Propositional Logic II: Meta-Theory

Type Theory and Mechanized Reasoning Lecture 5

Introduction

Administrivia

- Assignment 1 is officially due 11:59PM tomorrow.
- Assignment 2 will be released tomorrow as well.
- If you haven't gotten Agda set up on your machine, please do so ASAP.

Objectives

- 1. Use propositional logic as a setting for learning important concepts and terms in logic.
- 2. Look at what we can say *about* logic, not just with logic.
- 3. Being discussing normal forms, which are used for representing formulas in the context of CS.

Unicode cheatsheet

```
\rightarrow is \->
```

v is \or

N is \bN

¬ is \neg

x is \times

_p is \^p

∧ is \and

:: is \::

λ is \lambda

⊎ is \uplus

Practice Problem

```
Write a function replace-implies which recursively replaces A \rightarrow B in Form with \neg A \lor B.
```

(see the starter code for more details)

Agda Tutorial: Pattern Matching

Standard Pattern Matching

```
my-pred: \mathbb{N} \to \mathbb{N}
my-pred zero = zero
my-pred (suc n) = n
```

We define a function in multiple lines in which the argument varies by pattern.

We can automatically split on a variable using C-c C-c.

Anonymous Functions

```
foo : List \mathbb{N}
foo = map (\lambda { x \rightarrow x * 10 }) (1 :: 2 :: 3 :: [])
my-pred : \mathbb{N} \rightarrow \mathbb{N}
my-pred = \lambda { zero \rightarrow zero
; (suc n) \rightarrow n }
```

As in most languages, we have lambdas.

We can even pattern match within lambdas.

Case Expressions

```
case_of_: {A B : Set} \rightarrow A \rightarrow (A \rightarrow B) \rightarrow B case x of f = f x
```

Because of this, we don't need special case notation.

Casing is just function application(!)

Simple Example

```
my-pred : N → N
my-pred n = case n of λ
{ zero → zero
; (suc n) → n }
```

Because of this, we can get OCaml-like function definitions.

This is nice for simple functions, but it has its drawbacks...

With Abstractions

The slightly more canonical way of pattern matching on intermediate values is by using with abstractions.

demo

Understanding Check

Write a function split of type

```
\{A : Set\} \rightarrow \{n : \mathbb{N}\} \rightarrow Vec A n \rightarrow Fin n \rightarrow List A \times List A
```

which split a vector into to lists given an index.

(the function Data. Vec. toList will be useful here.)

Propositional Logic: Recap

Recall: Boolean Connectives

Propositional logic is the study of logical (Boolean) connectives.

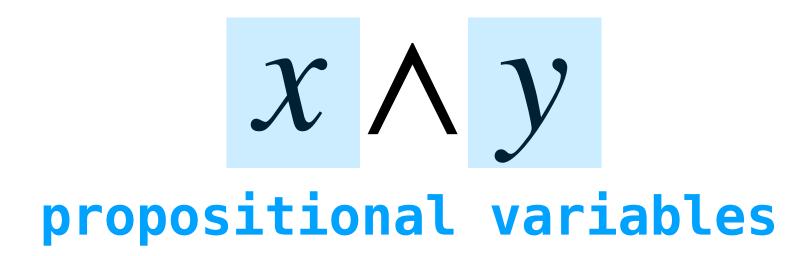
- » What do connectives mean? How do they affect truth?
- » What connectives exists? How many do we need?
- » How do connectives interact?

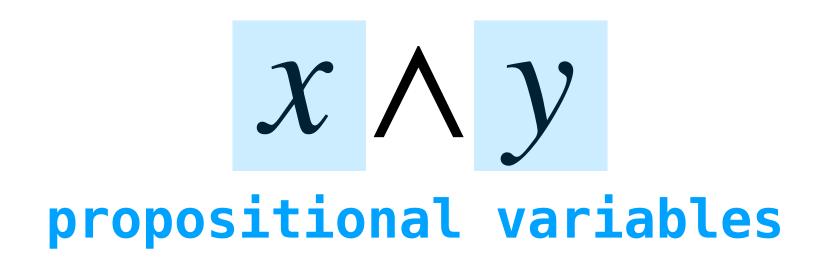
Recall: Programming Conditionals

```
if is_raining and not is_warm:
    # some code
```

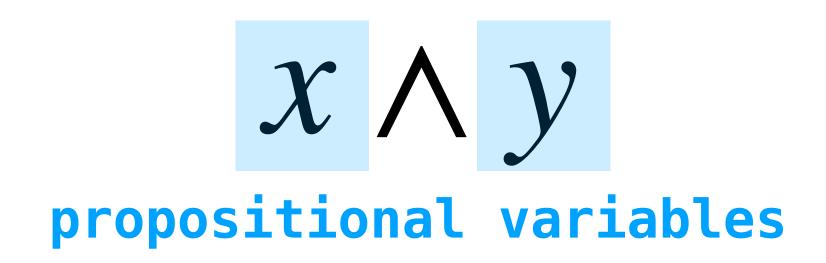
Propositional logic is the study of Bools.

- » When can I replace one conditional with another?
- » Why is there only and, or, and not?
- » Why can conditionals short-circuit?



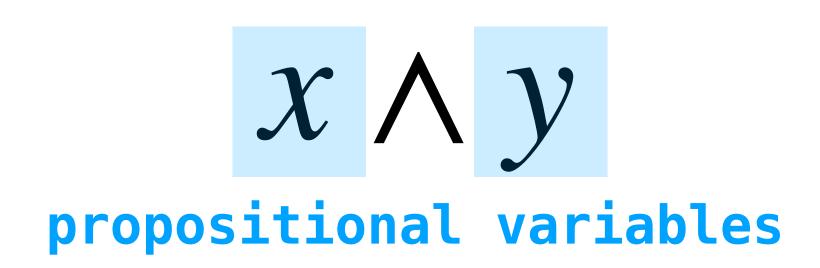


We're interested in how propositions *interact* with respect to connectives.



We're interested in how propositions *interact* with respect to connectives.

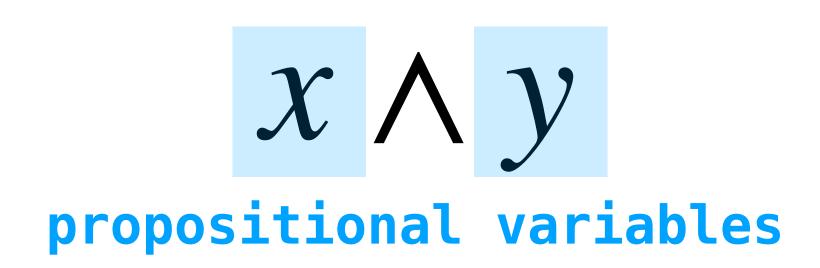
We don't care what the actual propositions are.



We're interested in how propositions *interact* with respect to connectives.

We don't care what the actual propositions are.

We think of these variables in the same was as we think of variables in algebra.*



We're interested in how propositions *interact* with respect to connectives.

We don't care what the actual propositions are.

We think of these variables in the same was as we think of variables in algebra.*

*We will assume a countable number of variable symbols like

in algebra

Syntax: In Agda

```
data Formula : Set where
   _p : String → Formula
   _p_ : Formula → Formula
   _np_ : Formula → Formula → Formula
   _vp_ : Formula → Formula → Formula
   _p_ : Formula → Formula → Formula
```

The tree structure of formulas is implicit in it being an ADT.

Given an expression $3x + (2y \times 6h^2)$ and values for x, y, and h, we can compute the value of the entire expression.

Given an expression $3x + (2y \times 6h^2)$ and values for x, y, and h, we can compute the value of the entire expression.

Given a formula $\neg(x \land y) \lor (x \to \neg z)$ if we know the values of x, y, and z, we can compute the value of the the formula.

Given an expression $3x + (2y \times 6h^2)$ and values for x, y, and h, we can compute the value of the entire expression.

Given a formula $\neg(x \land y) \lor (x \to \neg z)$ if we know the values of x, y, and z, we can compute the value of the the formula.

We think of a propositional variable as having the values of either **true** or **false**.

Definition. A valuation is a function from all possible propositional values to {true, false}.

Definition. A valuation is a function from all possible propositional values to {true, false}.

A valuation v is like a state of affairs.

Definition. A valuation is a function from all possible propositional values to {true, false}.

A valuation v is like a state of affairs.

The idea. If we know the state of affairs, we can determine the truth or falsity of any statement.

Partial Valuations

$$v(x) = \text{true}$$

$$v(y) = false$$

$$(x \wedge y) \vee z$$

$$v(z) = false$$

$$v(\underline{\hspace{0.1cm}}) = false$$

Note. We will typically only care about a small collection of variables.

We can assume all unspecified variables are assigned to be false.

A valuation function can be lifted from propositional variables to arbitrary formulas.

A valuation function can be lifted from propositional variables to arbitrary formulas.

This is where we give a formula meaning with respect to a state of the affairs.

A valuation function can be lifted from propositional variables to arbitrary formulas.

This is where we give a formula meaning with respect to a state of the affairs.

We have to say what we want the truth of a statement to be based on its constituent parts.

demo

The evaluation function we wrote in Agda for a valuation v is written \overline{v} .

The evaluation function we wrote in Agda for a valuation v is written \overline{v} .

We say that a valuation v makes ϕ true if $\overline{v}(\phi) = \text{true}$.

The evaluation function we wrote in Agda for a valuation v is written \overline{v} .

We say that a valuation v makes ϕ true if $\overline{v}(\phi) = \text{true}$.

I leave it as an exercise to write out a "mathy" definition, but the shape will be similar, e.g.,

$$\overline{v}(\neg P) = \begin{cases} \text{false} & \overline{v}(P) = \text{true} \\ \text{true} & \text{otherwise} \end{cases}$$

Propositional Logic: Semantic Notions

Definition. A formula ϕ is **valid** (written $\models \phi$) if *every* valuation v makes ϕ true, e.g.

$$x \vee \neg x$$

Definition. A formula ϕ is **valid** (written $\models \phi$) if *every* valuation v makes ϕ true, e.g.

$$\chi \vee \neg \chi$$

We say that ϕ is a tautology.

Definition. A formula ϕ is **valid** (written $\models \phi$) if *every* valuation v makes ϕ true, e.g.

$$x \vee \neg x$$

We say that ϕ is a tautology.

Definition. A formula ϕ is **satisfiable** if there is *some* valuation which makes ϕ true, e.g.

$$x \vee y$$

Example

Proposition. $A \lor \neg A$ is a tautology for any formula A.

Proof. Let's try it.

Theorem. ϕ is a tautology if and only if $\neg \phi$ is unsatisfiable.

Proof. Let's try it.

Entailment

Definition. A set of formulas Γ **entails** ϕ (written $\Gamma \models \phi$) if *every* valuation which make *every* formula in Γ true, also makes ϕ true.

Example. $\{x \rightarrow y, y \rightarrow z\} \models x \rightarrow z$

The Deduction Theorem

Theorem. $\Gamma \cup \{\phi\} \models \psi$ if and only if $\Gamma \models \phi \rightarrow \psi$.

Proof. Let's try it.

Understanding Check

True or False: If $\Gamma \models A \lor B$ then $\Gamma \models A$ or $\Gamma \models B$.

Definition. Formulas ϕ and ψ are **logically equivalent** (written $\phi \equiv \psi$) if $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$. That is, all valuations agree on ϕ and ψ .

Definition. Formulas ϕ and ψ are **logically equivalent** (written $\phi \equiv \psi$) if $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$. That is, all valuations agree on ϕ and ψ .

Example. $x \wedge y$ is logically equivalent to $y \wedge x$.

Definition. Formulas ϕ and ψ are **logically equivalent** (written $\phi \equiv \psi$) if $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$. That is, all valuations agree on ϕ and ψ .

Example. $x \wedge y$ is logically equivalent to $y \wedge x$.

Logical equivalence captures when boolean conditionals express the same thing.*

Definition. Formulas ϕ and ψ are **logically equivalent** (written $\phi \equiv \psi$) if $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$. That is, all valuations agree on ϕ and ψ .

Example. $x \wedge y$ is logically equivalent to $y \wedge x$.

Logical equivalence captures when boolean conditionals express the same thing.*

Understanding Check

Show that $A \wedge B \equiv \neg(\neg A \vee \neg B)$.

Standard Logical Equivalences

Distribution:

$$A \lor (B \land C) = (A \lor C) \land (B \lor C)$$

$$A \land (B \lor C) = (A \land C) \lor (B \land C)$$

$$A \wedge (B \vee C) = (A \wedge C) \vee (B \wedge C)$$

DeMorgan's Law:

$$A \wedge B \equiv \neg(\neg A \vee \neg B)$$
 $A \vee B \equiv \neg(\neg A \vee \neg B)$

$$A \vee B \equiv \neg(\neg A \vee \neg B)$$

<u>Double Negation Elimination:</u> $\neg(\neg A) \equiv A$

Law of the Excluded Middle: $A \lor \neg A$

And many more...