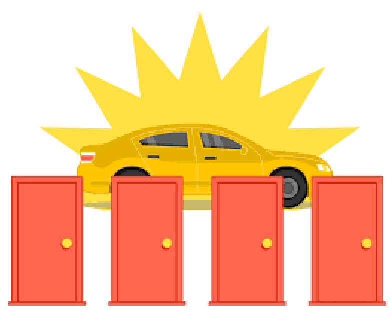
American University of Armenia

College of Science and Engineering

Capstone

Monty Hall Problem



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Abstract

Monty Hall Problem became known from the game show called Let's Make a Deal where the host suggests closed doors and only one entry contains a car. The player will win a car if he/she can guess behind which door the car is. A probabilistic theory presents the solution to the problem and suggests the dominant strategy for winning. Simulations by R also provide clarification and proof of the accuracy of the approaches used in mathematical analyses by R which justify the logical procedures used. Except for the central statement of the problem paper also confers the most widely recruited and used versions of the problem.

Keywords: Monty Hall Problem, Switching, Probability, Let's Make a Deal, choose the door

Contents

Introduction

The Monty Hall problem, known from the American television game show named Lets Make a Deal is a logical game based on probabilistic theory. The name of the problem is recalled, and it is stated initially in a scientific context by Steve Selvin in the scientific journal called American Statistician in 1975. However, it became famous in 1990 from a readers letter quoted in Ask Marilyn column authored by Marilyn Vos Savant in the Parade magazine (see [2]).

Problem Setting and Description

The player participates in a game show when he/she needs to choose between three doors. Behind one door is a car, behind the other two doors, are goats. The player needs to pick a gate (e.g., door 3) after which host opens another entry which does not contain a car (e.g., door 2). Then the host suggests changing a door and choosing the other (e.g., door 1) closed door. The problem is: Is it better to switch the door or not for winning the car?

Here are some variations of the problem for example:

- The host always suggests to change the choice;
    - Including the case when host suggests N doors instead of 3;
- The host offers to switch the door 100% of times when the player picked the car from the first time and offers 50% of the time to change the choice if the player did not choose the door holding the car (see [1]).

The Structure of the Capstone Paper

The paper consists of four main parts. The first section of it gives problem statement and description, the overall summary of the paper structure, historical framework of the problem, and information related to the development of the problem and theories starting from 1975. The second part gives detailed information and clarification of mathematical background and presents solutions to variations of the puzzle. The next section provides results based on codes with R and mathematical knowledge by comparing and contrasting outcomes. The last one gives a brief description of the JavaScript game representing two cases of the problem.

Historical Background

The Monty Hall problem same as Monty Hall paradox became known from the American television game show named Lets Make a Deal famous in the 1960s. The show was organized in the following order, so that host shows three large doors behind two of which player can find a goat and behind the other car. The aim of playing the game is winning the car. Monty Hall (Monty Halparin, birth date: August 21; 1921, death date: September 30, 2017) was a long-running host of the show, and due to that fact, the problem was formulated in mathematics with the name of Monty Hall problem. He was a game show host, producer, and philanthropist.

Bio-statistician Steve Selvin was the first person who affirmed the problem in Mathematical context and suggested a solution to it presenting it under the name “Monty Hall Problem”.

Based on the examinations of Vos Savant strongly believed that switching the choice will give a higher probability to win. It thus bases on the standard assumptions of Vos Savant changing the door gives 2/3 likelihood to win the game in contrast to the not switching which gives 1/3 chance to win. However, this claim by Vos Savant was not accepted by the other mathematicians who responded with letters to the publisher by trying to contradict the theory and the logic used by Vos Savant. One of the correspondents of the publication was also Monty Halparin himself who was not happy with the statement of the problem. He clarified that he always makes suggestions of switching or not switching based on the initial decision of the player and also considers the psycho type and behavior of the player while making his suggestions. In addition to this, there were also responses sent from mathematicians which theories firmly stated that switching or not switching does not change the probability of winning and losing and it stays just 1/2 as in this case there are only two closed doors each with 1/2 chance to win. There were long discussions and argument within professors and in the opponent's side of the logic of Vos Savant was included Paul Erds(one of the most prolific mathematicians) who stayed unconvinced until seeing the computer simulation demonstrating the foretold events see ([3]).

Based on the assumptions and outcomes of the Monty Hall problem, it can be considered somehow identical to the Three Prisoners Problem stated by Martin Gardner in 1959. It is also important to mention that on the other hand there is an older problem stated in the probability theory called Bertrand's box paradox which was asserted before 1889 and is based on the same logic which Monty Hall and Three Prisoners Problem state see ([4])

1 Mathematical Implementation

1.1 Intuition of the problem

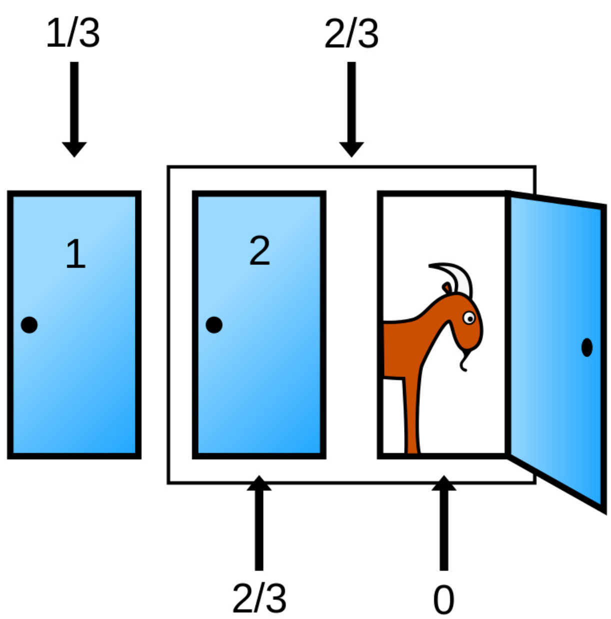
First let us discuss better known Three Prisoners Problem which has similar intuition as Monty Hall. The problem is stated in the following way: There are A, B, and C prisoners in the separate cells, and they are sentenced to death. The governor randomly selects one of them to be excused. The guard has information about who will be pardoned but is not allowed to share the information with others(for more details check [4]). Prisoner A asks the guard to give him information about who will be excused by the governor. The prisoner tells the guard, If B is to be justified, give me Cs name. If C is to be excused, give me Bs name. And if Im to be justified, flip a coin to decide whether to name B or C. The policeman tells the A prisoner that B will be pardoned. News makes A to feel safer as he considers that his probability of being excused became higher from 1/3 to 1/2 as now there is a chance of surviving between him and C prisoner. This information A also passes to the C prisoner. Information makes C happy because he considers that now A has 1/3 probability of surviving and he has higher chances 2/3 possibility to gain life. Here one needs to decide which one is the correct answer for this situation. As an answer to this problem, one can consider the following logic:

- A knew that policeman would not name his name in any case;
- Before getting any information from the guard as a chance of surviving is considered equal to the other possibilities which is 1/3;
- Policeman gives the name of B, which means either C or A will be pardoned. A will be forgiven with the probability of 1/3 and coin showed the B's side which is $1/3 \times 1/2 = 1/6$;
- After hearing B's name as an executed the C's chances go twice high and it becomes $2 \times 1/6 = 1/3$;
- After hearing that B will be executed, the estimate of A's chance of being pardoned is half that of C. This means his chances of being pardoned, now knowing B is not, again are 1/3, but C has a 2/3 chance of being pardoned (see [4]).

Returning back to the intuition of the Monty Hall problem, Vos Savant's theory stated that switching the choice will make the chances of winning twice higher from not turning the option. Thus, one of the intuitive reasoning in solving the problem is the following:

- With 1/3 probability, the contestant chooses any of the closed doors
- Suppose the contestant chooses the door 1 and the likelihood that car will be behind the door 1 in 1/3
- Host offers the contestant as the choice between his initial choice door 1, or doors 2 and 3 together
- Consequently, the chance that the car will be behind door 2 or door 3 is 1/3 +1/3 =2/3
- When the host opens the door 2, and he knows that car is not behind the door 2 probability that the car will be behind the door 3 will become 2/3
- Henceforth switching the door gives twice high chances of winning.

Here is the visualization of the explanation which will make things more understandable:



Here is also table presenting the real picture of cases from which one can see that winning the car after switching will be 2/3 and winning the car without turning the car is 1/3 as initially, the probability that the player will choose goat containing door is 2/3 (see [3]).

Behind door 1	Behind door 2	Behind door 3	Result if staying at door #1	Result if switching to the door offered
Goat	Goat	**Car**	Wins goat	**Wins car**
Goat	**Car**	Goat	Wins goat	**Wins car**
Car	Goat	Goat	**Wins car**	Wins goat

1.2 Variations / Cases of the problem

The paper discusses two possible variations of the problem, which are the following:

- Case 1: The host provides tree doors and always suggests to change the door
  - The host suggests N doors and always suggests to change the initial choice. Here the player chooses one door then host suggests to reverse the decision after opening p doors, then the last step becomes either player picks the right door or losses the game.
- Case 2: The game is played with tree doors. The host opens a door and makes the offer to switch 100% of the time if the contestant initially picked the car, and 50% the time otherwise.

1.3 Bayes' Theorem basics

First and the most famous variation of the Monty Hall problem is the following: Host always knows behind which door is the car hidden and he always suggests to change the choice of the door. Following problem can be solved based on the Bayes's theorem of conditional probabilities. Bayes' theorem is stated mathematically as the following equation:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}, \tag{1}$$

where A and B are events and $P(B) \neq 0$.

$$P(A \mid B) \tag{2}$$

is a conditional probability, which is the likelihood of event A given that event B is true. $P(B \mid A)$ is also a conditional probability: the likelihood of event B occurring given that A is true. P(A) and P(B) are the probabilities of observing A and B independently from each other; this is known as the marginal probability (see [7]).

1.4 Classical Case

Now let's discuss the first variation of the problem in a mathematical context. For the beginning the contestant needs to choose any initial door to start the game by guessing where is the car hidden. Therefore as he/she knows that from three doors only one contains the car, there is only 1/3 chance for each door to contain a car. Thus, here are the probabilities for each door to contain a car:

$$P(1) = \frac{1}{3}, P(2) = \frac{1}{3}, P(3) = \frac{1}{3}. \tag{3}$$

Assume that contestant first choose door 1. Now Host needs to decide which door to open for the player to reveal a goat. Here notable fact is that host always knows behind which door is the car hidden (see [7]). So,

- the probability that host will open the door 3 knowing that contestant choose door 1 and car is behind that door: $P(III \mid 1) = 1/2$;
- the probability that host will open door 3 knowing that contestant choose door 1 and car is behind door 2: $P(III \mid 2) = 1$;
- the probability that host will open door 3 knowing that contestant choose door 1 and car is behind door 3: $P(III \mid 3) = 0$;

Now passing to the Bayes' theorem we need to calculated values for $P(A_k \mid B)$ where $\{A_k\}$ is the event for the car to be behind K-th door and B is the event for the host opening one of the doors containing goat.

$$P(A_k \mid B) = \frac{P(B \mid A_k)P(A_k)}{P(B)} \tag{4}$$

Where on the other hand

$$\begin{multline} P(B) = \sum_{n=1}^{\infty} P(B \mid A_n)P(A_n) = \\ P(III \mid 1)P(1) + P(III \mid 2)P(2) + P(III \mid 3)P(3) = \\ \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3} = \frac{1}{2} \end{multline} \tag{5}$$

Thus, the probability that car will be initially behind the door 2 and after switching player will win is

$$P(2 \mid III) = \frac{P(III \mid 2)P(2)}{P(III)} = \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}. \tag{6}$$

The probability that car will be initially behind the door 1 and with not switching the player will win is

$$P(1 \mid III) = \frac{P(III \mid 1)P(1)}{P(III)} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}. \tag{7}$$

1.4.1 N door case

Later in 1975 D.L.Ferguson in the letter to Selvin suggested the generalized version of the game when the player is implied to play with N doors, and after the initial choice of the player, the host decides to open p number of entries (see [6]) Thus, the probability that after changing the door player will find car behind it is:

$$\frac{1}{N} \cdot \frac{N-1}{N-p-1}. \tag{8}$$

This probability is always greater than $\frac{1}{N}$, therefore switching always brings an advantage.

1.5 Reviewed Case

The second case of the problem presents the following algorithm:

- Contestant needs to choose any initial door. Thus the chance that the contestant will be correct, considering the fact that car is behind one of 3 doors is 1/3

- Then host has 2 versions of behavior. If the initial choice is right he/she always suggests to change the choice. If the initial choice is wrong he/she suggests to change the choice with 1/2 probability. If the initial choice is wrong, and host decides not to suggest change he/she finishes the game by opening all doors and player looses game just after the first choice.

Thus, the probability that host will reveal the other door given that the initial choice was right is 1.

$$P(Reveal \mid Correct) = 1, P(Correct) = \frac{1}{3} \tag{9}$$

Therefore the Probability that the contestant will choose a right door given that the host revealed other door will be:

$$P(Correct \mid Reveal) = \frac{P(Reveal \mid Correct) \times P(Correct)}{P(Reveal)} \tag{10}$$

$$P(Reveal) = P(Reveal \mid Correct)P(Correct) + P(Reveal \mid incorrect)P(incorrect) \tag{11}$$

$$P(Reveal) = 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{2}{3} \tag{12}$$

$$P(Correct \mid Reveal) = \frac{1 \times \frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \tag{13}$$

Thus if Monty chooses to reveal a goat, there is a 50 percent chance that players initial choice was correct. Therefore chances of success are equal under the decision to switch or not to switch in this case (see [8]).

2 Computational Results

In the table 1, R simulation results of the classical case are provided. The number of doors is 3. For understanding the table, the first row of it will be now discussed. Here "Number of Iterations" is showing how many games are played. The next two columns show how many games out of 10 won after switching and after not changing respectively. The last two columns show the probability of winning in both cases based on the total number of iterations.

Number of Iterations	Wins After Switching	Wins After Not Switching	Probability of Wins After Switching	Wins After Not Switching
10	5	5	0.5	0.5
200	131	69	0.665	0.345
3000	1967	1033	0.655	0.344
40 000	26747	13253	0.668	0.331
100 000	66648	33352	0.666	0.333

Table 1: Classical case with three doors

2.0.1 Classical Case with N doors

In the next two tables of this section are presented couple of variations of N door cases. Table 2 is showing 10 door case when host opens just one door after the player picks any door out of 10. To better understand the relation between theoretical calculations and R results some calculations now will be presented. From the N door case. For the beginning consider 10 door case when host opens just one door. Here for the probability of winning after switching we will have:

$$\frac{1}{10} \cdot \frac{10-1}{10-1-1} = 0.112. \tag{14}$$

The table 3 presents 100 door case, but here host opens 3 doors instead of 1. Here the follower can see that as the number of iterations grows the results reach

Number of Iterations	Wins After Switching	Wins After Not Switching	Probability of Wins After Switching	Probability of Wins After Not Switching
10	3	1	0.3	0.1
200	25	15	0.125	0.075
3000	336	295	0.112	0.0.098
40 000	4545	4088	0.113	0.102
100 000	11214	9911	0.112	0.099

Table 2: Classical case with ten doors

closer to the mathematical calculations and probability of winning after switching is always staying larger than 1/100. In this case while substituting numbers in the formula the probability of winning after switching will be following:

$$\frac{1}{100} \cdot \frac{100-1}{10-3-1} = 0.011. \tag{15}$$

Number of Iterations	Wins After Switching	Wins After Not Switching	Probability of Wins After Switching	Probability of Wins After Not Switching
10	0	0	0	0
200	2	3	0.01	0.015
3000	35	32	0.011	0.01
40 000	438	432	0.01	0.01
100 000	1026	1021	0.011	0.01

Table 3: Classical case with hundred doors when host opens 3 doors

2.1 Reviewed case

Here the hosts behaviour changes based on the previously presented Reviewed Case section. Therefore, to calculate the probability of winning after switching or probability of winning after not switching one needs to use the following formulas:

$$ProbabilityofWinsAfterSwitching = \frac{WinsAfterSwitching}{WinsAfterSwitching + WinsAfterNotSwitching} \tag{16}$$

$$ProbabilityofWinsAfterNotSwitching = \frac{WinsAfterNotSwitching}{WinsAfterSwitching + WinsAfterNotSwitching} \tag{17}$$

Thus, the results of those calculations are presented in 4.

Number of Iterations	Wins After Switching	Wins After Not Switching	Probability of Wins After Switching	Probability of Wins After Not Switching
10	4	4	0.5	0.5
200	65	56	0.537	0.462
3000	1014	1001	0.503	0.496
40 000	13303	13419	0.497	0.502
100 000	33122	33308	0.498	0.501

Table 4: Reviewed case with three doors

References

[1] Bell, W. Comment on Morgan et al. *American Statistician*,(46), 1992, 241-242.

[2] vos Savant, Mailyn Ask Marilyn *Parade Magazine*,9 September 1990,p. 16.

[3] vos Savant, Mailyn Letter to the editor *American Statistician*,(45)347, 1991, pp. 347-348.

[4] Shimojo S., Ichikawa S. Intuitive reasoning about probability: Theoretical and experimental analyses of the problem of three prisoners. *Cognition*,(32), 1989, pp. 1-24.

[5] Whitmeyer M. Intuitive reasoning about probability: Theoretical and experimental analyses of the problem of three prisoners. *Games*,(8)3, 2017, p. 31.

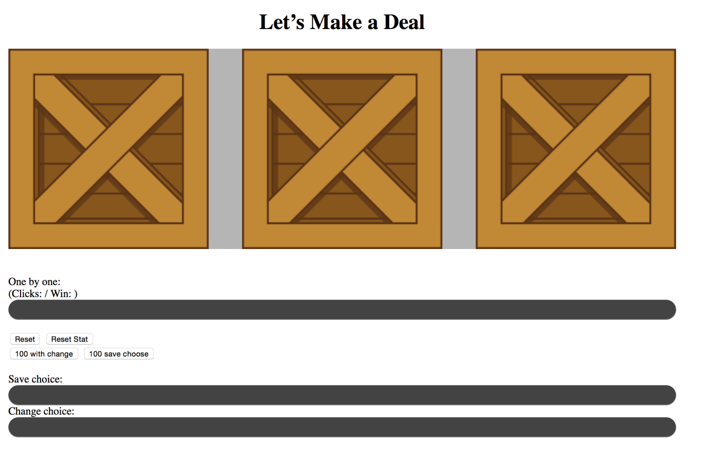
[6] Whitmeyer M. A Logical Analysis of Monty Hall and Sleeping Beauty *Studia Logica*,(106)6, 2018, pp. 1123-1162.

[7] Whitmeyer M. The Monty Hall Problem as a Bayesian Game *Games*,(8)3, 2017, p.31.

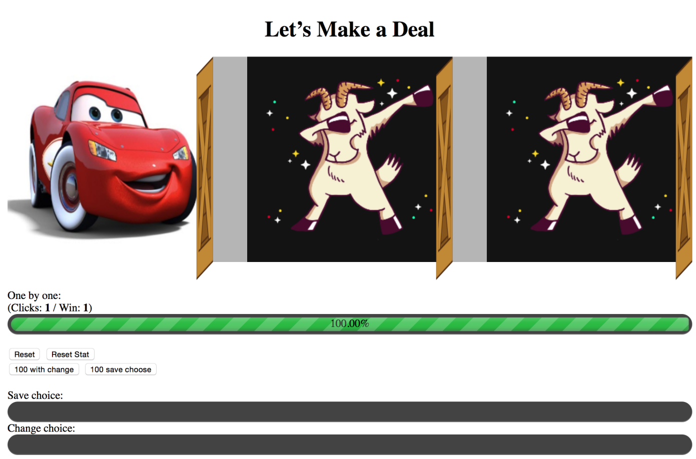
[8] Whitmeyer M. The Monty Hall Conundra Revisited *SSRN Electronic Journal* 2009.

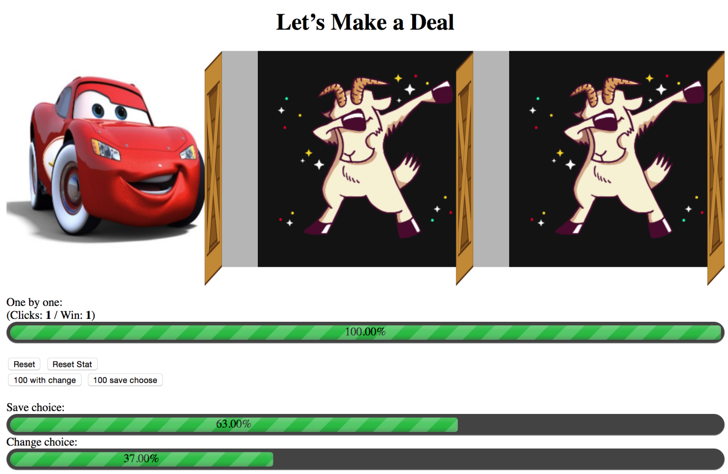
A Playing the Game

This section is particularly created for those who are still suspicious and do not rely on calculations presented on the paper. Based on the paper and investigations, game simulation is built with HTML,JavaScript and CSS with the help of which participants can play and check the statistics of their play. Game has two versions: Classical and Reviewed. Here are some pictures from the game playing process:


Let's Make a Deal
One by one:
(Clicks: / Win:)
Reset
Reset Stat
100 with change
100 save choose
Save choice:
Change choice:


Let's Make a Deal
One by one:
(Clicks: / Win:)
Reset
Reset Stat
100 with change
100 save choose
Save choice:
Change choice:


Let's Make a Deal
One by one:
(Clicks: 1 / Win: 1)
100.00%
Reset
Reset Stat
100 with change
100 save choose
Save choice:
Change choice:


Let’s Make a Deal
One by one:
(Clicks: 1 / Win: 1)
100.00%
Reset
Reset Stat
100 with change
100 save choose
Save choice:
63.00%
Change choice:
37.00%