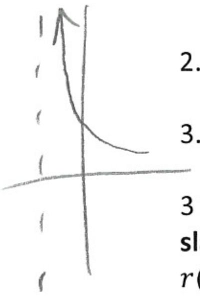
Section 3.4—Limits at Infinity; Horizontal Asymptotes

- Recall the symbol for infinity $\pm\infty$ does not represent a real number. It is used to describe the behavior of a function when its x or y —values get large without bound or get increasingly small without bound.
- Recall rules for finding horizontal asymptotes of rational functions. $\frac{2x^3-5x}{x^3+1}$ $y=2$

  Compare the degrees of the numerator and the denominator. There are 3 cases:

  1. If the degree of the numerator and the degree of the denominator are equal, then there is a horizontal asymptote at the line $y = \frac{a_n}{b_n}$, where a_n is the leading coefficient of the numerator and b_n is the leading coefficient of the denominator. why?
  2. If the degree of the numerator is less than the degree of the denominator, then there is a horizontal asymptote at the line $y = 0$. $\frac{2x^2}{2x^3}$ H.A. at $y=0$ why?
  3. If the degree of the numerator is larger than the degree of the denominator then there is not a horizontal asymptote. why?

  3 ½. If the degree of the numerator is exactly **1 degree greater** than the degree of the denominator, there is a **slant** asymptote. To find the slant asymptote, divide the numerator by the denominator to get $f(x) = Q(x) + r(x)$, where $Q(x)$ is the quotient and $r(x)$ is the remainder. The line $y = Q(x)$ is the slant asymptote. why



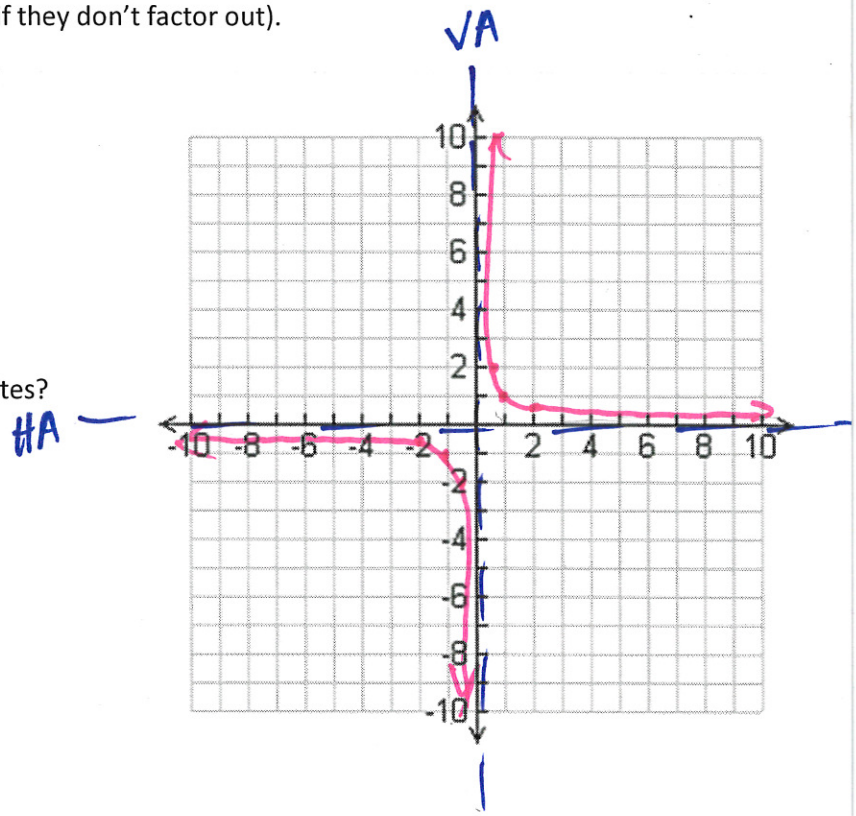
- Vertical Asymptotes: Factor the numerator and the denominator. Any zeros of the denominator are either holes in the graph (if they factor out) or vertical asymptotes (if they don't factor out).

Example 1: Let $f(x) = \frac{1}{x}$. $\lim_{x\to\infty} \frac{1}{x} = 0^+$

a) Investigate $\lim_{x\to\infty} \frac{1}{x}$ $=0^+$ and $\lim_{x\to-\infty} \frac{1}{x}$. $= 0^-$

b) Investigate $\lim_{x\to 0^-} \frac{1}{x}$ $= -\infty$ and $\lim_{x\to 0^+} \frac{1}{x}$? $= \infty$

Are there any horizontal asymptotes? Vertical Asymptotes?



1—Finite Limits as $x \to \pm\infty$ and Horizontal Asymptotes

Intuitive Definition of a Limit at $\pm$ Infinity

- Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x\to\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

- Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x\to-\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Definition:

A line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if either:

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Theorem

All the Limit Laws we have previously learned are true when we replace $\lim\limits_{x\to c}$ with $\lim\limits_{x\to\infty}$ *or* $\lim\limits_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

In particular, by combining some of the previous limit laws with the results we observed in Example 1 above, notice that, if $r > 0$ is a rational number such that x^r is defined for all x, then $\lim\limits_{x\to\pm\infty} \frac{1}{x^r} = 0$

Example 2: Find $\lim\limits_{x\to\infty}(-37 + \frac{1}{x})$

$\frac{1}{x}$

$= \lim\limits_{x\to\infty} -37 + \lim\limits_{x\to\infty} \frac{1}{x}$

$= -37 + 0$

$= -37$

Example 3: Find $\lim\limits_{\theta\to\infty} \frac{\sin\theta}{\theta}$

$\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$\frac{-1}{\theta} \le \frac{\sin\theta}{\theta} \le \frac{1}{\theta}$

$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 0 \quad 0$

$\lim\limits_{\theta\to\infty} -\frac{1}{\theta} = 0$

$\lim\limits_{\theta\to\infty} \frac{1}{\theta} = 0$

By the Squeeze Theorem,

$\lim\limits_{\theta\to\infty} \frac{\sin\theta}{\theta} = 0$

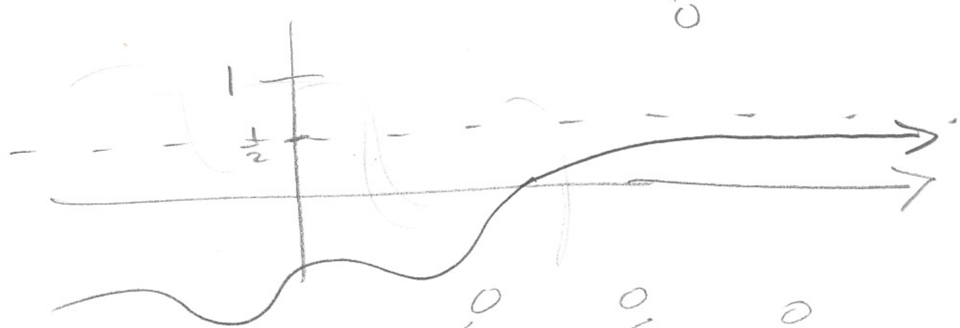
2--Rational Functions—Type $\frac{\infty}{\infty}$

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator.

Type $\frac{\infty}{\infty}$

Example 4: Find $\lim_{x\to\infty} \frac{x^2+3}{2x^2+1}$.

$$\lim_{x\to\infty} \frac{x^2+3}{2x^2+1} = \lim_{x\to\infty} \frac{\cancel{x^2}/\cancel{x^2} + 3/x^2}{2\cancel{x^2}/\cancel{x^2} + 1/x^2} = \lim_{x\to\infty} \frac{1 + 3/x^2}{2 + 1/x^2} = \frac{1+0}{2+0} = \frac{1}{2}$$



Example 5: Find $\lim_{x\to\infty} \frac{3x^2+7x-13}{x^6+5x+1}$.

$$= \lim_{x\to\infty} \frac{3x^2/x^6 + 7x/x^6 - 13/x^6}{x^6/x^6 + 5x/x^6 + 1/x^6} = \lim_{x\to\infty} \frac{\cancel{3/x^4} + \cancel{7/x^5} - \cancel{13/x^6}}{1 + \cancel{5/x^5} + \cancel{1/x^6}} \qquad \frac{3}{\infty^4}$$

$$= \frac{0+0-0}{1+0+0} = 0$$

3--Limits of Differences of Infinity—Type $\infty - \infty$

Example 6: Find $\lim_{x\to-\infty} (\sqrt{x^2+3} + x)$ $\qquad \infty - \infty$

$$\lim_{x\to-\infty} \frac{\left(\sqrt{x^2+3} + x\right)}{1} \cdot \frac{\sqrt{x^2+3} - x}{\sqrt{x^2+3} - x} = \lim_{x\to-\infty} \frac{\cancel{x^2}+3-\cancel{x^2}}{\sqrt{x^2+3} - x}$$

$$= \lim_{x\to-\infty} \frac{3}{\underbrace{\sqrt{x^2+3} - x}_{\infty + \infty}} = \left(\frac{3}{\infty}\right) = 0$$

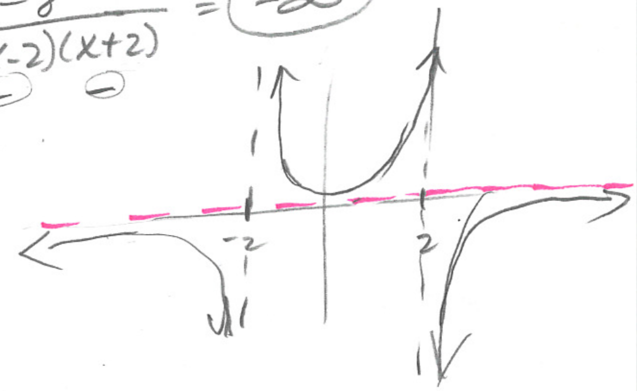
$3 \cdot \frac{1}{\infty}$ $\qquad$ as $x \to -\infty$, $f(x) \to 0$

Example 7: Find the horizontal and vertical asymptotes of the graph of the function $g(x) = -\frac{8}{x^2-4}$.

$$\frac{-8}{x^2-4} = \frac{-8}{(x-2)(x+2)} \leftarrow \text{V.A. at } \boxed{x=2 \text{ \& at } x=-2}$$

$$\lim_{x\to 2^+} \frac{-8}{(x-2)(x+2)} = -\infty \qquad \lim_{x\to -2^+} \frac{-8}{(x-2)(x+2)} = \infty$$

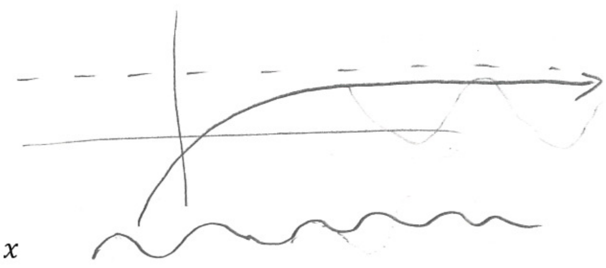
$$\lim_{x\to 2^-} \frac{-8}{(x-2)(x+2)} = \infty \qquad \lim_{x\to -2^-} \frac{-8}{(x-2)(x+2)} = -\infty$$



H.A.

$$\lim_{x\to\infty} \frac{-8}{x^2+4} = 0$$

$$\lim_{x\to -\infty} \frac{-8}{x^2+4} = 0 \qquad \text{H.A. at } y=0.$$



Example 8: Evaluate the following limits.

a) $\lim_{x\to\infty} \sin\frac{1}{x}$

Let $t = \frac{1}{x}$ As $x \to \infty$, $t \to 0$

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0} \sin t = 0$$

$$\lim_{x\to\infty} \sin\frac{1}{x} = \sin\left(\lim_{x\to\infty} \frac{1}{x}\right)$$

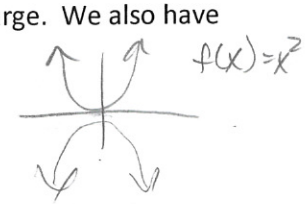
b) $\lim_{x\to\infty} \sin x$

$= \text{dne}$

The values are oscillating between -1 & 1.

4—Infinite Limits at Infinity

The notation $\lim_{x\to\infty} f(x) = \infty$ is used to indicate that the values of $f(x)$ become large as x becomes large. We also have the following: $\lim_{x\to\infty} f(x) = -\infty$ $\quad \lim_{x\to-\infty} f(x) = \infty$ $\quad \lim_{x\to-\infty} f(x) = -\infty$



Example 9: Find $\lim_{x\to\infty} f(x) = (x^2 - x)$ Type $\infty - \infty$

$$= \lim_{x\to\infty} x(x-1) = \infty \cdot \infty = \infty$$

Example 10: Find $\lim\limits_{x\to\infty} \frac{2x^2-5}{x+3}$.   Type $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{2x^2/x - 5/x}{x/x + 3/x} = \lim_{x\to\infty} \frac{2x - 5/x}{1 + 3/x} = \lim_{x\to\infty} \frac{2x - 0}{1+0} = \infty$$

∞, $\to 0$, $\to 0$

$$\frac{2x^2-5}{x+3} \cdot \frac{x-3}{x-3} = \frac{(2x^2-5)(x-3)}{x^2-9}$$

The next example shows that by using infinite limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial without computing derivatives.

Example10: Sketch the graph of $y = (x - 2)^4(x+1)^3(x-1)$ by finding its intercepts and its limits as x approaches positive and negative infinity.

y-int: $(-2)^4(-1) = -16$ $(0,-16)$

x-int: $(x-2)^4(x+1)^3(x-1) \overset{\text{set}}{=} 0$

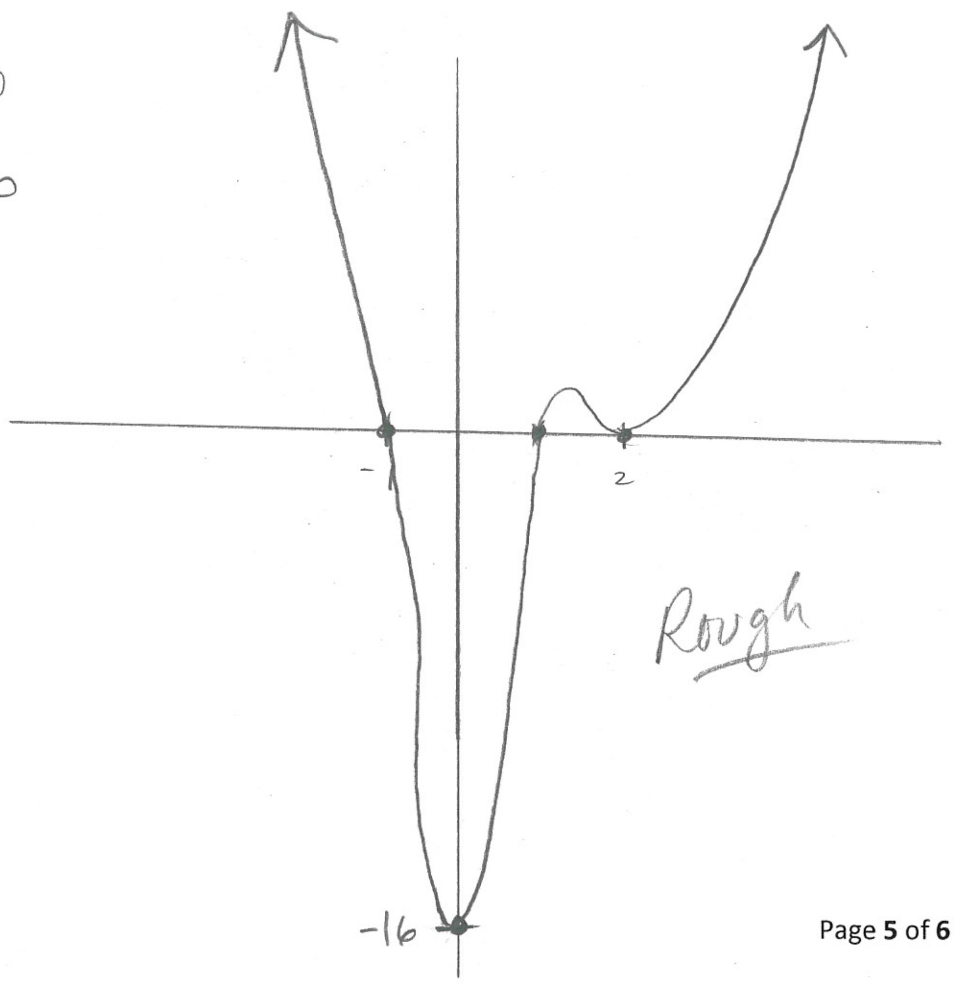
$x=2$ $x=-1$ $x=1$

touches   cross

$$\lim_{x\to\infty} (x-2)^4(x+1)^3(x-1) = \infty$$

$$\lim_{x\to-\infty} (x-2)^4(x+1)^3(x-1) = \infty$$

$+ \quad - \cdot -$



4—Formal Definition **See Text for definition of limits at $-\infty$.

Precise Definition of a Limit at Infinity

Let f be a function defined on some interval (a, b). Then

$$\lim_{x\to\infty} f(x) = L$$

Means that for every $\varepsilon > 0$ there is a corresponding number N such that

If $x > N$ then $|f(x) - L < \varepsilon|$.

Homework: 3, 4, 7-29(odd), 14, 18, 22, 36-38, 45-55(odd), 56