

# Extra exercises – Week 14

## Graphs

**Note:** These exercises are for extra practice only. Do not submit them as they will not be graded. The exercises to be presented in the live class will be located in a separate document.

### Questions

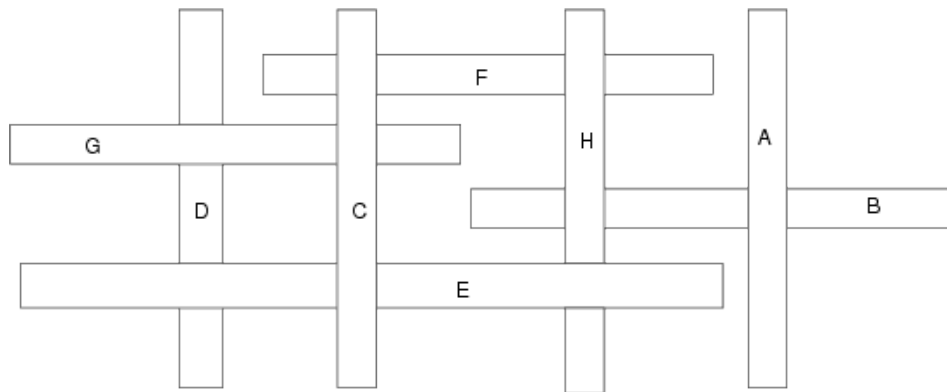
1. Would you use the adjacency list structure or the adjacency matrix structure in each of the following cases? Justify your choice.
  - (a) The graph has 10,000 vertices and 20,000 edges and it is important to use as little space as possible.
  - (b) The graph has 10,000 vertices and 20,000,000 edges, and it is important to use as little space as possible.
  - (c) You need to answer the query `areAdjacent()` as quickly as possible, no matter how much space you use. (Two vertices  $u$  and  $v$  are adjacent if either  $(u, v)$  or  $(v, u)$  is an edge in the graph.)
  - (d) You need to perform operation `insertVertex`.
  - (e) You need to perform operation `removeVertex`.
2. A *weighted graph* is a graph such that each edge has a weight (a number). The weights can be represented with an adjacency matrix, whose elements contain the weights (instead of boolean values). Note that this representation does not distinguish the case that there is an edge with weight 0 from the case that there is no edge. For this reason, we interpret a 0 in the matrix to mean that there is no edge.

Let  $A$  be the adjacency matrix of a weighted graph,  $G$ , with the columns and rows labelled, in order, by the vertices of the set  $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ . That is, the first row of the matrix represents the adjacencies of vertex  $v_0$ , the second row those of vertex  $v_1$ , etc. Note that there are some non-zero diagonals which means that there are some edges of the form  $(v, v)$ .

$$A = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 2 \\ 1 & 1 & 0 & 4 & 5 & 0 \\ 3 & 0 & 0 & 5 & 4 & 0 \\ 0 & 4 & 5 & 2 & 0 & 6 \\ 0 & 5 & 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 6 & 2 & 3 \end{bmatrix}$$

- (a) Write down the adjacency list for the graph, such that the edges in the list are ordered *by increasing weight*. (If two edges have the same weight, then order them by their index order.)

- (b) Perform a (preorder) depth first search on  $G$ , beginning at vertex  $v_0$ . Use the ordering in the adjacency list to determine the ordering of vertices visited. Give solutions both using the recursive algorithm and using a stack.
- (c) Perform a breadth first search of the graph,  $G$ , again beginning at  $v_0$  and using the ordering in the adjacency list.
3. Explain why the depth first traversal runs in  $O(n^2)$  time on an  $n$ -vertex graph that is represented with an adjacency matrix structure.
4. The figure below shows a set of rectangles whose overlap relationships can be represented with a directed graph. Each rectangle is represented by one vertex, and there is a directed edge whenever one rectangle overlaps another rectangle. For example, there is an edge  $(A,B)$  but no edge  $(B,A)$ .



- (a) Give the adjacency list for this graph. *The vertices must be ordered alphabetically.*
- (b) Give the ordering of vertices visited in a *breadth first* traversal of the graph, starting from vertex C. *Show the BFT tree.*  
NOTE: In this question and the next, there is only one answer since you must order the vertices alphabetically.
- (c) Give the ordering of vertices visited in a *recursive preorder depth first search* traversal, starting from vertex C. *Show the DFT tree.*
- (d) Give a *new* example having four rectangles I,J,K,L, such that corresponding graph contains a *cycle*. Draw the rectangles and the graph.
5. The depth first traversal algorithms that I presented in class were all *preorder* since vertices were visited before their children. Is a postorder recursive graph traversal possible? (Here we are assuming that the from-to relation on an edge should be interpreted as parent-child.) If so, how? If not, why not?

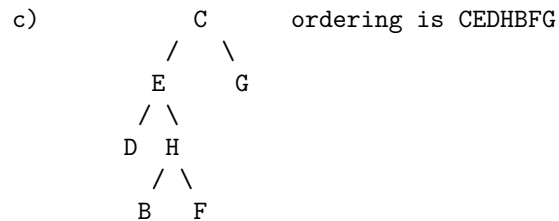
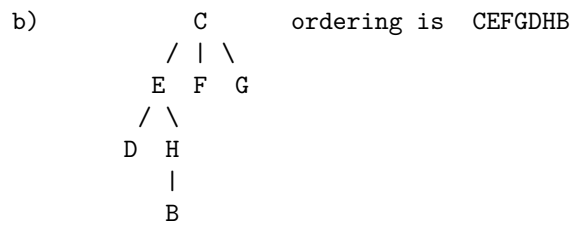
## Solutions

1. (a) An adjacency list would be better. Why? An adjacency list would have about 20,000 nodes, whereas the adjacency matrix would require  $10,000 \times 10,000 = 100,000,000$  booleans. A boolean is still typically represented with a byte (and the value would be either 0 or 1), so the matrix would require more space. Even if the booleans were packed more efficiently – 8 booleans packed into a byte – it would still require over 10,000,000 bytes.
- (b) There is no clear winner. The exact space usage of the two structures depends on the implementation details, in particular, of a node. It also depends on how the booleans are represented (see (a)).
- (c) Adjacency matrix. To answer `areAdjacent(i,j)` one need only read off matrix entries  $(i,j)$  and  $(j,i)$ , a constant time i.e.  $O(1)$  operation. In an adjacency list, one would need to search along row  $i$  to find  $j$ , and linear search is  $O(n)$ .
- (d) The adjacency list structure is much better for `insertVertex`. If you use a linked list or if you used a hash table for the vertices, the insert would be  $O(1)$ . By contrast, for an adjacency matrix representation of the edges, you would need to build a new (larger by 1 row and 1 column) matrix and copy all entries from old to new. This would be  $O(n^2)$ .
- (e) For `removeVertex()`, if you used a hash table to represent the vertices, then the vertex could be removed from the hash table in time  $O(1)$ , and the edges *from* the vertex could be removed in time  $O(1)$  also, just by deleting the adjacency list for that vertex.

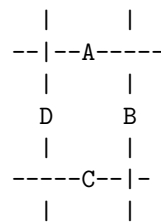
But what about the edges *to* the vertex that you want to remove? If you want a `removeVertex()` method to be fast, then you need to maintain a second adjacency list for each vertex that represents edges *to* each of the vertices. This would require doubling the space needed. Note that this doesn't change the  $O(1)$  behavior.

If you use an adjacency matrix, and you don't want rows and columns to continue representing edges that involved deleted vertices, then you would need to remake the matrix every time you do a remove. Just like in (d), this would be expensive:  $O(n^2)$ .

2. (a)  $v_0 - v_1, v_5, v_2$   
 $v_1 - v_0, v_1, v_3, v_4$   
 $v_2 - v_0, v_4, v_3$   
 $v_3 - v_3, v_1, v_2, v_5$   
 $v_4 - v_5, v_2, v_1$   
 $v_5 - v_4, v_5, v_3$
- (b) deep first traversal using recursion:  $v_0, v_1, v_3, v_2, v_4, v_5$   
 deep first traversal using stack:  $v_0, v_1, v_5, v_2, v_4, v_3$
- (c) breath first traversal:  $v_0, v_1, v_5, v_2, v_3, v_4$
3. A depth first traversal will consider every reachable vertex of the graph, eventually. For each vertex that it considers, it must consider the set of neighbours of the vertex – in order to do so, it must examine the entire row of the adjacency matrix corresponding to the vertex. Thus, for each of  $n$  rows/vertices, it must consider the  $n - 1$  non-diagonal entries, which makes for  $O(n^2)$  operations.
4. a) A - B  
 B -  
 C - E,F,G  
 D -  
 E - D,H  
 F -  
 G - D  
 H - B,F



- d) Use the overlap method for the four (folding) top pieces of a packing box. i.e. (A,B), (B,C), (C,D), (D,A).



5. Yes, it is possible. However, we cannot just move the `v.visited = true` assignment after the `depthFirst` call because we could get stuck in an infinite loop, in the case of a cycle. The following will NOT work, for example, if the graph consists of a single cycle:

```

depthFirst(v){
  for each w such that (v,w) is in E
    if !w.visited
      depthFirst(w)
  v.visited = true
}

```

To make a post-order traversal work, we need to distinguish between “reaching” a vertex and “visiting” a vertex. Reaching a vertex could just mean that we access the adjacency list of the vertex, whereas visiting it might mean that we write into a field in the vertex. The following works – in the sense that it avoids the infinite loop from a cycle, and also allows us to visit a vertex after all the adjacent vertices have been visited.

```

depthFirst(v){
  v.reached = true
  for each w such that (v,w) is in E
    if !w.reached
      depthFirst(w)
  v.visited = true
}

```

Notice that for the above algorithms to work properly, the fields `visited` and `reached` should both

be initialized to false. Also note that with a postorder traversal, you would get the same tree, but you would visit the nodes in a different order. For the above example, you would *reach* the nodes in the same order as with the preorder traversal, but you would *visit* the nodes in a different order.