.

We can even try to fill in a value (which can have it's own holes).

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4. (C–c C–SPACE) write a definition and try to fill it in.
5. Rinse and repeat.

Interaction Cheat Sheet

`C-c C-l`	`load file`
`C-c C-,`	`check type in hole`
`C-c C-c`	`pattern match within hole`
`C-c C-SPACE`	`try to fill in hole`
`C-c C-n`	`compute value of term`

demo

Propositional Logic: Motivation

What is logic?

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semantics	what do those things mean?

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(this is also what you need for a programming language)

What does syntax mean?

$$\begin{tikzcd}
\boxed{\begin{matrix}\texttt{sequence of}\\\texttt{characters}\end{matrix}} \arrow[r] & \boxed{\begin{matrix}\texttt{sequence of}\\\texttt{lexemes}\end{matrix}} \arrow[r] & \boxed{\begin{matrix}\texttt{sequence of}\\\texttt{tokens}\end{matrix}} \arrow[r] & \boxed{\begin{matrix}\texttt{parse}\\\texttt{tree}\end{matrix}}
\end{tikzcd}$$

We're not going to focus on *syntax or parsing* in this course.

We will almost always presume we already have a parse tree.

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```
[(, X, □, a, n, d, □, Y, )]
```

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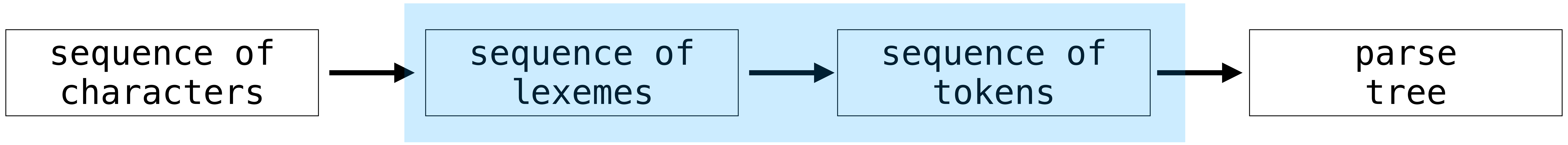

```
[sym '(', var 'X', con 'and', var 'Y', sym ')']
```

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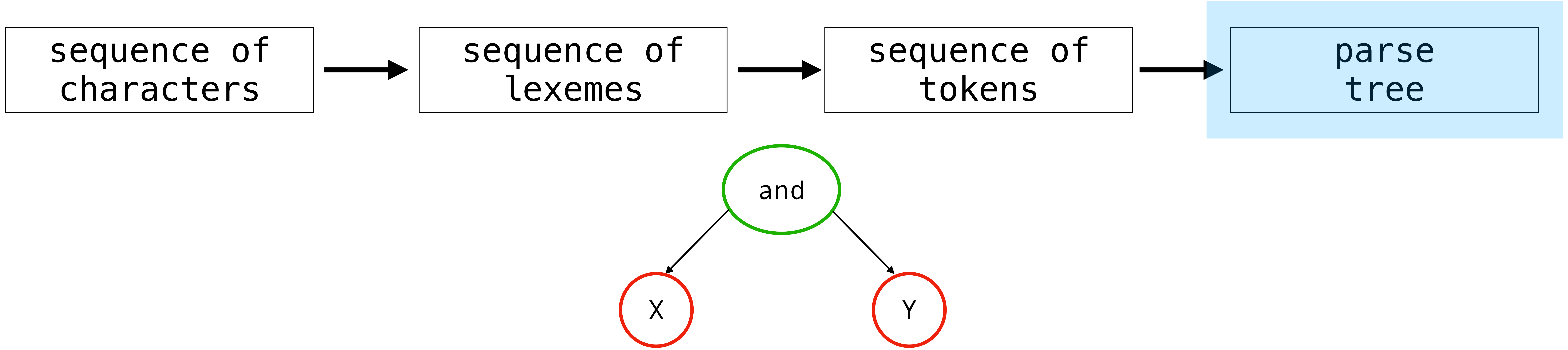

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What does meaning mean?

$$\begin{matrix} v(x) = \text{true} \\ v(y) = \text{true} \end{matrix} \quad \Longrightarrow \quad v(x \wedge y) = \text{true}$$

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In classical logic (what we will consider today) meaning will mean ***truth***.

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`Very interesting question from a philosophical perspective.`

`In classical logic (what we will consider today) meaning will mean` ***truth***`.`

`In intuitionistic logic, meaning will be something different.`

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» It is cold.

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Some facts seem contingent on the world:

» It is raining.
» It is cold.

Some facts seem unassailable:

» $\underbrace{\texttt{If}}_{\text{connective}}\ \overbrace{\texttt{it is raining}}^{\text{proposition}}\ \underbrace{\texttt{and}}_{\text{connective}}\ \overbrace{\texttt{it is cold}}^{\text{proposition}}\ \underbrace{\texttt{then}}_{\text{connective}}\ \overbrace{\texttt{it is raining}}^{\text{proposition}}.$

Why is this?

Boolean Connectives

$$x \land y$$

conjunction (and)

Propositional logic is the study of logical (Boolean) connectives.

- *What do connectives mean? How do they affect truth?*
- *What connectives exists? How many do we need?*
- *How do connectives interact?*

Propositional Logic and Programming Conditionals

```
if is_raining and not is_warm:
    # some code
```

Propositional logic is the study of **Bools.**

- » *When can I replace one conditional with another?*
- » *Why is there only **and**, **or**, and **not**?*
- » *Why can conditionals short-circuit?*

Propositional Logic: Syntax

Propositional Variables

$$\boxed{x} \land \boxed{y}$$

propositional variables

Propositional Variables

$$\boxed{x} \wedge \boxed{y}$$

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*We will assume a countable number of variable symbols like in algebra

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- If P is a formula then so it $\neg P$.*
- If P and Q are formulas then so are $P \wedge Q$ and $P \vee Q$ and $P \rightarrow Q$.

*P is a *meta-variable*. It refers to an arbitrary formula. It is not the same as a propositional variable.

Examples

$$P \wedge (Q \wedge \neg Z)$$

$$A \rightarrow (B \rightarrow (\neg C \vee B))$$

$$A \wedge (A \wedge (A \wedge A))$$

Note. Parentheses are *meta-syntactical*. Remember that these formulas are shorthand for trees.

Example (More formally)

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P is a variable so P is a formula.

P is a formula and $Q \land \neg Z$ is a formula so $P \land (Q \land \neg Z)$ is a formula.

Informal Meaning

$\neg$	$\equiv$	"not"
$\wedge$	$\equiv$	"and"
$\vee$	$\equiv$	"or"
$\rightarrow$	$\equiv$	"implies"

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$\neg \equiv$	"not"
$\wedge \equiv$	"and"
$\vee \equiv$	"or"
$\rightarrow \equiv$	"implies"

R	It's raining
C	It's cold
$(R \wedge C) \rightarrow R$	If it's raining and cold then it's raining

Understanding Check:
English to Formula

Syntax: In Agda

```
data Formula : Set where
  _ᵖ : String → Formula
  ¬ᵖ_ : Formula → Formula
  _∧ᵖ_ : Formula → Formula → Formula
  _∨ᵖ_ : Formula → Formula → Formula
  _→ᵖ_ : Formula → Formula → Formula
```

The tree structure of formulas is implicit in it being an ADT.

Syntax: What's Next?

Remember. We haven't actually given formulas *meaning*. That is, we haven't given a semantics.

But we can ask about:

» Function on formulas
» Transformations of formulas

Example: Depth (In Mathematical English)

Definition. The **depth** of a formula is defined as the depth of its corresponding tree:

$$\begin{aligned} d(x) &= 0 \\ d(\neg P) &= 1 + d(P) \\ d(P \mathbin{\square} Q) &= \max(d(P), d(Q)) + 1 \end{aligned}$$

where $\square$ is any one of '$\wedge$' or '$\vee$' or '$\rightarrow$'.

Example: Depth (In Agda)

```
Let's do a demo.
```

Propositional Logic: Semantics

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Given an expression $3x + (2y \times 6h^2)$ and values for x, y, and h, we can compute the value of the entire expression.

Given a *formula* $\neg(x \land y) \lor (x \to \neg z)$ if we know the values of x, y, and z, we can compute the value of the the formula.

We think of a propositional variable as having the values of either **true** or **false**.

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A valuation v is like a state of affairs.

The idea. *If we know the state of affairs, we can determine the truth or falsity of any statement.*

Partial Valuations

$$v(x) = \text{true}$$
$$v(y) = \text{false}$$
$$v(z) = \text{false}$$
$$v(\_) = \text{false}$$

$$(x \wedge y) \vee z$$

Note. We will typically only care about a small collection of variables.

We can assume all unspecified variables are assigned to be false.

Valuation (In Agda)

```
Let's do a demo.
```

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This is where we give a formula meaning with respect to a state of the affairs.

We have to say what we want the truth of a statement to be based on its constituent parts.

Evaluation: In Agda

```
Let's do a demo.
```