Section 2.9—Linear Approximations and Differentials

1—Linear Approximations

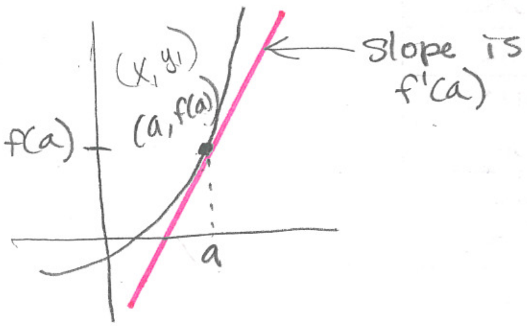
Near the point of tangency, a tangent line looks very much like the curve. (See Desmos.com)

We can use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. Write the equation of this tangent line at the point $(a, f(a))$.

point-slope eqn: $y - y_1 = m(x - x_1)$

eqn of tangent line: $y - f(a) = f'(a)(x - a)$

$\boxed{y = f(a) + f'(a)(x-a)}$



If we use the tangent line to approximate f, then the equation will look like:

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \text{ near } a$$

we give this approximation a special name: $L(x)$

This equation is called the **linear approximation** or **tangent line approximation** of f at a.

> **Definition**
>
> Then the linear function whose graph is this tangent line,
>
> $$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$
>
> Is called the **linearization** of f at a.

Example 1: a) Find the linearization of the function $f(x) = \sqrt{x+3}$ $\left((x+3)^{1/2}\right)$ at $a = 1$.

$L(x) = f(1) + f'(1)(x-1)$

- $f(1) = \sqrt{1+3} = 2$
- $f'(x) = \frac{1}{2}(x+3)^{-1/2}$
- $f'(1) = \frac{1}{2}(1+3)^{-1/2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

- $L(x) = 2 + \frac{1}{4}(x-1)$

$= 2 + \frac{1}{4}x - \frac{1}{4}$

$L(x) \boxed{= \frac{7}{4} + \frac{1}{4}x}$

b) Use the linearization found in part (a) to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. That is, use it to approximate the function at $x = .98$ and at $x = 1.05$. Are these approximations overestimates or underestimates?

$f(.98) \approx L(.98) = \frac{7}{4} + \frac{.98}{4} = 1.995 \implies \sqrt{3.98} \approx 1.995 \qquad \sqrt{3.98} = 1.994993$

$f(1.05) \approx L(1.05) = \frac{7}{4} + \frac{1.05}{4} = 2.0125 \implies \sqrt{4.05} \approx 2.0125 \qquad \sqrt{4.05} = 2.01246118$

overestimates

*Notice that the linear approximation gives an approximation *over an entire interval.*

The following table compares the estimates from the linear approximation in Example 1 with the true values. What do you notice about the accuracy of the estimates?

$\sqrt{x+3}$

x	$f(x)$	$L(x)$	Actual Value
0.9	$\sqrt{3.9}$	1.975	1.97484176...
0.98	$\sqrt{3.98}$	1.995	1.99499373...
1	$\sqrt{4}$	2	2.000000000...
1.05	$\sqrt{4.05}$	2.0125	2.01246117...
1.1	$\sqrt{4.1}$	2.025	2.02484567...
2	$\sqrt{5}$	2.25	2.23606797...
3	$\sqrt{6}$	2.5	2.44948974...

We can determine what level of accuracy we want our approximation(s) to have. The following example shows how.

Example 2: For what values of x is the linear approximation $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ accurate to within 0.5? How about within 0.1?

This means the difference between the 2 functions should be less than 0.5

$$\left|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)\right| < 0.5$$

OR

$$\sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5$$

If $-2.6 < x < 8.6$, then our approx. is accurate to within 0.5.

If $-1.1 < x < 3.9$ approximation is accurate to within 0.1

Applications of linear approximations are used in physics and engineering and will be studied further in Chapter 11.

2—Differentials

Definition

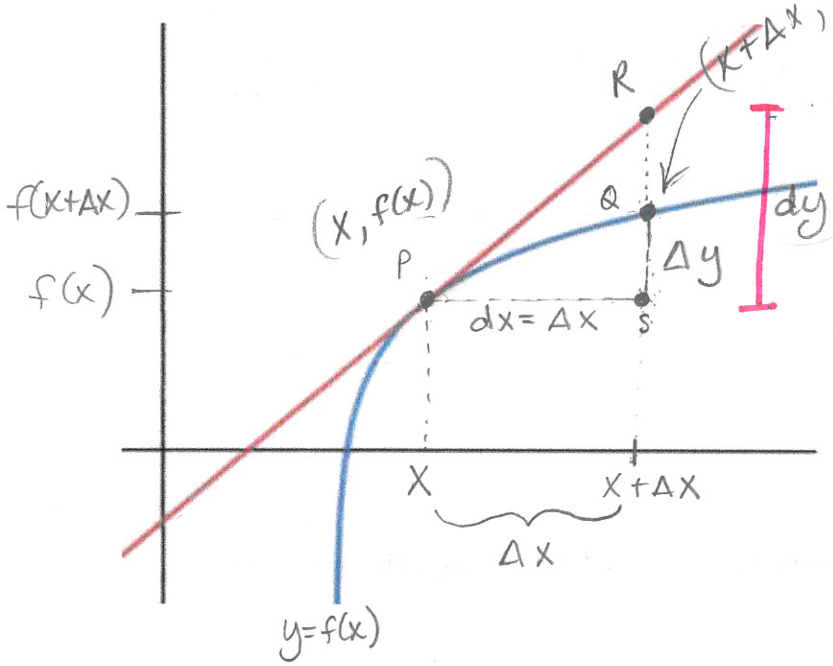
If $y = f(x)$ is a differentiable function, then the **differential** dx is an independent variable and the **differential** dy is a dependent variable (depends on x and dx) and is defined by the equation:

$$dy = f'(x)dx$$

$y_2 - y_1$

Where $dx = \Delta x$, and $\Delta y = f(x + \Delta x) - f(x)$ is the change in y of the *curve*, and dy is the change in y of the *tangent line*, or the change in the *linearization*.

Let's examine the meaning of differentials geometrically:



- For a function $y=f(x)$ for some change in x, Δx, the corresponding change in y will be Δy.
- Notice $\boxed{\Delta y = f(x+\Delta x) - f(x)}$
- The slope of the tangent line $\frac{dy}{dx} = f'(x)$
- to find dy = the rise

$m = \frac{\text{rise}}{\text{run}} = \frac{dy}{dx} = f'(x)$

$\boxed{dy = f'(x)\,dx}$

Example 3: (See Figure 6, pg. 191 in text)

a) Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05.

$x=2 \quad \Delta x = .05 \quad x+\Delta x = 2.05$

$\Delta y = f(x+\Delta x) - f(x)$
$= f(2.05) - f(2)$
$= 9.717625 - 9$
$= .717625$ ← Actual change in y for the curve (function)

$dy = f'(x)dx$
$= f'(2)(.05)$
$= 14(.05) = .7$ ← The rise of the tangent line

(An approx. of the rise of the curve)

$f'(x) = 3x^2 + 2x - 2$
$f'(2) = 3(2)^2 + 2(2) - 2$
$= 12 + 4 - 2 = \boxed{14}$

$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = \boxed{9.717625}$
$f(2) = 2^3 + 2^2 - 2(2) + 1 = 8 + 4 - 4 + 1 = \boxed{9}$

$x = 2 \quad \Delta x = .01$

b) Compare the values of Δy and dy for the same function in part (a) if x changes from 2 to 2.01.

$\Delta y = f(x+\Delta x) - f(x)$
$= f(2.01) - f(2)$
$= 9.140701 - 9$
$= .140701$

rise in the curve

$dy = f'(x)dx$
$= f'(2)(.01)$
$= 14(.01)$
$= .14$

rise of the tangent line

dy is an underestimate.

$f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1$
$= 9.140701$

In some cases it may be impossible to compute Δy exactly. In that case, the approximation by differentials is very useful.

Homework: 1-4, 7, 9, 11-18, 23, 26, 28