Упражнения после 2.6

Упражнение 1

$(K\downarrow CRng) \simeq K\text{-alg}$?

$\varphi:\ K\downarrow CRng \xrightarrow{\sim} K\text{-alg}$

$$\begin{tikzcd} K \arrow[d, "f"] \\ R \end{tikzcd} \longmapsto R, \text{ где } \begin{array}{c} K\times R \longrightarrow R \\ (k,r)\longmapsto f(k)\cdot r \end{array}$$

Так определённая операция задаёт структуру K-модуля.

1) $a(x+y) = f(a)(x+y) = f(a)x + f(a)y = ax+ay$
2) $(a+b)x = f(a+b)x = f(a)x + f(b)x = ax+bx$
3) $(ab)x = f(ab)x = f(a)f(b)\cdot x = f(a)\cdot f(b)x = a(bx)$
4) $1x = f(1)x = 1\cdot x = x$

Аксиома алгебры:

$a\cdot xy = f(a)\cdot xy = f(a)x\cdot y = x\cdot f(a)y = ax\cdot y = x\cdot ay$

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ R \arrow[rr, dashed, "h"'] & & R' \end{tikzcd} \longmapsto \begin{tikzcd} R \arrow[d, "h"] \\ R' \end{tikzcd} \quad \begin{aligned} h(kx) &= h(f(k)x) = \\ &= hf(k)\cdot h(x) = \\ &= f'(k)\cdot h(x) = k\cdot h(x) \end{aligned}$$

$\varphi^{-1}:\ K\text{-alg} \xrightarrow{\sim} K\downarrow CRng$

$$R \longmapsto \begin{tikzcd} K \arrow[d, "f"] \\ R \end{tikzcd}, \text{ где } f(k) = k\cdot 1_R$$

$f(k+k') = (k+k')1 = k\cdot 1 + k'\cdot 1 = f(k)+f(k')$

$f(1) = 1\cdot 1_R = 1_R$

$f(kk') = kk'\cdot 1 = k1\cdot k'1 = f(k)\cdot f(k')$

$$\begin{tikzcd} R \arrow[d, "h"] \\ R' \end{tikzcd} \qquad \begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ R \arrow[rr, dashed, "h"'] & & R' \end{tikzcd} \qquad h\circ f(k) = h(k\cdot 1_R) = k\cdot h(1_R) = k\cdot 1_{R'} = f'(k)$$

Упражнение 2

t - универсальный притягивающий элемент в категории C. $(C\downarrow t) \simeq C$?

$$\begin{tikzcd} c \arrow[d, "f"] \\ t \end{tikzcd} \in Ob(C\downarrow t)$$

$$\begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "f"'] & & c' \arrow[dl, "f'"] \\ & t & \end{tikzcd} \in Hom_{C\downarrow t}(c,c')$$

$\varphi:\ C\downarrow t \xrightarrow{\sim} C$

$$\begin{tikzcd} c \arrow[d, "f"] \\ t \end{tikzcd} \longmapsto c$$

$$\begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "f"'] & & c' \arrow[dl, "f'"] \\ & t & \end{tikzcd} \longmapsto c \xrightarrow{h} c'$$

$\varphi^{-1}:\ C \xrightarrow{\sim} C\downarrow t$

$$c \longmapsto \begin{tikzcd} c \arrow[d, "!f"] \\ t \end{tikzcd}$$

$$c \xrightarrow{h} c' \longmapsto \begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "!"'] & & c' \arrow[dl, "!"] \\ & t & \end{tikzcd}$$

Упражнение 3

$$\begin{tikzcd} Te \arrow[r, "Th"] \arrow[d, "f"'] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sk"'] & Sd' \end{tikzcd}$$

$$\begin{tikzcd} e \arrow[d, "h"] \\ e' \end{tikzcd} \longmapsto \begin{tikzcd} Te \arrow[d, "Th"] \\ Te' \end{tikzcd} \longleftarrow \begin{array}{c} \tau(0) = Th \\ \tau(1) = Sk \end{array} \longmapsto \begin{tikzcd} Sd \arrow[d, "Sk"] \\ Sd' \end{tikzcd} \longleftarrow \begin{tikzcd} d \arrow[d, "k"] \\ d' \end{tikzcd}$$

Упражнение 4

$D \xrightarrow{T} C \xleftarrow{S} D$

$\tau: T \dashrightarrow S \overset{?}{=} \tau: D \longrightarrow (T\downarrow S)$, где $P\tau = Q\tau = id_D$

$$\begin{tikzcd} & & T\downarrow S \arrow[dll, "P"'] \arrow[d, "R"] \arrow[drr, "Q"] & & \\ D \arrow[r] & C & C^2 \arrow[l] \arrow[r] & C & D \arrow[l] \end{tikzcd}$$

$\tau(d) = \langle d, d, \tau d\rangle$

$$\begin{tikzcd} d \arrow[d, "h"] \\ d' \end{tikzcd} \rightsquigarrow \begin{tikzcd} Td \arrow[r, "Th"] \arrow[d, "\tau d"'] & Td' \arrow[d, "\tau d'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd}$$

Тогда $P\tau = Q\tau = id_D$

$$d \mapsto \langle d, d, \tau d\rangle \begin{array}{l} \nearrow d \\ \searrow d \end{array}$$

$$\begin{tikzcd} d \arrow[d, "h"] \\ d' \end{tikzcd} \mapsto \begin{tikzcd} Td \arrow[r, "Th"] \arrow[d, "\tau d"'] & Td' \arrow[d, "\tau d'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd} \begin{array}{l} \nearrow h \\ \searrow h \end{array}$$

Упражнение 5

$$\begin{tikzcd}
 & & X \arrow[lld, "P'"'] \arrow[d, dashed, "\varphi"] \arrow[rrd, "Q'"] \arrow[dd, bend left, "R'"] & & \\
 & & T\downarrow S \arrow[llu, phantom] \arrow[d, "R"] & & \\
E \arrow[r, "T"'] & C & C^2 \arrow[l, "C^0"] \arrow[r, "C^1"'] & C & D \arrow[l, "S"]
\end{tikzcd}$$

($P: T\downarrow S \to E$, $Q: T\downarrow S \to D$)

$P\circ\varphi = P'$; $Q\circ\varphi = Q'$; $R\circ\varphi = R'$

$\varphi: X \longrightarrow T\downarrow S$ - функтор

$x \longmapsto (P'x,\ Q'x,\ R'x)$

$$\begin{tikzcd}
x \arrow[d, "f"] \\ x'
\end{tikzcd}
\longmapsto
\begin{tikzcd}
TP'x \arrow[r, "TP'f = R'f(0)"] \arrow[d, "R'x"'] & TP'x' \arrow[d, "R'x'"] \\
SQ'x \arrow[r, "SQ'f = R'f(1)"'] & SQ'x'
\end{tikzcd}$$

Упражнение 6

(а) C, D, E - фиксированные категории

$(C^E)^{op} \times (C^D) \longrightarrow Cat$

$\langle T, S\rangle \rightsquigarrow (T\downarrow S)$

$$\begin{tikzcd}
\langle T, S\rangle \arrow[d, "{\langle \tau^{op}, \sigma\rangle}"] & \rightsquigarrow & (T\downarrow S) \arrow[d] \\
\langle T', S'\rangle & & (T'\downarrow S')
\end{tikzcd}$$

$\langle e, d, f: Te \to Sd\rangle$

$\downarrow$

$\langle e, d, \sigma f \tau : T'e \to S'd\rangle$

$$\begin{tikzcd}
Te \arrow[r, "Th"] \arrow[d, "f"'] & Te' \arrow[d, "f'"] \\
Sd \arrow[r, "Sk"'] & Sd'
\end{tikzcd}
\longmapsto
\begin{tikzcd}
T'e \arrow[r, "T'h"] \arrow[d, "\tau"'] \arrow[ddd, bend right] & T'e' \arrow[d, "\tau"] \arrow[ddd, bend left] \\
Te \arrow[r, "Th"] \arrow[d, "f"'] \arrow[u, phantom] & Te' \arrow[d, "f'"] \\
Sd \arrow[r, "Sk"] \arrow[d, "\sigma"'] & Sd' \arrow[d, "\sigma"] \\
S'd \arrow[r, "S'k"'] & S'd'
\end{tikzcd}$$

(б) ?