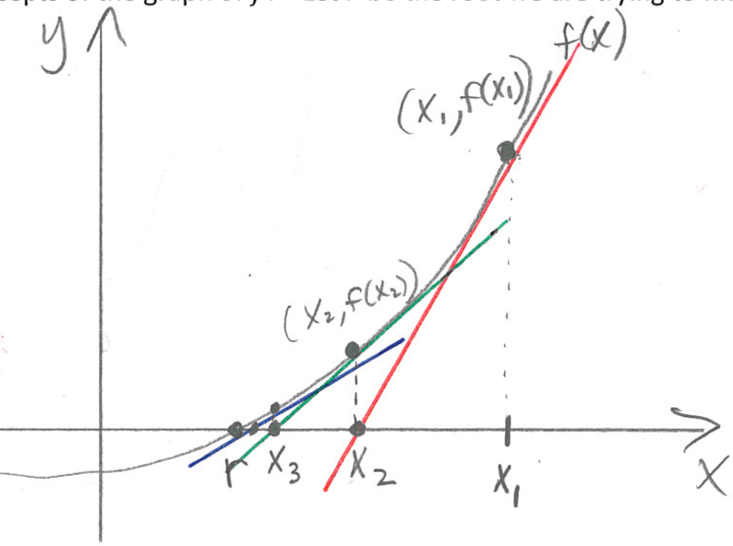
Section 3.8—Newton's Method

We know how to solve equations of the form $ax^2 + bx + c = 0$. There are also formulas for 3rd and 4th degree polynomials, though they are complicated. There is no formula for solving polynomials of degree 5 or higher. Likewise, there is not a formula for solving transcendental equations such as $\sin x = x$.

In this section we learn how calculators and computers solve these types of equations (at least in part).

1—Newton's Method

Suppose we wish to solve an equation of the form $f(x) = 0$. Then the **roots** of this equation correspond to the x-intercepts of the graph of f. Let r be the root we are trying to find, and begin with a guess that you think is near r, call it x_1.



Write the equation of the tangent line to the curve at x_1.

$$y - f(x_1) = f'(x_1)(x - x_1)$$

How can we use the equation to find x_2?

We know $(x_2, 0)$ is on the tangent line. So it is a solution to the equation of the tangent line.

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$\frac{0 - f(x_1)}{f'(x_1)} = \frac{f'(x_1)(x_2 - x_1)}{f'(x_1)}$$

$$\frac{-f(x_1)}{f'(x_1)} = x_2 - x_1$$

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x_2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

How would we find x_3?

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

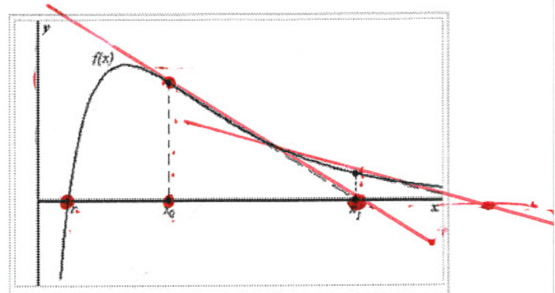
Newton's Method
If x_n is an approximate solution of $f(x) = 0$ and if $f'(x) \neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

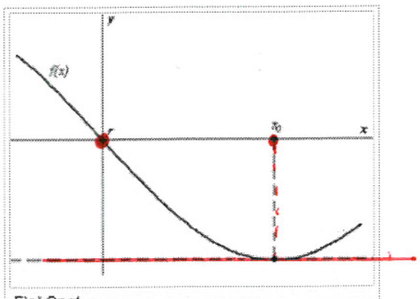
Counterexamples: Sometimes Newton's method works very slowly, or even fails.

Secondly, the starting point must be chosen carefully, and it is best chosen with an approximate idea of the graph of the function in mind. If chosen wrongly, one of the following three situations could happen:

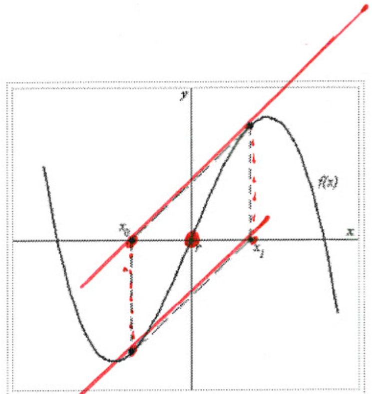
1. Runaway: In which Newton's Method leads away from the root instead of towards the root; the solution diverges rather than converges.
2. Flat Spot: In which the derivative of the graph at the starting point is 0, and thus the next iterative point occurs at infinity and cannot be used.
3. Cycle: In which the solutions cycle between two points, and never converges to the root.



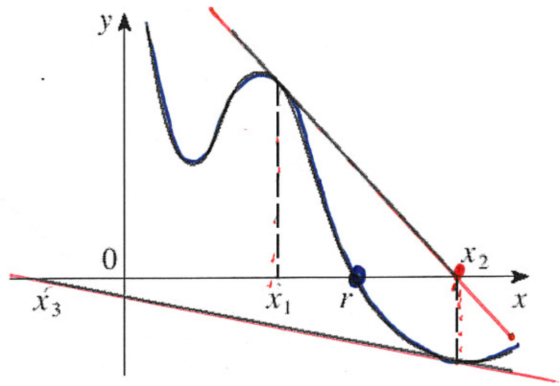
Runaway



Flat Spot



Cycle



Example 1: Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

$f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

$x_1 = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \left(\frac{-1}{10}\right)$

$= 2 + \frac{1}{10} = \boxed{2.1}$

$f(2) = 8 - 4 - 5 = -1$

$f'(2) = 12 - 2 = 10$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1 - \frac{f(2.1)}{f'(2.1)}$

$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2}$

$= 2.1 - \frac{.061}{11.23} = \boxed{2.094568121}$

*How do you know when to stop? How many times do you go through the process? A common way to know when you are finished is to continue until two successive approximations agree to a given number of decimal places.

Things to Remember when using Newton's Method:

1. In order to apply the method, we need to have equations of the form $f(x) = 0$.
2. You need to find an initial approximation. Try sketching the graph, or if you know there is a solution to the function in a certain interval, you can use the midpoint of the interval as x_1.
3. What level of accuracy you need.

Example 2: Use Newton's Method to determine an approximation to the solution to $\cos x = x$ that lies in the interval $[0,2]$. Find the approximation to six decimal places.

1. we need $f(x)=0$
   Use: $\cos x - x = 0$
2. Use $x_1 = 1$
3. six dec. places
4. $f(x) = \cos x - x$
   $f'(x) = -\sin x - 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{\cos(1) - 1}{-\sin(1) - 1}$$

$$= \boxed{.7503638678}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = .7503638678 - \frac{f(.7503638...)}{f'(.7503638...)}$$

$$= \text{Ans} - \frac{f(\text{ans})}{f'(\text{ans})}$$

$$= \text{Ans} - \frac{(\cos \text{Ans} - \text{Ans})}{(-\sin \text{Ans} - 1)}$$

$$= \boxed{.7391128909}$$

$$x_4 = .7390851334$$

$$x_5 = .7390851332$$

The root, correct to six decimal places is .739085

Example 3: Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

$x = \sqrt[6]{2}$

$x^6 = 2$

$x^6 - 2 = 0$

① use $f(x) = x^6 - 2$

$f'(x) = 6x^5$

② use $x_1 = 1$

③ 8 dec. places

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \left(\frac{-1}{6}\right)$$

$$= 1 + \frac{1}{6} = \boxed{1.166666667}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = Ans - \frac{f(Ans)}{f'(Ans)}$$

$$= Ans - \frac{(Ans^6 - 2)}{(6\,Ans^5)}$$

$$= \boxed{1.126443678}$$

$x_4 = 1.122497067$

$x_5 = 1.122462051$

$x_6 = 1.122462048$

$x_7 = 1.122462048$

root or $\sqrt[6]{2} \approx 1.12246205$ correct to 8 dec. places.

Homework: 4, 6-8, 11, 15-21 odd, 14, 22, 29