Section 3.1—Maximum and Minimum Values

1—Extreme Values and Local Extrema of Functions

Given a function, discuss the following in everyday language:

Extreme values of a function f — Known as extrema — high or low points maximum minimum

Global maxima or minima

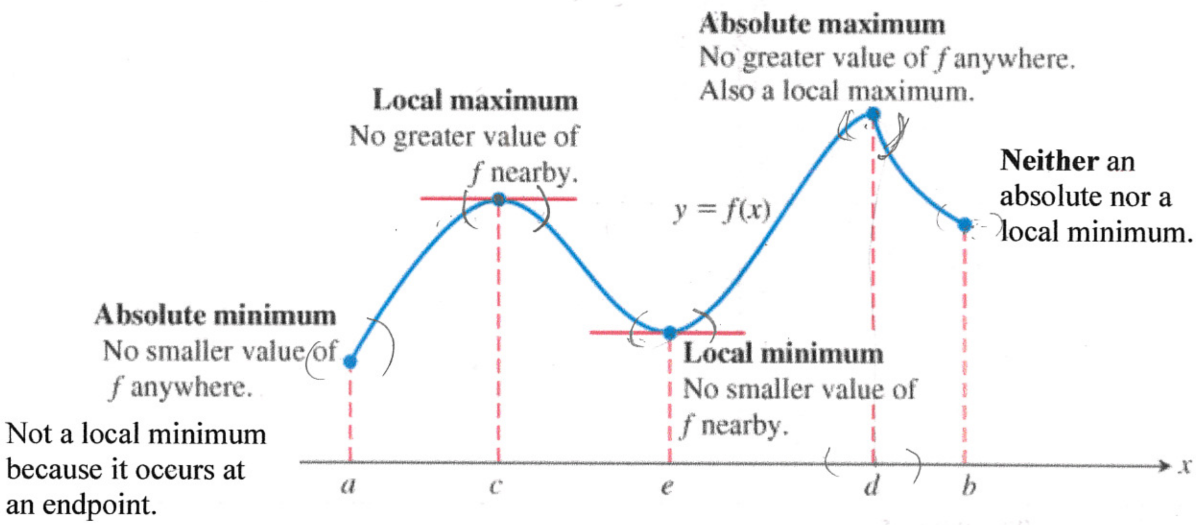
Absolute maxima or minima

Absolute highest or lowest point a function reaches — usually also a local max or min

Relative maxima or minima

Local maxima or minima

a high or low point that is the highest or lowest in a neighborhood – Relatively high or low.



Definitions of Extreme Values of Functions

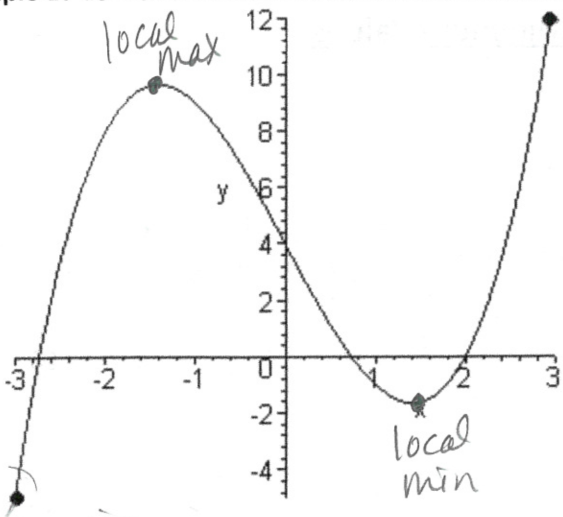
Let c be a number in the domain D of a function f. Then $f(c)$ is the

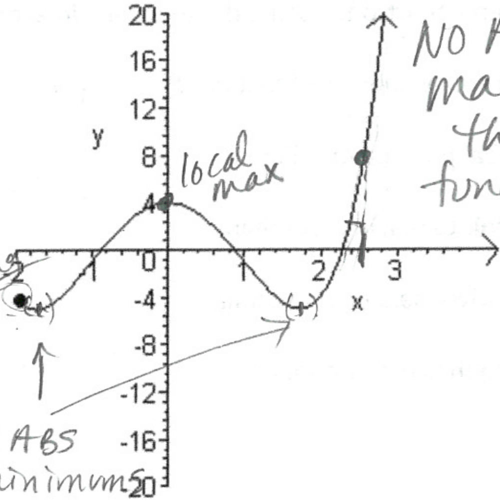
- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

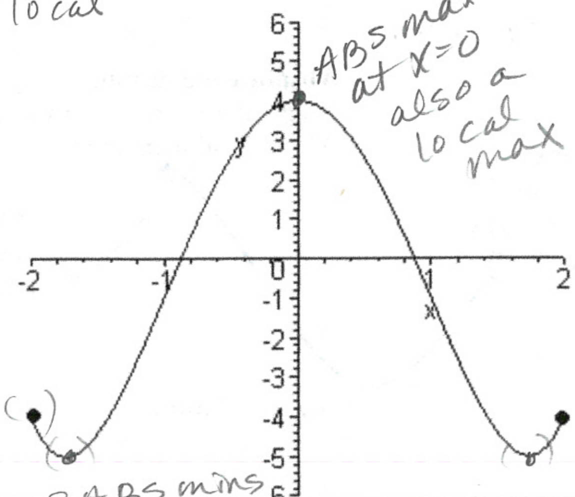
The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ for all x in some open interval (or neighborhood) containing c.
- **local minimum** value of f if $f(c) \leq f(x)$ for all x in some open interval (or neighborhood) containing c.

Example 1: Locate the local and absolute extrema of the graphs of the following functions.







Extreme Value Theorem

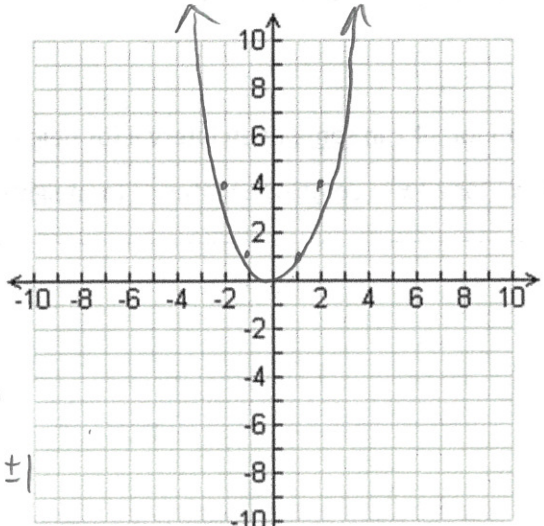
If f is continuous on a closed interval $[a, b]$, then f *will achieve* both an absolute maximum and an absolute minimum on $[a, b]$.

(Somewhere in the middle, or at one of the endpoints, a or b.)

Note that the requirements in the Theorem that the interval be closed and finite, and that the function be continuous, cannot be dropped.

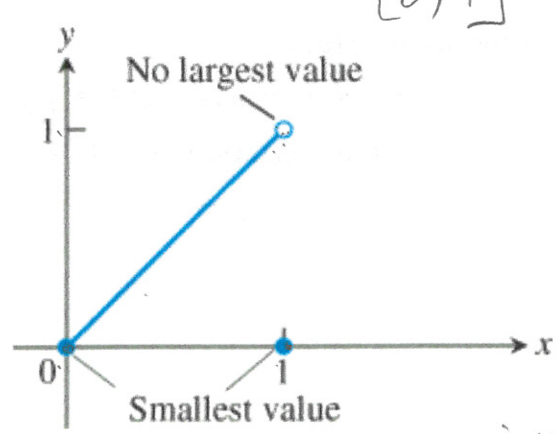
Examine the graph of $f(x) = x^2$ on the following intervals: $(-\infty, \infty)$, [-1, 1], (0,2]

- $(-\infty,\infty)$ – ABS min at $x=0$ NO ABS max
- $[-1,1]$ – ABS min at $x=0$ ABS max at $x=\pm1$
- $(0,2]$ = No ABS min ABS max at $x=2$



fn is continuous, but we don't have closed interval

$[0,1]$



- ABS min at $x=0$ & $x=1$
- ABS max? None

2— Finding Extrema

Definition

If c is in the domain of f and $f'(c) = 0$ OR if $f'(c)$ is undefined, then we call c a **critical number** of the function f.

$8 \cdot x^{3/5} \cdot x^{2/5}$

Example 2: Find the critical numbers for the following functions:

a) $g(x) = \cancel{x \ln x}$ $\quad g(x) = x^{3/5}(4 - x) = 4x^{3/5} - x^{8/5}$

$g'(x) = \frac{12}{5}x^{-2/5} - \frac{8}{5}x^{3/5} = \frac{12}{5x^{2/5}} - \frac{8x^{3/5}}{5} = \frac{12 - 8x}{5x^{2/5}} \overset{\text{set}}{=} 0$

$12 - 8x = 0$

$12 = 8x$

$x = \frac{12}{8} = \boxed{\frac{3}{2}}$

2 critical #'s

$\boxed{x \neq 0}$ is undefined

b) $s(x) = \sqrt[3]{x} = x^{1/3}$

$s'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \overset{\text{set}}{=} 0$ Never equals 0.

undefined at $x = 0$ $\leftarrow$ 1 critical number

If $f(c)$ is a local maximum or local minimum, then c is a critical number of f. This means we can check for critical points to find extrema of our function.

Fermat's Theorem

If f has a local maximum or minimum at a point c, and if $f'(c)$ exists, then $f'(c) = 0$.

That is, the slope of the curve (the derivative) at a local maximum or minimum is equal to 0 .
Thus, all maxima and minima are found at critical points. BUT, not all critical points are maxima or minima!

This means that the only domain points where a function can assume extreme values are at critical points and endpoints that are included. The result of this is that we only need to examine a few values to find a function's extrema—the critical points and the endpoints!

The Closed Interval Method

To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Evaluate f at the critical points. That is, check where $f'(x) = 0$ or where $f'(x)$ does not exist.
2. Evaluate f at endpoints. That is, find $f(a)$ and $f(b)$.
3. Take the largest and smallest of these values.

Example 3: Find the absolute maximum and minimum values of the following functions on the given intervals.

a) $f(x) = 3x^4 - 4x^3 - 12x^2$ on $[-2,3]$.

$f'(x) = 12x^3 - 12x^2 - 24x \overset{\text{set}}{=} 0$

$12x(x^2 - x - 2) = 0$

$12x(x+1)(x-2) = 0$

$x=0 \quad x=-1 \quad x=2$ ← 3 critical #'s

$f(0) = 0$

$f(-1) = -5$

$f(2) = -32$ ← ABS min of -32 at $x=2$ $(2,-32)$

$f(-2) = 32$ ← ABS max is 32 at $x=-2$. $(-2,32)$

$f(3) = 27$

b) $g(x) = x^{3/5}(2-x)$ on $[-1,2]$.

$f' = \frac{3}{5}x^{-2/5}$ $\quad g' = -1$

$g'(x) = (2-x)\left(\frac{3}{5}x^{-2/5}\right) - x^{3/5}$

$= \frac{6}{5}x^{-2/5} - \frac{3}{5}x^{3/5} - x^{3/5}$

$= \frac{6}{5}x^{-2/5} - \frac{8}{5}x^{3/5}$

$= \frac{6}{5x^{2/5}} - \frac{8x^{3/5}}{5} \overset{\text{set}}{=} 0$

$5x^{2/5} \cdot \frac{6}{5x^{2/5}} = \frac{8}{5}x^{3/5} \cdot 5x^{2/5}$

$6 = 8x \Rightarrow x = \frac{6}{8} = \frac{3}{4}$

$f(0) = 0$

$f\left(\frac{3}{4}\right) = 1.05$ ← ABS max is 1.05 at $x = 3/4$

$f(-1) = -3$ ← ABS min is -3 at $x=-1$

$f(2) = 0$

c) $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

Extra practice:

d) $f(x) = \frac{3x}{2x^2+2}$ on $[-4,4]$ $f' = 3$ $g' = 4x$

$f'(x) = \frac{3(2x^2+2) - 3x(4x)}{(2x^2+2)^2} = \frac{6x^2+6-12x^2}{(2x^2+2)^2} = \frac{-6x^2+6}{(2x^2+2)^2} \stackrel{set}{=} 0$

↑ never 0 so f' is never undefined

$-6x^2+6 = 0$

$-6(x^2-1) = 0$

$(x-1)(x+1) = 0$

$x = 1 \quad x = -1$ ← 2 critical #'s

$f(1) = \frac{3}{2+2} = \boxed{\frac{3}{4}} = \boxed{.75}$ ← ABS max is .75 at x = 1

$f(-1) = \boxed{\frac{-3}{4}} = \boxed{-.75}$ ← ABS min is -.75 at x = 1

$f(-4) = \boxed{\frac{-12}{34}} = \frac{-6}{17} \approx \boxed{-.35}$

$f(4) = \boxed{\frac{12}{34}} = \frac{6}{17} \approx \boxed{.35}$

e) $h(x) = x + 2\cos x$ on the closed interval $[0, \pi]$

$h'(x) = 1 - 2\sin x \stackrel{set}{=} 0$

$1 = 2\sin x$

$\sin x = \frac{1}{2}$

$x = \pi/6 \ \ \& \ \ \frac{5\pi}{6}$

$h(\pi/6) = \pi/6 + 2\cos \pi/6 = \pi/6 + \sqrt{3} \approx \boxed{2.256}$ ← ABS max is 2.256 at $x = \pi/6$

$h(5\pi/6) = 5\pi/6 + 2\cos 5\pi/6 = \frac{5\pi}{6} - \sqrt{3} \approx \boxed{.8859}$

ABS min is .8859 at $x = \frac{5\pi}{6}$

$h(0) = 0 + 2\cos 0 = 0 + 2(1) = \boxed{2}$

$h(\pi) = \pi + 2\cos\pi = \pi + 2(-1) = \pi - 2 \approx \boxed{1.1415}$

Homework: 1-6, 7-13 odd, 15-54 multiples of 3, 25, 46, 52, 55, 69