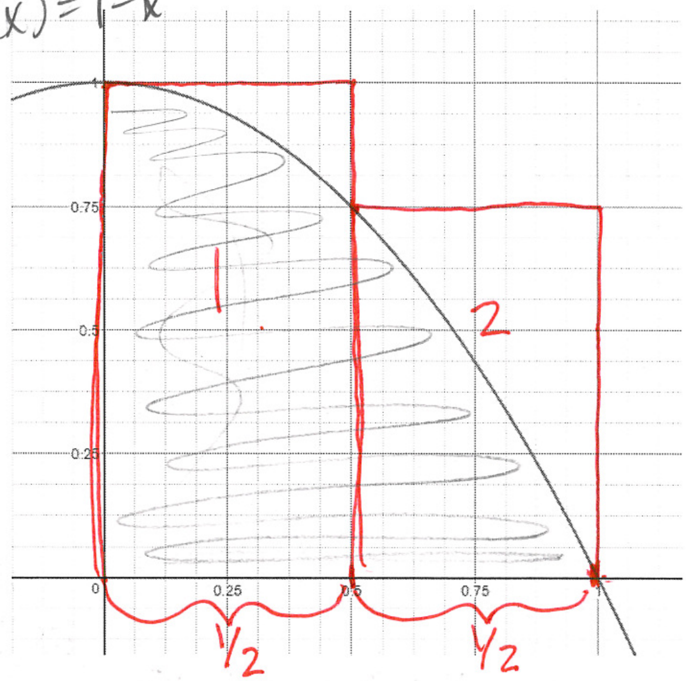
Section 4.1—Areas and Distances

1—Estimating Area using Finite Sums

- Approximate the area under the curve $f(x) = 1 - x^2$ on the interval $[0, 1]$ using an upper sum with 2 rectangles, an upper sum with 4 rectangles, a lower sum with 4 rectangles, and the midpoint rule using 4 rectangles.

- Upper Sum

$f(x) = 1 - x^2$

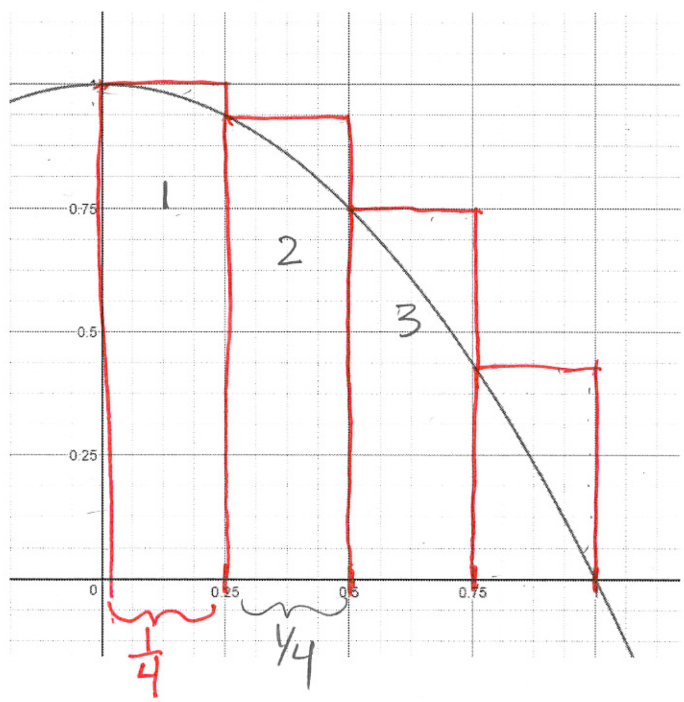


$$A_1 = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$A_2 = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

$$A_1 + A_2 = \frac{1}{2} + \frac{3}{8} = \frac{7}{8} \approx .875$$

overestimate



$$A = \left(\frac{1}{4} \cdot 1\right) + \left(\frac{1}{4} \cdot \frac{15}{16}\right) + \left(\frac{1}{4} \cdot \frac{12}{16}\right) + \left(\frac{1}{4} \cdot \frac{7}{16}\right)$$

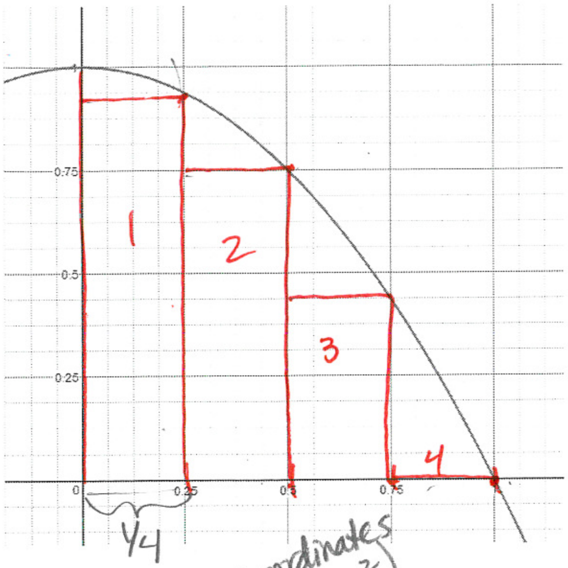
$$= \frac{1}{4}\left[1 + \frac{15}{16} + \frac{12}{16} + \frac{7}{16}\right]$$

$$= \frac{1}{4}\left[\frac{16 + 15 + 12 + 7}{16}\right]$$

$$= \frac{1}{\cancel{4}_{2}} \cdot \frac{\cancel{50}^{25}}{16} = \frac{25}{32} \approx .78125$$

Still an overestimate But better!

- Lower Sum



$$A = \frac{1}{4}\cdot\frac{15}{16} + \frac{1}{4}\cdot\frac{12}{16} + \frac{1}{4}\cdot\frac{7}{16} + \frac{1}{4}\cdot 0$$

$$= \frac{1}{4}\left[\frac{15}{16} + \frac{12}{16} + \frac{7}{16}\right]$$

$$= \frac{17}{32} \approx .53125$$

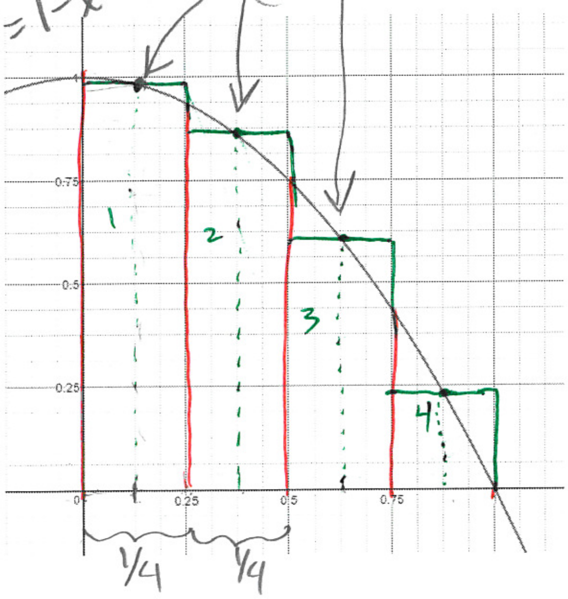
An underestimate

$$\boxed{\text{True Area lies between the Upper sum \& the Lower sum} \quad .53125 < A < .78125}$$

- Midpoint Sum

$f(x) = 1 - x^2$

Coordinates $(x, 1-x^2)$



$$A = \frac{1}{4}\cdot f\left(\tfrac{1}{8}\right) + \frac{1}{4}\cdot f\left(\tfrac{3}{8}\right) + \frac{1}{4}\cdot f\left(\tfrac{5}{8}\right) + \frac{1}{4}\cdot f\left(\tfrac{7}{8}\right)$$

$$= \frac{1}{4}\left(\frac{63}{64} + \frac{55}{64} + \frac{39}{64} + \frac{15}{64}\right)$$

$$= \frac{1}{4}\cdot\frac{172}{64} = \frac{43}{64} \approx .67187$$

better! over or under-estimate?

$f\left(\frac{1}{8}\right) = 1 - \left(\frac{1}{8}\right)^2 = 1 - \frac{1}{64} = \frac{63}{64}$

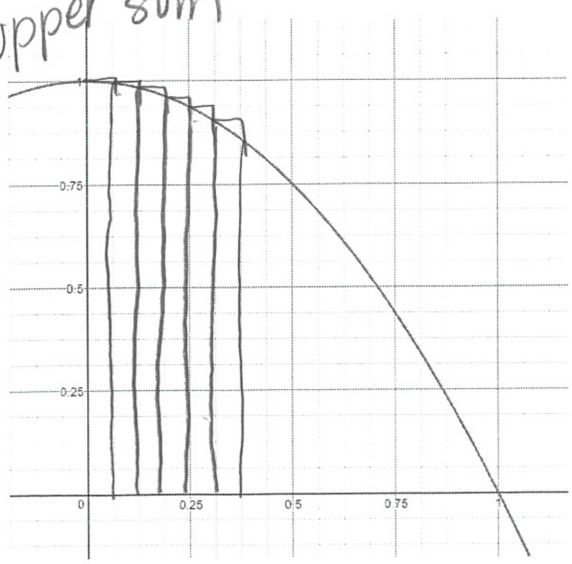
$f\left(\frac{3}{8}\right) = \frac{55}{64}$

$f\left(\frac{5}{8}\right) = \frac{39}{64}$

$f\left(\frac{7}{8}\right) = \frac{15}{64}$

- Dividing into more (smaller) intervals

upper sum



Lower sum

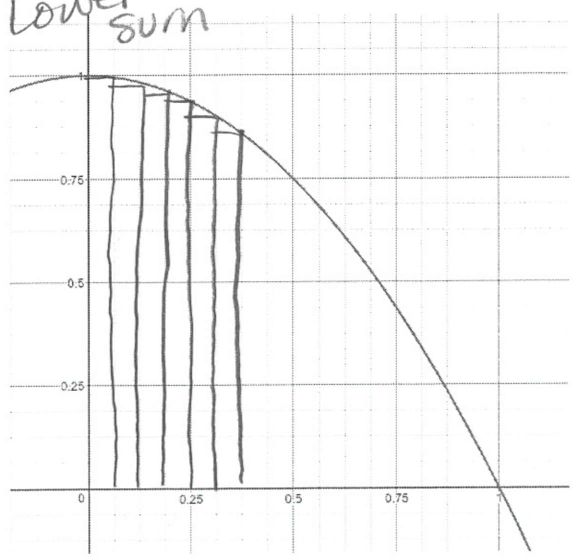


Table 1: Finite approximations for the area under the curve $f(x) = 1 - x^2$ between $x = 0$ and $x = 1$.

Number of Subintervals	Lower Sum	Midpoint Sum	Upper Sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

From the table it appears that, as n increases, both the Upper and the Lower sums (as well as the Midpoint sum) become better and better approximations to the area contained under the curve. Therefore, we *define* the area A to be the limit of the sums of the areas of the approximating rectangles.

We apply the previous ideas to a more general region, call it S, and subdivide S into n strips $S_1, S_2, \dots, S_n$ of equal width.

If we approximate the ith strip, S_i, by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoints. Then the area of the ith rectangle is $f(x_i) \cdot \Delta x$. Then what we think of as the area under the curve is approximated by the sum of the areas of these rectangles, which is

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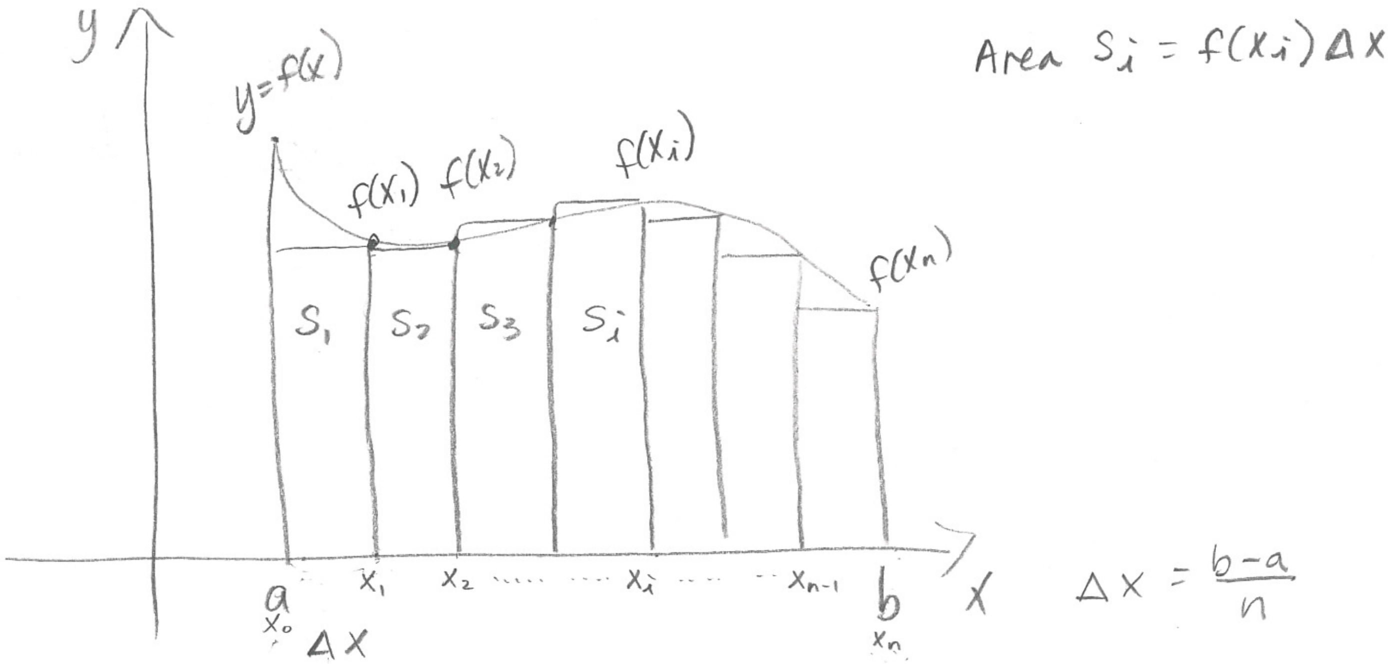
$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

This approximation will become better and better as the number of strips increases, that is, as $n \to \infty$.

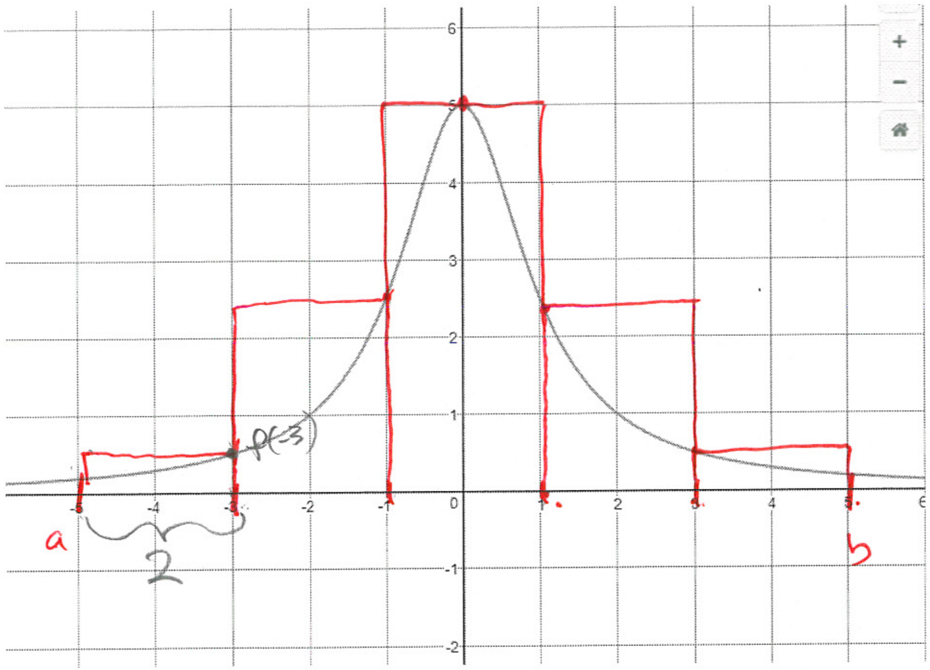


Definition

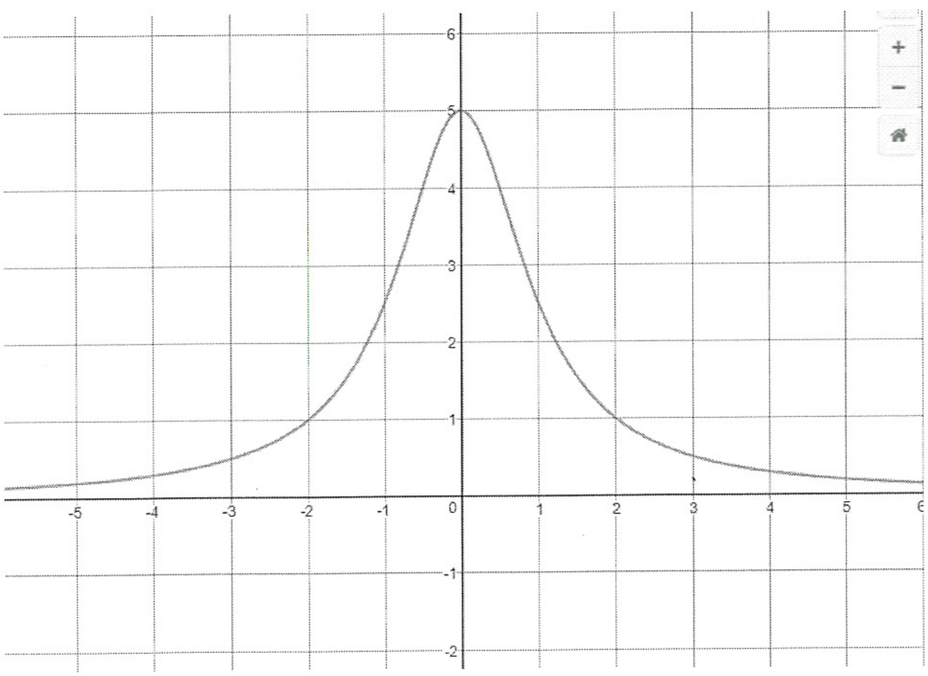
The **area** A of the region S that lies under the graph of the continuous function f is the limit fo the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} [\, f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

Example 1: (Practice) Approximate the area under the curve $f(x) = \frac{5}{x^2+1}$ on the interval $[-5, 5]$ using an upper sum with 5 rectangles. Then approximate the area using the midpoint rule with 5 rectangles of equal width.



$$A = 2\left[f(-3) + f(-1) + 5 + f(1) + f(3)\right]$$
$$= 2\left[\frac{1}{2} + \frac{5}{2} + 5 + \frac{5}{2} + \frac{1}{2}\right]$$
$$= 1 + 5 + 10 + 5 + 1$$
$$= 22$$



2—The Distance Problem

Now let's consider the *distance problem*, which is to find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. IF the velocity remains constant, then the distance problem is easy to solve by means of the formula distance = velocity x time. But if the velocity varies, it's not so easy to find the distance traveled.

$d = rt$

Example 2: Suppose the odometer on your car is broken and you want to estimate the distance driven over a 30-second time interval. You take the speedometer reading every five seconds and record them in a table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

First, convert the velocity readings to feet per second: (Note, 1mi/h = 5280/3600 ft/s)

Time (s)	0	5	10	15	20	25	30
Velocity (~~mi/h~~) ft/sec	25	31	35	43	47	45	41

a) Estimate the total distance traveled in 30 seconds using the velocity readings at the beginning of each time period.

$d = r \cdot t$

$25 \frac{ft}{sec} \cdot 5\ sec = 125\ ft$

$31 \frac{ft}{sec} \cdot 5\ sec = 155\ ft.$

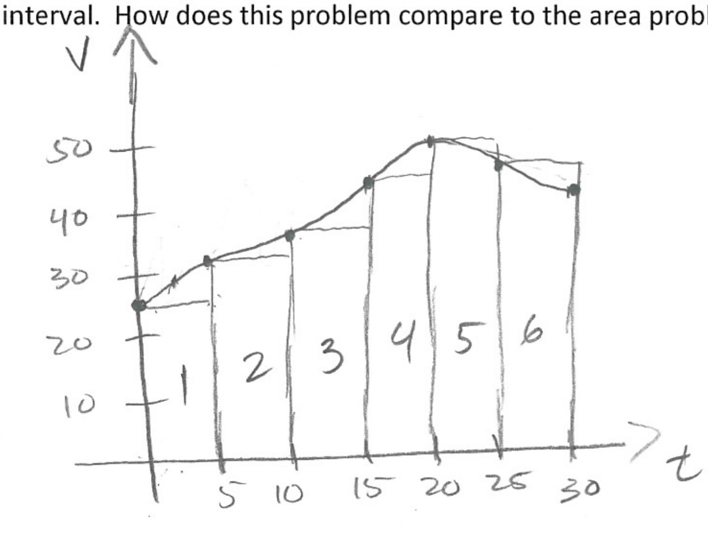
$(25 \cdot 5) + (31 \cdot 5) + (35 \cdot 5) + (43 \cdot 5) + (47 \cdot 5) + (45 \cdot 5)$

$5[25 + 31 + 35 + 43 + 47 + 45] = 1,130\ ft.$

b) Estimate the total distance traveled using the velocity at the of each time period.

$5[31 + 35 + 43 + 47 + 45 + 41] = 1,210\ ft.$

c) Sketch a graph of the velocity function and draw rectangles whose heights are the initial velocities for each time interval. How does this problem compare to the area problems we looked at earlier?



$d = 5 \cdot 25 = 125 \leftarrow$ dist. traveled in 1st 5 seconds

$A_1 = 5 \cdot 25 = 125$

d) What would happen if we took velocity readings more frequently?

our estimate gets better & better!

Homework: 1, 3-5, 13, 14, 17, 18, 21, 22, 24