CS3230
AY21/22 SEM 2
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01. COMPUTATIONAL MODELS

- **algorithm** $\rightarrow$ a well-defined procedure for finding the correct solution to the input
- **correctness**
  - **worst-case correctness** $\rightarrow$ correct on *every valid input*
  - other types of correctness: correct on random input/with high probability/approximately correct
- **efficiency** / **running time** $\rightarrow$ measures the number of steps executed by an algorithm as a function of the *input size* (depends on computational model used)
  - number input: typically the length of binary representation
  - **worst-case** running time $\rightarrow$ *max* number of steps executed when run on an input of size n

adversary argument $\rightarrow$ inputs are decided such that they have different solutions

Comparison Model

- algorithm can **compare** any two elements in one time unit ($x > y$, $x < y$, $x = y$)
- running time = number of pairwise comparisons made
- array can be manipulated at no cost

Decision Tree

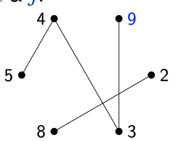
- each comparison represents the relationship between two elements
- each node is a comparison
- each branch is an outcome of the comparison
- log base is determined by the number of branches per node
- each leaf is a class label (decision after *all* comparisons)
- **lower bound of worst-case** runtime = height of tree
- # of leaves = # of permutations $\Rightarrow \lg(n!) = \Theta(n \lg n)$
- any decision tree that can sort n elements must have height $\Omega(n \lg n)$.

`Max` Problem

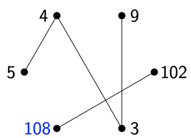
problem: find largest element in array A of n distinct elements

Proof. $n-1$ comparisons are needed

fix an algorithm M that solves the `Max` problem on all inputs using $< n-1$ comparisons. construct graph G where nodes i and j are adjacent iff M compares i & j.



A



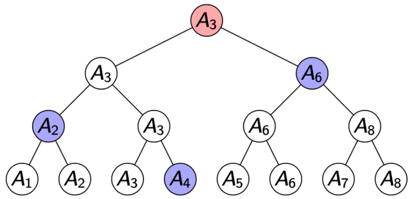
A'

M cannot differentiate A and A'.

Second Largest Problem

problem: find the second largest element in $< 2n-3$ comparisons (2x Maximum $\Rightarrow (n-1)+((n-1)-1) = 2n-3$)

- *solution*: **knockout tournament** $\Rightarrow n + \lceil \lg n \rceil - 2$



1. bracket system: $n-1$ matches
   - every non-winner has lost exactly once
2. then compare the elements that have lost to the largest
   - the 2nd largest element must have lost to the winner
   - compares $\lceil \lg n \rceil$ elements that have lost to the winner using $\lceil \lg n \rceil - 1$ comparisons

Sorting

Claim. there is a sorting algorithm that requires $\leq n \lg n - n + 1$ comparisons.

Proof. every sorting algorithm must make $\geq \lg(n!)$ comparisons.

1. let set $\mathcal{U}$ be the set of all permutations of the set $\{1, \ldots, n\}$ that the adversary could choose as array A. $|\mathcal{U}| = n!$
2. for each query "is $A_i > A_j$?", if $\mathcal{U}_{yes} = \{A \in \mathcal{U} : A_i > A_j\}$ is of size $\geq |\mathcal{U}|/2$, set $\mathcal{U} := \mathcal{U}_{yes}$. else: $\mathcal{U} := \mathcal{U} \backslash \mathcal{U}_{yes}$
3. the size of $\mathcal{U}$ decreases by at most half with each comparison
4. with $< \lg(n!)$ comparisons, $\mathcal{U}$ will still contain at least 2 permutations

$$n! \geq (\tfrac{n}{e})^n$$
$$\Rightarrow \lg(n!) \geq n \lg(\tfrac{n}{e}) = n \lg n - n \lg e$$
$$\approx n \lg n - 1.44n$$

$\Rightarrow$ roughly $n \lg n$ comparisons are **required** and **sufficient** for sorting n numbers

String Model

input	string of n bits
each query	find out **one bit** of the string

- n queries are **necessary** and **sufficient** to check if the input string is all 0s.
- **query complexity** $\rightarrow$ number of bits of the input string queried by the algorithm
- **evasive** $\rightarrow$ a problem requiring n query complexity

Graph Model

input	(symmetric) adjacency matrix of an n-node undirected graph
each query	find out if an edge is present between two chosen nodes (one entry of G)

- **evasive** $\rightarrow$ requires $\binom{n}{2}$ queries
- *Proof.* determining whether the graph is connected is evasive (requires $\binom{n}{2}$ queries)
  1. suppose M is an algorithm making $\leq \binom{n}{2}$ queries.
  2. whenever M makes a query, the algorithm tries not adding this edge, but adding all remaining unqueried edges.
     2.1. if the resulting graph is connected, M replies 0 (i.e. edge does not exist)
     2.2. else: replies 1 (edge exists)
  3. after $< \binom{n}{2}$ queries, at least one entry of the adjacency matrix is unqueried.

□

02. ASYMPTOTIC ANALYSIS

- **algorithm** $\rightarrow$ a *finite* sequence of well-defined instructions to solve a given computational problem
- **word-RAM model** $\rightarrow$ runtime is the total number of instructions executed
  - operators, comparisons, if, return, etc
  - each instruction operates on a *word* of data (limited size) $\Rightarrow$ fixed constant amount of time

Asymptotic Notations

upper bound ($\leq$): $f(n) = O(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq f(n) \leq cg(n)}$

lower bound ($\geq$): $f(n) = \Omega(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq cg(n) \leq f(n)}$

tight bound: $f(n) = \Theta(g(n))$
if $\exists c_1, c_2, n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)}$

o**-notation** ($<$): $f(n) = o(g(n))$
if $\forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq f(n) < cg(n)}$

ω**-notation** ($>$): $f(n) = \omega(g(n))$
if $\forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq cg(n) < f(n)}$

Proof. $(n+1)! \neq O(n!)$ since $\frac{(n+1)!}{n!} = (n+1) > c$

Limits

Assume $f(n), g(n) > 0$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \qquad \Rightarrow f(n) = o(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \qquad \Rightarrow f(n) = O(g(n))$$
$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \qquad \Rightarrow f(n) = \Theta(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \qquad \Rightarrow f(n) = \Omega(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \qquad \Rightarrow f(n) = \omega(g(n))$$

Proof.
1. Since $\lim_{n\to\infty} = 0$, we have for all $\epsilon > 0$, there exists $\delta > 0$ s.t $\frac{f(n)}{g(n)} < \epsilon$ for $n > \delta$
2. Set $c = \epsilon$ and $n_0 = \delta$
3. $\forall n \geq n_0$, $\frac{f(n)}{g(n)}$ <c
4. $\forall n \geq n_0$, $f(n) < cg(n)$
5. By definition, f(n)=o(g(n))

Properties of Big O

$\boxed{\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))}$

- **transitivity** - applies for $O, \Theta, \Omega, o, \omega$
  $f(n) = O(g(n)) \wedge g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
- **reflexivity** - for O, Ω, Θ, $\quad f(n) = O(f(n))$
- **symmetry** - $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$
- **complementarity** -
  - $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
  - $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$
- **misc**
  - if $f(n) = \omega(g(n))$, then $f(n) = \Omega(g(n))$
  - if $f(n) = o(g(n))$, then $f(n) = O(g(n))$

$\boxed{\log\log n < \log n < (\log n)^k < n^k < (n+1)! < k^n}$

insertion sort: $O(n^2)$ with worst case $\Theta(n^2)$

03. ITERATION, RECURSION, DIVIDE-AND-CONQUER

Iterative Algorithms

- **iterative** $\rightarrow$ loop(s), sequentially processing input elements
- **loop invariant** implies correctness if
  - *initialisation* - true before the first iteration of the loop
  - *maintenance* - if true before an iteration, it remains true at the beginning of the next iteration
  - *termination* - true when the algorithm terminates

examples

- **insertionSort**: with loop variable as j, $A[1..J-1]$ is sorted.
  1. A[1...i]=A'[1...i]. Elements not considered are unaffected.
  2. A[i+2...j]=A'[i+1...j-1]. Relative order of shifted elements is preserved.
  3. A[i+2...j] >key. Elements to its right are sorted and greater.
- **selectionSort**: with loop variable as j, the array $A[1..j-1]$ is sorted and contains the $j-1$ smallest elements of A.
- **Dijkstra's**:
  Proof. 1. **invariant 1** $\forall x \in R$: $dist[x] = \sigma(s, x)$
  2. **invariant 2** $\forall$ y neighbouring $x \in R$: $dist[y] = min_{x \in R} \sigma(s, x) + W(x, y)$

Recursive Algorithm

- **recursive** $\rightarrow$ solves sub problems
- Correctness is proven using **mathematical induction** on size of problem
- Use strong induction, prove base case, show algorithm works assuming it works for all smaller cases

Examples

Binary-Search $(A, a, b, x) \triangleright A[a\,..\,b]$
if $a > b$ **then**
  return *false*
else $mid = \lfloor (a+b)/2 \rfloor$
if $x == A[mid]$ **then**
  return *true*
if $x < A[mid]$ **then**
  return Binary-Search $(A, a, mid\text{-}1, x)$
else
  return Binary-Search $(A, mid{+}1, b, x)$

- **binary search**(A,a,b,x) returns the correct answer when b-a+1=n
- Base case: n=b-a+1=0, since a=b+1, A[a..b] is empty and the answer is false
- Inductive step: $n = b - a + 1 > 0$
- By strong induction, assume (A,a',b',x) returns the correct answer for all j s.t $0 \leq j \leq n - 1$ where $j = b' - a' + 1$
- By the algorithm, $mid = \lfloor \frac{a+b}{2} \rfloor$ and $a \leq mid \leq b$
- If $x == A[mid]$ then $x \in A[a..b]$ and the algorithm returns true correctly
- If $x < A[mid]$ then $x \in A[a..mid - 1] iff x \in A[a..b]$
- By the inductive hypothesis, (A,a,mid-1,x) is correct since $0 \leq (mid - 1) - a + 1 = mid - a \leq n - 1$
- The $x > A[mid]$ is similar

Divide-and-Conquer

powering a number

problem: compute $f(n, m) = a^n \pmod m$ for all $n, m \in \mathbb{Z}$

- observation: $f(x + y, m) = f(x, m) * f(y, m) \pmod m$
- **naive solution**: recursively compute and combine $f(n - 1, m) * f(1, m) \pmod m$
  - $T(n) = T(n - 1) + T(1) + \Theta(1) \Rightarrow T(n) = \Theta(n)$
- **better solution**: divide and conquer (only one sub problem computed)
  - divide: trivial
  - conquer: recursively compute $f(\lfloor n/2 \rfloor, m)$
  - combine:
    - $f(n, m) = f(\lfloor n/2 \rfloor, m)^2 \pmod m$ if n is even
    - $f(n, m) = f(1, m) * f(\lfloor n/2 \rfloor, m)^2 \pmod m$ if odd
  - $T(n) = T(n/2) + \Theta(1) \Rightarrow \Theta(\log n)$

Peak finding

problem: Find the peak element (no neighbours are greater) in 2D array

- ***Naive*** O(mn)
  - for the middle column, find the maximum element
  - return if it is peak
  - p1=Find2DPeak(left)
  - p2=Find2DPeak(right)
  - return p1 or p2 if one is a peak
- ***Divide and conquer*** O(mlogn), $T(n) = T(n/2) + O(1)$
  n is number of columns
-
  - find the middle column, find the maximum element
  - if it is a peak, return it
  - if not, recurse on the side with a larger element
- ***Optimised*** O(m+n)
  - find the middle column, find the maximum element
  - recurse on the quarter with the larger element

Solving Recurrences

for a sub-problems of size $\frac{n}{b}$ where $f(n)$ is the time to divide and combine,

$$T(n) = aT(\tfrac{n}{b}) + f(n)$$

Telescoping

Express $\frac{T(n)}{g(n)} as \frac{T(\frac{n}{b})}{g(\frac{n}{b})} + h(n)$

Choose the poly(n) part of f(n) as $\frac{f(n)}{b}$

1. Example: T(n) = 2T(n/2) + n
2. $\frac{T(n)}{n} = \frac{T(n/2)}{n/2} + 1$

$$\log n \left\{ \begin{aligned} \frac{T(n)}{n} &= \frac{T(n/\cancel{2})}{\cancel{n}/2} + 1 \\ \frac{T(n/\cancel{2})}{\cancel{n}/2} &= \frac{T(n/\cancel{4})}{\cancel{n}/4} + 1 \\ \frac{T(n/\cancel{4})}{\cancel{n}/4} &= \frac{T(n/\cancel{8})}{\cancel{n}/8} + 1 \\ &\dots \\ \frac{T(\cancel{2})}{\cancel{2}} &= \frac{T(1)}{1} + 1 \end{aligned} \right. \Rightarrow \frac{T(n)}{n} = \frac{T(1)}{1} + \log n$$

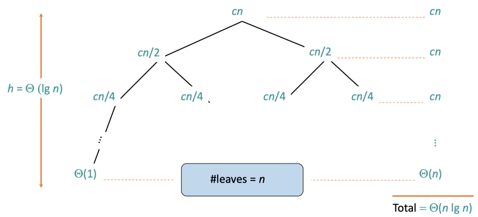
Hence, T(n) = O(n log n).

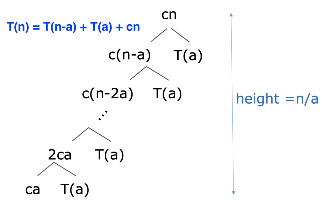
If g(n)=n, we need a=b since g(n) = $n^{\log_b(a)}$

Recursion tree

total = height $\times$ number of leaves

- each node represents the cost of a single subproblem
- height of the tree = longest path from root to leaf





Master method

- $a \geq 1, b > 1,$ and f is asymptotically positive.
- a (number of sub problems), b(size of sub problems), f(time to divide and combine)

$T(n) = aT(\frac{n}{b}) + f(n) =$

$$\begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) < n^{\log_b a} \text{ polynomially} \\ \Theta(n^{\log_b a} \log n) & \text{if } f(n) = n^{\log_b a} \\ \Theta(f(n)) & \text{if } f(n) > n^{\log_b a} \text{ polynomially} \end{cases}$$

three common cases

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$,
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ by n^ϵ factor.
   - then $T(n) = \Theta(n^{\log_b a})$.
   - This is when overhead at leaf $>$ overhead at root
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some $k \geq 0$,
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.
   - then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$,
   - and $f(n)$ satisfies the **regularity condition**
     - $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n
     - this guarantees that the sum of subproblems is smaller than $f(n)$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ by n^ϵ factor
   - then $T(n) = \Theta(f(n))$.
   - This is when the root (splitting) $>$ leaf

Substitution method

1. guess that $T(n) = O(f(n))$.
2. verify by induction:
   2.1. to show that for $n \geq n_0, T(n) \leq c \cdot f(n)$
   2.2. set $c = \max\{2, q\}$ and $n_0 = 1$
   2.3. verify base case(s): $T(n_0) = q$
   2.4. recursive case ($n > n_0$):
   - by strong induction, assume $T(k) \leq c \cdot f(k)$ for $n > k \geq n_0$
   - T(n) = ¡recurrence¿ ... $\leq c \cdot f(n)$

   2.5. hence $T(n) = O(f(n))$.

! may not be a tight bound!

example

Proof. $T(n) = 4T(n/2) + n^2/\lg n \Rightarrow \Theta(n^2 \lg\lg n)$

$$\begin{aligned} T(n) &= 4T(n/2) + \frac{n^2}{\lg n} \\ &= 4(4T(n/4) + \frac{(n/2)^2}{\lg n - \lg 2}) + \frac{n^2}{\lg n} \\ &= 16T(n/4) + \frac{n^2}{\lg n - \lg 2} + \frac{n^2}{\lg n} \\ &= \sum_{k=1}^{\lg n} \frac{n^2}{\lg n - k} \\ &= n^2 \lg\lg n \text{ by approx. of harmonic series } (\textstyle\sum \frac{1}{k}) \end{aligned}$$

Can also be solved via telescoping using g(n)=n^2

Proof. $T(n) = 4T(n/2) + n \Rightarrow O(n^2)$

To show that for all $n \geq n_0, T(n) \leq c_1 n^2 - c_2 n$

1. Set $c_1 = q + 1, c_2 = 1, n_0 = 1$.
2. Base case ($n = 1$): subbing into $c_1 n^2 - c_2 n$, $T(1) = q \leq (q + 1)(1)^2 - (1)(1)$
3. Recursive case ($n > 1$):

- by strong induction, assume $T(k) \leq c_1 \cdot k^2 - c_2 \cdot k$ for all $n > k \geq 1$
- $T(n) = 4T(n/2) + n$
  $$\begin{aligned} &= 4(c_1(n/2)^2 - c_2(n/2)) + n \\ &= c_1 n^2 - 2c_2 n + n \\ &= c_1 n^2 - c_2 n + (1 - c_2)n \\ &= c_1 n^2 - c_2 n \quad \text{since } c_2 = 1 \Rightarrow 1 - c_2 = 0 \end{aligned}$$

$\square$

Non-comparison sort

Counting sort

Set C[i] be the number of elements equal i.

for $i \leftarrow 1$ **to** k
  do $C[i] \leftarrow 0$
for $j \leftarrow 1$ **to** n
  do $C[A[j]] \leftarrow C[A[j]] + 1 \quad \triangleright C[i] = |\{\text{key} = i\}|$

Set C[i] be the number of elements smaller than or equal i.

for $i \leftarrow 2$ **to** k prefix sum
  do $C[i] \leftarrow C[i] + C[i-1] \quad \triangleright C[i] = |\{\text{key} \leq i\}|$

Move elements equal i to B[C[i-1]+1..C[i]].

for $j \leftarrow n$ **downto** 1
  do $B[C[A[j]]] \leftarrow A[j]$
    $C[A[j]] \leftarrow C[A[j]] - 1$

- O(n+k), where k is the number of elements
- Stable

Radix sort

Suppose there are T digits.

for $t \leftarrow 1$ **to** T
  do sort by the t^{th} least significant digit using a **stable** sorting algorithm;

3 2 9	7 2 0	7 2 0	3 2 9
4 5 7	3 5 5	3 2 9	3 5 5
6 5 7	4 3 6	4 3 6	4 3 6
8 3 9	4 5 7	8 3 9	4 5 7
4 3 6	6 5 7	3 5 5	6 5 7
7 2 0	3 2 9	4 5 7	7 2 0
3 5 5	8 3 9	6 5 7	8 3 9

Example: We can use counting sort as the stable sorting algorithm.

- Each pass processes r digits at $O(n + 2^r)$
- $T(n, b) = \theta(\frac{b}{2}(n + 2^r))$ where b is the number of bits in the key
- Optimal r=$\lg n$ T(n,b)=$\theta(\frac{bn}{\lg n})$
- Fast when the number of passes ($\frac{b}{r}$) is small
- Little locality of reference, not cache friendly

04. AVERAGE-CASE ANALYSIS & RANDOMISED ALGORITHMS

- **average case** $A(n) \rightarrow$ expected running time when the input is chosen uniformly at random from the set of all $n!$ permutations
  - $A(n) = \frac{1}{n!} \sum_\pi Q(\pi)$ where $Q(\pi)$ is the time complexity when the input is permutation π.
  - $A(n) = \underset{x \sim \mathcal{D}_n}{\mathbb{E}}$ [Runtime of Alg on x]
    - $\mathbb{E}_{x \sim \mathcal{D}_n}$ is a probability distribution on U restricted to inputs of size n.

Quicksort Analysis

- divide & conquer, linear-time $\Theta(n)$ partitioning subroutine
- assume we select the first array element as pivot
- $T(n) = T(j) + T(n - j - 1) + \Theta(n)$
  - if the pivot produces subarrays of size j and $(n - j - 1)$
- **worst-case**: $T(n) = T(0) + T(n - 1) + \Theta(n) \Rightarrow \Theta(n^2)$

Proof. for quicksort, $A(n) = O(n \log n)$.

let $P(i)$ be the set of all those permutations of elements $\{e_1, e_2, \dots, e_n\}$ that begins with e_i.

Let $G(n, i)$ be the average running time of quicksort over $P(i)$. Then

$$\begin{aligned} G(n) &= A(i - 1) + A(n - i) + (n - 1). \\ A(n) &= \frac{1}{n} \sum_{i=1}^{n} G(n, i) \\ &= \frac{1}{n} \sum_{i=1}^{n} (A(i - 1) + A(n - i) + (n - 1)) \\ &= \frac{2}{n} \sum_{i=1}^{n} A(i - 1) + n - 1 \end{aligned}$$

$= O(n \log n)$ by taking it as area under integration

quicksort vs mergesort

	average	best	worst
quicksort	$1.39n \lg n$	$n \lg n$	$n(n-1)$
mergesort	$n \lg n$	$n \lg n$	$n \lg n$

- disadvantages of mergesort:
  - overhead of temporary storage
  - cache misses
- advantages of quicksort
  - in place
  - reliable (as $n \uparrow$, chances of deviation from avg case $\downarrow$)
- issues with quicksort
  - **distribution-sensitive** $\rightarrow$ time taken depends on the initial (input) permutation. Resolved with median pivot or randomised partitions

Randomised Algorithms

- **randomised algorithms** $\rightarrow$ output and running time are **functions** of the **input** and **random bits chosen**
  - vs non-randomised: output & running time are functions of the *input only*
- expected running time = worst-case running time = $E(n) = \max_{\text{input } x \text{ of size } n} \mathbb{E}[\text{Runtime of RandAlg on } x]$
- **randomised quicksort**: choose pivot at random
  - probability that the runtime of *randomised* quicksort exceeds average by $x\% = n^{-\frac{x}{100} \ln \ln n}$
  - P(time takes at least double of the average) = 10^{-15}
  - distribution insensitive

Balls into bins - Indicator Random Variable

There are n balls and m bins. Each ball is placed into a bin at random. How many empty bins?

- X_i = 1 if ball i is in bin j, 0 otherwise
- $E(X_i) = 1 * P(i^{th} bin_{empty}) + 0 * P(...)$=$(1 - \frac{1}{n})^m$
- $E(X) = E(X_1) + E(X_2) + .. + E(X_n) = n(1 - \frac{1}{n})^m$

Randomised Quicksort Analysis

$T(n) = n - 1 + T(q-1) + T(n-q)$

Let $A(n) = \mathbb{E}[T(n)]$ where the expectation is over the randomness in expectation.

Taking expectations and applying linearity of expectation:

$A(n) = n - 1 + \frac{1}{n}\sum_{q=1}^{n}(A(q-1) + A(n-q))$

$= n - 1 + \frac{2}{n}\sum_{q=1}^{n-1} A(q)$

$A(n) = n \log n$ $\Rightarrow$ same as average case quicksort

Randomised Quickselect

- $O(n)$ to find the k^{th} smallest element
- randomisation: unlikely to keep getting a bad split

Types of Randomised Algorithms

- randomised **Las Vegas** algorithms
  - output is always correct
  - runtime is a *random variable*
  - e.g. randomised quicksort, randomised quickselect
- randomised **Monte Carlo** algorithms
  - output may be incorrect with some small probability
  - runtime is *deterministic*

Examples

- *smallest enclosing circle:* given n points in a plane, compute the smallest radius circle that encloses all n points
  - best **deterministic** algorithm: $O(n)$, but complex
  - Las Vegas: average $O(n)$, simple solution
- *minimum cut:* given a connected graph G with n vertices and m edges, compute the smallest set of edges whose removal would disconnect G.
  - best **deterministic** algorithm: $O(mn)$
  - **Monte Carlo**: $O(m \log n)$, error probability n^{-c} for any c
- *primality testing:* determine if an n bit integer is prime
  - best **deterministic** algorithm: $O(n^6)$
  - **Monte Carlo**: $O(kn^2)$, error probability 2^{-k} for k checks

Geometric Distribution

Let X be the number of trials repeated until success. X is a random variable and follows a geometric distribution with probability p.

Expected number of trials, $E[X] = \frac{1}{p}$

$$Pr[X = k] = q^{k-1}p$$

Linearity of Expectation

For any two events X, Y and a constant a,

$$E[X + Y] = E[X] + E[Y]$$
$$E[aX] = aE[X]$$

Coupon Collector Problem

n types of coupon are put into a box and randomly drawn with replacement. What is the expected number of draws needed to collect at least one of each type of coupon?

- let T_i be the time to collect the i-th coupon after the $i-1$ coupon has been collected.
  - Probability of collecting a new coupon, $p_i = \frac{(n-(i-1))}{n}$
  - T_i has a **geometric distribution**
  - $E[T_i] = 1/p_i$
- total number of draws, $T = \sum_{i=1}^{n} T_i$
- $E[T] = E[\sum_{i=1}^{n} T_i] = \sum_{i=1}^{n} E[T_i]$ by linearity of expectation
  $= \sum_{i=1}^{n} \frac{n}{n-(i-1)} = n \cdot \sum_{i=1}^{n} \frac{1}{i} = \Theta(n \lg n)$

05. HASHING

Dictionary ADT

- different types:
  - **static** - fixed set of inserted items; only care about queries
  - **insertion-only** - only insertions and queries
  - **dynamic** - insertions, deletions, queries
- implementations
  - sorted list (static) - $O(\log N)$ query
  - balanced search tree (dynamic) - $O(\log N)$ all operations
  - direct access table
    - $\times$ needs items to be represented as non-negative integers (**prehashing**)
    - $\times$ huge space requirement
- using $\mathcal{H}$ for dictionaries: need to store both the hash table and the matrix A.
  - additional storage overhead $= \Theta(\log N \cdot \log |U|)$, if $M = \Theta(N)$
  - other universal hashing constructions may have more efficient hash function evaluation
- *associative array* - has both key and value (dictionary in this context has only key)

Hashing

- **hash function**, $h : U \rightarrow \{1, \dots, M\}$ gives the location of where to store in the hash table
  - notation: $[M] = \{1, \dots, M\}[M] = \{1, \dots, M\}$
  - storing N items in hash table of size M
- **collision** $\rightarrow$ for two different keys x and y, $h(x) = h(y)$
  - resolve by **chaining**, **open addressing**, etc
- desired properties
  - ✓ minimise collisions - `query(x)` and `delete(x)` take time $\Theta(|h(x)|)$
  - ✓ minimise storage space - aim to have $M = O(N)$
  - ✓ function h is easy to compute (assume constant time)
- if $|U| \geq (N-1)M + 1$, for any $h : U \rightarrow [M]$, there is a set of N elements having the same hash value.
  - *Proof*: pigeonhole principle
- use **randomisation** to overcome the adversary
  - e.g. randomly choose between two *deterministic* hash functions h_1 and h_2
  - $\Rightarrow$ for any pair of keys, with probability $\geq \frac{1}{2}$, there will be no collision

Universal Hashing

Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$.

$\mathcal{H}$ is **universal** if $\forall x \neq y$, $\frac{|h \in \mathcal{H} : h(x) = h(y)|}{|H|} \leq \frac{1}{M}$

or $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$

- aka: for any $x \neq y$, if h is chosen uniformly at random from a universal $\mathcal{H}$, then there is at most $\frac{1}{M}$ probability that $h(x) = h(y)$
- probability where h is sampled uniformly from $\mathcal{H}$
- aka: for any $x \neq y$, the fraction of hash functions with collisions is at most $\frac{1}{M}$.

Properties of universal hashing

Collision Analysis

- for any N elements $x_1, \dots, x_N \in \mathcal{U}$, the **expected number of collisions** between x_N and other elements is $< N/M$.
  - it follows that for K operations, the expected cost of the last operation is $< K/M = O(1)$ if $M > K$.

Proof. by definition of Universal Hashing, each element $x_1, \dots, x_{N-1} \in \mathcal{U}$ has at most $\frac{1}{M}$ probability of collision with x_N (over random choice of h).

by indicator r.v., $E[A_i] = P(A_i = 1) \leq \frac{1}{M}$. expected number of collisions = $(N-1) \cdot \frac{1}{M} < \frac{N}{M}$.

- if $x_1, \dots, x_N$ are added to the hash table, and $M > N$, the expected **number of pairs** (i, j) with collisions is $< 2N$.

Proof. let A_{ij} be an indicator r.v. for collision.

$$\mathbb{E}[\sum_{1 \leq i,j \leq N} A_{ij}] = \sum_{i=1}^{N} \mathbb{E}[A_{ii}] + \sum_{i \neq j} \mathbb{E}[A_{ij}]$$
$$\leq N \cdot 1 + N(N-1) \cdot \frac{1}{M} < 2N$$

Expected Cost

- for any sequence of N operations, if $M > N$, then the **expected total cost** for executing the sequence is $O(N)$.

Proof. linearity of expectation; sum up expected costs

Construction of Universal Family

Obtain a universal family of hash functions with $M = O(N)$.

- Suppose U is indexed by u-bit strings and $M = 2^m$.
- For any $m \times u$ binary matrix A, $h_A(x) = Ax \pmod 2$
  - each element `x => x % 2`
  - x is a $u \times 1$ matrix $\Rightarrow Ax$ is $m \times 1$
- *Claim*: $\{h_A : A \in \{0,1\}^{m \times u}\}$ is universal
- e.g. $U = \{00, 01, 10, 11\}, M = 2$
  - h_{ab} means $A = [a\ b]$

	00	01	10	11
h_{00}	0	0	0	0
h_{01}	0	1	0	1
h_{10}	0	0	1	1
h_{11}	0	1	1	0

Proof. Let $x \neq y$. Let $z = x - y$. We know $z \neq 0$.

Collision: $P(Ax{=}Ay){=}P[A(x{-}y){=}0]{=}P(Az{=}0)$.

To show $P(Az = 0) \leq \frac{1}{M}$.

Special case - Suppose z is 1 at the i-th coordinate but 0 everywhere else. Then Az is the i-th column of A. Since the i-th column is uniformly random, $P(Az = 0) = \frac{1}{2^m} = \frac{1}{M}$.

General case - Suppose z is 1 at the i-th coordinate. Let $z = [z_1\ z_2\ \dots\ z_u]^T$. $A = [A_1\ A_2\ \dots\ A_u]$ hence A_k is the k-th column of A.

Then $Az = z_1A_1 + z_2A_2 + \cdots + z_uA_u$.

$Az = 0 \Rightarrow z_1A_1 = -(z_2A_2 + \cdots + z_uA_u) \quad (*)$

We fix z_1A_1 to be an arbitrary $m \times 1$ matrix of 1s and 0s. The probability that $(*)$ holds is $\frac{1}{2^m}$.

Perfect Hashing

static case - N fixed items in the dictionary $x_1, x_2, \dots, x_N$

To perform `Query` *in $O(1)$ worst-case time.*

Quadratic Space: $M = N^2$

if $\mathcal{H}$ is universal and $M = N^2$, and h is sampled uniformly from $\mathcal{H}$, then the expected number of collisions is < 1.

Proof. for $i \neq j$, let indicator r.v. A_{ij} be equal to 1 if $h(x_i) = h(x_j)$, or 0 otherwise.

By universality, $E[A_{ij}] = P(A_{ij} = 1) \leq 1/N^2$

$E[\#\text{ collisions}] = \sum_{i<j} E[A_{ij}] \leq \binom{N}{2}\frac{1}{N^2} < 1$

It follows that there exists $h \in \mathcal{H}$ causing no collisions (because if not, $\mathbb{E}$[#collisions] would be ≥ 1).

2-Level Scheme: $M = N$

- No collision and less space needed

Construction

Choose $h : U \to [N]$ from a universal hash family.

- Let L_k be the number of x_i's for which $h(x_i) = k$.
- Choose $h_1, \ldots, h_N$ **second-level** hash functions $h_k : [N] \to [(L_k)^2]$ s.t. there are no collisions among the L_k elements mapped to k by h.
  - quadratic second-level table $\to$ ensures no collisions using quadratic space

Analysis

if $\mathcal{H}$ is universal and h is sampled uniformly from $\mathcal{H}$, then

$$E\left[\sum_k L_k^2\right] < 2N$$

Proof. For $i, j \in [1, N]$, define indicator r.v. $A_{ij} = 1$ if $h(x_i) = h(x_j)$, or 0 otherwise.

A_{ij} = # possible collisions = # pairs * 2 = L_k^2
Hence $\sum_k L_k^2 = \sum_{i,j} A_{ij}$

$$E[\sum_{i,j} A_{ij}] = \sum_i E[A_{ii}] + \sum_{i \neq j} E[A_{ij}]$$
$$\leq N \cdot 1 + N(N-1) \cdot \frac{1}{N}$$
$$< 2N$$

Hash Table Resizing

- when number of inserted items, N is not known
  - **rehashing** - choose a new hash function of a larger size and re-hash all elements
  - costly but infrequent $\Rightarrow$ amortize

06. FINGERPRINTING & STREAMING

String Pattern Matching

problem: does the pattern string P occur as a substring of the text string T?

m = length of P, n = length of T, ℓ = size of alphabet

- assumption: operations on strings of length $O(\log n)$ can be executed in $O(1)$ time. (word-RAM model)
- naive solution: $\Theta(n^2)$

Fingerprinting approach (Karp-Rabin)

- faster string equality check:
  - for substring X, check $h(X) == h(P)$ for a hash function $h \Rightarrow \Theta(1)$ + cost of hashing instead of $\Theta(|X|)$
- **Rolling Hash**: $O(m + n)$
  - update the hash from what we already have from the previous hash - $O(1)$
  - compute $n - m + 1$ hashes in $O(n)$ time
  - *Monte Carlo* algorithm

Division Hash

Choose a random **prime** number p in the range $\{1, \ldots, K\}$.
For integer x, $h_p(x) = x \pmod p$

- if p is small and x is b-bits long in binary, hashing $\Rightarrow O(b)$
- hash family $\{h_p\}$ is approximately universal
- if $0 \leq x < y < 2^b$, then $\underset{h}{Pr}[h_p(x) = h_p(y)] < \frac{b \ln K}{K}$

Proof. $h_p(x) = h_p(y)$ when $y - x = 0 \pmod p$.
Let $z = y - x$.
Since $z < 2^b$, then z can have at most b distinct prime factors.
p divides z if p is one of these $\leq b$ prime factors.
number of primes in range $\{1, \ldots, K\}$ is $> \frac{K}{\ln K}$,
hence the probability is $b / \frac{K}{\ln K} = \frac{b \ln K}{K}$

values of K

- higher K = lower probability of false positive
  - for $\delta = \frac{1}{100n}$, P(false positive) ¡ 1%.

$\forall \delta > 0$, if $X \neq Y$ and $K = \frac{2m}{\delta} \cdot \lg \ell \cdot \lg(\frac{2m}{\delta} \lg \ell)$, then $Pr[h(X) = h(Y)] < \delta$

Streaming

problem: Consider a sequence of insertions or deletions of items from a large universe $\mathcal{U}$. At the end of the stream, the *frequency* f_i of item i is its net count.

Let M be the sum of all frequencies at the end of stream.

naive solutions

- direct access table - $\Omega(U)$ space
- sorted list - $\Omega(M)$ space, no $O(1)$ update
- binary search tree - $O(M)$ space

Frequency Estimation

an approximation $\hat{f}_i$ is **ϵ-approximate** if

$$f_i - \epsilon M \leq \hat{f}_i \leq f_i + \epsilon M$$

Using Hash Table

$$f_i \leq \mathbb{E}[\hat{f}_i] \leq f_i + M/k$$

- increment/decrement $A[h(j)]$ on an empty table A of size k
- collision $\Rightarrow$ false positives $\Rightarrow$ may give overestimate of f_i
  - $A[h(i)] = \sum_{j: h(j) = h(i)} f_j \geq f_i$
- if h is drawn from a universal family, overestimate, $\mathbb{E}[A[h(i)] - f_i] \leq M/k$
- space: $O(\frac{1}{\epsilon} \cdot \lg M + \lg U \cdot \lg M)$
  let $k = \frac{1}{\epsilon}$ for some $\epsilon > 0$.
  - number of rows $= O(\frac{1}{\epsilon})$
  - size of each row $= O(\lg M)$
  - size of hash function (using universal hash family from ch.05) $= O(\lg U \cdot \lg M)$
- **Count-Min Sketch** $\to$ gives a bound on the probability that $\hat{f}_i$ deviates from f_i instead of a bound on the expectation of the gap

07. AMORTIZED ANALYSIS

- **amortized analysis** $\to$ guarantees the *average* performance of each operation in the *worst case*.
- total amortized cost provides an *upper bound* on the total true cost
- For a sequence of n operations $o_1, o_2, \ldots, o_n$,
  - let $t(i)$ be the time complexity of the i-th operation o_i
  - let $f(n)$ be the *worst-case* time complexity for *any* of the n operations
  - let $T(n)$ be the time complexity of all n operations

$$T(n) = \sum_{i=1}^{n} t(i) = n f(n)$$

Types of Amortized Analysis

Aggregate method

- look at the whole sequence, sum up the cost of operations and take the average - simpler but less precise
- e.g. binary counter - amortized $O(1)$
- e.g. queues (with `INSERT` and `EMPTY`) - amortized $O(1)$
- Find (a) The number of operations and (b) the upperbound of each operation
  - $a = n$
  - $b = \sum_{i=1}^{n} t(i) = n f(n)$

Accounting method

- charge the i-th operation a fictitious amortized cost $c(i)$
  - **amortized cost** $c(i)$ is a fixed cost for each operation
  - **true cost** $t(i)$ depends on when the operation is called
- amortized cost $c(i)$ must satisfy:

$$\sum_{i=1}^{n} t(i) \leq \sum_{i=1}^{n} c(i) \text{ for all } n$$

- take the extra amount for cheap operations early on as "credit" paid in advance for expensive operations
  - **invariant**: bank balance never drops below 0
- the total amortized cost provides an **upper bound** on the total true cost

Potential method

- ϕ : potential function associated with the algo/DS
- $\phi(i)$: potential at the end of the i-th operation
- c_i : amortized cost of the i-th operation
- t_i : true cost of the i-th operation

$$c_i = t_i + \phi(i) - \phi(i-1)$$
$$\sum_{i=1}^{n} c_i = \phi(n) - \phi(0) + \sum_{i=1}^{n} t_i$$

- hence as long as $\phi(n) \geq 0$, then amortized cost is an upper bound of the true cost.

$$\sum_{i=1}^{n} c_i \geq \sum_{i=1}^{n} t_i$$

- **Validity** $\phi(0) = 0$ and $\phi(i) \geq 0$ for all i
- **e.g.** for queue:
  - let $\phi(i)$ = # of elements in queue after the i-th operation
  - amortized cost for insert:
    $c_i = t_i + \phi(i) - \phi(i-1) = 1 + 1 = 2$
  - amortized cost for empty (for k elements):
    $c_i = t_i + \phi(i) - \phi(i-1) = k + 0 - k = 0$
- try to keep $c(i)$ small: using $c(i) = t(i) + \Delta\phi_i$
  - if $t(i)$ is small, we want $\Delta\phi_i$ to be positive and small
  - if $t(i)$ is large, we want $\Delta\phi_i$ to be negative and large

e.g. Dynamic Table (insertion only)

Aggregate method

cost of n insertions =
$$\sum_{i=1}^{n} t(i) \leq n + \sum_{j=1}^{\lfloor \log(n-1) \rfloor} 2^j \leq 3n$$

i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
$t(i)$	1	1	1	1	1	1	1	1	1	1
		1	2		4				8	

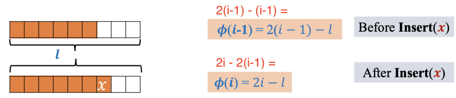
Cost for i th insert
Cost for copying due to overflow

Accounting method

- charge \$3 per insertion
  - \$1 for insertion itself
  - \$1 for moving itself when the table expands
  - \$1 for moving one of the existing items when the table expands

Potential method

let $\phi(i) = 2i - size(T)$



Operation Insert(x)	Actual Cost	$\Delta\phi_i$	Amortized Cost
Case 1: when table is not full	1	2	3
Case 2: when table is already full	i	$3 - i$	3

Amortized cost of n insertions = $3n = O(n)$

Actual cost of n insertions = $O(n)$

- show that SUM of amortized cost $\geq$ SUM of actual cost
- conclude that sum of amortized cost is $O(f(n)) \Rightarrow$ sum of actual cost is $O(f(n))$

08. DYNAMIC PROGRAMMING

- **cut-and-paste proof** $\to$ proof by contradiction - suppose you have an optimal solution. Replacing ("cut") subproblem solutions with this subproblem solution ("paste" in) should improve the solution. If the solution doesn't improve, then it's not optimal (contradiction).
- **overlapping subproblems** - recursive solution contains a small number of distinct subproblems repeated many times
- **optimal substructure** - optimal solution to a problem contains optimal solutions to subproblems

Longest Common Subsequence

- for sequence $A : a_1, a_2, \ldots, a_n$ stored in array
- C is a **subsequence** of $A \to$ if we can obtain C by removing zero or more elements from A.

problem: given two sequences $A[1..n]$ and $B[1..m]$, compute the *longest* sequence C such that C is a subsequence of A and B.

brute force solution

- check *all* possible subsequences of A to see if it is also a subsequence of B, then output the longest one.
- analysis: $O(m2^n)$
  - checking each subsequence takes $O(m)$
  - 2^n possible subsequences

recursive solution

let $LCS(i, j)$: longest common subsequence of $A[1..i]$ and $B[1..j]$

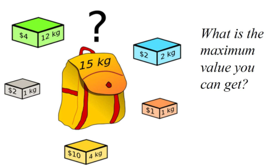
- base case: $LCS(i, 0) = \emptyset$ for all i, $LCS(0, j) = \emptyset$ for all j
- general case:
  - if last characters of A, B are $a_n = b_m$, then $LCS(n, m)$ must terminate with $a_n = b_m$
    - the optimal solution will match a_n with b_m
  - if $a_n \neq b_m$, then either a_n or b_m is not the last symbol
- **optimal substructure**: (general case)
  - if $a_n = b_m$,
    $LCS(n, m) = LCS(n-1, m-1) :: a_n$
  - if $a_n \neq b_m$,
    $LCS(n, m) = LCS(n-1, m) \mid\mid LCS(n, m-1)$

- **simplified problem**:
  - $L(n,m) = 0$ if $n = 0$ or $m = 0$
  - if $a_n = b_m$, then $L(n,m) = L(n-1, m-1) + 1$
  - if $a_n \neq b_m$, then $L(n,m) = \max(L(n, m-1), L(n-1, m))$

analysis
  - number of distinct subproblems $= (n+1) \times (m+1)$
  - to use $O(\min\{m,n\})$ space: bottom-up approach, column by column
  - memoize for DP $\Rightarrow$ makes it $O(mn)$ instead of exponential time

Knapsack Problem

- input: $(w_1, v_1), (w_2, v_2), \dots, (w_n, v_n)$ and capacity W
- output: subset $S \subseteq \{1, 2, \dots, n\}$ that maximises $\sum_{i \in S} v_i$ such that $\sum_{i \in S} w_i \leq W$



- 2^n subsets $\Rightarrow$ naive algorithm is costly
- **recursive solution**:
  - let $m[i,j]$ be the maximum value that can be obtained using a subset of items $\{1, 2, \dots, i\}$ with total weight no more than j.
  - $m[i,j] = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \\ \max\{m[i-1, j-w_i] + v_i, m[i-1,j]\}, & \text{if } w_i \leq j \\ m[i-1, j], & \text{otherwise} \end{cases}$
- **analysis**: $O(nW)$
  - **!** $O(nW)$ is **not** a polynomial time algorithm
  - not polynomial in input bitsize
    - W can be represented in $O(\lg W)$ bits
    - n can be represented in $O(\lg n)$ bits
  - polynomial time is strictly in terms of the number of bits for the input

Changing Coins

problem: use the fewest number of coins to make up n cents using denominations $d_1, d_2, \dots, d_n$. Let $M[j]$ be the fewest number of coins needed to change j cents.

- **optimal substructure**:
  - $M[j] = \begin{cases} 1 + \min_{i \in [k]} M[j - d_i], & j > 0 \\ 0, & j = 0 \\ \infty, & j < 0 \end{cases}$

  Proof. Suppose $M[j] = t$, meaning $j = d_{i_1} + d_{i_2} + \dots + d_{i_t}$ for some $i_1, \dots, i_t \in \{1, \dots, k\}$.
  Then, if $j' = d_{i_1} + d_{i_2} + \dots + d_{i_{t-1}}$, $M[j'] = t - 1$, because otherwise if $M[j'] < t - 1$, by **cut-and-paste** argument, $M[j] < t$.
- runtime: $O(nk)$ for n cents, k denominations

Dijsktra's Algorithm

- **property 1**: The nearest neighbor is also the vertex nearest to s
- Violated by **negative edges** since there can exist a shorter path to a vertex that is not the nearest neighbor
- **property 2**: Optimal subpath property
- Violated when there are **negative cycles** since the final parth can consist of a subparth that is not the optimal subpath

09. GREEDY ALGORITHMS

- solve only one subproblem at each step
- beats DP and divide-and-conquer when it works
- **greedy-choice property** $\rightarrow$ a locally optimal choice is globally optimal
- **Note:** This is not true for DP, a clear counter-example is that the LCS of a substring is not necessarily in the LCS of the entire string

Examples

Fractional Knapsack

- $O(n \log n)$
- **greedy-choice property**: let j^* be the item with *maximum* value/kg, v_j / w_i. Then there exists an optimal knapsack containing $\min(w_{j^*}, W)$ kg of item j^*.
- **optimal substructure**: if we remove w kg of item j from the optimal knapsack, then the remaining load must be the optimal knapsack weighing at most $W - w$ kgs that one can take from $n - 1$ original items and $w_j - w$ kg of item j.

Proof. cut-and-paste argument

Suppose the remaining load after removing w kgs of item j was *not* the optimal knapsack weighing ...

Then there is a knapsack of value $> X - v_j \cdot \frac{w}{w_j}$ with weight ...

Combining this knapsack with w kg of item j gives a knapsack of value $> X \Rightarrow$ contradiction!

Minimum Spanning Trees

for a connected, undirected graph $G = (V, E)$, find a spanning tree T that connects all vertices with minimum weight. Weight of spanning tree T, $w(T) = \sum_{(u,v) \in T} w(u,v)$.

- **optimal substructure**: let T be a MST. remove any edge $(u,v) \in T$. then T is partitioned into T_1, T_2 which are MSTs of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

  Proof. cut-and-paste: $w(T) = w(u,v) + w(T_1) + w(T_2)$

  if $w(T_1') < w(T_1)$ for G_1, then $T' = \{(u,v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G.

  $\Rightarrow$ contradiction, T is the MST
- **Prim's algorithm** - at each step, add the least-weight edge from the tree to some vertex outside the tree
- Exchange Argument:
- *Proof.* Let T' be the optimal solution without edge (u,v). Swap (u,v) with the first edge along the simple path from u to v in T'. Clearly, a lighter weight spanning tree than T' results.
- **Kruskal's algorithm** - at each step, add the least-weight edge that does *not* cause a cycle to form

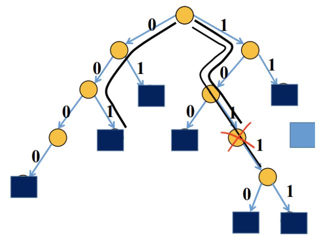
Binary Coding

Given an alphabet set $A : \{a_1, a_2, \dots, a_n\}$ and a text file F (sequence of alphabets), how many bits are needed to encode a text file with m characters?

- **fixed length encoding**: $m \cdot \lceil \log_2 n \rceil$
  - encode each alphabet to unique binary string of length $\lceil \log_2 n \rceil$
  - total bits needed for m characters $= m \cdot \lceil \log_2 n \rceil$
- **variable length encoding**
  - different characters occur with different frequency $\Rightarrow$ use fewer bits for *more frequent* alphabets
  - average bit length, $ABL(\gamma) = \sum_{x \in A} f(x) \cdot |\gamma(x)|$
  - BUT overlapping prefixes cause indistinguishable characters
  - E.g. $0|10|10|111, 0|101|0|111$

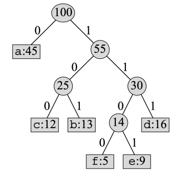
Prefix coding

Problem: Given a set A of n alphabets and their frequencies, compute coding γ such that:
- γ is a prefix code
- ABL(γ) is minimum
- a coding $\gamma(A)$ is a **prefix coding** if $\nexists x, y \in A$ such that $\gamma(x)$ is a prefix of $\gamma(y)$.
- **labelled binary tree**: $\gamma(A)$ = label of path from root



- for each prefix code A of n alphabets, there exists a binary tree T on n leaves such that there is a **bijective mapping (1-1)** between the alphabets and the leaves
- $ABL(\gamma) = \sum_{x \in A} f(x) \cdot |\gamma(x)| = \sum_{x \in A} f(x) \cdot |depth_T(x)|$
- the binary tree corresponding to an *optimal* prefix coding must be a **full binary tree**.
  - every internal node has degree exactly 2
  - multiple possible optimal trees - most optimal depends on alphabet frequencies
- accounting for alphabet **frequencies**:
  - let $a_1, a_2, \dots, a_n$ be the alphabets of A in non-decreasing order of their frequencies.
  - a_1 must be a leaf node; a_2 *can* be a sibling of a_1.
  - If a_1 swapping a_1 with a higher (frequency) none-leaf node results in a less optimal solution
  - there exists an optimal prefix coding in which a_1 and a_2 are siblings
- **Greedy algorithm** for prefix coding:
  - A = $\{a_1, a_2, \dots, a_n\}$ in ascending order of freq
  - A' = $\{a_3, a_4, \dots, a - i \dots a_n\}$ in ascending order of freq
  - f(a') = f(a_1) + f(a_2)
  - $OPT_ABL(A) = OPT_ABL(A') + f(a_1) + f(a_2)$
- derivation of optimal prefix coding: **Huffman's algorithm**
  - keep merging the two least frequent items

```
Huffman(C):
  Q = new PriorityQueue(C)
  while Q:
    allocate a new node z
    z.left = x = extractMin(Q)
    z.right = y = extractMin(Q)
    z.val = x.val + y.val
    Q.add(z)
  return extractMin(Q) // root
```

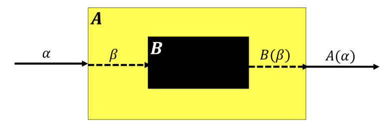


10. REDUCTIONS & INTRACTABILITY

Reduction

Consider two problems A and B, A can be solved as follows:

1. convert instance α of A to an instance of β in B
2. solve β to obtain a solution
3. based on the solution of β, obtain the solution of α.
4. $\Rightarrow$ then we say A **reduces** B.



instance $\rightarrow$ another word for input

e.g. Matrix Multiplication & Squaring

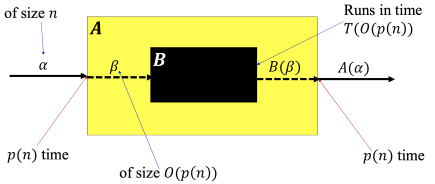
- MAT-MULTI: matrix multiplication
  - *input*: two $N \times N$ matrices A and B.
  - *output*: $A \times B$
- MAT-SQR: matrix squaring
  - *input*: one $N \times N$ matrix C. *output*: $C \times C$
- MAT-SQR can be reduced to MAT-MULTI
  - *Proof.* Given input matrix C for MAT-SQR, let $A = C$ and $B = C$ be inputs for MAT-MULTI. Then $AB = C^2$.
- MAT-MULTI can also be reduced to MAT-SQR!
  - *Proof.* let $C = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ $\Rightarrow C^2 = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}$

T-Sum

- 0-SUM: given array A, output $i, j \in (1, n)$ such that $A[i] + A[j] = 0$
- T-SUM: given array B, output $i, j \in (1, n)$ such that $B[i] + B[j] = T$
- **reduce T-SUM to 0-SUM**:
  - given array B, define array A s.t. $A[i] = B[i] - T/2$.
  - if i, j satisfy $A[i] + A[j] = 0$, then $B[i] + B[j] = T$.

$p(n)$-time Reduction

- $p(n)$**-time Reduction** $\rightarrow$ if for any instance α of problem A of size n,
  - an instance β for B can be constructed in $p(n)$ time
  - a solution to problem A for input α can be recovered from a solution to problem B for input β in time $p(n)$.
- **!** n is in **bits**!
- if there is a $p(n)$-time reduction from problem A to B and a $T(n)$-time algorithm to solve problem B, then there is a $T(O(p(n))) + O(p(n))$ time algorithm to solve A.



- $A \leq_P B$ $\rightarrow$ if there is a $p(n)$-time reduction from A to B for some polynomial function $p(n) = O(n^c)$ for some constant c. ("A is a special case of B")
  - if B has a polynomial time algorithm, then so does A
  - "polynomial time" $\approx$ reasonably efficient
- $A \leq_P B, B \leq_P C \Rightarrow A \leq_P C$

Polynomial Time

- **polynomial time** $\rightarrow$ runtime is polynomial in the **length of the encoding** of the problem instance
- **"standard" encodings**
  - binary encoding of integers
  - list of parameters enclosed in braces (graphs/matrices)
- **pseudo-polynomial** algorithm $\rightarrow$ runs in time polynomial in the **numeric value** if the input but is **exponential** in the **length** of the input
  - e.g. DP algo for Knapsack since W is in numeric value
- Knapsack is NOT polynomial time: $O(nW \log M)$ but W is not the number of bits
- Fractional Knapsack is polynomial time: $O(n \log n \log W \log M)$

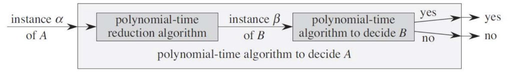
Decision Problems

- **decision problem** $\rightarrow$ a function that maps an instance space I to the solution set $\{YES, NO\}$
- decision vs optimisation problem:
  - **decision problem**: given a directed graph G, *is there* a path from vertex u to v of length $\leq k$?
  - **optimisation problem**: given ..., what is the *length* of the shortest path ... ?
  - convert from **decision** $\rightarrow$ **optimisation**: given an instance of the optimisation problem and a number k, is there a solution with value $\leq k$?
- the decision problem is *no harder than* the optimisation problem.
  - given the optimal solution, check that it is $\leq k$.
  - if we cannot solve the decision problem quickly $\Rightarrow$ then we cannot solve the optimisation problem quickly
- decision $\leq_P$ optimisation

Reductions between Decision Problems

given two decision problems A and B, a polynomial-time reduction from A to B denoted $A \leq_P B$ is a **transformation** from instances α of A and β of B such that

1. α is a *YES*-instance of A $\iff$ β is a *YES*-instance of B
2. the transformation takes polynomial time in the size of α



Examples

- Independent-Set: given a graph $G = (V, E)$ and an integer k, is there a subset of $\leq k$ vertices such that no 2 are adjacent?
- Vertex-Cover: given a graph $G = (V, E)$ and an integer k, is there a subset of $\leq k$ vertices such that each edge is incident to *at least one* vertex in this subset?
- Independent-Set $\leq_P$ Vertex-Cover
  - *Reduction*: to check whether G has an independent set of size k, we check whether G has vertex cover of size $n - k$.

Proof. If Independent-Set, then Vertex-Cover.

Suppose (G, k) is a *YES*-instance of Indep-Set. Then there is subset S of size $\geq k$ that is an independent set.

$V - S$ is a vertex cover of size $\leq n - k$. Proof: Let $(u, v) \in E$. Then $u \notin S$ or $v \notin S$.

So either u or v is in $V - S$, the vertex cover.

Proof. If Vertex-Cover, then Independent-Set.

Same as above, but flip IS and VC

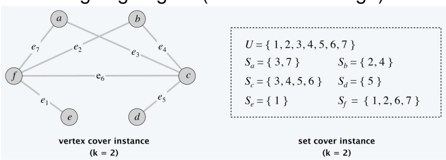
e.g. Set-Cover

Given integers k and n, and collection $\mathcal{S}$ of subsets of $\{1, \ldots, n\}$, are there $\leq k$ of these subsets whose union equals $\{1, \ldots, n\}$?

Claim: **Vertex-Cover** $\leq_P$ **Set-Cover**

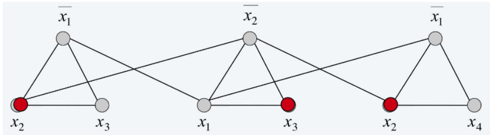
Reduction: given (G, k) instance of Vertex-Cover, generate an instance $(n, k', \mathcal{S})$ of Set-Cover.

Proof. For each node v in G, construct a set S_v containing all its outgoing edges. (Number each edge)



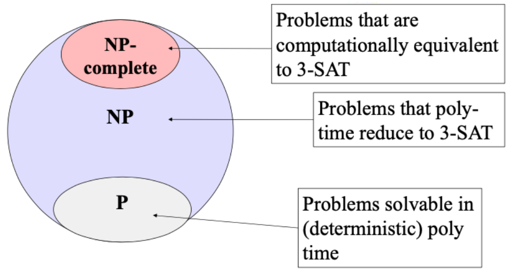
e.g. 3-SAT

- **SAT**: given a CNF formula Φ, does it have a satisfying truth assignment?
  - literal: a boolean variable or its negation $x, \bar{x}$
  - clause: a disjunction (OR) of literals
  - conjunctive normal form (CNF): formula Φ that is a conjunction (AND) of clauses
- **3-SAT** $\rightarrow$ SAT where each clause contains exactly 3 literals
- **3-SAT** $\leq_P$ **Independent-Set**
  - *Reduction*: Construct an instance (G, k) of Indep-Set s.t. G has an independent set of size k $\iff$ Φ is satisfiable
  - node: each literal term
  - edge: connect 3 literals in a clause in a triangle
  - edge: connect literal to all its negations
  - reduction runs in polynomial time
- $\Rightarrow$ for k clauses, connecting k vertices form an independent set in G.



11. NP-COMPLETENESS

- **P** $\rightarrow$ the class of *decision* problems solvable in (deterministic) polynomial time
- **NP** $\rightarrow$ the class of *decision* problems for which polynomial-time verifiable **certificates** of YES-instances exist.
  - aka *non-deterministic polynomial*
  - i.e. no poly-time algo, but verification can be poly-time
  - **certificate** $\rightarrow$ result that can be checked in poly-time to verify correctness
- $P \subseteq NP$: any problem in **P** is in **NP**.
  - if $P = NP$, then all these algos can be solved in poly time



NP-Hard and NP-Complete

- a problem A is said to be **NP-Hard** if for *every* problem $B \in NP$, $B \leq_P A$.
  - aka A is at least as hard as every problem in **NP**.
- a problem A is said to be **NP-Complete** if it is in **NP** and is also **NP-Hard**
  - aka the hardest problems in NP.
- **Cook-Levin Theorem** $\rightarrow$ every problem in NP-Hard can be poly-time *reduced* to 3-SAT. Hence, **3-SAT is NP-Hard and NP-Complete**.
- NP-Complete problems can still be approximated in poly-time! (e.g. greedy algorithm gives a 2-approximation for Vertex-Cover)

showing NP-Completeness

1. show that X is in NP. $\Rightarrow$ a YES-instance has a certificate that can be verified in polynomial time
2. show that X is NP-hard
   - by giving a poly-time reduction from another NP-hard problem A to X. $\Rightarrow$ X is at least as hard as A
   - reduction should *not* depend on whether the instance of A is a YES- or NO-instance
3. show that the reduction is valid
   3.1. reduction runs in poly time
   3.2. if the instance of A is a YES-instance, then the instance of X is also a YES-instance
   3.3. if the instance of A is a NO-instance, then the instance of X is also a NO-instance

```
def INDEPENDENT-SET(G, k) -> bool:
    G', k' = reduction(G, k)
    yes_or_no: bool = CLIQUE(G', k') # magically given
    return yes_or_no
```

What to show for a correct reduction:

- `(G, k)` is YES-instance → `(G', k')` is also a YES-instance
- `(G', k')` is YES-instance → `(G, k)` is also a YES-instance
- The transformation takes polynomial time in the size of `(G, k)`

showing NP-HARD

1. take any **NP-Complete** problem A
2. show that $A \leq_P X$

helpful approximations

stirling's approximation: $T(n) = \sum_{i=0}^{n} \log(n-i) = \log \prod_{i=0}^{n}(n-i) = \Theta(n \log n)$

$loglog(2) + loglog(4) + \cdots + loglog(n) = log(n)loglog(n)$

harmonic number, $H_n = \sum_{k=1}^{n} \frac{1}{k} = \Theta(\lg n)$

basel problem: $\sum_{n=1}^{N} \frac{1}{n^2} \leq 2 - \frac{1}{N} \xrightarrow{N \to \infty} 2$

because $\sum_{n=1}^{N} \frac{1}{N^2} \leq 1 + \sum_{x=2}^{\log_3 n} \frac{1}{(x-1)x} = 1 + \sum_{n=2}^{N}(\frac{1}{n-1} - \frac{1}{n}) = 1 + 1 - \frac{1}{N} = 2 - \frac{1}{N}$

number of primes in range $\{1, \ldots, K\}$ is $> \frac{K}{\ln K}$

asymptotic bounds

$1 < \log n < \sqrt{n} < n < n \log n < n^2 < n^3 < 2^n < 2^{2n}$
$\log_a n < n^a < a^n < n! < n^n$
for any $a, b > 0$, $\quad \log_a n < n^b$

multiple parameters

for two functions $f(m,n)$ and $g(m,n)$, we say that $f(m,n) = O(g(m,n))$ if there exists constants c, m_0, n_0 such that $0 \leq f(m,n) \leq c \cdot g(m,n)$ for all $m \geq m_0$ or $n \geq n_0$.

set notation

$O(g(n))$ is actually a *set of functions*. $f(n) = O(g(n))$ means $f(n) \in O(g(n))$

- $O(g(n)) = \{f(n) : \exists c, n_0 > 0 \mid \forall n \geq n_0,\ 0 \leq f(n) \leq cg(n)\}$
- $\Omega(g(n)) = \{f(n) : \exists c, n_0 > 0 \mid \forall n \geq n_0,\ 0 \leq cg(n) \leq f(n)\}$
- $\Theta(g(n)) = \{f(n) : \exists c_1, c_2, n_0 > 0 \mid \forall n \geq n_0, \quad 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\} = O(g(n)) \cap \Omega(g(n))$
- $o(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0 \mid \forall n \geq n_0, \quad 0 \leq f(n) < cg(n)\}$
- $\omega(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0 \mid \forall n \geq n_0, \quad 0 \leq cg(n) < f(n)\}$

example proofs

Proof. that $2n^2 = O(n^3)$

let $f(n) = 2n^2$. then $f(n) = 2n^2 \leq n^3$ when $n \geq 2$.
set $c = 1$ and $n_0 = 2$.
we have $f(n) = 2n^2 \leq c \cdot n^3$ for $n \geq n_0$. □

Proof. $n = o(n^2)$

For any $c > 0$, use $n_0 = 2/c$.

Proof. $n^2 - n = \omega(n)$

For any $c > 0$, use $n_0 = 2(c+1)$.

Example. let $f(n) = n$ and $g(n) = n^{1+\sin(n)}$.

Because of the oscillating behaviour of the sine function, there is no n_0 for which f dominates g or vice versa.
Hence, we cannot compare f and g using asymptotic notation.

Example. let $f(n) = n$ and $g(n) = n(2 + \sin(n))$.

Since $\frac{1}{3}g(n) \leq f(n) \leq g(n)$ for all $n \geq 0$, then $f(n) = \Theta(g(n))$. (note that limit rules will not work here)

mentioned algorithms

- ch.3 - **Misra Gries** - space-efficient computation of the majority bit in array A
- ch.3 - **Euclidean** - efficient computation of GCD of two integers
- ch.3 - **Tower of Hanoi** - $T(n) = 2^n - 1$
  1. move the top $n-1$ discs from the first to the second peg using the third as temporary storage.
  2. move the biggest disc directly to the empty third peg.
  3. move the $n-1$ discs from the second peg to the third using the first peg for temporary storage.
- ch.3 - **MergeSort** - $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n)$
- ch.3 - **Karatsuba Multiplication** - multiply two n-digit numbers x and y in $O(n^{\log_2 3})$
  - worst-case runtime: $T(n) = 3T(\lceil n/2 \rceil) + \Theta(n)$

uncommon notations

- $\perp$ - false

NP-Complete problems

1. Circuit-SAT
2. CNF-SAT
3. 3-SAT
4. Independent Set
5. Vertex Cover
6. Max-Clique
7. Hamiltonian Cycle (both directed and undirected)
8. Hamiltonian Path (undirected)
9. Traveling Sales Person
10. Subset Sum
11. Knapsack
12. Hitting Set