ST233
AY22/23 Sem 2
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Chapter 1: Basic Concepts of Probability

Definition 1 (Experiment, Sample space, Event)
*A **statistical experiment** is any procedure that produces data or observations.*

*The **sample space**, denoted by S, is the set of all possible outcomes of a statistical experiment. The sample space depends on the problem of interest!*

*A **sample point** is an outcome (element) in the sample space.*

*An **event** is a subset of the sample space.*

Union: $A \cup B = x : x \in A \vee x \in B$
Union of n events: $\cup_{i=1}^{n} = x : x \in A_1 \vee x \in A_2 ...$
Union: $A \cap B = x : x \in A \wedge x \in B$
Interection of n events: $\cap_{i=1}^{n} = x : x \in A_1 \wedge x \in A_2 ...$
Complement: $A' = x : x \in S \wedge x \notin A$
Mutually Exclusive: $A \cap B = \emptyset$

More Event Operations
(a) $A \cap A' = \emptyset$
(b) $A \cap \emptyset = \emptyset$
(c) $A \cup A' = S$
(d) $(A')' = A$
(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(g) $A \cup B = A \cup (B \cap A')$
(h) $A = (A \cap B) \cup (A \cap B')$

De Morgan's Law
For any n events $A_1, A_2, \ldots, A_n$,
(i) $(A_1 \cup A_2 \cup \ldots \cup A_n)' = A_1' \cap A_2' \cap \ldots \cap A_n'$.
A special case: $(A \cup B)' = A' \cap B'$.
(j) $(A_1 \cap A_2 \cap \ldots \cap A_n)' = A_1' \cup A_2' \cup \ldots \cup A_n'$.
A special case: $(A \cap B)' = A' \cup B'$.

Multiplication
Sequential experiments
$n_1 * n_2 * ... * n_r$
outcomes

Addition
Independent experiments
$n_1 + n_2 + ... + n_r$
outcomess

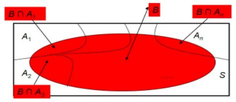
Permutation: $P_r^n = \frac{n!}{(n-r)!} = n(n-1)...(n-(r-1))$
Selection and arrangement of r objects out of n. Order is considered.
Combination: $C_r^n = \frac{n!}{r!(n-r)!} = \frac{n(n-1)...(n-(r-1))}{r!}$
Selection of r objects out of n. Order is not considered.
$C_r^n = C_{n-r}^n$
Axioms and Properties of Probability

1. $0 \leq P(A) \leq 1$
2. P(S)=1
3. $P(A \cup B) = P(A) + P(B)$ if A and B are **mutex** (not to be confused with independence)
4. P($\emptyset$)=0
5. P($A_1 \cup A_2 \cup ... A_n$)=$\sum_{i=1}^{n} A_i$
6. P(A')=$1 - P(A)$
7. $P(A \cup B) = P(A) + P(B) - P(AB)$
8. P(A)=$P(A \cap B) + P(A \cap B')$
9. $A \subset B \rightarrow P(A) \leq P(B)$
10. $P(A_1) = P(A_2) = ... = P(A_k) \rightarrow for B \subset S, P(B) = \frac{|B|}{|S|}$

Conditional Probability (B occurs given A):
$P(B|A) = \frac{P(A \cap B)}{P(A)}$
Multiplication rules
$P(A \cap B) = P(A)P(B|A)$ if $P(A)! = 0$
$P(A \cap B) = P(B)P(A|B)$ if $P(B)! = 0$
Inverse Probability Formula: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
Indepenence $A \perp B iff P(A \cap B) = P(A)P(B)$ This implies $P(A|B) = P(A)$ and $P(B|A) = P(B)$
Partition
$A_1, A_2 ... A_r are mutually exclusive and \sum_{i=1}^{n} A_i = S$

Theorem 11 (Law of Total Probability)
Suppose $A_1, A_2, \ldots, A_n$ is a partition of S. Then for any event B, we have

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(A_i) P(B|A_i).$$



$P(B) = P(A)P(B|A) + P(A')P(B|A')$
Bayes' theorem
$P(A_k|B = \frac{P(B|A)P(A_k)}{P(A)P(B|A) + P(A')P(B|A')})$
Bayes' Theorem (n=2): $P(A|B) = \frac{P(B|A)P(A)}{\sum_{i=1}^{n} P(A_i)P(B|A_k)}$
Denom is P(B)

L2:Random Variables

Random variable X is a function from S to R
Uppercase letters denote random variables
Lowercase letters denote observed values of random variables

Definition 3 (Probability Mass Function)
For a discrete random variable X, define

$$f(x) = \begin{cases} P(X = x), & \text{for } x \in R_X; \\ 0, & \text{for } x \notin R_X. \end{cases}$$

*Then $f(x)$ is known as the **probability function (pf)**, or **probability mass function (pmf)** of X.*

*The collection of pairs $(x_i, f(x_i)), i = 1, 2, 3, \ldots$, is called the **probability distribution** of X.*

Properties of the Probability Mass Function
The probability mass function $f(x)$ of a discrete random variable **must** satisfy:
(1) $f(x_i) \geq 0$ for all $x_i \in R_X$;
(2) $f(x) = 0$ for all $x \notin R_X$;
(3) $\sum_{i=1}^{\infty} f(x_i) = 1$, or $\sum_{x_i \in R_X} f(x_i) = 1$.

For any set $B \subset \mathbb{R}$, we have

$$P(X \in B) = \sum_{x_i \in B \cap R_X} f(x_i).$$

Definition 4 (Probability Density Function)
*The **probability density function** of a continuous random variable X, denoted by $f(x)$, is a function that satisfies:*

(1) $f(x) \geq 0$ for all $x \in R_X$; and $f(x) = 0$ for $x \notin R_X$;

(2) $\int_{R_X} f(x)\, dx = 1$;

(3) For any a and b such that $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

note: $P(a < X < b) = P(a \leq X \leq B)$
To chcek that a function is a PDF, chcek conditions 1 and 2
Cumulative Distribution Function: $F(x) = P(X \leq x)$