Lecture 17: 2D Elliptic Unified CG/DG

Let's consider the PDE:

$$\nabla^2 q(\bar{x}) = f(\bar{x}) \quad \text{with} \quad q\big|_{\Gamma_D} = g(x) \quad \&$$

$$\left(\hat{n}\cdot\bar{\nabla} q\right)\Big|_{\Gamma_N} = h(\bar{x})$$

Now, let us merge the ideas from Lecture 15 (for CG) & Lecture 16 (for DG) to find a unified approach. However, we have to start w/ the flux formulation in order to use DG.

Flux Formulation

As in the LDG method, let's start with the flux formulation:

$$\bar{Q}(\bar{x}) = \bar{\nabla} q(\bar{x})$$

$$\bar{\nabla}\cdot\bar{Q}(\bar{x}) = f(\bar{x})$$

Discretizing, we write:

$$(17.1.1)\quad \int_{\Omega_e} \bar{\tau}_i \cdot \bar{Q}_N^{(e)}\, d\Omega_e = \int_{\Omega_e} \bar{\tau}_i \cdot \bar{\nabla} q_N^{(e)}\, d\Omega_e$$

(17.1.2) $$\int_{\Omega_e} \psi_i \, \overline{\nabla} \cdot \overline{Q}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i \, f_N^{(e)} \, d\Omega_e$$

Integrating Eq. (17.1) by parts we get:

(17.2.1) $$\int_{\Omega_e} \overline{\tau}_i \cdot \overline{Q}_N^{(e)} \, d\Omega_e = \int_{\Gamma_e} \hat{n} \cdot \left(\overline{\tau}_i \, q_N^{(e)}\right)^{(*)} d\Gamma_e - \int_{\Omega_e} \overline{\nabla} \cdot \overline{\tau}_i \, q_N^{(e)} \, d\Omega_e$$

(17.2.2) $$\int_{\Gamma_e} \hat{n} \cdot \left(\psi_i \, \overline{Q}_N^{(e)}\right)^{(*)} d\Gamma_e - \int_{\Omega_e} \overline{\nabla}\psi_i \cdot \overline{Q}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i \, f_N^{(e)} \, d\Omega_e$$

Primal Formulation

If we let $\overline{\tau} = \psi \, \overline{\overline{I}}$ we recover the LDG method, with the appropriate fluxes.

How about if we choose $\overline{\tau} = \overline{\nabla}\psi$? Let's see what happens. We get:

(17.3.1) $$\int_{\Omega_e} \overline{\nabla}\psi_i \cdot \overline{Q}_N^{(e)} \, d\Omega_e = \int_{\Gamma_e} \hat{n} \cdot \left(\overline{\nabla}\psi_i \, q_N^{(e)}\right)^{(*)} d\Gamma_e - \int_{\Omega_e} \nabla^2\psi_i \, q_N^{(e)} \, d\Omega_e$$

$$(17.3.2) \quad \int_{\Gamma_e} \hat{n} \cdot \left(\psi_i \bar{Q}_N^{(e)} \right)^{(*)} d\Gamma_e - \int_{\Omega_e} \bar{\nabla} \psi_i \cdot \bar{Q}_N^{(e)} \, d\Omega_e$$

$$= \int_{\Omega_e} \psi_i f_N^{(e)} \, d\Omega_e$$

Subbing 1^{st} term in (17.3.1) into 2^{nd} term in (17.3.2) gives:

$$(17.4) \quad \int_{\Gamma_e} \hat{n} \cdot \left(\psi_i \bar{Q}_N^{(e)} \right)^{(*)} d\Gamma_e - \int_{\Gamma_e} \hat{n} \cdot \left(\bar{\nabla} \psi_i q_N^{(e)} \right)^{(*)} d\Gamma_e$$

$$+ \int_{\Omega_e} \underbrace{\nabla^2 \psi_i \, q_N^{(e)}} \, d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} \, d\Omega_e$$

We need to apply IBP here.

Let's replace:

$$(17.5) \quad \int_{\Omega_e} \nabla^2 \psi_i \, q_N^{(e)} \, d\Omega_e = \underbrace{\int_{\Gamma_e} \hat{n} \cdot \left(\bar{\nabla} \psi_i \, q_N^{(e)} \right) d\Gamma_e} - \int_{\Omega_e} \bar{\nabla} \psi_i \cdot \overbrace{\bar{\nabla} q_N^{(e)}}^{= \bar{Q}_N^{(e)}} \, d\Omega_e$$

no numerical flux here since we already introduced one.

Subbing (17.5) into (17.4) gives:

$$(17.6) \quad \int_{\Gamma_e} \hat{n} \cdot \left(\psi_i \bar{Q}_N^{(e)} \right)^{(*)} d\Gamma_e + \int_{\Gamma_e} \hat{n} \cdot \left[\left(\bar{\nabla} \psi_i \, q_N^{(e)} \right) - \left(\bar{\nabla} \psi_i \, q_N^{(e)} \right)^{(*)} \right] d\Gamma_e$$

$$- \int_{\Omega_e} \vec{\nabla}\psi_i \cdot \vec{Q}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} \, d\Omega_e$$

Assuming that $\psi^{(*)} = \psi$ & $\vec{\nabla}\psi^{(*)} = \vec{\nabla}\psi$ we get:

$$(17.7) \quad \int_{\Gamma_e} \psi_i \, \hat{n} \cdot \vec{Q}_N^{(*)} \, d\Gamma_e + \int_{\Gamma_e} \hat{n} \cdot \vec{\nabla}\psi_i \left[q_N^{(e)} - q_N^{(*,e)} \right] d\Gamma_e$$

$$- \int_{\Omega_e} \vec{\nabla}\psi_i \cdot \vec{Q}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} \, d\Omega_e$$

Replacing $\vec{Q}_N^{(e)} = \vec{\nabla} q_N^{(e)}$ yields:

$$(17.8) \quad \int_{\Gamma_e} \psi_i \, \hat{n} \cdot \vec{\nabla} q_N^{(*)} \, d\Gamma_e + \int_{\Gamma_e} \hat{n} \cdot \vec{\nabla}\psi_i \left[q_N^{(e)} - q_N^{(*,e)} \right] d\Gamma_e$$

$$- \int_{\Omega_e} \vec{\nabla}\psi_i \cdot \vec{\nabla} q_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} \, d\Omega_e$$

With appropriate numerical fluxes, Eq. (17.8) represents the Symmetric Interior Penalty Galerkin (SIPG) method.

SIPG Method

To define the SIPG method, we need to state the numerical fluxes.

The classical fluxes are the following:

$$q_N^{(*)} = \left\{ q_N^{(e,k)} \right\} \equiv \frac{1}{2} \left(q_N^{(e)} + q_N^{(k)} \right)$$

For the gradient, we can define the flux:

$$\vec{\nabla} q_N^{(*)} = \left\{ \vec{\nabla} q_N^{(e,k)} \right\} - \hat{n}\,\mu \,[\![q^{(e,k)}]\!]$$

where μ plays the role of $|\lambda|$ as in the Rusanov flux &

$$[\![q^{(e,k)}]\!] = \left(q^{(k)} - q^{(e)} \right)$$

Subbing these fluxes into Eq. (17.8) yields:

$$(17.9)\quad -\int_{\Omega_e} \vec{\nabla}\psi_i \cdot \vec{\nabla} q_N^{(e)}\, d\Omega_e + \int_{\Gamma_e} \psi_i\, \hat{n} \cdot \left[\left\{ \vec{\nabla} q_N^{(e,k)} \right\} - \hat{n}\,\mu\, [\![q^{(e,k)}]\!] \right] d\Gamma_e$$

$$+ \int_{\Gamma_e} \hat{n} \cdot \vec{\nabla}\psi_i \left[q_N^{(e)} - \left\{ q_N^{(e,k)} \right\} \right] d\Gamma_e = \int_{\Omega_e} \psi_i f_N^{(e)}\, d\Omega_e$$

The only term that has not yet been defined is μ. We can let $\mu = 0$ but, another approach, is to let

$$\mu = \mu_c \frac{N(N+1)}{2} \frac{J^{(f)}}{J^{(e)}}$$

where $J^{(e)}$ & $J^{(f)}$ are the element & face Jacobians & μ_c is a constant. The larger it is, the better we can satisfy Dirichlet BCs.

$\boxed{\text{Summary}}$ Eq. (17.9) can be used for both CG & DG. With this eq. we can write one unified method.