Gene Golub SIAM Summer School 2012
Lecture 2: 1D Application of DG Methods

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Summary of DG Lecture 1

- We covered the background (theoretical) for DG methods.
- We saw that DG uses an integral (weak) form.
- We saw that DG requires good interpolation and good integration.
- One can construct a modal DG approach (using orthogonal polynomials) or a nodal DG approach (using Lagrange polynomials).

Outline

- Problem Statement
- 1D Matrices
- Resulting Element Equations
- Numerical Flux
- Analysis of Discretized Spatial Operators
- Results for the 1D Equations

Problem Statement

- The 1D wave equation in conservation (flux) form is

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad \text{where} \quad f = qu$$

and u is the velocity.

- Taking the weak integral form in Ω_e we get

$$\int_{\Omega_e} \psi \left(\frac{\partial q_N}{\partial t} + \frac{\partial f_N}{\partial x} \right) d\Omega_e = 0.$$

- Expanding the variables using a linear nodal approximation $(N = 1)$

$$q_N(x,t) = \sum_{j=0}^{1} q_j(t)\psi_j(x), \quad f_N(x,t) = f(q_N) = uq_N$$

Problem Statement

- where $f_N = \sum_{j=0}^{1} f_j(t)\psi_j(x)$ for a linear function f, yields
- The weak integral form is

$$\int_{\Omega_e} \psi \left(\frac{\partial q_N}{\partial t} + \frac{\partial f_N}{\partial x} \right) d\Omega_e = 0.$$

- Integrating by parts gives

$$\int_{\Omega_e} \psi \frac{\partial q_N}{\partial t} d\Omega_e + \int_{\Omega_e} \frac{\partial}{\partial x} (\psi f_N)\, d\Omega_e - \int_{\Omega_e} \frac{d\psi}{dx} f_N\, d\Omega_e = 0.$$

- Integrating the second term yields

$$\int_{\Omega_e} \psi \frac{\partial q_N}{\partial t} d\Omega_e + [\psi f_N]_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi}{dx} f_N\, dx = 0.$$

- In matrix form

$$M_{ij} \frac{dq_j}{dt} + F_{ij} f_j - \widetilde{D}_{ij} f_j = 0$$

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Element Matrices

- The equation in matrix-vector form is

$$M_{ij}\frac{dq_j}{dt} + F_{ij}f_j - \widetilde{D}_{ij}f_j = 0$$

- The matrices

$$\begin{aligned} M_{ij}^{(e)} &= \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e, \\ \widetilde{D}_{ij}^{(e)} &= \int_{\Omega_e} \frac{d\psi_i}{dx} \psi_j \, d\Omega_e, \\ F_{ij} &= \left[\psi_i(x)\psi_j(x) \right]_{\Gamma_e} \end{aligned}$$

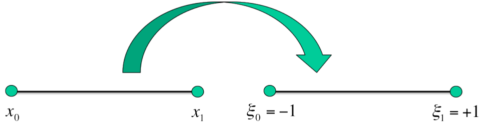
are the mass, differentiation, and flux matrices.

Basis Functions

- For 2 grid points per element (linear) let

$$q = \begin{cases} q_0 & \text{for} \quad x = x_0 \\ q_1 & \text{for} \quad x = x_1 \end{cases}.$$

- We first map from physical space $x \in [x_0, x_1]$ to computational space $\xi \in [-1, +1]$



- This mapping is now used as follows

$$q = \begin{cases} q_0 & \text{for} \quad \xi = \xi_0 = -1 \\ q_1 & \text{for} \quad \xi = \xi_1 = +1 \end{cases}.$$

- with the basis functions

$$\psi_0 = \frac{1}{2}(1-\xi) \quad \text{and} \quad \psi_1 = \frac{1}{2}(1+\xi)$$

Basis Functions

- where they have been obtained from general Lagrange polynomial formula

$$\psi_i(\xi) = \prod_{\substack{j=0,\\ j\neq i}}^{1} \left(\frac{\xi - \xi_j}{\xi_i - \xi_j} \right)$$

Exercise

Using the above relation and the interpolations points $\xi_0 = -1$ and $\xi_1 = +1$ build the two basis functions

$$\psi_0 = \frac{1}{2}(1-\xi) \quad \textit{and} \quad \psi_1 = \frac{1}{2}(1+\xi).$$

Basis Functions

- The reason we map from $x \to \xi$ is that this change of variable simplifies the construction of local element-based Galerkin (EBG) methods because we need not solve matrices for every single element in our grid.
- We only do it for the reference element and then use metric terms to scale the reference element to the true size.
- We can now approximate the coordinates of the element by the expansion

$$x = \sum_{j=0}^{1} x_j \psi_j(x) = \frac{1}{2}(1-\xi)x_0 + \frac{1}{2}(1+\xi)x_1 \quad \text{and} \quad dx = \frac{\Delta x}{2} d\xi$$

Basis Functions

- Conversely
$$\xi = \frac{2(x - x_0)}{x_1 - x_0} - 1 \quad \text{and} \quad \frac{d\xi}{dx} = \frac{2}{\Delta x}.$$
- Let us review the reference element in one-dimension which is simply the line $\xi \in [-1, +1]$.
- Note that we can now construct all the relevant matrices required by our partial differential equations in terms of this reference element.
- The construction of these matrices can be done somewhat independently (we will explain the caveat when we introduce two-dimensional problems) of the size of the physical domain, say $x \in [-L, +L]$, and also quite independently of how small or how large each of our elements are.

Mass Matrix

- Mapping from the physical space $x \in [x_0, x_1]$ to computational space $\xi \in [-1, +1]$ we get for the mass matrix

$$M_{ij}^{(e)} = \int_{x_0}^{x_1} \psi_i(x)\psi_j(x)\, dx = \int_{-1}^{+1} \psi_i(\xi)\psi_j(\xi)\frac{\Delta x}{2} d\xi$$

which in matrix form is

$$M_{ij}^{(e)} = \frac{\Delta x}{2} \int_{-1}^{+1} \begin{pmatrix} \psi_0\psi_0 & \psi_0\psi_1 \\ \psi_1\psi_0 & \psi_1\psi_1 \end{pmatrix} d\xi.$$

- Substituting ψ (for $N = 1$) we get

$$M_{ij}^{(e)} = \frac{\Delta x}{2} \int_{-1}^{+1} \begin{pmatrix} \frac{1}{2}(1-\xi)\frac{1}{2}(1-\xi) & \frac{1}{2}(1-\xi)\frac{1}{2}(1+\xi) \\ \frac{1}{2}(1+\xi)\frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi)\frac{1}{2}(1+\xi) \end{pmatrix} d\xi.$$

Mass Matrix

- Integrating analytically

$$M_{ij}^{(e)} = \frac{\Delta x}{8} \left(\begin{array}{cc} \xi - \xi^2 + \frac{1}{3}\xi^3 & \xi - \frac{1}{3}\xi^3 \\ \xi - \frac{1}{3}\xi^3 & \xi + \xi^2 + \frac{1}{3}\xi^3 \end{array} \right) \Bigg|_{-1}^{+1}$$

and evaluating at the limits of integration gives

$$M_{ij}^{(e)} = \frac{\Delta x}{8} \left(\begin{array}{cc} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{array} \right).$$

- Factoring out the term $\frac{4}{3}$ yields

$$M_{ij}^{(e)} = \frac{\Delta x}{6} \left(\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right).$$

Mass Matrix

Exercise

Derive the mass matrix for $N = 1$ Lagrange polynomials (based on Lobatto points) on your own. Feel free to use the slides to guide you along.

Mass Matrix

- This matrix is clearly full. However, if we decided to use numerical integration we can use one of two options.
- We can either mimic the analytic integration and use $Q = N + 1$ (for Lobatto Points).
- This is exact for the mass matrix because the mass matrix is a $2N$ polynomial and so using $Q = N + 1$ integration (Lobatto) points will integrate exactly $2(N + 1) - 1$ points.
- Using $Q = N + 1$, which is $Q = 2$ for $N = 1$, Legendre-Gauss-Lobatto (LGL) yields

$$
\begin{aligned}
M_{ij}^{(e)} &= \int_{x_0}^{x_1} \psi_i(x)\psi_j(x)\, dx \\
&= \int_{-1}^{+1} \psi_i(\xi)\psi_j(\xi)\frac{\Delta x}{2} d\xi = \frac{\Delta x}{2} \sum_{k=0}^{2} w_q \psi_i(\xi_k)\psi_j(\xi_k)
\end{aligned}
$$

Mass Matrix

- Where the quadrature weights are

$$w_{0,1,2} = \frac{1}{3}, \frac{4}{3}, \frac{1}{3}$$

- and the quadrature roots are

$$\xi_{0,1,2} = -1, 0, +1.$$

- Substituting these values yields

$$M_{ij}^{(e)} = \frac{\Delta x}{2} \sum_{k=0}^{2} w_k \psi_i(\xi_k) \psi_j(\xi_k) = \frac{\Delta x}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

- Alternatively we could take $Q = N$, which is inexact in this case, which then yields

$$M_{ij}^{(e)} = \int_{-1}^{+1} \psi_i(\xi) \psi_j(\xi) \frac{\Delta x}{2} d\xi = \frac{\Delta x}{2} \sum_{k=0}^{1} w_k \psi_i(\xi_k) \psi_j(\xi_k)$$

Mass Matrix

- Where the quadrature weights are now

$$w_{0,1} = 1, 1$$

- and the quadrature weights are

$$\xi_{0,1} = -1, +1.$$

- Substituting these values yields

$$M_{ij}^{(e)} = \frac{\Delta x}{2} \sum_{k=0}^{1} w_k \psi_i(\xi_k) \psi_j(\xi_k) = \frac{\Delta x}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  which is a diagonal matrix.

- However, we would not want to do this for $N < 4$ because the accuracy is compromised.

Differentiation Matrix

- The differentiation matrix in computational space becomes

$$\widetilde{D}_{ij} = \int_{x_0}^{x_1} \frac{d\psi_i}{dx}(x)\,\psi_j(x)dx \; = \int_{-1}^{+1} \left(\frac{d\psi_i}{d\xi}(\xi)\frac{d\xi}{dx} \right) \psi_j(\xi) \left(\frac{dx}{d\xi} d\xi \right)$$

  where $\widetilde{D}$ is the weak form differentiation matrix.
- Substituting ψ (for $N = 1$) we get in matrix form

$$\widetilde{D}_{ij} = \frac{2}{\Delta x} \frac{\Delta x}{2} \int_{-1}^{+1} \begin{pmatrix} \frac{1}{2}(-1)\frac{1}{2}(1-\xi) & \frac{1}{2}(-1)\frac{1}{2}(1+\xi) \\ \frac{1}{2}(+1)\frac{1}{2}(1-\xi) & \frac{1}{2}(+1)\frac{1}{2}(1+\xi) \end{pmatrix} d\xi.$$

Differentiation Matrix

- Integrating

$$\widetilde{D}_{ij} = \frac{1}{4} \begin{pmatrix} -\xi + \frac{1}{2}\xi^2 & -\xi - \frac{1}{2}\xi^2 \\ \xi - \frac{1}{2}\xi^2 & \xi + \frac{1}{2}\xi^2 \end{pmatrix} \Bigg|_{-1}^{+1}$$

- and evaluating at the integration limits gives

$$\widetilde{D}_{ij} = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}.$$

Flux Matrix

- The flux matrix (from the boundary integral term) is given as

$$F_{ij} = \left[\psi_i(x)\psi_j(x)\right]_{x_0}^{x_1} = \left[\psi_i(\xi)\psi_j(\xi)\right]_{-1}^{+1} = \left.\begin{pmatrix} \psi_0\psi_0 & \psi_0\psi_1 \\ \psi_1\psi_0 & \psi_1\psi_1 \end{pmatrix}\right|_{-1}^{+1}$$

- Substituting ψ (again, for $N = 1$) we get

$$F_{ij} = \left.\begin{pmatrix} \frac{1}{2}(1-\xi)\frac{1}{2}(1-\xi) & \frac{1}{2}(1-\xi)\frac{1}{2}(1+\xi) \\ \\ \frac{1}{2}(1+\xi)\frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi)\frac{1}{2}(1+\xi) \end{pmatrix}\right|_{-1}^{+1}.$$

- Evaluating at the limits of integration gives

$$F_{ij} = \frac{1}{4}\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}.$$

- Factoring out the term $\frac{1}{4}$ yields

$$F_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Resulting Equations

- The resulting equation which must be satisfied within each DG element is

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix}^{(e)} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}^{(e)} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}^{(*)} = 0$$

  where the subscript 0 and 1 denote the current element's grid points.
- This, in fact, is nothing more than the matrix problem

$$M_{ij}\frac{dq_j^{(e)}}{dt} - \widetilde{D}_{ij} f_j^{(e)} + F_{ij} f_j^{(*)} = 0$$

  where $f^{(*)}$ denotes the numerical flux function.
- First we shall assume the numerical flux function to be $f^{(e)}$, that is, the flux within the element.

Resulting Equations

- Let us now construct the element equations for the center element having gridpoints (I-1,I).
- The following figure illustrates the contribution of the left and right elements to the center element.

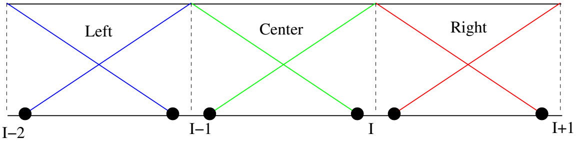

Figure: Contribution to the gridpoint I from the left $(I-2, I-1)$, center $(I-1, I)$ and right $(I, I+1)$ elements.

- The interface gridpoints between the left and center at $I-1$ and those for the right and center at I are purposely not touching in order to denote the discontinuity across the element interfaces.

Left Element

- Since the following is the left element equation

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_{I-2}^{(L)} \\ q_{I-1}^{(L)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_{I-2}^{(L)} \\ f_{I-1}^{(L)} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_{I-2}^{(L)} \\ f_{I-1}^{(L)} \end{pmatrix} = 0$$

then the contribution to $I-2$ is

$$\frac{1}{3}\left(2\frac{dq_{I-2}^{(L)}}{dt} + \frac{dq_{I-1}^{(L)}}{dt}\right) + \frac{1}{\Delta x}\left(f_{I-1}^{(L)} - f_{I-2}^{(L)}\right) = 0$$

and the contribution to $I-1$ is

$$\frac{1}{3}\left(\frac{dq_{I-2}^{(L)}}{dt} + 2\frac{dq_{I-1}^{(L)}}{dt}\right) + \frac{1}{\Delta x}\left(f_{I-1}^{(L)} - f_{I-2}^{(L)}\right) = 0.$$

Center Element

- Since the following is the center element equation

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_{I-1}^{(C)} \\ q_I^{(C)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_{I-1}^{(C)} \\ f_I^{(C)} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_{I-1}^{(C)} \\ f_I^{(C)} \end{pmatrix} = 0$$

then the contribution to $I-1$ is

$$\frac{1}{3}\left(2\frac{dq_{I-1}^{(C)}}{dt} + \frac{dq_i^{(C)}}{dt}\right) + \frac{1}{\Delta x}\left(f_I^{(C)} - f_{I-1}^{(C)}\right) = 0$$

and the contribution to I is

$$\frac{1}{3}\left(\frac{dq_{I-1}^{(C)}}{dt} + 2\frac{dq_I^{(C)}}{dt}\right) + \frac{1}{\Delta x}\left(f_I^{(C)} - f_{I-1}^{(C)}\right) = 0.$$

Right Element

- For the right element, we have

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_I^{(R)} \\ q_{I+1}^{(R)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_I^{(R)} \\ f_{I+1}^{(R)} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_I^{(R)} \\ f_{I+1}^{(R)} \end{pmatrix} = 0$$

then the contribution to I is

$$\frac{1}{3}\left(2\frac{dq_I^{(R)}}{dt} + \frac{dq_{I+1}^{(R)}}{dt}\right) + \frac{1}{\Delta x}\left(f_{I+1}^{(R)} - f_I^{(R)}\right) = 0.$$

and the contribution to $I+1$ is

$$\frac{1}{3}\left(\frac{dq_I^{(R)}}{dt} + 2\frac{dq_{I+1}^{(R)}}{dt}\right) + \frac{1}{\Delta x}\left(f_{I+1}^{(R)} - f_I^{(R)}\right) = 0.$$

Total Contribution

- By looking at the left, center, and right contributions we see immediately that all three elements are completely decoupled from each other.
- To further emphasize the point, note that the interface values between the center element and its two neighbors ($I-1$ and I) have different values depending on which element you are referencing.
- For example, the value for the left element at $I-1$ and the value for the center element at $I-1$ are not equal!
- Therefore we have

$$q_{I-1}^{(C)} \neq q_{I-1}^{(L)}$$

with the analogous situation for the grid point I, namely

$$q_{I}^{(C)} \neq q_{I}^{(R)}.$$

Total Contribution

- However, while it is physically possible to have discontinuities it is not possible for parcels of air (i.e., elements) to be completely decoupled from the rest of the domain; fortunately, we have only come up with this decoupling due to our treatment of the flux matrices.
- In the above equations we used the interface values specific to the element we were evaluating without regard for its neighbors - we did this to show how decoupled the elements can be from each other in DG.
- However, in reality we want information from contiguous elements to propagate across neighbors.

Total Contribution

- Since at the element interfaces we have discontinuities (from the contribution of the left, center, and right elements having different solutions) then we need to use averaged values at the interface.
- The mathematical argument for using averaged values stems from well-posedness conditions.
- For example, in the extreme case where we only have one DG element, then ignoring the boundary conditions would violate well-posedness and will result in an ill-posed problem.

Total Contribution

- Analogously, for a multi-element problem we must impose the neighbor values as boundary conditions via the flux terms to ensure well-posedness.

Remark

In fact, only through the flux integrals do contiguous elements talk to each other. Therefore the only interprocessor communication for DG occurs in the computation of the flux integrals. The remainder of the operations occur completely on processor because they are purely element computation and, as we have seen, the element integrals only require information specific to that element. This gives DG an advantage on MPP computers.

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- Numerical Flux
  - Centered Flux
  - Rusanov Flux
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Centered Flux

- If we now chose to define the numerical flux $f^{(*)}$ as the mean value of the left and right elements defines the following centered flux
$$f^{(*)} = \frac{1}{2}\left(f^{(e)} + f^{(k)}\right)$$
where the superscripts e and k denote the element and its neighbor, respectively.
- Using this centered flux gives for the center element equation

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_{I-1}^{(C)} \\ q_I^{(C)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_{I-1}^{(C)} \\ f_I^{(C)} \end{pmatrix}$$
$$+ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\left(f_{I-1}^{(C)} + f_{I-1}^{(L)}\right) \\ \frac{1}{2}\left(f_I^{(C)} + f_I^{(R)}\right) \end{pmatrix} = 0$$

Centered Flux

- This results in the following equation

$$\frac{1}{3}\left(\frac{dq_{I-1}^{(C)}}{dt}+2\frac{dq_{I}^{(C)}}{dt}\right)+\frac{1}{\Delta x}\left(f_I^{(C)}-f_{I-1}^{(C)}\right)+\frac{1}{\Delta x}\left(f_I^{(R)}-f_I^{(C)}\right)=0$$

for the center element at the gridpoint I.

- Note that for the right element, we get the two equations

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_I^{(R)} \\ q_{I+1}^{(R)} \end{pmatrix}-\frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_I^{(R)} \\ f_{I+1}^{(R)} \end{pmatrix}$$

$$+\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\left(f_I^{(R)}+f_I^{(C)}\right) \\ \frac{1}{2}\left(f_{I+1}^{(R)}+f_{I+1}^{(C)}\right) \end{pmatrix}=0.$$

- For the right element at the gridpoint I we get

$$\frac{1}{3}\left(2\frac{dq_{I}^{(R)}}{dt}+\frac{dq_{I+1}^{(R)}}{dt}\right)+\frac{1}{\Delta x}\left(f_{I+1}^{(R)}-f_{I}^{(R)}\right)+\frac{1}{\Delta x}\left(f_I^{(R)}-f_I^{(C)}\right)=0.$$

Centered Flux

- The term $\frac{1}{\Delta x}\left(f_I^{(R)} - f_I^{(C)}\right)$ is nothing more than the jump condition between the center and right elements.
- Note that if we diagonalize the mass matrix then we get for the center element at I

$$\frac{dq_I^{(C)}}{dt} + \frac{f_I^{(R)} - f_{I-1}^{(C)}}{\Delta x} = 0$$

and for the right element at I

$$\frac{dq_I^{(R)}}{dt} + \frac{f_{I+1}^{(R)} - f_I^{(C)}}{\Delta x} = 0$$

for I which look like upwinding/downwinding schemes.

Centered Flux

- In fact, if we assume that $q \in C^0$ then we would indeed recover (for the center element)

$$\frac{dq_I}{dt} + \frac{f_I - f_{I-1}}{\Delta x} = 0$$

and (for the right element)

$$\frac{dq_I}{dt} + \frac{f_{I+1} - f_I}{\Delta x} = 0$$

Rusanov Flux

- The single most common numerical flux function is the Rusanov (or local Lax-Friedrichs) flux which is a generalized upwinding method.
- The Rusanov flux is defined as

$$f^{(*,k)} = \frac{1}{2}\left[f^{(e)} + f^{(k)} - \widehat{\mathbf{n}}_{\Gamma_e}^{(e,k)}|\lambda_{max}|\left(q^{(k)} - q^{(e)}\right)\right]$$

  where $\widehat{\mathbf{n}}_{\Gamma_e}^{(e,k)}$ denotes the outward pointing normal to the interface of the element e and its neighbor k, and λ_{max} is the maximum wave speed of your system. In our example, $\lambda_{max} = u$ but in general it represents the maximum eigenvalue of the Jacobian matrix of the governing equations of motion.
- The Rusanov flux is just the average value between the two elements sharing an edge with the addition of a dissipation term ($|\lambda|$).

Rusanov Flux

- This dissipation term will allow the flux function to modify itself based on the flow conditions in order to construct an upwind-biased method.
- Let's consider the following figure to see what the Rusanov flux would look like for the center element.

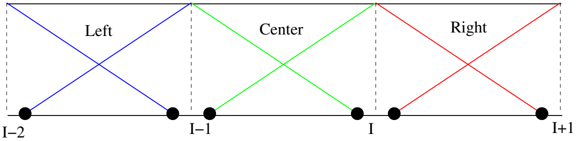


Figure: Contribution to the gridpoint I from the left $(I-2, I-1)$, center $(I-1, I)$ and right $(I, I+1)$ elements.

- Since $f = qu$ we can rewrite the Rusanov flux as

$$f^{(*,k)} = \frac{1}{2}\left[f^{(e)} + f^{(k)} - \widehat{\mathbf{n}}^{(e,k)}_{\Gamma_e}\left(f^{(k)} - f^{(e)}\right)\right]$$

Rusanov Flux

- At the interface $(e,k)=(C,L)$, that is, between the center and left elements, the outward pointing normal vector from C to L is $\widehat{\mathbf{n}}_{\Gamma_e}^{(e,k)}=-1$.
- This gives

$$f^{(*,C,L)}=\frac{1}{2}\left[f^{(C)}+f^{(L)}-(-1)\left(f^{(L)}-f^{(C)}\right)\right]$$

that can be simplified to

$$f^{(*,C,L)}=f^{(L)}.$$

- At the interface $(e,k)=(C,R)$, that is, between the center and right elements, the outward pointing normal vector from C to R is $\widehat{\mathbf{n}}_{\Gamma_e}^{(e,k)}=+1$.

Rusanov Flux

- This gives

$$f^{(*,C,R)} = \frac{1}{2}\left[f^{(C)} + f^{(R)} - (+1)\left(f^{(R)} - f^{(C)}\right)\right]$$

that can be simplified to

$$f^{(*,C,L)} = f^{(C)}.$$

- Using these flux values in our canonical equation gives for the center element equation

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_{I-1}^{(C)} \\ q_I^{(C)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_{I-1}^{(C)} \\ f_I^{(C)} \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_{I-1}^{(L)} \\ f_I^{(C)} \end{pmatrix} = 0$$

Rusanov Flux

- This results in the following equation at the gridpoint I

$$\frac{1}{3}\left(\frac{dq_{I-1}^{(C)}}{dt}+2\frac{dq_{I}^{(C)}}{dt}\right)+\frac{1}{\Delta x}\left(f_{I}^{(C)}-f_{I-1}^{(C)}\right)=0.$$

- For the right element, we get the equations

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q_{I}^{(R)} \\ q_{I+1}^{(R)} \end{pmatrix}-\frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_{I}^{(R)} \\ f_{I+1}^{(R)} \end{pmatrix}$$

$$+\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_{I}^{(C)} \\ f_{I+1}^{(R)} \end{pmatrix}=0.$$

- For the right element at the gridpoint I we get

$$\frac{1}{3}\left(2\frac{dq_{I}^{(R)}}{dt}+\frac{dq_{I+1}^{(R)}}{dt}\right)+\frac{1}{\Delta x}\left(f_{I+1}^{(R)}-f_{I}^{(C)}\right)+\frac{1}{\Delta x}\left(f_{I}^{(R)}-f_{I}^{(C)}\right)=0.$$

Rusanov Flux

- Diagonalizing the mass matrix (via lumping), yields for the center element at I

$$\frac{dq_I^{(C)}}{dt} + \frac{f_I^{(C)} - f_{I-1}^{(C)}}{\Delta x} = 0$$

and for the right element at I

$$\frac{dq_I^{(R)}}{dt} + \frac{f_{I+1}^{(R)} - f_I^{(C)}}{\Delta x} + \frac{f_I^{(R)} - f_I^{(C)}}{\Delta x} = 0$$

that shows that for the center element, we indeed get an upwinding stencil and for the right element we get a downwinding stencil but with a dissipation term.

Rusanov Flux

- We can rewrite the equation for the right element at I as follows:

$$\frac{dq_I^{(R)}}{dt} + \frac{f_{I+1}^{(R)} - f_I^{(R)}}{\Delta x} + 2\left(\frac{f_I^{(R)} - f_I^{(C)}}{\Delta x}\right) = 0$$

that shows the derivative in terms of the variables of the local element (right) and the definition of the jump term.

- Let us now derive the differencing stencil but this time let us use the full mass matrix. The inverse of the mass matrix for this particular case ($N = 1$) is:

$$M^{-1} = \frac{2}{\Delta x}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Rusanov Flux

- Left multiplying the element equations yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} q_{I-1}^{(C)} \\ q_I^{(C)} \end{pmatrix} - \frac{1}{\Delta x} \begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix} \begin{pmatrix} f_{I-1}^{(C)} \\ f_I^{(C)} \end{pmatrix}$$

$$+ \quad \frac{1}{\Delta x} \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} f_{I-1}^{(L)} \\ f_I^{(C)} \end{pmatrix} = 0,$$

  with a similar relation obtained for the right element.
- From this relation we can now show the differencing stencil for the grid point I.
- For the center element, we get

$$\frac{dq_I^{(C)}}{dt} + \frac{f_I^{(C)} - f_{I-1}^{(C)}}{\Delta x} - 2 \left(\frac{f_{I-1}^{(C)} - f_{I-1}^{(L)}}{\Delta x} \right) = 0$$

Rusanov Flux

- and for the right element:

$$\frac{dq_I^{(R)}}{dt} + \frac{f_{I+1}^{(R)} - f_I^{(R)}}{\Delta x} + 4\left(\frac{f_I^{(R)} - f_I^{(C)}}{\Delta x}\right) = 0.$$

Remark

For $N = 1$ the difference between the diagonal and full mass matrices is felt only in the jump terms.

Outline

Matrix Properties

- Recall that the DG elemental equation for the 1D wave equation is given by

$$M_{ij}\frac{dq_j}{dt} - \widetilde{D}_{ij} f_j + F_{ij} f_j^{(*)} = 0$$

- To get the gridpoint representation, we left-multiply this equation by M^{-1} to get

$$\frac{dq_i}{dt} = M_{ik}^{-1}\widetilde{D}_{kj} f_j - M_{ik}^{-1} F_{kj} f_j^{(*)}$$

  where we have now replaced the flux f_j on the far right by the numerical flux in order to denote that we are in fact using some *smart* numerical flux representation.

- Letting

$$\widehat{F} = M^{-1}F \text{ and } \widehat{\widetilde{D}} = M^{-1}\widetilde{D}$$

Matrix Properties

- allows us to write

$$\frac{dq_i}{dt} = \widehat{\widetilde{D}}_{ij} f_j - \widehat{F}_{ij} f_j^{(*)}$$

that can be further simplified, at least for the 1D wave equation with constant speed u, as

$$\frac{dq_i}{dt} = \widehat{D}_{ij}^{DG} f_j$$

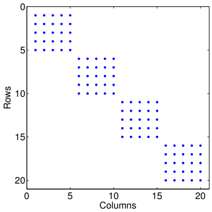
or

$$M_{ij} \frac{dq_i}{dt} = D_{ij}^{DG} f_j.$$

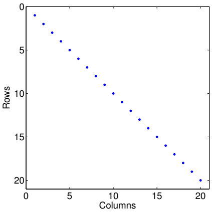
- Note that $D^{DG} = \widetilde{D} - F$ denotes the DG representation of the differentiation matrix, while $\widehat{D}^{DG} = M^{-1} D^{DG}$ represents the right-hand-side matrix for DG.
- Let us now analyze the properties of these matrices.

Matrix Properties

- For solving the 1D wave equation, we will assume periodic boundary conditions and that we shall use a total of $N_p = N_e(N+1) = 20$ gridpoints (for $N_e = 4$ and $N = 4$) to completely cover the domain
- we shall define the initial value problem explicitly in the following section.
- The figure below shows the sparsity pattern for the mass matrix M for both exact ($Q = N + 1$) and inexact ($Q = N$) integration.



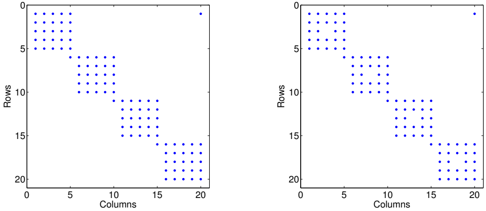
a) Exact



b) Inexact

Matrix Properties

- The figure below shows the sparsity pattern for the differentiation matrix D with Rusanov flux (i.e., upwind-biased numerical flux).



a) Exact b) Inexact

- We can see the outline of the elements in the differentiation matrix.
- Note that there is no change in the structure of this matrix going from exact to inexact integration. Why is that?

Eigenvalues

- Let us now rewrite the equation

$$\frac{dq_i}{dt} = \widehat{D}^{DG}_{ij} f_j$$

as follows

$$\frac{dq_I}{dt} = R_{IJ} q_J$$

where $R = \widehat{D}^{DG}$.

- We can now replace the matrix R by its eigenvalues, that is,

$$Rx = \lambda x$$

to arrive at

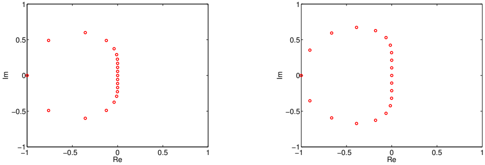
$$\frac{dq_I}{dt} = \lambda_I q_I.$$

- This equation has the analytic (exact) solution

$$q_I = q_0 e^{\lambda_I t}$$

Eigenvalues

- This solution will be bounded (less than infinity) for $Re(\lambda) \leq 0$.
- Let us now look at the eigenvalues of R for the example 1D wave equation problem that we defined earlier ($N_p = 20$ gridpoints).
- The figure below shows the eigenvalues of the right-hand-side matrix R.



a) Exact b) Inexact

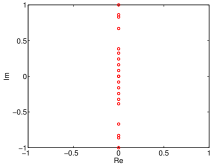
- This matrix is a representation of the complete discretization of the spatial operators.
- Note that the real part of the eigenvalues is very near zero.

Eigenvalues

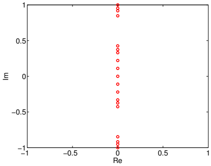
- In fact, it is zero up to machine double precision; the rest of the eigenvalues are located away from the imaginary axis and in the negative real axis (left-hand plane).
- Therefore, this method is quite stable since only a few eigenvalues are near $Re = 0$.
- For comparison, let us now plot the eigenvalues of R using a centered numerical flux.

Eigenvalues

- The figure below shows the eigenvalues of the right-hand-side matrix R using a centered flux.



a) Exact

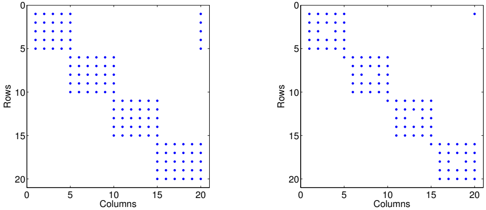


b) Inexact

- This figure shows that the eigenvalues look very similar to those for CG or FD; however, in this particular case, the maximum real eigenvalue is slightly greater than 0 (it is almost at machine double precision).
- This does mean, however, that this method can become unstable.

Matrix Structure

- Before closing this subsection, let us look at the structure of the matrix R.



a) Exact b) Inexact

- Note that unlike CG, the DG right-hand-side matrix is not full (even for exact integration).
- This is because unlike the CG method, the DG method is truly local; that is, the governing equations are satisfied element-wise and the resulting matrix problem is indeed a local one.

Matrix Structure

- The only change in R going from exact to inexact integration is in the boundary conditions (top right corner of the matrix).
- This is due to the differences in the structure of the mass matrices.

Outline

- Problem Statement
- 1D Matrices
- Resulting Element Equations
- Numerical Flux
- Analysis of Discretized Spatial Operators
- Results for the 1D Equations

Results for 1D Wave Equation

- Suppose we wish to solve the continuous partial differential equation
$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0 \qquad \forall x \in [-1, +1]$$
where $f = qu$ and $u = 2$ is a constant.
- Thus, an initial wave $q(x, 0)$ will take exactly $t = 1$ time in order to complete one full revolution (loop) of the domain.
- Since the governing PDE is a hyperbolic system, then this problem represents an initial value problem (IVP or Cauchy Problem).
- We, therefore, need an initial condition.
- Let it be the following Gaussian

$$q(x, 0) = e^{-cx^2}$$

Boundary Conditions and Norms

- This problem also requires a boundary condition: let us impose periodic boundary conditions, meaning that the domain at $x = +1$ should wrap around and back to $x = -1$.
- Let us define the normalized L^2 error norm as follows

$$L^2 = \sqrt{\frac{\sum_{k=1}^{N_p} \left(q_k^{numerical} - q^{exact}(x_k)\right)^2}{\sum_{k=1}^{N_p} q^{exact}(x_k)^2}}$$

where $k = 1, ..., N_p$ are $N_p = N_e(N+1)$ global gridpoints and $q^{numerical}$ and q^{exact} are the numerical and exact solutions after one full revolution of the wave.

- Note that the wave should just stop where it began without changing shape (in a perfect world).

Time-Integration

- To solve the time-dependent portion of the problem we use the 2nd order Runge-Kutta (RK) method: for $\frac{dq}{dt} = R(q)$ let

$$q^{n+1/2} = q^n + \frac{\Delta t}{2} R(q^n)$$

$$q^{n+1} = q^n + \frac{\Delta t}{2} \left(R(q^{n+1/2}) + R(q^n) \right)$$

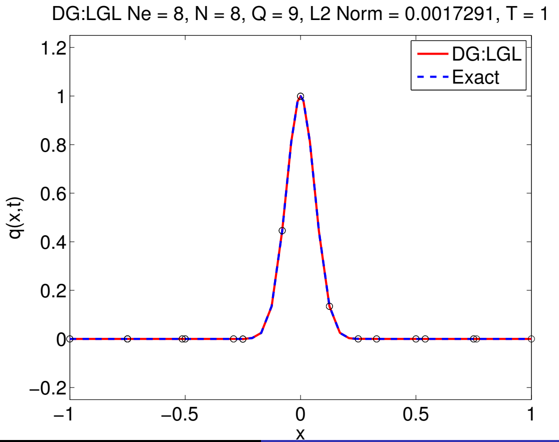
  but of course one can use other methods (i.e., higher order RK methods).
- Recall that the Courant number

$$C = u \frac{\Delta t}{\Delta x}$$

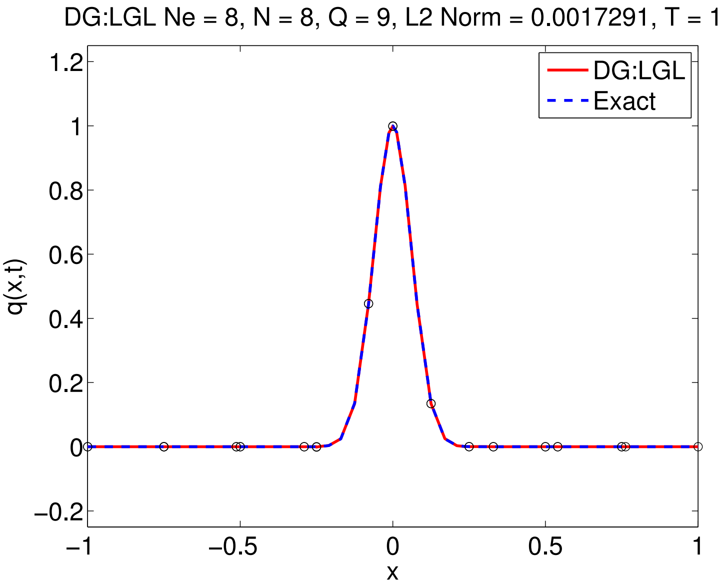
  must be within a certain value for stability.
- For the 2nd order RK method we use time-steps such that $C \leq \frac{1}{4}$.
- For Δx we take the minimum value of $x_{I+1} - x_I$ for all points $I = 1, ..., N_p - 1$ (since the grid spacing is non-uniform).

Solution Accuracy

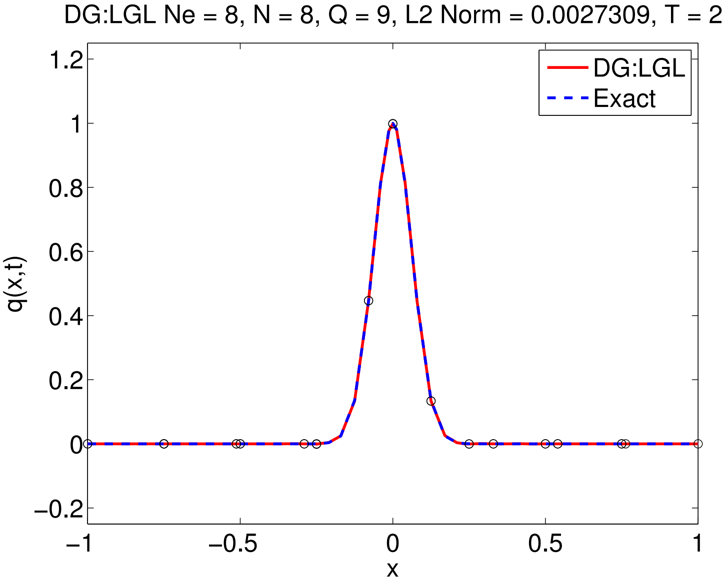
- The figure below shows the snapshot of the exact and DG numerical solutions (with Rusanov flux) after one revolution ($t = 1$) using $N = 8$ order polynomials and $N_e = 8$ elements for a total of $N_p = 72$ gridpoints.



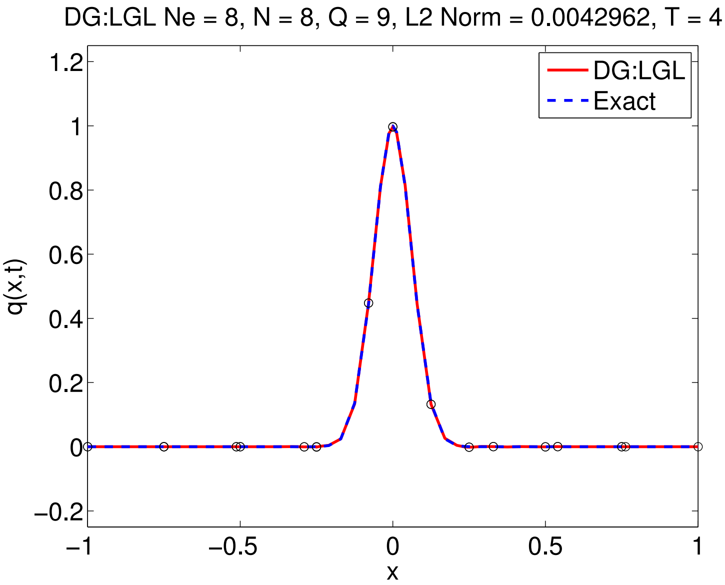
Movie: 1 Revolution



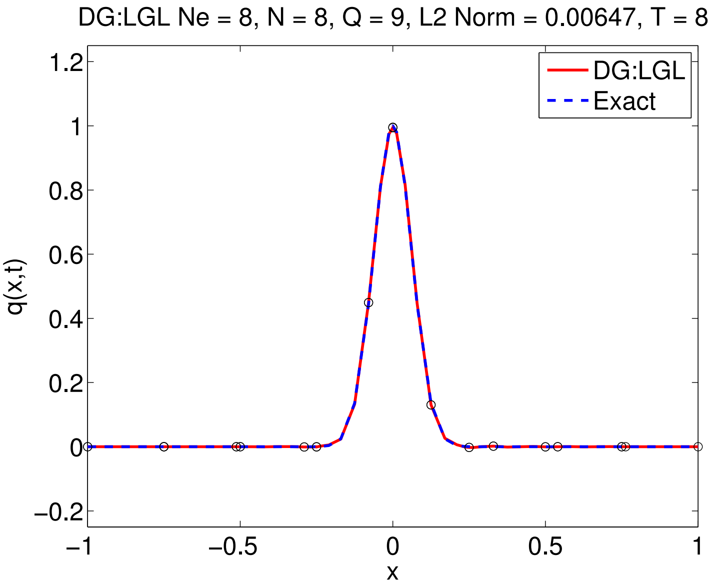
Movie: 2 Revolutions



Movie: 4 Revolutions

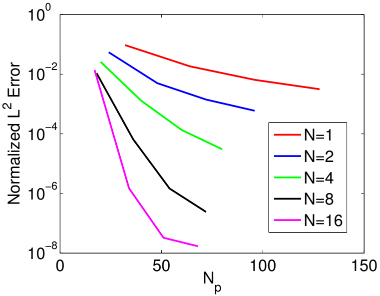


Movie: 8 Revolutions



Convergence Rates: Lobatto Points

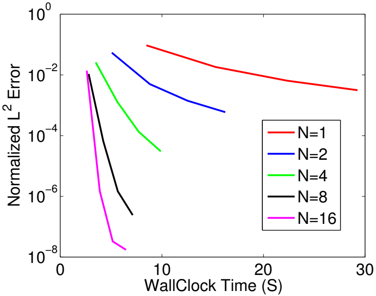
- There is very little difference between the exact and numerical solutions.
- This is corroborated by the convergence rates shown in the following figure for various polynomial orders, N, for a total number of gridpoints N_p where, for 1D, $N_p = N_e(N+1)$.



- The question we need to answer is whether using high-order is more efficient than low-order.

Convergence Rates

- The figure below shows the L^2 error norm as a function of wallclock time in seconds.



- The high-order methods are, in fact, more efficient to reach a certain level of accuracy than the low-order methods.
- To achieve an accuracy of 10^{-4} or 10^{-8} is most efficiently reached with $N = 16$; $N = 1$ and $N = 2$ would require prohibitively large computational times to achieve these levels of accuracy.

1D Shallow Water Equations

- Let us consider the 1D shallow water equations written in conservation (flux) form

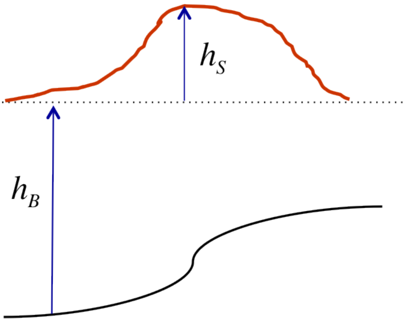
$$\frac{\partial h_S}{\partial t} + \frac{\partial}{\partial x}(U) = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(\frac{U^2}{h} + \frac{1}{2}gh_S^2\right) = -gh_B\frac{\partial h_S}{\partial x}$$

where $h = h_S + h_B$ is the total height of the fluid, h_S is the surface height measured from the mean level, and h_B is the distance from the mean level to the bottom, g is the gravitational constant, and $U = hu$ is the momentum where u is the velocity.

1D Shallow Water Equations

- Where h_B and h_S are defined as follows



- The shallow water equations can be rewritten in the following compact vector form

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q})}{\partial x} = S(\mathbf{q})$$

1D Shallow Water Equations

- where

$$\mathbf{q} = \begin{pmatrix} h_S \\ U \end{pmatrix}, \quad \mathbf{F}(\mathbf{q}) = \begin{pmatrix} U \\ \frac{U^2}{h} + \frac{1}{2} g h_S^2 \end{pmatrix}, \quad S(\mathbf{q}) = \begin{pmatrix} 0 \\ -h_B \frac{\partial h_S}{\partial x} \end{pmatrix}$$

- Inserting the basis function expansion into the compact vector form of the equations, multiplying by a test function ψ and integrating within each element Ω_e yields

$$\int_{\Omega_e} \psi \left(\frac{\partial \mathbf{q}_N^{(e)}}{\partial t} + \frac{\partial \mathbf{F}\left(\mathbf{q}_N^{(e)}\right)}{\partial x} \right) d\Omega_e = \int_{\Omega_e} \psi S \left(\mathbf{q}_N^{(e)}\right) d\Omega_e.$$

- Integrating by parts gives

$$\int_{\Omega_e} \psi \frac{\partial \mathbf{q}_N^{(e)}}{\partial t} d\Omega_e + \int_{\Omega_e} \frac{d}{dx} \left(\psi \mathbf{F}\left(\mathbf{q}_N^{(e)}\right) \right) d\Omega_e$$
$$- \int_{\Omega_e} \frac{d\psi}{dx} \mathbf{F}\left(\mathbf{q}_N^{(e)}\right) d\Omega_e = \int_{\Omega_e} \psi S \left(\mathbf{q}_N^{(e)}\right) d\Omega_e.$$

1D Shallow Water Equations

- Evaluating the second integral yields

$$\int_{\Omega_e} \psi \frac{\partial \mathbf{q}_N^{(e)}}{\partial t} d\Omega_e + \sum_{k=1}^{N_{faces}} \left[\widehat{\mathbf{n}}_{\Gamma_e}^{(e,k)} \psi \mathbf{F} \left(\mathbf{q}_N^{(e,k)} \right) \right]_{\Gamma_e}$$
$$- \int_{\Omega_e} \frac{d\psi}{dx} \mathbf{F} \left(\mathbf{q}_N^{(e)} \right) d\Omega_e = \int_{\Omega_e} \psi S \left(\mathbf{q}_N^{(e)} \right) d\Omega_e$$

where $\mathbf{F}\left(\mathbf{q}_N^{(e,k)}\right)$ denotes the numerical flux (Riemann solver); for simplicity we can assume that it is the Rusanov flux defined as in the 1D wave equation.

- Writing this eq. in matrix-vector form yields

$$M_{ij}^{(e)} \frac{d\mathbf{q}_j^{(e)}}{dt} + \sum_{k=1}^{N_{faces}} F_{ij}^{(e,k)} \mathbf{F} \left(\mathbf{q}_j^{(e,k)} \right) - \widetilde{D}_{ij}^{(e)} \mathbf{F} \left(\mathbf{q}_j^{(e)} \right) = M_{ij}^{(e)} S \left(\mathbf{q}_j^{(e)} \right)$$

where $S_i^{(e)}$ is the source function vector.

1D Shallow Water Equations

- At this point we have already seen every term here except for $\mathbf{q}_j^{(e)}$, $\mathbf{F}\left(\mathbf{q}_j^{(e)}\right)$, $\mathbf{F}\left(\mathbf{q}_j^{(e,k)}\right)$, and $S\left(\mathbf{q}_j^{(e)}\right)$; let us now explicitly write these terms.
- Beginning with the vector $\mathbf{q}_j^{(e)}$ we note that it is nothing more than the expansion coefficients but now defined for the 1D shallow water equations defined as

$$\mathbf{q}_j^{(e)} = \begin{pmatrix} h_{S,j}^{(e)} \\ \\ U_j^{(e)} \end{pmatrix}.$$

- The term $S\left(\mathbf{q}_j^{(e)}\right)$ is expressed as follows

$$S\left(\mathbf{q}_j^{(e)}\right) = -\begin{pmatrix} 0 \\ h_{B,j}^{(e)} \left(\sum_{k=0}^{N} \left(\frac{d\psi_k}{d\xi}\frac{d\xi}{dx}\right) h_{S,k}^{(e)}\right) \end{pmatrix}.$$

1D Shallow Water Equations

- The term $\mathbf{F}\left(\mathbf{q}_j^{(e)}\right)$ is a bit more complicated and is expressed as follows

$$\mathbf{F}\left(\mathbf{q}_j^{(e)}\right) = \left(\begin{array}{c} U_j^{(e)} \\ \frac{U_j^{(e)} U_N^{(e)}}{h_N^{(e)}} + \frac{1}{2} g h_{S,j}^{(e)} h_{S,N}^{(e)} \end{array} \right).$$

- To define the numerical flux term let us first write the flux in using the following notation

$$\mathbf{F}\left(\mathbf{q}_j^{(e)}\right) = \left(\begin{array}{c} F_h \\ F_U \end{array} \right).$$

- which now allows us to define the numerical flux (Rusanov) as follows

$$\mathbf{F}\left(\mathbf{q}_j^{(e,k)}\right) = \frac{1}{2} \left(\begin{array}{c} \{F_h\}^{(e,k)} - \widehat{\mathbf{n}}_{\Gamma_e}^{(e,k)} \mid \lambda_{max} \mid [\![h_{S,j}]\!]^{(e,k)} \\ \{F_U\}^{(e,k)} - \widehat{\mathbf{n}}_{\Gamma_e}^{(e,k)} \mid \lambda_{max} \mid [\![U_j]\!]^{(e,k)} \end{array} \right).$$

1D Shallow Water Equations

- where we have used classical DG notation with the above delimiters defined as follows:

$$\{F\}^{(e,k)} = F^{(e)} + F^{(k)},$$

$$[\![\mathbf{q}]\!]^{(e,k)} = \mathbf{q}^{(k)} - \mathbf{q}^{(e)}$$

- and λ_{max} is the maximum eigenvalue of the 1D shallow water equations which is in fact $|u| + \sqrt{gh}$; this term represents the maximum propagation speed of all possible waves in the system.

1D Linearized Shallow Water Equations

- Suppose we wish to solve the one-dimensional linearized shallow water equations

$$\frac{\partial}{\partial t}\begin{pmatrix} h_S \\ U \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} U \\ g h_B h_S \end{pmatrix} = \begin{pmatrix} 0 \\ h_S \frac{\partial h_B}{\partial x} \end{pmatrix}$$

- where $h = h_S + h_B$ is the total height of the water column, and h_S is the height of the fluid from mean (sea) level to the surface of the wave, h_B is the depth of the bathymetry, g is the gravitational constant, and $U = hu$ is the momentum.
- Since the governing PDE is a hyperbolic system, then this problem represents an initial value problem (IVP or Cauchy Problem).
- We, therefore, need an initial condition.

1D Linearized Shallow Water Equations

- Note that setting $g = h_B = 1$ the following relations

$$h_S(x,t) = \frac{1}{2}\cos c\pi x \cos c\pi t \quad U(x,t) = \frac{1}{2}\sin c\pi x \sin c\pi t$$

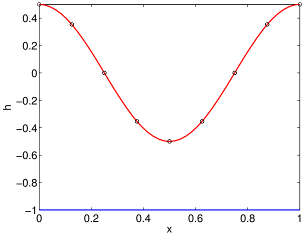
  satisfy an analytic solution to this linear system for any constant c where the domain is defined to be $(x,t) \in [0,1]^2$.
- From this relation we can produce the following initial conditions

$$h_S(x,0) = \frac{1}{2}\cos c\pi x, \quad U(x,t) = 0$$

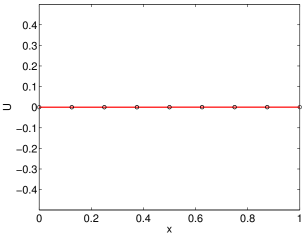
  that we can use to begin the numerical solution; a homogeneous Dirichlet boundary condition for the momentum is only satisfied for integer values of c.
- All the results shown in the next section assume a value of $c = 2$.

1D Linearized Shallow Water Equations

- The analytic solution at time $T = 1$ using $c = 2$



a) h



b) U

- Let us impose no-flux boundary conditions which are satisfied by the analytic solution given previously, i.e., at $x = 0$ and $x = 1$, the momentum is zero.
- Let us define the L^2 error norm as follows

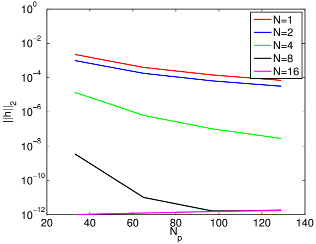
$$L^2 = \sqrt{\sum_{e=1}^{N_e} \sum_{i=0}^{N} \left(q_{N,i}^{(e)} - q_{E,i}^{(e)} \right)^2}$$

1D Linearized Shallow Water Equations

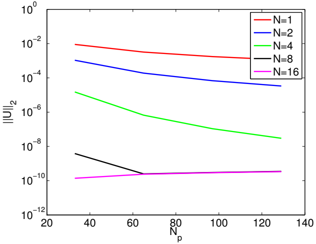
- where $e = 1, \ldots, N_e$ are the number of elements and $i = 0, \ldots, N$ are the interpolation points and q_N and q_E denote the numerical and exact solutions.
- In addition, let us define the mass conservation measure as follows
$$\Delta M = \mid \mathrm{Mass(t)} - \mathrm{Mass(0)} \mid$$
where $\mathrm{Mass(t)}$ is the total mass at time t and $\mathrm{M(0)}$ the mass at the initial time.
- The mass is defined as follows:
$$\mathrm{Mass(t)} = \sum_{e=1}^{N_e} \sum_{i=0}^{N} \left(h_{S,i}^{(e)}(t) + h_{B,i}^{(e)} \right).$$

1D Linearized Shallow Water Equations

- The figures below shows the convergence rates for various polynomial orders, N, for a total number of gridpoints N_p where, for 1D, $N_p = N_e\,(N+1)$.



a) h error

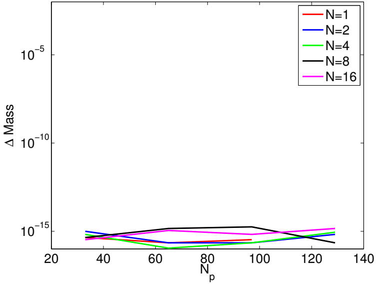


b) U error

- In this example, an RK4 time step of $\Delta t = 1 \times 10^{-3}$ is used for all the simulations and the norms are computed at a final time of $t = 1$ using exact integration with LGL interpolation points.

1D Linearized Shallow Water Equations

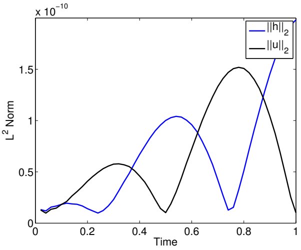
- The figure below shows the mass loss

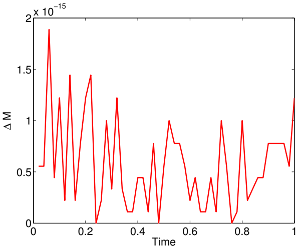


- Let us now look at one specific simulation comprised of $N_e = 8$ $N = 8$ $Q = 9$ and $\Delta t = 1 \times 10^{-4}$.

1D Linearized Shallow Water Equations

- The figure below shows the time history of the h_S and U L^2 error norms (left panel) and the change in mass conservation (right panel).





Summary of DG Lecture 2

- We covered 1D Mass, Differentiation, and Flux Matrices.
- We discussed the Resulting Element Equations.
- We covered Numerical Flux.
- We Analyzed the Spatial Operators.
- We showed some results for the 1D Wave and Shallow Water Equations.
- DG Lecture 3 (NMW Lecture 6) will cover extensions to 2D.
- NMW Lecture 7 (by Shiva Gopalakrishnan) will cover wetting and drying with DG methods.
- NMW Lecture 8 (by Michal Kopera) will cover Adaptive DG methods.