Gene Golub SIAM Summer School 2012
Lecture 3: 2D Application of DG Methods

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Modal Interpolation

- In 1D, we already saw that interpolation can be defined in either modal or nodal space.
- When modal functions are used this means that we are representing any function as the sum of multiple kinds of waves having $N+1$ distinct frequencies and amplitudes.
- For example, the modal interpolation of a function $f(x)$ is given as

$$f(x) = \sum_{i=0}^{N} \phi_i(x) \widetilde{f}_i.$$

- The modal functions can be defined to be either the simple monomials $\phi_i^{1D}(x) = x^i \ \forall \ i = 0, ..., N$ or the Jacobi polynomials that are orthogonal in the interval $x \in [-1, +1]$ $\phi_i^{1D}(x) = P_i^{(\alpha,\beta)}(x) \ \forall \ i = 0, ..., N$.

Nodal Interpolation

- In 2D, we simply apply a tensor product of the 1D modal functions as follows

$$\phi_{ij}^{2D}(x,y) = \phi_i^{1D}(x)\phi_j^{1D}(y), \; \forall \; (i,j) = 0, ..., N.$$

- Recall that when nodal interpolation is used, we actually use the value of the function at specific interpolation points as our expansion coefficients.
- For example, the nodal interpolation of a function $f(x)$ is given as

$$f(x) = \sum_{i=0}^{N} L_i(x) f_i$$

where L are Lagrange polynomials and $f_i = f(x_i)$ is the value of the function $f(x)$ evaluated at $x = x_i$.

Quadrilateral Elements

- Using the same idea that we just introduced for the construction of 2D modal interpolation functions on the quadrilateral, we can write the tensor product of the 1D Lagrange polynomials as follows

$$L_{ij}^{2D}(x,y) = L_i^{1D}(x) L_j^{1D}(y), \ \forall\ (i,j) = 0, ..., N.$$

- It becomes immediately obvious why the quadrilateral is the element of choice in most high-order element-based Galerkin models.
- The popularity of quadrilateral elements is very much a direct consequence of the ease with which interpolation functions can be constructed.
- However, another reason for their ubiquity is due to their computational efficiency.

Quadrilateral Elements

- Methods for multi-dimensions that are based on tensor products tend to be very fast because a very nice trick (called sum factorization) can be used to reduce the cost of the method from $\mathcal{O}(N^{d+2})$ to $\mathcal{O}(N^{d+1})$ where N is the order of the interpolation functions and d is the dimension of the space.
- For non-tensor product based methods the cost is $\mathcal{O}(N^{d+2})$.
- Let us take a close look at nodal basis functions on the quadrilateral based on Lobatto points.

Quadrilateral Elements

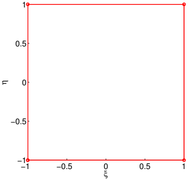
- Let us now look at what the Lagrange polynomial (nodal) basis functions look like in two-dimensions (2D).
- To simplify the exposition, let us renumber the 2D basis functions as follows: let

$$\psi_i(\xi,\eta) = L_j^{1D}(\xi) L_k^{1D}(\eta), \; \forall \, (j,k) = 0,...,N, \; i = 1,...,(N+1)^2$$

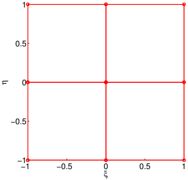
where i is associated with j,k as follows: $i = kN + j + 1$.

Quadrilateral Elements

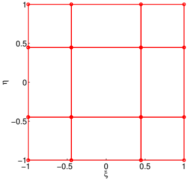
- Lobatto grids of different orders



$N = 1$



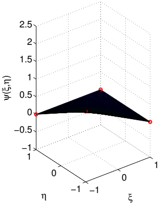
$N = 2$



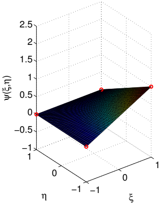
$N = 3$

Quadrilateral Elements

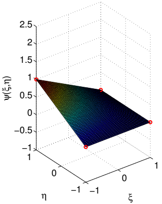
- The figure below shows the $N = 1$ basis functions associated with the four interpolation points (vertices)



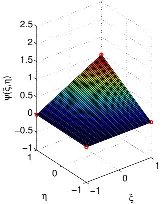
ψ_1



ψ_2



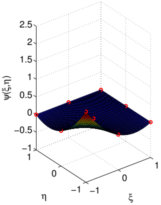
ψ_3



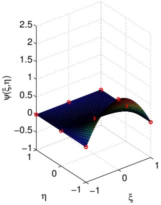
ψ_4

Quadrilateral Elements

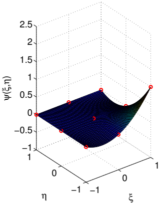
- The figure below shows the first four basis functions for $N = 2$



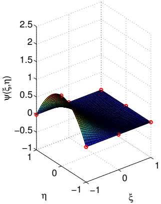
ψ_1



ψ_2



ψ_3



ψ_4

Outline

Modal Interpolation

- Previously we saw that the modal functions in the 1D space $x \in [-1,+1]$ are obtained by solving the singular Sturm-Liouville eigenvalue problem.
- In 2D, we solve the exact same eigenvalue problem but in the domain $T = (x,y) | x, y \geq -1; x + y \leq 0$ which is nothing more than (x,y) values that lie in the right-triangle given in the following figure:

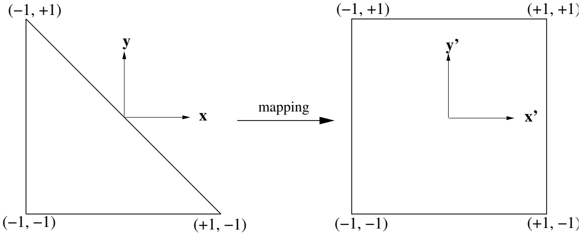


Figure: The mapping from the canonical right triangle (x,y) to the canonical quadrilateral (x',y').

Modal Interpolations

- It turns out that these eigenfunctions are in fact

$$\phi_{ij}(x,y) = \sqrt{\frac{(2i+1)(i+j+1)}{2}} \left(\frac{1-y'}{2}\right)^i P_i^{(0,0)}(x') P_j^{(2i+1,0)}(y')$$

$\forall\ (i,j) \geq 0; i+j \leq N$

- where $P_i^{(0,0)}$ are Legendre polynomials and $P_j^{(2i+1,0)}$ are Jacobi polynomials with $\alpha = 2i+1$ and $\beta = 0$, and

$$(x', y') = \left(\frac{2x+y+1}{1-y}, y\right)$$

represents the mapping of (x, y) on the triangle to the unit square $(x', y') \in [-1, +1]^2$.

- These polynomials are orthonormal on the unit triangle and are known as the *Proriol-Koornwinder-Dubiner* (PKD) polynomials; they are the natural basis on the triangle and are analogous to the Jacobi polynomials on the line $x \in [-1, +1]$.

Modal Interpolation

- Furthermore, the Jacobi polynomials are defined as follows:

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}$$

- where, for integer positive integer values, the Gamma function yields

$$\Gamma(n) = (n-1)!$$

- These are the modal polynomials that we shall use to construct the Lagrange polynomials on the triangle.

Nodal Interpolation

- Previously we saw that we can write the Lagrange polynomials as
$$\phi_i(x) = \sum_{j=0}^{N} \phi_i(x_j) L_j(x)$$
where $x_j \; \forall \; j = 0, ..., N$ are any set of interpolation points at which the modal functions, $\phi(x)$, have been sampled.
- In a similar fashion, we can represent the PKD polynomials as expansions of the Lagrange polynomials but before doing so, let us rewrite the PKD polynomials as follows
$$\phi_{ij}(x,y) = \sqrt{\frac{(2i+1)(i+j+1)}{2}} \left(\frac{1-y'}{2}\right)^i P_i^{(0,0)}(x') P_j^{(2i+1,0)}(y')$$
$\forall \; (i,j) \geq 0; i+j \leq N$ with $k = i + j(N+1) + 1$ that will now allow us to take the 2D polynomials as one long vector (with dependence only on k).

Nodal Interpolation

- Using this form of the PKD polynomials, we can expand them using the Lagrange polynomials that we are looking for as follows
$$\phi_i(x,y) = \sum_{j=1}^{M} \phi_i(x_j,y_j) L_j(x,y)$$
where $M = \frac{1}{2}(N+1)(N+2)$ are the number of points on the triangle required to form an Nth order interpolant.
- Taking advantage of the fact that $V_{ij} = \phi_i(x_j,y_j)$ is the generalized *Vandermonde* matrix allows us to write
$$V_{ij} L_j(x,y) = \phi_i(x,y)$$
and left-multiplying by the inverse of V yields
$$L_i(x,y) = V_{ij}^{-1} \phi_j(x,y).$$
- However, we know from Linear Algebra that the inverse of a matrix is never actually computed unless absolutely necessary.

Nodal Interpolation

- We merely write the Lagrange polynomial in this form for convenience but will compute them differently.
- Using Eq. (17) we can see that for specific interpolation points (x_j, y_j) for $j = 1, ..., M$ we get

$$\begin{pmatrix} \phi_1(x,y) \\ \phi_2(x,y) \\ \vdots \\ \phi_M(x,y) \end{pmatrix} = \begin{pmatrix} \phi_1(x_1,y_1) & \phi_1(x_2,y_2) & \dots & \phi_1(x_M,y_M) \\ \phi_2(x_1,y_1) & \phi_2(x_2,y_2) & \dots & \phi_2(x_M,y_M) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_M(x_1,y_1) & \phi_M(x_2,y_2) & \dots & \phi_M(x_M,y_M) \end{pmatrix}$$

- Using Cramer's rule, we can solve for L_i as follows

$$L_i(x,y) = \frac{\det(V_i(x,y))}{\det(V)}$$

Nodal Interpolation

- where

$$V = \begin{pmatrix} \phi_1(x_1,y_1) & \dots & \phi_1(x_{i-1},y_{i-1}) & \phi_1(x_i,y_i) & \phi_1(x_{i+1},y_{i+1}) & \dots & \phi_1(x_M,y_M) \\ \phi_2(x_1,y_1) & \dots & \phi_2(x_{i-1},y_{i-1}) & \phi_2(x_i,y_i) & \phi_2(x_{i+1},y_{i+1}) & \dots & \phi_2(x_M,y_M) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_M(x_1,y_1) & \dots & \phi_M(x_{i-1},y_{i-1}) & \phi_M(x_i,y_i) & \phi_M(x_{i+1},y_{i+1}) & \dots & \phi_M(x_M,y_M) \end{pmatrix}$$

is the Vandermonde matrix written in a slightly different way and

$$V_i(x,y) = \begin{pmatrix} \phi_1(x_1,y_1) & \dots & \phi_1(x_{i-1},y_{i-1}) & \phi_1(x,y) & \phi_1(x_{i+1},y_{i+1}) & \dots & \phi_1(x_M,y_M) \\ \phi_2(x_1,y_1) & \dots & \phi_2(x_{i-1},y_{i-1}) & \phi_2(x,y) & \phi_2(x_{i+1},y_{i+1}) & \dots & \phi_2(x_M,y_M) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_M(x_1,y_1) & \dots & \phi_M(x_{i-1},y_{i-1}) & \phi_M(x,y) & \phi_M(x_{i+1},y_{i+1}) & \dots & \phi_M(x_M,y_M) \end{pmatrix}$$

is the Vandermonde matrix (V) with the ith column replaced by $\phi_i(x,y)$.

Nodal Interpolation

- This equation tells us everything that we need to know about Lagrange polynomials.

$$L_i(x,y) = \frac{\det(V_i(x,y))}{\det(V)}$$

- For example, we see that for $(x,y) = (x_i, y_i)$, $L_i(x_i, y_i) = 1$ since $V_i(x_i, y_i) = V$, that is, the numerator and denominator are equal.
- Similarly, we can see that if we pick another (x,y) that is in the set of (x_j, y_j) $j = 1, \ldots, M$ this means that two columns of $V_i(x_j, y_j)$ will be exactly the same; in other words the matrix $V_i(x_j, y_j)$ will be singular and $\det(V_i(x_j, y_j)) = 0$ which will then enforce $L_i(x_j) = 0$ for $i \neq j$.
- This proves that $L_i(x,y)$ must be cardinal.

Nodal Interpolation

- This equation also tells us that to have a well-behaved Lagrange polynomial, L_i, we must be sure that the Vandermonde matrix, V, is well-conditioned (i.e., non-singular).
- There are two possibilities for L to become ill-behaved - this can happen if V becomes singular or numerically ill-conditioned (with finite precision mathematics, a very small number can be interpreted as machine zero on a computer).
- This situation arises when we choose modal functions, $\phi_i(x, y)$, that are not linearly independent.
- Recall that linear dependence means the following: the basis vectors ϕ are said to be linearly dependent if

$$\sum_{i=0}^{N} a_i \phi_i = 0$$

can be satisfied for a_i not all equal to zero.

Nodal Interpolation

- This means that you can form one of the functions, say ϕ_k, as a linear combination of the other functions ϕ_i.
- The implication of this is the following: if we applied row-reduction (e.g., Gaussian elimination) to V, then one of the rows will be all zeros meaning that this row is redundant (contains no information); a matrix with either a row or a column containing all zeros is singular and therefore $\det(V) = 0$ in this case.
- The second possibility for L to become ill-behaved occurs when $\det(V(x, y))$ achieves large values that are then not *normalized* by V.
- Previously we saw that the Lagrange polynomials produce large values near the endpoints when equi-spaced points are used.

Nodal Interpolation

- The reason for this is partially due to the fact that the $\det(V)$ decreases with N for equi-spaced points and therefore cannot normalize the large values of $\det(V(x,y))$.
- The reason why this does not occur with the Lobatto points is because these points maximize $\det(V)$ and hence then normalizes $\det(V(x,y))$ in potentially troublesome regions of the interpolation region (such as at the endpoints).

Quality of Nodal Interpolation

- For a Lagrange interpolation function a good measure of its interpolation strength is the Lebesgue (pronounced Lebeck) function

$$\Lambda_N(x,y) = \sum_{i=0}^{N} |L_i(x,y)|$$

- and its associated Lebesgue constant defined as

$$\Lambda_N = max \left(\sum_{i=0}^{N} |L_i(x,y)| \right)$$

where the maximum is obtained for the entire domain $x \in D$ where D is the domain of interest.

Quality of Nodal Interpolation

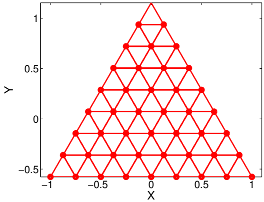
- Let's consider the interpolation of the 2D Runge function

$$f(x,y) = \frac{1}{1+50\left((x+1/2)^2+(y+1/2)^2\right)}.$$

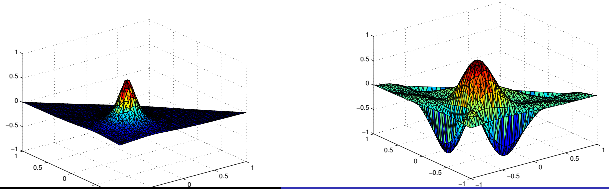
defined on the canonical right-triangle
$T = (x,y)|x,y \geq -1; x+y \leq 0.$

Quality of Nodal Interpolation

- The figure below shows the 8th order ($N = 8$) equally-spaced points on the canonical equilateral triangle.

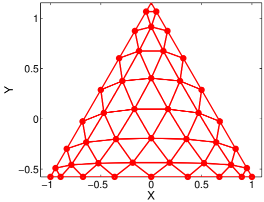


- This figure shows the Runge function (left panel) and 8th order equally-space approximation (right panel)

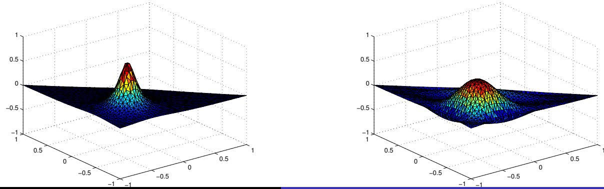


Quality of Nodal Interpolation

- This figures shows the set of 8th order Fekete points on the canonical equilateral triangle.



- This figure shows the Runge function (left panel) and 8th order Fekete points approximation (right panel).

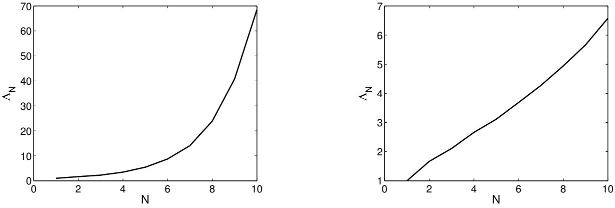


Quality of Nodal Interpolation

- These points are designed to be analogous to the Lobatto points; in fact, they are exactly the Lobatto points along the edges but are optimized in a specific fashion in the interior.
- Clearly, the Lagrange interpolation based on 8th order Lobatto-like (Fekete) points do not exhibit the severe oscillations that are exhibited by the equally-spaced points.
- To understand why the Lobatto-like points work better, let us compute their respective Lebesgue constants.

Quality of Nodal Interpolation

- This figure shows the Lebesgue constants for various Lagrange polynomial orders for equally-spaced points (a) and the Fekete points (b).

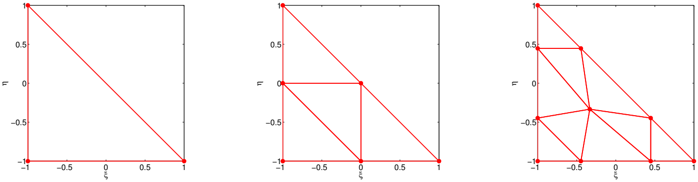


a) Equally-Spaced b) Fekete

- This figure shows that the Lebesgue constant for the equally-spaced points increases exponentially while for the Lobatto-like points, the Lebesgue constant does not grow at such an exaggerated rate.

Examples of Triangular Basis Functions

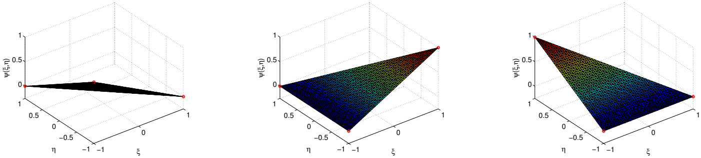
- Let us now look at what the Lagrange polynomial (nodal) basis functions look like for the triangle.
- As we did for the quadrilateral, we shall rename the Lagrange polynomials as follows $\psi_i(\xi, \eta)$ where (ξ, η) are the points in the computational (canonical) space $\{-1 \le \xi \le 1, -1 \le \xi + \eta \le 0\}$.
- The figure below shows the interpolation points for elements of orders $N = 1, 2$, and 3.



a) $N = 1$ b) $N = 2$ c) $N = 3$

Examples of Triangular Basis Functions

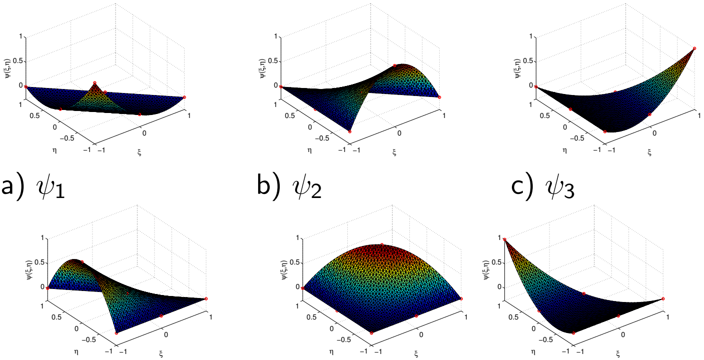
- The figure below shows the $N = 1$ basis functions associated with the three interpolation points (vertices)



a) ψ_1 b) ψ_2 c) ψ_3

Examples of Triangular Basis Functions

- The figure below shows the $N = 2$ basis functions associated with the six interpolation points



a) ψ_1 b) ψ_2 c) ψ_3

d) ψ_4 e) ψ_5 f) ψ_6

Outline

Problem Statement

- Let us begin our discussion with the two-dimensional wave equation which can be written in the following form

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0 \qquad (1)$$

- where $q = q(\mathbf{x}, t)$ is our scalar solution variable, $\nabla = \left(\frac{\partial}{\partial x} \widehat{\mathbf{i}} + \frac{\partial}{\partial y} \widehat{\mathbf{j}} \right)$ is the 2D gradient operator, $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$ are the directional unit vectors in 2D Cartesian space.
- The form described in Eq. (1) is known as the advection or non-conservation form of the equation.
- The alternate form

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\mathbf{u}) = q\nabla \cdot \mathbf{u} \qquad (2)$$

Problem Statement

- Note that we can now recast this equation in general form

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{f} = S(q)$$

- where $\mathbf{f} = (q\mathbf{u})$ and $S(q) = q\boldsymbol{\nabla} \cdot \mathbf{u}$.
- For the case that the velocity field is divergence-free ($\boldsymbol{\nabla} \cdot \mathbf{u} = 0$) we obtain the conservation law

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{f} = 0.$$

- To simplify the discussion of the DG method we will assume that the velocity is indeed divergence-free.
- We will discuss the construction of the DG method for the wave equation in conservation form.
- The reason why we use the conservation form and not the advection form is that DG requires the equations in strict conservation form.

Weak Integral Form

- To construct the weak integral form, we begin by expanding the solution variable q as follows

$$q_N^{(e)}(\mathbf{x},t) = \sum_{j=1}^{M_N} \psi_j(\mathbf{x}) q_j^{(e)}(t)$$

with

$$\mathbf{u}_N^{(e)}(\mathbf{x}) = \sum_{j=1}^{M_N} \psi_j(\mathbf{x}) \mathbf{u}_j^{(e)}$$

and $\mathbf{f}_N^{(e)} = q_N^{(e)} \mathbf{u}_N^{(e)}$ where $\mathbf{u}(\mathbf{x}) = u(\mathbf{x})\widehat{\mathbf{i}} + v(\mathbf{x})\widehat{\mathbf{j}}$ is the velocity vector with u and v components along $\widehat{\mathbf{i}}$ (x-direction) and $\widehat{\mathbf{j}}$ (y-direction).

Weak Integral Form

- The integer M_N in the basis function approximation

$$q_N^{(e)}(\mathbf{x},t) = \sum_{j=1}^{M_N} \psi_j(\mathbf{x}) q_j^{(e)}(t)$$

  is a function of N which determines the number of points inside each element Ω_e.
- For the case of quadrilaterals (which we use in this talk) $M_N = (N+1)^2$.
- For the case of triangles $M_N = \frac{1}{2}(N+1)(N+2)$.

Problem Statement

- Next, we substitute $q_N^{(e)}$, $\mathbf{u}_N^{(e)}$, and $\mathbf{f}_N^{(e)}$ into the PDE, multiply by a test function and integrate within the local domain Ω_e yielding the weak integral form: find $q \in L^2$ such that

$$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t}\, d\Omega_e + \int_{\Gamma_e} \psi_i \widehat{\mathbf{n}}^{(e,k)} \cdot \mathbf{f}_N^{(*,k)}\, d\Gamma_e - \int_{\Omega_e} \boldsymbol{\nabla}\psi_i \cdot \mathbf{f}_N^{(e)}\, d\Omega_e = 0$$

$$\forall \psi \in L^2$$

- where $= \mathbf{f}^{(*,k)}$ is the numerical flux function which we will assume to be the Rusanov flux

$$\mathbf{f}^{(*,k)} = \frac{1}{2}\left[\mathbf{f}^{(k)} + \mathbf{f}^{(e)} - \mid \lambda \mid \widehat{\mathbf{n}}^{(e,k)} \left(q^{(k)} - q^{(e)}\right)\right]$$

Problem Statement

- where the superscript (k) denotes the side/edge neighbor of (e), $\widehat{\mathbf{n}}^{(e,k)}$ is the unit normal vector of the edge shared by elements e and k, and λ is the maximum wave speed of the system; for the wave equation λ is just the maximum normal velocity $\widehat{\mathbf{n}} \cdot \mathbf{u}|_{\Gamma_e}$.

2D Basis Functions

- Once again, we construct the 2D basis functions on the reference elements as a tensor product of the 1D basis functions such as

$$\psi_i(\xi,\eta) = h_j(\xi) \otimes h_k(\eta)$$

- where h are the 1D basis functions we have discussed already, $\otimes$ is the tensor product, and the 1D indices vary as follows $j,k = 0,...,N$ with the 2D index varying as $i = 1,...,M_N$ where $M_N = (N+1)^2$.
- To get from the 1D local indices (j,k) to the 2D local index i requires the mapping $i = k(N+1) + j + 1$.
- With this definition in place, we can now expand the solution variable q as follows

$$q_N^{(e)}(\mathbf{x},t) = \sum_{j=1}^{M_N} \psi_j(\mathbf{x}) q_j^{(e)}(t)$$

2D Basis Functions

- which implies the approximation of the gradient operator to be

$$\nabla q_N^{(e)}(\mathbf{x},t) = \sum_{j=1}^{M_N} \nabla \psi_j(\mathbf{x}) q_j^{(e)}(t)$$

- where the partial derivatives are defined as follows

$$\frac{\partial q_N^{(e)}(\mathbf{x},t)}{\partial x} = \sum_{j=1}^{M_N} \frac{\partial \psi_j(\mathbf{x})}{\partial x} q_j^{(e)}(t)$$

- and

$$\frac{\partial q_N^{(e)}(\mathbf{x},t)}{\partial y} = \sum_{j=1}^{M_N} \frac{\partial \psi_j(\mathbf{x})}{\partial y} q_j^{(e)}(t).$$

- Since we will perform all of our computations in the reference element with coordinates (ξ,η) then we must transform the derivatives from (x,y) to (ξ,η).

2D Metric Terms

- Using the chain rule, we write the derivatives of the basis functions as

$$\frac{\partial\psi(x(\xi,\eta),y(\xi,\eta))}{\partial x}=\frac{\partial\psi(\xi,\eta)}{\partial\xi}\frac{\partial\xi(x,y)}{\partial x}+\frac{\partial\psi(\xi,\eta)}{\partial\eta}\frac{\partial\eta(x,y)}{\partial x}$$

- and

$$\frac{\partial\psi(x(\xi,\eta),y(\xi,\eta))}{\partial y}=\frac{\partial\psi(\xi,\eta)}{\partial\xi}\frac{\partial\xi(x,y)}{\partial y}+\frac{\partial\psi(\xi,\eta)}{\partial\eta}\frac{\partial\eta(x,y)}{\partial y}$$

- where we must now define the derivatives $\frac{\partial\psi}{\partial\boldsymbol{\xi}}$ and the metric terms $\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{x}}$.
- This mapping now allows us to construct basis function derivatives in terms of the reference element (ξ,η) as follows

$$\frac{\partial\psi}{\partial\mathbf{x}}=\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial\mathbf{x}}+\frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial\mathbf{x}}$$

2D Map

- Since we are assuming $\mathbf{x} = \mathbf{x}(\xi, \eta)$ then from the chain rule we get

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi} d\xi + \frac{\partial \mathbf{x}}{\partial \eta} d\eta$$

which can be written in the following matrix form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

where

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \tag{3}$$

is the Jacobian of this transformation,

2D Map

- with determinant

$$\det(J) = |J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}.$$

- Alternatively, we can write the derivatives of $\boldsymbol{\xi}(x, y)$ to get

$$d\boldsymbol{\xi} = \frac{\partial \boldsymbol{\xi}}{\partial x} dx + \frac{\partial \boldsymbol{\xi}}{\partial y} dy$$

which, in matrix form, yields

$$\begin{pmatrix} d\xi \\ \\ d\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ \\ dy \end{pmatrix}$$

where

$$J^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \tag{4}$$

2D Map

- Again, using the chain rule, we can write the derivatives in the global space in terms of the local surface element space as
$$\frac{\partial}{\partial \mathbf{x}} = \frac{\partial \xi}{\partial \mathbf{x}} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial \mathbf{x}} \frac{\partial}{\partial \eta}.$$
- Expanding into its components, we see that the derivatives are in fact given by
$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{pmatrix} = J^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \\ \frac{\partial}{\partial \eta} \end{pmatrix}$$
where the inverse Jacobian matrix is
$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix} \tag{5}$$
and has been obtained by inverting Eq. (3).

2D Map

- This mapping now allows us to construct basis function derivatives in terms of the reference element (ξ, η) as follows

$$\frac{\partial \psi}{\partial \mathbf{x}} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial \mathbf{x}} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial \mathbf{x}}$$

where from equating Eqs. (4) and (5) we see that the metric terms are in fact defined explicitly as

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta}, \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta},$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi}, \qquad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi}.$$

2D Map

- The final step required to compute these metric terms is to approximate the derivatives $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ using the basis functions.
- Let us approximate the physical coordinates by the basis function expansion

$$\mathbf{x}_N(\xi,\eta) = \sum_{j=0}^{N} \psi_j(\xi,\eta)\mathbf{x}_j$$

which then yields the derivatives

$$\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}(\xi,\eta) = \sum_{j=0}^{N} \frac{\partial \psi_j}{\partial \boldsymbol{\xi}}(\xi,\eta)\mathbf{x}_j.$$

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Resulting Element Equations

- The equation that we need to solve is the element equation

$$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t} \, d\Omega_e + \int_{\Gamma_e} \psi_i \widehat{\mathbf{n}} \cdot \mathbf{f}_N^{(*)} \, d\Gamma_e - \int_{\Omega_e} \boldsymbol{\nabla} \psi_i \cdot \mathbf{f}_N^{(e)} \, d\Omega_e = 0$$

$$\forall \psi \in L^2$$

- At this point we have to decide whether or not we are going to evaluate the integrals using co-located quadrature (which results in inexact integration) or non-colocated quadrature (that, for a specific number of quadrature points, results in exact integration).
- Let us first see what happens when we use exact integration.

Exact Integration

- Let $M_Q = (Q+1)^2$ where $Q = \frac{3}{2}N + \frac{1}{2}$ will integrate $3N$ polynomials exactly (note that the advection term is a 3N degree polynomial).
- This quadrature rule yields the following matrix-vector problem

$$M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + \sum_{k=1}^{N_{face}} \left(\mathbf{F}_{ij}^{(e,k)}\right)^{\mathcal{T}} \mathbf{f}_j^{(*,k)} - \left(\widetilde{\mathbf{D}}_{ij}^{(e)}\right)^{\mathcal{T}} \mathbf{f}_j^{(e)} = 0$$

- where the above matrices are defined as follows:

$$M_{ij}^{(e)} = \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e = \sum_{k=1}^{M_Q} w_k |J_k| \psi_{ik} \psi_{jk}$$

is the mass matrix where w_k and $|J_k|$ are the quadrature weight and determinant of the Jacobian evaluated at the quadrature point ξ_k;

Exact Integration

- the flux matrix is defined as follows

$$\mathbf{F}_{ij}^{(e,k)} = \int_{\Gamma_e} \psi_i \psi_j \widehat{\mathbf{n}}^{(e,k)} \, d\Gamma_e = \sum_{l=0}^{Q} w_l^{(k)} |J_l^{(k)}| \psi_{il} \psi_{jl} \widehat{\mathbf{n}}_l^{(e,k)}$$

  where the superscript (k) denotes the side/edge variables,
- and

$$\widetilde{\mathbf{D}}_{ij}^{(e)} = \int_{\Omega_e} \boldsymbol{\nabla} \psi_i \psi_j \, d\Omega_e = \sum_{k=1}^{M_Q} w_k |J_k| \boldsymbol{\nabla} \psi_{ik} \psi_{jk}$$

  is the differentiation matrix, which is nothing more than the 2D version of the weak form differentiation matrix we have already seen.
- There is no problem integrating the above matrices exactly, except for the issue of having to deal with a non-diagonal mass matrix.

Exact Integration

- However, in the DG method having a non-diagonal mass matrix poses very little difficulty since this matrix is small and local.
- Inverting the mass matrix yields the final matrix problem

$$\frac{dq_i^{(e)}}{dt} + \sum_{k=1}^{N_{face}} \left(\widehat{\mathbf{F}}_{ij}^{(e,k)}\right)^{\mathcal{T}} \mathbf{f}_j^{(*,k)} - \left(\widehat{\widetilde{\mathbf{D}}}_{ij}^{(e)}\right)^{\mathcal{T}} \mathbf{f}_j^{(e)} = 0$$

- where

$$\widehat{\mathbf{F}}_{ij}^{(e,k)} = \left(M_{il}^{(e)}\right)^{-1} \mathbf{F}_{lj}^{(e,k)}$$

and

$$\widehat{\widetilde{\mathbf{D}}}_{ij}^{(e)} = \left(M_{ik}^{(e)}\right)^{-1} \widetilde{\mathbf{D}}_{kj}^{(e)}$$

are the flux and differentiation matrices premultiplied by the inverse mass matrix.

Inexact Integration

- Looking at the mass, flux, and differentiation matrices we see that they represent the integrals of polynomials of degree: 2N for mass and flux matrices and 3N for the differentiation matrix.
- Note that the differentiation matrix is written above as a 2N polynomial but in reality, due to the nonlinear nature of the flux **f** we end up with a 3N polynomial matrix; however, this is of little importance in the following discussion as we shall concentrate on the effects of inexact integration on the mass and flux matrices.
- If we use inexact integration, that is, 2N-1 integration, we only commit a small numerical crime for the mass and flux matrices; for the differentiation matrix, this crime is a bit larger, however.
- Let us now use inexact integration and see what the resulting matrix-vector problem looks like.

Inexact Integration

- In this case, we let $M_Q = (Q+1)^2 = M_N$ where $Q = N$ to get the following matrix-vector problem

$$M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + \sum_{k=1}^{N_{face}} \left(\mathbf{F}_{ij}^{(e,k)}\right)^{\mathcal{T}} \mathbf{f}_j^{(*,k)} - \left(\widetilde{\mathbf{D}}_{ij}^{(e)}\right)^{\mathcal{T}} \mathbf{f}_j^{(e)} = 0 \qquad (6)$$

  where changes in our matrix-vector problem from exact to inexact integration occur in all the matrices.
- The mass matrix simplifies to

$$M_{ij}^{(e)} = w_i |J_i| \delta_{ij},$$

- the flux matrix becomes

$$\mathbf{F}_{ij}^{(e,k)} = w_i^{(k)} |J_i^{(k)}| \widehat{\mathbf{n}}_i^{(e,k)} \delta_{ij}$$

Inexact Integration

- where the flux matrix only affects the boundary of the element
- and the differentiation matrix is

$$\widetilde{\mathbf{D}}_{ij}^{(e)} = \sum_{k=1}^{M_N} w_k |J_k| \nabla \psi_{ik} \psi_{jk}.$$

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Problem Statement

- Suppose we wish to solve the continuous first order differential equation

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} q = 0 \qquad \forall (x,y) \in [-1,1]^2$$

- where $q = q(x,y,t)$ and $\mathbf{u} = \mathbf{u}(x,y)$ with $\mathbf{u} = (u,v)^T$. Let the velocity field be

$$u(x,y) = y \qquad \text{and} \qquad v(x,y) = -x$$

- which forms a velocity field that rotates fluid particles in a clockwise direction.
- Note that this velocity field is divergence-free, that is, that the following condition is satisfied

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0.$$

Problem Statement

- Clearly, this problem represents a 2D wave equation that is hyperbolic and is thereby an initial value problem that requires an initial condition.
- Let that initial condition be the Gaussian

$$q(x,y,0) = e^{-\left[(x-x_c)^2+(y-y_c)^2\right]/(2\sigma_c^2)}$$

- where $(x_c, y_c) = (-0.5, 0)$ is the initial center of the Gaussian and $\sigma_c = \frac{1}{8}$ controls the shape (steepness) of the Gaussian wave.
- The analytic solution to this problem is given as

$$q(x,y,t) = q(x-ut, y-vt, 0)$$

where periodicity is enforced at all four boundaries.

Problem Statement

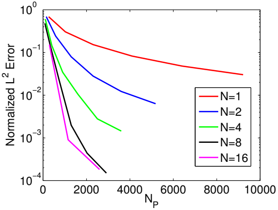
- We define the normalized L^2 error norm as follows

$$L^2 = \sqrt{\frac{\sum_{e=1}^{N_e} \sum_{i=1}^{M_N} \left(q_{e,i}^{numerical} - q_{e,i}^{exact} \right)^2}{\sum_{e=1}^{N_e} \sum_{i=1}^{M_N} \left(q_{e,i}^{exact} \right)^2}} \tag{7}$$

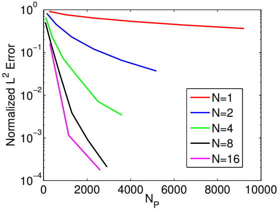
- where N_e represents the number of elements and M_N are the number of interpolation points within each element Ω_e and $q^{numerical}$ and q^{exact} are the numerical and exact solutions.

Solution Accuracy

- This figure shows the convergence rates for various polynomial orders, N, for a total number of gridpoints N_p where, for 2D quadrilaterals, $N_p = (N_e^{(x)}(N+1))(N_e^{(y)}(N+1))$, where $N_e^{(s)}$ denotes the number of elements along the coordinate direction (s).



a) Exact Integration



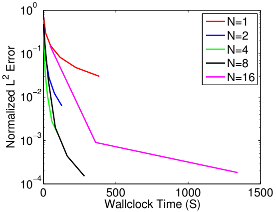
b) Inexact Integration

Solution Accuracy

- The previous figure shows that there is little difference between exact and inexact integration in terms of their rates of convergence.
- The main differences occur for low values of N (≤ 4), however, as N increases (beyond $N = 8$) the difference between exact and inexact integration all but disappear.
- Let us see if high-order is indeed worthwhile to pursue.

Solution Accuracy

- This figure shows the computational cost (wallclock time in seconds) for various polynomial orders, N.



a) Exact Integration b) Inexact Integration

Figure: Convergence rates as a function of wallclock time (in seconds) of DG (Lobatto Points) for the 2D hyperbolic equation for polynomial orders $N = 1$, $N = 2$, $N = 4$, $N = 8$, and $N = 16$ using a total number of gridpoints N_p for a) exact integration ($Q = N + 1$) and b) inexact integration ($Q = N$).

Solution Accuracy

- The previous figure shows that for exact integration, the highest order $N = 16$ is far too expensive.
- For exact integration, we see that $N = 8$ achieves an error accuracy of of 10^{-3} most efficiently with $N = 4$ a close second.
- However, if we insist on more accuracy, then we see that $N = 8$ easily wins.
- This conclusion is more obvious in the case of inexact integration where $N = 8$ and $N = 16$ win even at accuracies near 10^{-2}.

Movie: 1/4 Revolution

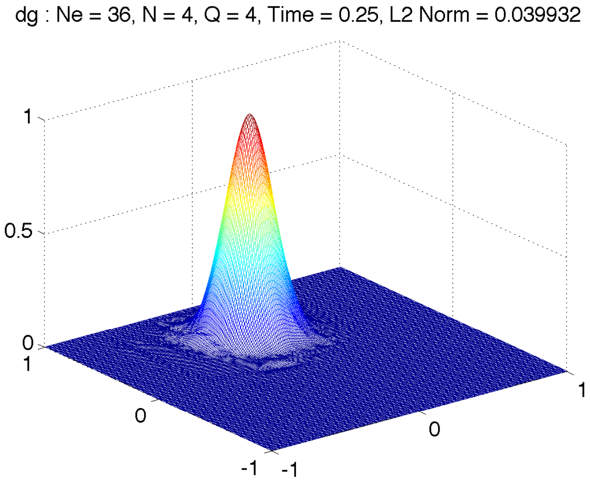


Figure: DG solution for $N_e = 6$ and $N = 4$, and $Q = 5$ at $t = 0.25$.

Movie: 1/2 Revolution

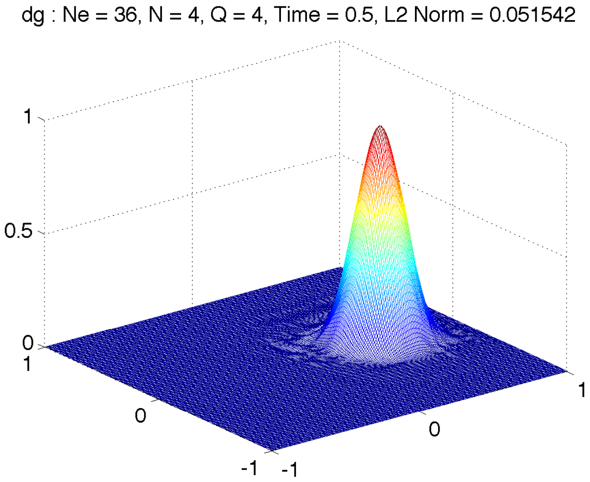


Figure: DG solution for $N_e = 6$ and $N = 4$, and $Q = 5$ at $t = 0.5$.

Movie: 3/4 Revolution

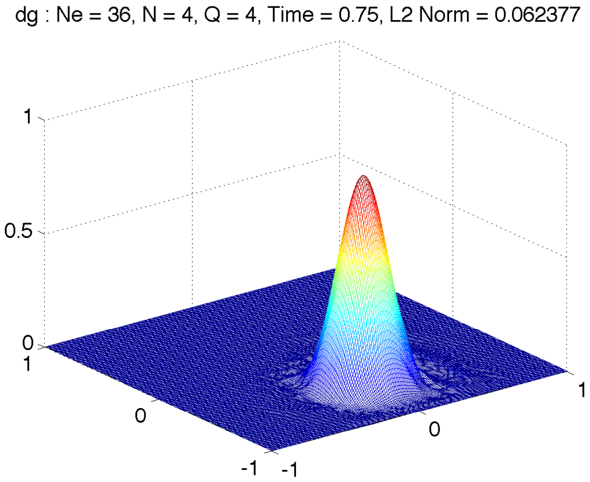


Figure: DG solution for $N_e = 6$ and $N = 4$, and $Q = 5$ at $t = 0.75$.

Movie: 1 Revolution

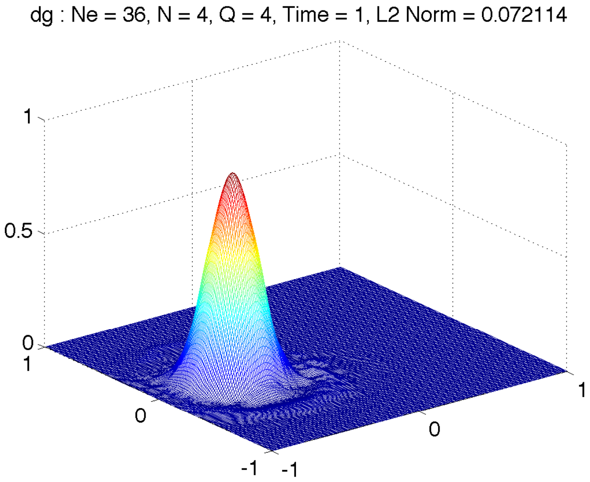


Figure: DG solution for $N_e = 6$ and $N = 4$, and $Q = 5$ at $t = 1.0$.

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Summary of DG Lecture 3

- We covered interpolation on the Quadrilateral and the Triangle.
- We discussed the Resulting Element Equations.
- We covered the basic mappings required to handle unstructured/general grids; we use the basis function for everything.
- We showed some results for the 2D Wave Equation.
- NMW Lecture 7 (by Shiva Gopalakrishnan) will cover wetting and drying with DG methods. This will include some 2D shallow water simulations using DGCOM. If anyone is interested in a tutorial on using/running DGCOM please let me know).
- NMW Lecture 8 (by Michal Kopera) will cover non-conforming Adaptive DG methods.

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Exercises: 2D DG Linear Elements

- Let us construct the matrix problem for the 2D advection equation using linear elements $N = 1$.
- The local matrix problem is then

$$M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + \sum_{k=1}^{N_{face}} \left(\mathbf{F}_{ij}^{(e,k)} \right)^{\mathcal{T}} \mathbf{f}_j^{(*,k)} - \left(\widetilde{\mathbf{D}}_{ij}^{(e)} \right)^{\mathcal{T}} \mathbf{f}_j^{(e)} = 0$$

- where $\mathbf{f} = q\mathbf{u}$

Basis Functions

- Recall that for linear elements, the 1D basis function defined at the element grid point "i" is

$$h_i(\xi) = \frac{1}{2}(1 + \xi_i \xi)$$

- where $\xi \in [-1, +1]$. In a similar vane we can define

$$h_i(\eta) = \frac{1}{2}(1 + \eta_i \eta)$$

  where $\eta \in [-1, +1]$.
- In these two expressions the index is defined as follows: $i = 1, ..., N + 1$.
- If we wish to use the 1D basis functions to construct 2D functions, then we take the the tensor product which amounts to just defining $i = 1, ..., (N + 1)^2$ where we now have to define ξ_i and η_i not just at two grid points as in 1D but now four points in 2D.

Basis Functions

- These four points are the vertices (corners) of the quadrilateral reference element.
- At this point we can now define the 2D basis functions as

$$\psi_i(\xi,\eta) = \frac{1}{2}\left(1+\xi_i\xi\right)\frac{1}{2}\left(1+\eta_i\eta\right)$$

or, more compactly as

$$\psi_i(\xi,\eta) = \frac{1}{4}\left(1+\xi_i\xi\right)\left(1+\eta_i\eta\right).$$

- From this expression we can now define the derivatives as follows

$$\frac{\partial\psi_i}{\partial\xi}(\xi,\eta) = \frac{1}{4}\xi_i\left(1+\eta_i\eta\right)$$

and

$$\frac{\partial\psi_i}{\partial\eta}(\xi,\eta) = \frac{1}{4}\eta_i\left(1+\xi_i\xi\right).$$

Metric Terms

- We need the metric terms in order to represent the derivatives in the reference element to the physical derivatives which are computed as follows

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

- and

$$\frac{\partial \psi_i}{\partial y} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

- where the metric terms are defined as

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta}, \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta},$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi}, \qquad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi}.$$

- with

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}.$$

Metric Terms

- To compute these terms, we expand the coordinates as follows:

$$\mathbf{x}_N(\xi,\eta) = \sum_{j=1}^{M_N} \psi_j(\xi,\eta)\mathbf{x}_j.$$

- Taking the derivative yields:

$$\frac{\partial \mathbf{x}_N}{\partial \xi}(\xi,\eta) = \sum_{j=1}^{M_N} \frac{\partial \psi_j}{\partial \xi}(\xi,\eta)\mathbf{x}_j$$

- where, substituting $\frac{\partial \psi}{\partial \xi}$ yields

$$\frac{\partial \mathbf{x}_N}{\partial \xi}(\xi,\eta) = \sum_{j=1}^{M_N} \frac{1}{4}\xi_j\left(1+\eta_j\eta\right)\mathbf{x}_j.$$

Metric Terms

- For linear elements we have

$$\frac{\partial \mathbf{x}_N}{\partial \xi}(\xi,\eta) = \frac{1}{4}\left(\xi_1\left(1+\eta_1\eta\right)\mathbf{x}_1 + \xi_2\left(1+\eta_2\eta\right)\mathbf{x}_2\right) + \frac{1}{4}\left(\xi_3\left(1+\eta_3\eta\right)\mathbf{x}_3 + \xi_4\left(1+\eta_4\eta\right)\mathbf{x}_4\right).$$

- Let us assume that x is along the ξ direction and y is along η.
- This means that since $\xi_i = (-1,+1,-1,+1)$ and $\eta_i = (-1,-1,+1,+1)$ then $x_1 = x_3$ and $x_2 = x_4$ and similarly $y_1 = y_2$ and $y_3 = y_4$.
- Let us call $x_1 = x_o$ and $x_2 = x_o + \Delta x$.
- We can now show that $\frac{\partial x}{\partial \xi}$ is

$$\frac{\partial x_N}{\partial \xi}(\xi,\eta) = \frac{1}{4}\left(-\left(1-\eta\right)x_o + \left(1-\eta\right)\left(x_o+\Delta x\right)\right) + \frac{1}{4}\left(-\left(1+\eta\right)x_o + \left(1+\eta\right)\left(x_o+\Delta x\right)\right)$$

Metric Terms

- which simplifies to
$$\frac{\partial x_N}{\partial \xi} = \frac{\Delta x}{2}.$$
- If we did this for the other terms we would find that
$$\frac{\partial x_N}{\partial \eta} = \frac{\partial y_N}{\partial \xi} = 0$$
- and
$$\frac{\partial y_N}{\partial \eta} = \frac{\Delta y}{2}.$$
- With these terms known, we can now evaluate the determinant of the Jacobian to yield
$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = \frac{\Delta x \Delta y}{4}.$$

Metric Terms

- The derivatives in physical coordinates are defined as

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

- and

$$\frac{\partial \psi_i}{\partial y} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

- where, when we substitute the metric terms and the derivatives of the basis functions in the reference element yield

$$\frac{\partial \psi_i}{\partial x} = \frac{1}{4}\xi_i \left(1 + \eta_i \eta\right) \frac{2}{\Delta x}$$

- and

$$\frac{\partial \psi_i}{\partial y} = \frac{1}{4}\eta_i \left(1 + \xi_i \xi\right) \frac{2}{\Delta y}.$$

Mass Matrix

- The mass matrix is defined as

$$M_{ij}^{(e)} = \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e$$

- which, after using the definition of the basis functions, we can write it as

$$M_{ij}^{(e)} = \frac{1}{16} \int_{-1}^{+1} \int_{-1}^{+1} (1+\xi_i\xi)(1+\eta_i\eta)(1+\xi_j\xi)(1+\eta_j\eta) \frac{\Delta x \Delta y}{4} \, d\xi \, d\eta.$$

- Integrating yields the following expression

$$M_{ij}^{(e)} = \frac{\Delta x \Delta y}{64} \left(\xi + \frac{1}{2}(\xi_i + \xi_j)\xi^2 + \frac{1}{3}\xi_i\xi_j\xi^3 \right)_{-1}^{+1} \left(\eta + \frac{1}{2}(\eta_i + \eta_j)\eta^2 + \frac{1}{3}\eta_i\eta_j\eta^3 \right)_{-1}^{+1}.$$

Mass Matrix

- Evaluating the integrals yields

$$M_{ij}^{(e)} = \frac{\Delta x \Delta y}{64}\left(2+\frac{2}{3}\xi_i\xi_j\right)\left(2+\frac{2}{3}\eta_i\eta_j\right).$$

- Introducing the values of the reference coordinates as follows where $(\xi,\eta)_1 = (-1,-1)$, $(\xi,\eta)_2 = (+1,-1)$, $(\xi,\eta)_3 = (-1,+1)$, $(\xi,\eta)_4 = (+1,+1)$
- yields

$$M_{ij}^{(e)} = \frac{\Delta x \Delta y}{36}\begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix}.$$

Differentiation Matrix

- The weak form differentiation matrix is written as

$$\widetilde{\mathbf{D}}_{ij}^{(e)} = \int_{\Omega_e} \nabla \psi_i \psi_j \, d\Omega_e.$$

- Beginning with

$$\widetilde{D}_{ij}^{(e,x)} = \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} \psi_j \, d\Omega_e \qquad D_{ij}^{(e,y)} = \int_{\Omega_e} \frac{\partial \psi_i}{\partial y} \psi_j \, d\Omega_e$$

- we find that

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial x},$$
$$\frac{\partial \psi_i}{\partial y} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial y},$$

- where, since we are assuming that x is along ξ and y is along y, we get the metric terms

$$\frac{\partial \xi}{\partial x} = \frac{2}{\Delta x} \qquad \frac{\partial \eta}{\partial y} = \frac{2}{\Delta y} \qquad \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = 0$$

Differentiation Matrix

- and $|J| = \frac{\Delta x \Delta y}{4}$.
- This now allows us to write the differentiation matrices in terms of the reference element coordinates

$$\widetilde{D}_{ij}^{(e,x)} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{4}\xi_i\left(1+\eta_i\eta\right)\frac{1}{4}\left(1+\xi_j\xi\right)\left(1+\eta_j\eta\right)\frac{2}{\Delta x}\frac{\Delta x \Delta y}{4}\, d\xi\, d\eta$$

$$\widetilde{D}_{ij}^{(e,y)} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{4}\eta_i\left(1+\xi_i\xi\right)\frac{1}{4}\left(1+\xi_i\xi\right)\left(1+\eta_i\eta\right)\frac{2}{\Delta y}\frac{\Delta x \Delta y}{4}\, d\xi\, d\eta.$$

- Simplifying and integrating yields

$$\widetilde{D}_{ij}^{(e,x)} = \frac{\Delta y}{32}\left[\xi_i\xi + \frac{1}{2}\xi_i\xi_j\xi^2\right]_{-1}^{+1}\left[\eta + \frac{1}{2}(\eta_i+\eta_j)\eta^2 + \frac{1}{3}\eta_i\eta_j\eta^3\right]_{-1}^{+1}$$

$$\widetilde{D}_{ij}^{(e,y)} = \frac{\Delta x}{32}\left[\eta_i\eta + \frac{1}{2}\eta_i\eta_j\eta^2\right]_{-1}^{+1}\left[\xi + \frac{1}{2}(\xi_i+\xi_j)\xi^2 + \frac{1}{3}\xi_i\xi_j\xi^3\right]_{-1}^{+1}.$$

Differentiation Matrix

- Evaluating the matrices at the bounds of integration and simplifying yields the compact form

$$\widetilde{D}_{ij}^{(e,x)} = \frac{\Delta y}{24}\xi_i(3+\eta_i\eta_j) \qquad \widetilde{D}_{ij}^{(e,y)} = \frac{\Delta x}{24}\eta_i(3+\xi_i\xi_j).$$

- Substituting for the values of the reference element coordinates (ξ, η) yields the matrix form

$$\widetilde{D}_{ij}^{(e,x)} = \frac{\Delta y}{12}\begin{pmatrix} -2 & -2 & -1 & -1 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & -2 & -2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

- and

$$\widetilde{D}_{ij}^{(e,y)} = \frac{\Delta x}{12}\begin{pmatrix} -2 & -1 & -2 & -1 \\ -1 & -2 & -1 & -2 \\ 2 & -1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}.$$

Flux Matrix

- Recall that the flux matrix is defined as

$$\mathbf{F}^{(e,k)} = \int_{\Gamma_e} \psi_i \psi_j \widehat{\mathbf{n}}^{(e,k)} \, d\Gamma_e$$

- where k is the side/edge neighbor of e and $\widehat{\mathbf{n}}^{(e,k)}$ is the outward pointing unit normal vector of the side/edge shared by the elements e and k, pointing from e to k.
- Since we are assuming linear elements then our basis functions are $\psi_i(\xi,\eta) = \frac{1}{4}\left((1+\xi_i\xi)\left(1+\eta_i\eta\right)\right.$.
- Substituting this definition into the flux matrix yields

$$\mathbf{F}^{(e,k)} = \int_{-1}^{+1} \frac{1}{16} \left(1+\xi_i\xi\right)\left(1+\eta_i\eta\right)\left(1+\xi_j\xi\right)\left(1+\eta_j\eta\right)\left|J(\xi,\eta)\right| \, ds$$

where ds is either $d\xi$ or $d\eta$, depending on which of the four sides of the element e we are evaluating.

Flux Matrix

- Let us use Fig. 7 to discuss the edge components of the flux.

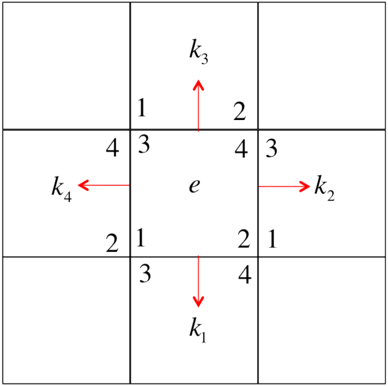

Figure: The element (e) and its four edge neighbors (k_1), (k_2), (k_3), and (k_4).

Side 1

- Looking at Fig. 7 we see that for edge 1, the one shared by elements (e) and (k_1), the integral is along the ξ direction where $\eta = -1$ so that we get the following integral

$$\mathbf{F}^{(e,k_1)} = \int_{-1}^{+1} \frac{1}{16} \left[1 + (\xi_i + \xi_j)\,\xi + \xi_i\xi_j\xi^2\right](1-\eta_i)(1-\eta_j)\,\frac{\Delta x}{2}\widehat{\mathbf{n}}^{(e,k_1)}\,d\xi.$$

- Integrating yields

$$\mathbf{F}^{(e,k_1)} = \frac{\Delta x\,\widehat{\mathbf{n}}^{(e,k_1)}}{32}\left[\xi + \frac{1}{2}(\xi_i + \xi_j)\,\xi^2 + \frac{1}{3}\xi_i\xi_j\xi^3\right]_{-1}^{+1}(1-\eta_i)(1-\eta_j)$$

- which, upon evaluating at the bounds of integration, yields

$$\mathbf{F}^{(e,k_1)} = \frac{\Delta x\,\widehat{\mathbf{n}}^{(e,k_1)}}{48}(3 + \xi_i\xi_j)(1-\eta_i)(1-\eta_j).$$

Side 1

- Substituting in the values of $(\xi, \eta)_i$ and $(\xi, \eta)_j$ yields the matrix

$$\mathbf{F}^{(e,k_1)} = \frac{\Delta x \, \widehat{\mathbf{n}}^{(e,1)}}{48} \begin{pmatrix} 16 & 8 & 0 & 0 \\ 8 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- which is simplified to its final form

$$\mathbf{F}^{(e,k_1)} = \frac{\Delta x \, \widehat{\mathbf{n}}^{(e,k_1)}}{6} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- which is nothing more than a 1D mass matrix for the edge (e, k_1). For this edge, its normal vector is defined as follows $\widehat{\mathbf{n}}^{(e,k)} = 0\widehat{\mathbf{i}} - 1\widehat{\mathbf{j}}$ where $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$ are the usual Cartesian coordinate directional vectors.

Side 2

- For edge 2, the one shared by elements (e) and (k_2), the integral is along the η direction where $\xi = +1$ so that we get

$$\mathbf{F}^{(e,k_2)} = \int_{-1}^{+1} \frac{1}{16} \left(1+\xi_i\right)\left(1+\xi_j\right)\left[1+\left(\eta_i+\eta_j\right)\eta+\eta_i\eta_j\eta^2\right] \frac{\Delta y}{2} \widehat{\mathbf{n}}^{(e,k_2)}\, d\eta.$$

- Integrating yields

$$\mathbf{F}^{(e,k_2)} = \frac{\Delta y\, \widehat{\mathbf{n}}^{(e,k_2)}}{32} \left(1+\xi_i\right)\left(1+\xi_j\right)\left[\eta+\frac{1}{2}\left(\eta_i+\eta_j\right)\eta^2+\frac{1}{3}\eta_i\eta_j\eta^3\right]_{-1}^{+1}$$

- which we then evaluate to yield

$$\mathbf{F}^{(e,k_2)} = \frac{\Delta y\, \widehat{\mathbf{n}}^{(e,k_2)}}{48} \left(1+\xi_i\right)\left(1+\xi_j\right)\left(3+\eta_i\eta_j\right).$$

Side 2

- Substituting the values for $(\xi,\eta)_i$ and $(\xi,\eta)_j$ yields the matrix

$$\mathbf{F}^{(e,k_2)} = \frac{\Delta y \; \widehat{\mathbf{n}}^{(e,k_2)}}{6} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

- where the normal vector here is defined as $\widehat{\mathbf{n}}^{(e,k_2)} = 1\widehat{\mathbf{i}} + 0\widehat{\mathbf{j}}$.

Side 3

- For edge 3, the one shared by elements (e) and (k_3), the integral is along the ξ direction where $\eta = +1$ so that we get the following integral

$$\mathbf{F}^{(e,k_3)} = \int_{-1}^{+1} \frac{1}{16} \left[1 + (\xi_i + \xi_j)\,\xi\right](1+\eta_i)(1+\eta_j) \frac{\Delta x}{2} \widehat{\mathbf{n}}^{(e,k_3)}\, d\xi.$$

- Integrating yields

$$\mathbf{F}^{(e,k_3)} = \frac{\Delta x\, \widehat{\mathbf{n}}^{(e,k_3)}}{32} \left[\xi + \frac{1}{2}(\xi_i + \xi_j)\,\xi^2 + \frac{1}{3}\xi_i\xi_j\xi^3\right]_{-1}^{+1} (1+\eta_i)(1+\eta_j)$$

- which, upon evaluating at the bounds of integration, yields

$$\mathbf{F}^{(e,k_3)} = \frac{\Delta x\, \widehat{\mathbf{n}}^{(e,k_3)}}{48} (3 + \xi_i\xi_j)(1+\eta_i)(1+\eta_j).$$

Side 3

- Substituting in the values of $(\xi, \eta)_i$ and $(\xi, \eta)_j$ yields the matrix

$$\mathbf{F}^{(e,k_3)} = \frac{\Delta x \ \widehat{\mathbf{n}}^{(e,k_3)}}{6} \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

where the normal vector here is defined as $\widehat{\mathbf{n}}^{(e,k_3)} = 0\widehat{\mathbf{i}} + 1\widehat{\mathbf{j}}$.

Side 4

- Finally, for edge 4, the one shared by elements (e) and (k_4), the integral is along the η direction where $\xi = -1$ so that we get

$$\mathbf{F}^{(e,k_4)} = \int_{-1}^{+1} \frac{1}{16} \left(1 - \xi_i\right)\left(1 - \xi_j\right)\left(1 + \eta_j \eta\right)\left[1 + \left(\eta_i + \eta_j\right)\eta\right] \frac{\Delta y}{2} \widehat{\mathbf{n}}^{(e,k_4)} \, d\eta.$$

- Integrating yields

$$\mathbf{F}^{(e,k_4)} = \frac{\Delta y \ \widehat{\mathbf{n}}^{(e,k_4)}}{32} \left(1 - \xi_i\right)\left(1 - \xi_j\right) \left[\eta + \frac{1}{2}\left(\eta_i + \eta_j\right)\eta^2 + \frac{1}{3}\eta_i \eta_j \eta^3\right]_{-1}^{+1}$$

- which we then evaluate to yield

$$\mathbf{F}^{(e,k_4)} = \frac{\Delta y \ \widehat{\mathbf{n}}^{(e,k_4)}}{48} \left(1 - \xi_i\right)\left(1 - \xi_j\right)\left(3 + \eta_i \eta_j\right).$$

Side 4

- Substituting the values for $(\xi, \eta)_i$ and $(\xi, \eta)_j$ yields the matrix

$$\mathbf{F}^{(e,k_4)} = \frac{\Delta y \; \widehat{\mathbf{n}}^{(e,k_4)}}{6} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- where the normal vector here is defined as $\widehat{\mathbf{n}}^{(e,k_4)} = -1\widehat{\mathbf{i}} + 0\widehat{\mathbf{j}}$.

Numerical Flux

- In order to solve the weak form local matrix problem written as follows:

$$M_{ij}^{(e)} \frac{\partial q_j^{(e)}}{\partial t} + \sum_{l=1}^{4} \left(\mathbf{F}_{ij}^{(e,k)} \right)^{\mathcal{T}} \mathbf{f}_j^{(*,l)} - \left(\widetilde{\mathbf{D}}_{ij}^{(e)} \right)^{\mathcal{T}} \mathbf{f}_j^{(e)} = 0$$

  requires us to know what exactly is the value of the term $\mathbf{f}^{(*,l)}$.
- Recall that we are using the Rusanov flux which is defined as

$$\mathbf{f}^{(*,l)} = \frac{1}{2} \left[\mathbf{f}^{(l)} + \mathbf{f}^{(e)} - \mid \lambda \mid \widehat{\mathbf{n}}^{(e,k)} \left(q^{(l)} - q^{(e)} \right) \right].$$

- To see the structure of the product

$$\left(\mathbf{F}_{ij}^{(e,k)} \right)^{\mathcal{T}} \mathbf{f}_j^{(*,l)}$$

  let us first extract the normal vector from the flux matrix and write this product as follows

$$F_{ij}^{(e,k)} \left(\widehat{\mathbf{n}}^{(e,k)} \cdot \mathbf{f}_j^{(*,l)} \right)$$

Side 1

- For side 1 (see Fig. 7), we then have the following numerical flux function

$$\widehat{\mathbf{n}}^{(e,1)} \cdot \mathbf{f}^{(*,1)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,1)} \cdot \left(\mathbf{f}^{(1)} + \mathbf{f}^{(e)}\right) - \mid \lambda \mid \left(q^{(1)} - q^{(e)}\right)\right].$$

- More specifically, for $j = 1, 2$ we get

$$\widehat{\mathbf{n}}^{(e,1)} . \mathbf{f}_1^{(*,1)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,1)} \cdot \left(\mathbf{f}_3^{(1)} + \mathbf{f}_1^{(e)}\right) - \mid \lambda \mid_{1,3}^{(e,1)} \left(q_3^{(1)} - q_1^{(e)}\right)\right],$$

$$\widehat{\mathbf{n}}^{(e,1)} . \mathbf{f}_2^{(*,1)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,1)} \cdot \left(\mathbf{f}_4^{(1)} + \mathbf{f}_2^{(e)}\right) - \mid \lambda \mid_{2,4}^{(e,1)} \left(q_4^{(1)} - q_2^{(e)}\right)\right],$$

  and for $j = 3, 4$ we do not need to define the flux function since it will be multiplied by rows 3 and 4 of $F^{(e,1)}$ which are all zero.
- Substituting the values for $\widehat{\mathbf{n}}^{(e,1)}$ and $\mathbf{f}$ yields

$$\widehat{\mathbf{n}}^{(e,1)} . \mathbf{f}_1^{(*,1)} = \frac{1}{2}\left[-\left((qv)_3^{(1)} + (qv)_1^{(e)}\right) - v_{1,3}^{(e,1)} \left(q_3^{(1)} - q_1^{(e)}\right)\right],$$

Side 1

- which, after simplifying, yields

$$\widehat{\mathbf{n}}^{(e,1)} \cdot \mathbf{f}_1^{(*,1)} = -(qv)_3^{(1)},$$

- and

$$\widehat{\mathbf{n}}^{(e,1)} \cdot \mathbf{f}_2^{(*,1)} = -(qv)_4^{(1)}$$

- where we have taken $|\lambda|_{1,3}^{(e,1)} = |\widehat{\mathbf{n}}^{(e,1)} \cdot \mathbf{u}^{(e,1)}|_{1,3} = v_{1,3}^{(e,1)}$ and $|\lambda|_{2,4}^{(e,1)} = |\widehat{\mathbf{n}}^{(e,1)} \cdot \mathbf{u}^{(e,1)}|_{2,4} = v_{2,4}^{(e,1)}$ and have assumed v to be continuous.

Side 2

- For side 2, we then have the following numerical flux function

$$\widehat{\mathbf{n}}^{(e,2)} \cdot \mathbf{f}^{(*,2)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,2)} \cdot \left(\mathbf{f}^{(2)} + \mathbf{f}^{(e)}\right) - \mid \lambda \mid \left(q^{(2)} - q^{(e)}\right)\right].$$

- For $j = 2, 4$ we get

$$\widehat{\mathbf{n}}^{(e,2)}.\mathbf{f}_2^{(*,2)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,2)} \cdot \left(\mathbf{f}_1^{(2)} + \mathbf{f}_2^{(e)}\right) - \mid \lambda \mid_{2,1}^{(e,2)} \left(q_1^{(2)} - q_2^{(e)}\right)\right],$$

$$\widehat{\mathbf{n}}^{(e,2)}.\mathbf{f}_4^{(*,2)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,2)} \cdot \left(\mathbf{f}_3^{(2)} + \mathbf{f}_4^{(e)}\right) - \mid \lambda \mid_{4,3}^{(e,2)} \left(q_3^{(2)} - q_4^{(e)}\right)\right],$$

- and for $j = 1, 3$ we do not need to define the flux function since it will be multiplied by rows 1 and 3 of $F^{(e,2)}$ which are all zero.
- Substituting the values for $\widehat{\mathbf{n}}^{(e,2)}$ and $\mathbf{f}$ yields

$$\widehat{\mathbf{n}}^{(e,2)} \cdot \mathbf{f}_2^{(*,1)} = \frac{1}{2}\left[\left((qu)_1^{(2)} + (qu)_2^{(e)}\right) - u_{2,1}^{(e,2)} \left(q_1^{(2)} - q_2^{(e)}\right)\right],$$

Side 2

- which, after simplifying, yields

$$\widehat{\mathbf{n}}^{(e,2)} \cdot \mathbf{f}_2^{(*,2)} = (qu)_2^{(e)},$$

- and

$$\widehat{\mathbf{n}}^{(e,2)} \cdot \mathbf{f}_4^{(*,2)} = (qu)_4^{(e)}$$

- where we have taken $|\,\lambda\,|_{2,1}^{(e,2)} = |\,\widehat{\mathbf{n}}^{(e,2)} \cdot \mathbf{u}^{(e,2)}\,|_{2,1} = u_{2,1}^{(e,2)}$ and $|\,\lambda\,|_{4,3}^{(e,2)} = |\,\widehat{\mathbf{n}}^{(e,2)} \cdot \mathbf{u}^{(e,2)}\,|_{4,3} = u_{4,3}^{(e,2)}$ and have assumed u to be continuous.

Side 3

- For side 3, we have the following numerical flux function

$$\widehat{\mathbf{n}}^{(e,3)} \cdot \mathbf{f}^{(*,3)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,3)} \cdot \left(\mathbf{f}^{(3)} + \mathbf{f}^{(e)}\right) - \mid \lambda \mid \left(q^{(3)} - q^{(e)}\right)\right].$$

- For $j = 3, 4$ we get

$$\widehat{\mathbf{n}}^{(e,3)} . \mathbf{f}_3^{(*,3)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,3)} \cdot \left(\mathbf{f}_1^{(3)} + \mathbf{f}_3^{(e)}\right) - \mid \lambda \mid_{3,1}^{(e,3)} \left(q_1^{(3)} - q_3^{(e)}\right)\right],$$

$$\widehat{\mathbf{n}}^{(e,3)} . \mathbf{f}_4^{(*,3)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,3)} \cdot \left(\mathbf{f}_2^{(3)} + \mathbf{f}_4^{(e)}\right) - \mid \lambda \mid_{4,2}^{(e,3)} \left(q_2^{(3)} - q_4^{(e)}\right)\right],$$

- and for $j = 1, 2$ we do not need to define the flux function since it will be multiplied by rows 1 and 2 of $F^{(e,3)}$ which are all zero.
- Substituting the values for $\widehat{\mathbf{n}}^{(e,3)}$ and $\mathbf{f}$ yields

$$\widehat{\mathbf{n}}^{(e,3)} \cdot \mathbf{f}_3^{(*,3)} = \frac{1}{2}\left[\left((qv)_1^{(3)} + (qv)_3^{(e)}\right) - v_{3,1}^{(e,3)} \left(q_1^{(3)} - q_3^{(e)}\right)\right],$$

Side 3

- which, after simplifying, yields

$$\widehat{\mathbf{n}}^{(e,3)} \cdot \mathbf{f}_3^{(*,3)} = (qv)_3^{(e)},$$

- and

$$\widehat{\mathbf{n}}^{(e,3)} \cdot \mathbf{f}_4^{(*,3)} = (qv)_4^{(e)}$$

- where we have taken $\mid \lambda \mid_{3,1}^{(e,3)} = \mid \widehat{\mathbf{n}}^{(e,3)} \cdot \mathbf{u}^{(e,3)} \mid_{3,1} = v_{3,1}^{(e,3)}$ and $\mid \lambda \mid_{4,2}^{(e,3)} = \mid \widehat{\mathbf{n}}^{(e,3)} \cdot \mathbf{u}^{(e,3)} \mid_{4,2} = v_{4,2}^{(e,3)}$ and have assumed v to be continuous.

Side 4

- For side 4, we get

$$\widehat{\mathbf{n}}^{(e,4)} \cdot \mathbf{f}^{(*,4)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,4)} \cdot \left(\mathbf{f}^{(4)} + \mathbf{f}^{(e)}\right) - \mid \lambda \mid \left(q^{(4)} - q^{(e)}\right)\right].$$

- For $j = 1, 3$ we get

$$\widehat{\mathbf{n}}^{(e,4)} . \mathbf{f}_1^{(*,4)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,4)} \cdot \left(\mathbf{f}_2^{(4)} + \mathbf{f}_1^{(e)}\right) - \mid \lambda \mid_{1,2}^{(e,4)} \left(q_2^{(4)} - q_1^{(e)}\right)\right],$$

$$\widehat{\mathbf{n}}^{(e,4)} . \mathbf{f}_3^{(*,4)} = \frac{1}{2}\left[\widehat{\mathbf{n}}^{(e,4)} \cdot \left(\mathbf{f}_4^{(4)} + \mathbf{f}_3^{(e)}\right) - \mid \lambda \mid_{3,4}^{(e,4)} \left(q_4^{(4)} - q_3^{(e)}\right)\right],$$

- and for $j = 2, 4$ we do not need to define the flux function since it will be multiplied by rows 2 and 4 of $F^{(e,4)}$ which are all zero.
- Substituting the values for $\widehat{\mathbf{n}}^{(e,4)}$ and $\mathbf{f}$ yields

$$\widehat{\mathbf{n}}^{(e,4)} . \mathbf{f}_1^{(*,4)} = \frac{1}{2}\left[-\left((qu)_2^{(4)} + (qv)_1^{(e)}\right) - u_{1,2}^{(e,4)} \left(q_2^{(4)} - q_1^{(e)}\right)\right],$$

Side 4

- which, after simplifying, yields

$$\widehat{\mathbf{n}}^{(e,4)} \cdot \mathbf{f}_1^{(*,4)} = -(qu)_2^{(4)},$$

- and

$$\widehat{\mathbf{n}}^{(e,4)} \cdot \mathbf{f}_3^{(*,4)} = -(qu)_4^{(4)}$$

- where we have taken $\mid \lambda \mid_{1,2}^{(e,4)} = \mid \widehat{\mathbf{n}}^{(e,4)} \cdot \mathbf{u}^{(e,4)} \mid_{1,2} = u_{1,2}^{(e,4)}$
- and $\mid \lambda \mid_{3,4}^{(e,4)} = \mid \widehat{\mathbf{n}}^{(e,4)} \cdot \mathbf{u}^{(e,4)} \mid_{3,4} = u_{3,4}^{(e,4)}$ and have assumed u to be continuous.