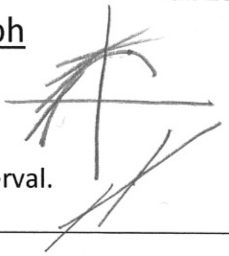
Section 3.3—Derivatives and the Shape of the Graph



1—Increasing Functions and Decreasing Functions

What does f' say about f?

A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

The Increasing/Decreasing Test

If $f'(x) > 0$ for every x in an interval (a, b), then f is strictly increasing on the interval (a, b).

If $f'(x) < 0$ for every x in an interval (a, b), then f is strictly decreasing on the interval (a, b).

Example 1: Find the intervals on which the function $f(x) = x^3 + 6x^2 - 15x + 7$ is increasing and decreasing.

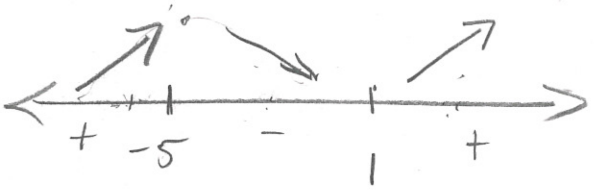
Find critical points of f.

$f'(x) = 3x^2 + 12x - 15 \overset{\text{Set}}{=} 0$

$3(x^2 + 4x - 5) = 0$

$3(x-1)(x+5) = 0$

2 crit. pts → $\boxed{x = 1 \quad x = -5}$



$f'(-6) = + \cdot - \cdot - = +$

$f'(0) = + \cdot - \cdot + = -$

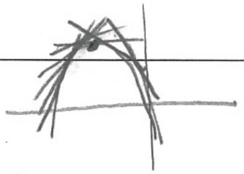
$f'(2) = + \cdot + \cdot + = +$

f is increasing on $(-\infty, -5)$ and $(1, \infty)$

f is decreasing on $(-5, 1)$

2—First Derivative Test

If you are looking for a local min or a local max of a given function, you always start by finding the critical numbers of the function. After you find the critical numbers, how do you determine if those critical numbers represent local mins, local maxs, or neither?



First Derivative Test for Local Extrema

Suppose that c is a critical number of a continuous function f.

- If f' changes from positive to negative at c then f has a local maximum at c.
- If f' changes from negative to positive at c then f has a local minimum at c.
- If f'does not change sign at c then f has no local extremum at c.

In other words: Once you have determined the critical numbers of f, use f' to determine if a function changes from decreasing to increasing (a minimum) or from increasing to decreasing (a maximum) at those critical numbers.

Example 2: Find the intervals on which the function $f(x) = 4x^3 + 3x^2 - 2x$ is increasing and decreasing. Find the function's local and absolute extreme values.

$f'(x) = \boxed{12x^2 + 6x - 2} \overset{\text{set}}{=} 0$

$2(\boxed{6x^2 + 3x - 1}) = 0$

$x = \frac{-3 \pm \sqrt{9 - 4(6)(-1)}}{12}$

$= \frac{-3 \pm \sqrt{9 + 24}}{12}$

$= \frac{-3 \pm \sqrt{33}}{12}$

2 critical numbers

$\boxed{x = .23 \text{ or } x = -.73}$

$f'(-1) = 12 - 6 - 2 > 0$

$f'(0) = -2 < 0$

$f'(1) = 12 + 6 - 2 > 0$

f is: Increasing on $(-\infty, -.73)$ and $(.23, \infty)$

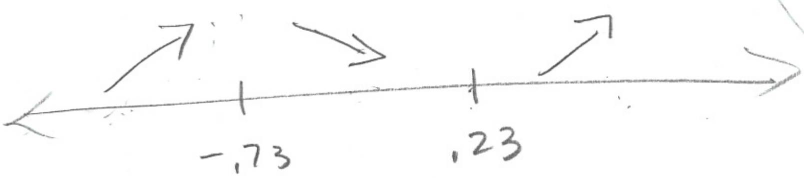
Decreasing on $(-.73, .23)$

Local max at $x = -.73$

$f(-.73) = 1.5$

Local min at $x = .23$

$f(.23) = -.25$



Example 3: Given $f(x) = x + 2\sin x$ find the function's local and absolute extreme values on $0 \leq x \leq 2\pi$.

$f'(x) = 1 + 2\cos x \overset{\text{set}}{=} 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$

2 criti. pts. → $\boxed{x = \frac{2\pi}{3} \quad x = \frac{4\pi}{3}}$

$f'(0) = 1 + 2\cos 0 = 3 > 0$

$f'(\pi) = 1 + 2\cos\pi = 1 - 2 = -1 < 0$

$f'(2\pi) = 1 + 2\cos 2\pi = 3 > 0$

f is: Increasing on $(-\infty, \frac{2\pi}{3})$ &, $(\frac{4\pi}{3}, \infty)$

Decreasing on $(\frac{2\pi}{3}, \frac{4\pi}{3})$

Local max at $x = \frac{2\pi}{3}$

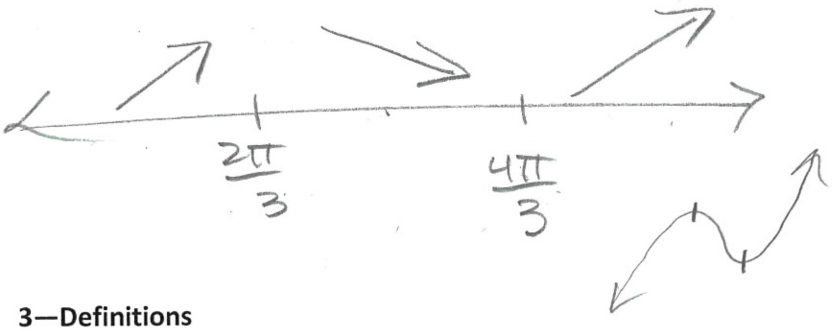
$2\pi/3 + \sqrt{3}$

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = 3.83$

Local min at $x = 4\pi/3$

$f(\frac{4\pi}{3}) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3}$

$= \frac{4\pi}{3} - \sqrt{3} \approx 2.46$

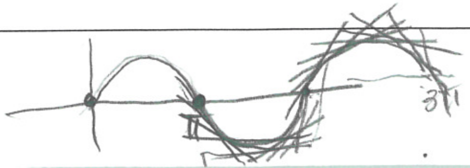


3—Definitions

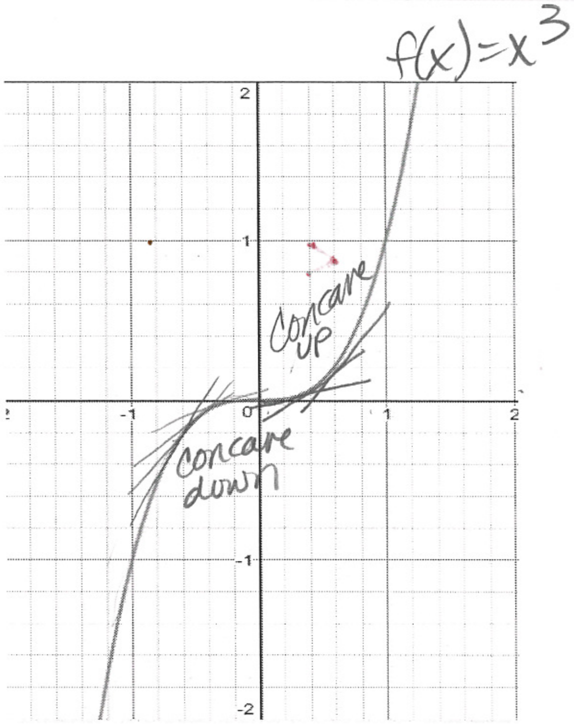
What does the second derivative tell us about a function? That is, what does f'' say about f?

Definitions

- **Concave Up:** If $f'(x)$ is increasing on the interval, then the function f is concave up on that interval.
- **Concave Down:** If $f'(x)$ is decreasing on an the interval, then the function f is concave down on that interval.
- **Point of Inflection:** An **inflection point** is a point at which a function changes concavity.



Examine the graph of $f(x) = x^3$. Notice that,



$f'(x) = 3x^2$ and $f''(x) = 6x$

$f''(x) > 0$ for $x > 0$ f is concave up for $x>0$

$f''(x) < 0$ for $x < 0$ f is concave down for $x<0$

f' is increasing on $(0, \infty)$ f is concave up

f' is decreasing on $(-\infty, 0)$ f is concave down

f'' changes sign at $x = 0$ $f'(0)=0$ $f''(0)=0$ ☆

Test for Concavity

- If $f'' > 0$ on the interval (a, b), then the function f is **concave up**.
- If $f'' < 0$ on the interval (a, b) then the function f is **concave down**.

Finding Inflection Points

- At a point of inflection, either $f''(c) = 0$ or $f''(c)$ fails to exist.

$gf'+fg'$

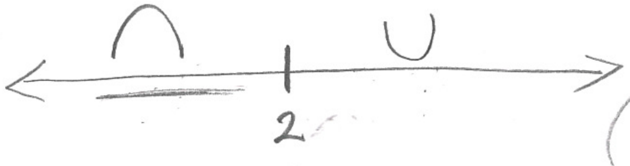
f g $f'=1$ $g'=2(6-2x)(-2) = -4(6-2x)$

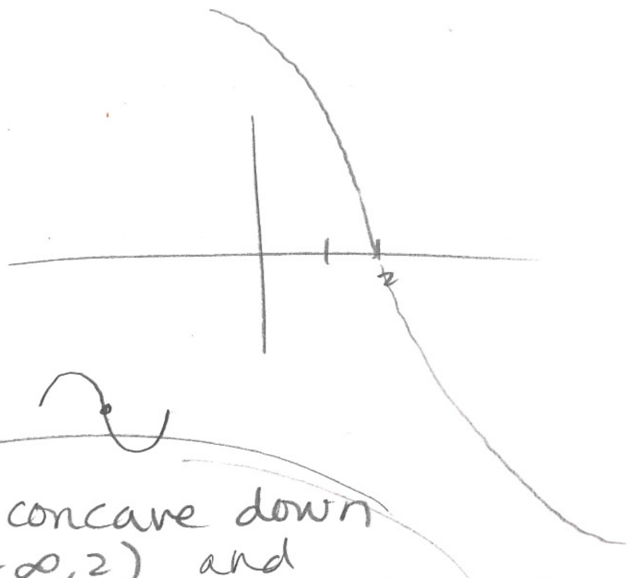
Example 4: Let $f(x) = x(6-2x)^2$. Find the points of inflection, and identify the intervals where it is concave up or concave down.

$$f'(x) = (6-2x)^2 + x(-4)(6-2x)$$
$$= 36-24x+4x^2-4x(6-2x)$$
$$= 36-24x+4x^2-\cancel{24}x+8x^2$$
$$= 36-48x+12x^2$$
$$f''(x) = -48+24x \overset{set}{=} 0$$
$$24x = 48$$
$$\boxed{x=2} \leftarrow \text{1 Inf. point}$$

$f''(0) = -48 < 0$

$f''(3) = > 0$





f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$

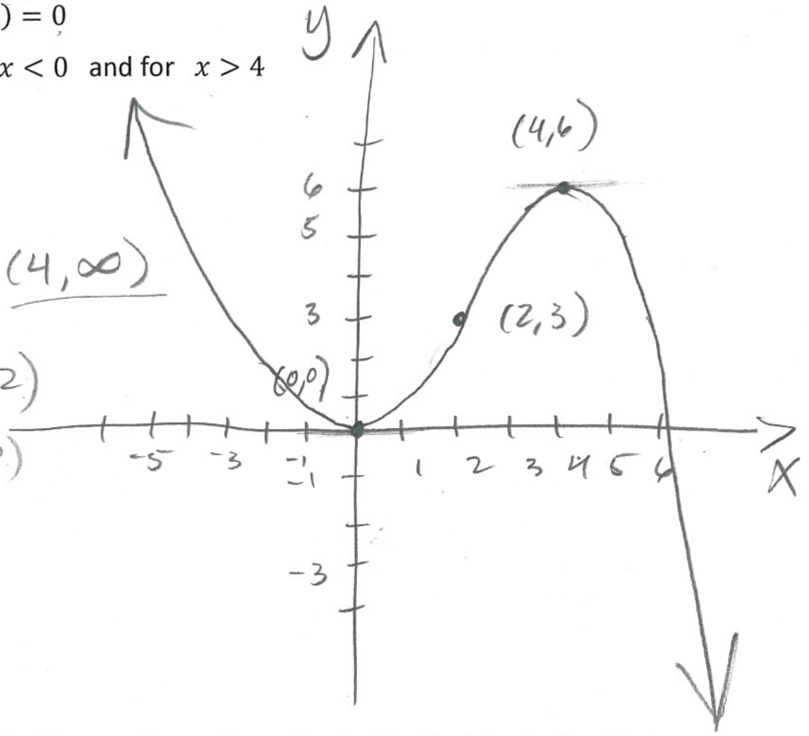
X=0 & X=4 are crit. pts.

Example 5: Sketch a possible graph of a function f that satisfies the following conditions:

1. $f(0) = 0$, $f(2) = 3$, $f(4) = 6$, $f'(0) = f'(4) = 0$
2. $f'(x) > 0$ for $0 < x < 4$, $f'(x) < 0$ for $x < 0$ and for $x > 4$
3. $f''(x) > 0$ for $x < 2$, $f''(x) < 0$ for $x > 2$

f is increasing on (0,4)
f is decreasing (-∞,0) & (4,∞)

f is concave up on (-∞,2)
concave down on (2,∞)
Inf. pt. at X = 2



4—The Second Derivative Test

The Second Derivative Test is a consequence of the Concavity Test and is an alternative to the First Derivative Test.

2nd Derivative Test

If c is a critical point of a function f:

1. If $f'(c) = 0$ *and* $f''(c) > 0$, then f has a local minimum at $x = c$. ∪
2. If $f'(c) = 0$ *and* $f''(c) < 0$, then f has a local maximum at $x = c$. ∩
3. If $f'(c) = 0$ *and* $f''(c) = 0$ then the test fails. The function f may have a local max, a local min, or neither at ~~x~~= c.

Example 6: Use the second derivative test to find the local extrema of the function $f(x) = x^4 - 4x^3$.

$f'(x) = 4x^3 - 12x^2 \overset{set}{=} 0$

$4x^2(x - 3) = 0$

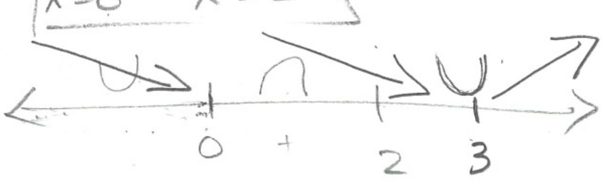
$\boxed{x=0 \quad x=3}$ ← 2 crit. pts.

$f''(3) = + \cdot + = + > 0$ So x=3 is a local min.

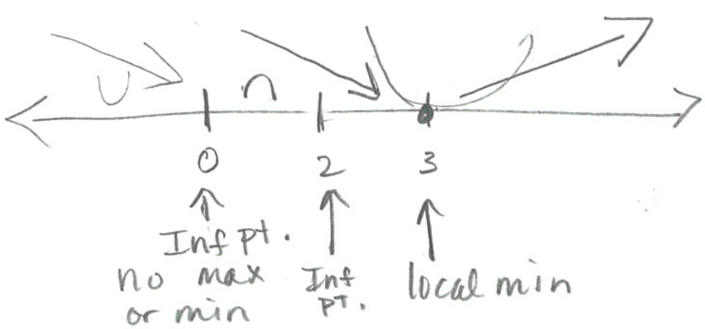
$f''(x) = 12x^2 - 24x \overset{set}{=} 0$

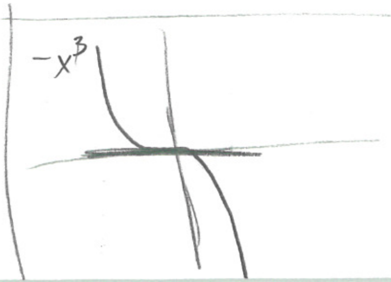
$12x(x-2) = 0$

$\boxed{x=0 \quad x=2}$ ← possible Inf. pts.



$f'(-1) = + \cdot - = -$





5—Using the First and Second Derivative Tests to Graph Functions

Sketching graphs of functions using the First and Second Derivative Tests

1. Determine the domain.
2. Find critical points of f by finding f' and determining where it equals 0 or is undefined.
3. Determine the behavior of f at the critical points using either the first or second derivative test.
4. Find inflection points of f by finding f'' and determining where it equals 0 or is undefined.
5. Determine the concavity of f over the intervals defined by the inflection points.
6. Determine horizontal and vertical asymptotes and holes
7. Plot strategic points: critical points, inflection points
8. Fill in with concavity of .

$\frac{1}{3}x^3$

Example 7: Sketch the graphs of the following functions.

a) $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$ Domain: $\mathbb{R}$

$f'(x) = x^2 - x - 2 \overset{\text{set}}{=} 0$

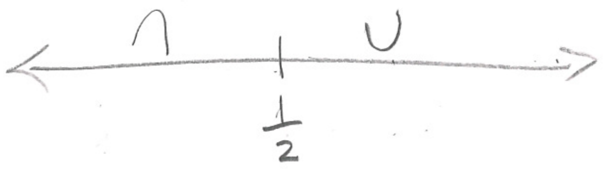
$(x+1)(x-2) = 0$

$\boxed{x = -1 \quad x = 2}$ ← 2 critical points

$f''(x) = 2x - 1 \overset{\text{set}}{=} 0$

$2x = 1$

$\boxed{x = \frac{1}{2}}$ ← 1 Inflection point



x = -1 is a local max

x = 2 is a local min

$f''(-1) = -3 < 0$

$f''(2) = 3 > 0$

concave down on $(-\infty, \frac{1}{2})$

concave up $(\frac{1}{2}, \infty)$

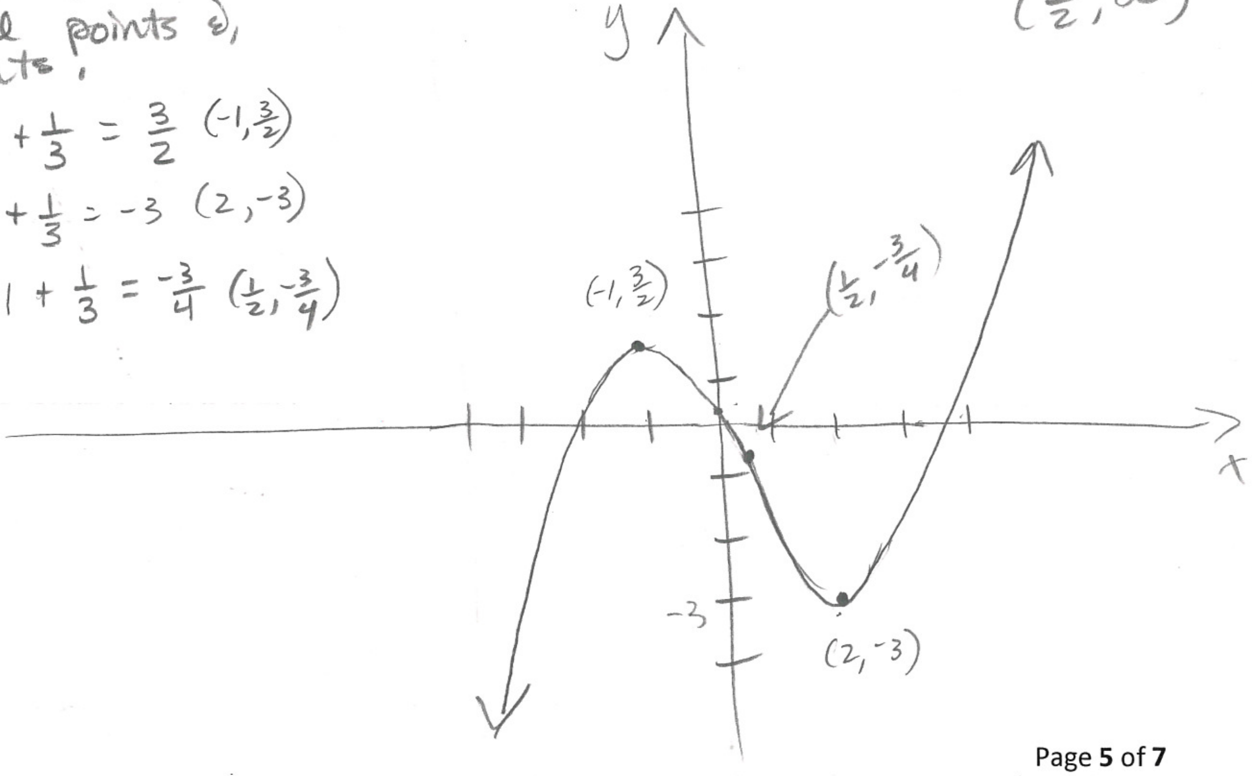
No H.A. or V.A.

we need the values of f at the critical points & Inflection points.

$f(-1) = \frac{-1}{3} - \frac{1}{2} + 2 + \frac{1}{3} = \frac{3}{2}$ $(-1, \frac{3}{2})$

$f(2) = \frac{8}{3} - 2 - 4 + \frac{1}{3} = -3$ $(2, -3)$

$f(\frac{1}{2}) = \frac{1}{24} - \frac{1}{8} - 1 + \frac{1}{3} = \frac{-3}{4}$ $(\frac{1}{2}, \frac{-3}{4})$



b) $g(x) = 2x - 3x^{\frac{2}{3}}$ Domain: $\mathbb{R}$

$g'(x) = 2 - 2x^{-1/3} \overset{set}{=} 0$

$2 - \frac{2}{x^{1/3}} = 0$ ← undefined for $x = 0$

$2 = \frac{2}{x^{1/3}}$

$2x^{1/3} = 2$

$(x^{1/3})^3 = (1)^3$

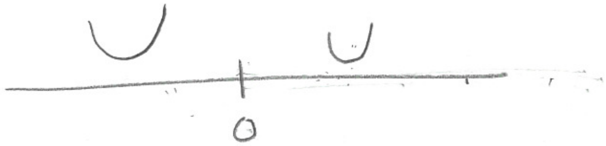
$\boxed{x = 1}$

$\cancel{(-1)^{4/3} = [(-1)^4]^{1/3}}$

$g''(x) = \frac{2}{3}x^{-4/3}$

$= \frac{2}{3x^{4/3}} \overset{set}{=} 0$

Never equals 0, but it is undefined at $\boxed{x = 0}$ ← possible Inflection pt.



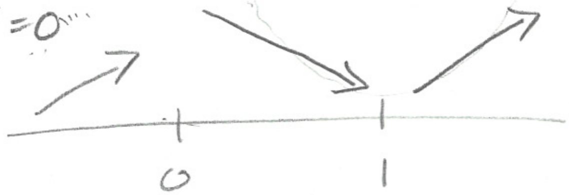
$g''(1) = >0$

$g''(-1) = >0$

crit. pts at

$x = 1$ ← $x = 1$ is a local min

$x = 0$

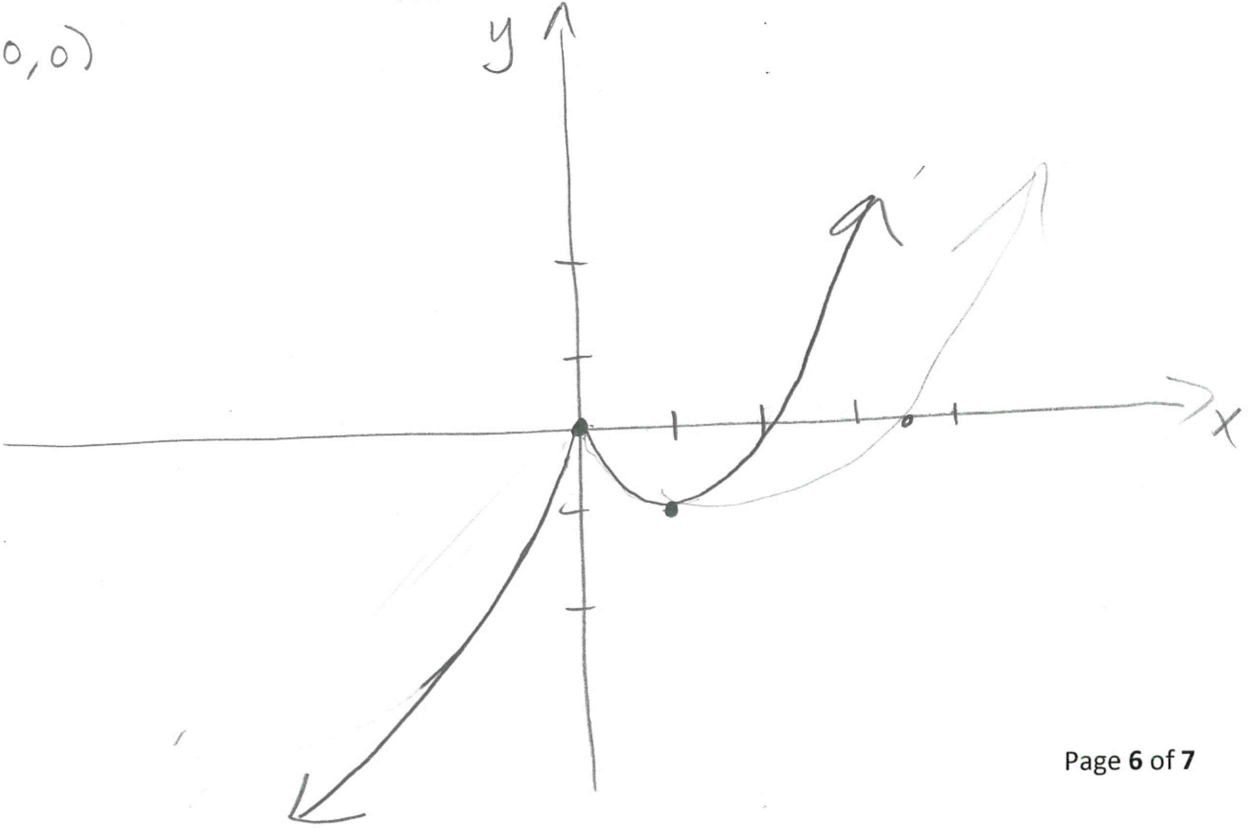


$f'(-300) = >0$

Find values of f at cr. pts. at at

$g(1) = 2 - 3 = -1$ $(1, -1)$ ← local min

$g(0) = 0$ $(0, 0)$



$(2x^{1/3} - 3) = 0$

$2x^{1/3} = 3$

$x^{1/3} = \frac{3}{2}$

$x = \frac{27}{8}$

c) $y = x^4 - 4x^3 + 10$

Homework: 1-8, 9-29(odd), 12, 14, 16, 33-44