Section 4.2—The Definite Integral

1—Finite Sums & Sigma Notation

Suppose we want to add up many numbers such as:

$$1+2+3+4+\ldots+99+100 \qquad \text{or} \qquad 1+2+3+4+\ldots+(n-1)+n$$

$$\text{Or} \qquad f(1)+f(2)+f(3)+\ldots+f(1000)$$

We can use a sort of shorthand notation called **Sigma notation** to write sums with many terms.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

where index i ends

Formula for the i th term of the sum

Summation Greek letter sigma corresponds to our letters.

$$\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4$$

lower index where index i begins

Example 1: Evaluate the following sums:

$$\sum_{i=1}^{5} i = 1+2+3+4+5 = 15$$

$$\sum_{i=1}^{4} (-1)^i i = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4 = -1+2-3+4 = 2$$

$$\sum_{i=1}^{3} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

$$\sum_{i=3}^{6} \frac{i}{i+1} = \frac{3}{3+1} + \frac{4}{4+1} + \frac{5}{5+1} + \frac{6}{6+1} = \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7}$$

$$= \frac{1361}{420} \approx 3.24$$

Example 2:

a) Find the sum of $1 + 2 + 3 + 4 + \ldots + 99 + 100$

$$\begin{array}{l} 100 + 99 + 98 + 97 + \ldots + 2 + 1 \\ \hline \underbrace{101 + 101 + 101 + \cdots + 101 + 101}_{100 \text{ of them!}} \end{array} = \frac{100 \cdot 101}{2} = 5{,}050$$

b) Find the sum of $1 + 2 + 3 + 4 + \ldots + (n-1) + n$

$$\begin{array}{l} 1 + 2 + 3 + \ldots + (n-1) + n \\ n + (n-1) + (n-2) + \ldots + 2 + 1 \\ \hline \underbrace{(n+1) + (n+1) + (n+1) + \ldots (n+1) + (n+1)}_{n \text{ of them!}} \end{array} = \frac{n(n+1)}{2}$$

Many formulas have been discovered for the values of finite sums. Here are a few :

The first n integers:	$1 + 2 + 3 + 4 + \ldots + (n-1) + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
The first n squares:	$1^2 + 2^2 + 3^2 + \ldots + k^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
The first n cubes:	$1^3 + 2^3 + 3^3 + \ldots + k^3 + \ldots + n^3 = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 3: Evaluate

$$\sum_{i=1}^{50} i^2 = \frac{50 \cdot 51 \cdot 101}{6} = 42{,}925$$

Example 4: Express the following sums in sigma notation:

a) $4 + 9 + 16 + 25 + 36$

$2^2 + 3^2 + 4^2 + 5^2 + 6^2$

$$= \sum_{i=2}^{6} i^2$$

b) $10 + 100 + 1000 + 10{,}000$

$10^1 + 10^2 + 10^3 + 10^4$

$$= \sum_{i=1}^{4} 10^i$$

c) $\frac{4^2}{4-1} + \frac{5^2}{5-1} + \frac{6^2}{6-1}$

$$= \sum_{i=4}^{6} \frac{i^2}{i-1}$$

d) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{77}$

$\frac{1}{1}$

$$= \sum_{i=1}^{77} (-1)^{i+1} \frac{1}{i}$$

$$\sum_{i=1}^{77} (-1)^i \frac{-1}{i}$$

2— Algebra Rules for Finite Sums

Algebra Rules for Finite Sums

1. *Sum Rule:* $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. *Difference Rule:* $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. *Constant Multiple Rule:* $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c) $ca_1 + ca_2 + ca_3 \cdots ca_n = c[a_1 + a_2 + a_3 \cdots + a_n]$

4. *Constant Value Rule:* $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

$\sum_{i=1}^{4} c = c + c + c + c = 4c$

Example 5: Evaluate the following sums.

$\frac{n^2(n+1)^2}{4}$

a) $\sum_{i=1}^{10} i^3 = \frac{100 \cdot 121}{4} = 3,025$

b) $\sum_{i=1}^{100} 2 = 100 \cdot 2 = 200$

c) $\sum_{i=1}^{10} 3i^2 + 7 = = 3\sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} 7 = 3\left(\frac{10 \cdot 11 \cdot 21}{6}\right) + 70 = 1,225$

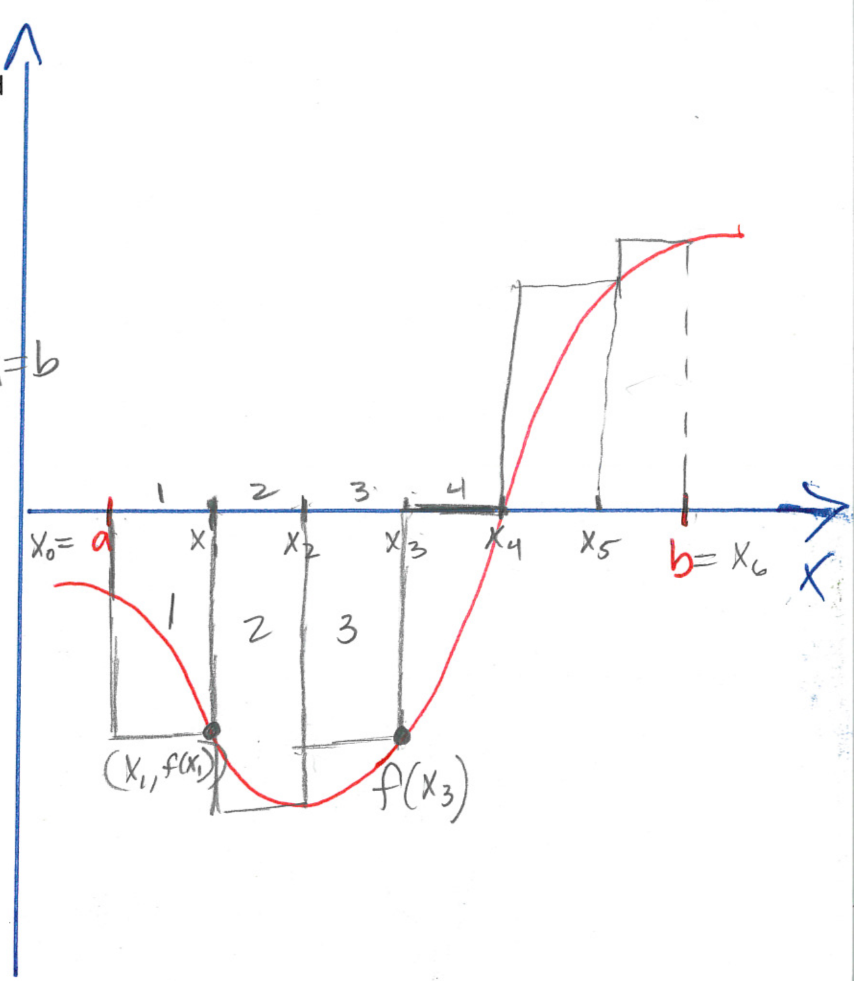
d) $\sum_{i=1}^{5} \frac{\pi i}{15} =$

e) $\sum_{i=1}^{13} \frac{3}{n} =$

f) $\sum_{i=1}^{7} i(2\cancel{k}i - 1) =$

3—Riemann Sums

1. Begin with an arbitrary bounded function f defined on a closed interval $[a, b]$.



2. Divide the interval $[a, b]$ into n equal subintervals .

the points of the subintervals are ordered so that

$a = X_0 < X_1 < X_2 < X_3 \dots < X_{n-1} < X_n = b$

3. We then have the following closed subintervals.

$[X_0, X_1]$ $[X_1, X_2]$ $\dots$ $[X_{n-1}, X_n]$

The i^{th} **subinterval** is:

$[X_{i-1}, X_i]$ for i an integer between 1 & n

if $i=3$, $[X_2, X_3]$

4. The width of the subintervals is $\frac{b-a}{6}$.

$\Delta X = X_1 - X_0$

$$\Delta X = \frac{b-a}{n}$$

5. Select the right endpoint of each subinterval. Then the chosen point for the i^{th} subinterval is simply x_i.

6. Draw a rectangle over each subinterval with height $f(x_i)$.

7. Form the product $f(x_i) \cdot \Delta x$. What do you notice about this product? What does it mean if the product is positive? Negative?

$f(X_3) \cdot \Delta X$ = Area of rectangle width ΔX & height $f(X_3)$

8. Sum the products to get the **Riemann Sum:**

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(X_1)\Delta X + f(X_2)\Delta X + f(X_3)\Delta X + \dots + f(X_n)\Delta X$$

as $n \to \infty$, $\Delta X \to 0$

9. What happens if we take the limit of this sum as $\to \infty$, Δx goes to ________.

10. We could do the same thing using left endpoints, or midpoints (book uses $\bar{x}_i$ to represent a midpoint) of each subinterval. Interestingly enough, we don't actually have to choose the same point in every subinterval! It is sufficient to take *any* sample point in each subinterval, call it x_i^*. So long as you have chosen intervals of equal width, the result will be the same when you let $n \to \infty$. (Why?)

4—Definition of the Definite Integral

The Definite Integral

Let f be a function defined on a closed interval $[a, b]$ and divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $a = x_0 < x_1 < \ldots < x_{k-1} < x_k < \ldots < x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the **definite integral** of f from $[a, b]$, denoted by $\int_a^b f(x)dx$, is defined to be the **number**, call it J, which is the limiting value of the Riemann Sum for f:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = J$$

We read this as, "the integral of f from a to b with respect to x".

If this limit exists, we say that f is **integrable** on $[a, b]$.

- As long as the ~~norm of the partition~~ width of the subinterval is small enough, then it doesn't matter what point you choose from each subinterval.
- The definite integral notation was developed by Leibniz, and it reflects its construction as a limit of Riemann sums.

Formula for ith term

width of subintervals

upper limit of integration

tells us that x is the variable of integration

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \;=\; \int_a^b f(x)dx$$

add up for all intervals from 1 to n

lower limit of integration

Theorem 4 (Definition of Definite Integral Using right endpoints)

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Example 6: Express the limits below as definite integrals on the given interval:

a) $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 - x_i + 1)\Delta x$, on $[0, 3]$

$= \int_0^3 (x^2 - x + 1)\,dx$

b) $\lim_{n\to\infty} \sum_{i=1}^{n}(\sin x_i + 2x_i)\Delta x$, on $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$= \int_{-\pi/2}^{\pi/2} (\sin x + 2x)\,dx$

> **Theorem**
>
> If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$ and the definite integral $J = \int_a^b f(x)dx$ exists.

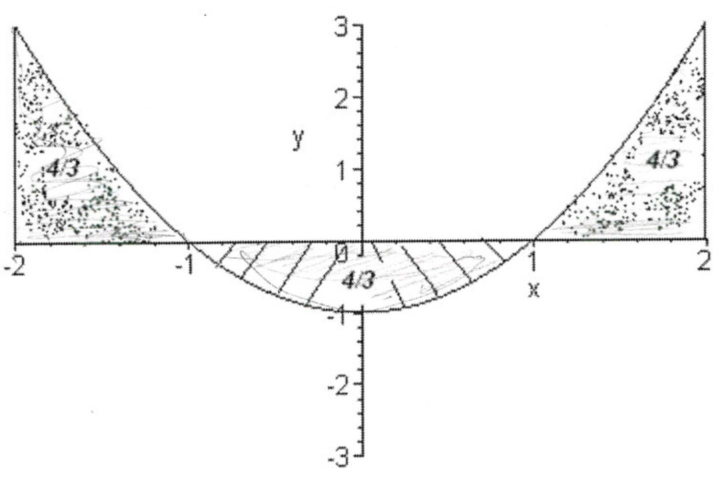
Notice….

- If f is a positive function on the interval $[a, b]$ then $\int_a^b f(x)dx =$ Area of the region below the curve $y = f(x)$ and above the x-axis on the interval $[a, b]$.
- If f is a negative function on the interval $[a, b]$ then $\int_a^b f(x)dx = -$(Area of the region above the curve $y = f(x)$ and below the x-axis on the interval $[a, b]$) .
- If f is a function that is sometimes positive and sometimes negative on the interval $[a, b]$ then $\int_a^b f(x)dx =$ (The area of the region below the curve where $f(x) \ge 0$ and above the x-axis) – (The area that lies below the x-axis and above the curve where $f(x) \le 0$).

Example 7: Let $f(x) = x^2 - 1$. What is the integral of f on the interval $-2 \le x \le 2$?

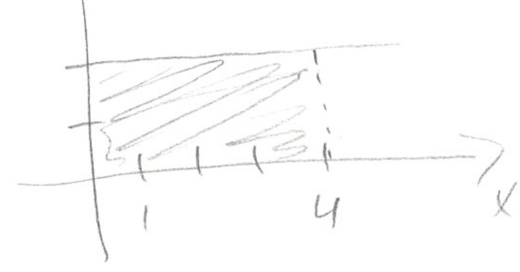
$\int_{-2}^{2} (x^2 - 1)\,dx$

$= \frac{4}{3} - \frac{4}{3} + \frac{4}{3} = \frac{4}{3}$

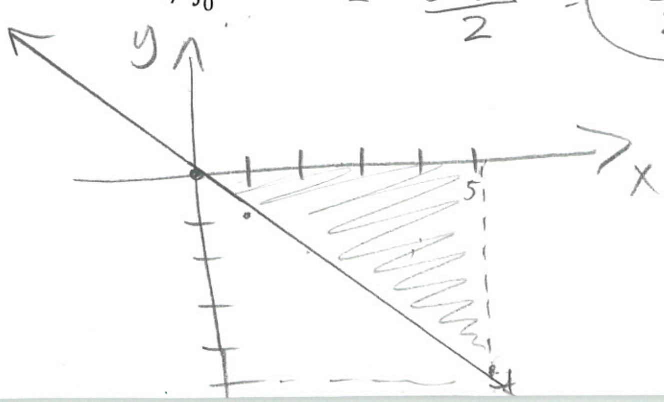


Example 8: Find the exact values of the following integrals without using a calculator.

a) $\int_0^4 2\,dx$ $= 4 \cdot 2 = 8$



b) $\int_0^5 -x\,dx$ $= -\frac{5(5)}{2} = -\frac{25}{2}$



c) $\int_0^3 (2x - 2)\, dx$

5—Properties of Definite Integrals

When we defined $\int_a^b f(x)dx$, we assume that $a < b$, and we move from left to right across the interval $[a, b]$. What if we put b in the lower position and a in the upper position? That is, move from right to left from b to a? The integral would be:

What about $\int_a^a f(x)dx$?

Some more properties of definite integrals:

TABLE 5.6 Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b - a).$$

7. *Domination:* $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0$ (Special case)

Example 9:

Suppose that : $\int_{-1}^{3} f(x)dx = 1$ $\int_{3}^{7} f(x)dx = -2$ and $\int_{-1}^{3} g(x)dx = 3$

Evaluate the following integrals.

a) $\int_{7}^{3} f(x)dx$

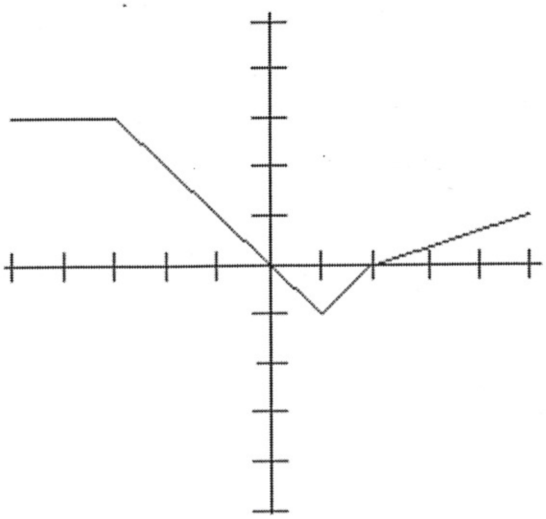
b) $\int_{-1}^{3} [2f(x) + 5g(x)]dx$

c) $\int_{-1}^{7} f(x)dx$

d) $\int_{-1}^{3} \frac{2}{3} g(x)dx$

e) $\int_{-1}^{3} \left[\frac{f(x)}{\sqrt{3}} - 4g(x)\right] dx$

f) Find the exact value of $\int_{-5}^{5} f(x)dx$ where f is the function whose graph is given below.



Homework: 1, 5ab, 12, 17, 18, 21, 22, 33-36, 47-50, 59-61