Lecture 16: 2D Elliptic DG

Let's consider the PDE:

(16.1) $$\nabla^2 q(\bar{x}) = f(\bar{x}) \quad \text{w/ appropriate BCs.}$$

Recall that DG is very good for solving 1^{st} order operators so a possible solution strategy is to use the so-called flux form as follows:

(16.2.1) $$\bar{Q} = \bar{\nabla} q$$

(16.2.2) $$\bar{\nabla} \cdot \bar{Q} = f(\bar{x})$$

Let's define the basis function expansions:

(16.3.1) $$q_N^{(e)}(\bar{x}) = \sum_{j=1}^{M_N} \psi_j(\bar{x})\, q_j^{(e)}$$

(16.3.2) $$\bar{Q}_N^{(e)}(\bar{x}) = \sum_{j=1}^{M_N} \psi_j(\bar{x})\, \bar{Q}_j^{(e)}$$

We can now discretize Eq. (16.2) as follows:

(16.4.1) $$\int_{\Omega_e} \bar{Q}_N^{(e)} \cdot \bar{\tau}_i \, d\Omega_e = \int_{\Omega_e} \bar{\nabla} q_N^{(e)} \cdot \bar{\tau}_i \, d\Omega_e \quad \forall\ \bar{\tau} \in L^2$$

(16.4.2) $$\int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \bar{Q}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i \, f_N^{(e)} \, d\Omega_e \quad \forall\ \psi \in L^2$$

where $\psi_i(\xi,\eta) = h_j(\xi) \otimes h_k(\eta)$ $\quad j,k = 0,\ldots,N$

$i = j+1+k(N+1)$

&

$$\bar{\bar{\tau}} = \psi \,\bar{\bar{I}}_d = \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix} \quad \text{for } d=2$$

First Step: Evaluating Q

Starting w/ Eq. (16.4.1):

$$\int_{\Omega_e} \bar{Q}_N^{(e)} \cdot \bar{\bar{\tau}}_i \, d\Omega_e = \int_{\Omega_e} \bar{\nabla} q_N^{(e)} \cdot \bar{\bar{\tau}}_i \, d\Omega_e$$

We note that:

$\bar{Q} = Q^{(x)}\hat{i} + Q^{(y)}\hat{j}$ & so:

$$(16.5) \quad \bar{Q}_N^{(e)} \cdot \bar{\bar{\tau}}_i \equiv \left(Q_N^{(x)} \quad Q_N^{(y)} \right) \begin{pmatrix} \psi_i & 0 \\ 0 & \psi_i \end{pmatrix}$$

$$= \left(\psi_i Q_N^{(x)} \,,\; \psi_i Q_N^{(y)} \right) \rightarrow 2 \text{ eqs.}$$

Next, note that:

$$(16.6) \quad \bar{\nabla} \cdot \left(q_N^{(e)} \bar{\bar{\tau}}_i \right) = \bar{\nabla} q_N^{(e)} \cdot \bar{\bar{\tau}}_i + q_N^{(e)} \bar{\nabla} \cdot \bar{\bar{\tau}}_i$$

or, rearranging:

$$\bar{\nabla} f_N^{(e)} \cdot \bar{\tau}_i = \bar{\nabla} \cdot \left(f_N^{(e)} \bar{\tau}_i \right) - f_N^{(e)} \bar{\nabla} \cdot \bar{\tau}_i$$

Subbing into (16.4.1) gives:

(16.7) $$\int_{\Omega_e} \bar{Q}_N^{(e)} \cdot \bar{\tau}_i \, d\Omega_e = \int_{\Gamma_e} \left(\hat{n} \cdot \bar{\tau}_i \right) f_N^{(e)} \, d\Gamma_e - \int_{\Omega_e} \left(\bar{\nabla} \cdot \bar{\tau}_i \right) f_N^{(e)} \, d\Omega_e$$

where we already know that:

$$\bar{Q}_N^{(e)} \cdot \bar{\tau}_i = \left(\psi_i Q_N^{(x,e)} , \psi_i Q_N^{(y,e)} \right)$$

similarly,

$$\hat{n} \cdot \bar{\tau}_i = \left(n_x \psi_i , n_y \psi_i \right)$$

$$\bar{\nabla} \cdot \bar{\tau}_i = \left(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} \right) \begin{pmatrix} \psi_i & 0 \\ 0 & \psi_i \end{pmatrix} = \left(\frac{\partial \psi_i}{\partial x} , \frac{\partial \psi_i}{\partial y} \right)$$

& so Eq. (16.7) becomes:

(16.8.1) $$\int_{\Omega_e} \psi_i \bar{Q}_N^{(x,e)} \, d\Omega_e = \int_{\Gamma_e} n_x \left(\psi_i f_N \right)^{(x)} d\Gamma_e$$

$$- \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} \zeta_N^{(e)} \, d\Omega_e$$

(16.8.2) $$\int_{\Omega_e} \psi_i \, \bar{Q}_N^{(y,e)} \, d\Omega_e = \int_{\Gamma_e} n_y \left(\psi_i \, \zeta_N^{(e)}\right)^{(*)} d\Gamma_e$$

$$- \int_{\Omega_e} \frac{\partial \psi_i}{\partial y} \zeta_N^{(e)} \, d\Omega_e$$

But writing it compactly in matrix form yields:

(16.9) $$M_{ij}^{(e)} \left(\bar{Q}_j^{(e)} \cdot \bar{\bar{I}}_2 \right) = \left(F_{ij}^{(e)} \right)^T \left(\zeta_j^{(*,e)} \bar{\bar{I}}_2 \right)$$

$$- \tilde{D}_{ij}^{(e)} \zeta_j^{(e)}$$

where:

$$M_{ij}^{(e)} = \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e$$

$$F_{ij}^{(e)} = \int_{\Gamma_e} \psi_i \psi_j \, \hat{n}(\bar{x}) \, d\Gamma_e$$

$$\tilde{D}_{ij}^{(e)} = \int_{\Omega_e} \bar{\nabla} \psi_i \, \psi_j \, d\Omega_e$$

We have already seen these matrices before. However, we have not yet specified our numerical flux.

Numerical flux

To derive Eq. (16.9) we assumed the following:

$$\left(\psi_i \, \zeta_N^{(e)}\right)^{(*)} = \psi_i \, \zeta_N^{(*,e)}$$

which is reasonable because we have the power to choose ψ.

Although many options exist, we can write:

$$\zeta_N^{(*,e)} = \alpha \, \zeta_N^{(e)} + (1-\alpha)\, \zeta_N^{(k)}$$

where k is the neighbor of e which shares the unit normal vector $\hat{n}$

The choice $\alpha = \frac{1}{2}$ yields the Bassi-Rebay flux.

Using this flux & corresponding Dirichlet BCs, we can use (16.9) to solve for $\overline{Q}$.

Second Step: Evaluating ζ

Assuming that we know $\overline{Q}$ & appropriate Neumann BCs we now seek to solve:

$$(16.10) \quad \int_{\Omega_e} \psi_i \, \overline{\nabla} \cdot \overline{Q}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i \, f_N^{(e)} \, d\Omega_e$$

Using the Product Rule:

$$\bar{\nabla}\cdot(\psi_i \bar{Q}) = \bar{\nabla}\psi_i \cdot \bar{Q} + \psi_i \bar{\nabla}\cdot\bar{Q}$$

we rewrite (16.10) as follows:

$$(16.11) \quad \int_{\Gamma_e} \hat{n}\cdot\left(\psi_i \bar{Q}_N^{(e)}\right)^{(*)} d\Gamma_e - \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{Q}_N^{(e)} d\Omega_e$$

$$= \int_{\Omega_e} \psi_i f_N^{(e)} d\Omega_e$$

In matrix form, we write:

$$(16.12) \quad \left(F_{ij}^{(e)}\right)^T \bar{Q}_j^{(*,e)} - \left(\tilde{D}_{ij}^{(e)}\right)^T \bar{Q}_j^{(e)} = M_{ij}^{(e)} f_j^{(e)}$$

where:

$$\bar{Q}_N^{(*,e)} = (1-\alpha)\bar{Q}_N^{(e)} + \alpha\, \bar{Q}_N^{(k)}$$ → Flip-flops the flux for $f_N^{(e)}$

Solution Strategy

The readings discusses two approaches. Here, we will only discuss the recommended approach.

Starting w/ (16.9) for $\bar{Q}$ we write:

$$(16.13) \quad \bar{Q}_i^{(e)} \cdot \bar{\bar{I}}_2 = \left(\hat{F}_{ij}^{(e)}\right)^T \left(q_j^{(*,e)} \bar{\bar{I}}_2\right) - \hat{D}_{ij}^{(e)} q_j^{(e)}$$

where:

$$\hat{F}_{ij}^{(e)} = \left(M_{ik}^{(e)}\right)^{-1} \bar{F}_{kj}^{(e)} \quad \& \quad \hat{D}_{ij}^{(e)} = \left(M_{ik}^{(e)}\right)^{-1} \tilde{D}_{ik}$$

Let us further simplify the presentation by introducing a new flux matrix such that:

$$\left(\bar{F}_{ij}^{(e)}\right)^T \left(f_j^{(*,e)} \bar{\bar{I}}_2\right) = \left(\bar{F}_{ij}^{(*)}\right) f_j^{(e)}$$

This allows us to write (16.13) as follows:

(16.16) $$\bar{Q}^{(e)} \cdot \bar{\bar{I}}_2 = M^{-1} \left(\bar{F}_f^{(*)} - \tilde{D}^{(e)}\right) f^{(e)}$$

where $\bar{F}_f^{(*)}$ contains Dirichlet BC data.

Similarly, Eq. (16.12) can be written as:

(16.17) $$\left(\bar{F}_Q^{(*)} - \tilde{D}^{(e)}\right)^T \bar{Q} = M f$$

where $\bar{F}_Q^{(*)}$ contains Neumann BC data.

Now, let $\hat{D}_Q = \bar{F}_Q^{(*)} - \tilde{D}^{(e)}$

Expanding the LHS of (16.17) gives:

(16.18) $$\hat{D}_Q^{(x)} Q^{(x)} + \hat{D}_Q^{(y)} Q^{(y)} = M f$$

Subbing (16.16) into (16.18) gives:

(16.18) $$\left(\hat{D}_q^{(x)} M^{-1} \hat{D}_g^{(x)} + \hat{D}_q^{(y)} M^{-1} \hat{D}_g^{(y)}\right) g = Mf$$

$\therefore$ we see that the LDG Laplacian is:

$$L = \hat{D}_q^{(x)} M^{-1} \hat{D}_g^{(x)} + \hat{D}_q^{(y)} M^{-1} \hat{D}_g^{(y)}$$