Investing 100 dollars every month for 40 years

#finance #math_for_fun

Background

Consider the following equation which describes how an initial investment of P dollars grows after t years (the derivation can be found in Compound interest formula or on this Wikipedia page)

$$A(t) = P\left(\left(1+\frac{r}{N}\right)^{N}\right)^{t}$$

where N represents the number of times **per year** that interest will be compounded, $A(t)$ is the amount of money in the investment account after t years, and r is the interest rate written as a decimal (for example an interest rate of 5% would be written as $r = 0.05$). In the case of continuously compounding interest the formula is given by

$$A_{cont}(t) = Pe^{rt}.$$

For simplicity's sake we will use the continuously compounding interest formula since it is easier to work with and it gives an upper bound for the usual compound interest.

The actual question

Suppose that every month ($\frac{1}{12}$ of a year) you invest P dollars into an account that earns you continuously compounding interest with an interest rate of r. After the first month you will have

$$Pe^{\frac{r}{12}} + P$$

dollars, where the first term is the initial investment you made and the interest earned on it, and the second term is your second investment of P dollars. At the end of month two you would have

$$\left(Pe^{\frac{r}{12}} + P\right)e^{\frac{r}{12}} + P = P(e^{\frac{r}{12}})^2 + Pe^{\frac{r}{12}} + P$$

dollars in your account. Continuing like this we can see that (by letting $b = e^{\frac{r}{12}}$) we would get

$$\text{Amount after n months} = P\left(b^n + b^{n-1} + \cdots + b^2 + b + 1\right) = P\sum_{i=0}^{n} b^i.$$

From here we can use the closed form for the sum of the first n terms of a geometric series to get

$$\text{Amount after n months} = P\left(\frac{1-b^{n+1}}{1-b}\right).$$

Let's call the amount in the account after n months $S(n, P, r)$, where we have made S an explicit function of P and r to express the fact that these parameters change the outcome. Substituting $b = e^{\frac{r}{12}}$ back in we get

$$S(n, P, r) = \text{Amount after n months} = P\left(\frac{1-b^{n+1}}{1-b}\right) = P\left(\frac{1-e^{\frac{r(n+1)}{12}}}{1-e^{\frac{r}{12}}}\right).$$

Okay let's put some actual numbers in. Suppose the interest earned in the account is 4 percent annually, you invest 100 dollars every month, and you do so for 40 years = 480 months. Then we will plug $n = 480$, $P = 100$, and $r = 0.04$ into S. As can be seen in this desmos file, $S(480, 100, 0.04) \approx 118,888$ which is nowhere near 1 million dollars. We can, however, tweak the interest rate and see what happens. What if, for example, we take $r = 10.15\%$ which is the average annualized return of the S&P 500 from 1957 to 2022. Well then things become a little more interesting! Plugging this into the desmos file we get $S(480, 100, .1015) \approx 676,541$ which is over halfway to 1 million dollars, but still not 1 million dollars. It turns out that you would have to invest in a portfolio with a rate of return of almost 11.5% in order to have 1 million dollars at the end of 40 years, and keep in mind we are assuming interest compounded continuously which is an **overestimate**.

Why am I wasting my time on this anyway?

You might be wondering what got me started down this rabbit hole. It all started with a tweet my mom shared in our family group-chat. It was written by some guy named Dave Ramsey, who as far as I can tell is more of a scam artist than he is a financial advisor. If you haven't already checked it out, his tweet states the following as if it is undeniable fact:

> "$100 each month invested from age 25-65 is $1,176,000. YOU DO NOT HAVE TO RETIRE BROKE."

It is followed by some other fluff along with a link to what I assume is a website that funnels cash from well meaning people to Dave Ramsey. What irks me the most about his tweet is that it is stated with such baseless authority. Picking $1,176,000 is a very precise choice for such a vague claim. What does he mean by, "invest"? Why are you assuming I was planning on retiring penniless? WHY ARE YOU YELLING AT ME? Anyway, as we saw earlier, without a specific interest rate (and compounding frequency for that matter) this claim is meaningless.

Depending on the interest rate you could have 50 thousand dollars or 50 million dollars after 40 years.