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P1.

i). For paraxial Helmholtz equation:

$$\begin{cases} \nabla_T^2 A - j2k\dfrac{\partial A}{\partial z} = 0 \\ \nabla_T^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \end{cases}$$

For $A(r) = (\frac{A_0}{z})\exp[-jk(x^2+y^2)/2z]$,

i.e $\dfrac{\partial^2 A(r)}{\partial x^2} = \dfrac{A_0}{z^3} k^2 x^2 \exp[-jk(x^2+y^2)/2z]$

~~i.e~~ $\cancel{\nabla_T^2}$ Similarly $\dfrac{\partial^2 A(r)}{\partial y^2} = \dfrac{A_0}{z^3} k^2 y^2 \exp[-jk(x^2+y^2)/2z]$

i.e. $\nabla_T^2 = \dfrac{\partial^2 A(r)}{\partial x^2} + \dfrac{\partial^2 A(r)}{\partial y^2} = \dfrac{A_0}{z^3} k^2 (x^2+y^2)\exp[-jk(x^2+y^2)/2z]$

$$\frac{\partial A(r)}{\partial z} = \left[-\frac{A_0}{z^2} + \frac{A_0 jk(x^2+y^2)}{2z^3}\right]\exp[-jk(x^2+y^2)/2z]$$

$$= \frac{jkA_0}{z^3}\exp\left[-jk\frac{x^2+y^2}{2z}\right](x^2+y^2)$$

$\cancel{\text{i.e. } \nabla_T^2 A = \frac{\partial A}{\partial z}}$

i.e. $\nabla_T^2 A - j2k\dfrac{\partial A}{\partial z} = \dfrac{A_0}{z^3} k^2 (x^2+y^2)\exp[-jk(x^2+y^2)/2z]$

$$- \frac{A_0}{z^3} k^2 \exp[-jk(x^2+y^2)/2z](x^2+y^2) = 0$$

Hence $A(r)$ satisfies the paraxial Helmholtz equation

ii). For $A(r) = \dfrac{A}{q(z)}\exp\left[-jk\dfrac{x^2+y^2}{2q(z)}\right]$

Similarly to i). $\dfrac{\partial A}{\partial x} = \dfrac{-jkAx}{q^2(z)}\exp\left[-jk\dfrac{x^2+y^2}{2q(z)}\right]$

$$\frac{\partial^2 A}{\partial x^2} = -A\,\frac{k^2x^2 + jkq(z)}{q^3(z)}\exp\left[-jk\frac{x^2+y^2}{2q(z)}\right]$$

$$\frac{\partial^2 A}{\partial y^2} = -A\,\frac{k^2y^2 + jkq(z)}{q^3(z)}\exp\left[-jk\frac{x^2+y^2}{2q(z)}\right]$$

$$\frac{\partial A}{\partial z} = A\cdot\frac{2jk-4}{4q^2(z)}\exp\left[-jk\frac{x^2+y^2}{2q(z)}\right]$$

$\rightarrow \nabla_T^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = -A\left[\frac{k^2(x^2+y^2)+2jkq(z)}{q^3(z)}\right]\exp\left[-jk\frac{x^2+y^2}{2q(z)}\right]$

i.e. $\nabla_T^2 A + 2jk\frac{\partial A}{\partial z} = \left[\exp\left[-jk\frac{x^2+y^2}{2q(z)}\right]\right]\left[-A\frac{k^2(x^2+y^2)+2jkq(z)}{q^3(z)} + A\frac{k^2(x^2+y^2)+2jkq(z)}{q^3(z)}\right]$

i.e. It also satisfies the paraxial Helmholtz equation

P2. For $U(r) = \frac{A_0}{r}\exp(-jkr)$

$\frac{\partial U}{\partial r} = -\frac{A_0}{r^2}\exp(-jkr) + \frac{A_0}{r}(-jk)\exp(-jkr)$

$\frac{\partial^2 u}{\partial r} = -\frac{1}{r^2}[-A_0 jk\exp(-jkr) - A_0 jk^2 r\exp(-jkr) + A_0 jk\exp(-jkr)]$

$= \frac{A_0}{r}k^2\exp(-jkr)$

i.e. $\nabla^2 u + k^2 u$

$= -\frac{A_0}{r}k^2\exp(-jkr) + \frac{A_0}{r}k^2\exp(-jkr) = 0$

i.e. $U(r)$ satisfies the Helmholtz equation.

P3. Since the law of reflection is $k_1 r = k_2 r$, the orientation of the reflected wave is turned by π from the original orientation of the spherical wave.

P4. Suppose that $c(x,y,z)$ is the incident wave's complex amplitude.

$\frac{C(x,y,d_0)}{C(x,y,0)} = \frac{C(x,y,\Sigma d_q)}{C(x,y,0)} = e^{-j\Sigma d_q n_q k_0}$

When $n=1$, $e^{-j\Sigma d_q n_q k_0} = e^{-jdk_0}$

i.e. $\Sigma d_q n_q k_0 = dk_0$

i.e. $d = \Sigma n_q d_q$

P5. Suppose that

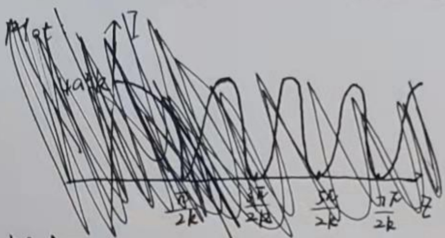
$$\begin{cases} y_1 = a\sin(\omega t + kz) \\ y_2 = a\sin(\omega t - kz) \end{cases}$$

$$y = y_1 + y_2 = a\sin(\omega t + kz) + a\sin(\omega t - kz) = 2a\sin\omega t\cos kz$$

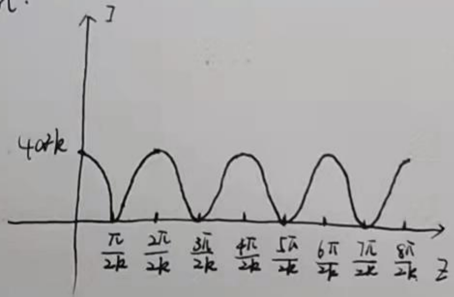
$$= (2a\cos kz)\sin\omega t$$

$$A = 2a\cos kz$$

$$I = kA^2 = 4a^2\cos^2 kz \cdot k = \boxed{4a^2 k\cos^2 kz}$$



Plot:



P6.

a) For $z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi \cdot (10^{-3})^2 \times \frac{1}{20^2}}{633 \times 10^{-9}} = 0.0124\,m$.

i.e. $\theta_0 = \frac{w_0}{z_0} = \frac{\frac{1}{20} \times 10^{-3}}{0.0124} = 4.0 \times 10^{-3}$ rad

$P_f = 2z_0 = 0.0248\,m$

$d = 2w = 2\theta_0 z = 2.82 \times 10^{-6}\,m$

b) $R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right] = z + \frac{z_0^2}{z}$

i). When $z = 0$, $R(z) \to \infty$

ii) When $z = z_0$. $R(z) = z_0 + z_0 = 2z_0 = 0.0248\,m$

iii) When $z = 2z_0$ $R(z) = 2z_0 + \frac{z_0}{2} = 0.0318\,m$.

c) When $\rho = 0$, $z = 0$

P7. i)

Since $R = z\left[1+\left(\frac{z_0}{z}\right)^2\right]$

i.e. $W = w_0 \sqrt{\left[1+\left(\frac{z_0}{z}\right)^2\right]}$

$$W^2 = w_0^2\left[1+\left(\frac{z_0}{z}\right)^2\right]$$

$$= \frac{\lambda z_0}{\pi}\left[1+\left(\frac{z_0}{z}\right)^2\right]$$

i.e. $\frac{R}{z} = \frac{W^2\pi}{\lambda z_0}$

i.e. $\frac{z_0}{z} = \frac{R\lambda}{\pi W^2} = \frac{z\left[1+\left(\frac{R\lambda}{\pi W^2}\right)^2\right]\lambda}{\pi W^2}$

i.e. $z = \frac{R}{1+\frac{\lambda^2 R^2}{\pi^2 W^4}} = \frac{R\pi^2 W^4}{\pi^2 W^4 + \lambda^2 R^2} = \frac{R}{1+(\lambda R/\pi W^2)^2}$

Hence eq. 3.1-24 is proved.

ii). According to i).

$$z = \frac{R}{1+(\lambda R/\pi W^2)^2}$$

$$W = w_0\sqrt{1+\left(\frac{z}{z_0}\right)^2}$$

i.e. $w_0 = \frac{W}{\left[1+(\pi W^2/\lambda R)^2\right]^{\frac{1}{2}}}$

Hence, eq. 3.1-25 is proved.

P8. a). Since $A = 0.1 \times 0.1 = 0.01\,mm^2$

$$S = \frac{P}{A} = 10^8\,W/m^2$$

For $S = \frac{P}{A} = \frac{EB}{\mu} = \frac{E_{rms}^2}{\mu c} = 10^8$

i.e $E_{rms}^2 = \mu c \cdot 10^8 = 1.256 \times 10^{-6} \times 3 \times 10^8 \times 10^8$

$= 3.768 \times 10^2 \times 10^8 = 3.768 \times 10^{10}$

i.e $E_{rms} = \sqrt{3.768 \times 10^{10}}$

$E_{peak} = \sqrt{2}\,E_{rms} = \sqrt{7.536 \times 10^{10}} = 2.745 \times 10^5\,V/m$

b). $A = \pi \cdot W_0^2 \approx 3.14 \times 10^{-8}\,m^2$

$$S = \frac{P}{A} \approx 3.18 \times 10^7\,W/m^2$$

Similarly $E_{rms} = \sqrt{S \mu c} \approx 1.1 \times 10^5\,V/m$

$E_{peak} = \sqrt{2}\,E_{rms} \approx 1.55 \times 10^5\,V/m.$

P9. i) According to Maxwell equation:

$$\begin{cases} \nabla \times \vec{H} = \sigma \frac{\partial \vec{E}}{\partial t} \\ \nabla \times \vec{E} = -\mu_0 \frac{\partial H}{\partial t} \end{cases}$$

$$\nabla \times \vec{B} = \frac{jE_0 k}{c} e^{j(\omega t - kx)} \hat{y} = \mu_0 \sigma \vec{E} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$= \mu_0 \sigma E_0 e^{j(\omega t - kx)} \hat{y} - \mu_0 \epsilon j \omega E_0 e^{j(\omega t - kx)} \hat{y}$$

$$= (\mu_0 \sigma E_0 - \mu_0 \epsilon j \omega E_0)\, e^{j(\omega t - kx)} \hat{y}.$$

i.e $\frac{jE_0 k}{c} = \mu_0 \sigma E_0 - \mu_0 \epsilon j \omega E_0$

i.e $k = \mu_0 c (\epsilon \omega - j\sigma)$

ii). $\alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$ => $\boxed{\alpha = \omega \sqrt{\frac{\mu\epsilon}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - \frac{\mu\epsilon}{2}}}$

P10. i)

$$J_\alpha = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \quad \text{Suppose that } J(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

When $\alpha = 0$, $J_\alpha = \begin{bmatrix} \cos 0 & \sin 0 \\ \sin 0 & -\cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$J_{out} = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) \\ \sin(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = J(-\theta)$$

Hence the transmitted light is linearly polarized at an angle $-\theta$.

ii) The linearly polarized light rotates twice ~~the~~ of the angles when transmits through a halfwave retarder to the transmission through a polarization rotator.

P11 i)

$$\cos\theta_2 = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1} = \sqrt{1 - \left(\frac{1}{1.5}\right)^2 \sin^2(45^\circ)} = 0.88$$

i.e. $\theta_2 = \cos^{-1}(0.88) = 28.1^\circ$

$$r_x = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 - n_2\cos\theta_2} = \frac{\cos 45^\circ - 1.5\cos(28.1^\circ)}{\cos 45^\circ + 1.5\cos(28.1^\circ)} = -0.303$$

$$r_y = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2} = \frac{1.5\cos 45^\circ - \cos(28.1^\circ)}{1.5\cos 45^\circ + \cos(28.1^\circ)} = 0.092$$

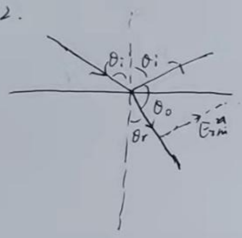
i.e. $R_x = |r_x|^2 = 0.092 \qquad R_y = |r_y|^2 = 0.0084$

i.e. The intensity reflectances of TE wave is 0.092

- - — — — - — . — TM wave is 0.0084.

ii). The average reflectance is $\left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1 - 1.5}{1 + 1.5}\right)^2 = 0.04$

P12.



Since $\frac{n_1}{n_2} = \frac{\cos\theta_i}{\cos\theta_r}$ ①

According to Snell's Law

$$\frac{n_1}{n_2} = \frac{\sin\theta_r}{\sin\theta_i} \quad ②$$

①②,

$$\frac{\cos\theta_i}{\cos\theta_r} = \frac{\sin\theta_r}{\sin\theta_i}$$

i.e. $\theta_i + \theta_r = \frac{\pi}{2}$

Hence. $\theta_0 = \frac{\pi}{2}$

i.e. The direction of the reflected and refracted waves are orthogonal.

Since $\hat{E}_{TM} \perp$ the direction of the refracted wave, which means, the electric field of the reflected TM wave is parallel to the direction of the reflected wave,