Machine Learning 1 EX 11

19.01.2016

1. **Kernology**

(a)

i. $\sum_{i=1}^{n}\sum_{j=1}^{n} C_i C_j k(x_i, x_j)$

$= a(c_1(c_1 + c_2 + ... + c_n) + c_2(c_1 + c_2 + ... + c_n) + ... + c_n(c_1 + c_2 + ... + c_n))$

$= a(c_1 + c_2 + ... + c_n)(c_1 + c_2 + ... + c_n)$

$= a(c_1 + c_2 + ... + c_n)^2 \geq 0$

Therefore $k(x, x') = a$ is a Mercer kernel.

ii. According to the definition of inner product, we can get the following properties:

· Conjugate symmetry:

$< x, y >= \overline{< y, x >}$;

· Linearity in the first argument:

$< ax, y >= a < x, y >$;

$< x + y, z >=< x, z > + < y, z >$;

· Positive-definiteness:

$< x, x >\geq 0$.

When $F = \mathbb{R}$, the following equations are also satisfied:

$< x, ay >= \overline{< ay, x >} = \overline{a}\overline{< y, x >} = \overline{a} < x, y >$;

$< x, y + z >= \overline{< y + z, x >} = \overline{< y, x >} + \overline{< z, x >} =< x, y > + < x, z >$.

Therefore:

$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i, x_j)$

$= c_1^2 < x_1, x_1 > + c_1c_2 < x_1, x_2 > + ... + c_1c_n < x_1, x_n > + c_2c_1 < x_2, x_1 > + c_2^2 < x_2, x_2 > + ... + c_2c_n < x_2, x_n > + ... + c_nc_1 < x_n, x_1 > + c_nc_2 < x_n, x_2 > + ... + c_n^2 < x_n, x_n >$

$= c_1^2 < x_1, x_1 > + \frac{x_2}{x_1}c_1c_2 < x_1, x_2 > + ... + \frac{x_n}{x_1}c_1c_n < x_1, x_n > + \frac{x_2}{x_1}c_2c_1 < x_2, x_1 > + (\frac{x_2}{x_1})^2c_2^2 < x_1, x_1 > + ... + \frac{x_2}{x_1}\cdot\frac{x_n}{x_1}c_2c_n < x_1, x_1 > + ... + \frac{x_n}{x_1}c_nc_1 < x_1, x_1 > + \frac{x_n}{x_1}\cdot\frac{x_2}{x_1}c_nc_2 < x_1, x_1 > + ... + (\frac{x_n}{x_1})^2c_n^2 < x_1, x_1 >$

$=< x_1, x_1 > \cdot(c_1(c_1 + \frac{x_2}{x_1}c_2 + \frac{x_3}{x_1}c_3 + ... + \frac{x_n}{x_1}c_n) + \frac{x_2}{x_1}c_2(c_1 + \frac{x_2}{x_1}c_2 + \frac{x_3}{x_1}c_3 + ... + \frac{x_n}{x_1}c_n) + ... + \frac{x_n}{x_1}c_n(c_1 + \frac{x_2}{x_1}c_2 + \frac{x_3}{x_1}c_3 + ... + \frac{x_n}{x_1}c_n))$

$=< x_1, x_1 > \cdot (c_1 + \frac{x_2}{x_1}c_2 + ... + \frac{x_n}{x_1}c_n) \cdot (c_1 + \frac{x_2}{x_1}c_2 + ... + \frac{x_n}{x_1}c_n)$

$=< x_1, x_1 > \cdot (c_1 + \frac{x_2}{x_1}c_2 + ... + \frac{x_n}{x_1}c_n)^2 \geq 0.$

Hence $k(x, x') =< x, x' >$ is a Mercer kernel.

iii. $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j k(x_i, x_j)$

$= c_1^2 f^2(x_1) + c_1c_2 f(x_1)f(x_2) + ... + c_1c_n f(x_1)f(x_n) + c_2c_1 f(x_2)f(x_1) + c_2^2 f^2(x_2) + ... + c_2c_n f(x_2)f(x_n) +$
$... + c_nc_1 f(x_n)f(x_1) + c_nc_2 f(x_n)f(x_2) + ... + c_n^2 f^2(x_n)$

$= c_1 f(x_1)(c_1 f(x_1) + c_2 f(x_2) + ... + c_n f(x_n)) + c_2 f(x_2)(c_1 f(x_1) + c_2 f(x_2) + ... + c_n f(x_n)) + ... +$
$c_n f(x_n)(c_1 f(x_1) + c_2 f(x_2) + ... + c_n f(x_n))$

$= (c_1 f(x_1) + c_2 f(x_2) + ... + c_n f(x_n)) \cdot (c_1 f(x_1) + c_2 f(x_2) + ... + c_n f(x_n))$

$= (c_1 f(x_1) + c_2 f(x_2) + ...c_n f(x_n))^2 \geq 0.$

Therefore $k(x, x') = f(x) \cdot f(x')$ is a Mercer kernel.

(b)

i. $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j k(x_i, x_j)$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j (k_1(x_i, x_j) + k_2(x_i, x_j))$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j k_1(x_i, x_j) + \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j k_2(x_i, x_j) \geq 0.$

Therefore $k(x, x') = k_1(x, x') + k_2(x, x')$ is a Mercer kernel.

ii. as $\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j k_1(x_i, x_j) \geq 0$, $\sum_{f=1}^{n} \sum_{g=1}^{n} b_f b_g k_2(x_f, x_g) \geq 0$

holds for any coefficients a_i, a_j, b_f, b_g,

so, $(\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j k_1(x_i, x_j)) \cdot (\sum_{f=1}^{n} \sum_{g=1}^{n} b_f b_g k_2(x_f, x_g))$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{f=1}^{n} \sum_{g=1}^{n} a_i a_j b_f b_g (k_1(x_i, x_j) k_2(x_f, x_g)) \geq 0,$

also holds for any coefficients a_i, a_j, b_f, b_g,

we set :

$$\begin{cases} b_f = 1, b_g = 1 & f = i, g = j \\ b_f = 0, b_g = 0 & f \neq i, or, g \neq j \end{cases}$$

so,

$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{f=1}^{n} \sum_{g=1}^{n} a_i a_j b_f b_g (k_1(x_i, x_j) k_2(x_f, x_g))$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j k_1(x_i, x_j) k_2(x_i, x_j) \geq 0.$

Therefore $k(x, x') = k_1(x, x') \cdot k_2(x, x')$ is a Mercer kernel.

(c) $k_1(x, x') =< x, x' > \cdots\cdots$ a Mercer kernel as (a)ii has proved.

$k_2(x, x') = \theta \cdots\cdots$ a Mercer kernel as (a)i has proved.

$k_3(x, x') = k_1(x, x') + k_2(x, x') =< x, x' > +\theta \cdots\cdots$ a Mercer kernel as (b)i has proved.

$k_4(x, x') = k_3(x, x') k_3(x, x') = (< x, x' > +\theta)^2 \cdots\cdots$ a Mercer kernel as (b)ii has proved.

$k_5(x, x') = k_4(x, x') k_3(x, x') = (< x, x' > +\theta)^3 \cdots\cdots$ a Mercer kernel as (b)ii has proved.

$\vdots$

$k_{d+2}(x, x') = k_{d+1}(x, x') k_3(x, x') = (< x, x' > +\theta)^d \cdots\cdots$ a Mercer kernel as (b)ii has proved.

Therefore $k(x, x') = (< x, x' > +\theta)^d$ is a Mercer kernel.

(d) It can be taken as known that: $exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

Hence $exp(k(x, x'))$ is an infinite sum of kernels. Using (a)i and (b)i, we can get that:

$exp(k(x, x'))$ should be Mercer kernels, if k(x,x') is Mercer kernel.

as $k(x, x') = -\frac{xx'}{2\sigma}$ is a Mercer kernel, so $exp(-\frac{xx'}{2\sigma})$ should also be a Mercer kernel.

by (a)iii: let $f(x) = exp(-\frac{x^2}{2\sigma})$

$k(x, x') = f(x) \cdot f(x') = exp(-\frac{x^2}{2\sigma}) \cdot exp(-\frac{x'^2}{2\sigma})$ should be Mercer kernel.

so, k(x,x') $= exp(-\frac{(x-x')^2}{2\sigma}) = exp(-\frac{x^2}{2\sigma}) \cdot exp(-\frac{x'^2}{2\sigma}) \cdot exp(-\frac{xx'}{2\sigma})$ is also a Merner kernel.

2. **The Feature Map**

(a) $k(x, y) = (\sum_{i=1}^{2} x_i y_i) = (x_i + y_i)^2 = x_1^2 y_1^2 + x_2^2 + y_2^2 + 2x_1 y_1 x_2 y_2$

$$\varphi\binom{x_1}{x_2} \cdot \varphi\binom{y_1}{y_2} = \begin{pmatrix} x_1^2 \\ \sqrt{2} x_1 x_2 \\ x_2^2 \end{pmatrix} \cdot \begin{pmatrix} y_1^2 \\ \sqrt{2} y_1 y_2 \\ y_2^2 \end{pmatrix}$$

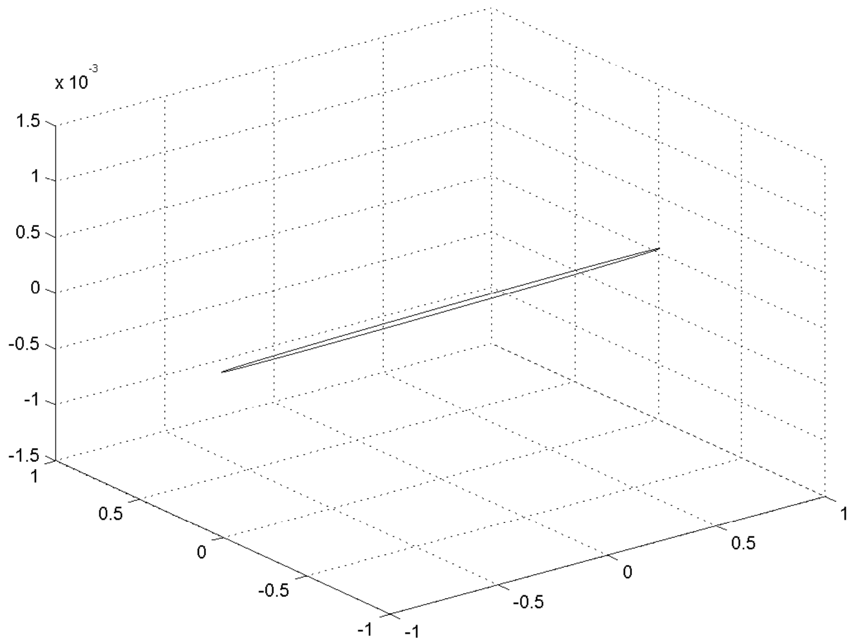
$= x_1^2 y_1^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_2^2$

$= k(x, y).$

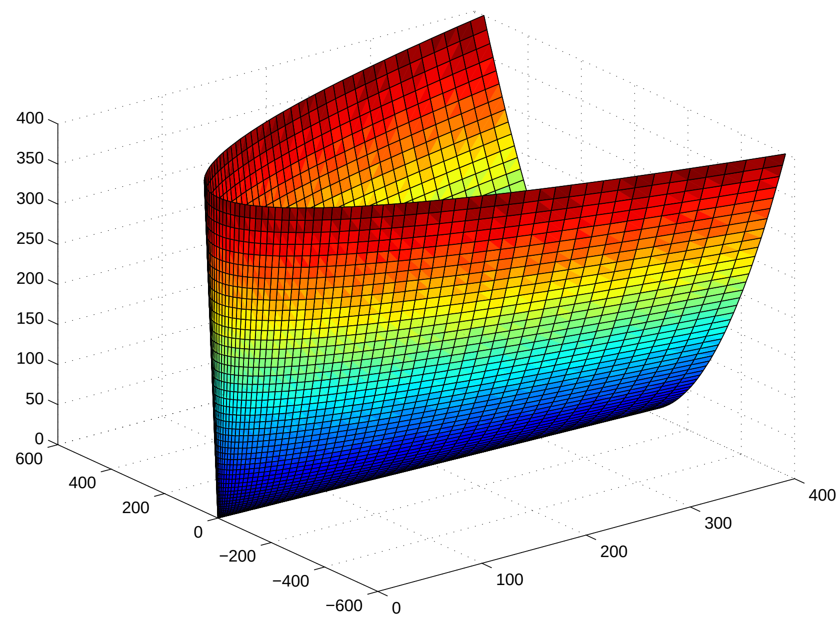
Therefore, F and φ are possible choices for feature space and feature map.

(b)

i. Under φ the circle maps to a closed curve which looks like a narrow ellipse whose major axe is much longer than the minor axe. The following is the graph of function calculated by matlab:



ii. Under φ the plane maps to a curved surface which looks like an open briefcase. The following is the graph of function calculated by matlab:



(c) The plane passes three points $(1, 0, 1), (0, 0, 1), (\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2})$. Hence the new plane is vertical to plane formed by $X - Z$ axes.

(d) Point $(0,1,0)$ is not contained in $\varphi(A)$ because we cannot find the corresponding x_1, x_2 that satisfies

$$\begin{cases} x_1^2 = 0 \\ \sqrt{2}x_1x_2 = 1 \\ x_2^2 = 0 \end{cases}$$