Section 3.5—Summary of Curve Sketching

Guidelines for sketching a Curve

The following checklist is intended as a guide to sketching a curve by hand without a calculator. Not every item is relevant to every function. But the guidelines provide all the information needed to make a sketch that displays the most important aspects of the function.

1. Domain
2. *x* and *y*-intercept(s)
3. Symmetry / Periodicity
4. Horizontal, Vertical, and Slant Asymptotes
5. Intervals of Increase or Decrease
6. Local Maximum and Minimum Values
7. Concavity and Points of Inflection

Once you have completed the checklist, sketch the graph.

Even if $f(-x) = f(x)$

Odd if $f(-x) = -f(x)$

Example 1: Sketch the graphs of the following functions by hand (no calculator!) When you think you are finished, you may check your graph with a calculator or desmos. Keep practicing until you can graph functions confidently without relying on a graphing utility.

a) $f(x) = x + 2\sin x$ on $[0,2\pi]$

① Domain: $[0, 2\pi]$

② y-int: $(0,0)$

③ Periodic

④ $\lim_{x\to\infty} x + 2\sin x = \infty$

⑤ $\boxed{f'(x) = 1 + 2\cos x} \overset{set}{=} 0$

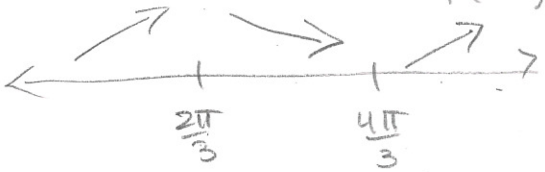
⑥ $2\cos x = -1$

max $\cos x = -\frac{1}{2}$

$\boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ min}}$ ← 2 crit. pts.

$f'(0) = 3 > 0$ $f'(\pi) = 1 + 2\cos\pi = -1 < 0$

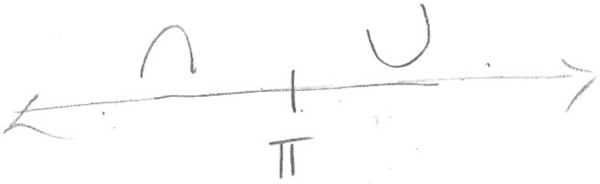
$f'(2\pi) > 0$



⑦ $f''(x) = -2\sin x \overset{set}{=} 0$

$\sin x = 0$

$\boxed{x = 0, \pi, 2\pi}$ ← 1 Inf. pts.



$f''(\pi/2) = -2\sin(\frac{\pi}{2}) = -2 < 0$

$f''(\frac{3\pi}{2}) = -2\sin(\frac{3\pi}{2}) = 2 > 0$

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + \cancel{2}\frac{\sqrt{3}}{\cancel{2}} = \frac{2\pi}{3} + \sqrt{3}$

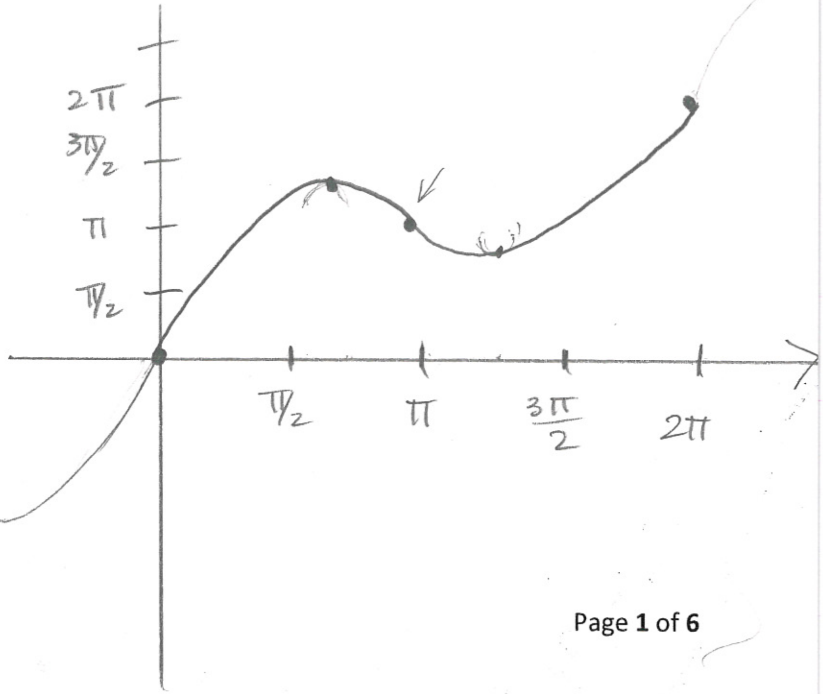
≈ 3.826 ← max

$f(\frac{4\pi}{3}) = 2.456$ ← min

$f(\pi) = \pi + 2\sin\pi = \pi$ (π, π)

$f(0) = 0 + 2\sin 0 = 0$ $(0,0)$

$f(2\pi) = 2\pi + 2\sin 2\pi = 2\pi$ $(2\pi, 2\pi)$



b) $f(x) = \frac{4x}{x^2+4}$

① Domain : $\mathbb{R}$

② y-int : (0,0)

X-int : (0,0) ← 1 x-int !

③ Even or odd ?

$f(-x) = \frac{4(-x)}{(-x)^2+4} = \frac{-4x}{x^2+4}$ Neither

No symmetry

④ Look for H.A.

$\lim_{x \to \infty} \frac{4x}{x^2+4} = \lim_{x \to \infty} \frac{4x/x^2}{x^2/x^2 + 4/x^2} = \lim_{x \to \infty} \frac{4/x}{1 + 4/x^2} = 0$

H.A. at line $\boxed{y = 0}$

⑤ $f'(x) = \frac{(x^2+4)(4) - 4x(2x)}{(x^2+4)^2}$

$= \frac{4x^2+16-8x^2}{(x^2+4)^2}$

$\boxed{= \frac{-4x^2+16}{(x^2+4)^2}} \overset{\text{set}}{=} 0$

$-4x^2+16 = 0$

$-4(x^2-4) = 0$

$-4(x-2)(x+2) = 0$

$\boxed{\underset{\text{max}}{x = 2} \quad \underset{\text{min}}{x = -2}}$ ← 2 crit. pts.

⑥ $f''(x) = \frac{(x^2+4)^{\cancel{2}}(-8x) - (-4x^2+16)(2)(\cancel{x^2+4})(2x)}{(x^2+4)^{\cancel{4}}}$

$= \frac{(x^2+4)(-8x) - 4x(-4x^2+16)}{(x^2+4)^3}$

$= \frac{-\cancel{8x^3} - 32x + \cancel{16}x^3 - 64x}{(x^2+4)^3}$

$= \frac{8x^3 - 96x}{(x^2+4)^3}$

$\boxed{= \frac{8x(x^2-12)}{(x^2+4)^3}} \overset{\text{set}}{=} 0$

$8x(x^2-12) = 0$

$\boxed{x = 0}$ $\quad x^2 = 12$

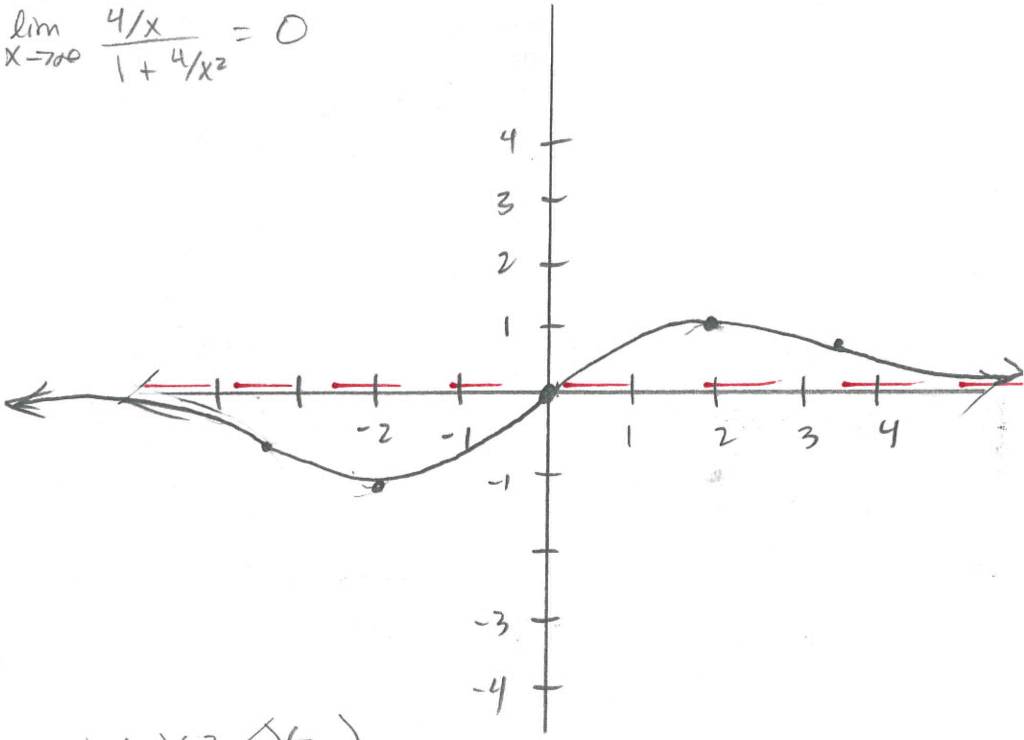
$\boxed{x = \pm 2\sqrt{3} \approx \pm 3.46}$ ← 3 pts of inflection

$f(-2) = \frac{-8}{8} = -1$

$f(2) = \frac{8}{8} = 1$

$f(-2\sqrt{3}) = -\frac{\sqrt{3}}{2} \approx -.866$

$f(2\sqrt{3}) = \frac{\sqrt{3}}{2} \approx .866$

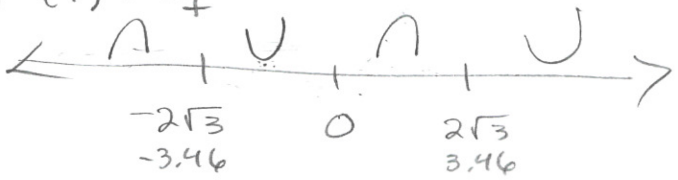


$f''(-4) = \frac{- \cdot +}{+} = -$

$f''(-1) = \frac{- \cdot -}{+} = +$

$f''(1) = \frac{+ \cdot -}{+} = -$

$f''(4) = \frac{+ \cdot +}{+} = +$



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Example 1: Sketch the graphs of the following functions by hand (no calculator!) When you think you are finished, you may check your graph with a calculator or desmos. Keep practicing until you can graph functions confidently without relying on a graphing utility.

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The following checklist is intended as a guide to sketching a curve by hand without a calculator. Not every item is relevant to every function. But the guidelines provide all the information needed to make a sketch that displays the most important aspects of the function.

1. Domain
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3. Symmetry
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Once you have completed the checklist, sketch the graph.

a) $f(x) = x + 2\sin x$ on $[0,2\pi]$

① Domain : $[0, 2\pi]$

② y-int : $(0,0)$

③ periodic

④ No V.A. No H.A. since on $[0, 2\pi]$

Notice $\lim_{x\to\infty} x+2\sin x = \infty$

⑤ $f'(x) = 1 + 2\cos x \overset{\text{set}}{=} 0$

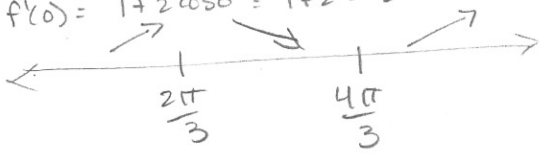
$2\cos x = -1$

$\cos x = -\frac{1}{2}$

$\boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3}}$ cr. pts.

$f'(\pi) = 1+2\cos\pi = 1-2 = -1 < 0$

$f'(0) = 1+2\cos 0 = 1+2 = 3 > 0$

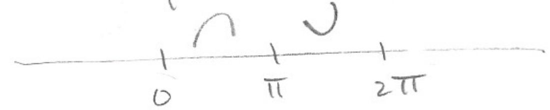


$f'(\frac{3\pi}{2}) = 1 + 2\cos\frac{3\pi}{2} = 1 > 0$

⑥ $f''(x) = -2\sin x \overset{\text{set}}{=} 0$

$\sin x = 0$

$\boxed{x = 0, \textcircled{\pi}, 2\pi}$ Inf. pt.



~~$f''(-\pi/2)\ -2\sin(-\frac{\pi}{2}) = (-2)(-1) = 2 > 0$~~

$f''(\pi/2) = -2\sin(\pi/2) = (-2)(1) = -2 < 0$

$f''(3\pi/2) = -2\sin(\frac{3\pi}{2}) = -2(-1) = 2 > 0$

local min @ $x = \frac{4\pi}{3}$

$f(\frac{4\pi}{3}) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3}$

$= \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{4\pi}{3} - \sqrt{3}$

$= 2.456$

$(4\pi/3, 2.456)$

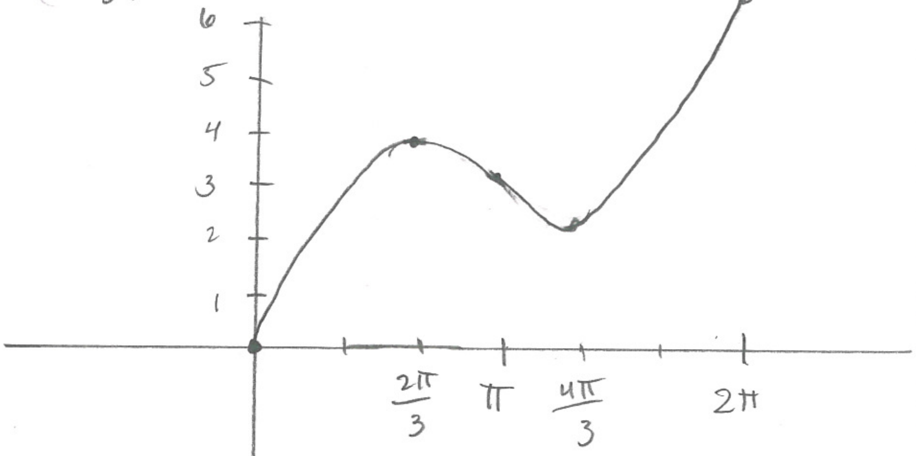
local max at $x = \frac{2\pi}{3}$

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}$

$= \frac{2\pi}{3} + 2\frac{\sqrt{3}}{2}$

$= 3,826$

$(\frac{2\pi}{3}, 3,826)$



$f(0) = 0 \quad (0,0)$

$f(\pi) = \pi + 2\sin\pi = \pi \quad (\pi, \pi)$

$f(2\pi) = 2\pi + 2\sin 2\pi$

$= 2\pi \quad (2\pi, 2\pi$

b) $f(x) = \frac{4x}{x^2+4}$

① Domain: $\mathbb{R}$

② y-int: $(0,0)$

x-int: $4x \overset{set}{=} 0$

$\boxed{x=0}$ 1 x-int

$f = 4x$ $g = x^2+4$

$f' = 4$ $g' = 2x$

③ Even or odd?

$f(-x) = \frac{4(-x)}{(-x)^2+4} = \frac{-4x}{x^2+4}$ Neither No symmetry

④ No V.A. — Look for H.A.

$$\lim_{x\to\infty} \frac{4x}{x^2+4} = \lim_{x\to\infty} \frac{4x/x^2}{x^2/x^2+4/x^2} = \lim_{x\to\infty} \frac{4/x}{1+4/x^2} = 0$$

$\boxed{y=0 \text{ is a H.A.}}$

⑤ $$f'(x) = \frac{(x^2+4)4 - 4x(2x)}{(x^2+4)^2}$$

$$= \frac{4x^2+16-8x^2}{(x^2+4)^2}$$

$$= \frac{-4x^2+16}{(x^2+4)^2} = \frac{-4(x^2-4)}{(x^2+4)^2} \overset{set}{=} 0$$

$x^2-4=0$

$(x-2)(x+2)=0$

$\boxed{x=2 \quad x=-2}$ 2 crit. pts.

local max local min

$f(2) = \frac{8}{8} = 1$

$f(-2) = \frac{-8}{8} = -1$

$f(0) = 0$

$f(2\sqrt{3}) = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2} \approx .866$

$f(-2\sqrt{3}) = \frac{-8\sqrt{3}}{16} = -\frac{\sqrt{3}}{2} \approx -.866$

$(3.46, .866)$

$(-3.46, -.866)$

⑥ $$f''(x) = \frac{(x^2+4)^2(-8x) - (-4x^2+16)(2)(x^2+4)(2x)}{(x^2+4)^4}$$

$$= \frac{-8x(x^2+4)^2 - 4x(-4x^2+16)(x^2+4)}{(x^2+4)^4}$$

$$= \frac{(-4x)\cancel{(x^2+4)}\left[2(x^2+4) + (-4x^2+16)\right]}{(x^2+4)^{\cancel{4}}}$$

$$= \frac{-4x\left[2x^2+8-4x^2+16\right]}{(x^2+4)^3}$$

$$= \frac{-4x\left[-2x^2+24\right]}{(x^2+4)^3}$$

$$= \frac{8x(x^2-12)}{(x^2+4)^3} \overset{set}{=} 0$$

↑ Always positive

⑦ $\boxed{x=0}$ or $x^2 = 12$

$\boxed{x = \pm 2\sqrt{3}}$ $\approx \pm 3.46$

3 pts. of Inflection

∩ ∪ ∩ ∪

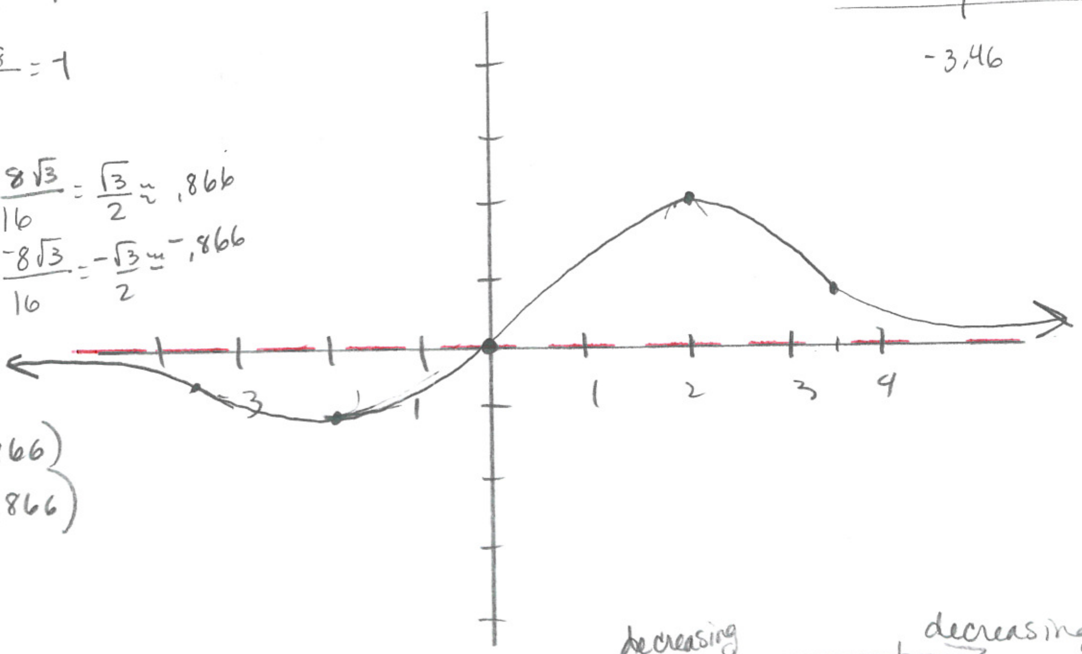
-3.46 0 3.46

$f''(-4) = \frac{- \cdot +}{+} = -$

$f''(-1) = \frac{- \cdot -}{+} = +$

$f''(1) = \frac{+ \cdot -}{+} = -$

$f''(4) = \frac{+ \cdot +}{+} = +$



decreasing ← -3.46 3.46 → decreasing

$f'(-4) = \frac{-}{+} = -$

$f'(4) = \frac{-}{+} = -$

c) $f(x) = \frac{2x^2}{x^2-1}$

① Domain: $x \neq \pm 1$

② y-int: $(0,0)$

x-int: same ← just one!

③ Even - so will be symmetrical across the y-axis only need to graph $(0-\infty)$

④ V.A at $x = 1$ & $x = -1$

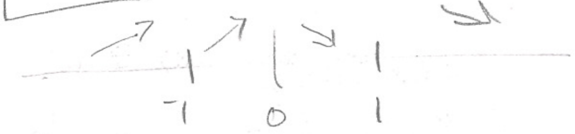
H.A. at $y = 2$

⑤ $f'(x) = \frac{(x^2-1)(4x) - 2x^2(2x)}{(x^2-1)^2}$

$= \frac{\cancel{4x^3} - 4x - \cancel{4x^3}}{(x^2-1)^2}$

$\boxed{= \frac{-4x}{(x^2-1)^2} \overset{set}{=} 0}$ ← Undefined for $x = \pm 1$

$\boxed{\begin{matrix} x=0 \\ x=1 \\ x=-1 \end{matrix}}$ ← 3 cri points!



$f'(-\frac{1}{2})$: $\frac{+}{+} > 0$

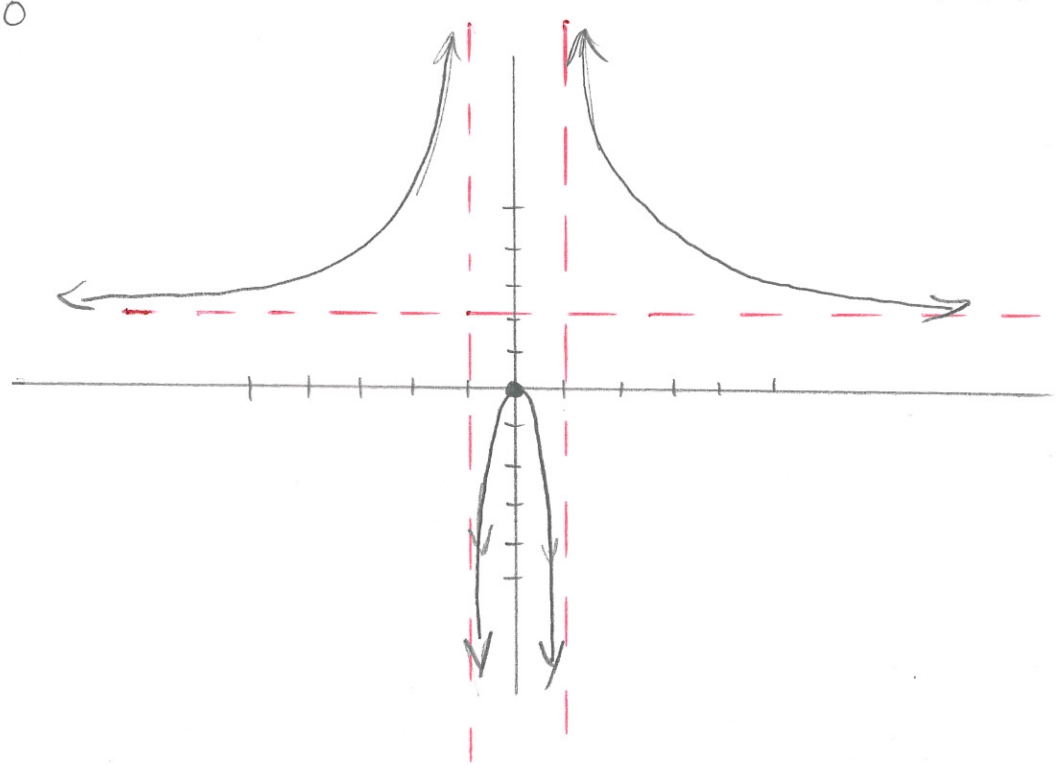
$f'(\frac{1}{2})$: $\frac{-}{+} < 0$

local max at $x=0$

$f'(-2)$: $\frac{+}{+} > 0$

$f'(2)$: $\frac{-}{+} < 0$

$f(0) = 0$



⑥ $f''(x) = \frac{(x^2-1)^2(-4) - (-4x)(2)(x^2-1)(2x)}{(x^2-1)^4}$

$= \frac{-4(x^2-1)^2 + 8x^2(x^2-1)}{(x^2-1)^4}$

$= \frac{-4(x^2-1) + 8x^2}{(x^2-1)^3} = \frac{-4x^2+4+8x^2}{(x^2-1)^3}$

$\boxed{= \frac{4x^2+4}{(x^2-1)^3} \overset{set}{=} 0}$

$4x^2+4=0$

$4(x^2+1)=0$ ← Never 0

$(x^2-1)^3 \overset{set}{=} 0$ ← Undefined for $\boxed{x = \pm 1}$

2 possible Inflection pts.

$f''(-2)$: $\frac{+}{+} = +$

$f''(0)$: $\frac{+}{-} = -$

$f''(2)$: $\frac{+}{+} = +$

$\lim_{x\to1^+} \frac{\overset{+}{2x^2}}{\underset{+}{x^2-1}} = \infty$

$\lim_{x\to1^-} \frac{\overset{+}{2x^2}}{\underset{-}{x^2-1}} = -\infty$

$\lim_{x\to-1^+} \frac{\overset{+}{2x^2}}{\underset{-}{x^2-1}} = -\infty$

$\lim_{x\to-1^-} \frac{\overset{+}{2x^2}}{\underset{+}{x^2-1}} = \infty$

① Domain $x+1 \geq 0$ $[-1,\infty)$

$x \geq -1$

d) $f(x) = \frac{x^2}{\sqrt{x+1}}$

② y-int: $(0,0)$ x-int: $(0,0)$ ← only 1!

③ V.A. at $\boxed{x=-1}$

$\lim_{x \to -1^+} \frac{x^{2^+}}{\sqrt{x+1}^{\,+}} = \infty$

$\lim_{x \to -1^-} \cancel{\frac{x^{2^+}}{\sqrt{x+1}}}$ = dne not in domain

$\boxed{\text{No H.A.}}$

④ No symmetry

⑤ $x^2(x+1)^{-1/2}$

$f = x^2$ $g = (x+1)^{-1/2}$

$f' = 2x$ $g' = -\frac{1}{2}(x+1)^{-3/2}$

$$f'(x) = (x+1)^{-1/2}(2x) + x^2\left(-\tfrac{1}{2}\right)(x+1)^{-3/2}$$

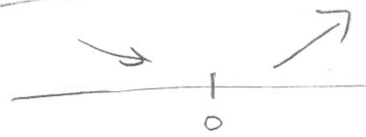
$$= \frac{2x}{\sqrt{x+1}} - \frac{x^2}{2(x+1)^{3/2}} \overset{\text{set}}{=} 0$$

$$\frac{4x(x+1) - x^2}{2(x+1)^{3/2}} = 0$$

$$\frac{4x^2+4x-x^2}{2(x+1)^{3/2}} = 0$$

$$\frac{3x^2+4x}{2(x+1)^{3/2}} = \boxed{\frac{x(3x+4)}{2(x+1)^{3/2}} = 0}$$

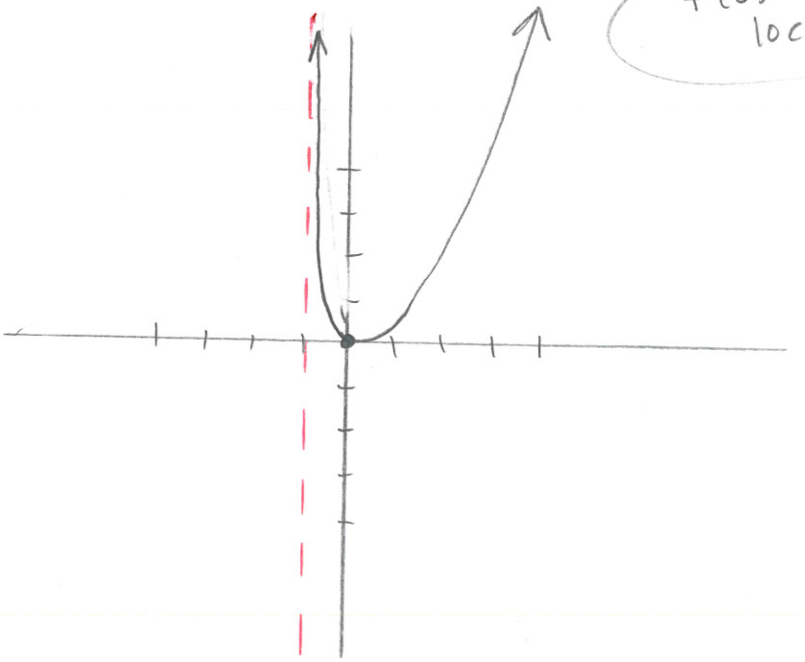
$\boxed{x=0 \quad x=\cancel{-4/3} \quad \& \quad x \cancel{=-1}}$ ← 1 critical points



$f'(-\tfrac{1}{2}) = \frac{- \cdot +}{+} = -$

$f'(1) = \frac{+ \cdot +}{+} = +$

$f(0) = 0$ is a local min



$f''(x) = \frac{3x^2+8x+8}{4(x+1)^{5/2}} \overset{\text{set}}{=} 0$

$3x^2+8x+8$ is never 0

undefined at $x=-1$ which is not in domain so there are no inflection pts./ It never changes concavity.

Numerator always >0 & so is denominator.

① Domain: sin x = −2 ← not in Range of sine function

ℝ

e) $f(x) = \frac{\cos x}{2+\sin x}$

② y-int: $\frac{\cos 0}{2+\sin 0} = \frac{1}{2}$ $\boxed{(0, \frac{1}{2})}$

x-int. $\cos x = 0$ $\boxed{x = \pi/2, \ 3\pi/2}$ etc. odd int. mult. of π/2

$(\pi/2, 0)$ $(3\pi/2, 0)$

③ periodic − graph 1 period & then repeat.

④ No V.A. No H.A.

⑤ $f'(x) = \frac{(2+\sin x)(-\sin x) - \cos x \cdot \cos x}{(2+\sin x)^2}$

$= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}$

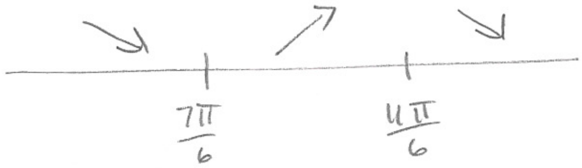
$= \frac{-(2\sin x + \sin^2 x + \cos^2 x)}{(2+\sin x)^2}$

$= \frac{-(2\sin x + 1)}{(2+\sin x)^2} \overset{set}{=} 0$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$

$\boxed{x = \frac{7\pi}{6} \quad x = \frac{11\pi}{6}}$ ↙ 2 crit pts.



$f'(\pi) = \frac{-}{+} < 0$

$f'(\frac{3\pi}{2}) = \frac{+}{+} > 0$

$f'(2\pi) = \frac{-}{+} < 0$

$f(7\pi/6) = \frac{\cos\frac{7\pi}{6}}{2+\sin 7\pi/6} = \frac{-\frac{\sqrt{3}}{2}}{2-1/2} = \frac{-\sqrt{3}}{2} \cdot \frac{2}{3} = \frac{-\sqrt{3}}{3} \approx \boxed{-.577}$

$f\left(\frac{11\pi}{6}\right) = \frac{\cos 11\pi/6}{2+\sin 11\pi/6} = \frac{\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{3} = \frac{\sqrt{3}}{3} \approx \boxed{.577}$

⑥ $f''(x) = \frac{(2+\sin x)^2(-2\cos x) + (2\sin x+1)(\cos x)2(2+\sin x)}{(2+\sin x)^4}$

$= \frac{(2+\sin x)(-2\cos x) + 2(2\sin x+1)(\cos x)}{(2+\sin x)^4}$

$= \frac{-4\cancel{\cos x} - 2\sin x\cos x + 4\cos x\sin x + 2\cancel{\cos x}}{(2+\sin x)^3}$

$= \frac{-2\cos x + 2\sin x\cos x}{(2+\sin x)^3}$

↑ always > 0

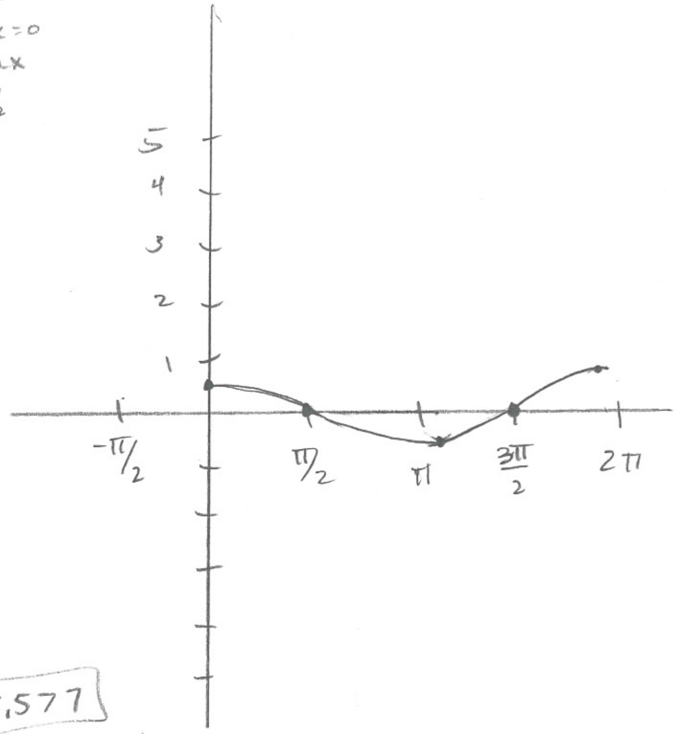
$-2\cos x + 2\sin x\cos x \overset{set}{=} 0$

$-2\cos x(1-\sin x) = 0$

$-2\cos x = 0$ $\quad 1-\sin x = 0$

$\cos x = 0$ $\quad 1 = \sin x$

$\boxed{x = \pi/2 \ \& \ 3\pi/2}$ $\quad x = \pi/2$



$f''(\pi/4) = \frac{-\cdot+}{+} < 0$

$f''(\pi) = \frac{+\cdot+}{+} > 0$

$f''(\frac{7\pi}{4}) = \frac{-\cdot+}{+} = < 0$

f) $f(x) = \frac{x^3}{x^2+1}$

Homework: 8, 13, 14, 16, 26, 29, 34, 40, 49, 51