ST233
AY22/23 Sem 2
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Chapter 1: Basic Concepts of Probability

Definition 1 (Experiment, Sample space, Event)
A statistical experiment is any procedure that produces data or observations.

The sample space, denoted by S, is the set of all possible outcomes of a statistical experiment. The sample space depends on the problem of interest!

A sample point is an outcome (element) in the sample space.

An event is a subset of the sample space.

Union: $A \cup B = x : x \in A \vee x \in B$
Union of n events: $\cup_{i=1}^{n} = x : x \in A_1 \vee x \in A_2 ...$
Union: $A \cap B = x : x \in A \wedge x \in B$
Interection of n events: $\cap_{i=1}^{n} = x : x \in A_1 \wedge x \in A_2 ...$
Complement: $A' = x : x \in S \wedge x \notin A$
Mutually Exclusive: $A \cap B = \emptyset$

More Event Operations
(a) $A \cap A' = \emptyset$
(b) $A \cap \emptyset = \emptyset$
(c) $A \cup A' = S$
(d) $(A')' = A$
(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(g) $A \cup B = A \cup (B \cap A')$
(h) $A = (A \cap B) \cup (A \cap B')$

De Morgan's Law
For any n events $A_1, A_2, \ldots, A_n$,
(i) $(A_1 \cup A_2 \cup \ldots \cup A_n)' = A_1' \cap A_2' \cap \ldots \cap A_n'$.
A special case: $(A \cup B)' = A' \cap B'$.
(j) $(A_1 \cap A_2 \cap \ldots \cap A_n)' = A_1' \cup A_2' \cup \ldots \cup A_n'$.
A special case: $(A \cap B)' = A' \cup B'$.

Multiplication
Sequential experiments
$n_1 * n_2 * ... * n_r$
outcomes

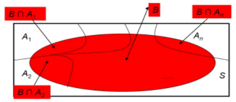
Addition
Independent experiments
$n_1 + n_2 + ... + n_r$
outcomess

Permutation: $P_r^n = \frac{n!}{(n-r)!} = n(n-1)...(n-(r-1))$
Selection and arrangement of r objects out of n. Order is considered.
Combination: $C_r^n = \frac{n!}{r!(n-r)!} = \frac{n(n-1)...(n-(r-1))}{r!}$
Selection of r objects out of n. Order is not considered.
$C_r^n = C_{n-r}^n$

Axioms and Properties of Probability
1 $0 \leq P(A) \leq 1$
2 P(S)=1
3 $P(A \cup B) = P(A) + P(B)$ if A and B are **mutex** (not to be confused with independence)
4 P($\emptyset$)=0
5 P($A_1 \cup A_2 \cup ... A_n$)=$\sum_{i=1}^{n} A_i$
6 P(A')=$1 - P(A)$
7 $P(A \cup B) = P(A) + P(B) - P(AB)$
8 P(A)=$P(A \cap B) + P(A \cap B')$
9 $A \subset B \rightarrow P(A) \leq P(B)$
10 $P(A_1) = P(A_2) = ... = P(A_k) \rightarrow for B \subset S, P(B) = \frac{|B|}{|S|}$

Conditional Probability (B occurs given A):
$P(B|A) = \frac{P(A \cap B)}{P(A)}$
Multiplication rules
$P(A \cap B) = P(A)P(B|A)$ if $P(A)! = 0$
$P(A \cap B) = P(B)P(A|B)$ if $P(B)! = 0$
Inverse Probability Formula: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
Indepedence $A \perp B iff P(A \cap B) = P(A)P(B)$ This implies $P(A|B) = P(A)$ and $P(B|A) = P(B)$
Partition $A_1, A_2 ... A_r$ are mutually exclusive and $\sum_{i=1}^{n} A_i = S$

Theorem 11 (Law of Total Probability)
Suppose $A_1, A_2, \ldots, A_n$ is a partition of S. Then for any event B, we have
$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(A_i)P(B|A_i).$$



$P(B) = P(A)P(B|A) + P(A')P(B|A')$
Bayes' theorem
$P(A_k|B) = \frac{P(B|A)P(A_k)}{P(A)P(B|A) + P(A')P(B|A')}$
Bayes' Theorem(n=2) $P(A|B) = \frac{P(B|A)P(A)}{\sum_{i=1}^{n} P(A_i)P(B|A_k)}$
Denom is P(B)
$P(A|B) = 1 - P(A'|B)$

L2:Random Variables

Random variable X is a function from S to R
Uppercase letters denote random variables
Lowercase letters denote observed values of random variables

Definition 3 (Probability mass function)
For a discrete random variable X, define
$$f(x) = \begin{cases} P(X = x), & \text{for } x \in R_X; \\ 0, & \text{for } x \notin R_X. \end{cases}$$
Then $f(x)$ is known as the probability function (pf), or probability mass function (pmf) of X.

The collection of pairs $(x_i, f(x_i)), i = 1, 2, 3, \ldots$, is called the probability distribution of X.

Properties of the Probability Mass Function
The probability mass function $f(x)$ of a discrete random variable X **must** satisfy:
(1) $f(x_i) \geq 0$ for all $x_i \in R_X$;
(2) $f(x) = 0$ for all $x \notin R_X$;
(3) $\sum_{i=1}^{\infty} f(x_i) = 1$, or $\sum_{x_i \in R_X} f(x_i) = 1$.
For any set $B \subset \mathbb{R}$, we have
$$P(X \in B) = \sum_{x_i \in B \cap R_X} f(x_i).$$

Definition 4 (Probability Density Function)
The probability density function of a continuous random variable X, denoted by $f(x)$, is a function that satisfies:
(1) $f(x) \geq 0$ *for all* $x \in R_X$; *and* $f(x) = 0$ *for* $x \notin R_X$;
(2) $\int_{R_X} f(x)\,dx = 1$;
(3) *For any a and b such that* $a \leq b$,
$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

note: $P(a < X < b) = P(a \leq X \leq B)$
To check that a function is a PDF, chcek conditions 1 and 2
Cumulative Distribution Function: $F(x) = P(X \leq x)$ and
$P(a \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a-)$
Deriving CDF from probability distribution

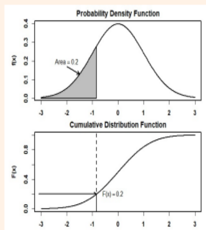
x	0	1	2
$f(x)$	1/4	1/2	1/4

We have
$$F(0) = f(0) = 1/4, F(1) = f(0) + f(1) = 3/4, F(2) = f(0) + f(1) + f(2) = 1.$$

Deriving probability distribution from cdf
a
$$F(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \leq x < 1 \\ 3/4, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases}.$$
- f(1)=P(X=0)=F(0)-F(0-)=$\frac{1}{4}$-0=$\frac{1}{4}$
- f(2)=P(X=1)=F(1)-F(1-)=$\frac{3}{4}$-$\frac{1}{4}$=$\frac{1}{2}$
- f(3)=P(X=2)=F(2)-F(2-)=1-$\frac{3}{4}$=$\frac{1}{4}$

CDF - Continuous Random Variable
- F(X) assumes different values in R_x when F(X)!=0
- F(X) is always non-decreasing (for discrete varaible too) $x_1 < x_2 \rightarrow F(x_1) < F(x_2)$
- Probability function and CDF is one-to-one and uniquely determined
- $0 < F(x) < 1$
- (discrete), $0 < f(x) < 1$
- (continuous), $f(x) \geq 0$ but $f(x)$ is not necessarily ≤ 1. While the PDF itself cannot exceed 1, there are points within its range that are greater than 1

CDF: Continuous Random Variable
If X is a continuous random variable,
$$F(x) = \int_{-\infty}^{x} f(t)dt,$$
and
$$f(x) = \frac{dF(x)}{dx}.$$
Further
$$P(a \leq X \leq b) = P(a < X < b) = F(b) - F(a).$$



Find CDF of a continuous random variable
$$f(x) = \begin{cases} \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \frac{t}{\theta^2} e^{-t^2/(2\theta^2)}\,dt$$
$$= \left[-e^{-t^2/(2\theta^2)}\right]_{t=0}^{x} = 1 - e^{-x^2/(2\theta^2)}.$$

Discrete: Expectation (mean) and Variance
- $E(X) = \sum_{x \in R_x} x_i f(x_i) = \mu_x$

Random: Expectation (mean) and Variance
- $E(X) = \mu_x = \int_{-\infty}^{\infty} x f(x) dx = \infty_{x \in R_x} x f(x) dx$
- The mean of X is not necessarily a possible value

Properties of Expectation
1 $E(aX + b) = aE(x) + b$
2 $E(X + Y) = E(X) + E(Y)$
3 (discrete) $E[g(x)] = \sum_{x \in R_x} g(x)f(x)$
3 (continuous) $E[g(x)] = \int_{x \in R_x} g(x)f(x)dx$
- Using properties 1, 2 we have $E(a_1X_1 + a_2X_2 + ... + a_kX + k) = a_1E(X_1) + ... + a_kE(X_k)$

Variance
- $Var(X) = \sigma_x^2 = E(X - \mu_x)^2 = E(X^2) - E(X)^2$
- (discrete) $Var(X) = \sum_{x \in R_x} (x - \mu_x)^2 f(x)$
- (continuous) $Var(X) = \int_{-\infty}^{\infty} (x - \mu_x)^2 f(x) dx$
- V(X) le 0 for any X, V(X)=0 when X is a constant

Joint Probability Distribution

Definition 5 (Discrete joint probability function)
*Let (X, Y) be a 2-dimensional **discrete** random variable. Its joint probability (mass) function is defined by*
$$f_{X,Y}(x,y) = P(X = x, Y = y),$$
for $(x,y) \in R_{X,Y}$.

Properties of the Discrete Joint Probability Function
The joint probability mass function has the following properties:
(1) $f_{X,Y}(x,y) \geq 0$ for any $(x,y) \in R_{X,Y}$.
(2) $f_{X,Y}(x,y) = 0$ for any $(x,y) \notin R_{X,Y}$.
(3) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{X,Y}(x_i,y_j) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(X = x_i, Y = y_j) = 1$.
Equivalently, $\sum\sum_{(x,y) \in R_{X,Y}} f(x,y) = 1$.
(4) Let A be any subset of $R_{X,Y}$, then
$$P((X,Y) \in A) = \sum\sum_{(x,y) \in A} f_{X,Y}(x,y).$$

Definition 6 (Continuous joint probability function)
Let (X,Y) be a 2-dimensional ***continuous*** *random variable. Its* ***joint probability (density) function*** *is a function $f_{X,Y}(x,y)$ such that*

$$P((X,Y)\in D)=\iint_{(x,y)\in D} f_{X,Y}(x,y)\,dy\,dx,$$

for any $D\subset\mathbb{R}^2$. More specifically,

$$P(a\le X\le b, c\le Y\le d)=\int_a^b\int_c^d f_{X,Y}(x,y)\,dy\,dx.$$

Properties of the continuous joint probability function
The joint probability density function has the following properties:

(1) $f_{X,Y}(x,y)\ge 0$, for any $(x,y)\in R_{X,Y}$.

(2) $f_{X,Y}(x,y)=0$, for any $(x,y)\notin R_{X,Y}$.

(3) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy=1$.

Equivalently, $\iint_{(x,y)\in R_{X,Y}} f_{X,Y}(x,y)\,dx\,dy=1$.

Marginal Probability Distribution

Definition 7 (Marginal probability distribution)
Let (X,Y) be a two-dimensional random variable with joint probability function $f_{X,Y}(x,y)$. We define the ***marginal distribution*** *of X as follows.*

If Y is a discrete random variable, then for any x, hold x to be constant

$$f_X(x)=\sum_y f_{X,Y}(x,y).$$

If Y is a continuous random variable, then for any x,

$$f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

x	y = 0	1	2	3	Row Total
0	0	3/84	6/84	1/84	10/84
1	4/84	24/84	12/84	0	40/84
2	12/84	18/84	0	0	30/84
3	4/84	0	0	0	4/84
Column Total	20/84	45/84	18/84	1/84	1

- Each cell is $P(X=x,Y=y)$
- Column total represents $f_Y(y)$
- Row total represents $f_X(x)$

Conditional Distribution

$f_{Y,X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}$, $f_{X,Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}$

- Only defined for $f_X(x)>0$ and $f_Y(y)>0$
- the summation of the PDF is 1
- $P(Y\le y|X=x)=\int_{-\infty}^{y} f_{Y|X}(y|x)dy$
- $E(Y|X=x)=\int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy$
- Replace bound with actual bounds

Normal Distribution

- $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$
- Shape is identical if two curves share the same σ
- Curve flattens as σ increases
- $\Phi(x)$ is the CDF of the normal distribution
- $\Phi(x)$ = $P(x<2)$, $\Phi(-x)=1-\Phi(x)$
- $P(|x|<2)=2\Phi(x)-1=p(-2<x<2)$
- $P(x_1<x<x_2)=\Phi(\frac{x_2-\mu}{\sigma})-\Phi(\frac{x_1-\mu}{\sigma})$
- $z=\frac{x-\mu}{\sigma}$
- $\phi(z)=f_Z(z)$
- $\Phi(z)=\int_{-\infty}^{z}(\phi(t))dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}(e^{-\frac{t^2}{2}})dt$

Properties of Independent Random Variables

Definition 9 (Independent Random Variables)
Random variables X and Y are ***independent*** *if and only if for any x and y,*

$$f_{X,Y}(x,y)=f_X(x)f_Y(y).$$

Random variables $X_1,X_2,\ldots,X_n$ are ***independent*** *if and only if for* ***any*** *$x_1,x_2,\ldots,x_n$,*

$$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots x_n)=f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n).$$

- $R_{x,y}$ must be a product space (not a square)
- Any zeroes in the table indicate that $R_{x,y}$ is not product space
- If f(x,y) can be easily factorised to $c*g(x)*g(y)$, it is a product space

Properties of independent random variables
Suppose X,Y are independent random variables.

(1) If A and B are arbitrary subsets of $\mathbb{R}$, the events $X\in A$ and $Y\in B$ are independent events in S. Thus

$$P(X\in A; Y\in B)=P(X\in A)P(Y\in B).$$

In particular, for any real numbers x,y,

$$P(X\le x; Y\le y)=P(X\le x)P(Y\le y).$$

(2) For arbitrary functions $g_1(\cdot)$ and $g_2(\cdot)$, $g_1(X)$ and $g_2(Y)$ are independent. For example,
- X^2 and Y are independent.
- $\sin(X)$ and $\cos(Y)$ are independent.
- e^X and $\log(Y)$ are independent.

(3) Independence is connected with conditional distribution.
- If $f_X(x)>0$, then $f_{Y|X}(y|x)=f_Y(y)$.
- If $f_Y(y)>0$, then $f_{X|Y}(x|y)=f_X(x)$.

Central Limit Theorem

- Under CLT, sample mean approaches population mean as n increases
- Law of large numbers: $P(|\bar{x}-\mu_x|>\epsilon)\to 0$ as n increases
- Symmetric no outliers: 15-20
- Skewed: 30-50, heavily skewed: >1000
- $\sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}\to$ large n, higher accuracy

Margin of Error

Different Cases

	Population	σ	n	Statistic	E	n for desired E_0 and α
I	Normal	known	any	$Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$	$z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}$	$\left(\frac{z_{\alpha/2}\cdot\sigma}{E_0}\right)^2$
II	any	known	large	$Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$	$z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}$	$\left(\frac{z_{\alpha/2}\cdot\sigma}{E_0}\right)^2$
III	Normal	unknown	small	$T=\frac{\bar{X}-\mu}{S/\sqrt{n}}$	$t_{n-1;\alpha/2}\cdot\frac{s}{\sqrt{n}}$	$\left(\frac{t_{n-1;\alpha/2}\cdot s}{E_0}\right)^2$
IV	any	unknown	large	$Z=\frac{\bar{X}-\mu}{S/\sqrt{n}}$	$z_{\alpha/2}\cdot\frac{s}{\sqrt{n}}$	$\left(\frac{z_{\alpha/2}\cdot s}{E_0}\right)^2$

Confidence Interval

Confidence Intervals for the Mean
The table below gives the $(1-\alpha)$ confidence interval (formulas) for the population mean.

Case	Population	σ	n	Confidence Interval
I	Normal	known	any	$\bar{x}\pm z_{\alpha/2}\cdot\sigma/\sqrt{n}$
II	any	known	large	$\bar{x}\pm z_{\alpha/2}\cdot\sigma/\sqrt{n}$
III	Normal	unknown	small	$\bar{x}\pm t_{n-1;\alpha/2}\cdot s/\sqrt{n}$
IV	any	unknown	large	$\bar{x}\pm z_{\alpha/2}\cdot s/\sqrt{n}$

Note that n is considered large when $n\ge 30$.

Alternative pooled variance:

$s_p^2=\frac{\sum_{n_1}^{i=1}(x_i-\bar{x})^2+\sum_{n_2}^{i=1}(y_i-\bar{y})^2}{n_1+n_2-2}$

Equal Variance Assumption: $\frac{1}{2}<\frac{s_1}{s_2}<2$

Assumptions

- Normal population (esp for small samples)
- Random and independent sampling

	Do not reject H_0	Reject H_0
H_0 is true	Correct Decision	**Type I error**
H_0 is false	**Type II error**	Correct Decision

Definition 1 (Type I vs Type II error)
The rejection of H_0 when H_0 is true is called a ***Type I error****.*

Not rejecting H_0 when H_0 is false is called a ***Type II error****.*

Definition 2 (Significance Level vs Power)
The probability of making a Type I error is called the ***level of significance****, denoted by α. That is,*

$$\alpha=P(\text{Type I error})=P(\text{Reject } H_0 \mid H_0 \text{ is true}).$$

Let

$$\beta=P(\text{Type II error})=P(\text{Do not reject } H_0 \mid H_0 \text{ is false}).$$

We define $1-\beta=P(\text{Reject } H_0 \mid H_0 \text{ is false})$ to be the ***power of the test****.*