Section 2.8—Related Rates

1—Related Rates Equations

Sometimes we need to know the rate at which some variable changes when it is known how the rate of some other related variable (or several variables) changes. Usually one rate is easier to measure than the other.

The problem of finding a rate of change from other known rates of change is called a ***related rates*** problem.

Example 1: Suppose we are pumping air into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm? $V = \frac{4}{3}\pi r^3$

*Key points:

- *Both the volume and the radius of the balloon are increasing over time.* In other words, **both V and r** are functions of t.
- Rates of change are derivatives!

What do we know? A constant rate

$\frac{dV}{dt} = 100 \text{ cm}^3/\text{sec}$

What are we asked to find?

Find $\frac{dr}{dt}$ when: diameter is 50 or when $\boxed{r = 25}$

Relate V & r: $V = \frac{4}{3}\pi r^3$

Differentiate w.r.t t: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Substitute: $100 = 4\pi (25)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{\cancel{100}^{25}}{\cancel{4}\pi\, 25^{\cancel{2}}} = \frac{1}{25\pi}$ cm/sec

$\approx .0127$ cm/sec

Related Rates Problem Strategy

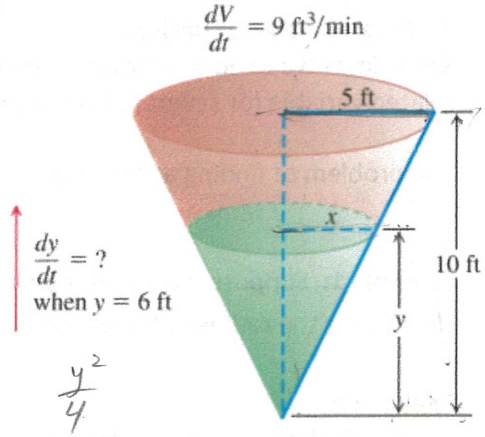
1. Draw a diagram. Use t for time. Assume that all variables are differentiable functions of t.
2. Identify the quantities associated with the given rates and the unknown rates and label them in the diagram.
3. Write down what you are asked to find (usually a rate, expressed as a derivative).
4. Write an equation that relates the quantities to each other. Make sure to distinguish between variable quantities and constant quantities.
5. Implicitly differentiate both sides with respect to t.
6. Evaluate: Substitute the known quantities and rates and solve for the unknown rate.

Example 2: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft. and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft. deep?

Given: $\frac{dV}{dt} = 9$ ft^3/min

Find: $\frac{dy}{dt}$

when: $y = 6$



Volume of a Cone: $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi x^2 y$

solve x in terms of y: $\frac{x}{y} = \frac{5}{10} = \frac{1}{2}$

$\boxed{x = \frac{y}{2}}$

$\frac{y^2}{4}$

$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y$

$\frac{1}{3} \cdot \frac{y^2}{4} = \frac{y^2}{12}$

$V = \frac{1}{12}\pi y^3$

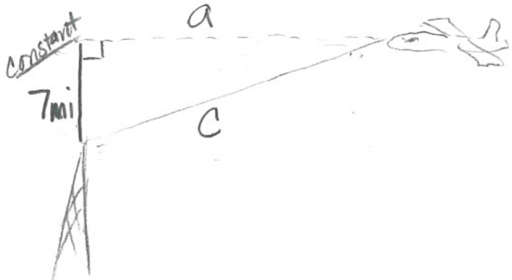
$\frac{1}{12} y^2$

$\frac{dV}{dt} = \frac{1}{4}\pi y^2 \frac{dy}{dt}$

$9 = \frac{1}{4}\pi (36) \frac{dy}{dt}$

$9 = 9\pi \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{\pi}$ ft/min

Example 3: A military plane is flying directly toward an air traffic control tower, maintaining an altitude of 7 miles above the tower. The radar detects the distance between the plane and the tower is 15 miles and that it is decreasing at a rate of 950 miles per hour. What is the ground speed of the plane?



Given: $\frac{dc}{dt} = -950$ mi/hr

at an instant in time, NOT a constant

Find: $\frac{da}{dt}$ when: $c = 15$ mi

$a^2 + 7^2 = c^2$

$\cancel{2}a \frac{da}{dt} = \cancel{2}c \frac{dc}{dt}$

-950×15

When $c = 15$

$a^2 + 7^2 = 15^2$

$a^2 + 49 = 225$

$a^2 = 176$

$a = \sqrt{176}$

$\sqrt{176} \frac{da}{dt} = 15 \cdot (-950)$

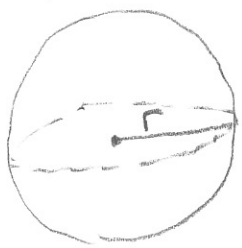
$\frac{da}{dt} = \frac{-14{,}250}{\sqrt{176}} = -1{,}074.13$ mi/hr

Speed $= 1{,}074.13$ mi/hr

Example 4: A spherical balloon is being filled with helium at a rate of 200 cm³/sec. At the moment when the radius is 20 cm, how fast is the surface area of the balloon increasing?

$V = \frac{4}{3}\pi r^3$ Given: $\frac{dV}{dt} = 200$ Find: $\frac{dS}{dt}$ when: $r = 20$ cm

$S = 4\pi r^2$



Relate S & r

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi\, 20 \frac{dr}{dt}$

$\frac{dS}{dt} = \cancel{8\pi}\, 20 \cdot \frac{1}{\cancel{8\pi}}$

$\frac{dS}{dt} = 20 \text{ cm}^2/\text{sec}$

Relate V & r

$V = \frac{4}{3}\pi r^3$

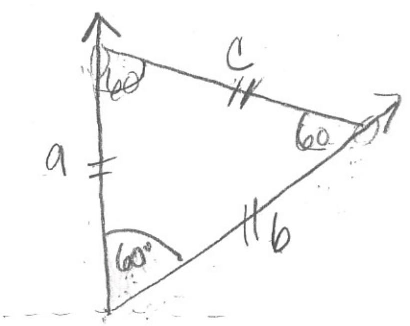
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$200 = 4\pi\, 20^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{200}{4\pi\, 20^2} = \frac{200}{1600\pi} = \frac{1}{8\pi}$

1 knot: 1 nautical mph 10 knots: 10 nautical mph

Example 5: At noon a ship sails due north from a certain point at 10 knots. Another ship leaves the same point at 1:00 p.m. on a course 60° East of North, sailing at 15 knots. How fast is the distance between the ships increasing at 3:00 p.m.?



Given: $\frac{da}{dt} = 10$ knots Find: $\frac{dc}{dt}$ at 3:00pm

$\frac{db}{dt} = 15$ knots

What is c at 3:00 pm?

at 3:00 pm

$a = 30$

$b = 30$

$c = 30$

How to relate a, b & c? Law of Cosines!

$c^2 = a^2 + b^2 - 2ab\underbrace{\cos 60°}_{1/2}$

$f\ g$ / $a\ b$ $\quad f' = \frac{da}{dt}$ $\quad g' = \frac{db}{dt}$

$c^2 = a^2 + b^2 - ab$

$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - \left[b\frac{da}{dt} + a\frac{db}{dt}\right]$

$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - b\frac{da}{dt} - a\frac{db}{dt}$

$60\frac{dc}{dt} = 60(10) + 60(15) - 30(10) - 30(15)$

$\frac{dc}{dt} = 12.5$ mph

Example 6: A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Example 7: A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Example 4: A spherical balloon is being filled with helium at a rate of 200 cm^3/sec. At the moment when the radius is 20 cm, how fast is the surface area of the balloon increasing?

$S = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

Given: $\frac{dV}{dt} = 200 \frac{cm^3}{sec}$ Find: $\frac{dS}{dt}$ When: $r = 20$ cm

Relate S & r

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{dS}{dt} = 160\pi \frac{dr}{dt}$ ← But what is $\frac{dr}{dt}$ when $r = 20$?

$\frac{dS}{dt} = 160\cancel{\pi} \cdot \frac{1}{8\cancel{\pi}}$

$\frac{dS}{dt} = 20$ cm^2/sec

Relate V & r:

$V = \frac{4}{3}\pi r^3$

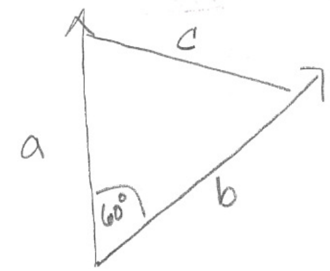
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$200 = 4\pi (20)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{200}{1600\pi} = \boxed{\frac{1}{8\pi}}$

A Knot: 1 nautical mph 10 knots: 10 nautical mph

Example 5: At noon a ship sails due north from a certain point at 10 knots. Another ship leaves the same point at 1:00 p.m. on a course 60° East of North, sailing at 15 knots. How fast is the distance between the ships increasing at 3:00 p.m.?

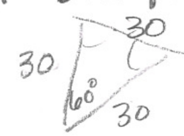


Given: $\frac{da}{dt} = 10$ knots

$\frac{db}{dt} = 15$ knots

Find: $\frac{dc}{dt}$ at 3:00 pm

what is c at 3:00 pm?



How to relate a, b & c? Law of cosines!

$c^2 = a^2 + b^2 - 2ab\cos \underbrace{60}_{1/2}$

$c^2 = a^2 + b^2 - ab$

at 3pm

$a = 30$

$b = 30$

$\Rightarrow c = 30!$

$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - \left[b\frac{da}{dt} + a\frac{db}{dt}\right]$

$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - b\frac{da}{dt} - a\frac{db}{dt}$

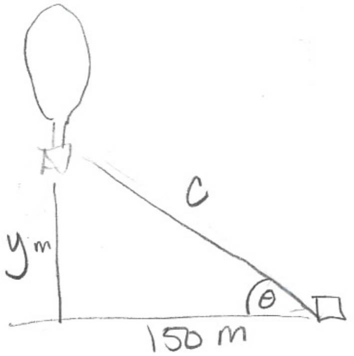
$60\frac{dc}{dt} = 60 \cdot 10 + 60 \cdot 15 - 30 \cdot 10 - 30 \cdot 15$

$60\frac{dc}{dt} = 600 + 900 - 300 - 450$

$60\frac{dc}{dt} = 750 \Rightarrow \frac{dc}{dt} = \frac{750}{60} =$ 12.5 ~~knots~~/hour

Example 6: A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

How fast balloon is rising



Find: $\frac{dy}{dt}$ When: $\frac{d\theta}{dt} = .14$ rad/min & $\theta = \frac{\pi}{4}$

Relate y & θ

$$\tan\theta = \frac{y}{150} = \frac{1}{150}y$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{150}\frac{dy}{dt}$$

$$\sec^2\frac{\pi}{4}(.14) = \frac{1}{150}\frac{dy}{dt}$$

$$\sec\frac{\pi}{4} = \frac{1}{\cos\pi/4} = \frac{2}{\sqrt{2}}$$

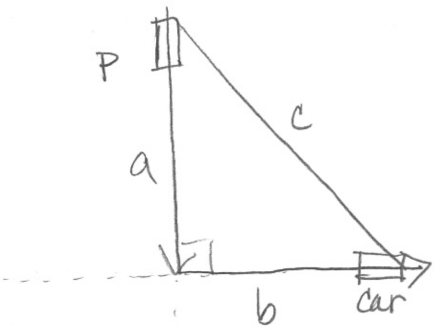
$$\left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2$$

$$150\sec^2\frac{\pi}{4}(.14) = \frac{dy}{dt}$$

$$150 \cdot 2(.14) = \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = 42 \text{ m/min}$$

Example 7: A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



Find: $\left|\frac{db}{dt}\right|$ When: $a = .6$ $b = .8$ $\frac{dc}{dt} = 20$

$\frac{da}{dt} = -60$

$$a^2 + b^2 = c^2$$

$$\cancel{2}a\frac{da}{dt} + \cancel{2}b\frac{db}{dt} = \cancel{2}c\frac{dc}{dt}$$

$$(.6)(-60) + (.8)\frac{db}{dt} = 1(20)$$

$$(.8)\frac{db}{dt} = 20 + 36$$

$$\frac{db}{dt} = \frac{56}{(.8)} = 70 \text{ mph}$$

$$a^2 + b^2 = c^2$$

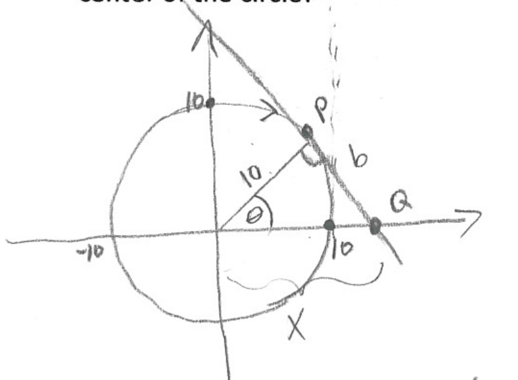
$$(.6)^2 + (.8)^2 = c^2$$

$$.36 + .64 = c^2$$

$$c^2 = 1$$

$$c = 1$$

Example 8: A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is (0, 10) on the y-axis, and its final destination is the point (10, 0) on the x-axis. Once the particle is in motion, the tangent line at P intersects the x-axis at a point Q (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point Q moving along the x-axis when it is 20 m from the center of the circle?



Find: $\frac{dx}{dt}$ when: $X = 20$ & $\frac{d\theta}{dt} = \frac{-\pi}{60}\frac{rad}{sec}$

Relate x & θ

$\cos\theta = \frac{10}{X}$

$\cos\theta = 10X^{-1}$

$-\sin\theta \frac{d\theta}{dt} = -10X^{-2}\frac{dx}{dt}$

$-\frac{\sqrt{300}}{20}\left(\frac{-\pi}{60}\right) = \frac{-10}{20^2}\frac{dx}{dt}$

$\frac{\sqrt{300}\,\pi}{1200} = \frac{-10}{400}\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{\sqrt{300}\,\pi}{\cancel{1200}_{3}} \cdot \frac{400}{-10}$ $\left(-\frac{\sqrt{300}\,\pi}{30}\right)$ or about -1.8 m/sec

sin θ when X = 20

$10^2 + b^2 = 20^2$

$b^2 = 300$

$b = \sqrt{300}$

$\sin\theta = \frac{\sqrt{300}}{20}$

It takes 30 seconds for θ to change $-\frac{\pi}{2}$ radians

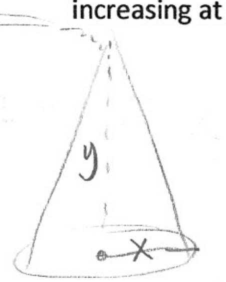
$\frac{d\theta}{dt} = \frac{-\pi/2 \text{ rads}}{30 \text{ seconds}} = \frac{-\pi}{60}\frac{\text{rads}}{\text{sec}}$

Angular Speed

Example 9: Corn is falling from the end of a conveyor belt at a rate of 10 m³/min and is forming a conical pile below. As the corn falls, the height of the pile is remaining equal to the diameter of the base. How fast is the height of the pile increasing at the moment when the pile is 4 m high?

Given: $\frac{dV}{dt} = 10$ m³/min

Find: $\frac{dy}{dt}$ when $y = 4$ m



diameter = 2X

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi x^2 y$

$y = 2X \Rightarrow X = \frac{y}{2}$

$V = \frac{1}{3}\pi x^2 y$

$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y$

$V = \frac{1}{3}\pi \frac{y^3}{4}$

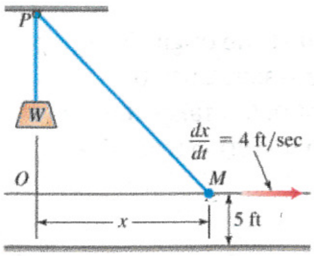
$V = \frac{1}{12}\pi y^3$

$\frac{dV}{dt} = \frac{1}{4}\pi y^2 \frac{dy}{dt}$

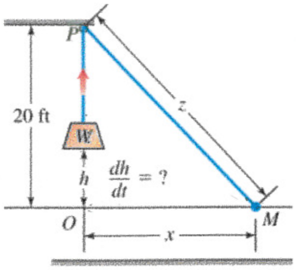
$10 = \frac{1}{\cancel{4}}\pi \cdot \cancel{16}^{4} \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{10}{4\pi} = \frac{5}{2\pi}$ m/min.

$\approx .7958$ or $.8$ m/min



(a)



(b)

Example 10: The figure to the left shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line PW at the rate of 4 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from PW?

Find: $\frac{dh}{dt}$ when: $x = 21$ feet and $\frac{dx}{dt} = 4$ ft/sec.

1 to find z

$$20^2 + 21^2 = z^2$$
$$400 + 441 = z^2$$
$$z^2 = 841$$
$$z = \sqrt{841} = 29$$

Relate h & x

$$20^2 + x^2 = z^2$$
$$\cancel{2}x \frac{dx}{dt} = \cancel{2} z \frac{dz}{dt}$$
$$21(4) = 29 \frac{dz}{dt}$$
$$84 = 29 \frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{84}{29} \text{ ft/sec} \approx 2.9 \text{ ft/sec}$$

2 Notice to find $\frac{dz}{dt}$

$$20 - h + z = 45$$
$$45 + h = 20 + z$$
$$z = 25 + h$$
$$\frac{dz}{dt} = \frac{dh}{dt}$$