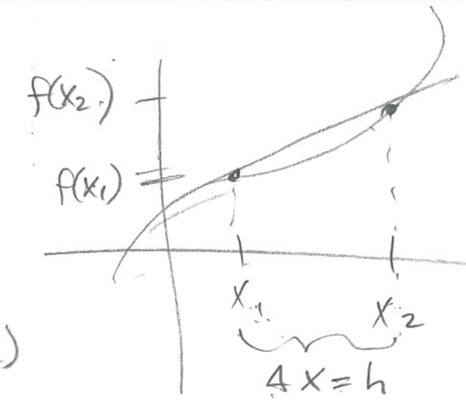
Section 2.7—Rates of Change



1—Instantaneous Rates of Change

If x changes from x_1 to x_2, then the change in x is: $\Delta x = x_2 - x_1$.

And the corresponding change in y is: $\Delta y = f(x_2) - f(x_1)$.

We have already learned that the difference quotient of a function given by:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h}$$

is the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$, and can be interpreted as the slope of the secant line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

When we take the limit as $\Delta x = h \to 0$, then we get the derivative, $f'(x_1)$, which we have interpreted as the **instantaneous rate of change of y with respect to x**, or the slope of the tangent line at $(x_1, f(x_1))$.

In Leibniz notation, this is written:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*We frequently just refer to the rate of change. By this we mean the *instantaneous* rate of change. We will specify when we mean average rate of change.

Example 1: The volume V of a spherical balloon is related to its radius by the equation $V = \frac{4}{3}\pi r^3$. How fast does the volume change with respect to the radius when the radius is 5 m?

$$V(r) = \frac{4}{3}\pi r^3$$

The rate of change of the Volume with respect to the radius is:

$$V' = \cancel{3} \cdot \frac{4}{\cancel{3}} \pi r^2 = 4\pi r^2$$

$$V'(5) = 4\pi 5^2 = 100\pi \ m^3/\text{unit of time}$$

2—Motion Along a Line: Displacement, Velocity, Speed, Acceleration & Jerk

Whenever the function $y = f(x)$ has a specific interpretation, then the derivative will have a specific interpretation as a rate of change. We already know at least two of these. What are they?

- Derivative of a position function is Velocity
- Derivative of a velocity function is acceleration.

Definitions

Velocity is the rate of change of displacement with respect to time. (Change of position w.r.t. time)

If $s = f(t)$ is the position function of a body at time t, then the body's **average velocity** over a time period Δt is:

$$v = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

And the **instantaneous velocity** or just simply **velocity** is the derivative of the position function and is given by:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Speed is the absolute value of velocity.

$$Speed = |v(t)| = \left|\frac{ds}{dt}\right|$$

Acceleration is the derivative of velocity with respect to time:

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

Distance given by a function	$s = f(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = s''(t)$
Jerk	$j(t) = s'''(t)$

Example 2: Suppose that the position of a particle is given by the function $s = t^3 - 9t^2 + 24t$ for $0 \le t \le 5$ where t is measured in seconds and distance is measured in feet.

a) Find the velocity at time t.

$v(t) = s'(t) = 3t^2 - 18t + 24$

b) What is the velocity after 3 seconds?

$v(3) = 3 \cdot 9 - 18 \cdot 3 + 24 = 27 - 54 + 24 = -3$ ft/sec.

c) When is the particle at rest? When velocity = 0 !

$v(t) = 3t^2 - 18t + 24 \overset{\text{set}}{=} 0$

$3(t^2 - 6t + 8) = 0$

$(t-2)(t-4) = 0$

$t = 2$ seconds $\quad t = 4$ seconds

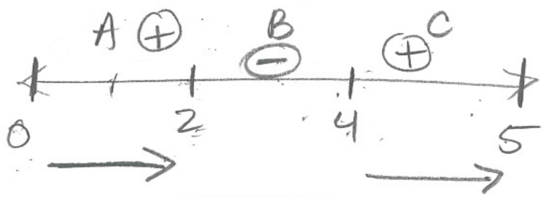
d) When is the particle moving forward? That is, in a positive direction? When the velocity IS positive!

$v(t) = 3t^2 - 18t + 24 > 0$?

$3t^2 - 18t + 24 = 0$

$3(t-2)(t-4) = 0$

$t = 2 \quad t = 4$



$v(0) = 24 > 0$

$v(3) = -3 <$

$v(4.5) =$

$t \in (0,2)$

The particle moves forward from $0 < t < 2$ seconds and $4 < t < 5$ seconds

e) What is the total distance traveled by the particle in 5 seconds? $\boxed{s(t) = t^3 - 9t^2 + 24t}$

*LOOK at the position in each time interval

$s(0) = 0$

$s(2) = 2^3 - 9 \cdot 4 + 48 = 8 - 36 + 48 = 20$ ft.

$s(4) = 4^3 - 9 \cdot 16 + 24 \cdot 4 = 64 - 144 + 96 = 16$

$s(5) = 5^3 - 9 \cdot 25 + 24(5) = 125 - 225 + 120 = 20$

$20 + 4 + 4 = 28$ feet total distance

f) When is the particle speeding up? When is it slowing down?

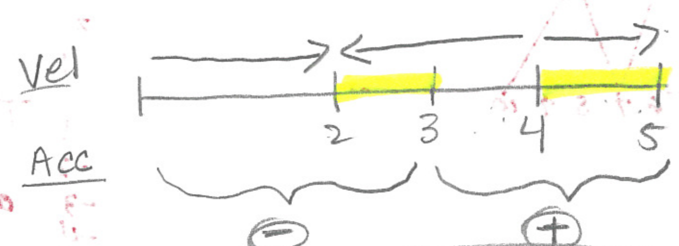
- When BOTH Velocity & Acceleration have the same sign, it is speeding up.
  - Both (+) - Speeding up & moving forward (or up)
  - Both (−) - speeding up Backwards (or down)
- Slows down when signs are opposite (Being pushed against the direction of movement.)

$a(t) = v'(t) = \boxed{6t - 18}$ ← Find where it changes sign

$6t - 18 = 0$

$6t = 18$

$\boxed{t = 3}$



$a(1) = 6 - 18 = -12$ (−)

$a(4) = 24 - 18 = 6$ (+)

Speeding up: $2 < t < 3$, $4 < t < 5$

Example 2: If an object is thrown into the air vertically with an initial velocity of 50 ft/s, then its height after t seconds is given by $h(t) = -16t^2 + 50t$.

a) When does the ball hit the ground?

$h(t) = -16t^2 + 50t \overset{\text{set}}{=} 0$

$t(-16t + 50) = 0$

$t = 0 \qquad -16t = -50$

$t = \frac{50}{16} = \frac{25}{8} \approx 3.125 \text{ sec}$

b) What is the velocity of the ball when it hits the ground?

$v(t) = -32t + 50$

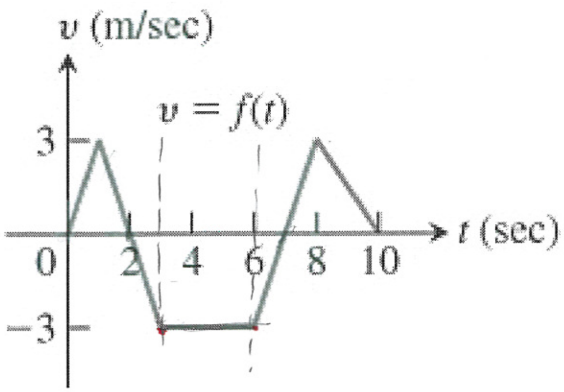
$v\left(\frac{25}{8}\right) = -\cancel{32}^{4}\left(\frac{25}{\cancel{8}}\right) + 50 = -100 + 50 = -50 \text{ ft/sec}$

c) What is the acceleration of the ball when it hits the ground?

$a(t) = -32 \text{ ft/sec}^2$

$a\left(\frac{25}{8}\right) = -32$

Example 3: The figure below shows the velocity $v = \frac{ds}{dt}$ of a body moving along a coordinate line (in meters per second).



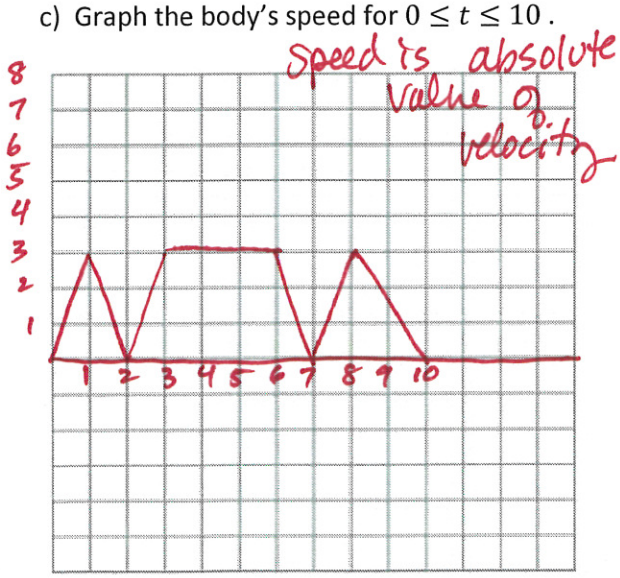
a) When does the body reverse direction?

at $t = 2$ and $t = 7$ seconds

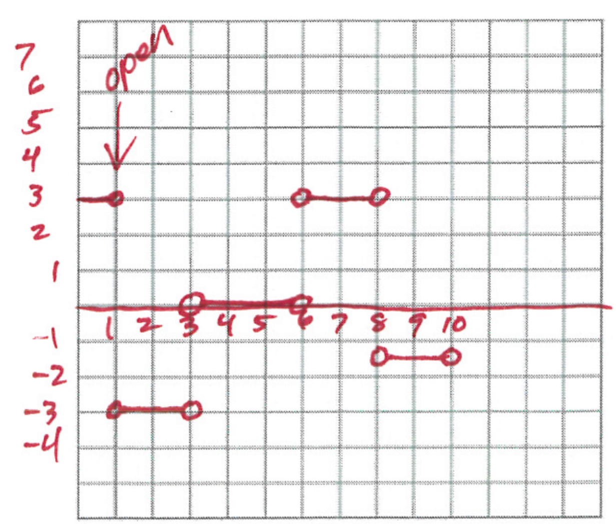
b) When (approximately) is the body moving at a constant speed?

$3 < t < 6$

c) Graph the body's speed for $0 \le t \le 10$.



d) Graph the acceleration, where defined.



Example 4: Interpret the derivatives of the following functions.

a) If $C(x)$ is a cost function which describes the total cost incurred by a company to produce x units of a commodity, how would we interpret $C'(x)$?

The Rate of change of the cost w.r.t. the number of items produced.

Marginal Cost

b) A current exists whenever electric charges move. If a function $y = Q(t)$ models the charge $Q(t)$ through a surface at a time t, how could we interpret $Q'(t)$?

Change of charge wrt time

Current

c) If a population of rabbits is governed by a function $P(t)$, where t is time, what is $P'(t)$?

rate of change population w.r.t.

Growth Rate

See pg. 177 in text

Homework: 1-13, 30