Thesis WIP with notes

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Abstract

TODO

Contents

1 Theoretical Exercise

A simplified version of the integral is:

$$I(q,p,m_\psi) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{\Big(k^2 - m_\psi^2 + i\epsilon\Big)\Big((k-q)^2 - m_\psi^2 + i\epsilon\Big)\Big((k+p)^2 - m_\psi^2 + i\epsilon\Big)} \tag{1.1}$$

$$= \int \frac{d^4k}{(2\pi)^4} i \prod_{i=1}^{3} \frac{1}{\Big((k-q_i)^2 - m_\psi^2 + i\epsilon\Big)} \tag{1.2}$$

with $q_i = (0, -q, p)_i$.

The pole conditions for each term in the product becomes

$$(k+q_i)^2 = m_\psi^2 - i\epsilon \quad \Longrightarrow \quad k^0 + q_i^0 = \pm\sqrt{(\vec{k}+\vec{q}_i)^2 + m_\psi^2} = \pm E_i \tag{1.3}$$

Choosing to close the contour below we can then apply the residue theorem. The integration over the semicircle vanishes in the limit $r \to \infty$ because the integrand is $\propto r^{-6}$.

$$\int_{\text{semi circle}} dk^0 r^{-6} \propto \int_\pi d\theta r^{-5} \propto r^{-5} \overset{r\to\infty}{\longrightarrow} 0 \tag{1.4}$$

The residues can be computed with l'Hospitals rule

$$\underset{E_i = k^0 + q_i^0}{\text{Res}} = \frac{1}{2E_i} \prod_{j\neq i} \frac{1}{\Big((k1q_i)^2 - m_\psi^2 + i\epsilon\Big)}\Bigg|_{k^0+q_i^0 = +E_1} \tag{1.5}$$

$$= \int_{k^0} dk\, \theta(k^0)\, \delta\left(k^2 - m_\psi^2\right) \prod_{j\neq i} \frac{1}{\Big((k+q_j)^2 - m_\psi^2 + i\epsilon\Big)} \tag{1.6}$$

Notice that the application of the delta exactly reproduces the derivatives, and the heaviside function ensures that the correct pole is selected. If we now define the shorthand notation:

$$\delta^+\left((k+q_i)^2 - m_\psi^2\right) = \theta(k^0 1 q_i^0)\, \delta\left((k+q_i)^2 - m_\psi^2\right) E_i \tag{1.7}$$

We can now apply the residue theorem, notice that we pick up an extra $-$ sign due to the clockwise direction of the semicircle

$$\begin{aligned}
I(q,p,m_\psi) &= \int \frac{d^3k}{(2\pi)^4} \int dk^0 i \prod_{i=1}^{3} \frac{1}{\Big((k-q_i)^2 - m_\psi^2 + i\epsilon\Big)} \\
&= \int \frac{d^3k}{(2\pi)^4} (2\pi i)\, i \left(- \sum_i \underset{E_i = k^0 + q_i^0}{\text{Res}} \right) \\
&= \int \frac{d^4k}{(2\pi)^3} \frac{\delta^+\Big(k^2 - m_\psi^2\Big) + \delta^+\Big((k+q)^2 - m_\psi^2\Big) + \delta^+\Big((k+p)^2 - m_\psi^2\Big)}{\Big(k^2 - m_\psi^2 + i\epsilon\Big)\Big((k+q)^2 - m_\psi^2 + i\epsilon\Big)\Big((k+p)^2 - m_\psi^2 + i\epsilon\Big)}
\end{aligned}$$

The execution of the k^0 integral is now a matter of inserting the correct values, we can introduce the notation $\bar{\eta}_{i,j}^{\pm_1\pm_2} = \pm_1 E_i \pm_2 E_j$ to keep the result shorter. This is procedure is especially easy (but tedious) when using the intermediary results from the previous step.

$$\begin{aligned}
I(q,p,m_\psi) = \int \frac{d^3k}{(2\pi)^3} \Bigg[& \\
& \frac{1}{2E_1} \frac{1}{(\bar{\eta}_{12}^{++} - q^0)(\bar{\eta}_{12}^{+-} - q^0)} \frac{1}{(\bar{\eta}_{13}^{++} + p^0)(\bar{\eta}_{13}^{+-} + p^0)} \\
& + \frac{1}{(\bar{\eta}_{21}^{++} + q^0)(\bar{\eta}_{21}^{+-} + q^0)} \frac{1}{2E_2} \frac{1}{(\bar{\eta}_{23}^{++} + p^0 + q^0)(\bar{\eta}_{23}^{+-} + p^0 + q^0)} \\
& + \frac{1}{(\bar{\eta}_{31}^{++} - p^0)(\bar{\eta}_{31}^{+-} - p^0)} \frac{1}{(\bar{\eta}_{32}^{++} - p^0 - q^0)(\bar{\eta}_{32}^{+-} - p^0 - q^0)} \frac{1}{2E_3} \Bigg].
\end{aligned}$$

We can also absorb the Energy shifts into the η coefficients by introducing the notation $\eta_{i,j}^{\pm_1 \pm_2} = \pm_1 E_i \pm_2 E_j \mp_1 (q_i^0 - q_j^0)$ and $q_i = (0, -q, p)$.

$$I(q,p,m_\psi) = \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2E_1} \frac{1}{\eta_{12}^{++}\,\eta_{12}^{+-}} \frac{1}{\eta_{13}^{++}\,\eta_{13}^{+-}} + \frac{1}{2E_2} \frac{1}{\eta_{21}^{++}\,\eta_{21}^{+-}} \frac{1}{\eta_{23}^{++}\,\eta_{23}^{+-}} + \frac{1}{2E_3} \frac{1}{\eta_{31}^{++}\,\eta_{31}^{+-}} \frac{1}{\eta_{32}^{++}\,\eta_{32}^{+-}} \right]. \tag{1.8}$$

We will now take a closer look at the singularities of this integral. We have a singularity exactly when one of the η coefficients is 0. It makes sense to split into 2 cases:

$$\eta_{ij}^{++} = E_i + E_j - (q_i^0 - q_j^0) \tag{1.9}$$
$$\eta_{ij}^{+-} = E_i - E_j + (q_i^0 - q_j^0) \tag{1.10}$$
$$\tag{1.11}$$

You can find an example here https://www.desmos.com/3d/7gzgrreciz First try finding an existence condition that any zero exists for η^{++}

$$0 = E_i + E_j - (q_i^0 - q_j^0) \tag{1.12}$$
$$= \sqrt{(\vec{k} + \vec{q}_i)^2 + m_\psi^2} + \sqrt{(\vec{k} + \vec{q}_j)^2 + m_\psi^2} - (q_i^0 - q_j^0) \tag{1.13}$$
$$\tag{1.14}$$

We can now simplify this expression by introducing $\vec{l} = \vec{k} + \vec{q}_i$

$$= \sqrt{\vec{l}^2 + m_\psi^2} + \sqrt{(\vec{l} + \vec{q}_j - \vec{q}_i)^2 + m_\psi^2} - (q_i^0 - q_j^0) \tag{1.15}$$

For sufficiently large values of $\vec{l}$ this is always positive. It is also easy to show, e.g. by taking the derivative, that the minimum of this expression is at $\vec{l} = \frac{1}{2}(\vec{q}_j - \vec{q}_i)$. Since the equation is continuous there exists a zero iff the minimum is ≤ 0.

$$\sqrt{(\frac{1}{2}(\vec{q}_j - \vec{q}_i))^2 + m_\psi^2} + \sqrt{(\frac{1}{2}(\vec{q}_j - \vec{q}_i))^2 + m_\psi^2} - (q_i^0 - q_j^0) \leq 0 \tag{1.16}$$
$$\implies \quad (\vec{q}_j - \vec{q}_i)^2 + 4m_\psi^2 \leq (q_i^0 - q_j^0)^2 \tag{1.17}$$
$$\implies \quad (q_i^0 - q_j^0)^2 - (\vec{q}_j - \vec{q}_i)^2 - \geq 4m_\psi^2 \tag{1.18}$$
$$m_S \geq 2m_\psi \tag{1.19}$$

Luckily the remaining η^{+-} singularities all cancel pairwise, we will show this by repeatedly applying the partial fractioning identity to (1.8)

$$\frac{1}{xy} = \frac{1}{x-y}\left(\frac{1}{y} - \frac{1}{x}\right) \tag{1.20}$$

First notice however, that

$$\eta_{ij}^{++} - \eta_{ij}^{+-} = 2E_j \tag{1.21}$$
$$\implies \quad \frac{1}{\eta_{ij}^{++}\eta_{ij}^{+-}} = \frac{1}{2E_j}\left(\frac{1}{\eta_{ij}^{+-}} - \frac{1}{\eta_{ij}^{++}}\right) \tag{1.22}$$
$$\implies \quad \frac{1}{\eta_{ij}^{+-}} = \frac{E_j}{\eta_{ij}^{++}\eta_{ij}^{+-}} + \frac{1}{\eta_{ij}^{++}} \tag{1.23}$$

This result can now be applied to each term in (1.8)

$$\begin{aligned} I(q,p,m_\psi) = \int d^3\vec{k} \frac{1}{(2\pi)^3} \frac{1}{(2E_1)(2E_2)(2E_3)} \Bigg[& \\ & \left(\frac{1}{\eta_{12}^{+-}} - \frac{1}{\eta_{12}^{++}}\right)\left(\frac{1}{\eta_{13}^{+-}} - \frac{1}{\eta_{13}^{++}}\right) \\ & + \left(\frac{1}{\eta_{21}^{+-}} - \frac{1}{\eta_{21}^{++}}\right)\left(\frac{1}{\eta_{23}^{+-}} - \frac{1}{\eta_{23}^{++}}\right) \\ & + \left(\frac{1}{\eta_{31}^{+-}} - \frac{1}{\eta_{31}^{++}}\right)\left(\frac{1}{\eta_{32}^{+-}} - \frac{1}{\eta_{32}^{++}}\right) \Bigg] \end{aligned}$$

We now have to factor out all the brackets.

$$
\begin{aligned}
I(q,p,m_\psi) = \int d^3\vec{k}\frac{1}{(2\pi)^3}\frac{1}{(2E_1)(2E_2)(2E_3)}\Bigg[& \\
& \frac{1}{\eta_{12}^{+-}\eta_{13}^{+-}} - \frac{1}{\eta_{12}^{+-}\eta_{13}^{++}} - \frac{1}{\eta_{12}^{++}\eta_{13}^{+-}} + \frac{1}{\eta_{12}^{++}\eta_{13}^{++}} \\
& + \frac{1}{\eta_{21}^{+-}\eta_{23}^{+-}} - \frac{1}{\eta_{21}^{+-}\eta_{23}^{++}} - \frac{1}{\eta_{21}^{++}\eta_{23}^{+-}} + \frac{1}{\eta_{21}^{++}\eta_{23}^{++}} \\
& + \frac{1}{\eta_{31}^{+-}\eta_{32}^{+-}} - \frac{1}{\eta_{31}^{+-}\eta_{32}^{++}} - \frac{1}{\eta_{31}^{++}\eta_{32}^{+-}} + \frac{1}{\eta_{31}^{++}\eta_{32}^{++}}\Bigg]
\end{aligned}
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\begin{aligned}
I(q,p,m_\psi) = \int d^3\vec{k}\frac{1}{(2\pi)^3}\frac{1}{(2E_1)(2E_2)(2E_3)}\Bigg[& \\
& \frac{1}{\eta_{21}^{++}\eta_{31}^{++}} + \frac{1}{\eta_{12}^{++}\eta_{13}^{++}} + \frac{1}{\eta_{12}^{++}\eta_{32}^{++}} + \frac{1}{\eta_{21}^{++}\eta_{23}^{++}} + \frac{1}{\eta_{13}^{++}\eta_{23}^{++}} + \frac{1}{\eta_{31}^{++}\eta_{32}^{++}}\Bigg]
\end{aligned}
$$