Section 3.7—Optimization Problems

A manufacturer wants to *minimize* production costs. What are the dimensions of a box with fixed volume having *minimum surface area*? What are the dimensions for the *least expensive* cylindrical can of a given volume? A businessman wants to *maximize* profits. How many items should be produced for the *most profitable* production run?

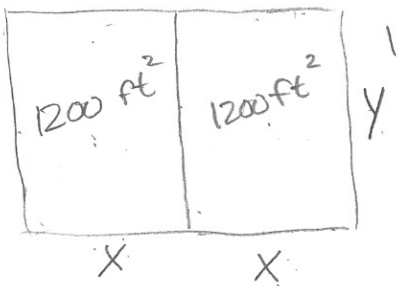
i.e. What is the *best* or *optimal* value of a given function? These types of problems are called *Optimization Problems.*

Solving Applied Optimization Problems

1. Understand what is given—Identify what you need to find/optimize.
2. Draw a picture!
3. Choose a variable(s).
4. Write an equation for the unknown quantity (obtain a mathematical description of the problem).
5. Do the math—test critical points and endpoints.
6. Interpret the results.

Take the Derivative

Example 1: A farmer needs to construct two adjoining (share one side) rectangular pens of identical areas. If each pen is to have an area of 1200 square feet, what dimensions will minimize the cost of fencing?



we want to minimize the perimeter.

$P = 4x + 3y$

$A = xy = 1200$

$y = \frac{1200}{x}$

$P = 4x + 3 \cdot \frac{1200}{x}$

$= 4x + 3600x^{-1}$

$4 - 3600x^{-2}$

$\frac{dP}{dx} = P' = \boxed{4 - \frac{3600}{x^2}} \overset{set}{=} 0$

$= \frac{1200}{30}$

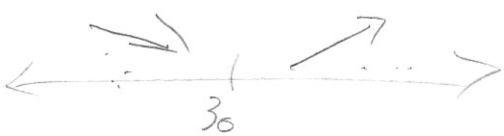
$\boxed{= 40}$

$4 = \frac{3600}{x^2}$

$4x^2 = 3600$

$x^2 = 900$

$\boxed{x = 30}$ ← minimum



$P'(1) = < 0$

$P'(31) = > 0$

Small pens: 30 x 40 feet

large pen: 60 x 40 feet

Example 2: You have been asked to design a one-liter (1000 cm^3) shaped like a right circular cylinder. What dimensions will use the least material? (Ignore the thickness of the material and the waste in manufacturing.)

we want to minimize Surface Area.

$$\boxed{A_{can} = 2\pi r^2 + 2\pi r h} \qquad V = \pi r^2 h = 1000$$

$$h = \frac{1000}{\pi r^2}$$

$$= 2\pi r^2 + 2\cancel{\pi} r\left(\frac{1000}{\cancel{\pi} r^{\cancel{2}}}\right)$$

$$= 2\pi r^2 + \frac{2000}{r} \quad (2000r^{-1})$$

$A_{Base} = \pi r^2$

$A_{side} = 2\pi r h$

$$\rightarrow A' = 4\pi r - \frac{2000}{r^2} \overset{set}{=} 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$4\pi r^3 = 2000$$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \left(\frac{2000}{4\pi}\right)^{1/3} \approx 5.42 \text{ cm}$$

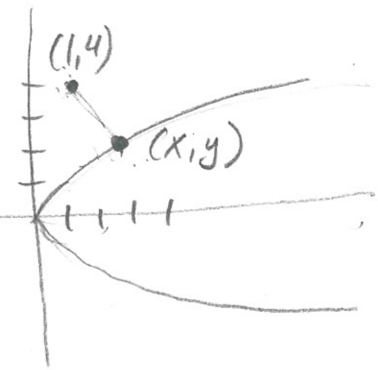
$A'(1) = <0$
$A'(6) = >0$

$$h = \frac{1000}{\pi\left[\left(\frac{2000}{4\pi}\right)^{1/3}\right]^2} = \frac{1000}{\pi\left(\frac{2000}{4\pi}\right)^{2/3}}$$

≈ 10.8385 cm height

$\sqrt{(x-x_1)^2+(y-y_1)^2}$

Example 3: Find the point on the parabola $y^2 = 2x$ that is closest to the point (1,4) .



Distance between 2 pts:

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

$$= \sqrt{\left(\tfrac{1}{2}y^2-1\right)^2 + (y-4)^2}$$

$$d^2 = \left(\tfrac{1}{2}y^2-1\right)^2 + (y-4)^2$$

$$f(y) = \left(\tfrac{1}{2}y^2-1\right)^2 + (y-4)^2$$

$$f'(y) = 2\left(\tfrac{1}{2}y^2-1\right)y + 2(y-4)$$

$$= y^3 - \cancel{2y} + \cancel{2y} - 8$$

$$= \boxed{y^3 - 8} \overset{set}{=} 0$$

$$y^3 = 8$$

$$x = \frac{y^2}{2} = \frac{1}{2}y^2$$

$$x = \frac{1}{2}(2^2) = \frac{4}{2}$$

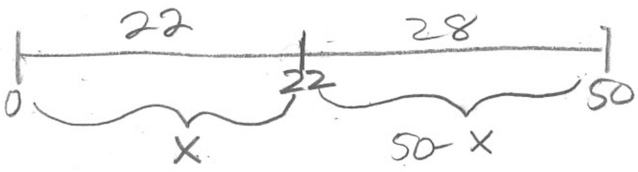
$f'(1) = <0$
$f'(3) = >0$

$\boxed{y = 2}$ $\boxed{x = 2}$

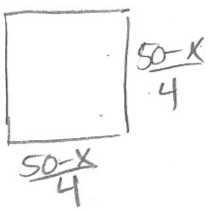
the point on the parabola that is closest to (1,4) is (2,2).

$$\frac{50}{4}-\frac{x}{4}=\frac{50}{4}-\frac{1}{4}x$$

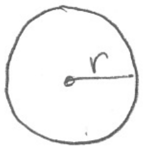
Example 4: A 50-inch piece of wire is cut into two pieces which are then bent into a square and a circle. Where should the wire be cut in order to minimize the sum of the areas?



$$A_T = A_{square} + A_{circle}$$
$$= \left(\frac{50-x}{4}\right)^2 + \frac{x^2}{4\pi} \qquad \frac{1}{4\pi}x^2$$



$$A = \left(\frac{50-x}{4}\right)^2$$



$$A = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2$$
$$= \cancel{\pi}\cdot\frac{x^2}{4\pi^{\cancel{2}}}$$
$$= \frac{x^2}{4\pi}$$

$$C = 2\pi r = x$$
$$r = \frac{x}{2\pi}$$

$$A_T' = 2\left(\frac{50-x}{4}\right)\left(-\frac{1}{4}\right) + \frac{1}{2\pi}x$$
$$= \left(-\frac{1}{2}\right)\left(\frac{50-x}{4}\right) + \frac{x}{2\pi}$$
$$\boxed{= -\left(\frac{50-x}{8}\right) + \frac{x}{2\pi}} \overset{\text{set}}{=} 0$$
$$\frac{x}{2\pi} = \frac{50-x}{8}$$
$$8x = \cancel{2}\pi(50-x)$$
$$4x = 50\pi - \pi x$$
$$4x + \pi x = 50\pi$$
$$x(4+\pi) = 50\pi$$
$$x = \frac{50\pi}{4+\pi} \approx 21.995$$

X is about 22 inches

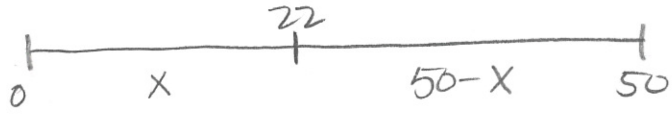
Example 5: If we want to make a rectangular box with a square bottom and no top that holds 32 cubic inches, and the construction material costs 3 cents per square inch, what are the dimensions and the cost of the least expensive box that can be made?

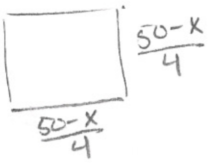
Example 6: A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Example 7: An open-top box is to be made by cutting small congruent squares from the corners of a 12-in. by 12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

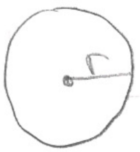
Homework: 2-4, 10-16, 18, 21, 22, 25, 29-38, 40

Example 4: A 50-inch piece of wire is cut into two pieces which are then bent into a square and a circle. Where should the wire be cut in order to minimize the sum of the areas?





$A = \left(\frac{50-x}{4}\right)^2$



$A = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2$

$= \cancel{\pi} \cdot \frac{x^2}{4\pi^2}$

$= \frac{x^2}{4\pi}$

$C = 2\pi r = x$

$r = \frac{x}{2\pi}$

$A_T = A_{square} + A_{circle}$

$A_T = \left(\frac{50-x}{4}\right)^2 + \frac{x^2}{4\pi}$

$A_T' = 2\left(\frac{50-x}{4}\right)\left(-\frac{1}{4}\right) + \frac{1}{2\pi}x$

$= \boxed{-\left(\frac{50-x}{8}\right) + \frac{x}{2\pi} \overset{set}{=} 0}$

$\frac{x}{2\pi} = \frac{50-x}{8}$

$8x = 2\pi(50-x)$

$4x = 50\pi - \pi x$

$4x + \pi x = 50\pi$

$x(4+\pi) = 50\pi$

$\longrightarrow x = \frac{50\pi}{4+\pi} \approx \boxed{21.995 \text{ inches}}$

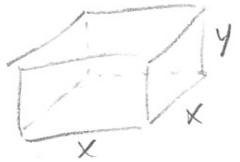
$A'(1) < 0$
$A'(34) > 0$ we have a minimum

Cut the wire at approximately 22 inches

2 pieces, 22 & 28 in long

Example 5: If we want to make a rectangular box with a square bottom and no top that holds 32 cubic inches, and the construction material costs 3 cents per square inch, what are the dimensions and the cost of the least expensive box that can be made?

We want to minimize the Surface Area of the box.



$V = 32 \text{ in}^3$

$x^2 y = 32$

$y = \frac{32}{x^2}$

$A = 4xy + x^2$

$= 4x \cdot \frac{32}{x^2} + x^2$

$= \frac{128}{x} + x^2$

$= 128x^{-1} + x^2$

$A' = \boxed{\frac{-128}{x^2} + 2x \overset{set}{=} 0}$

$2x = \frac{128}{x^2}$

$x^3 = 64$

$\boxed{x = 4}$

$y = \frac{32}{16} = \boxed{2}$

$A'(1) = < 0$

$A'(5) = > 0$

so x is a min

Dimensions: $\boxed{4 \times 4 \times 2 \text{ inches}}$

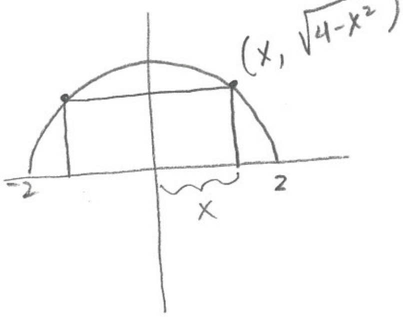
Cost: $A = 4xy + x^2$

$= 4(4)(2) + 16$

$= 32 + 16 = 48 \text{ in}^2$

$48(.03) = \$1.44$

Example 6: A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions? we want to maximize area



$$A = 2xy$$
$$= 2x\sqrt{4-x^2}$$
$$= 2x(4-x^2)^{1/2}$$
$$A' = (4-x^2)^{1/2}(2) + 2x\left(\tfrac{1}{2}\right)(4-x^2)^{-1/2}(-2x)$$
$$= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} \overset{\text{set}}{=} 0$$

undefined for $x = \pm 2$ why?

$$\cancel{2}\sqrt{4-x^2} = \frac{\cancel{2}x^2}{\sqrt{4-x^2}}$$
$$4 - x^2 = x^2$$
$$4 = 2x^2$$
$$x^2 = 2$$
$$\boxed{x = \sqrt{2}}$$

$A'(1) > 0$
$A'(1.5) < 0$
So we have a maximum

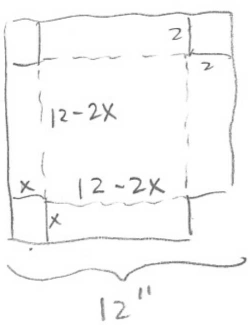
$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \sqrt{4-x^2}$$

$$y = \sqrt{4-(\sqrt{2})^2}$$
$$= \sqrt{4-2}$$
$$\boxed{= \sqrt{2}}$$

Dimensions are $2\sqrt{2} \times \sqrt{2}$

Area is: $2xy = 2\sqrt{2}\cdot\sqrt{2}$
$= 4$ units^2

Example 7: An open-top box is to be made by cutting small congruent squares from the corners of a 12-in. by 12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible? we want to maximize Volume.



$$V = l\cdot w\cdot h$$
$$= x(12-2x)^2$$
$$V' = (12-2x)^2 + x(2)(12-2x)(-2)$$
$$= (12-2x)^2 - 4x(12-2x)$$
$$= (12-2x)\left[(12-2x) - 4x\right]$$
$$\boxed{= (12-2x)(12-6x) \overset{\text{set}}{=} 0}$$

$12 - 2x = 0$
$12 = 2x$
$\cancel{x = 6}$

$12 - 6x = 0$
$12 = 6x$
$\boxed{x = 2}$

$V'(1)$: $+ \cdot + = + > 0$
$V'(3)$: $+ \cdot - = - < 0$
so we have a max at $x = 2$

Squares should be 2 inches square (2 inches on a side)

Homework: 2-4, 10-16, 18, 21, 22, 25,