ATTACKING (EC)DSA SCHEME WITH EPHEMERAL KEYS SHARING SPECIFIC BITS

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ABSTRACT. In this paper, we present a deterministic attack on (EC)DSA signature scheme, providing that several signatures are known such that the corresponding ephemeral keys share a certain amount of bits without knowing their value. By eliminating the shared blocks of bits between the ephemeral keys, we get a lattice of dimension equal to the number of signatures having a vector containing the private key. We compute an upper bound for the distance of this vector from a target vector, and next, using Kannan's enumeration algorithm, we determine it and hence the secret key. The attack can be made highly efficient by appropriately selecting the number of shared bits and the number of signatures.

1. Introduction - Statement of results

In August 1991, the U.S. government's National Institute of Standards and Technology (NIST) proposed an algorithm for digital signatures. The algorithm is known as DSA, for Digital Signature Algorithm [26, 22, 18]. It is an efficient variant of the ElGamal digital signature scheme [8] intended for use in electronic mail, electronic funds transfer, electronic data interchange, software distribution, data storage, and other applications which require data integrity assurance and data authentication. In 1998, an elliptic curve analogue called Elliptic Curve Digital Signature Algorithm (ECDSA) was proposed and standardized [16, 17, 18].

1.1. The (EC)DSA Signature Scheme. First, we recall the DSA schemes. The signer selects a prime p of size between 1024 and 3072 bits with increments of 1024, as recommended in FIPS 186-3 [9, page 15]. Also, he selects a prime q of size 160, 224 or 256 bits, with q|p-1 and a generator g of the unique order q subgroup G of the multiplicative group \mathbb{F}_p^* of the prime finite field \mathbb{F}_p . Furthermore, he selects a randomly $a \in \{1, \ldots, q-1\}$ and computes $R = g^a \mod p$. The public key of the signer is (p, q, g, R) and his private key a. He also publishes a hash function $h: \{0,1\}^* \to \{0,\ldots,q-1\}$. To sign a message $m \in \{0,1\}^*$, he selects randomly $k \in \{1,\ldots,q-1\}$ which is the ephemeral key, and computes $r = (g^k \mod p) \mod q$ and $s = k^{-1}(h(m) + ar) \mod q$. The signature of m is (r,s). The signature is accepted as valid if and only if the following holds:

$$r = ((g^{s^{-1}h(m) \bmod q} R^{s^{-1}r \bmod q}) \bmod p) \bmod q.$$

Next, let us recall the ECDSA scheme. The signer selects an elliptic curve E over \mathbb{F}_p , a point $P \in E(\mathbb{F}_p)$ with order a prime q of size at least 160 bits. Following

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FIPS 186-3, the prime p must belongs to the set $\{160, 224, 256, 512\}$. Further, he chooses randomly $a \in \{1, \ldots, q-1\}$ and computes Q = aP. Finally, he publishes a hash function $h: \{0,1\}^* \to \{0,\ldots,q-1\}$. The public key of the signer is (E,p,q,P,Q) and his private key a. To sign a message m, he selects randomly $k \in \{1,\ldots,q-1\}$ which is the ephemeral key and computes kP = (x,y) (where x and y are regarded as integers between 0 and p-1). He computes $r=x \mod q$ and $s=k^{-1}(h(m)+ar) \mod q$. The signature of m is (r,s). The verifier computes

$$u_1 = s^{-1}h(m) \mod q$$
, $u_2 = s^{-1}r \mod q$, $u_1P + u_2Q = (x_0, y_0)$.

He accepts the signature if and only if $r = x_0 \mod q$.

1.2. **Previous Results.** Researchers have explored various attacks on DSA schemes by analyzing the signature equation $s = k^{-1}(h(m) + ar) \mod q$ and using lattice reduction techniques such as LLL and CVP algorithms. One study focused on the use of a linear congruential pseudorandom number generator (LCG) for generating random numbers in DSA [3], showing that combining the DSA signature equations with LCG generation equations can lead to a system of equations that provide the secret key. To recover the secret key, several heuristic attacks have been proposed [15] in another study, which assume the revelation of a small fraction of the corresponding nonce k. However, these attacks are based on heuristic assumptions, making it difficult to make precise statements on their theoretical behavior.

The first rigorous lattice attack on (EC)DSA was presented in [27]. The authors successfully decreased the security of (EC)DSA to a Hidden Number Problem (HNP), which can then be further reduced to an approximation Closest Vector Problem (CVP) for a specific lattice. The signer's secret key a can be computed using this reduction in polynomial time. The attack was also adapted to the case of ECDSA, as described in [28].

The paper [4] describes an attack on DSA schemes that uses the LLL reduction method and requires one message. By computing two short vectors of a three-dimensional lattice, the attack derives two intersecting lines in (a, k), provided that a and k are sufficiently small, and the second shortest vector is sufficiently short. If two messages are available, the same attack can be applied to derive a linear congruence relating to the corresponding ephemeral keys.

The papers [29] and [6] describe attacks on DSA schemes using the LLL algorithm and one or two messages. In [29], the combination of LLL with algorithms for finding integral points of two classes of conics gives a, provided that at least one of the sets $\{a, k^{-1} \mod q\}$, $\{k, a^{-1} \mod q\}$, $\{a^{-1} \mod q, k^{-1} \mod q\}$ is sufficiently small. In [6], the Lagrange Reduction algorithm is applied on a 2-dimensional lattice defined by a signed message, and provides two straight lines intersecting at (a, k). Similar attacks can be applied to the pairs $(k^{-1} \mod q, k^{-1}a \mod q)$ and $(a^{-1} \mod q, a^{-1}k \mod q)$. If two signed messages are available, the above two attacks can be applied to the equation relating the two ephemeral keys.

The article [7] presents an attack using Coppersmith's method to compute the secret key a. The attack works when a and k satisfy a specific inequality, and in this case, the secret key a can be efficiently computed.

The article [30] describes an attack that involves constructing a system of linear congruences using signed messages. This system has at most one unique solution below a certain bound, which can be computed efficiently. Thus, if the length of a vector containing the secret and ephemeral keys of a signed message is quite small,

the secret key can be computed using the above system. The article [1] presents an improved version of this attack.

In [24, 25], the proposed attacks take advantage using of the bits in the ephemeral key and the Fast Fourier Transform.

In [32], it is shown that, using lattice reduction under some heuristic assumptions, that partial information about the nonces of multiple signatures can lead to recovery of the full private key. The original approach to doing so is based on discrete Fourier analysis techniques [5, 2].

A very important issue is the attacks on cryptosystems based on the malicious modification of memory registers. These attacks may affect the randomness of the secret parameters, and so, to force certain bits of the ephemeral key to be equal, without their values being known. In [19], it is discussed how such attacks could occur in a real-life scenario. Following the line of research from [19], the authors of [10] focus on an attack scenario where ephemeral keys share specific bits, such as the least significant bits (LSB) and/or most significant bits (MSB), either within multiple blocks. By eliminating the shared blocks of bits between the ephemeral keys, a lattice of dimension equal to the number of signatures is provided, which contains a quite short vector with components that reveal the secret key. Then, the LLL algorithm is used for the computation of this vector. Note that these attacks are based on heuristic assumptions. Later, in [11], the authors further improved upon the attack proposed in [10] by providing a probabilistic attack with a success probability approaching 1 when the pair (δ, n) is appropriately selected, where n represents the number of signatures, and δ represents the number of shared bits in the ephemeral keys. This attack relies on a mild assumption regarding the hash function used in (EC)DSA.

1.3. **Our Contribution.** Our study builds on the research presented in [10, 11], and we present a deterministic attack that, although not always polynomial in complexity, proves to be highly efficient in practical scenarios. Instead of using methods like LLL, approximate, or exact CVP, which were employed in previous attacks, we use enumeration on a suitable lattice to find lattice vectors that are close to a specific target vector. From these solutions, we can readily extract the secret key to the system.

It is important to highlight that the attacks presented in [10] rely on heuristics assumptions that aim to force the presence of a vector containing the private key as a solution to the Shortest Vector Problem (SVP) in a relatively large lattice. In [11], the authors provide a probabilistic approach to [10], where an assumption for the hash function is made and the attack is modelled as a Closest Vector Problem (CVP). Due to the computational complexity of finding such a vector using a deterministic algorithm, an approximation algorithm can be used instead.

Our approach takes a different path. We calculate a bound for the distance between the vector of the lattice containing the private key and a target vector. Then, we leverage Kannan's enumeration algorithm to determine this vector and, consequently, extract the secret key. Our experiments demonstrate that the attack can be made highly efficient by appropriately selecting values for δ and n. Finally, we improve the results provided in [11].

1.4. Our results. In the subsequent Theorem, we apply the framework suggested by [11, 10, 19], which presupposes that we have access to a collection of signed

messages with ephemeral keys that are shorter than q. These messages have some of their most and least significant bits in common, with a total of δ bits shared.

Theorem 1.1. Suppose we have a (EC)DSA scheme with a binary length ℓ prime number q and secret key a. Let m_j $(j=0,\ldots,n)$ be messages signed with this scheme, (r_j,s_j) their signatures, and $k_j = \sum_{i=1}^{\ell} k_{j,i} 2^{\ell-i}$ (where $k_{j,i} \in \{0,1\}$) are the corresponding ephemeral keys, respectively. Set $A_j = -r_j s_j^{-1} \mod q$. Suppose that $0 < k_j < q$ $(j=0,\ldots,n)$, and there are integers $\delta > 0$ and $0 \le \delta_L \le \delta$ such that the following conditions hold:

- (1) $k_{0,i+1} = \cdots = k_{n,i+1} \ (i = 1, \dots, \delta \delta_L, \ell \delta_L, \dots, \ell 1).$
- (2) For i = 0, ..., n, set $C_{i,j} = (A_{j-1} A_i)2^{-\delta_L} \mod q$, (j = 1, ..., i), and $C_{i,j} = (A_j A_i)2^{-\delta_L} \mod q$ (j = i+1, ..., n). The shortest vector of the lattice \mathcal{L}_i spanned by the vectors

$$(2^{\delta+1}q,0,\ldots,0),\ldots,(0,\ldots,0,2^{\delta+1}q,0),(2^{\delta+1}C_{i,1},\ldots,2^{\delta+1}C_{i,n},1)$$

has length

$$> \frac{1}{2} (2^{\delta+1}q)^{\frac{n}{n+1}}.$$

Then, the secret key a can be computed in

$$\mathcal{O}(2^{\ell-\delta n+2n} n ((n\ell)^c 2^{\mathcal{O}(n)} + \ell^4 2^n (n+1)^{\frac{n+1}{2}}))$$

bit operations, for some c > 0.

Remark 1.1. By the Gaussian heuristic [14, Section 6.5.3] the length of the vectors of the lattice \mathcal{L} is $> q^{n/(n+1)}$. Thus, the hypothesis (2) of Theorem 1.1 will very often be satisfied.

Remark 1.2. In the above complexity estimate, if $\ell \leq \delta n$, then the time complexity is polynomial in ℓ .

Roadmap. The paper is structured as follows: Section 2 presents an auxiliary lemma that will prove crucial in the proof of Theorem 1.1. Section 3 is dedicated to the proof of Theorem 1.1, providing a detailed explanation and justification. In Section 4, an attack on (EC)DSA, derived from Theorem 1.1, is presented. Additionally, several experiments are conducted to illustrate the effectiveness of the attack. Finally, Section 5 concludes the paper, summarizing the main findings and discussing potential avenues for future research.

2. Lattices

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subset \mathbb{Z}^n$ be a basis of \mathbb{R}^n . A *n-dimensional lattice* spanned by \mathcal{B} is the set

$$\mathcal{L} = \{ z_1 \mathbf{b}_1 + \dots + z_n \mathbf{b}_n / z_1, \dots, z_n \in \mathbb{Z} \}.$$

Recall that the scalar product of two vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in \mathbb{R} is the quantity $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \dots + u_n v_n$, and the *Euclidean norm* of a vector $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ the quantity

$$\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = (v_1^2 + \dots + v_n^2)^{1/2}.$$

The Gram-Schmidt orthogonalisation (GSO) of the basis \mathcal{B} is the orthogonal family $\{\mathbf{b}_1^{\star}, \ldots, \mathbf{b}_n^{\star}\}$ defined as follows:

$$\mathbf{b}_i^{\star} = \mathbf{b}_i - \sum_{j=0}^{i-1} \mu_{i,j} \mathbf{b}_j^{\star}, \quad \text{with} \quad \mu_{i,j} = \frac{\langle \mathbf{b}_i, \mathbf{b}_j^{\star} \rangle}{\|\mathbf{b}_j^{\star}\|^2} \quad (j = 0, \dots, i-1).$$

Let L be a lattice. If K is a convex body in \mathbb{R}^{n+1} symmetric about the origin, we denote by $\lambda_i(K, L)$ (i = 1, ..., n+1) the ith successive minimum of K with respect to L which it is defined as follows

$$\lambda_i(K, L) = \inf\{\lambda > 0 / (\lambda K) \cap L \text{ contains } i \text{ linearly independent points}\}.$$

Further, we denote by s(L) the length of a shortest vector in L.

Lemma 2.1. Let $B_{\mathbf{v}}(R)$ be the closest ball of center \mathbf{v} and radius R in \mathbb{R}^{n+1} and L a lattice. Then, we have:

$$|B_{\mathbf{v}}(R) \cap L| < \left(\frac{2R}{s(L)} + 1\right)^{n+1}.$$

Proof. Set

$$\mathcal{D}_{\mathbf{v}}(R) = \{ \mathbf{x} - \mathbf{y} / \ \mathbf{x}, \mathbf{y} \in B_{\mathbf{v}}(R) \}.$$

Then, $\mathcal{D}_{\mathbf{v}}(R)$ is a convex body, symmetric about the origin. Then [21] implies:

$$(2.1) |B_{\mathbf{v}}(R) \cap L| < \prod_{i=1}^{n+1} \left(\frac{1}{\lambda_i(\mathcal{D}_{\mathbf{v}}(R), L)} + 1 \right).$$

Let $\mathbf{x}, \mathbf{y} \in B_{\mathbf{v}}(R)$. Then, we have:

$$\|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{y}\| \le 2R.$$

It follows that $\mathcal{D}_{\mathbf{v}}(R) \subseteq B_{\mathbf{0}}(2R)$, and so we deduce

$$(2.2) \lambda_1(B_0(2R), L) < \lambda_i(\mathcal{D}_{\mathbf{v}}(R), L) \quad (i = 1, \dots, n).$$

Further, we have

(2.3)
$$\lambda_1(B_0(2R), L) > s(L)/2R$$
.

Combining the inequalities (2.1), (2.2) and (2.3), we obtain:

$$|B_{\mathbf{v}}(R) \cap L| < \left(\frac{2R}{s(L)} + 1\right)^{n+1}.$$

3. Proof of Theorem 1.1

Let a be the secret key and k_j , $j=0,\ldots,n$ the ephemeral keys. We put $A_j=-r_js_j^{-1} \mod q$ and $B_j=-h(m_j)s_j^{-1} \mod q$ for $j=0,\ldots,n$. The signing equation for (EC)DSA provides that,

(3.1)
$$k_j + A_j a + B_j \equiv 0 \pmod{q} \quad (j = 0, \dots, n).$$

Suppose first that $k_0 = \min\{k_0, \dots, k_n\}$. We set $\delta_M = \delta - \delta_L$. From the hypothesis of the Theorem we get

$$z_j = k_j - k_0 = \varepsilon 2^{\ell - \delta_M - 1} + \dots + \varepsilon' 2^{\delta_L},$$

for some $\varepsilon, \varepsilon' \in \{0,1\}$. Since $z_j > 0$ we get $0 < z_j < 2^{\ell - \delta_M}$ and there exists positive integer z_j' such that $z_j = 2^{\delta_L} z_j'$ Furthermore, we set $C_j = (A_j - A_0) 2^{-\delta_L} \mod q$ and $D_j = (B_j - B_0) 2^{-\delta_L} \mod q$. From (3.1) we have the congruences:

$$z'_j + C_j a + D_j \equiv 0 \pmod{q} \quad (j = 1, \dots, n).$$

Since z'_i is positive, there is a positive integer c_i such that

$$-C_i a - D_i + c_i q = z_i'.$$

Thus, we obtain:

$$0 < c_j q - C_j a - D_j < 2^{\ell - \delta}.$$

It follows that

$$-2^{\ell-\delta-1} < c_j q - C_j a - D_j - 2^{\ell-\delta-1} < 2^{\ell-\delta-1},$$

whence we get

$$0 < |c_i q - C_i a - D_i - 2^{\ell - \delta - 1}| < 2^{\ell - \delta - 1}.$$

Therefore, we have:

$$(3.2) 0 < |c_j q 2^{\delta+1} - C_j 2^{\delta+1} a - D_j 2^{\delta+1} - 2^{\ell}| < 2^{\ell}.$$

We consider the lattice \mathcal{L} spanned by the rows of the matrix

$$\mathcal{J} = \begin{pmatrix} 2^{\delta+1}q & 0 & 0 & \dots & 0 & 0\\ 0 & 2^{\delta+1}q & 0 & \dots & 0 & 0\\ 0 & 0 & 2^{\delta+1}q & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 2^{\delta+1}q & 0\\ 2^{\delta+1}C_1 & 2^{\delta+1}C_2 & 2^{\delta+1}C_3 & \dots & 2^{\delta+1}C_n & 1 \end{pmatrix}.$$

The vectors of the lattice \mathcal{L} are of the form

$$(2^{\delta+1}(qx_1+x_{n+1}C_1), 2^{\delta+1}(qx_2+x_{n+1}C_2), \dots, 2^{\delta+1}(qx_n+x_{n+1}C_n), x_{n+1}),$$

for some integers x_1, \ldots, x_{n+1} . By setting $(x_1, \ldots, x_{n+1}) = (c_1, \ldots, c_n, -a)$, we get the lattice vector

$$\mathbf{u} = (2^{\delta+1}(c_1q - C_1a), \dots, 2^{\delta+1}(c_nq - C_na), -a).$$

Further we consider the vector in the span of \mathcal{L} ,

$$\mathbf{v} = (D_1 2^{\delta+1} + 2^{\ell}, \dots, 2^{\delta+1} D_n + 2^{\ell}, 0).$$

Now, we have

$$\mathbf{u} - \mathbf{v} = (2^{\delta+1}(qc_1 - C_1a - D_1) - 2^{\ell}, \dots, 2^{\delta+1}(qc_n - C_na - D_n) - 2^{\ell}, -a),$$

and inequalities (3.2) yield:

$$\|\mathbf{u} - \mathbf{v}\| < 2^{\ell} \sqrt{n+1}.$$

Put $R = 2^{\ell} \sqrt{n+1}$. Then $\mathbf{u} \in B_{\mathbf{v}}(R)$.

Next, we compute a *LLL*-reduced basis for \mathcal{L} , say $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_{n+1}\}$. This can be done in time $\mathcal{O}(n^6(\log q)^3)$. By hypothesis (2) of Theorem, we have:

$$s(\mathcal{L}) > \frac{1}{2} (2^{\delta+1} q)^{\frac{n}{n+1}}.$$

Let $\{\mathbf{b}_1^*, \dots, \mathbf{b}_{n+1}^*\}$ the Gram-Schmidt orthogonalisation of \mathcal{B} . By [14, Theorem 6.66], we get:

$$4\|\mathbf{b}_{i}^{*}\|^{2} \ge 2\|\mathbf{b}_{i-1}^{*}\|^{2} \ge \|\mathbf{b}_{i-1}\|^{2} \ge s(L)^{2}$$

Thus, we obtain:

(3.4)
$$\frac{1}{4} (2^{\delta+1}q)^{\frac{n}{n+1}} \le ||\mathbf{b}_i^*|| \quad (i = 1, \dots, n+1).$$

Next, using Kannan's enumeration algorithm [12], we compute all the elements of $B_{\mathbf{v}}(R) \cap \mathcal{L}$. Combining [13, Theorem 5.1] with the inequality (3.4), we obtain that the bit complexity of the procedure is

$$(n\log q)^c \ 2^{\mathcal{O}(n)} \left(\frac{2^{\ell+2}}{(2^{\delta+1}q)^{\frac{n}{n+1}}}\right)^{n+1},$$

where c is a constant > 0. Then we check whether the last coefficient of $\mathbf{u} \in B_{\mathbf{v}}(R) \cap \mathcal{L}$ is the minus of the secret key $-a \mod q$. Every such operation needs $\mathcal{O}((\log q)^4)$ bit operations [33, Lemma 6.2, p.237]. If none of the elements of $\mathbf{u} \in B_{\mathbf{v}}(R) \cap \mathcal{L}$ gives the secret key, then we repeat the procedure assuming that $k_1 = \min\{k_0, \ldots, k_n\}$, and we continue until we found the secret key. By Lemma 2.1, we have:

$$|B_{\mathbf{v}}(R) \cap \mathcal{L}| < \left(\frac{2^{\ell+2}\sqrt{n+1}}{(2^{\delta+1}q)^{\frac{n}{n+1}}} + 1\right)^{n+1}.$$

Thus, the overall bit complexity of the computation of a is

$$\mathcal{O}\left(n(n\log q)^c \ 2^{\mathcal{O}(n)} \left(\frac{2^{\ell+2}}{(2^{\delta+1}q)^{\frac{n}{n+1}}}\right)^{n+1} + n\left(\frac{2^{\ell+2}\sqrt{n+1}}{(2^{\delta+1}q)^{\frac{n}{n+1}}} + 1\right)^{n+1} (\log q)^4\right),$$

whence the result.

4. The attack

The proof of Theorems 1.1 yields the following attack:

Algorithm 4.1. ATTACK-DSA

Input: Messages m_j $(j=0,\ldots,n)$ and (r_j,s_j) their (EC)DSA signatures and integers $\delta>0$ and $0\leq \delta_L\leq \delta$ and the public key (p,q,g,R) (resp. (E,p,q,P,Q)). Output: The private key a.

- (1) For j = 0, ..., n compute $A_j = -r_i s_i^{-1} \mod q$, $B_j = -h(m_j) s_i^{-1} \mod q$.
- (2) For i = 0, ..., n,
 - (a) For $j = 1, \ldots, i$ compute

$$C_{i,j} = (A_{j-1} - A_i)2^{-\delta_L} \mod q, \quad D_{i,j} = (B_{j-1} - B_i)2^{-\delta_L} \mod q,$$

and for $j = i + 1, \dots, n$ compute

$$C_{i,j} = (A_j - A_i)2^{-\delta_L} \mod q, \quad D_{i,j} = (B_j - B_i)2^{-\delta_L} \mod q.$$

(b) Consider the lattice \mathcal{L}_i spanned by the rows of the matrix

$$J_{i} = \begin{pmatrix} 2^{\delta+1}q & 0 & 0 & \dots & 0 & 0\\ 0 & 2^{\delta+1}q & 0 & \dots & 0 & 0\\ 0 & 0 & 2^{\delta+1}q & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 2^{\delta+1}q & 0\\ 2^{\delta+1}C_{i,1} & 2^{\delta+1}C_{i,2} & 2^{\delta+1}C_{i,3} & \dots & 2^{\delta+1}C_{i,n} & 1 \end{pmatrix}$$

and compute a LLL-basis \mathcal{B}_i for \mathcal{L}_i .

- (c) Consider the vector $\mathbf{v}_i = (2^{\delta+1}D_{i,1} + 2^{\ell}, \dots, 2^{\delta+1}D_{i,n} + 2^{\ell}, 0)$, and using Kannan's enumeration algorithm with basis \mathcal{B}_i , compute all $\mathbf{u} \in \mathcal{L}_i$ satisfying $\|\mathbf{u} \mathbf{v}_i\| < 2^{\ell}\sqrt{n+1}$.
- (d) Check whether the last coordinate of **u** say u_{n+1} satisfies $g^{-u_{n+1}} \equiv R \pmod{q}$ (resp. $P(-u_{n+1}) = Q$). If it is so, then return the secret key $-u_{n+1} \mod q = a$.

Remark 4.1. For the Pseudocode of Kannan's Enumeration Algorithm, one can see [13, Section 5.1, Algorithm 10].

Remark 4.2. Supposing that condition (2) is satisfied, taking n quite small and $n\delta \geq \ell$, Theorem 1.1 implies that the attack is polynomial in ℓ . Furthermore, if s(L) is closed to the Gauss heuristic, then the upper bound for the number of points of $B_{\mathbf{v}}(R) \cap \mathcal{L}$ will be the smaller possible, and so it is expect that the attack will be quite efficient.

Experiments. We conducted a thorough analysis of our experiments, and we compared our results with those presented by Gomez et al. [11]. Our findings indicate a significant improvement in almost all cases. Our experiments were conducted on a Linux machine with an i5-12400 CPU, using Sagemath 9.8 [31]. We made the assumption that we already knew the minimum ephemeral key. However, in the general case, where the minimum key is unknown, we would need to perform n executions, where n+1 represents the number of signatures. This worst-case scenario would require multiplying the execution time of each experiment by n. Overall, our results demonstrate a notable improvement compared to the previous findings (see the Table below). Finally, we have successfully found the secret key even when the shared bits in the ephemeral keys are only 5. Remarkably, in this case, we only needed a minimum of 58 signatures. It is worth noting that in [11], no successful attack was provided for the specific scenario where $\delta = 5$.

δ : shared bits	signatures ([11])	signatures (this paper)
5	?	58
6	≈ 50	40
8	≈ 27	25
10	≈ 20	18
12	≈ 17	14
14	≈ 14	12
16	≈ 12	11
18	≈ 11	9
20	≈ 10	8

TABLE 1. We considered $\ell=160$ and $R=2^\ell\sqrt{n+1}$. We found the private key in every experiment we executed. Instead of LLL we used BKZ with block size 8. For each row, we conducted 10 random experiments, and the results in the third column were computed on average in under two minutes and thirty seconds. For the cases, $\delta \in \{22, 24, 26, 28, 30\}$ we get the same number of signatures as in [11]. See https://github.com/drazioti/dsa/.

5. Conclusion

Attacks based on the malicious modification of memory registers is a topic of high importance, since it may affect the randomness of the secret parameters by forcing a limited number of bits to a certain value, which can be unknown to the attacker. In this context, we developed a deterministic attack on the DSA schemes, providing that several signatures are such that the corresponding ephemeral keys share a number of bits without knowing their value.

Our attack is deterministic, meaning that it always produces a result for a given input every time. However, it is important to note that while the attack is deterministic, it may not always be practical to execute. Deterministic attacks on the (EC)DSA are relatively rare, as they typically rely on heuristic assumptions.

While deterministic attacks on (EC)DSA, are rare, our attack demonstrates practical feasibility in specific scenarios, surpassing previous results, (see Table 1). However, it is important to note that the practicality and effectiveness of our attack may vary depending on the specific choice of (δ, n) .

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