# Section 3.9—Antiderivatives

## 1—Finding Antiderivatives

In many situations in physics, a scientist may know the velocity or acceleration of a particle and may want to know the position of the particle at a given time.

- A rock is dropped from the Golden Gate Bridge from a height of 220 feet. The acceleration of the rock is $-9.8$ meters/second/second. What is the velocity of the rock when it hits the water?

A microbiologist may know the rate at which a population of bacteria is growing and may want to know the number of bacteria in the population at a given time.

- By taking measurements, it is determined that a population of bacteria doubles every 132 minutes. Therefore, by the law of exponential growth, the rate at which an initial population of 1,000,000 bacteria are growing at $t$ number of hours is given by $\frac{5{,}000{,}000 \ln 2}{11} e^{\frac{\ln 2}{2.2} t}$. What is the size of the population after $t$ hours?

For each of these scenarios, the unknown that is sought for is a function whose derivative is the known function. To answer these questions we need to determine an **antiderivative** of the known functions.

| **Definition of Antiderivative** |
|---|
| A function $F$ is said to be an **antiderivative** of $f$ if $F'(x) = f(x)$. |
| In other words, an **antiderivative** of a function $f$ is a function whose derivative is $f$. |
| *Note: we generally use capital letters to represent antiderivatives. |

Given the following derivatives of a function, what is the original function? ( "The answer is _______. What is the question?" )

a) $\frac{dy}{dx}\left[\ \frac{x^8}{8} + C\ \right] = x^7$ $\qquad \frac{x^8}{8} + 47$

b) ~~$\frac{dy}{dx}[\qquad] = \frac{1}{x}$~~

$2 \sec\theta \sec\theta \tan\theta$

$3\cos\theta + 2\sec^2\theta \tan\theta$

c) $\frac{dy}{dx}\left[\ 3\sin\theta + \sec^2\theta + C\ \right] = 3\cos\cancel{2}\theta + 2\sec^2\theta\tan\theta$

$[\sec\theta]^2$

$2\sec\theta \sec\theta \tan\theta$

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called ***antidifferentiation.***

Recall an important consequence of the Mean Value Theorem:
If $f'(x) = g'(x)$ at each point $x \in (a, b)$, then $f(x) = g(x) + C$ for all $x \in (a, b)$.

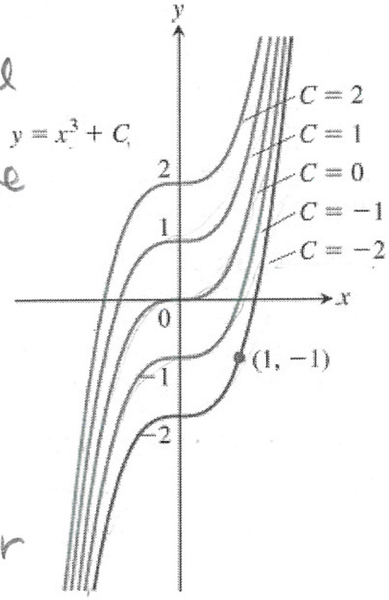
| **Theorem:** |
|---|
| If $F$ is an antiderivative of $f$ on an interval $I$, then $F(x) + C$, where $C$ is a constant, represents every antiderivative of $f$ on $I$. |
| We call $F(x) + C$ the **general antiderivative** of $f$. |

Notice that the general antiderivative of $f$ is a *family* of functions $F(x)+C$ whose graphs are vertical translations of one another.

**Example 1:** Find the antiderivative of $f(x)=3x^2$.

$\boxed{F(x) = x^3 + C}$ ← The general Anti-derivative

$F'(x) = 3x^2$




b) Find the antiderivative of $f$ that satisfies $F(1)=-1$ ?

$F(1) = 1^3 + C \overset{\text{must}}{=} -1$

$1 + C = -1$

$C = -2$

$\boxed{F(x) = x^3 - 2}$ ← The particular Solution

**Example 2:** Find the most general antiderivative of the following functions:

a) $f(x)=-\sin x$

$F(x) = \cos x + C$

b) $f(x)=x^5$

$F(x) = \frac{x^6}{6} + C$

If $F$ is an antiderivative of $f$ and $G$ is an antiderivative of $g$, then the following antiderivative laws are justified by derivative laws:

| **Function** | **General Antiderivative** |
|---|---|
| 1. $x^n \quad (n \neq 1)$ | $\frac{1}{n+1}x^{n+1}+C$ |
| 2. $k \cdot f(x)$ | $k \cdot F(x)+C$ |
| 3. $f(x) \pm g(x)$ | $F(x) \pm G(x)+C$ |
| 4. $\cos x$ | $\sin x\ +C$ |
| 5. $\sin x$ | $-\cos x+C$ |
| 6. $\sec^2 x$ | $\tan x+C$ |
| 7. $\sec x \tan x$ | $\sec x+C$ |
| 8. $\csc^2 x$ | $-\cot x+C$ |
| 9. $\csc x \cot x$ | $-\csc x+C$ |

***The derivative laws also tell us the following:

- $F(x)G(x)$ is **not** an antiderivative of $f(x)g(x)$.
- $\frac{F(x)}{G(x)}$ is **not** an antiderivative of $\frac{f(x)}{g(x)}$.

**Example 3:** Find all antiderivatives for the following functions.

a) $g(x) = 4x^2 + \cos x$

$G(x) = 4 \cdot \frac{1}{3} x^3 + \sin x + C$

$= \frac{4}{3} x^3 + \sin x + C$

b) $g(x) = \sqrt{2x} = \sqrt{2} \cdot \sqrt{x} = \sqrt{2}\, x^{1/2}$

$G(x) = \sqrt{2} \cdot \frac{2}{3} x^{3/2} + C$

$= \frac{2\sqrt{2}}{3} x^{3/2} + C$

c) $f(x) = -\frac{4+x^3}{\sqrt{x}} = \frac{-4-x^3}{\sqrt{x}}$

$= \frac{-4}{\sqrt{x}} - \frac{x^3}{\underbrace{\sqrt{x}}_{x^{1/2}}} = -4x^{-1/2} - x^{5/2}$

$F(x) = -4 \cdot 2x^{1/2} - \frac{2}{7} x^{7/2} + C$

$= -8x^{1/2} - \frac{2}{7} x^{7/2} + C$

d) $f(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x} = 4\sin x + 2x^4 - x^{-1/2}$

$F(x) = -4\cos x + 2 \cdot \frac{1}{5} x^5 - 2x^{1/2}$

$= -4\cos x + \frac{2}{5} x^5 - 2x^{1/2} + C$

**2— Initial Value Problems and Differential Equations**

The combination of a **differential equation** (a function which is a derivative of another function) and an initial condition is called an **initial value problem.** We can find the general antiderivative, and then plug in the initial condition (values at the time $t = 0$) to find the constant $C$, and thus find the **particular solution**, or the particular function which satisfies the equation.

**Example 4:** Given $f'(x) = 3x^{-\frac{2}{3}}$, *and* $f(-1) = -5$, find $f(x)$. *This is an example of an initial value problem.*

① $f(x) = 3 \cdot 3x^{1/3} + C$

$= \boxed{9x^{1/3} + C}$ ← Find Gen. Antiderivative

② Solve for C

$f(-1) = 9(-1)^{1/3} + C \overset{\text{must}}{=} -5$

$-9 + C = -5$

$\boxed{C = 4}$

③ $f(x) = 9x^{1/3} + 4$ ← write the particular solution

**Example 5:** A rock is dropped from the Golden Gate Bridge from a height of 220 ~~feet~~ meters. The acceleration of the rock is −9.8 meters/second/second. What is the position of the rock after $t$ seconds? What is the velocity of the rock when it hits the water? *This is an example of an initial value problem.*

position function

$\boxed{a(t) = -9.8} = v'(t)$

what is velocity of the rock at time $t=0$ $\quad V(0)=0$

$V(t) = -9.8t + C_1$

$V(0) = -9.8(0) + C_1 \overset{\text{must}}{=} 0$

$0 + C_1 = 0$

$\boxed{V(t) = -9.8t}$

what is $S(0)$ ? $= 220$

$S(t) = -9.8\left(\frac{1}{2}\right)t^2 + C_2$

$= -4.9t^2 + C_2$

$S(0) = -4.9(0)^2 + C_2 \overset{\text{must}}{=} 220$

$\boxed{S(t) = -4.9t^2 + 220}$

$-4.9t^2 + 220 \overset{\text{set}}{=} 0$

$\frac{220}{4.9} = \frac{4.9t^2}{4.9}$

$44.897 = t^2$

$t = \pm 6.7$

$t = 6.7$

$V(6.7) = -9.8(6.7) = -65.66 \frac{\text{meters}}{\text{sec}}$

**Gravity Constant**

An object near the surface of the earth is subject to a gravitational force that produces a downward acceleration denoted by $g$. For motion close to the ground we may assume that $g$ is constant, its value being about 9.8 m/s$^2$ or 32 ft/s$^2$.

**Example 6:** A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground.

a) Find its height above the ground $t$ seconds later.

$\boxed{a(t) = -32}$ ft/sec$^2$

$V(t) = -32t + C_1$

$V(0) = -32(0) + C_1 \overset{\text{must}}{=} 48$

$\boxed{V(t) = -32t + 48}$

$S(t) = -32\left(\frac{1}{2}\right)t^2 + 48t + C_2$

$= -16t^2 + 48t + C_2$

$S(0) = -16(0^2) + 48(0) + C_2 \overset{\text{must}}{=} 432$

$C_2 = 432$

$\boxed{S(t) = -16t^2 + 48t + 432}$

b) When does it reach its maximum height?

when the derivative equals 0 or when velocity = 0

$V(t) = -32t + 48 \overset{\text{set}}{=} 0$

$48 = 32t$

$t = 1.5$ seconds

c) When does it hit the ground? position is 0 when it hits the ground

$S(t) = -16t^2 + 48t + 432 \overset{\text{set}}{=} 0$

$-16(t^2 - 3t - 27) = 0$

$\sqrt{9 \cdot 13}$

$t = \frac{3 \pm \sqrt{9 - 4(1)(-27)}}{2} = \frac{3 \pm \sqrt{117}}{2}$

$= \frac{3 \pm 3\sqrt{13}}{2} \approx 6.908$, $\cancel{-3.908}$

when $t = 6.908$ seconds it hits the ground

## More Practice with Initial Value Problems

**Example 7:** A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

$\boxed{a(t) = 6t + 4}$

$V(t) = 6(\frac{1}{2})t^2 + 4t + c$

$= 3t^2 + 4t + c$

$V(0) = 0 + 0 + c = -6$

$c = -6$

$\boxed{V(t) = 3t^2 + 4t - 6} \longrightarrow 3(\frac{1}{3})t^3 + 4(\frac{1}{2})t^2 - 6t$

$s(t) = t^3 + 2t^2 - 6t + C$

$S(0) = 0 + 0 - 0 + C = 9$

$C = 9$

$\boxed{S(t) = t^3 + 2t^2 - 6t + 9}$

**Example 8:** Suppose that $f'(x) = \sin x$ for all $x$. Find $f(\pi)$ if $\boxed{f\left(\frac{\pi}{2}\right) = 5.}$

$f(x) = -\cos x + C$

$f(\frac{\pi}{2}) = -\cos\frac{\pi}{2} + C \overset{\text{must}}{=} 5$

$0 + C = 5$

$C = 5$

$\boxed{f(x) = -\cos x + 5} \leftarrow$ particular soln.

$f(\pi) = -\cos\pi + 5 = 1 + 5 = 6$

**Homework:** 3-18 multiples of 3, 24-42 multiples of 3, 26, 46, 53, 55, 56, 59, 62, 63, 68