Math 1210

Section 3.2—The Mean Value Theorem

1—Rolle's Theorem

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

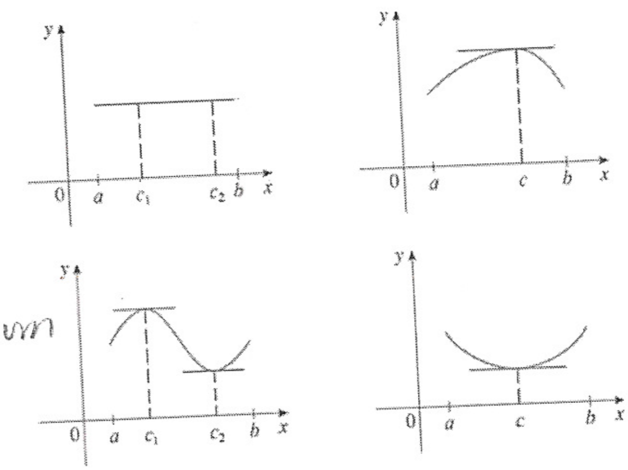
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

where The derivative $= 0$

Proof: There are 3 cases...

Case 1: $f(x) = K$, a constant.
Then $f'(x) = 0$, so c is any number in (a,b).



Case 2: There is an $x \in (a,b)$ such that $f(x) > f(a)$. Then by the Extreme Value Theorem, f has a maximum value in $[a,b]$. Since $f(a) = f(b)$, it must attain its maximum value at a # c in (a,b) ← open interval. Thus f has a local maximum at c and thus by Fermat's Theorem, $f'(c) = 0$

Case 3: $f(x) < f(a)$ for some x in (a,b).
Then by the E.V.T, f has a minimum value in $[a,b]$ and since $f(a) = f(b)$, it attains this minimum value at a # c in (a,b). By Fermat's Theorem, $f'(c) = 0$.

Example 1: Question from last week's worksheet: If a rock is thrown vertically upward from the surface of Mars with velocity 15 m/s, its height after t seconds is $h = 15t - 1.86t^2$.

b) What is the velocity of the rock when its height is 25 m on its way up? On its way down? at $t = 2.35$ & $t = .66$

What does Rolle's Theorem tell us about this situation?

The rock is in the same place at 2 different times: $t = .66$ & $t = 2$
so $f(.66) = f(2.35)$.
Rolle's Theorem says that there is some instant of time $t = c$ where c is between a & b [i.e. $c \in (a,b)$] when $f'(c) = 0$
That is, when the velocity is 0, At the vertex!

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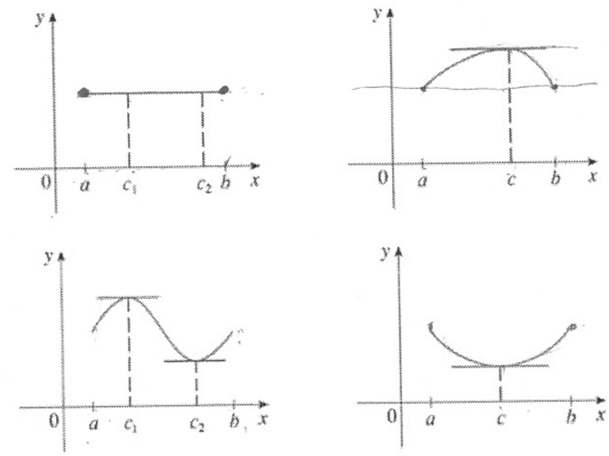
Proof:



Example 1: Question from last week's worksheet: If a rock is thrown vertically upward from the surface of Mars with velocity 15 m/s, its height after t seconds is $h = 15t - 1.86t^2$.

$t = 2.35$

$t = .66$

b) What is the velocity of the rock when its height is 25 m on its way up? On its way down?

What does Rolle's Theorem tell us about this situation?

③ $f(.66) = f(2.35)$ ② f is diff'ble on $(.66, 2.35)$ ① f is cont. everywhere

Rolle's Theorem tells us there is an instant of time $t = c$ where c is between .66 & 2.35 where $f'(c) = 0$. That is, when the velocity is 0. At the vertex!

Example 2: Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Use IVT to prove a real root exists.

$f(0) = -1 < 0$

$f(1) = 1 + 1 - 1 = 1 > 0$

So there is a root between 0 & 1.

To show there is no other real root, we argue by contradiction. Suppose there is another root. That is, suppose there are 2 roots - at a & at b. Then $f(a) = f(b) = 0$. f is cont. & diff'ble on $[a,b]$ & (a,b), respectively.

Then By Rolle's Theorem, there is a number c between a & b such that $f'(c) = 0$.

But $f'(x) = 3x^2 + 1$, and is never equal to 0, which is a contradiction.

So the function cannot have 2 real roots. ∎

2—The Mean Value Theorem

Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) where:

Slope of tang. line — Derivative — Inst. Rate of change

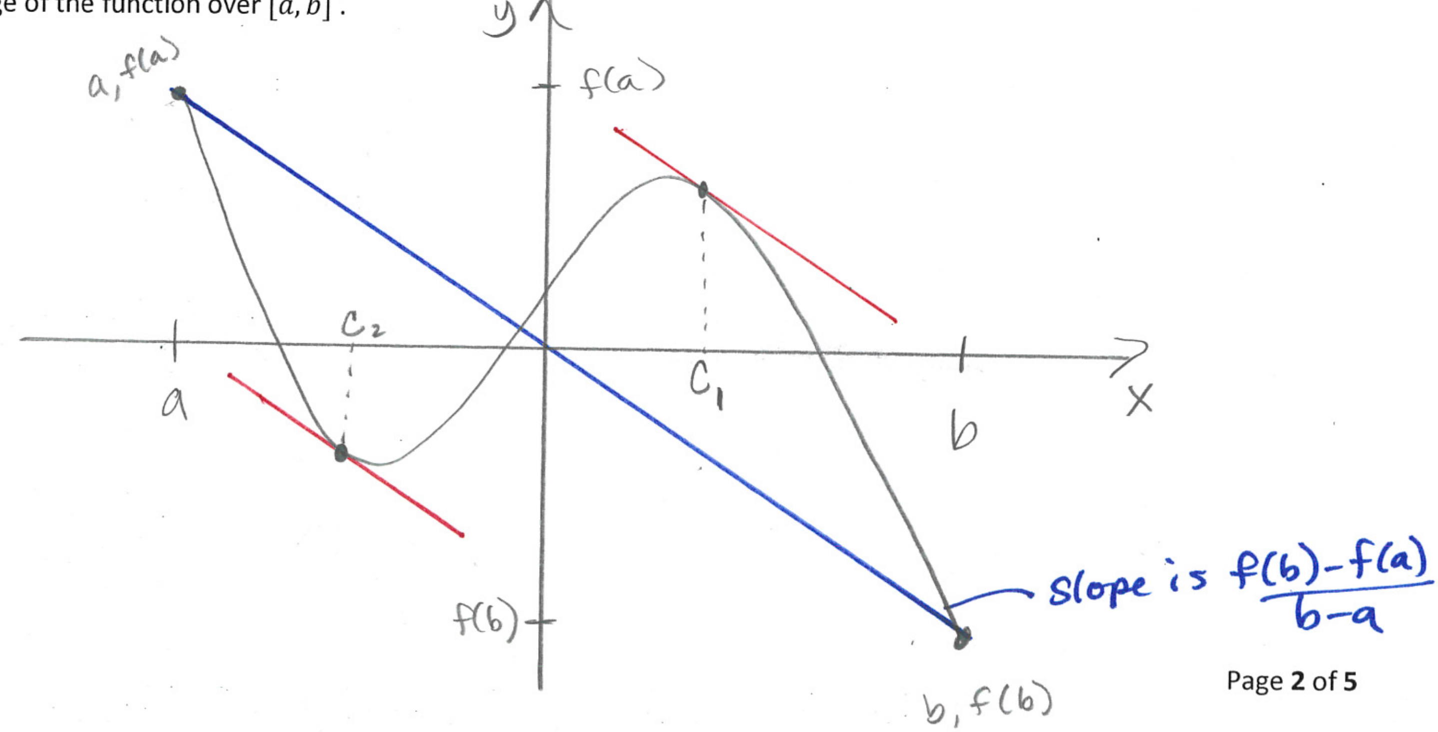
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

slope of secant line through $(a, f(a))$ & $(b, f(b))$ or Avg. rate of change

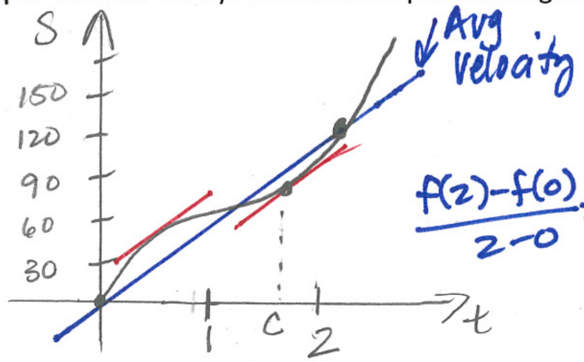
Or equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

This just says that there is a point c in (a, b) where the instantaneous rate of change is equal to the average rate of change of the function over $[a, b]$.



Example 3: A car accelerates from zero and takes 2 hours to go 120 miles. Then its average velocity for the 2 hours is 60 mph. The M.V.T. says that at some point during the trip the car must have been driving exactly 60 mph.



$a=0 \quad b=2$

$\frac{f(b)-f(a)}{b-a}$

$\frac{f(2)-f(0)}{2-0} = \frac{120-0}{2}$

$= 60$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Example 4: Find the values of c that satisfy the Mean Value Theorem for the following functions and intervals.

a) $f(x) = x^2 - 2x - 3$ on $[0,3]$ $a=0$ $b=3$

$\frac{f(3)-f(0)}{3-0} = \frac{0+3}{3} = 1$

$2c - 2 \overset{\text{must}}{=} 1$ by MVT

$2c = 3$

$c = \frac{3}{2}$

$f'(x) = 2x - 2$

$f'(c) = 2c - 2$

Check: $f'(\frac{3}{2}) = 2(\frac{3}{2}) - 2$
$= 3 - 2 = 1$ ✓

b) $f(x) = \sqrt{x-1}$ on $[1,3]$ $(x-1)^{1/2}$ $a=1$ $b=3$

$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{\sqrt{2}-0}{2} = \frac{\sqrt{2}}{2}$

$y - 0 = \frac{\sqrt{2}}{2}(x-1)$

$\boxed{y = \frac{\sqrt{2}}{2}(x-1)}$

$f'(x) = \frac{1}{2}(x-1)^{-1/2} = \frac{1}{2\sqrt{x-1}}$

$f'(c) = \frac{1}{2\sqrt{c-1}}$

$\Rightarrow 2 \cdot \frac{1}{\cancel{2}\sqrt{c-1}} = \frac{\sqrt{2}}{\cancel{2}} \cdot 2$

$\frac{1}{\sqrt{c-1}} = \sqrt{2}$

$\frac{1}{c-1} = 2$

$1 = 2(c-1)$

$1 = 2c - 2$

$3 = 2c$

$c = \frac{3}{2}$

$f(\frac{3}{2}) = \sqrt{\frac{1}{2}}$

$y - \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}(x - \frac{3}{2})$

Example 5: Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

f is diff'ble & thus continuous everywhere. So we can use the M.V.T. We will use the interval $[0,2]$. There exists a # c in between 0 & 2 such that:

$f(b) - f(a) = f'(c)(b-a)$

$f(2) - f(0) = f'(c)(2-0)$

$f(2) + 3 = 2f'(c)$

$f(2) = -3 + 2f'(c)$

$\leq -3 + 2(5)$

$= -3 + 10$

$= 7$

The largest f(2) can possibly be is 7.

Example 6: A truck driver is handed a ticket showing that in 2 hours he covered 159 miles on a toll road with a speed limit of 65 mph. Why was the driver cited?

$$\frac{159-0}{2-0} = \frac{159}{2} = 79.5$$

By the M.V.T. you were driving 79.5 mph at least once.

3—Mathematical Consequences of the Mean Value Theorem

Theorem 5
If $f'(x) = 0$ for every x in the interval (a, b), then f is a constant function on the interval (a, b).

(If the derivative is 0 everywhere (the slope is 0 everywhere), then the function is a constant function.)

It is implied that f is diff'ble & thus continuous on (a,b). and also given $f'(x)=0$ for every x in (a,b).
Choose 2 random points in (a,b), say x_1 & x_2 and $x_1 < x_2$
Then, by the M.V.T.,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \Rightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1} = 0 \Rightarrow f(x_2)-f(x_1)=0$$

$\Rightarrow f(x_2) = f(x_1)$ Thus, f is a constant function

Theorem 7
If $f'(x) = g'(x)$ for every x in an interval (a, b), then $f - g$ is constant on (a, b). That is, $f(x) = g(x) + C$ where C is a constant.

(If two functions have the same derivatives on an interval, then their graphs must be vertical translations of each other. In other words, the graphs have the same shape, but could be shifted up or down.)

Given $f'(x) = g'(x)$

$$\frac{d}{dx}(f-g) = \frac{d}{dx}f - \frac{d}{dx}g$$

Then $f'(x) - g'(x) = 0$

$$= \frac{d}{dx}(f(x)-g(x)) = 0$$

$\Rightarrow f(x)-g(x) = C$ By Theorem 5

$f(x) = g(x) + C$ $\boxed{}$

$f(x) = x^2+5$
$f'(x) = 2x$

$g(x) = x^2-37$
$g'(x) = 2x$

$$f(x)-g(x) = x^2+5-(x^2-37)$$
$$= \cancel{x^2}+5-\cancel{x^2}+37$$
$$= 42 \leftarrow \text{a constant}$$

Example 7: Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0,2)$.

$f'(x) = \sin x$

$f(x) = -\cos x + C$

$f(0) = -\cos 0 + C \overset{\text{must}}{=} 2$

$-1 + C = 2$

$C = 3$

$f(x) = -\cos x + 3$

Example 8: Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Is $f(x) = 3$ for all x? Why or why not?

$f'(x) = 0$ for all x. Then by Theorem 5, f is a constant function.

$f(x) = C$

Since $f(-1) = 3$, then $f(x) \overset{\text{must}}{=} 3$ for all x.

$(0,5)$

Example 9: Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Why or why not?

$f'(x) = 2$

$f(x) = 2x + C$

$f(0) = 2(0) + C \overset{\text{must}}{=} 5$

$C = 5$

$f(x) = 2x + 5$ ← By Theorem 7

Homework: 3, 5-10, 11-20, 26, 27, 29, 31, 32, 34, 35