# sheet03Klara_P_K

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## 1 Independent Component Analysis

In this exercise, you will implement the FastICA algorithm, and apply it to model independent components of a distribution of image patches. The description of the fastICA method is given in the paper *"A. Hyvärinen and E. Oja. 2000. Independent component analysis: algorithms and applications"* linked from ISIS, and we frequently refer to sections and equations in that paper.

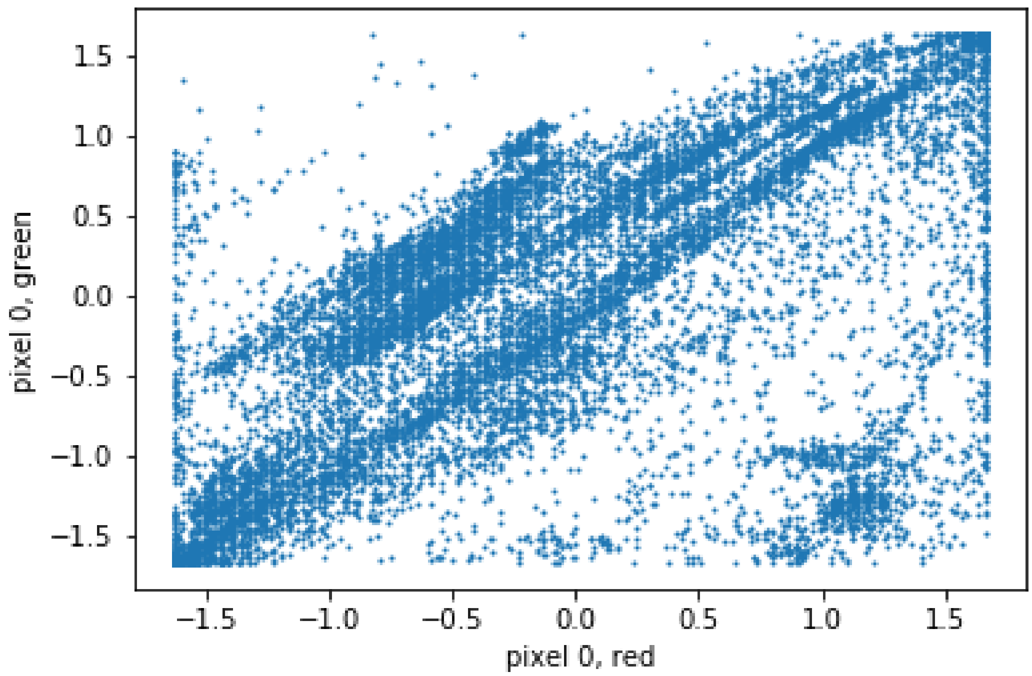
Three methods are provided for your convenience:

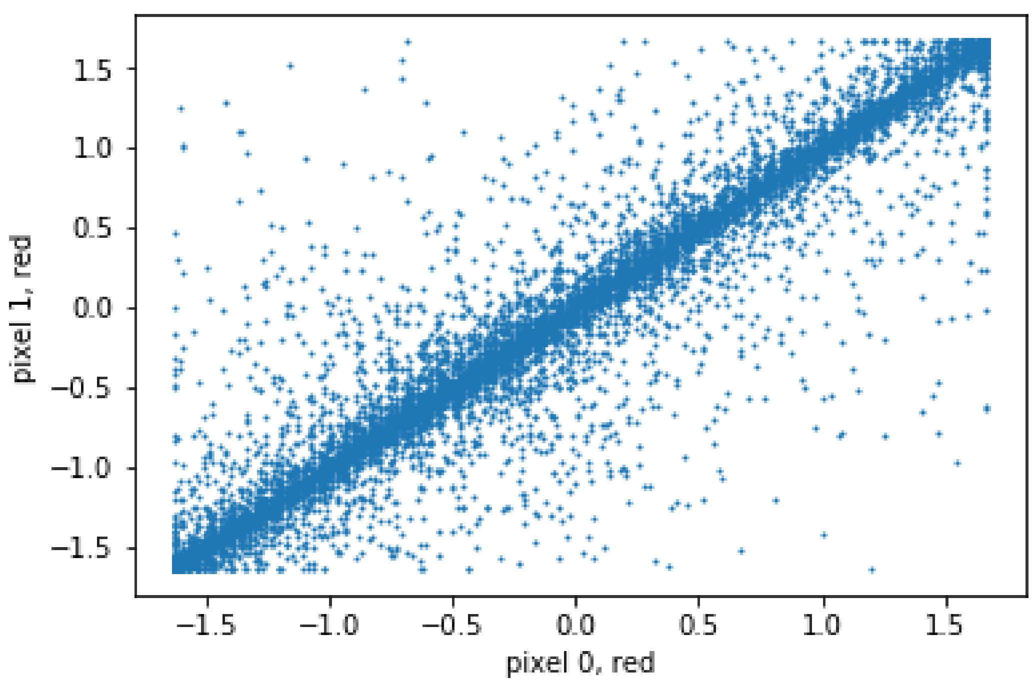
- **`utils.load()`** extracts a dataset of image patches from an collection of images (contained in the folder `images/` that can be extracted from the `images.zip` file). The method returns a list of RGB image patches of size $12 \times 12$, presented as a matrix of size $\#patches \times 432$. (Note that $12 \cdot 12 \cdot 3 = 432$).
- **`utils.scatterplot(...)`** produces a scatter plot from a two-dimensional data set. Each point in the scatter plot represents one image patch.
- **`utils.render(...)`** takes a matrix of size $\#patches \times 432$ as input and renders these patches in the IPython notebook.

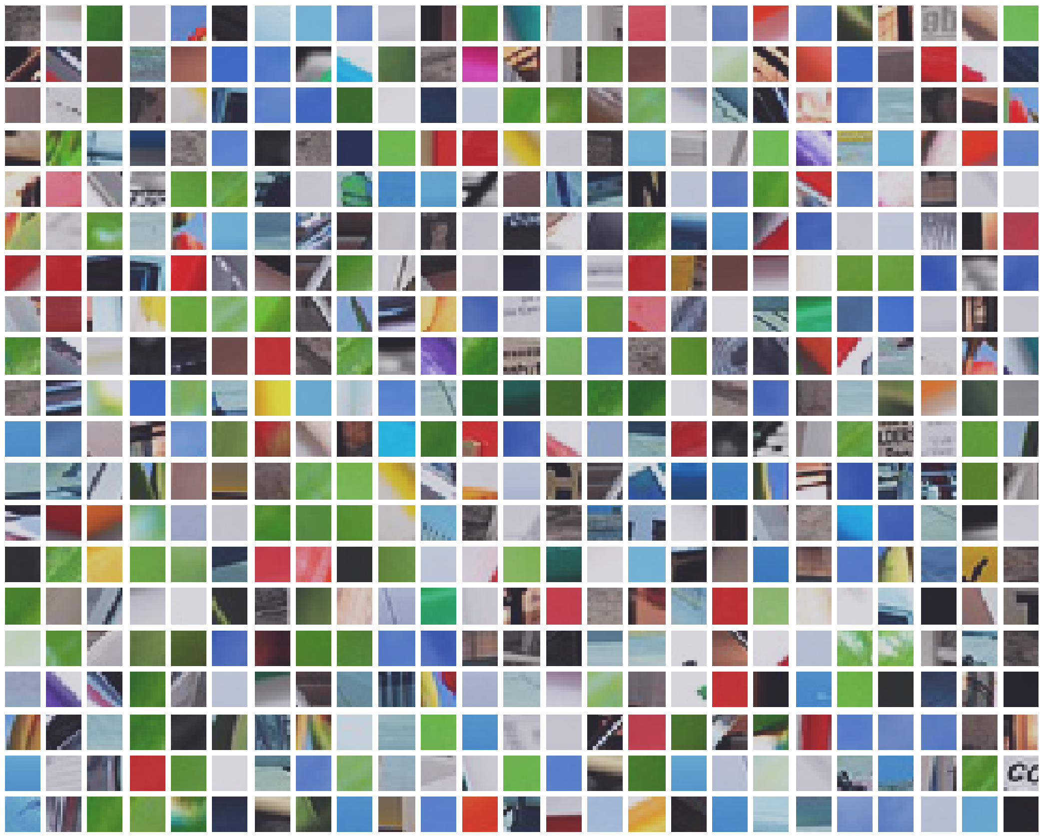
### 1.1 Demo code

A demo code that makes use of these three methods is given below. The code performs basic analysis such as loading the data, plotting correlations between neighboring pixels, or different color channels of the same pixel, and rendering some image patches.

In [1]:
```
import utils
%matplotlib inline

# Load the dataset of image patches
X = utils.load()

# Plot the red vs. green channel of the first pixel
utils.scatterplot(X[:,0],X[:,1],xlabel='pixel 0, red',ylabel='pixel 0, gree

# Plot the red channel of the first and second pixel
utils.scatterplot(X[:,0],X[:,3],xlabel='pixel 0, red',ylabel='pixel 1, red'

# Visualize 500 image patches from the image
utils.render(X[:500])
```


pixel 0, green
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
pixel 0, red


pixel 1, red
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
pixel 0, red




### 1.2 Whitening (10 P)

Independent component analysis applies whitening to the data as a preprocessing step. The whitened data matrix $\tilde{X}$ is obtained by linear projection of $X$, such data such that $\mathrm{E}[\tilde{x}\tilde{x}^\top] = I$, where $\tilde{x}$ is a row of the whitened matrix $\tilde{X}$. See Section 5.2 of the paper for a complete description of the whitening procedure.

**Tasks:**

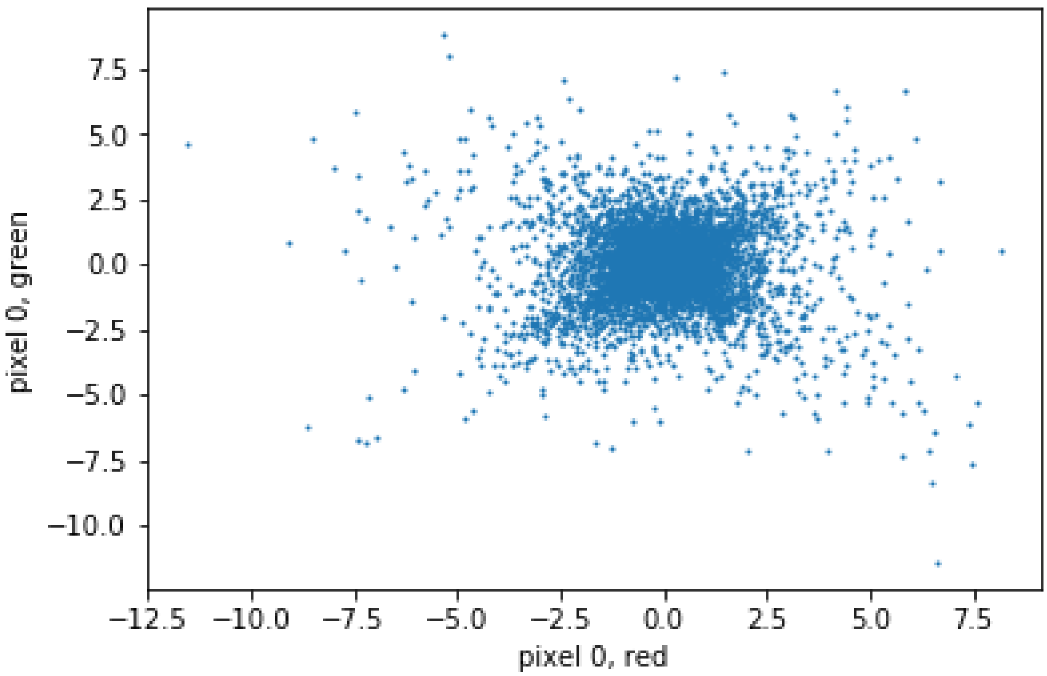
- **Implement a function that returns a whitened version of the data given as input.**
- **Add to this function a test that makes sure that $\mathrm{E}[\tilde{x}\tilde{x}^\top] \approx I$ (up to numerical accuracy).**
- **Reproduce the scatter plots of the demo code, but this time, using the whitened data.**
- **Render 500 whitened image patches.**

```
In [2]: ##### REPLACE BY YOUR CODE
        #import solutions
        #solutions.whitening()
        #####
        import numpy as np
```

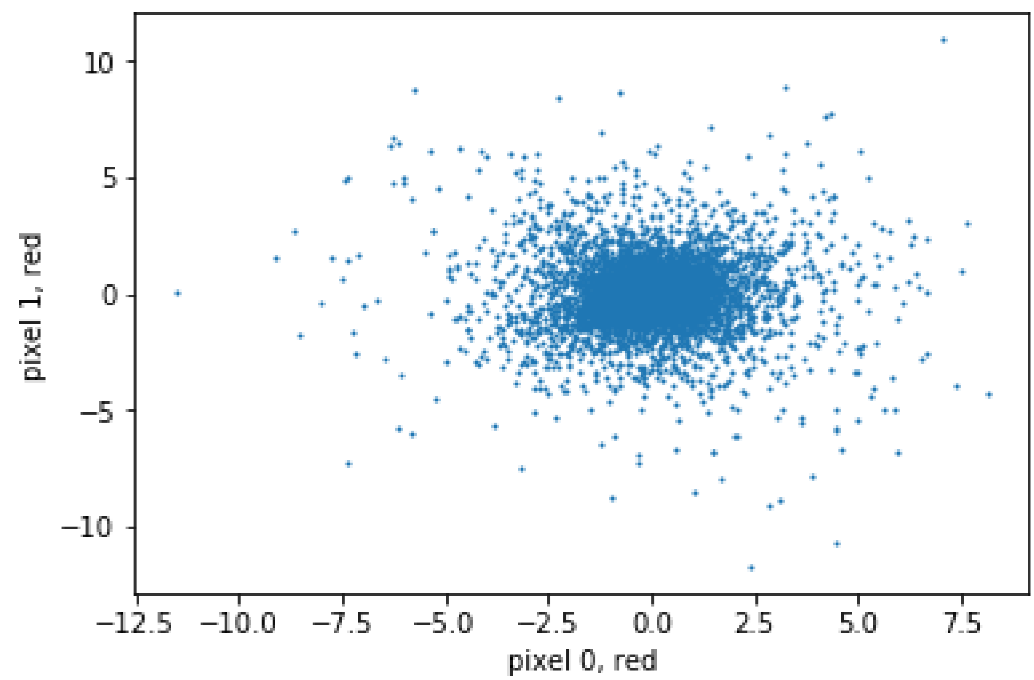
```
def centering(X):
    X = np.array(X)
    mean = X.mean(axis=0)
    X -= mean
    return X


def whiten(X):
    n,d = X.shape
    X = centering(X)
    C = 1.0/(n-1) *np.dot(X.T,X)
    d, E = np.linalg.eigh(C)
    D = np.diag(1./np.sqrt(abs(d)))
    XW = np.dot(E,np.dot(D, np.dot(E.T,X.T)))

    # test
    CW = np.cov(XW)
    d = CW.shape[0]
    print('test if whitened covariance equals identity matrix: ', np.sum(CW

    return XW.T
```

In [3]:
```
%matplotlib inline

# Load the dataset of image patches
XW = whiten(X)

# Plot the red vs. green channel of the first pixel
utils.scatterplot(XW[:,0],XW[:,1],xlabel='pixel 0, red',ylabel='pixel 0, gr

# Plot the red channel of the first and second pixel
utils.scatterplot(XW[:,0],XW[:,3],xlabel='pixel 0, red',ylabel='pixel 1, re

# Visualize 500 image patches from the image
utils.render(XW[:500])
```

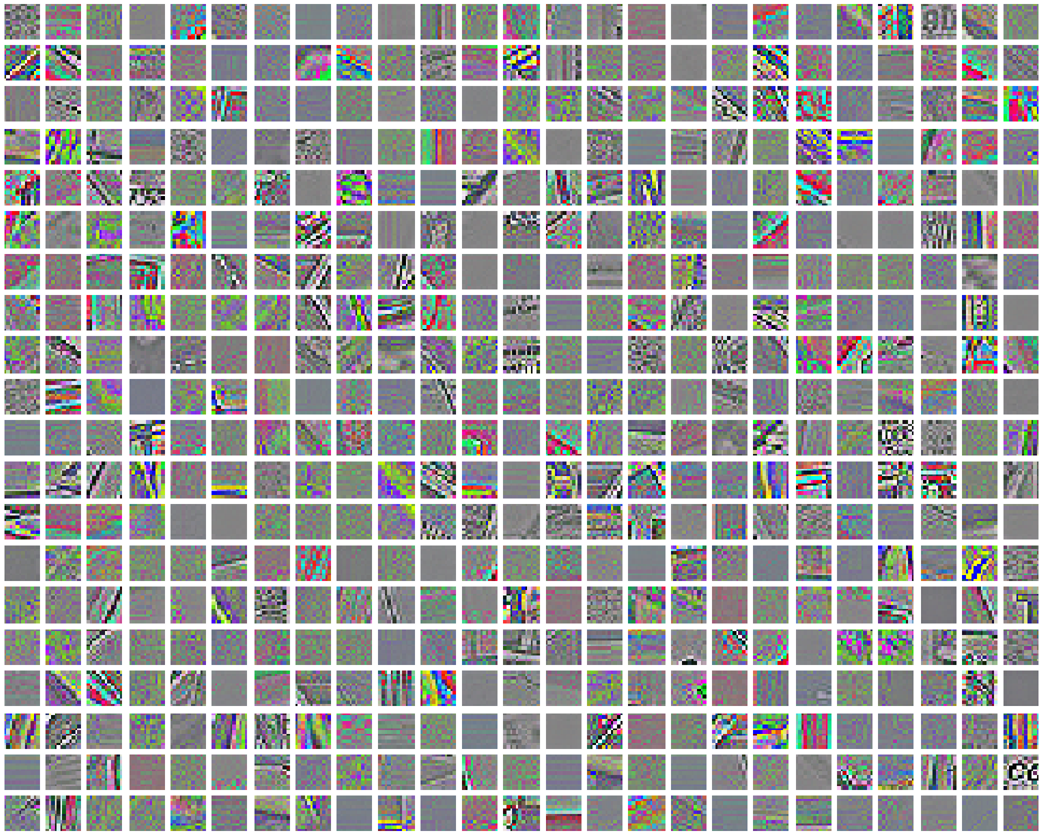
```
('test if whitened covariance equals identity matrix: ', -3.3962400029486829e-13)
```



7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
pixel 0, green
−12.5
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
pixel 0, red

10
5
0
−5
−10
pixel 1, red
−12.5
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
pixel 0, red



### 1.3 Implementing FastICA (20 P)

We now would like to learn 100 independent components of the distribution of whitened image patches. For this, we follow the procedure described in the Chapter 6 of the paper. Implementation details specific to this exercise are given below:

- **Nonquadratic function G**: In this exercise, we will make use of the nonquadratic function $G(x) = \frac{1}{a} \log \cosh(ax)$, proposed in Section 4.3.2 of the paper, with $a = 1.5$. This function admits as a derivative the function $g(x) = \tanh(ax)$, and as a double derivative the function $g'(x) = a \cdot (1 - \tanh^2(ax))$.

- **Number of iterations**: The FastICA procedure will be run for 64 iterations. Note that the training procedure can take a relatively long time (up to 5 minutes depending on the system). Therefore, during the developement phase, it is advised to run the algorithm for a fraction of the total number of iterations.

- **Objective function**: The objective function that is maximized by the ICA training algorithm is given in Equation 25 of the paper. Note that since we learn 100 independent components, the objective function is in fact the *sum* of the objective functions of each independent components.

- **Finding multiple independent components**: Conceptually, finding multiple independent components as described in the paper is equivalent to running multiple instances of FastICA (one per independent component), under the constraint that the components learned by these instances are decorrelated. In order to keep the learning procedure computationally affordable, the code must be parallelized, in particular, make use of numpy matrix multiplications instead of loops whenever it is possible.
- **Weight decorrelation**: To decorrelate outputs, we use the inverse square root method given in Equation 45.

**Tasks:**

- **Implement the FastICA method described in the paper, and run it for 64 iterations.**
- **Print the value of the objective function at each iteration.**
- **Create a scatter plot of the projection of the whitened data on two distinct independent components after 0, 1, 3, 7, 15, 31, 63 iterations.**
- **Visualize the learned independent components using the function `render(...)`.**

In [4]:
```
def G(x):
    a = 1.5
    return 1.0/a * np.log(np.cosh(a*x))

def g(x):
    a = 1.5
    return np.tanh(a*x)

def gprime(x):
    a = 1.5
    return a*(1-np.tanh(a*x)**2)

def objective(w):
    v = np.mean(G(np.random.normal(0,1,n)))
    J = (np.mean(G(np.dot(X,w))) -v)**2
    return J

def objectiveMulti(W):
    v = np.mean(G(np.random.randn(n,100)), axis =0)
    xtw = np.dot(X,W)
    y = np.mean(G(xtw), axis=0)
    J = np.sum((y-v)**2)
    return J

def decorrelate(W):
    #normalize W
    norms = np.linalg.norm(W, axis=0)
    W = W / norms
    #decorrelate W
```

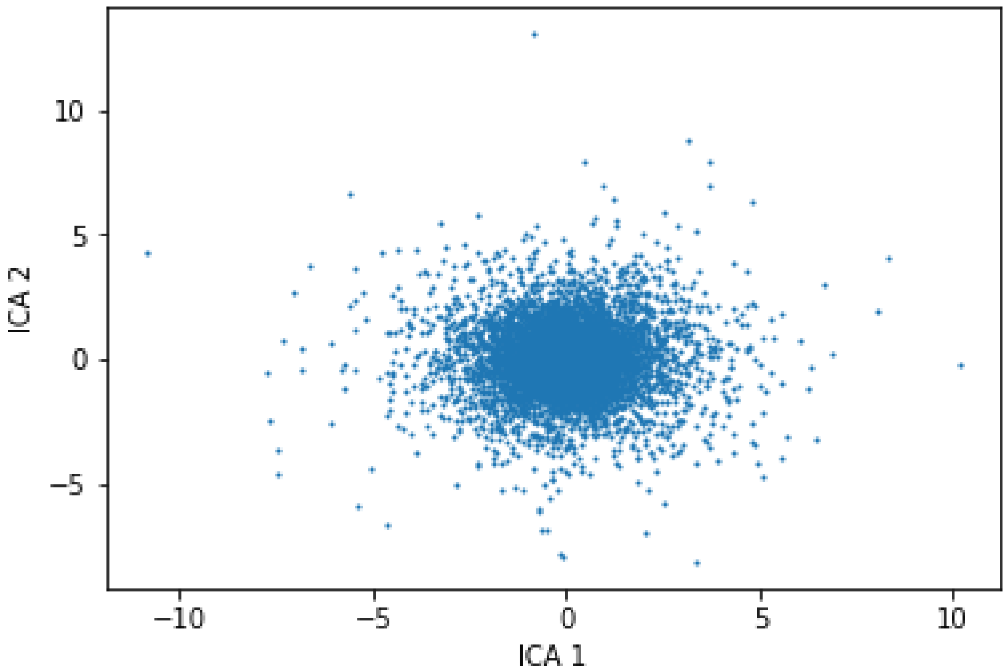
```python
    #W = W.T
    W2 = np.dot(W,W.T)
    eig,F = np.linalg.eigh(W2)
    eig = eig + 1.282566102764301e-25
    D = np.diag(1./np.sqrt(abs(eig)))
    Wsqrt = np.dot(F, np.dot(D,F.T))
    W = np.dot(Wsqrt, W)
    return W
```

In [5]:
```python
def singleUnitICA(X, iterations):
    it = iterations
    n,d = X.shape
    w = np.random.rand(d,1)
    for i in range(it):
        y = g(np.dot(X,w))
        Y = np.dot(X.T, y).T
        E1 = np.mean(Y, axis=0)
        E1 = E1.reshape(d,1)
        E2 = np.mean(g2(np.dot(X,w))) * w
        w = E1 -E2
        w = w/np.linalg.norm(w)
        print('Jsingle', objective(w))
    return w
```

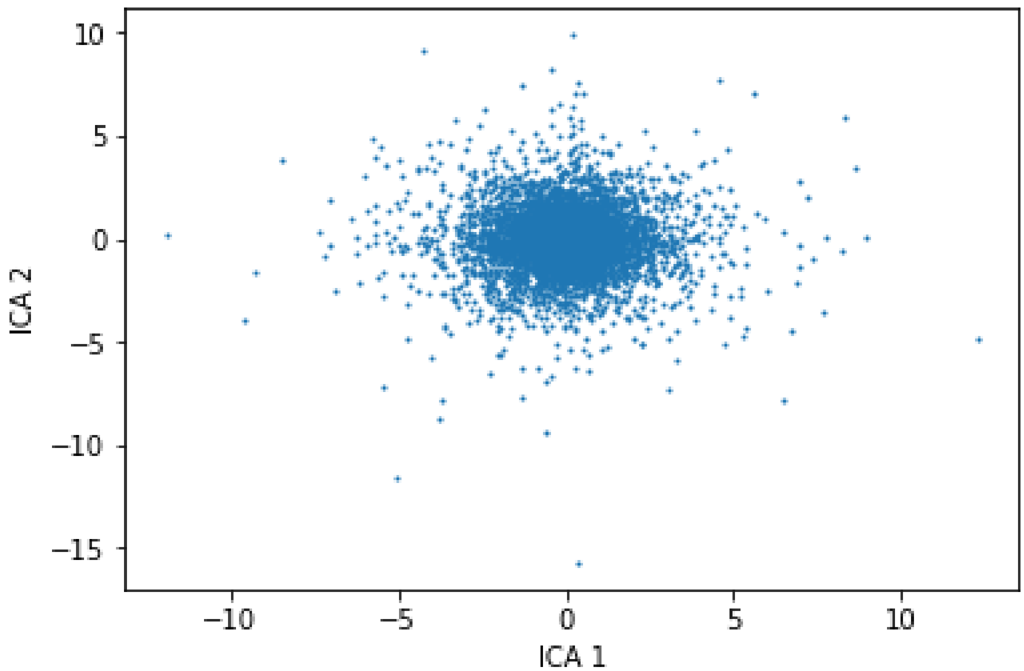
In [6]:
```python
def multiUnitICA(X, iterations, scatterplotMilestones):
    n,d = X.shape
    W = np.random.rand(d,100)
    norms = np.linalg.norm(W, axis=0)
    #W = W / norms.astype(float)

    it = iterations
    for i in range(it):
        wtx = np.dot(X,W)
        gwtx = g(wtx)
        g_wtx = gprime(wtx)
        W1 = np.dot(X.T, gwtx)/float(n-1)
        W2 = np.dot(W, np.diag(g_wtx.mean(axis=0)))
        W = W1- W2
        W = decorrelate(W)
        if i in scatterplotMilestones:
            firstProjection = np.dot(X,W[:,0] )
            secndProjection = np.dot(X,W[:,1] )
            utils.scatterplot(firstProjection, secndProjection, xlabel='ICA
        print "iteration ", i, ": ", objectiveMulti(W)

    W = decorrelate(W)

    return W
```

```
In [7]: X = utils.load()
        n,d = X.shape
        X = whiten(X)
        milestonesForScatterPlots = [0,1,3,7,15,31,63]
        W = multiUnitICA(X, 64, milestonesForScatterPlots)
        utils.render(W.T)
```

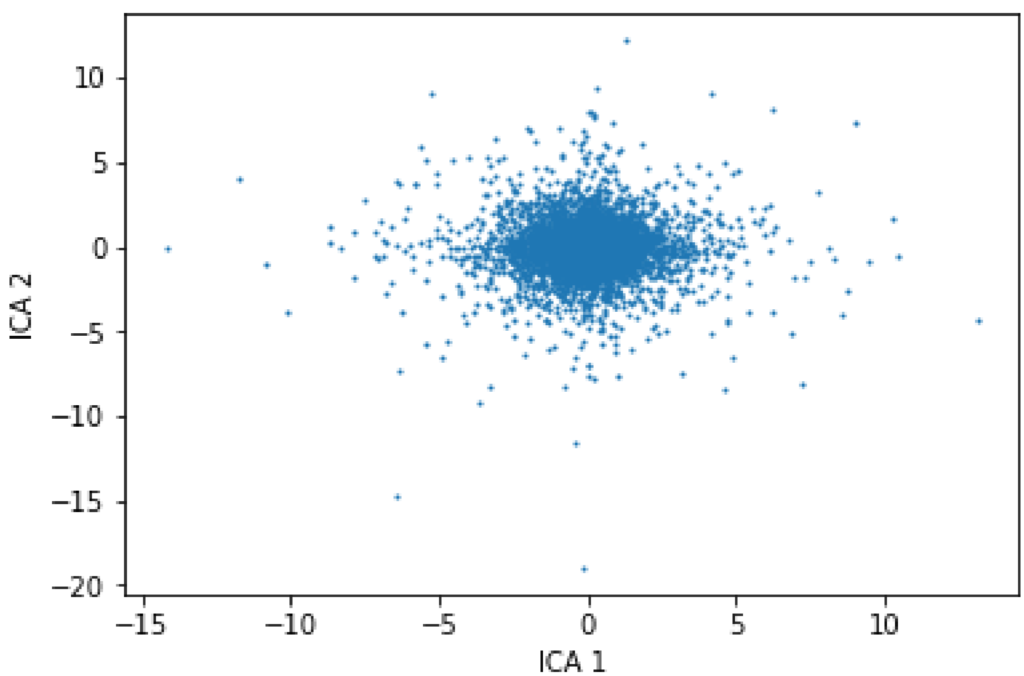
```
('test if whitened covariance equals identity matrix: ', -4.7584385060129601e-14)
```



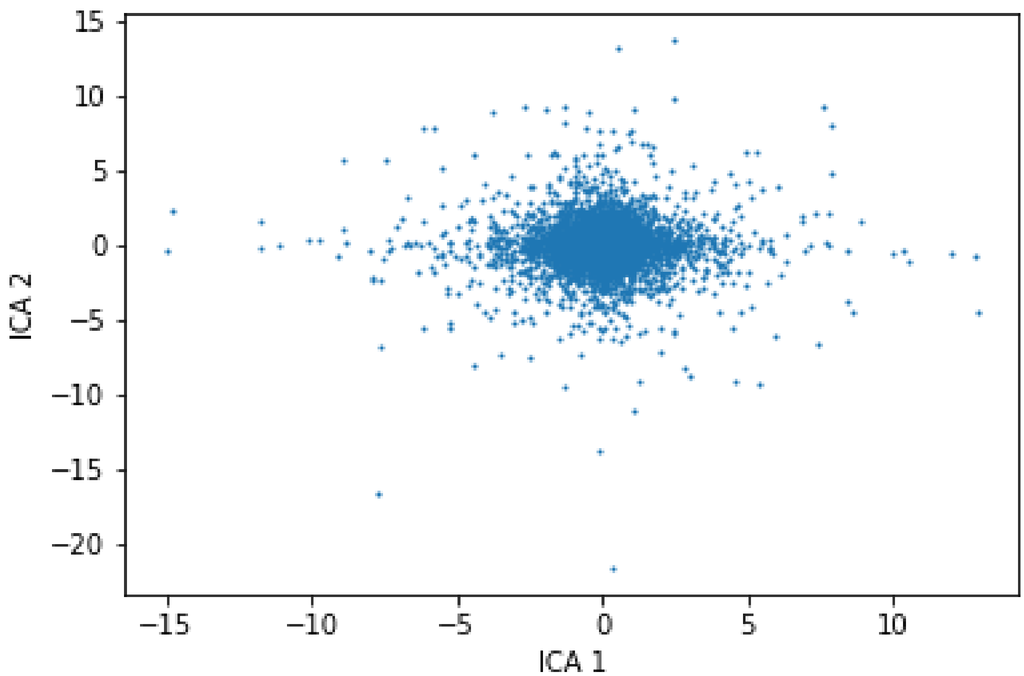

```
iteration  0 :  0.970347638315
```



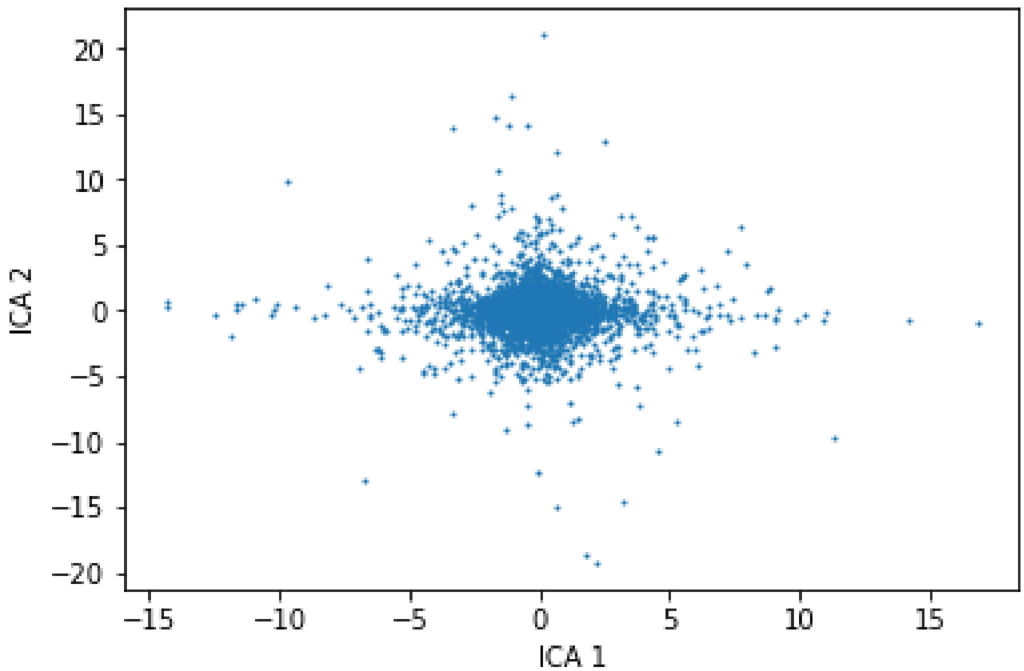

```
iteration  1 :  1.53530235025
iteration  2 :  2.00463305623
```



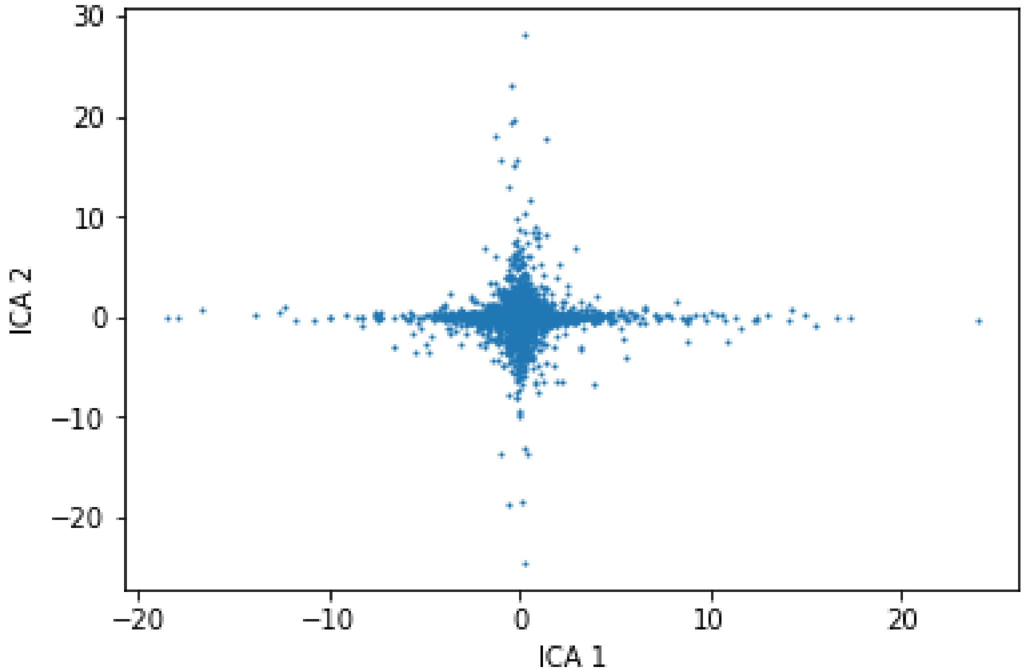
```
iteration  3 :  2.34493744882
iteration  4 :  2.57379447303
iteration  5 :  2.80825587887
iteration  6 :  2.97513582836
```




```
iteration  7 :  3.12959551206
iteration  8 :  3.24625795471
iteration  9 :  3.36947867887
iteration  10 :  3.50135163945
iteration  11 :  3.59975418306
iteration  12 :  3.72223747735
iteration  13 :  3.76344875783
iteration  14 :  3.87196757379
```

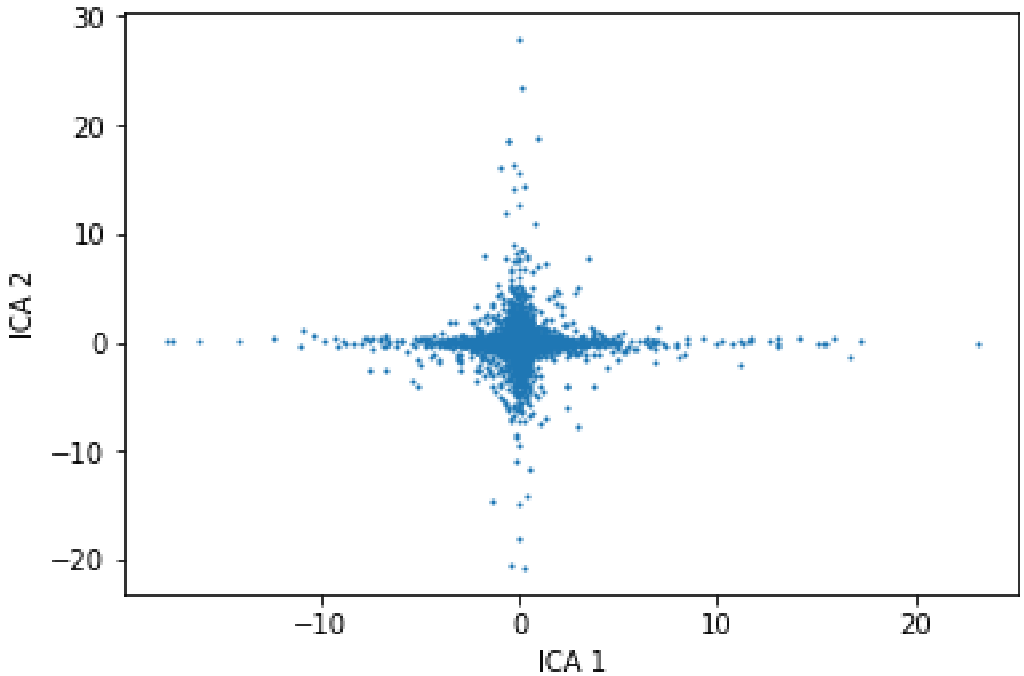



```
iteration  15 :  3.97837422924
iteration  16 :  4.11398202737
iteration  17 :  4.19871459356
iteration  18 :  4.29213966346
iteration  19 :  4.38171426273
iteration  20 :  4.50779570383
iteration  21 :  4.59560980441
iteration  22 :  4.6466710482
iteration  23 :  4.74574565017
iteration  24 :  4.86981959924
iteration  25 :  4.94239411574
iteration  26 :  5.05412737234
iteration  27 :  5.11701581337
iteration  28 :  5.19300869199
iteration  29 :  5.27115945714
iteration  30 :  5.35542589455
```

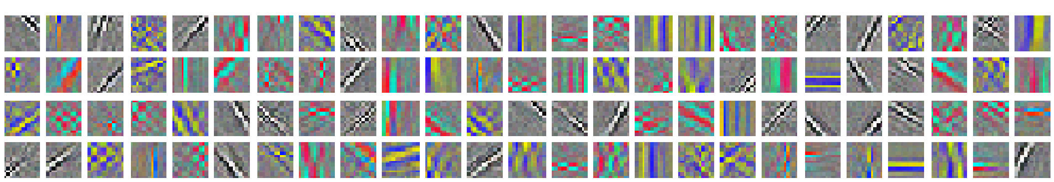



```
iteration  31 :  5.44379345219
iteration  32 :  5.54103125377
iteration  33 :  5.59018421188
iteration  34 :  5.64375452858
iteration  35 :  5.72292358346
iteration  36 :  5.76194355917
iteration  37 :  5.84766316504
iteration  38 :  5.91609277818
iteration  39 :  5.90713100546
iteration  40 :  5.988126507
iteration  41 :  6.0782805132
iteration  42 :  6.10589656348
iteration  43 :  6.16054497085
iteration  44 :  6.15765700894
iteration  45 :  6.21688612807
iteration  46 :  6.26983546475
iteration  47 :  6.29113137072
iteration  48 :  6.31757731599
iteration  49 :  6.34387750956
iteration  50 :  6.39165710387
iteration  51 :  6.37451922449
iteration  52 :  6.40455466829
iteration  53 :  6.42176711589
iteration  54 :  6.46761745824
```

```
iteration  55 :  6.45400135234
iteration  56 :  6.48903319649
iteration  57 :  6.50770475837
iteration  58 :  6.52161262778
iteration  59 :  6.58117312347
iteration  60 :  6.53759487996
iteration  61 :  6.59762136477
iteration  62 :  6.59180475656
```




```
iteration  63 :  6.6226784714
```



In [ ]: