Time Series Analysis

Discussion Section 02

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Organizational Issues

Please log in to **ISIS** (password: Zeit1718) and **download** the following files:

- GNP.dta
- acf_exercise.do

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Stationary Stochastic Processes

- Introduction
- **Identification**
  - Autocorrelation Function
  - Moving Average and Autoregressive Models
  - Partial Autocorrelation Function
  - ARMA Models
- Estimation
- Diagnostic Checking
- Forecasting

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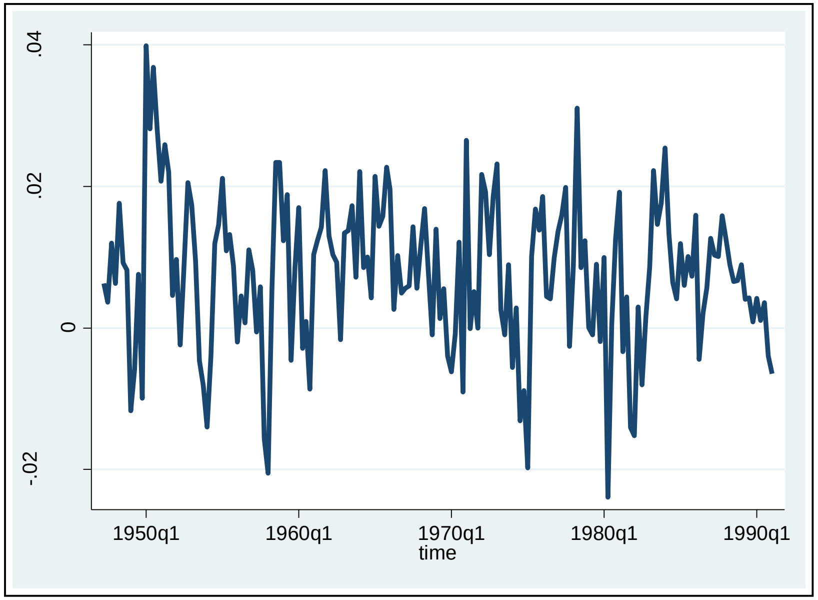
Exercise 2.1:

- What is meant by weak stationarity and what is the difference to strict stationarity?
- Does this series look stationary?
- Does it seem like there is serial dependence?



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Solution 2.1:

- What is meant by weak stationarity and what is the difference to strict stationarity?

A time series y_t (as a realization of a stochastic process, see Kirchgässner et al. p. 12) is said to be **strictly stationary** if the joint distribution of $(y_t,\ldots, y_{t+k})$ is identical to that of $(y_{t+\mathrm{m}},\ldots, y_{t+k+m})$ for all t, k, and m. A time series is **weakly stationary** if both the mean and the covariance are time-invariant.If y_t is strictly stationary, then y_t is also weakly stationary. The converse is not true in general. However, if the time series y_t is normally distributed, then weak stationarity is equivalent to strict stationarity.

- Does this series look stationary?

YES: There is no trend and the series seems to fluctuate around a constant mean. The (unconditional) variance is roughly constant. But it is hard to judge if the covariance is constant, too.

- Does it seem like there is serial dependence?

YES: The series seems not to fluctuate randomly around the mean, it looks rather smooth. If one observation is above the mean then the next is above the mean as well indicating a positive autocorrelation at small lags.

Exercise 2.2:

Create a scatter plot to identify if there is serial dependency for lag *k* = 1,2,3 in the `GNP` series. Add the means and the regression line.

Recall:

- Time-series operators: When a command allows a time-series *varlist*, you may include time-series operators.

Operator	Meaning
L.	lag x_{t-1}
L2.	2-period lag x_{t-2}
...	...

- `[twoway] scatter` *`varlist`* `[if] [in] [weight] [,options]` where *varlist* is `y_1 [y_2 [...]] x`
- **Twoway linear prediction plots**: `twoway lfit` calculates the prediction for *yvar* based on a linear regression of *yvar* on *xvar* and plots the resulting line (Sytax: `twoway lfit` *`yvar xvar`*)
- Use the **||-separator notation** to put on top of the scatter plot the prediction from a linear regression (`Syntax: [twoway] scatter ... [, scatter_options] || lfit ...`)

Exercise 2.2:

```
. describe

  obs:           176
 vars:             1                          21 Nov 2011 10:58
 size:           704
-------------------------------------------------------------------------------------------------------------------
              storage   display    value
variable name   type    format     label      variable label
-------------------------------------------------------------------------------------------------------------------
GNP             float   %9.0g                 Quarterly growth rate of U.S. real gross national product (1947q2 to
                                                1991q1)
-------------------------------------------------------------------------------------------------------------------
Sorted by:
```

Note:
We have **quarterly data** beginning in the **2nd** quarter of **1947** and ending in the **1st** quarter of **1991**.
However, there is no time variable in our data set.

=> create an appropriate time variable

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Solution 2.2-1:

```
. gen time = -52 + _n

. format time %tq

. tsset time

. summarize GNP
```

Alternative:

```
. gen time = tq(1947q2) + _n - 1
. format time %tq
. tsset time
```

```
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
         GNP |       176    .0077412    .0107275    -.02391     .03989
. return list

                 r(N) =  176
             r(sum_w) =  176
              r(mean) =  .0077412499703603
               r(Var) =  .000115080170731
                r(sd) =  .010727542623127 2
               r(min) =  -.023910000920295 7
               r(max) =  .039889998733997 3
               r(sum) =  1.362459994783421
```

```
. display r(mean)
.00774125
```

Solution 2.2-2:

```
. display r(mean)

.00774125

. local mean=r(mean)

. display `mean'

.00774125
```


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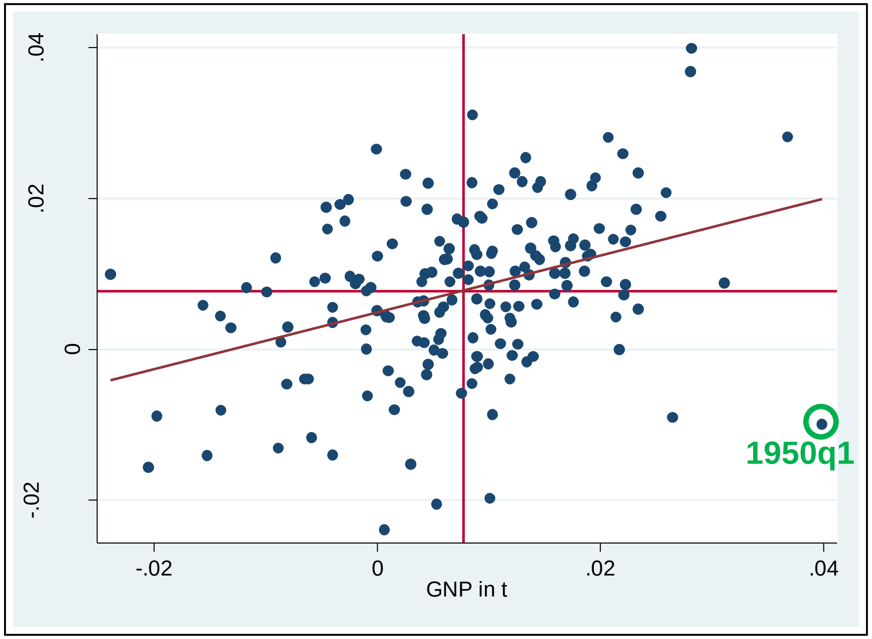
Solution 2.2-3:

Create a scatter plot to identify if there is serial dependency for lag $k = 1$ in the `GNP` series.

```
. twoway scatter L.GNP GNP, yline(`mean') xline(`mean') || lfit L.GNP
GNP, legend(off)
```



There are more points in the upper right corner (first quadrant) and the lower left corner (third quadrant). Therefore the regression line is upward sloping, indicating a positive autocovariance.

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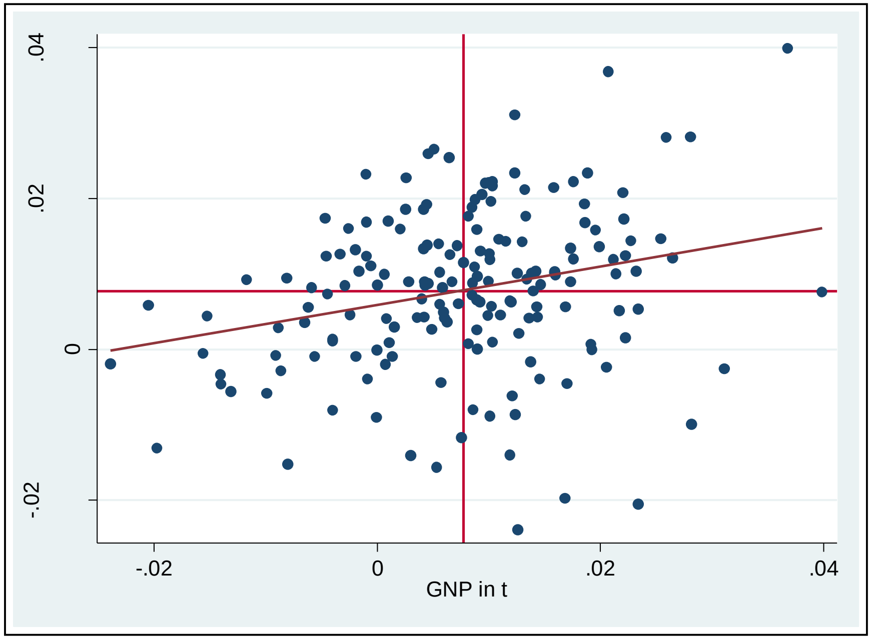
Solution 2.2-4:

Create a scatter plot to identify if there is serial dependency for lag k = 2 in the `GNP` series.

```
. twoway scatter L2.GNP GNP, yline(`mean') xline(`mean') || lfit L2.GNP GNP, legend(off)
```

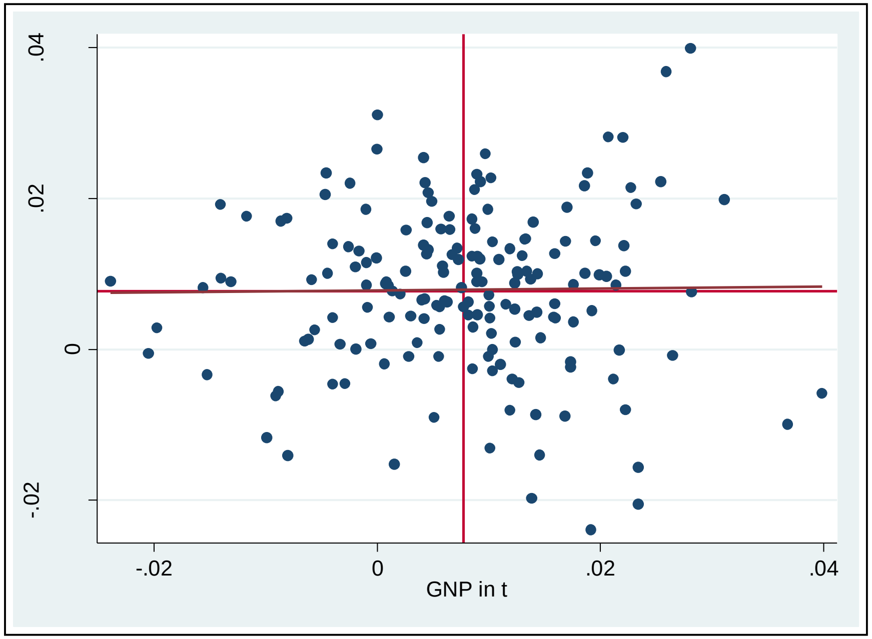


There are slightly more points in the upper right corner (first quadrant) and the lower left corner (third quadrant). Therefore the regression line is upward sloping, indicating a positive autocovariance.

Solution 2.2-5:

Create a scatter plot to identify if there is serial dependency for lag *k* = 3 in the `GNP` series.

```
. twoway scatter L3.GNP GNP, yline(`mean') xline(`mean') || lfit L3.GNP
GNP, legend(off)
```



There is about the same number of points in each quadrant. Therefore the regression line is roughly equivalent to the mean, indicating no autocovariance.

Exercise 2.3:

- What is the autocorrelation function and what is its purpose?
- Consider the formula of the Bravais-Pearson correlation coefficient and compare it to the formula for the autocorrelation coefficient at lag *k*. Explain under which assumptions both formulas yield the same result.

Bravais-Pearson correlation coefficient:

$$r = \frac{E\left[(x-\mu_x)(y-\mu_y)\right]}{\sqrt{E\left[(x-\mu_x)^2\right]E\left[(y-\mu_y)^2\right]}}$$

Autocorrelation coefficient at lag *k*:

$$\rho_k = \frac{E\left[(y_t-\mu_y)(y_{t+k}-\mu_y)\right]}{E\left[(y_t-\mu_y)^2\right]}$$

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Solution 2.3:

- What is the autocorrelation function and what is its purpose?

  Correlation between observations separated by *k* periods within the same time series. The estimated autocorrelation function (ACF) may suggest which of the time series models is suitable to reflect the dependence in the data.

- Consider the formula of the Bravais-Pearson correlation coefficient and compare it to the formula for the autocorrelation coefficient at lag *k*. Explain under which assumption(s) both formulas yield the same result.

  Weak stationarity: $\boxed{E(y_t) = \mu_y}$, which is a constant, and $\boxed{Cov(y_t, y_{t-k}) = \gamma_k}$, which only depends on *k*.

$$r = \frac{E\left[(x-\mu_x)(y-\mu_y)\right]}{\sqrt{E\left[(x-\mu_x)^2\right]E\left[(y-\mu_y)^2\right]}} \qquad \rho_k = \frac{E\left[(y_t - \boxed{\mu_y})(y_{t+k} - \boxed{\mu_y})\right]}{\boxed{E\left[(y_t-\mu_y)^2\right]}}$$

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Exercise 2.4:

- Calculate the values of the autocorrelation function for the **GNP** series.

  Note: `corrgram` *`varname`* tabulate autocorrelations

- Plot the autocorrelation function for the **GNP** series.

  Note: `ac` *`varname`* graph of autocorrelations

Solution 2.4-1:

```
. corrgram GNP
                                                -1       0       1 -1       0       1
 LAG       AC       PAC      Q     Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1        0.3769   0.3807   25.426  0.0000          |---               |---
2        0.2539   0.1344   37.034  0.0000          |--                |-
3        0.0125  -0.1443   37.062  0.0000          |                 -|
4       -0.0859  -0.0991   38.407  0.0000          |                  |
5       -0.1071  -0.0196   40.507  0.0000          |                  |
6       -0.0575   0.0352   41.116  0.0000          |                  |
7       -0.0182   0.0130   41.177  0.0000          |                  |
8       -0.0772  -0.1113    42.29  0.0000          |                  |
9       -0.0702  -0.0446   43.214  0.0000          |                  |
10       0.0104   0.0993   43.234  0.0000          |                  |
11      -0.0230  -0.0371   43.335  0.0000          |                  |
12      -0.0967  -0.1541   45.122  0.0000          |                 -|
13      -0.1011  -0.0503   47.085  0.0000          |                  |
14      -0.1145  -0.0222   49.622  0.0000          |                  |
15      -0.0747   0.0084   50.707  0.0000          |                  |
16       0.0292   0.0644   50.873  0.0000          |                  |
[...]
```

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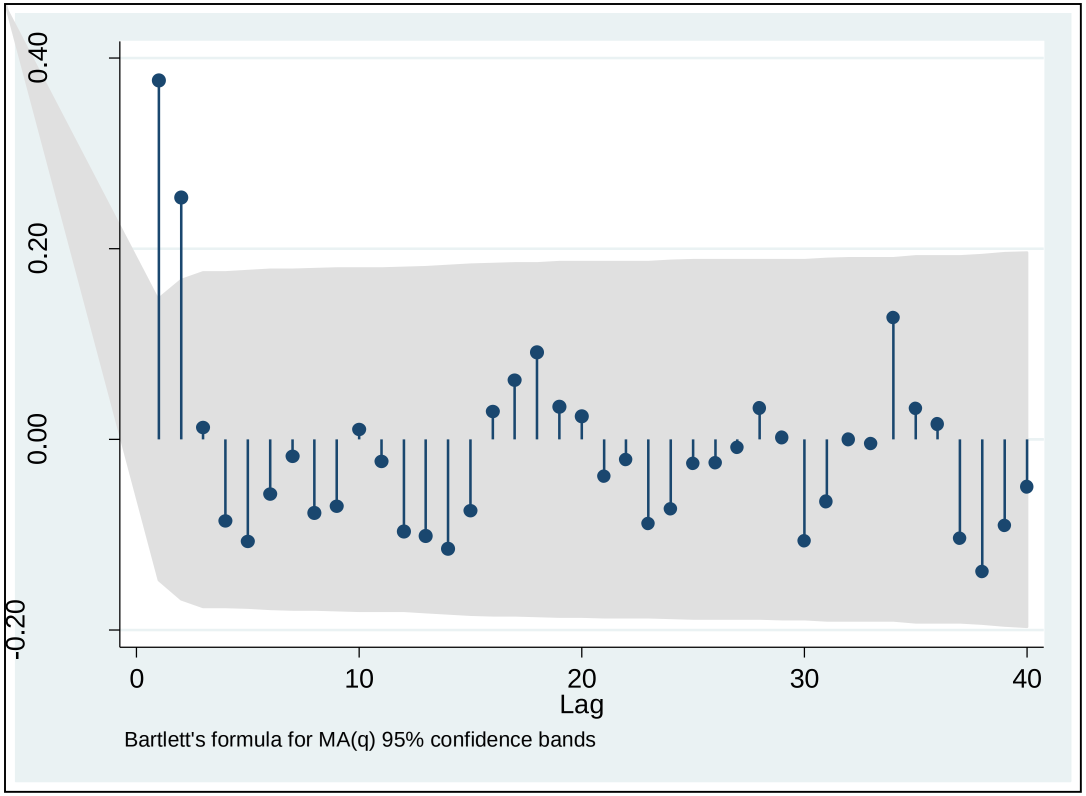
Solution 2.4-2:

```
. ac GNP
```



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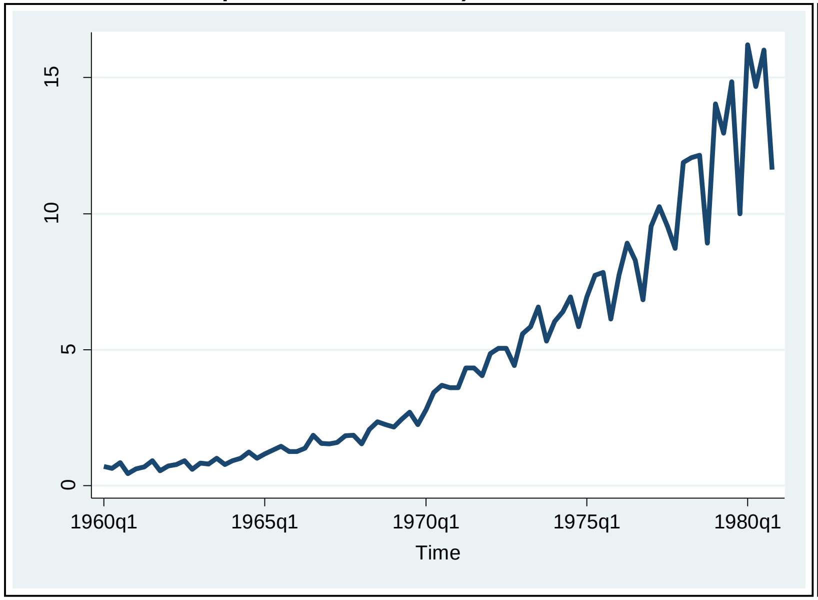
Exercise 2.5-1:

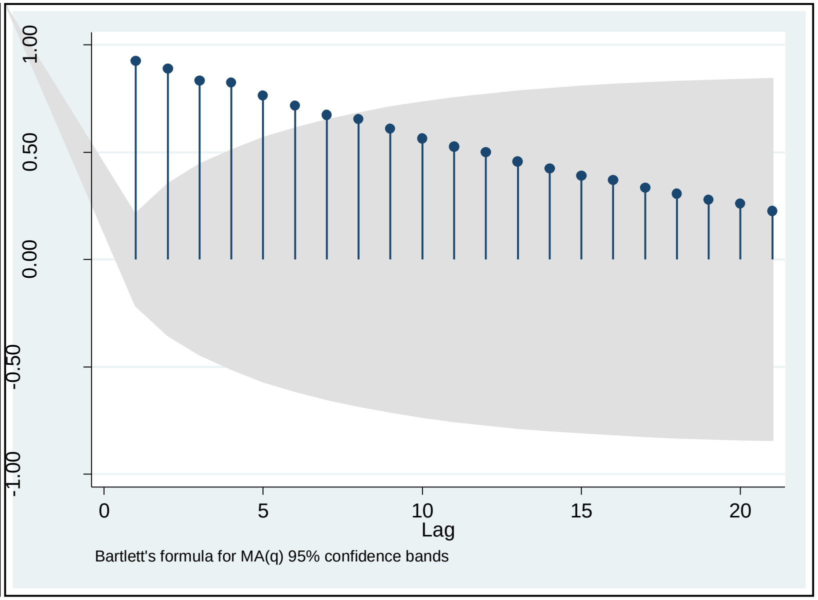
What should the ACF for this series look like?

Quarterly earnings per share for the U.S. company Johnson & Johnson (first quarter of 1960 to last quarter of 1980)



Solution 2.5-1:

The **ACF** for this series is rather slowly declining.



Shumway, Stoffer (2000) “Time series analysis and its applications”

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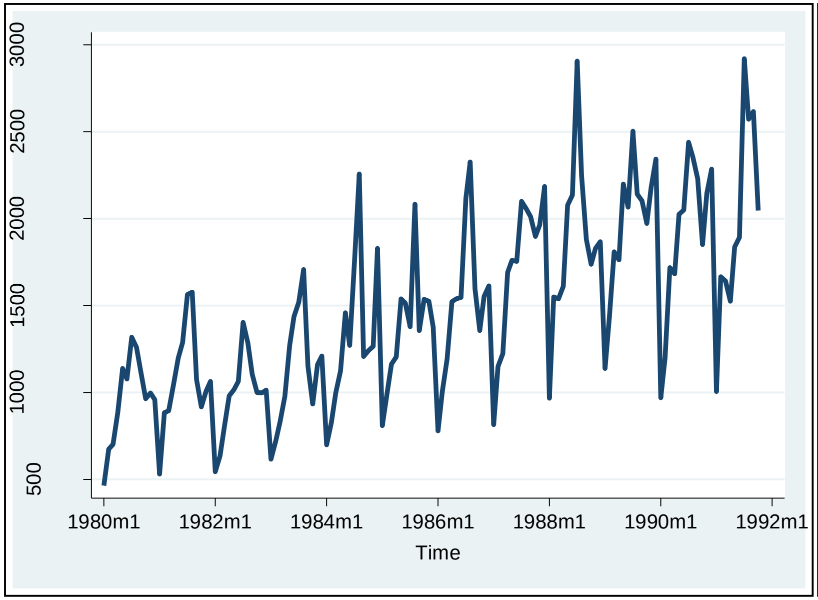
Exercise 2.5-2:

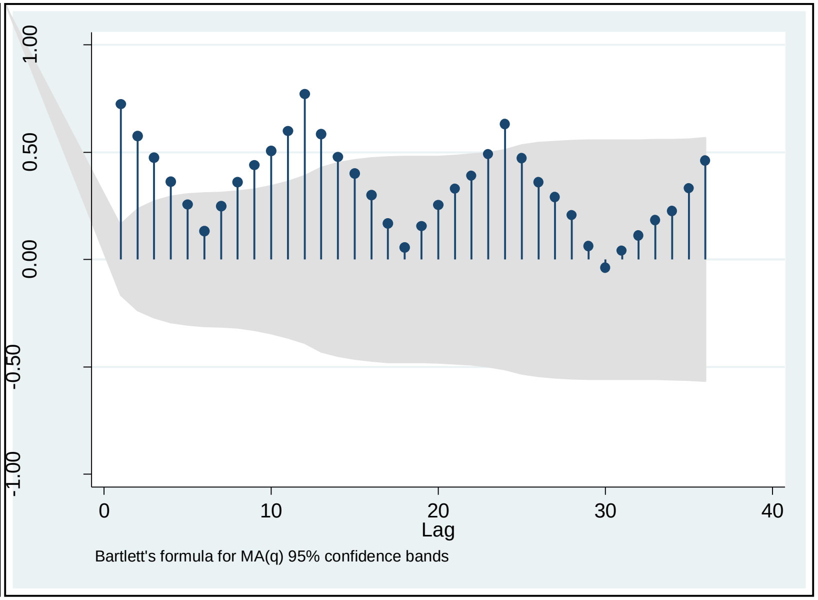
What should the ACF for this series look like?

Monthly Australian red wine sales (Jan. 1990 to Oct. 1991)

Solution 2.5-2:

The **ACF** for this series shows a seasonal pattern.





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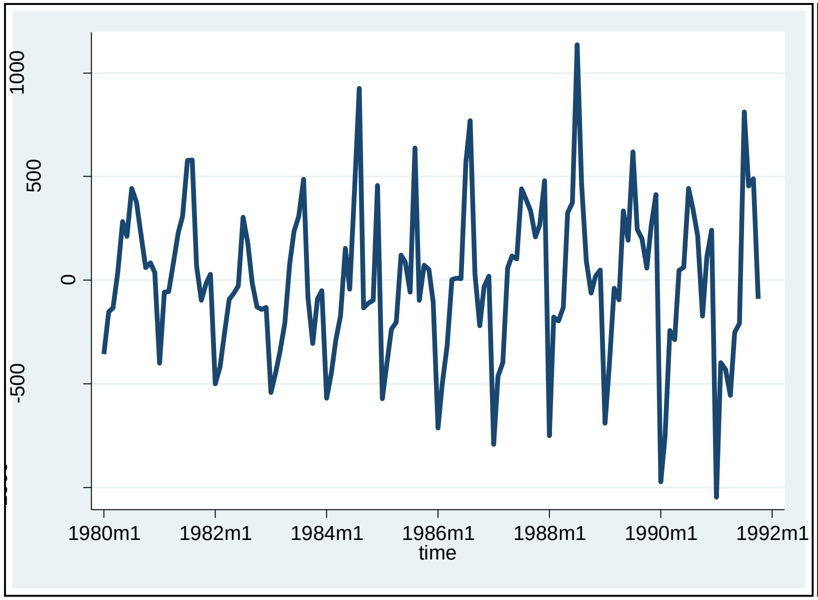
Exercise 2.5-3:

What should the ACF for this series look like?

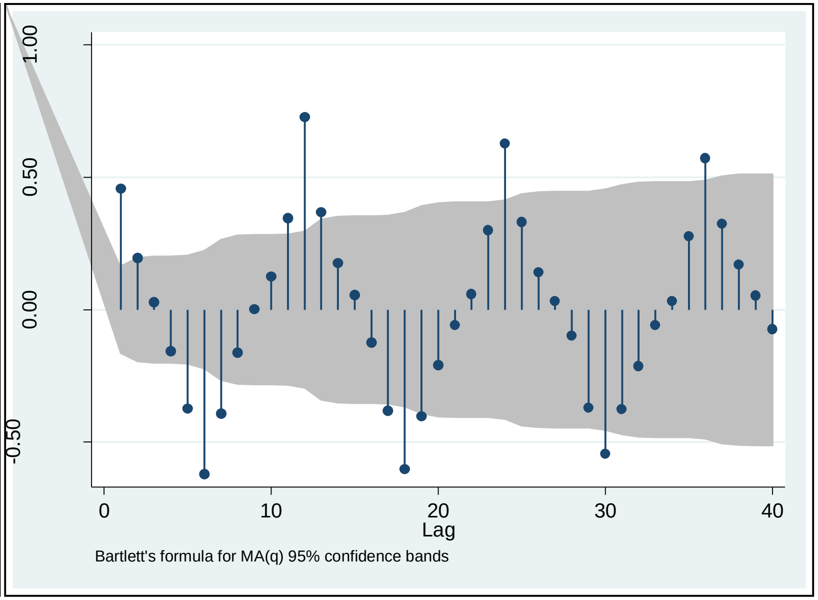
Monthly Australian red wine sales (Jan. 1990 to Oct. 1991)

Detrended



Solution 2.5-3:

The **ACF** for this series shows a seasonal detrended pattern.



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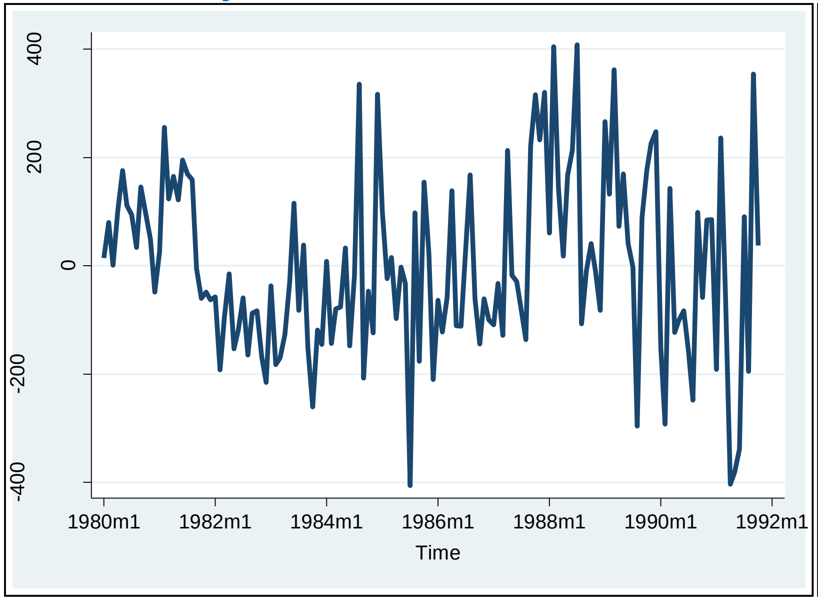
Exercise 2.5-4:

What should the ACF for this series look like?

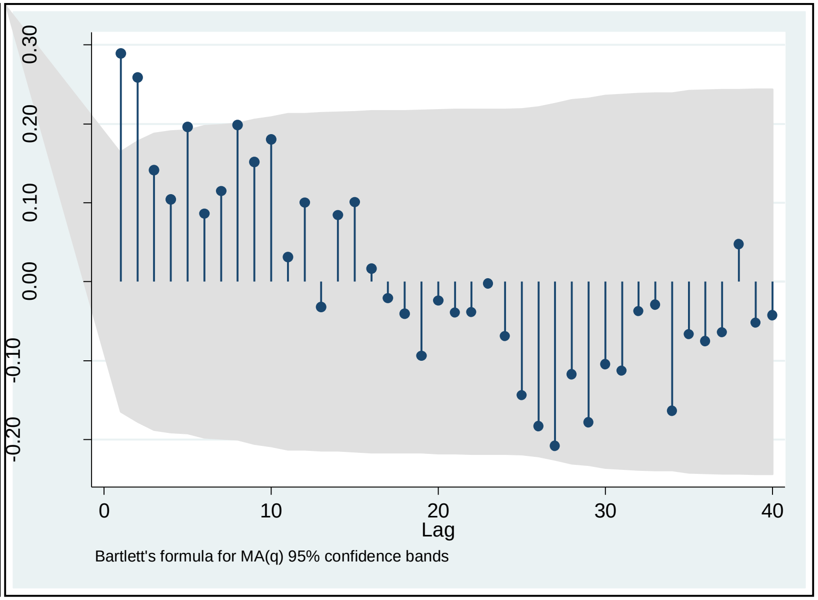
Monthly Australian red wine sales (Jan. 1990 to Oct. 1991)

Seasonal adjusted and detrended



Solution 2.5-4:

The **ACF** for this series is rather quickly declining, showing some serial correlation at lower lag order.



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Exercise 2.6:

Write down the following models in our general notation:

- MA(1)
- MA(2)
- AR(1)
- AR(2).

Calculate the values of the theoretical autocorrelation function for an MA(1) process with $\mu = 0$ for lags $k = 1, 2$.

Calculate the values of the theoretical autocorrelation function for an AR(1) process with $\delta = 0$ for lags $k = 1, 2$.

Hint: $$\rho_k := \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)} = \frac{E[(y_t - \mu)(y_{t-k} - \mu)]}{E[(y_t - \mu)^2]}$$

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Solution 2.6-1:

Write down the following models in our general notation:

MA(1): $y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}$

MA(2): $y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

AR(2): $y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \delta + \varepsilon_t$

with ε_t~ i.i.d. $E(\varepsilon_t) = 0, Var(\varepsilon_t) = \sigma_\varepsilon^2$

These conditions imply:

- $E(\varepsilon_t \varepsilon_{t-k}) = E(\varepsilon_t) E(\varepsilon_{t-k}) = 0$ *for any t and any* $k \neq 0$
- $Var(\varepsilon_t) = E\left[(\varepsilon_t - E(\varepsilon_t))^2\right] = E\left[(\varepsilon_t - 0)^2\right] = E(\varepsilon_t{}^2) = \sigma_\varepsilon^2$ *for any t*

  that means e.g. $E(\varepsilon_{t-1}^2) = \sigma_\varepsilon^2, \quad E(\varepsilon_{t-2}^2) = \sigma_\varepsilon^2, \quad E(\varepsilon_{t-3}^2) = \sigma_\varepsilon^2,$ *etc.*

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Recall:

$$\rho_k := \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)} = \frac{E[(y_t-\mu)(y_{t-k}-\mu)]}{E\left[(y_t-\mu)^2\right]}$$

Solution 2.6-2:

Theoretical ACF for a **MA(1)** process with $\boldsymbol{\mu = 0}$ for lags $\boldsymbol{k = 1, 2}$.

MA(1): $y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}$

with $\boxed{\varepsilon_t \sim i.i.d. \quad E(\varepsilon_t)=0, \quad Var(\varepsilon_t)=\sigma_\varepsilon^2}$

These conditions imply:

- $E(\varepsilon_t \varepsilon_{t-k}) = 0$ *for any t and any* $k \neq 0$
- $E(\varepsilon_t^2) = \sigma_\varepsilon^2$ *for any t*

$$\boxed{\gamma_0} = Var(y_t) = E\left[(y_t-\mu)^2\right] = E(y_t^2) = E\left[(\varepsilon_t - \theta_1\varepsilon_{t-1})^2\right] = E(\varepsilon_t^2 - 2\theta_1\varepsilon_t\varepsilon_{t-1} + \theta_1^2\varepsilon_{t-1}^2)$$
$$= E(\varepsilon_t^2) - 2\theta_1 E(\varepsilon_t\varepsilon_{t-1}) + \theta_1^2 E(\varepsilon_{t-1}^2) = \sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2 = \boxed{(1+\theta_1^2)\sigma_\varepsilon^2}$$

$$\boxed{\gamma_1} = Cov(y_t, y_{t-1}) = E[(y_t-\mu)(y_{t-1}-\mu)] = E(y_t y_{t-1}) = E[(\varepsilon_t - \theta_1\varepsilon_{t-1})(\varepsilon_{t-1} - \theta_1\varepsilon_{t-2})]$$
$$= E(\varepsilon_t\varepsilon_{t-1} - \theta_1\varepsilon_t\varepsilon_{t-2} - \theta_1\varepsilon_{t-1}^2 + \theta_1^2\varepsilon_{t-1}\varepsilon_{t-2}) = -\theta_1 E(\varepsilon_{t-1}^2) = \boxed{-\theta_1\sigma_\varepsilon^2}$$

$$\boxed{\gamma_2} = Cov(y_t, y_{t-2}) = E[(y_t-\mu)(y_{t-2}-\mu)] = E(y_t y_{t-2}) = E[(\varepsilon_t - \theta_1\varepsilon_{t-1})(\varepsilon_{t-2} - \theta_1\varepsilon_{t-3})]$$
$$= E(\varepsilon_t\varepsilon_{t-2} - \theta_1\varepsilon_t\varepsilon_{t-3} - \theta_1\varepsilon_{t-1}\varepsilon_{t-2} + \theta_1^2\varepsilon_{t-1}\varepsilon_{t-3}) = \boxed{0}$$

Solution 2.6-3:

Theoretical acf for a MA(1) process with $\mu = 0$ for lags $k = 1, 2$.

MA(1): $y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}$

with ε_t~ i.i.d. $E(\varepsilon_t) = 0, Var(\varepsilon_t) = \sigma_\varepsilon^2$

$$\gamma_0 = \left(1 + \theta_1^2\right)\sigma_\varepsilon^2$$

$$\gamma_1 = -\theta_1 \sigma_\varepsilon^2$$

$$\gamma_2 = 0$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1 \sigma_\varepsilon^2}{\left(1 + \theta_1^2\right)\sigma_\varepsilon^2} = \frac{-\theta_1}{\left(1 + \theta_1^2\right)}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0}{\left(1 + \theta_1^2\right)\sigma_\varepsilon^2} = 0$$

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Recall:

$$\rho_k := \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)} = \frac{E[(y_t-\mu)(y_{t-k}-\mu)]}{E[(y_t-\mu)^2]}$$

Solution 2.6-4:

Theoretical ACF for an **AR(1)** process with **δ = 0** for lags ***k* = 1, 2**.

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

with $\boxed{\varepsilon_t \sim i.i.d. \quad E(\varepsilon_t)=0, \quad Var(\varepsilon_t)=\sigma_\varepsilon^2}$

These conditions imply:

- $E(\varepsilon_t \varepsilon_{t-k}) = 0 \quad$ *for any t and any* $k \neq 0$
- $E(\varepsilon_t^2) = \sigma_\varepsilon^2 \quad$ *for any t*

Moreover:

- stationarity implies

$$E(y_t) = \varphi_1 E(y_{t-1}) + \delta$$
$$\mu = \varphi_1 \mu + \delta$$
$$\mu = \delta/(1-\varphi_1)$$

therefore

$$\delta = 0 \;\Rightarrow\; \mu = 0$$

- stationarity implies

$$Var(y_t) = Var(y_{t-k}) \quad \textit{for any } k$$

- by definition and using $\mu=0$

$$Var(y_t) = E[(y_t-\mu)^2] = E(y_t^2)$$

therefore

$$E(y_t^2) = E(y_{t-k}^2) =: \gamma_0 \quad \textit{for any } k,t$$

for any t

- $E(y_t \varepsilon_t) = E[(\varphi_1 y_{t-1} + \varepsilon_t)\varepsilon_t]$
$= \varphi_1 E[y_{t-1}\varepsilon_t] + E[\varepsilon_t^2]$
$= 0 + \sigma_\varepsilon^2 = \sigma_\varepsilon^2$

for any t and k≠0

- $E(y_{t-k}\varepsilon_t) = E[(\varphi_1 y_{t-k-1} + \varepsilon_{t-k})\varepsilon_t]$
$= \varphi_1 E[y_{t-k-1}\varepsilon_t] + E[\varepsilon_{t-k}\varepsilon_t]$
$= 0 + 0 = 0$

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Solution 2.6-4:

Theoretical acf for an AR(1) process with $\delta = 0$ for lags $k = 1, 2$.

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

with ε_t~ i.i.d. $E(\varepsilon_t) = 0, Var(\varepsilon_t) = \sigma_\varepsilon^2$

$$\boxed{\gamma_0} = \mathrm{Var}(y_t) = \mathrm{E}(y_t^{\,2}) = \mathrm{E}(\varphi_1 y_{t-1} + \varepsilon_t)^2 = \mathrm{E}(\varphi_1^2 y_{t-1}^2 + \varepsilon_t^2 + 2\varphi_1 y_{t-1}\varepsilon_t)$$

$$= \varphi_1^2 \mathrm{E}(y_{t-1}^2) + \mathrm{E}(\varepsilon_t^2) + 2\varphi_1 \mathrm{E}(y_{t-1}\varepsilon_t) = \varphi_1^2 \gamma_0 + \sigma_\varepsilon^2 \Rightarrow \boxed{\gamma_0 = \frac{\sigma_\varepsilon^2}{1-\varphi_1^2}}$$

$$\boxed{\gamma_1} = Cov(y_t, y_{t-1}) = E(y_t y_{t-1}) = E[(\varphi_1 y_{t-1} + \varepsilon_t) y_{t-1}] = E(\varphi_1 y_{t-1}^2 + \varepsilon_t y_{t-1})$$

$$= \varphi_1 E(y_{t-1}^2) + E(\varepsilon_t y_{t-1}) = \varphi_1 \gamma_0 = \boxed{\frac{\varphi_1 \sigma_\varepsilon^2}{1-\varphi_1^2}}$$

$$\boxed{\gamma_2} = Cov(y_t, y_{t-2}) = E[(\varphi_1 y_{t-1} + \varepsilon_t) y_{t-2}] = E[(\varphi_1(\varphi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t) y_{t-2}]$$

$$= E[(\varphi_1^2 y_{t-2} + \varphi_1 \varepsilon_{t-1} + \varepsilon_t) y_{t-2}] = \varphi_1^2 \gamma_0 = \boxed{\frac{\varphi_1^2 \sigma_\varepsilon^2}{1-\varphi_1^2}}$$

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Solution 2.6-5:

Theoretical acf for an AR(1) process with δ = 0 for lags k = 1, 2.

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

with ε_t~ i.i.d. $E(\varepsilon_t) = 0, Var(\varepsilon_t) = \sigma_\varepsilon^2$

$$\gamma_0 = \frac{\sigma_\varepsilon^2}{1-\varphi_1^2}$$

$$\gamma_1 = \frac{\varphi_1 \sigma_\varepsilon^2}{1-\varphi_1^2}$$

$$\gamma_2 = \frac{\varphi_1^2 \sigma_\varepsilon^2}{1-\varphi_1^2}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \varphi_1$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \varphi_1^2$$

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Exercise 2.7:

Simulate the following processes with 200 observations and $\varepsilon_t \sim \text{i.i.d.}\, N(0, \sigma_\varepsilon^2 = 1)$:

- White noise
- MA(1) with θ_1 = -0.8
- MA(2) with θ_1 = -0.6 and θ_2 = 0.3

Plot in each case the ACF of the series with the help of the do-file (acf_exercise.do) and compare to its theoretical ACF you would expect.

Notice:

ACF MA(1)

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{-\theta_1}{1+\theta_1^2} & k = 1 \\ 0 & k > 1 \end{cases}$$

ACF MA(2)

$$\rho_1 = \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} \quad \rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$\rho_k = 0 \quad \text{for } k > 2$$

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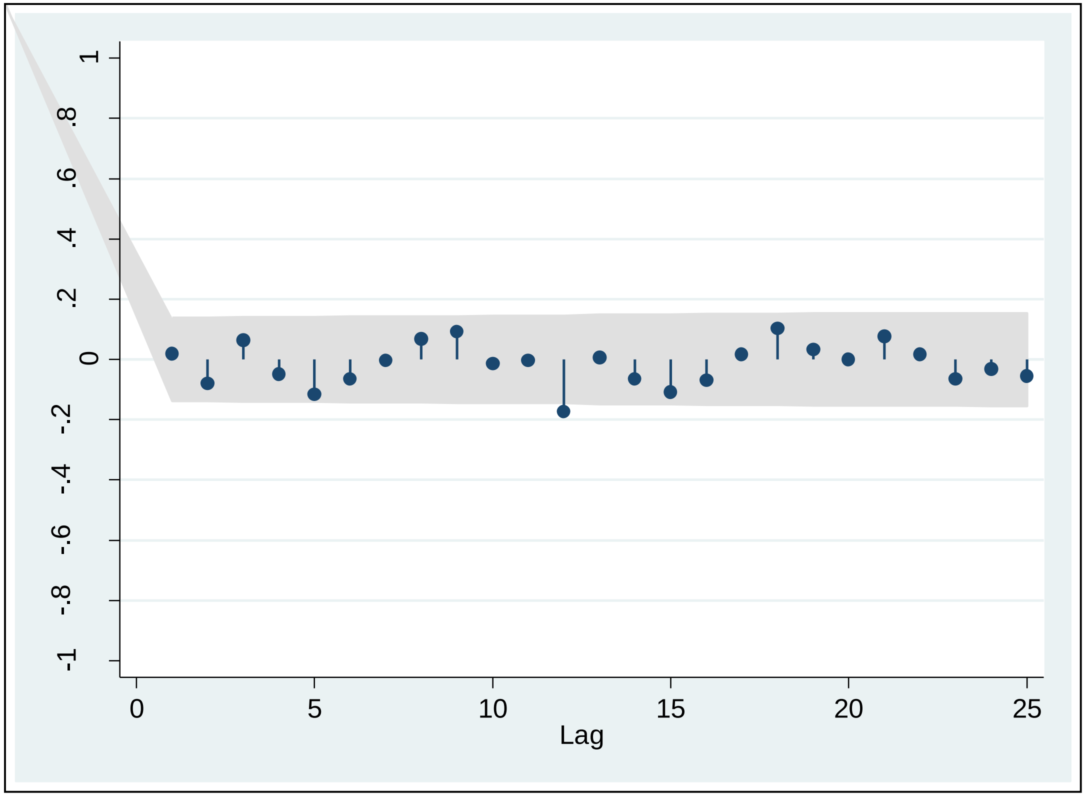
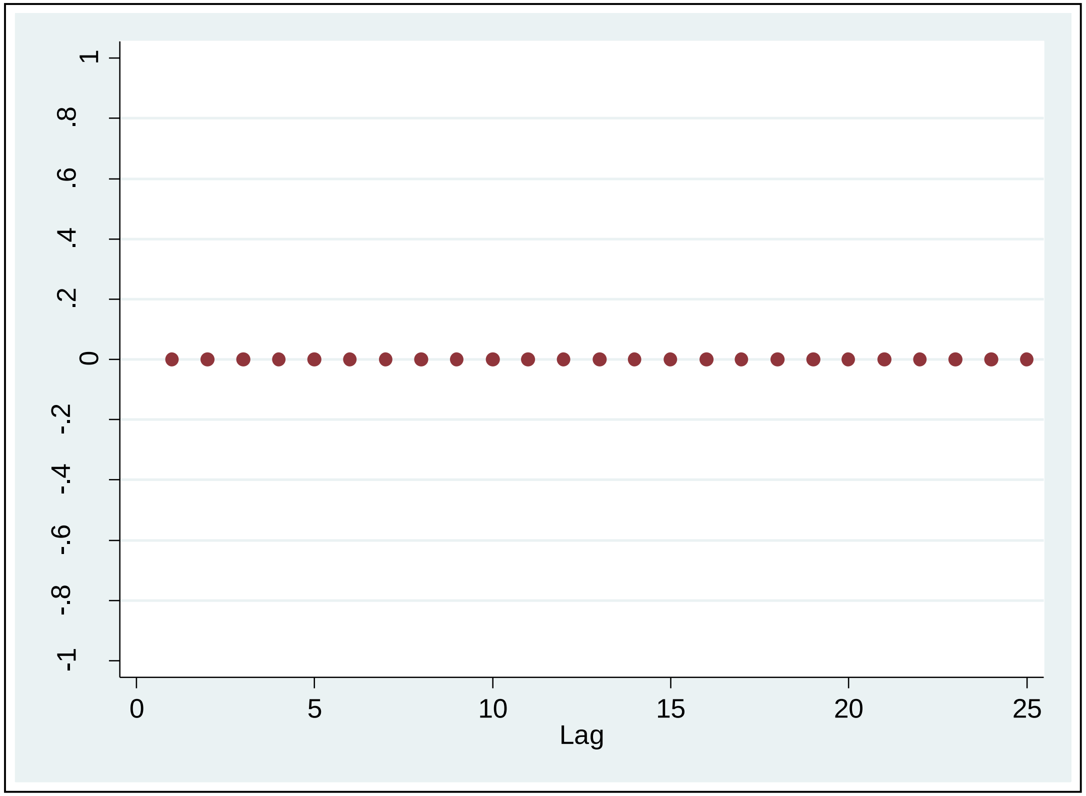
Solution 2.7-1:

ACF for **simulated** white noise



Solution 2.7-2:

Theoretical ACF for white noise process



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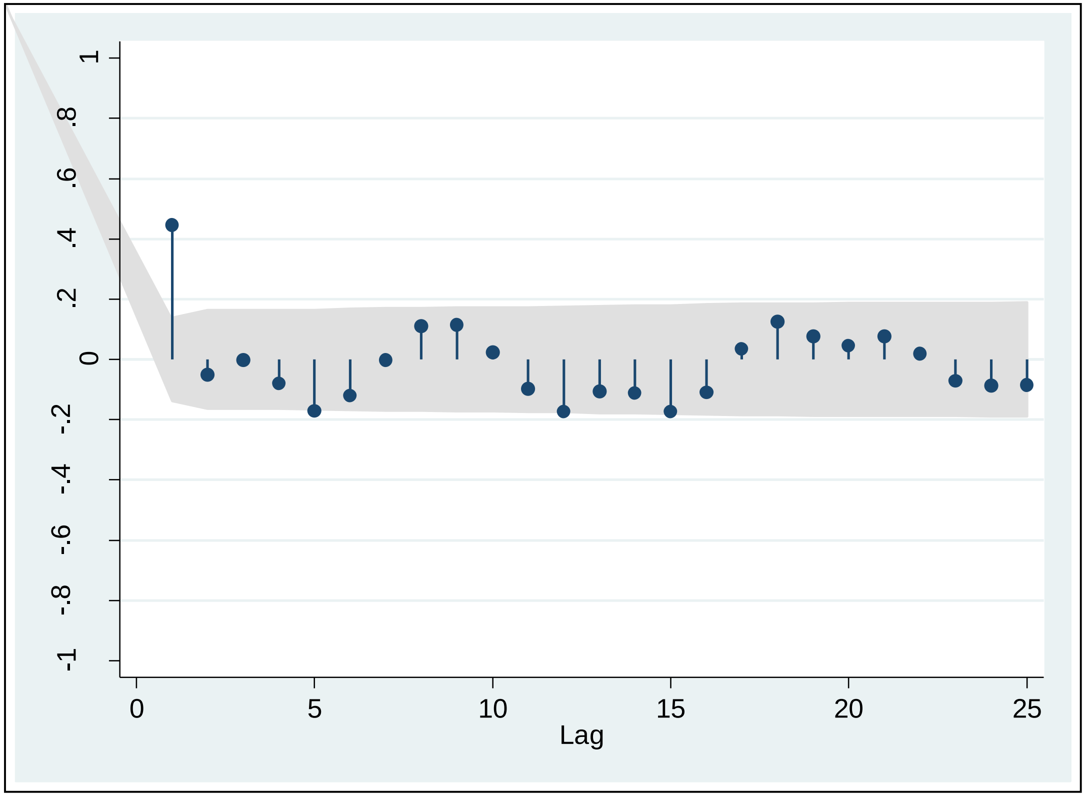
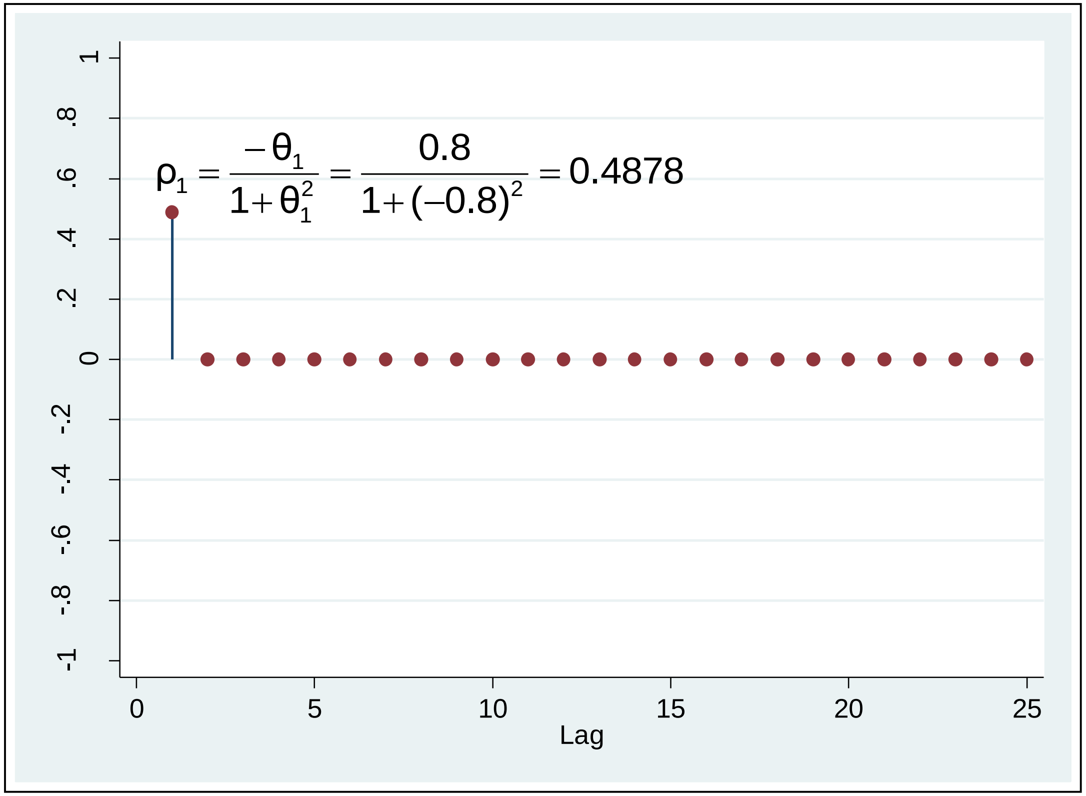
Solution 2.7-3:

ACF for **simulated** MA(1) with θ_1 = -0.8



Solution 2.7-4:

Theoretical ACF for MA(1) with θ_1 = -0.8



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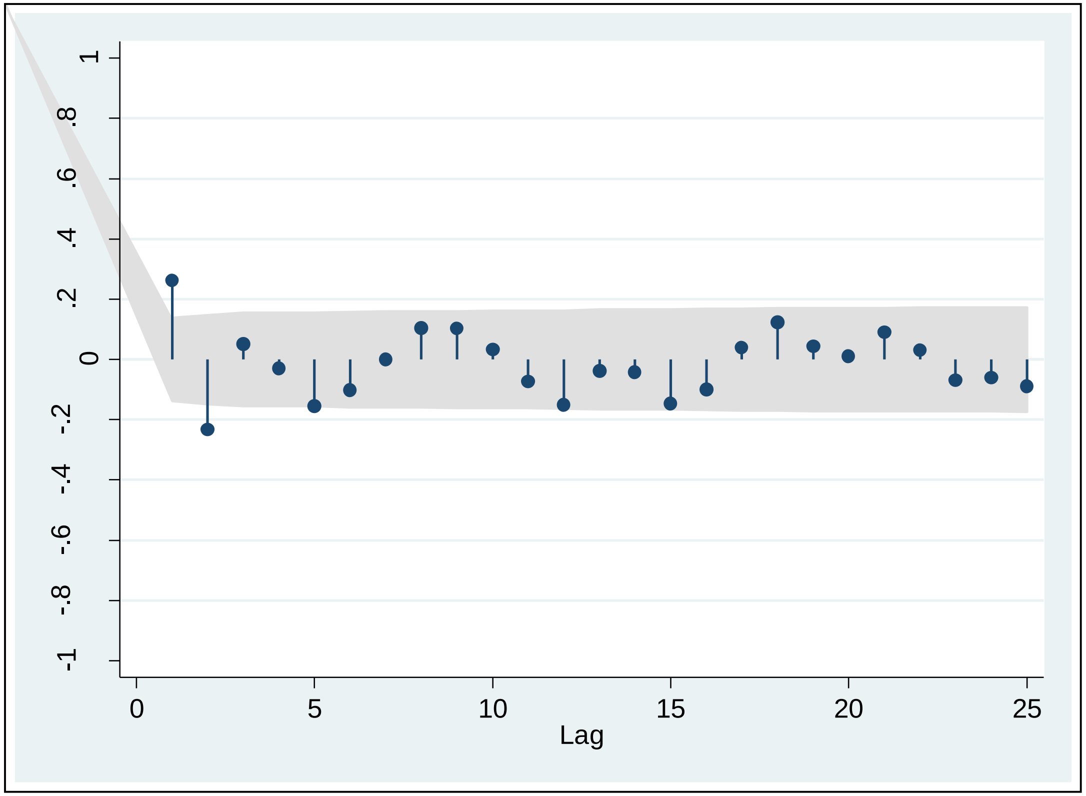
Solution 2.7-5:

ACF for **simulated** MA(2) with θ_1 = -0.6, θ_2 = 0.3



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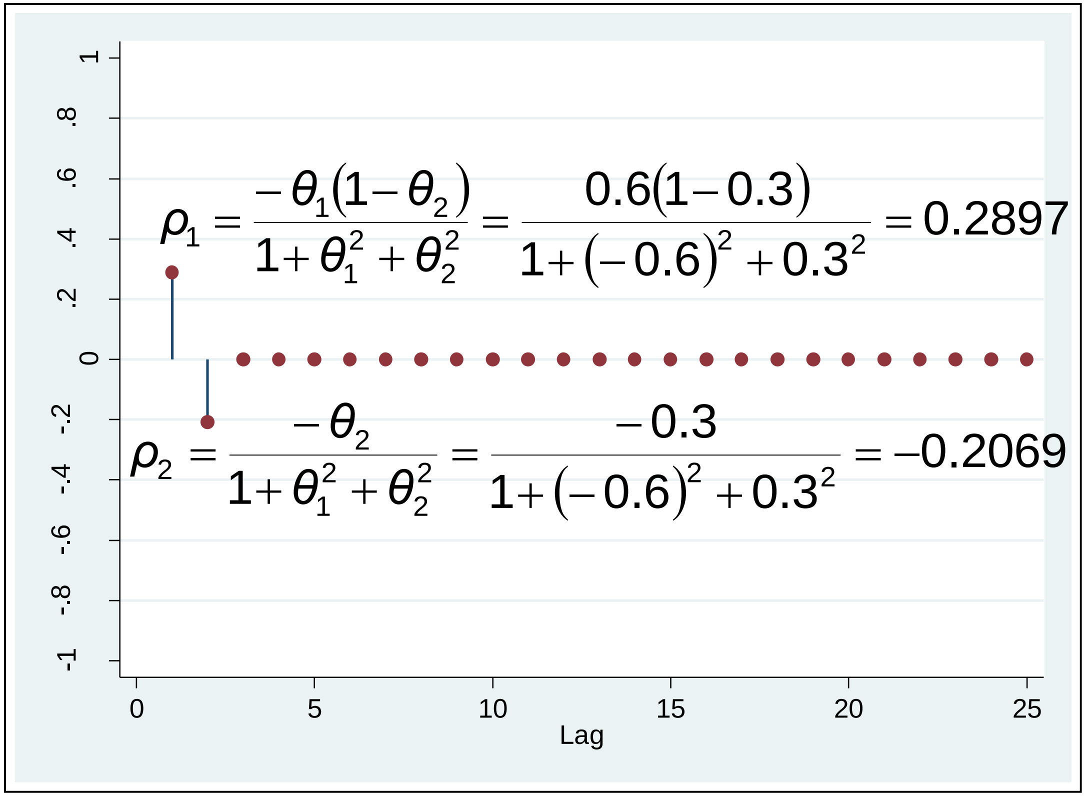
Solution 2.7-6:

Theoretical ACF for MA(2) with θ_1 = -0.6, θ_2 = 0.3



$$\rho_1 = \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} = \frac{0.6(1-0.3)}{1+(-0.6)^2+0.3^2} = 0.2897$$

$$\rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2} = \frac{-0.3}{1+(-0.6)^2+0.3^2} = -0.2069$$

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Exercise 2.8:

Simulate the following processes with 2000 observations and $\varepsilon_t \sim \text{i.i.d.}\, N\left(0, \sigma_\varepsilon^2 = 1\right)$:

- AR(1) with $\varphi_1 = 0.9$
- AR(2) with $\varphi_1 = 0.9$ und $\varphi_2 = -0.7$

Plot in each case the ACF of the series with the help of the do-file (acf_exercise.do) for the last 200 observations and compare to its theoretical ACF you would expect.

Notice:

ACF AR(1)

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \varphi_1^k$$

ACF AR(2)

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\varphi_1}{1-\varphi_2}, \quad \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\varphi_1^2}{1-\varphi_2} + \varphi_2$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \varphi_1 \rho_{k-1} + \varphi_2 \rho_{k-2}$$

Stochastic first-order difference equation

Date	Equation
1	$y_1 = \varphi_1 \cdot y_0 + (\delta + \varepsilon_1)$
2	$y_2 = \varphi_1 \cdot y_1 + (\delta + \varepsilon_2)$
	$= \varphi_1 \cdot [\varphi_1 \cdot y_0 + (\delta + \varepsilon_1)] + (\delta + \varepsilon_2)$
	$= \varphi_1^2 \cdot y_0 + (\varphi_1 + 1)\delta + \varphi_1 \varepsilon_1 + \varepsilon_2$
3	$y_3 = \varphi_1 \cdot y_2 + (\delta + \varepsilon_3)$
	$= \varphi_1^3 \cdot y_0 + (\varphi_1^2 + \varphi_1 + 1)\delta + \varphi_1^2 \varepsilon_1 + \varphi_1 \varepsilon_2 + \varepsilon_3$
	M
t	$y_t = \varphi_1^t \cdot y_0 + \left(\varphi_1^{t-1} + \varphi_1^{t-2} + \ldots + \varphi_1 + 1\right) \cdot \delta + \sum_{j=1}^{t} \varphi_1^{t-j} \cdot \varepsilon_j$

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AR(1) as First-order difference equation

$$y_t = \varphi_1^t \cdot y_0 + \left(\varphi_1^{t-1} + \varphi_1^{t-2} + \ldots + \varphi_1 + 1\right) \cdot \delta + \sum_{j=1}^{t} \varphi_1^{t-j} \cdot \varepsilon_j$$

$$E[y_t] = \varphi_1^t \cdot y_0 + \left(\varphi_1^{t-1} + \varphi_1^{t-2} + \ldots + \varphi_1 + 1\right) \cdot \delta$$

$$E[y_{t+s}] = \varphi_1^{t+s} \cdot y_0 + \left(\varphi_1^{t+s-1} + \varphi_1^{t+s-2} + \ldots + \varphi_1^{t-1} + \ldots + \varphi_1 + 1\right) \cdot \delta$$

$$E[y_t] \neq E[y_{t+s}]$$

To get stationarity, we need to impose conditions.

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AR(1) as First-order difference equation:

$$y_t = \varphi_1^t \cdot y_0 + \left(\varphi_1^{t-1} + \varphi_1^{t-2} + \ldots + \varphi_1 + 1\right) \cdot \delta + \sum_{j=1}^{t} \varphi_1^{t-j} \cdot \varepsilon_j$$

If $t \to \infty$ and $|\varphi_1|<1$

$$\lim y_t = \delta \big/ (1-\varphi_1) + \sum_{j=0}^{\infty} \varphi_1^j \cdot \varepsilon_{t-j} \qquad \Rightarrow E[y_t] = \delta \big/ (1-\varphi_1)$$

$t \to \infty$ means: the process has started a long time ago "**stochastic initial conditions**" ➔ That is why we only use the last 200 observations from our simulated 2000 data points!

$|\varphi_1|<1$ means: dependence can't be too strong

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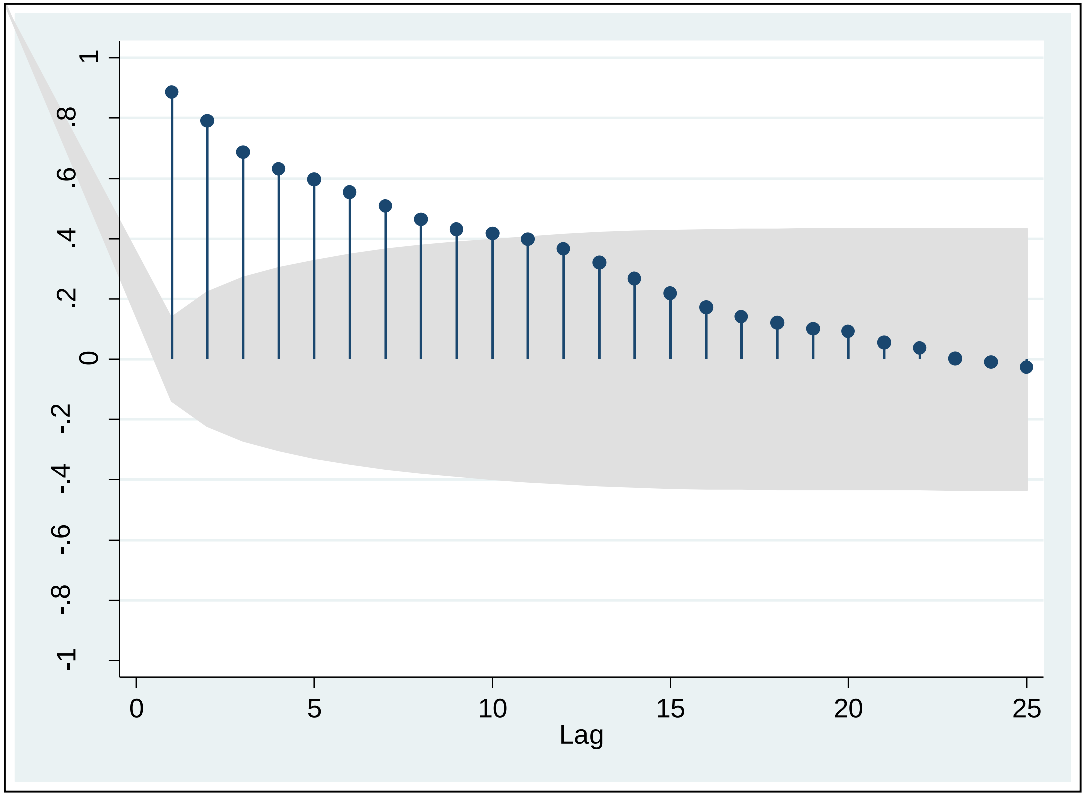
Solution 2.8-1:

ACF for **simulated** AR(1) (last 200 observations) with $\varphi_1 = 0.9$



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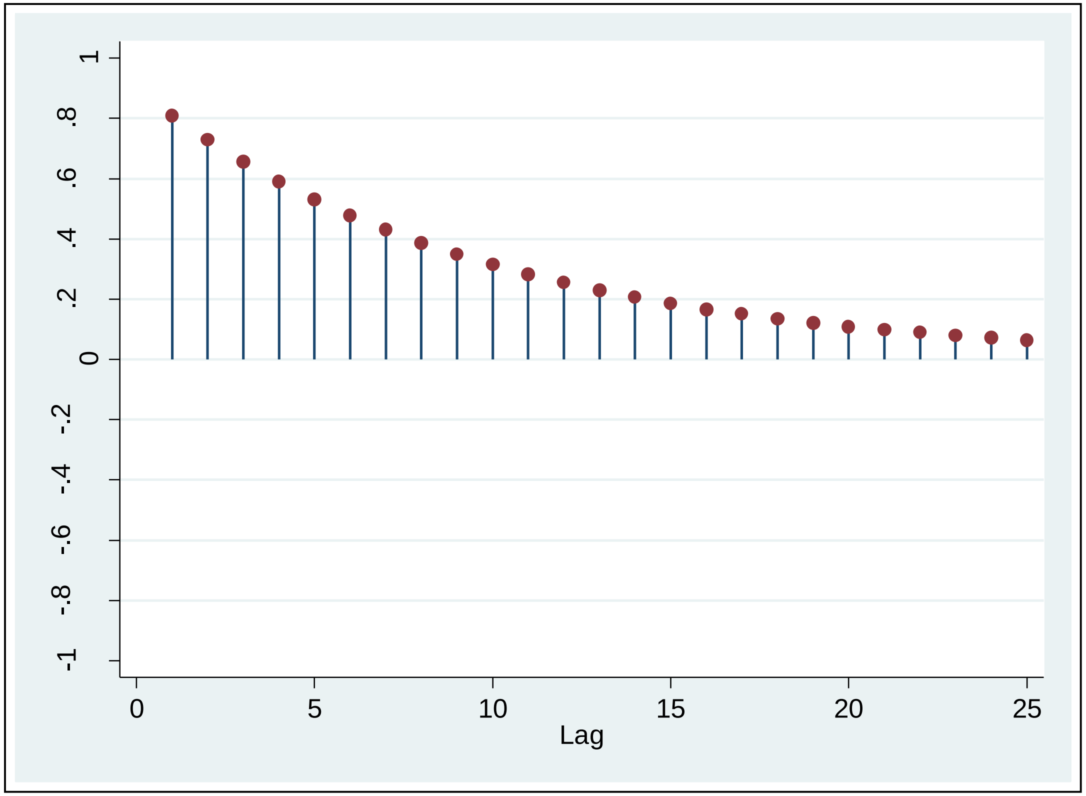
Solution 2.8-2:

Theoretical ACF for an AR(1) with $\varphi_1 = 0.9$



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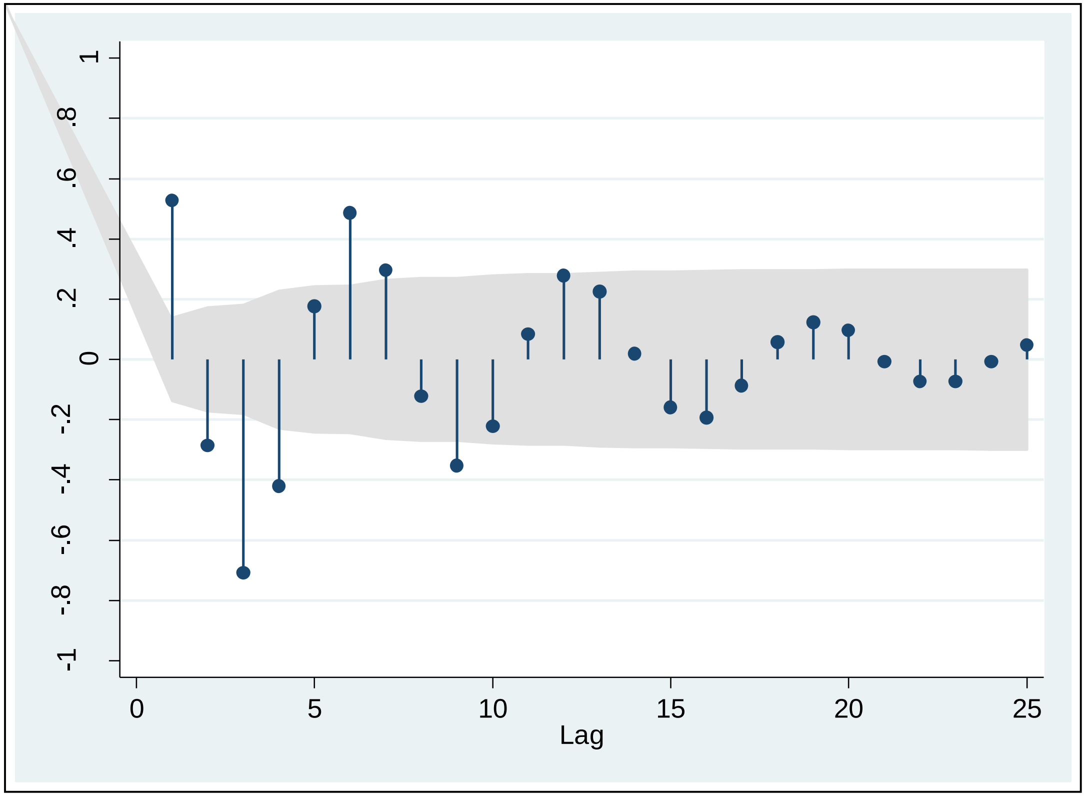
Solution 2.8-3:

ACF for **simulated** AR(2) (last 200 observations) with $\varphi_1 = 0.9$ and $\varphi_2 = -0.7$



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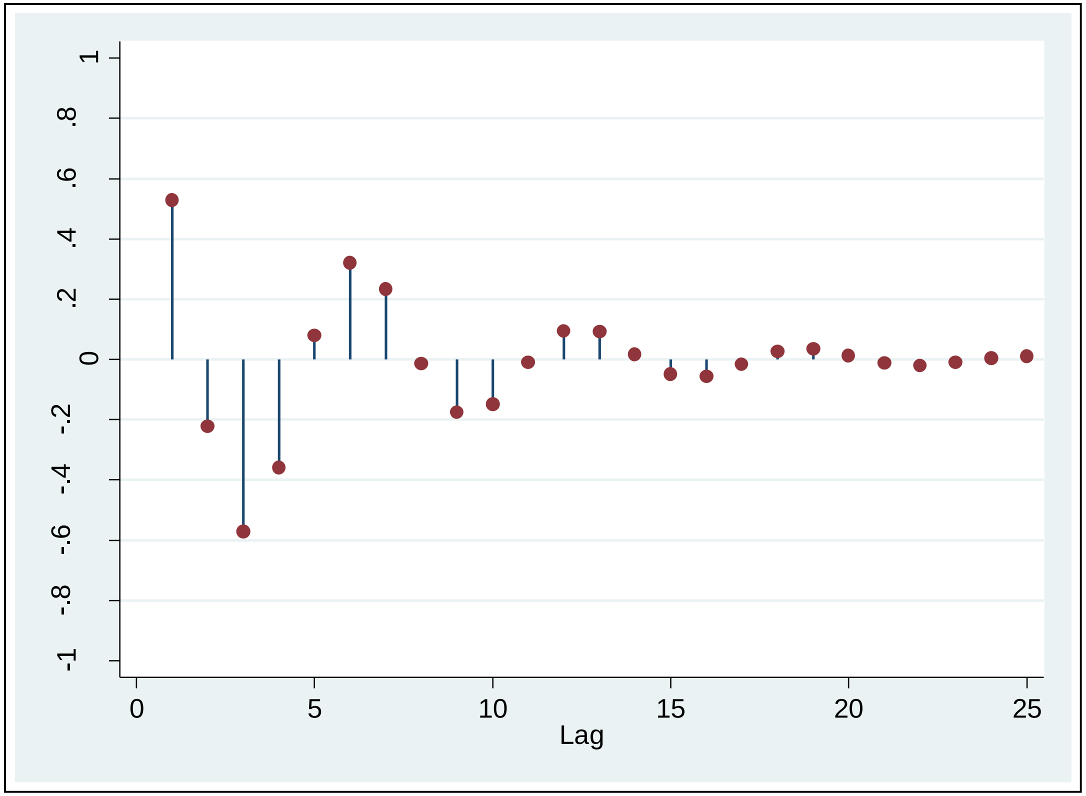
Solution 2.8-4:

Theoretical ACF for AR(2) with φ_1 = 0.9 and φ_2 = -0.7



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Exercise 2.9:

- What is the **partial** autocorrelation function (PACF) and what is its purpose?
- Which regression should you run to estimate the partial autocorrelation coefficient for y_t at lag $k = 3$?
- What should the PACFs for the simulated processes from **Exercise 2.7** and **Exercise 2.8** look like?

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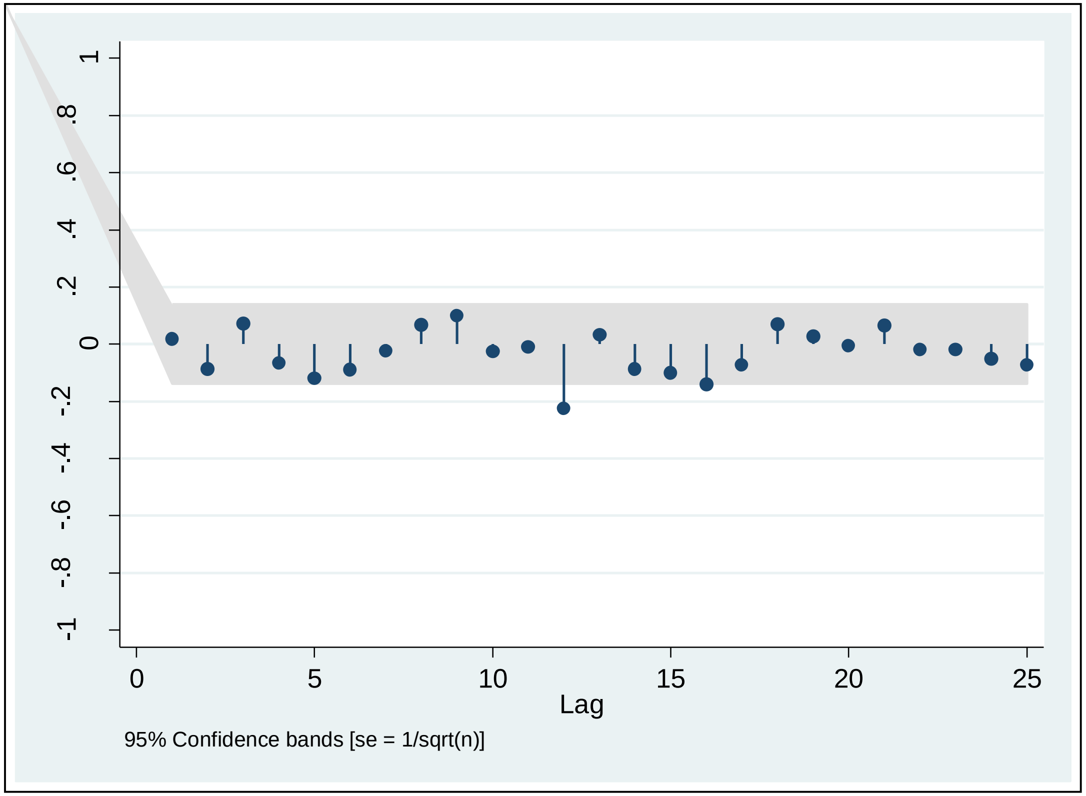
Solution 2.9-1:

- It measures how y_t and y_{t+k} are related, but **with the effects of the intervening *y*'s accounted for**.
- The PACF is useful for determining the order of an AR process.
- Regress y_t on a constant and y_{t-1}, y_{t-2} and y_{t-3}. The last coefficient is the partial autocorrelation coefficient for lag $k = 3$.

Solution 2.9-2:

PACF for **simulated** white noise



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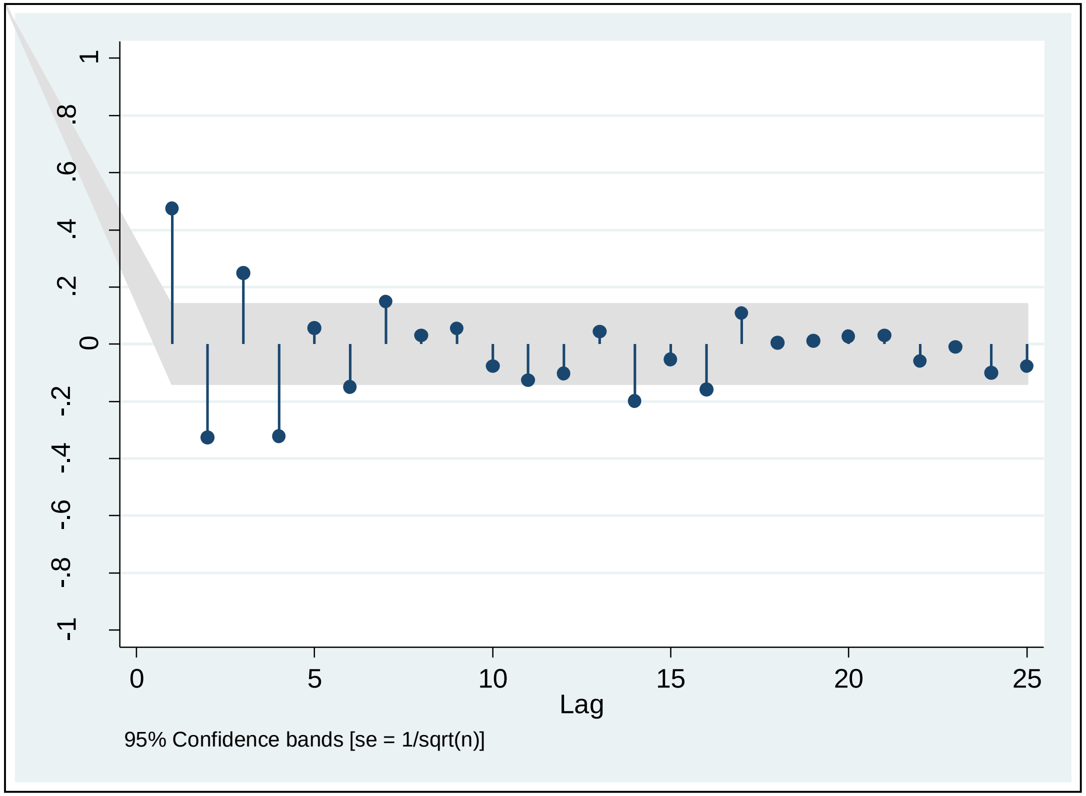
Solution 2.9-3:

PACF for **simulated** MA(1) with θ_1 = -0.8



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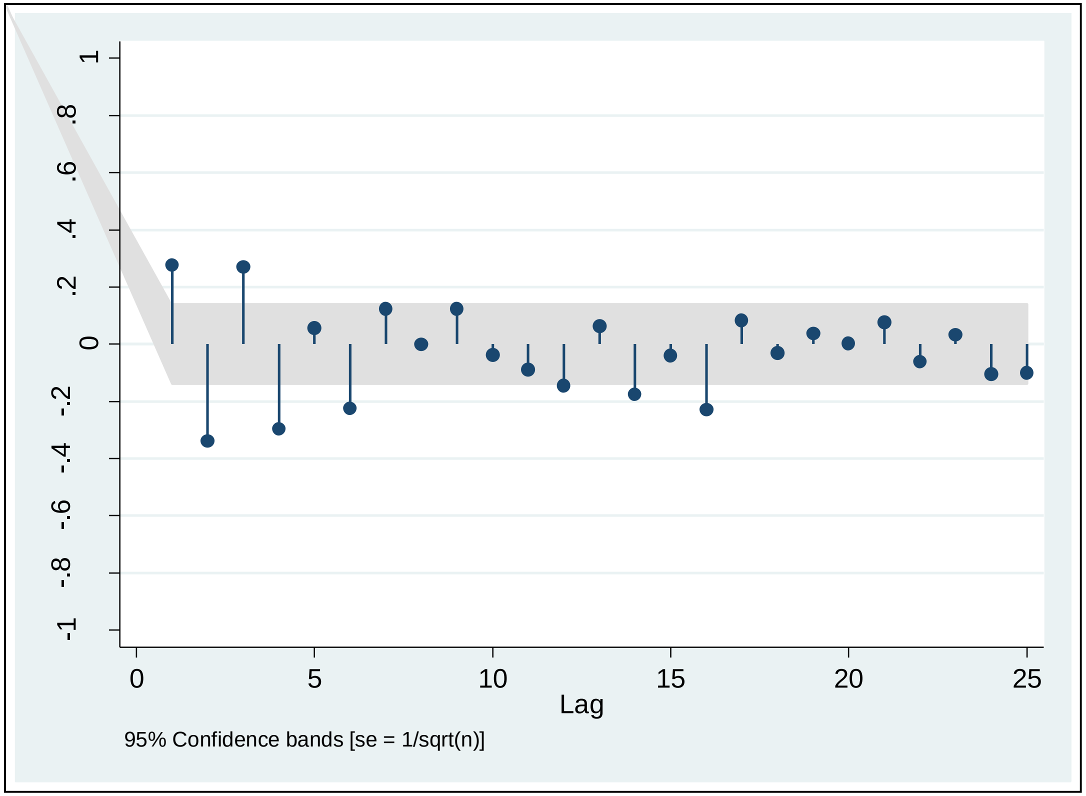
Solution 2.9-4:

PACF for **simulated** MA(2) with θ_1 = -0.6, θ_2 = 0.3



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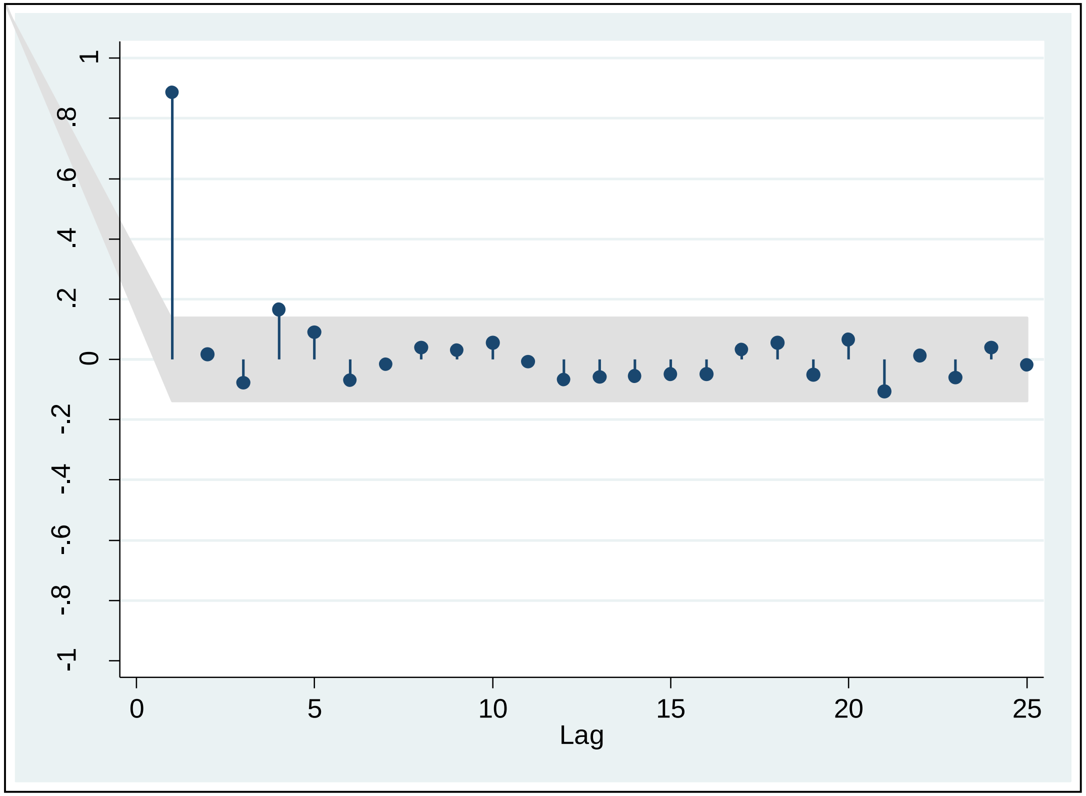
Solution 2.9-5:

PACF for **simulated** AR(1) with $\varphi_1 = 0.9$



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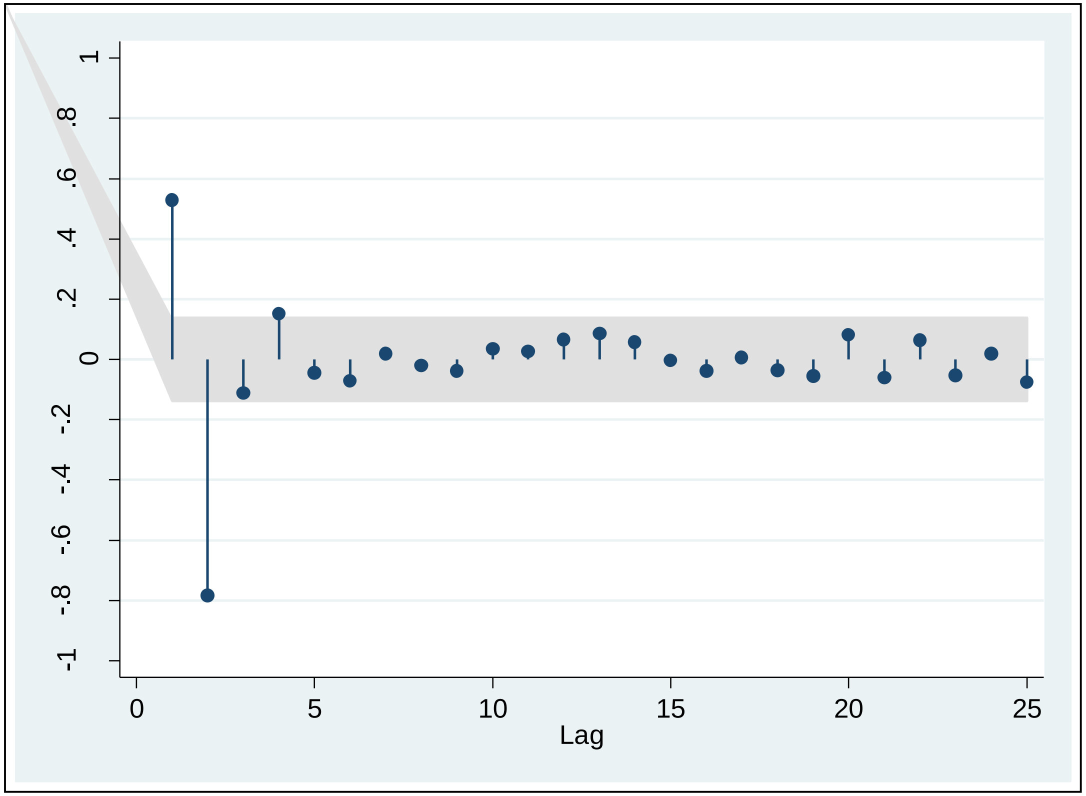
Solution 2.9-6:

ACF for **simulated** AR(2) with φ_1 = 0.9 and φ_2 = -0.7



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Exercise 2.10:

- Summarize the stylized shapes for the autocorrelation and partial autocorrelation functions for the following stochastic processes: white noise, AR(1), AR(2), MA(1) and MA(2).
- Which model in general is more applicable to model strong serial dependence?

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Solution 2.10-1:

Process	ACF	PACF
white noise	no spikes	no spikes
AR(1)	Exponential decay • on the positive side if $\varphi_1 > 0$ • alternating in sign if $\varphi_1 < 0$	Spike at lag 1, then cuts off to zero • spike is positive if $\varphi_1 > 0$ • spike is negative if $\varphi_1 < 0$
AR(2)	A mixture of exponential decays or a damped sine wave	Spikes at lags 1 and 2, then cuts off to zero
MA(1)	Spike at lag 1, then cuts off to zero • spike is positive if $\theta_1 < 0$ • spike is negative if $\theta_1 > 0$	Damps out exponentially • alternating in sign, starting on the positive side, if $\theta_1 < 0$ • starting on the negative side, if $\theta_1 > 0$
MA(2)	Spike at lag 1 and 2, then cuts off to zero	A mixture of exponential decays or a damped sine wave

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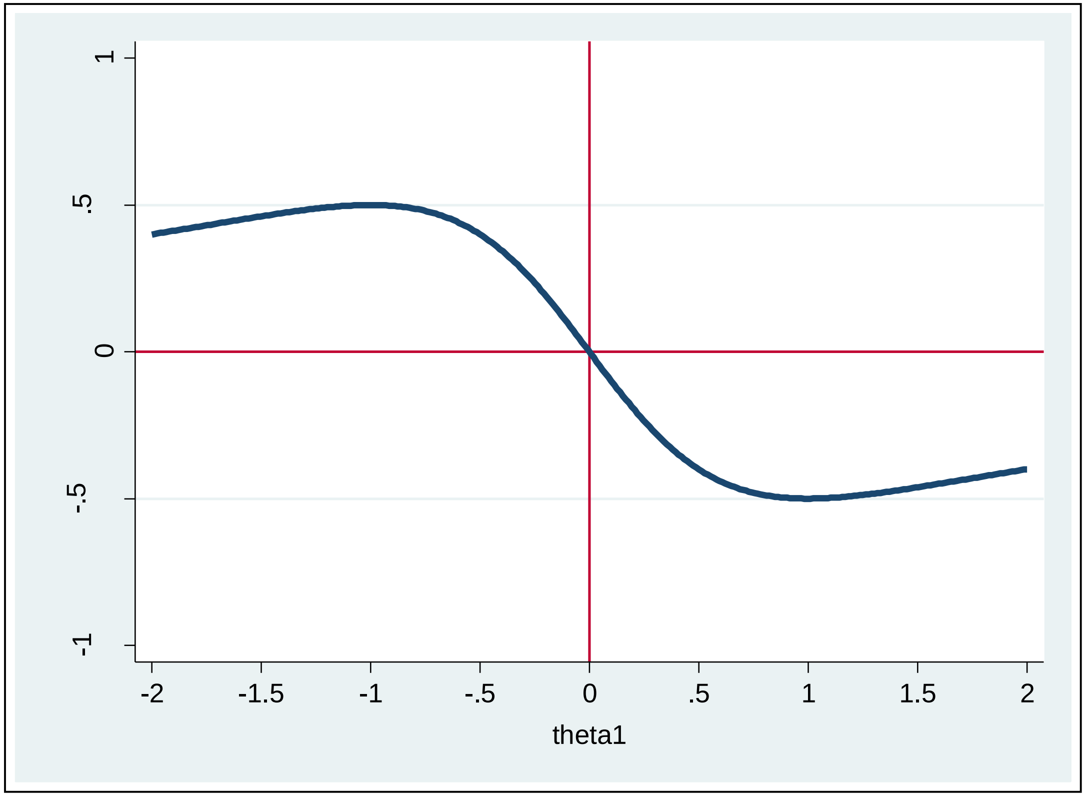
Solution 2.10-2:

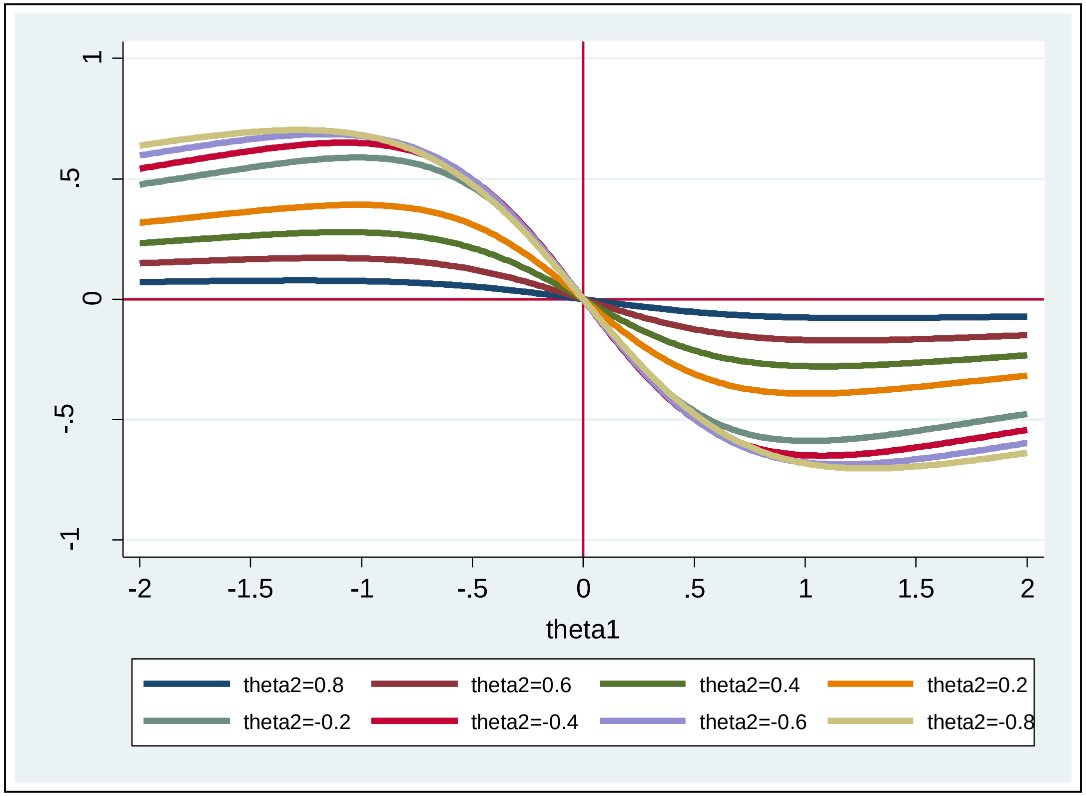


MA(1):

$$y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$\rho_1 = \frac{-\theta_1}{1+\theta_1^2}$$

Solution 2.10-3:

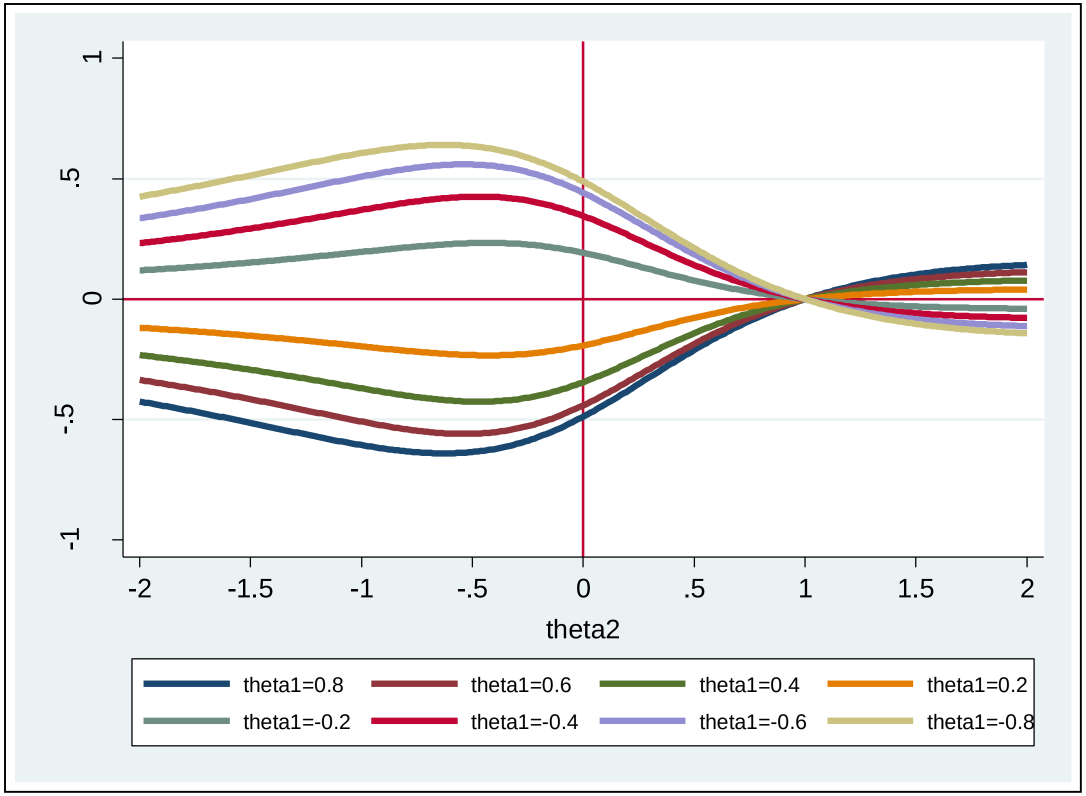


MA(2):

$$y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

$$\rho_1 = \frac{-\theta_1 (1 - \theta_2)}{1 + \theta_1^2 + \theta_2^2}$$

Solution 2.10-4:



MA(2):

$$y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

$$\rho_1 = \frac{-\theta_1 (1 - \theta_2)}{1 + \theta_1^2 + \theta_2^2}$$

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Organizational Issues

Please log in to **ISIS** (password: Zeit1718) and **download** the following file:

- inventories.dta

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Univariate Box-Jenkins models for stationary time series

General Procedure:

1. Identification
2. Estimation
   - Solution of the Yule-Walker equations (AR processes)
   - Least Squares Estimation (AR processes)
   - Maximum Likelihood Estimation
   - Conditional Maximum Likelihood Estimation
3. Diagnostic Checking
4. Forecasting

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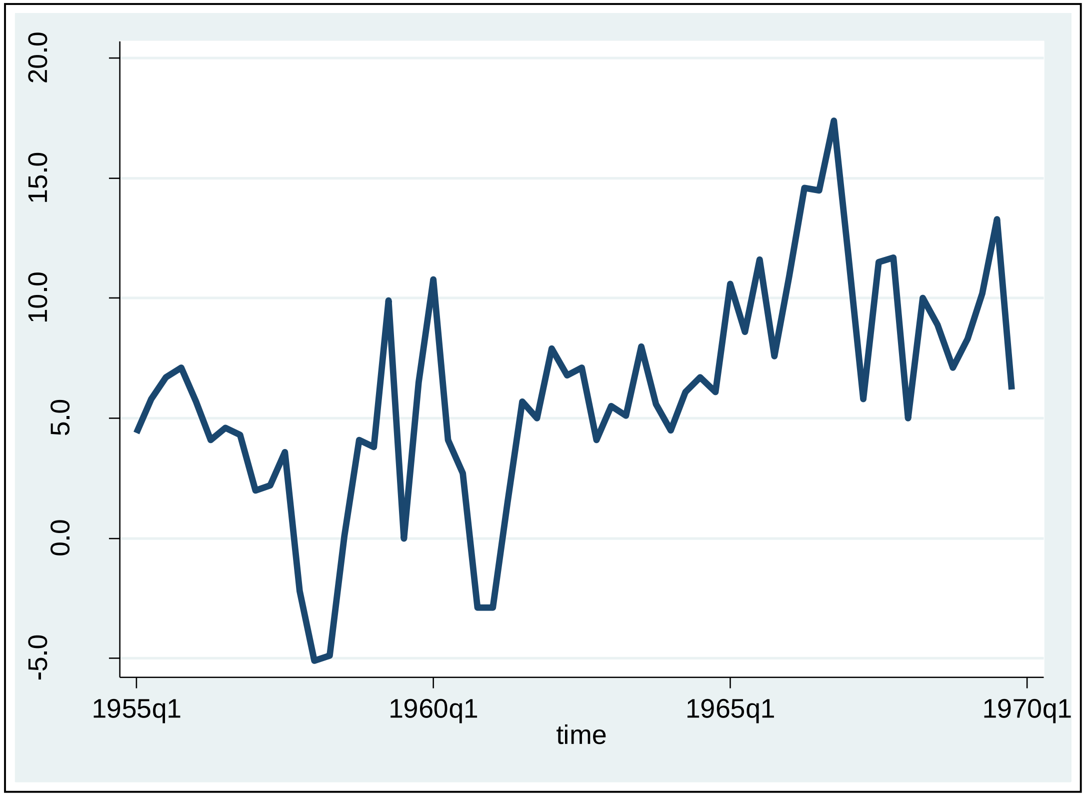
Business Inventories – Original Series

Y: “Quarterly change in business inventories”



- 60 observations from 1955q1 trough 1969q4
- the data have been seasonally adjusted

Pankratz (1983) “Forecasting with univariate Box-Jenkins models”

Exercise 2.11: Identification

- What is the first step in the Box-Jenkins methodology?

  Examine the **original series**. Is the series stationary?

  - The observations seem to fluctuate around a constant mean.
  - The variance seems to be constant over time.

- What is the next step in the Box-Jenkins methodology?

  Plot the **ACF** and **PACF** and describe their pattern. Use about one-forth of the number of observations. Identify the most appropriate time series process for this data.

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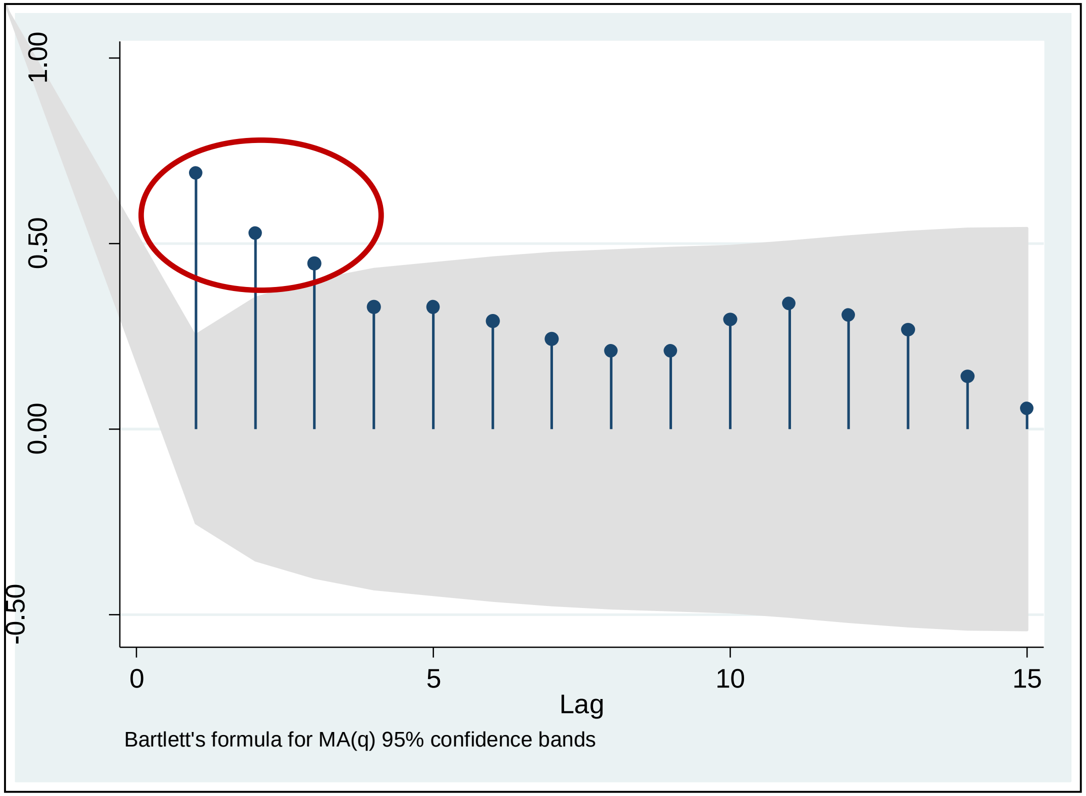
Solution 2.11-1: Identification

ACF: Is the series stationary? Which model is appropriate?

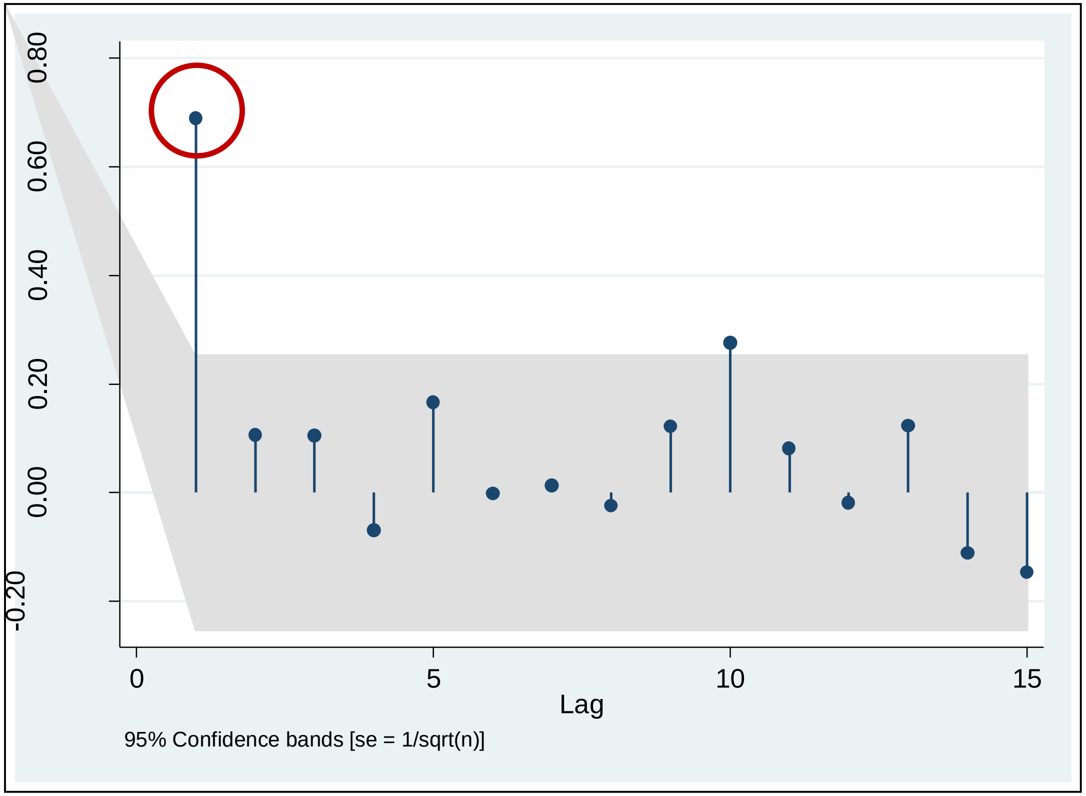


- Only the first three autocorrelations are significantly different from zero.
- The autocorrelations decay to statistical insignificance rather quickly.
- An AR or an ARMA model seems appropriate.
- The estimated PACF should help us make the decision.

Exercise (inventories.dta)

Solution 2.11-2: Identification

PACF: Which model is appropriate?



- It has one spike at lag 1 which is significantly different from zero then it cuts off to zero.
- The estimated PACF suggests an **AR(1)**.
- $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

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Exercise 2.12: Estimation

Estimate the parameter of the

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

- Solution of the Yule-Walker equations
- Least Squares Estimation
- Maximum Likelihood Estimation
- Conditional Maximum Likelihood Estimation

Estimation – Solution of the Yule-Walker equations

Yule-Walker equations:

$$\rho_1 = \varphi_1 \rho_0 + \varphi_2 \rho_1 + \ldots + \varphi_p \rho_{p-1}$$

M

$$\rho_p = \varphi_1 \rho_{p-1} + \varphi_2 \rho_{p-2} + \ldots + \varphi_p$$

If ρ_1, ρ_2, ..., ρ_p are known, the equation can be solved for the autoregressive parameters φ_1, φ_2, ..., φ_p.

For an **AR(1)** process it reduces to:

$$\rho_1 = \varphi_1 \rho_0$$

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Solution 2.12-1: Estimation (Solution of the Yule-Walker equations)

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

For an **AR(1)** process: $\hat{\rho}_1 = \hat{\varphi}_1 \hat{\rho}_0$

```
. corrgram inventories
                                                          -1       0       1 -1       0       1
 LAG        AC        PAC        Q      Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1         0.6897   0.6898   29.996   0.0000          |-----             |-----
2         0.5274   0.1067   47.837   0.0000          |----              |
[...]
```

$$\hat{\varphi}_1 = \frac{\hat{\rho}_1}{\hat{\rho}_0} = 0.6897$$

```
. sum inventories
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
 inventories |        60       6.095    4.597362       -5.1       17.4
```

$$\hat{\delta} = \hat{\mu}(1 - \hat{\varphi}_1) = 6.095(1 - 0.6897) = 1.8913$$

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Estimation – Least Squares Estimation

We can estimate δ, φ_1, φ_2, ..., φ_p by ordinary least squares (these estimates minimize the sum of squared residuals):

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \ldots + \varphi_p y_{t-p} + \delta + \varepsilon_t$$

Under the following **assumption** OLS provides consistent estimators:

$$E(y_{t-j}\varepsilon_t) = 0 \quad \text{for } j = 1,2,3,\ldots,p$$

For an **AR(1)** process it reduces to:

$$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$$

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Solution 2.12-2: Estimation (Least Squares Estimation)

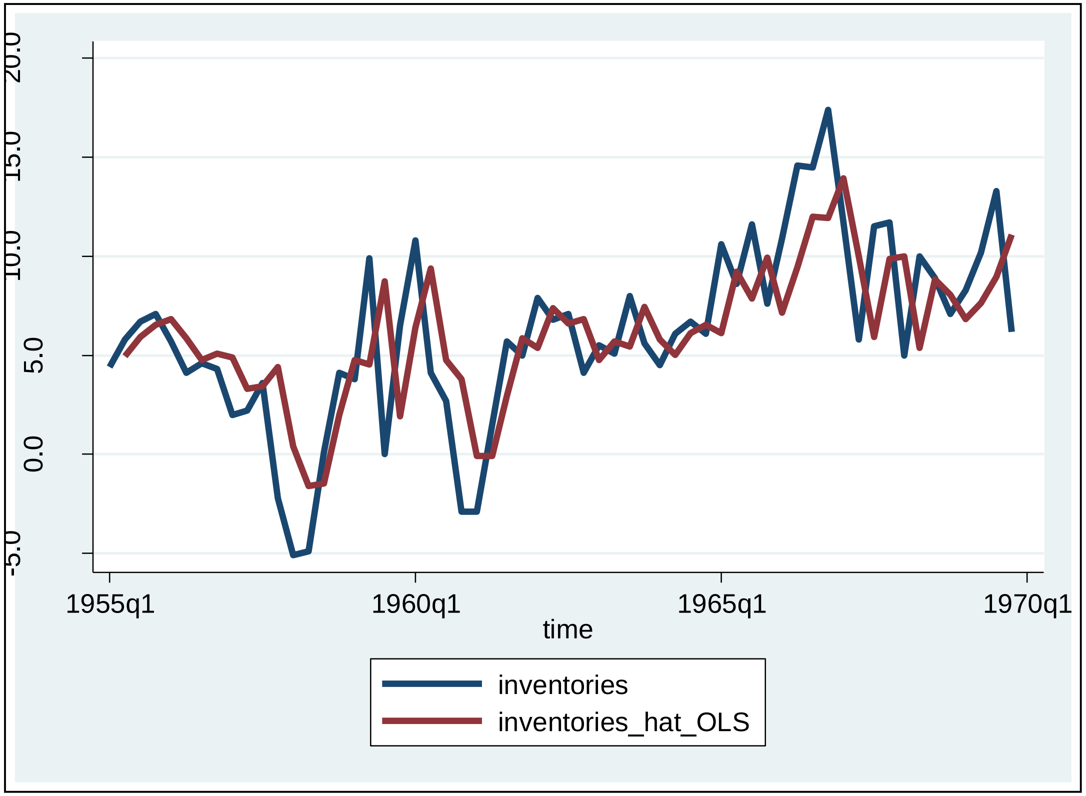
AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

```
. regress inventories L1.inventories

      Source |       SS       df       MS              Number of obs =      59
-------------+------------------------------           F(  1,    57) =   51.96
       Model |   593.26957     1   593.26957           Prob > F      =  0.0000
    Residual |  650.817213    57  11.4178458           R-squared     =  0.4769
-------------+------------------------------           Adj R-squared =  0.4677
       Total |  1244.08678    58  21.4497721           Root MSE      =   3.379

------------------------------------------------------------------------------
 inventories |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
 inventories |
         L1. |   .6897525   .0956884     7.21   0.000     .4981398    .8813652
       _cons |   1.920915   .7303908     2.63   0.011     .4583325    3.383497
------------------------------------------------------------------------------
```

$$\hat{\varphi}_1 = 0.69 \quad \hat{\mu} = \frac{\hat{\delta}}{(1-\hat{\varphi}_1)} = \frac{19209}{1-0.6898} = 6.1925 \quad \hat{y}_t = \hat{\varphi}_1 y_{t-1} + \hat{\delta} = 0.69 y_{t-1} + 1.92$$

Solution 2.12-3: Estimation (Least Squares Estimation)

AR(1): $\hat{y}_t = \hat{\varphi}_1 y_{t-1} + \hat{\delta} = 0.69 y_{t-1} + 1.92$




```
+------------------------------+
|    time      inv~s     inv~S |
|------------------------------|
|  1955q1        4.4         . |
|  1955q2        5.8      4.96 |
|  1955q3        6.7      5.92 |
|  1955q4        7.1      6.54 |
|  1956q1        5.7      6.82 |
|------------------------------|
|  1956q2        4.1      5.85 |
|  1956q3        4.6      4.75 |
|  1956q4        4.3      5.09 |
+------------------------------+
```

$$\boxed{\begin{aligned}\hat{y}_2 &= 0.69 y_1 + 1.92 \\ &= 0.69 \cdot 4.4 + 1.92 \\ &= 4.96\end{aligned}}$$

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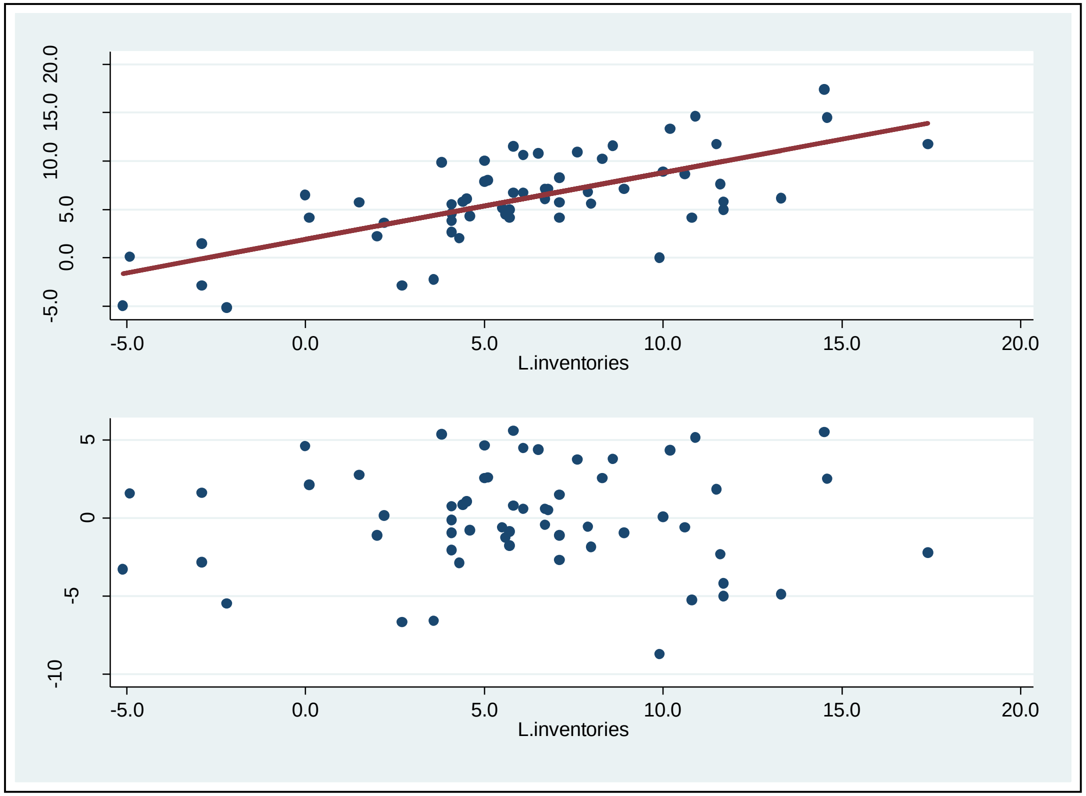
Solution 2.12-4: Estimation (Least Squares Estimation)

AR(1): $\hat{y}_t = \hat{\varphi}_1 y_{t-1} + \hat{\delta} = 0.69 y_{t-1} + 1.92 \qquad \hat{\varepsilon}_t = y_t - \hat{y}_t$



$$\hat{\sigma}_\varepsilon^2 = \frac{1}{(T^* - p - 1)} \sum_{t=1}^{T-p} \hat{\varepsilon}_t^2 = 11.4178$$

$$\boxed{\hat{\sigma}_\varepsilon = 3.379}$$

```
Number of obs =       59
F(  1,     57) =    51.96
Prob > F      =   0.0000
R-squared     =   0.4769
Adj R-squared =   0.4677
Root MSE      =    3.379
```

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Estimation – Maximum Likelihood Estimation

Calculate the probability density:

$$f_{Y_T, Y_{T-1}, \ldots, Y_1}(y_T, y_{T-1}, \ldots, y_1; \boldsymbol{\theta})$$

"probability of having observed the particular sample y_1, y_2, ..., y_T"

The maximum likelihood estimate (MLE) of $\boldsymbol{\theta}$ is the value of $\boldsymbol{\theta}$ that maximizes this probability.

Assumption: $\varepsilon_t \sim \text{i.i.d.}\, N\left(0, \sigma_\varepsilon^2\right)$

The joint density is a product of **conditional densities:**

$$f_{Y_T, Y_{T-1}, \ldots, Y_1}(y_T, y_{T-1}, \ldots, y_1 | \boldsymbol{\theta}) = f_{Y_1}(y_1; \boldsymbol{\theta}) \cdot \prod_{t=2}^{T} f_{Y_t | Y_{t-1}, Y_{t-2}, \ldots, Y_1}(y_t | y_{t-1}, y_{t-2}, \ldots, y_1; \boldsymbol{\theta})$$

For an **AR(1)** process each factor (t = 2, ..., T) reduces to:

$$f_{Y_t | Y_{t-1}, Y_{t-2}, \ldots, Y_1}(y_t | y_{t-1}, y_{t-2}, \ldots, y_1; \boldsymbol{\theta}) = f_{Y_t | Y_{t-1}}(y_t | y_{t-1}; \boldsymbol{\theta})$$

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Estimation – Maximum Likelihood Estimation

Likelihood function for an **AR(1)** process:

$$f_{Y_T,Y_{T-1},\ldots,Y_1}(y_T, y_{T-1},\ldots,y_1;\boldsymbol{\theta}) = f_{Y_1}(y_1;\boldsymbol{\theta}) \cdot \prod_{t=2}^{T} f_{Y_t|Y_{t-1}}(y_t \mid y_{t-1};\boldsymbol{\theta})$$

Log likelihood function for an **AR(1)** process:

$$L(\boldsymbol{\theta}) = \log\left[f_{Y_1}(y_1;\boldsymbol{\theta})\right] + \sum_{t=2}^{T} \log\left[f_{Y_t|Y_{t1}}(y_t \mid y_{t-1};\boldsymbol{\theta})\right]$$

$$L(\boldsymbol{\theta}) = log\left[\frac{1}{\sqrt{2\pi}\sqrt{\sigma_\varepsilon^2/(1-\varphi_1^2)}} exp\left[\frac{-\{y_1 - [\delta/1-\varphi_1]\}^2}{2\sigma_\varepsilon^2/(1-\varphi_1^2)}\right]\right] + \sum_{t=2}^{T} log\left[\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} exp\left[\frac{-\{y_t - \delta - \varphi_1 y_{t-1}\}^2}{2\sigma_\varepsilon^2}\right]\right]$$

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Estimation – Maximum Likelihood Estimation

Log likelihood function for an **AR(1)** process:

$$L(\boldsymbol{\theta}) = log\left[\frac{1}{\sqrt{2\pi}\sqrt{\sigma_\varepsilon^2/(1-\varphi_1^2)}}exp\left[\frac{-\{y_1-[\delta/1-\varphi_1]\}^2}{2\sigma_\varepsilon^2/(1-\varphi_1^2)}\right]\right] + \sum_{t=2}^{T} log\left[\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}exp\left[\frac{-\{y_t-\delta-\varphi_1 y_{t-1}\}^2}{2\sigma_\varepsilon^2}\right]\right]$$

$$L(\boldsymbol{\theta}) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left[\sigma_\varepsilon^2/(1-\varphi_1^2)\right] - \frac{\{y_1-[\delta/1-\varphi_1]\}^2}{2\sigma_\varepsilon^2/(1-\varphi_1^2)} - \left[\frac{(T-1)}{2}\right]\cdot\log(2\pi) - \left[\frac{(T-1)}{2}\right]\cdot\log(\sigma_\varepsilon^2) - \sum_{t=2}^{T}\left[\frac{\{y_t-\delta-\varphi_1 y_{t-1}\}^2}{2\sigma_\varepsilon^2}\right]$$

In principle, the maximization requires differentiating and setting the result equal to zero. In praxis, it requires iterative or numerical procedures.

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Estimation – Maximum Likelihood Estimation

`arima` *`varname`*`, ar(`*`numlist`*`) ma(`*`numlist`*`)` estimates an AR(*p*) MA(*q*) model using the maximum likelihood method.

Example:

`arima` *`varname`*`, ar(1) ma(1/3)` estimates an ARMA(1,3) model, because *`numlist 1/3`* denotes three numbers: 1, 2, 3

Notation in Stata:

$$y_t = \sum_{i=1}^{p} \varphi_i y_{t-i} \boxed{+} \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$$

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Solution 2.12-5: Estimation (Maximum Likelihood Estimation)

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

```
. arima inventories, ar(1)

(setting optimization to BHHH)
Iteration 0:   log likelihood = -157.05066
Iteration 1:   log likelihood = -157.04196
Iteration 2:   log likelihood = -157.04145
Iteration 3:   log likelihood = -157.04135
Iteration 4:   log likelihood = -157.04132
(switching optimization to BFGS)
Iteration 5:   log likelihood = -157.04132

ARIMA regression

Sample:  1955q1 to 1969q4                        Number of obs      =         60
                                                 Wald chi2(1)       =      60.23
Log likelihood = -157.0413                       Prob > chi2        =     0.0000

[...]
```

Solution 2.12-6: Estimation (Maximum Likelihood Estimation)

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

```
[...]
------------------------------------------------------------------------------
             |                 OPG
 inventories |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
inventories  |
       _cons |   6.040731   1.379228     4.38   0.000     3.337493    8.743969
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .6803345   .0876654     7.76   0.000     .5085135    .8521554
-------------+----------------------------------------------------------------
      /sigma |   3.297332   .3331842     9.90   0.000     2.644303    3.950361
------------------------------------------------------------------------------
```

$\hat{\varphi}_1 = 0.6803$

$\hat{\delta} = \hat{\mu}(1 - \hat{\varphi}_1) = 6.0407(1 - 0.6803) = 1.93$

$\hat{y}_t = \hat{\varphi}_1 y_{t-1} + \hat{\delta} = 0.68 y_{t-1} + 1.93$

$\hat{\sigma}_\varepsilon^2 = 3.2973^2 = 10.8722$

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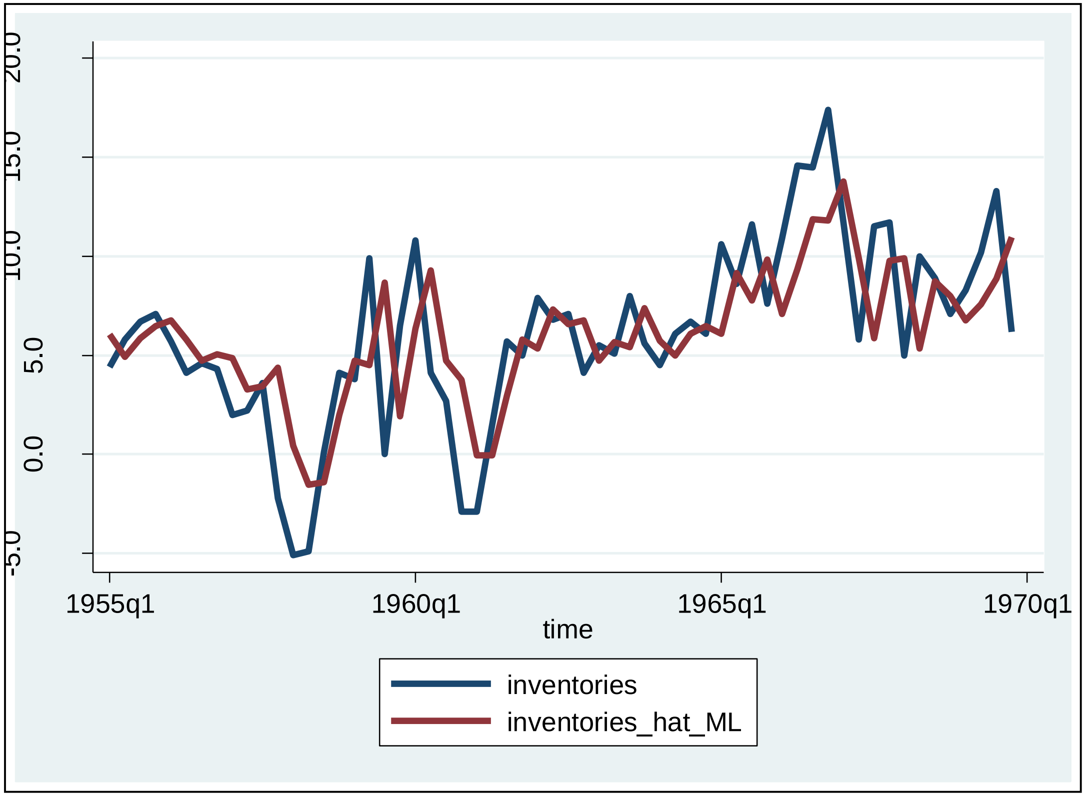
Solution 2.12-7: Estimation (Maximum Likelihood Estimation)

AR(1): $\hat{y}_t = \hat{\varphi}_1 y_{t-1} + \hat{\delta} = 0.68 y_{t-1} + 1.93$

$\hat{\mu} = 6.04$



time	inv~s	inv~L
1955q1	4.4	6.04
1955q2	5.8	4.92
1955q3	6.7	5.88
1955q4	7.1	6.49
1956q1	5.7	6.76
1956q2	4.1	5.81
1956q3	4.6	4.72
1956q4	4.3	5.06

$$\begin{aligned}\hat{y}_2 &= 0.68 y_1 + 1.93 \\ &= 0.68 \cdot 4.4 + 1.93 \\ &= 4.92\end{aligned}$$

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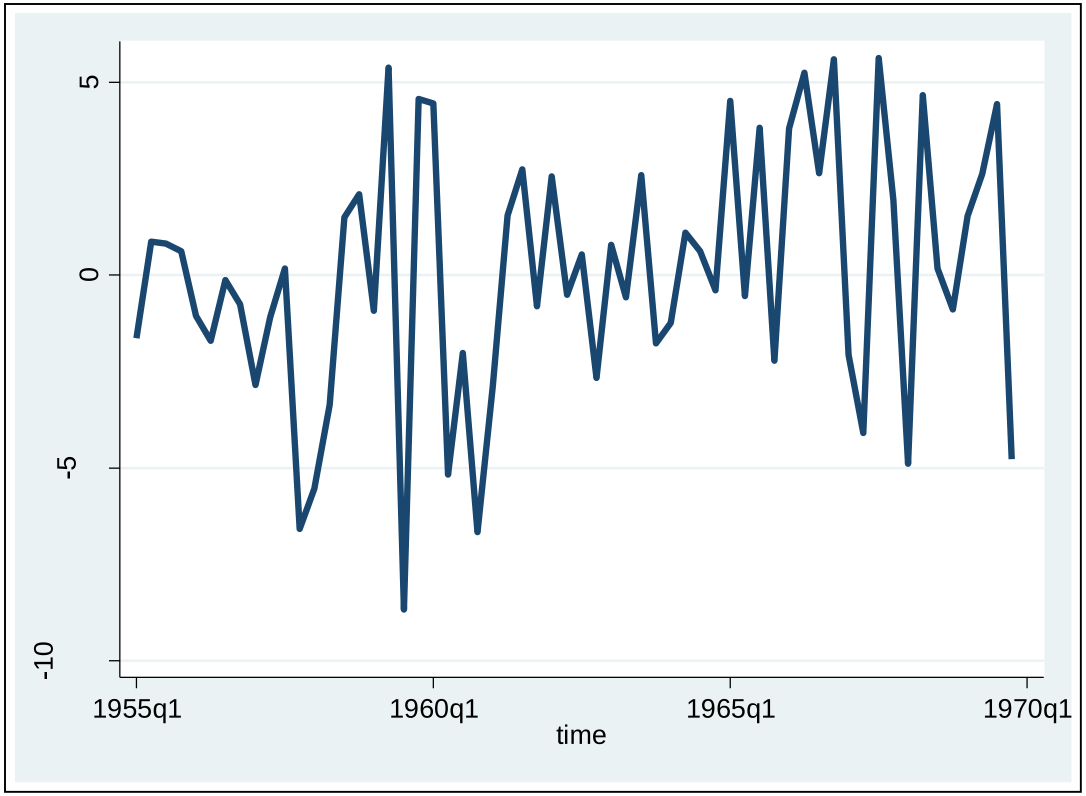
Solution 2.12-8: Estimation (Maximum Likelihood Estimation)

AR(1): $\hat{y}_t = \hat{\varphi}_1 y_{t-1} + \hat{\delta} = 0.68 y_{t-1} + 1.93 \qquad \hat{\varepsilon}_t = y_t - \hat{y}_t$



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Estimation – Conditional ML Estimation

Treat the value of y_1 as deterministic and maximize the likelihood conditioned of the first observation:

$$f_{Y_T,Y_{T-1},\dots,Y_2|Y_1}\left(y_T,y_{T-1},\dots,y_2|y_1;\boldsymbol{\theta}\right)$$

For an **AR(1)** process:

$$L\left(y_T,y_{T-1},\dots,y_2|y_1;\boldsymbol{\theta}\right)=-\left[\frac{(T-1)}{2}\right]\cdot\log(2\pi)-\left[\frac{(T-1)}{2}\right]\cdot\log\left(\sigma_\varepsilon^2\right)-\sum_{t=2}^{T}\left[\frac{\left\{y_t-\delta-\varphi_1 y_{t-1}\right\}^2}{2\sigma_\varepsilon^2}\right]$$

Maximization with respect to δ and φ_1 is equivalent to minimize:

$$\sum_{t=2}^{T}\left\{y_t-\delta-\varphi_1 y_{t-1}\right\}^2$$

This can be achieved by an OLS regression of y_t on a constant and it's own lagged values.

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Estimation – Conditional ML Estimation

For an **AR(1)** process:

$$L\left(y_T, y_{T-1}, \ldots, y_2 | y_1; \boldsymbol{\theta}\right) = -\left[\frac{(T-1)}{2}\right] \cdot \log(2\pi) - \left[\frac{(T-1)}{2}\right] \cdot \log\left(\sigma_\varepsilon^2\right) - \sum_{t=2}^{T}\left[\frac{\left\{y_t - \delta - \varphi_1 y_{t-1}\right\}^2}{2\sigma_\varepsilon^2}\right]$$

The conditional MLE of the variance is found by differentiating the log likelihood with respect to σ_ε^2:

$$\hat{\sigma}_\varepsilon^2 = \sum_{t=2}^{T}\left[\frac{\left\{y_t - \hat{\delta} - \hat{\varphi}_1 y_{t-1}\right\}^2}{T-1}\right]$$

The conditional MLE is the average squared residuals from the OLS regression.

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Exercise 2.13: Diagnostic Checking

Do diagnostic checking for the ML estimation of the

AR(1): $y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$

- Is the series stationary?
- Are the estimated coefficients significant?
- Is the AR(1) model appropriate?

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Solution 2.13-1: Diagnostic Checking

```
. arima inventories, ar(1)
[...]
------------------------------------------------------------------------------
             |                 OPG
 inventories |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
inventories  |
       _cons |   6.040731   1.379228     4.38   0.000     3.337493    8.743969
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .6803345   .0876654     7.76   0.000     .5085135    .8521554
-------------+----------------------------------------------------------------
      /sigma |   3.297332   .3331842     9.90   0.000     2.644303    3.950361
------------------------------------------------------------------------------
```

Is the series stationary? **YES** $|\hat{\varphi}_1| < 1$

Are the estimated coefficients significant? **YES** p-value ≤ 0.05

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Solution 2.13-2: Diagnostic Checking

```
. arima inventories, ar(1/2)
[...]
------------------------------------------------------------------------------
             |                 OPG
 inventories |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
inventories  |
       _cons |    6.07296   1.531381     3.97   0.000     3.071507    9.074412
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .6080156   .1097436     5.54   0.000     .3929222     .823109
         L2. |   .1031826   .1166612     0.88   0.376     -.125469    .3318343
-------------+----------------------------------------------------------------
      /sigma |   3.280592   .3626976     9.04   0.000     2.569718    3.991466
------------------------------------------------------------------------------
```

If you want to save the coefficients you can do so after you have run the ML-Estimation:

```
. generate AR_phi1=_b[ARMA:L1.ar]
. display  AR_phi1
.68033445
. generate AR_delta=_b[_cons]
. display  AR_delta
6.040731
```

Diagnostic Checking – Residual ACF and PACF

How should the residuals "behave"?

- They are estimates of the "true" residuals.
- The "true" residuals ε_t of an ARMA model are white noise, i.e. their ACF and PACF are zero for all lags. Hence, if we estimated the correct model for y_t, the ACF and PACF of the estimated residuals should be zero.

Test $\rho_k = 0$ for a particular k

Bartlett's result: For a white noise process, the sample (partial) autocorrelation coefficients (for $k > 0$) are distributed approximately according to a normal distribution with zero mean and variance:

$$Var(\hat{\rho}_k) = \begin{cases} \dfrac{1}{T} & k = 1 \\ \dfrac{1}{T}\left\{1 + 2\sum_{i=1}^{k-1}\hat{\rho}_i^2\right\} & k > 1 \end{cases} \qquad Var(\hat{\rho}_k) \approx \frac{1}{T}$$

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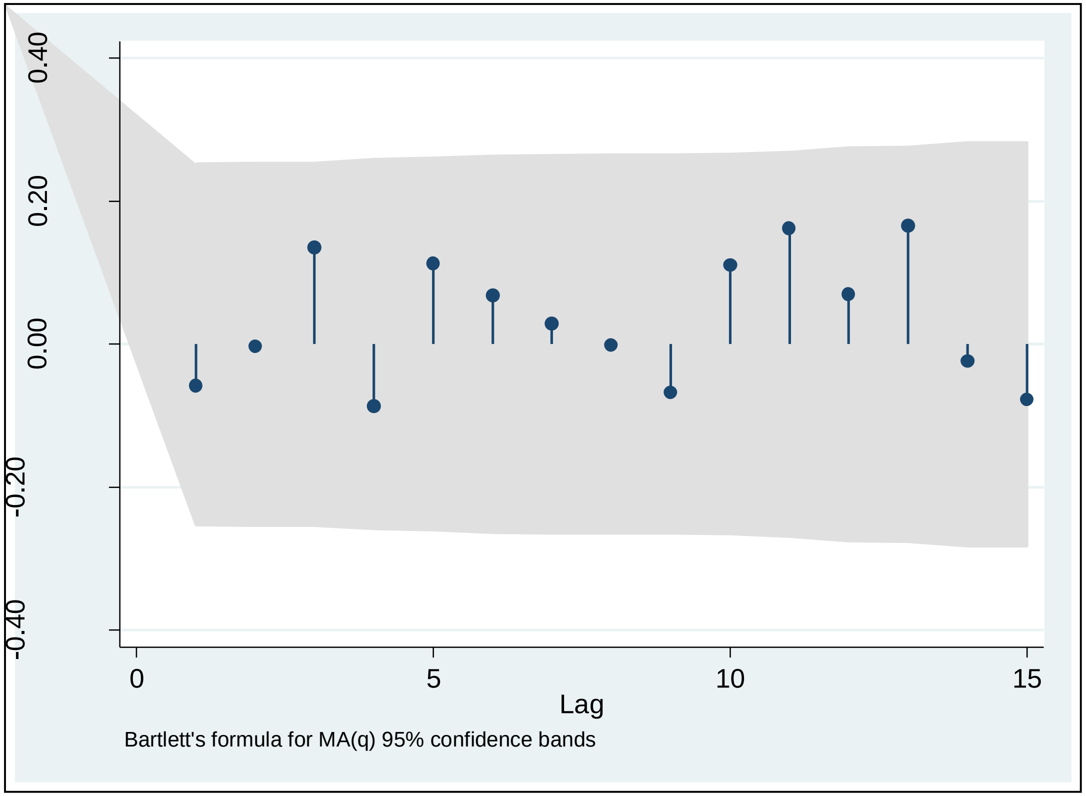
Solution 2.13-3: Diagnostic Checking

ACF of the residuals



- None of the autocorrelations of the residuals is significantly different from zero.

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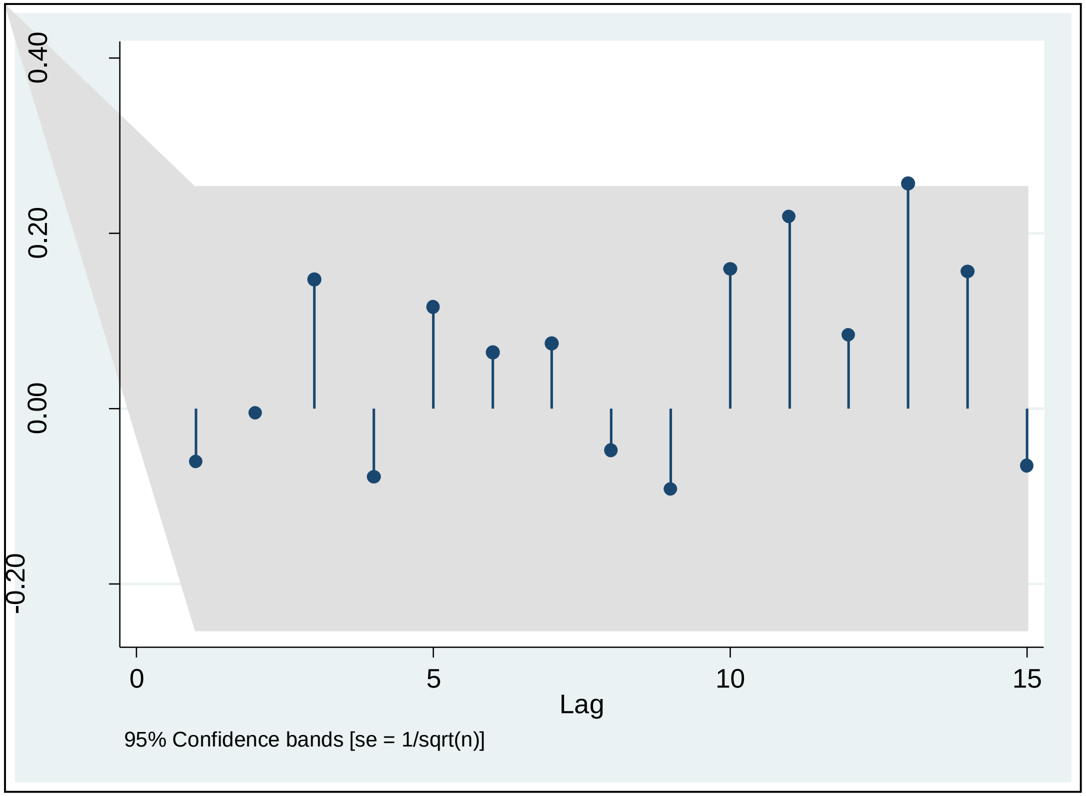
Solution 2.13-4: Diagnostic Checking

PACF of the residuals



- None of the partial autocorrelations of the residuals at lower lags is significantly different from zero.

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Diagnostic Checking – Joint Hypothesis Test

H_0: All autocorrelation coefficients are zero

Box and Pierce

$$Q = T\sum_{k=1}^{K}\hat{\rho}_k^2 \quad \sim\chi^2 \quad \text{with } K-p-q \text{ degrees of freedom}$$

Box and Ljung (refined test)

$$Q = T(T+2)\sum_{k=1}^{K}\frac{1}{T-k}\hat{\rho}_k^2 \quad \sim\chi^2 \quad \text{with } K-p-q \text{ degrees of freedom}$$

Solution 2.13-5: Diagnostic Checking

```
. corrgram res_AR1
                                               -1       0       1 -1       0       1
 LAG       AC       PAC      Q     Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1       -0.0580  -0.0604   .21226  0.6450          |                  |
2       -0.0031  -0.0049   .21288  0.8990          |                  |
3        0.1351   0.1474   1.4044  0.7045          |-                 |-
[...]
14      -0.0238   0.1566   8.9945  0.8314          |                  |-
15      -0.0771  -0.0653   9.4865  0.8507          |                  |
```

```
. di 60*62*((1/59)*(-0.0580)^2+(1/58)*(-0.0031)^2)
.21271942

. di 1-chi2(2,.21288)
.89902899
```

$$Q = T(T+2)\sum_{k=1}^{K}\frac{1}{T-k}\hat{\rho}_k^2 \quad \sim \chi^2 \quad \text{with } K-p-q \text{ degrees of freedom}$$

```
. di 1-chi2(1,.21288)
.64451939
```

```
. wntestq  res_AR1, lags(15)

Portmanteau test for white noise
---------------------------------------
 Portmanteau (Q) statistic =     9.4865
 Prob > chi2(15)           =     0.8507

. di 1-chi2(15,9.4865)
.85073869

. di 1-chi2(14,9.4865)
.79868101
```


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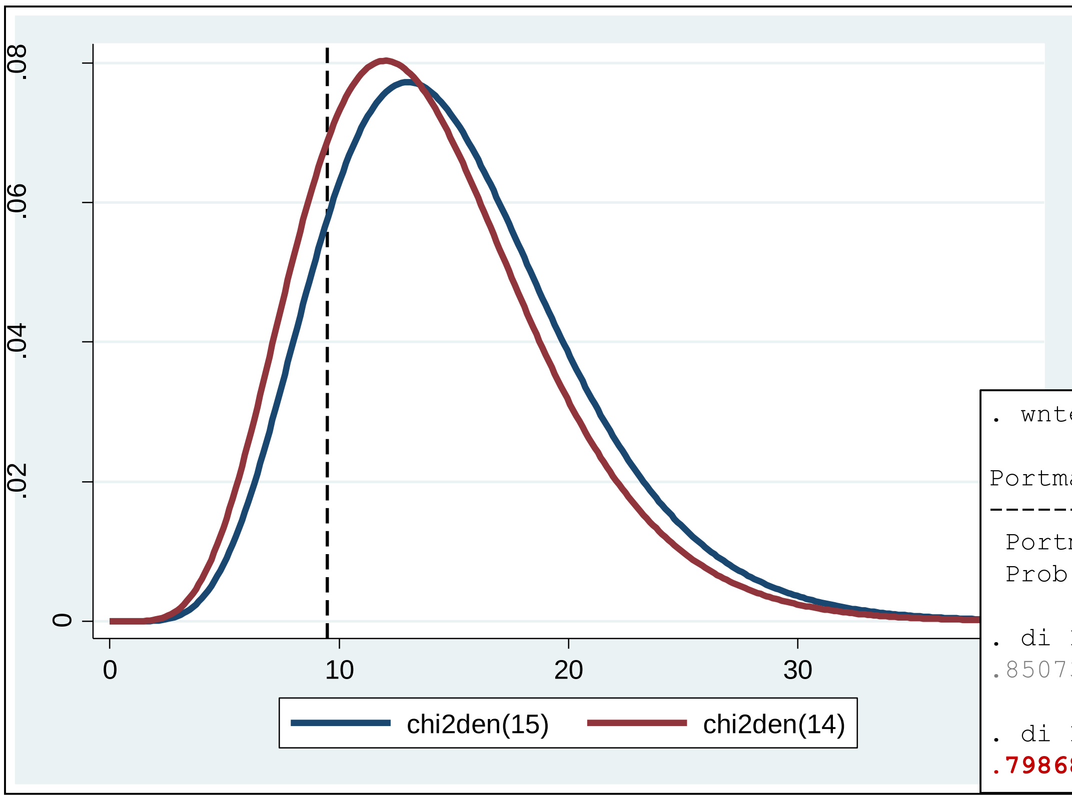
Solution 2.13-6: Diagnostic Checking




```
. wntestq  res_AR1, lags(15)

Portmanteau test for white noise
---------------------------------------
 Portmanteau (Q) statistic =     9.4865
 Prob > chi2(15)           =     0.8507

. di 1-chi2(15,9.4865)
.85073869

. di 1-chi2(14,9.4865)
.79868101
```

Exercise 2.14: Forecasting

- Estimate the AR(1) model by OLS with only the first 56 observations.
- What is the optimal forecast? In which sense is it optimal? What “assumptions” are part of the information set?
- Calculate forecasts for 1969q1 to 1969q4.
- Compare these forecasts to the actual values and compute percentage forecast errors.
- What is the forecast for December 2010?

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Solution 2.14-1: Forecasting

```
. regress inventories L1.inventories if time <= 35
      Source |       SS       df       MS              Number of obs =      55
-------------+------------------------------           F(  1,     53) =   50.34
       Model |  568.936612     1  568.936612           Prob > F      =  0.0000
    Residual |  598.977207    53  11.3014567           R-squared     =  0.4871
-------------+------------------------------           Adj R-squared =  0.4775
       Total |  1167.91382    54  21.6280337           Root MSE      =  3.3618
------------------------------------------------------------------------------
 inventories |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
 inventories |
         L1. |   .6977864   .0983462     7.10   0.000     .5005288     .895044
       _cons |   1.810721   .7308342     2.48   0.016     .3448536    3.276589
------------------------------------------------------------------------------
```

$\hat{\varphi}_1 = 0.6977864 \quad \hat{\delta} = 1.810721$

Recall:

```
. regress inventories L1.inventories
[...]
inventories |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
 inventories |
         L1. |   .6897525   .0956884     7.21   0.000     .4981398    .8813652
       _cons |   1.920915   .7303908     2.63   0.011     .4583325    3.383497
------------------------------------------------------------------------------
```

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Solution 2.14-2: Forecasting

The optimal predictor (minimal *MSE*) is the **conditional mean**

$$\tilde{y}_{T+1} \mid \Omega_T = E(Y_{T+1} \mid \Omega_T)$$

it minimizes the expected squared forecast error

$$\min_{\tilde{y}_{T+l}} MSE(\tilde{y}_{T+l}) = E\left[(y_{T+l} - \tilde{y}_{T+l})^2 \mid \Omega_T\right]$$

Information set Ω_T at period T:

- true model
- known parameters
- all past observations $y_T, \ldots, y_2, y_1, y_0, y_{-1}, \ldots$

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Forecasting - AR(1)

Forecasting an **AR(1) Process** 1 and 2 periods ahead

$$\tilde{y}_{T+l} = E\left(y_{T+l} | \Omega_T\right)$$

$$y_{T+1} = \varphi_1 y_T + \delta + \varepsilon_{T+1}$$

$$\begin{aligned}\tilde{y}_{T+1} &= E\left(y_{T+1} | \Omega_T\right) = E\left(\varphi_1 y_T + \delta + \varepsilon_{T+1} | \Omega_T\right)\\ &= \varphi_1 E\left(y_T | \Omega_T\right) + E\left(\delta | \Omega_T\right) + E\left(\varepsilon_{T+1} | \Omega_T\right)\\ &= \varphi_1 y_T + \delta + \underbrace{E\left(\varepsilon_{T+1}\right)}_{=0} = \varphi_1 y_T + \delta\end{aligned}$$

$$\begin{aligned}\tilde{y}_{T+2} &= E\left(y_{T+2} | \Omega_T\right) = E\left(\varphi_1 y_{T+1} + \delta + \varepsilon_{T+2} | \Omega_T\right)\\ &= \varphi_1 E\left(y_{T+1} | \Omega_T\right) + \delta + \underbrace{E\left(\varepsilon_{T+2}\right)}_{=0} = \varphi_1 \tilde{y}_{T+1} + \delta\end{aligned}$$

$$\tilde{y}_{T+1} = \varphi_1 y_T + \delta + \underbrace{E\left(\varepsilon_{T+1}\right)}_{=0} \quad \text{and} \quad \tilde{y}_{T+2} = \varphi_1 \tilde{y}_{T+1} + \delta + \underbrace{E\left(\varepsilon_{T+2}\right)}_{=0}$$

Solution 2.14-3: Forecasting

One-period ahead forecast of an AR(1): $\hat{\varphi}_1 = 0.6977864 \quad \hat{\delta} = 1.810721$

$$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t \quad \Rightarrow y_{T+1} = \varphi_1 y_T + \delta + \varepsilon_{T+1}$$

$$\tilde{y}_{T+1} = E\left(y_{T+1} | \Omega_T\right)$$

$$\tilde{y}_{T+1} = E\left(\varphi_1 y_T + \delta + \varepsilon_{T+1} \,|\, \Omega_T\right)$$

$$\tilde{y}_{T+1} = \varphi_1 y_T + \delta$$

$$\Omega_T = \left\{y_T, \ldots, y_1; y_t = \varphi_1 y_{t-1} + \delta + \varepsilon\right\}$$

$$\hat{y}_t = 0.6977864 y_{t-1} + 1.810721$$

$$\tilde{y}_{1969q1} = 1.810721 + 0.6977864 \cdot y_{1968q4}$$

$$\tilde{y}_{1969q1} = 1.810721 + 0.6977864 \cdot 7.1$$

$$\tilde{y}_{1969q1} = 6.765004$$

```
. list time inventories in 52/60

     +--------------------+
     |   time   invent~s |
     |--------------------|
 52. | 1967q4       11.7 |
 53. | 1968q1        5.0 |
 54. | 1968q2       10.0 |
 55. | 1968q3        8.9 |
 56. | 1968q4        7.1 |
     |--------------------|
 57. | 1969q1        8.3 |
 58. | 1969q2       10.2 |
 59. | 1969q3       13.3 |
 60. | 1969q4        6.2 |
     +--------------------+
```


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Solution 2.14-4: Forecasting

Two-period ahead forecast of an AR(1): $\hat{\varphi}_1 = 0.6977864 \quad \hat{\delta} = 1.810721$

$$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t \quad \Rightarrow y_{T+2} = \varphi_1 y_{T+1} + \delta + \varepsilon_{T+2}$$

$$\tilde{y}_{T+2} = E\left(y_{T+2} | \Omega_T\right)$$

$$\tilde{y}_{T+2} = E\left(\varphi_1 y_{T+1} + \delta + \varepsilon_{T+2} \,|\, \Omega_T\right)$$

$$\tilde{y}_{T+2} = \varphi_1 \tilde{y}_{T+1} + \delta$$

$$\hat{y}_t = 0.6977864 y_{t-1} + 1.810721$$

$$\tilde{y}_{1969q2} = 1.810721 + 0.6977864 \cdot \tilde{y}_{1969q1}$$

$$\tilde{y}_{1969q2} = 1.810721 + 0.6977864 \cdot 6.765004$$

$$\tilde{y}_{1969q2} = 6.531249$$

```
. list time inventories in 52/60

     +--------------------+
     |   time   invent~s |
     |--------------------|
 52. | 1967q4       11.7 |
 53. | 1968q1        5.0 |
 54. | 1968q2       10.0 |
 55. | 1968q3        8.9 |
 56. | 1968q4        7.1 |
     |--------------------|
 57. | 1969q1        8.3 |
 58. | 1969q2       10.2 |
 59. | 1969q3       13.3 |
 60. | 1969q4        6.2 |
     +--------------------+
```

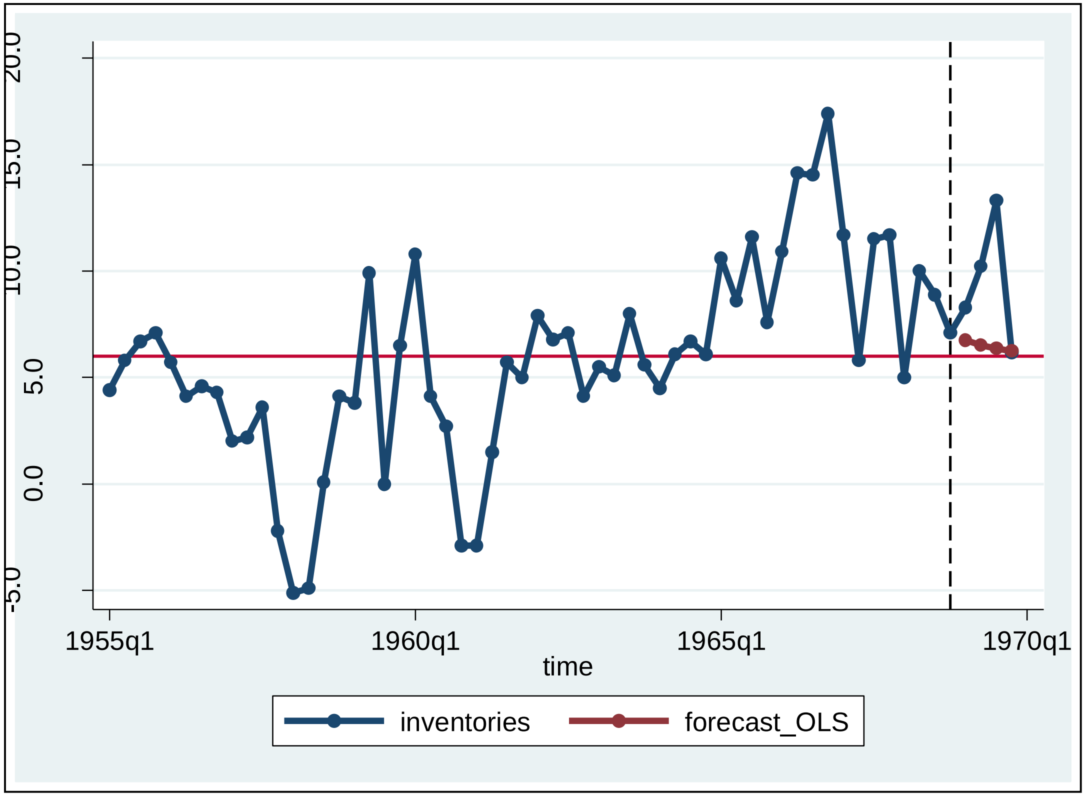
Solution 2.14-5: Forecasting

```
. list time inventories forecast_OLS in 57/60
     +------------------------------+
     |   time   invent~s   foreca~S |
     |------------------------------|
 57. | 1969q1        8.3   6.765004 |
 58. | 1969q2       10.2   6.531249 |
 59. | 1969q3       13.3   6.368138 |
 60. | 1969q4        6.2   6.254321 |
     +------------------------------+
. di (8.3-6.7650)/8.3
.18493976
```

Time	Inventories	Forecast	Percent Forecast Error
1969q1	8.3	6.7650	18.49%
1969q2	10.2	6.5312	35.97%
1969q3	13.3	6.3681	52.12%
1969q4	6.2	6.2543	-0.88%

Solution 2.14-6: Forecasting

The forecast **decays geometrically** toward *μ* as *l* increases.



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Organizational Issues

Please log in to **ISIS** (password: Zeit1718) and **download** the following file:

- coal_production.dta

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Univariate Box-Jenkins models for stationary time series

General Procedure:

1. Identification
2. Estimation
   - Solution of the Yule-Walker equations
   - Least Squares Estimation
   - Maximum Likelihood Estimation
3. Diagnostic Checking
   - Significance of estimated coefficients
   - Test of individual values of residual ACF and PACF
   - Joint test that residuals are white noise (Box and Ljung)
   - Comparison of different candidate models (AIC and BIC)
4. Forecasting

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Some helpful formulas

Yule-Walker equations:

$$\begin{aligned}
\rho_1 &= \varphi_1\rho_0 + \varphi_2\rho_1 + \ldots + \varphi_p\rho_{p-1} \\
&\vdots \\
\rho_p &= \varphi_1\rho_{p-1} + \varphi_2\rho_{p-2} + \ldots + \varphi_p
\end{aligned}$$

Box and Ljung (refined test):

$$Q = T(T+2)\sum_{k=1}^{K}\frac{1}{T-k}\hat{\rho}_k^2 \quad \sim\chi^2 \quad \text{with } K-p-q \text{ degrees of freedom}$$

Akaike's Information Criterion (AIC):

$$AIC = \log\hat{\sigma}^2 + 2\frac{p+q}{T}$$

Bayesian Information Criterion (BIC):

$$BIC = \log\hat{\sigma}^2 + \frac{p+q}{T}\log T$$

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Some helpful Stata commands

- `generate` ***newvar* = *exp***
  creates a new variable, the values of the variable are specified by =*exp*. It allows you to create a new variable that is an algebraic expression of other variables.
- `_n`
  contains the number of the current observation; it can be used with mathematical operators.
- `format` ***timevar %fmt***
  allows you to specify the display format for variables.

Format (`fmt`)	Description	Coding
%td	daily	0 = 01jan1960, 1 = 02jan1960
%tw	weekly	0 = 1960w1, 1 = 1960w2
%tm	monthly	0 = 1960m1, 1 = 1960m2
%tq	quarterly	0 = 1960q1, 1 = 1960q2
%th	halfyearly	0 = 1960h1, 1 = 1960h2
%ty	yearly	1960 = 1960, 1961 = 1961

- `tsset` ***timevar***
  declares the data to be a time series and designates that *timevar* represents time.
- `tsline` ***varname***
  draws line plots for time-series data.

Some helpful Stata commands

- **`ac *varname*, lags(#)`**
  produces a graph of # autocorrelations with pointwise confidence intervals based on Bartlett's formula.
- **`pac *varname*, lags(#)`**
  produces a graph of # partial autocorrelations with confidence intervals calculated using a standard error of 1/sqrt(n).
- **`corrgram *varname*, lags(#)`**
  produces a table of # autocorrelations, partial autocorrelations, and Portmanteau (Q) statistics.
- **`summarize *varlist*`**
  calculates and displays a variety of univariate summary statistics.
- **`display`**
  can be used as a substitute for a hand calculator.
- **`regress *depvar* [*indepvars*]`**
  fits a model of *depvar* on *indepvars* using linear regression.
- **`L.*varname*`**
  refers to the lagged value of variable *varname*.

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Some helpful Stata commands

- **`arima`** ***`varname,`*** **`ar(`*****`numlist`*****`)`** **`ma(`*****`numlist`*****`)`**
  estimates an AR(p) MA(q) model using the maximum likelihood method.
- ***`numlist`***
  is a list of numbers; example: *1/3* three numbers: 1, 2, 3.
- **`predict`** ***`newvar,`*** **`residuals`**
  predicts the residuals from the last estimation.
- **`chi2(`*****`n`*****`,`*****`x`*****`)`**
  returns the cumulative chi-squared distribution with n degrees of freedom for n > 0.
- **`log(`*****`x`*****`)`**
  returns the natural logarithm of x if x>0. This is a synonym for ln(x).

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Exercise 2.15: Coal Production

Follow the **Univariate Box-Jenkins models for stationary time series** to estimate an **appropriate** model (if necessary consider different candidate models).

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Describe the data

```
. describe

 obs:                96
 vars:                1                          1 Dec 2011 13:59
 size:              384
--------------------------------------------------------------------------------
              storage   display    value
variable name   type    format     label      variable label
--------------------------------------------------------------------------------
coal_production long    %12.0g                Monthly bituminous coal
                                              production in the US
                                              (Jan. 1952 to Dec. 1959)
--------------------------------------------------------------------------------
Sorted by:
```

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Generate time variable

```
. generate time = -97 + _n
. format time %tm
```

Alternative:

```
. generate time=tm(1952m1)+_n-1
. format time %tm
. tsset time
```

Declare data to be time-series data

```
. tsset time
```

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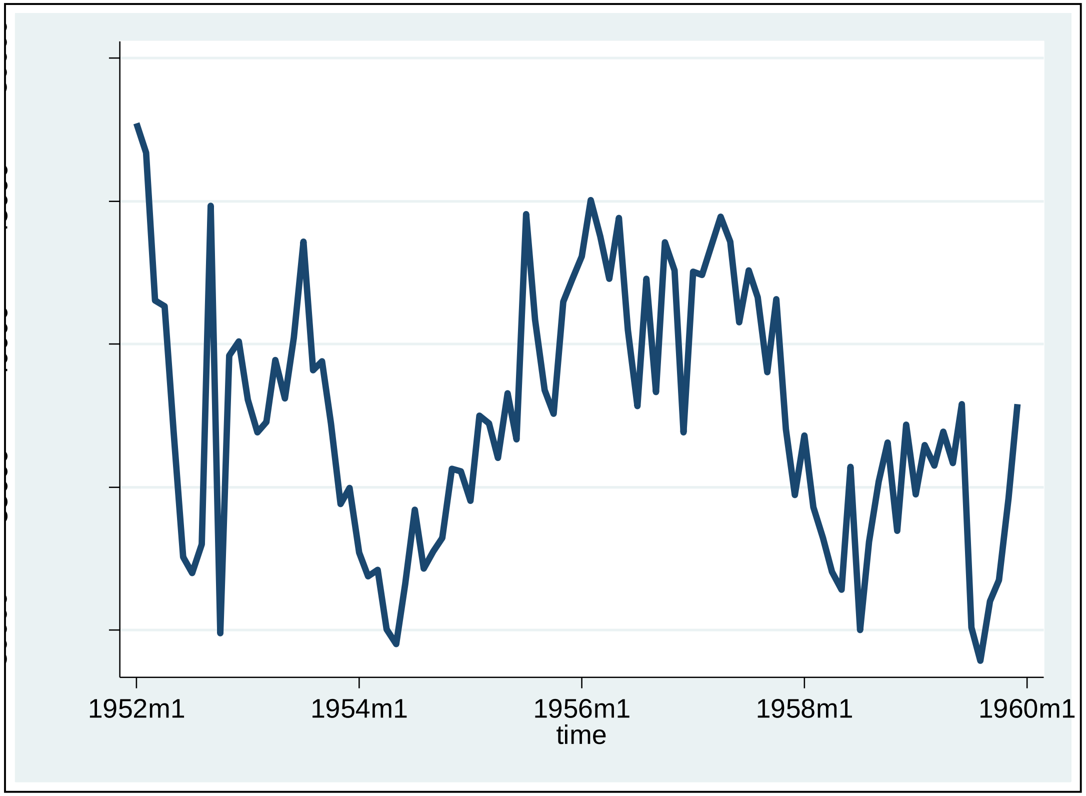
Coal Production – Original Series

```
. tsline coal_production, lwidth(thick)
```

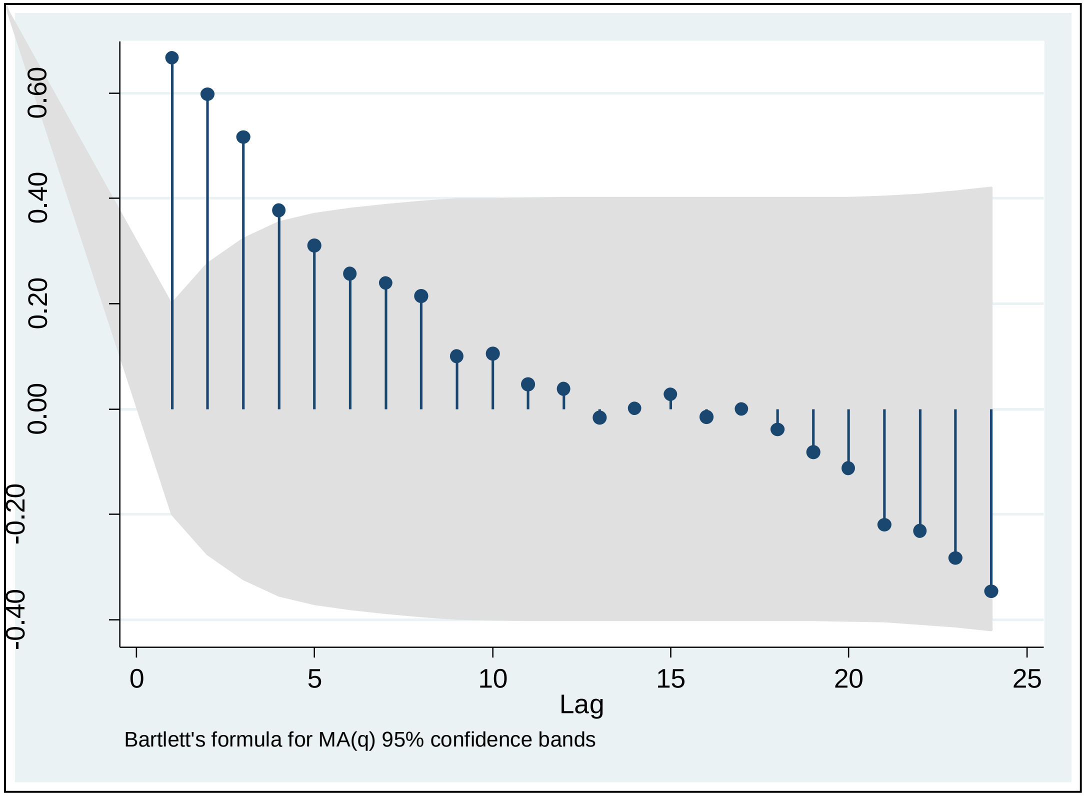


- "Monthly bituminous coal production in the US"
- 96 observations from January 1952 through December 1959
- the data have been seasonally adjusted
- Is the series stationary?

Pankratz (1983) "Forecasting with univariate Box-Jenkins models"

Identification

ACF: `. ac coal_production, lags(24)`



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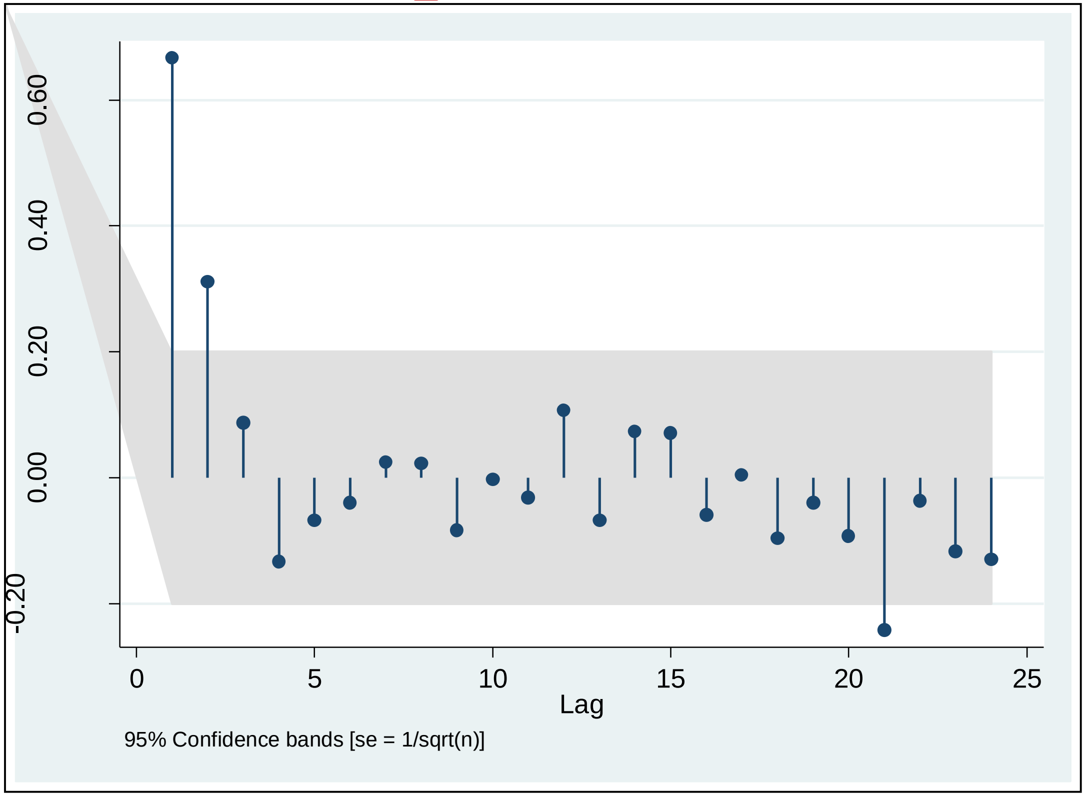
Identification

PACF: `. pac coal_production, lags(24)`



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Estimation: Solution of the **Yule-Walker** equations

In general:

$$\rho_1 = \varphi_1\rho_0 + \varphi_2\rho_1 + \ldots + \varphi_p\rho_{p-1}$$
$$\vdots$$
$$\rho_p = \varphi_1\rho_{p-1} + \varphi_2\rho_{p-2} + \ldots + \varphi_p$$

In case of an AR(2) process: $y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \delta + \varepsilon_t$

$$\rho_1 = \varphi_1\rho_0 + \varphi_2\rho_1$$
$$\rho_2 = \varphi_1\rho_1 + \varphi_2\rho_0$$

Solving this system of equations for $\boldsymbol{\varphi_1}$ and $\boldsymbol{\varphi_2}$ yields:

$$\hat{\varphi}_2 = \frac{\hat{\rho}_2 - \hat{\rho}_1^2}{\left(1 - \hat{\rho}_1^2\right)} \qquad \textit{and} \qquad \hat{\varphi}_1 = \hat{\rho}_1\left(1 - \frac{\hat{\rho}_2 - \hat{\rho}_1^2}{\left(1 - \hat{\rho}_1^2\right)}\right)$$

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Estimation – Solution of the Yule-Walker equations

AR(2) process

$$\hat{\varphi}_2 = \frac{\hat{\rho}_2 - \hat{\rho}_1^2}{\left(1-\hat{\rho}_1^2\right)} \quad \hat{\varphi}_1 = \hat{\rho}_1\left(1-\frac{\hat{\rho}_2 - \hat{\rho}_1^2}{\left(1-\hat{\rho}_1^2\right)}\right)$$

```
. corrgram coal_production, lags(24)
                                          -1       0       1 -1       0       1
 LAG       AC       PAC      Q     Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1        0.6663   0.6663   43.961  0.0000          |-----             |-----
2        0.5977   0.3113   79.711  0.0000          |----              |--
```

$$\hat{\varphi}_1 = 0.4821 \quad \hat{\varphi}_2 = 0.2765$$

```
. sum coal_production
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
coal_produ~n |        96    37469.72    4468.253      28931      47730
```

$$\hat{\delta} = \hat{\mu}\left(1-\hat{\varphi}_1 - \hat{\varphi}_2\right) = 9045.19$$

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Estimation – Least Squares Estimation

AR(2) process

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \delta + \varepsilon_t$$

```
. regress coal_production L1.coal_production L2.coal_production
      Source |       SS       df       MS              Number of obs =      94
-------------+------------------------------           F(  2,    91) =   46.12
       Model |   856796006     2   428398003           Prob > F      =  0.0000
    Residual |   845317521    91  9289203.52           R-squared     =  0.5034
-------------+------------------------------           Adj R-squared =  0.4925
       Total |  1.7021e+09    93    18302296           Root MSE      =  3047.8
------------------------------------------------------------------------------
coal_produ~n |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
coal_produ~n |
         L1. |   .4317794   .0992874     4.35   0.000     .2345572    .6290015
         L2. |   .3112845   .0966694     3.22   0.002     .1192626    .5033064
       _cons |   9461.016     2913.8     3.25   0.002      3673.11    15248.92
------------------------------------------------------------------------------
```

$\hat{\varphi}_1 = 0.4318 \quad \hat{\varphi}_2 = 0.3113 \quad \hat{\delta} = 9461.016 \quad \hat{\mu} = \dfrac{\hat{\delta}}{(1 - \hat{\varphi}_1 - \hat{\varphi}_2)} = 36822.447$

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Estimation – Maximum Likelihood Estimation

AR(2) process

```
. arima coal_production, ar(1/2)
[...]
Sample:  1952m1 to 1959m12                        Number of obs      =         96
------------------------------------------------------------------------------
             |                 OPG
coal_produ~n |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
coal_produ~n |
       _cons |   37981.21   1409.507    26.95   0.000     35218.63    40743.79
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .4839235   .0815506     5.93   0.000     .3240873    .6437597
         L2. |   .3223401   .0719627     4.48   0.000     .1812958    .4633844
-------------+----------------------------------------------------------------
      /sigma |    3066.34   186.4115    16.45   0.000      2700.98      3431.7
------------------------------------------------------------------------------
```

$$\hat{\varphi}_1 = 0.4839 \quad \hat{\varphi}_2 = 0.3223 \quad \hat{\mu} = 37981.21 \quad \hat{\sigma}_\varepsilon = 3066.34$$

$$\hat{\delta} = \hat{\mu}\left(1 - \hat{\varphi}_1 - \hat{\varphi}_2\right) = 7358.3429$$

Estimation – Maximum Likelihood Estimation

ARMA(1,1) process

```
. arima coal_production, ar(1) ma(1)
[...]
------------------------------------------------------------------------------
             |                 OPG
coal_produ~n |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
coal_produ~n |
       _cons |   37982.71   1463.531    25.95   0.000     35114.24    40851.17
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .8860966   .0591459    14.98   0.000     .7701727     1.00202
          ma |
         L1. |  -.3690676   .0956238    -3.86   0.000    -.5564868   -.1816484
-------------+----------------------------------------------------------------
      /sigma |   3084.761   196.4155    15.71   0.000     2699.794    3469.729
------------------------------------------------------------------------------
```

$\hat{\varphi}_1 = 0.8861 \quad \hat{\theta}_1 = 0.3691 \quad \hat{\mu} = 37982.71 \quad \hat{\sigma}_\varepsilon = 3084.761$

$\hat{\delta} = \hat{\mu}(1 - \hat{\varphi}_1) = 4326.3598$

Estimation – Summary

AR(2) process:

$$\hat{y}_t = \hat{\varphi}_1 y_{t-1} + \hat{\varphi}_2 y_{t-2} + \hat{\delta}$$

- Solution of the Yule-Walker equations
  $\hat{y}_t = 0.4821 y_{t-1} + 0.2765 y_{t-2} + 9045.19$
- Least Squares Estimation
  $\hat{y}_t = 0.4318 y_{t-1} + 0.3113 y_{t-2} + 9461.016$
- Maximum Likelihood Estimation
  $\hat{y}_t = 0.4839 y_{t-1} + 0.3223 y_{t-2} + 7358.3429$

ARMA(1,1) process:

$$\hat{y}_t = \hat{\varphi}_1 y_{t-1} - \hat{\theta}_1 \varepsilon_{t-1} + \hat{\delta}$$

- Maximum Likelihood Estimation
  $\hat{y}_t = 0.8861 y_{t-1} - 0.3691 \varepsilon_{t-1} + 4326.3598$

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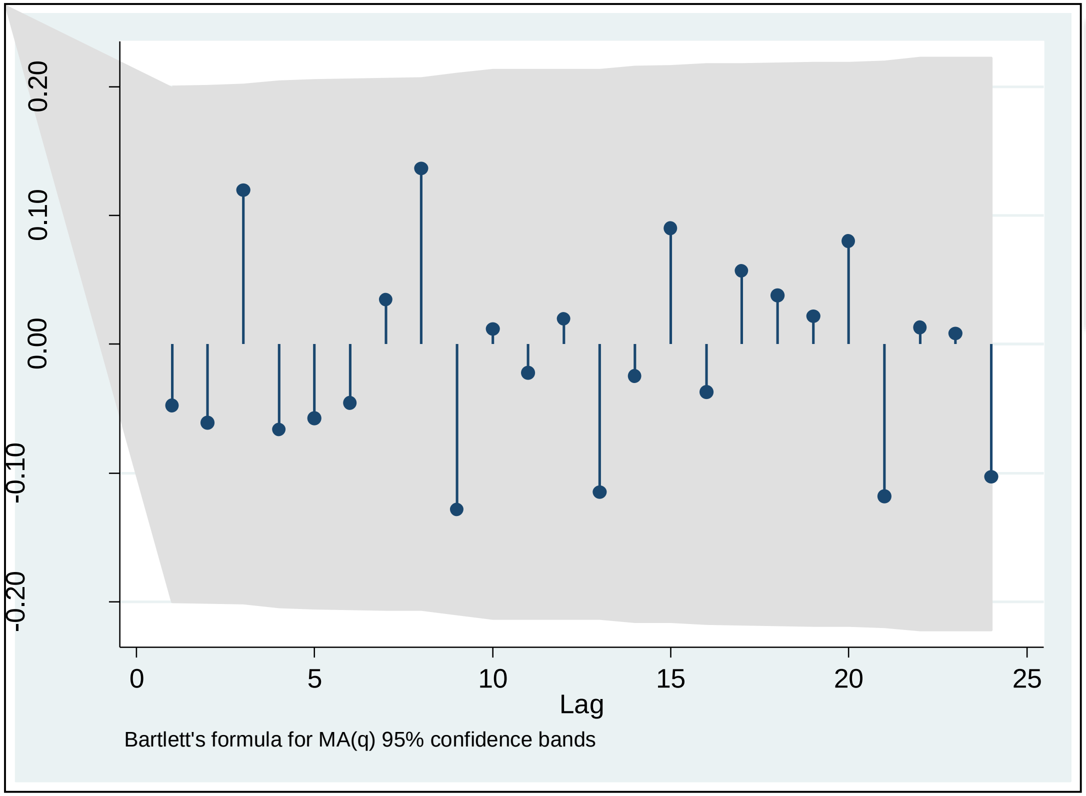
Diagnostic Checking

ACF of the AR(2) residuals



ACF of the AR(2) residuals, whereas the AR(2) was estimated by the ML Method.

```
. arima coal-production, ar(1/2)
. predict res_AR2, residuals
. ac res_AR2, lags(24)
```

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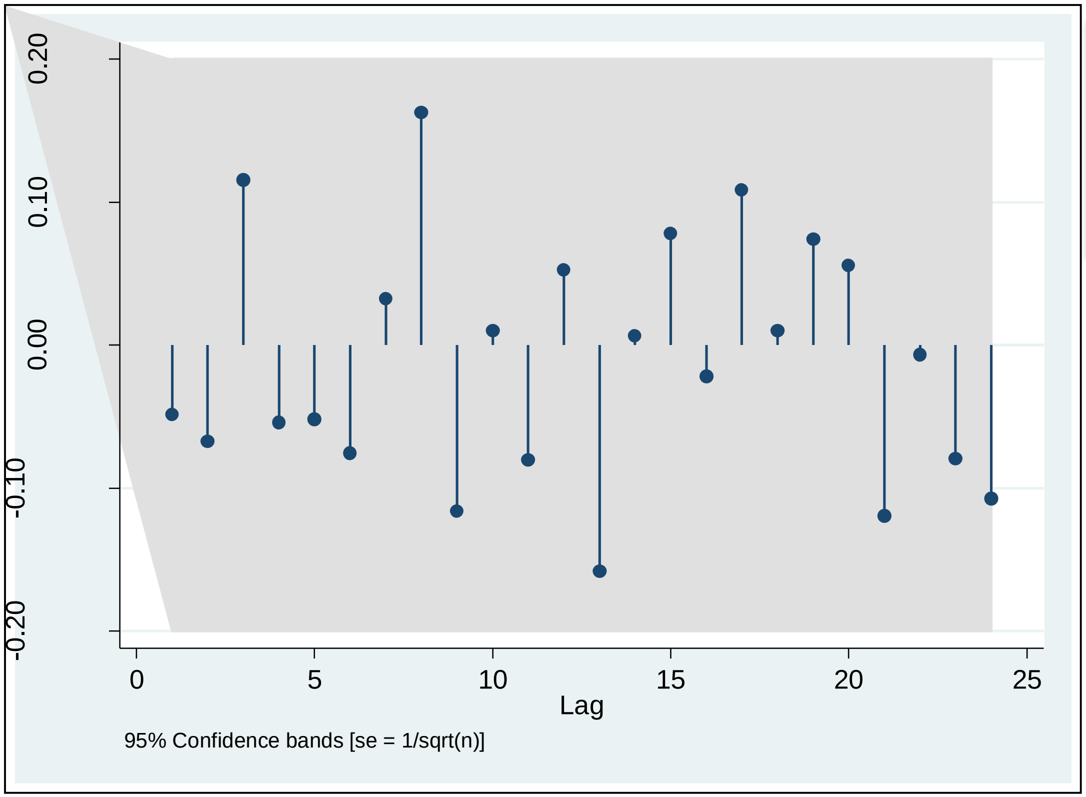
Diagnostic Checking

PACF of the AR(2) residuals



PACF of the AR(2) residuals, whereas the AR(2) was estimated by the ML Method.

```
. pac res_AR2, lags(24)
```

Remember:
H_1: at least one AC-coefficient of the residuals is not equal to zero, i.e. formally

$$\rho_1 \neq 0 \;\; \textit{or} \;\; \rho_2 \neq 0 \;\; \textit{or} \;\; \ldots \;\; \textit{or} \;\; \rho_{24} \neq 0$$

Diagnostic Checking

Box-Ljung test for AR(2) residuals

H_0: All autocorrelation coefficients up to lag K=24 are zero, i.e. formally

$$\rho_1 = \rho_2 = \ldots = \rho_{24} = 0$$

```
. corrgram res_AR2, lags(24)
                                                    -1       0       1 -1       0       1
 LAG       AC       PAC      Q      Prob>Q  [Autocorrelation]  [Partial Autocor]
--------------------------------------------------------------------------------
1       -0.0473  -0.0484   .22189  0.6376
2       -0.0609  -0.0676   .59365  0.7432
[...]
23       0.0082  -0.0795   12.942  0.9532
24      -0.1033  -0.1071   14.337  0.9387
```

Here we test if the **first 24** autocorrelation coefficients **of the residuals of the AR(2)** model are equal to zero.
Rule of thumb: K=(T/4)=96/4=24

```
. di 96*98*(((1/95)*(0.0473^2))+((1/94)*(0.0609^2)))
.592759
```

Difference in Q-Values du to rounding, STATA uses more decimal place than given in output.

```
. di 1-chi2(24, 14.337)
.93866913

. di 1-chi2(22, 14.337)
.88908811
```

On **α=5%(α=0.05)** with the correctly calculated p-value we **can not reject** the null hypothesis as the p-value of **0.88908811** is greater than **0.05**.

Percentiles of the chi-squared distribution

Percentiles of the χ^2 Distribution										
	Percent									
df	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
1	0.000039	0.000157	0.000982	0.003932	0.015791	2.705544	3.841459	5.023886	6.634897	7.879439
2	0.010025	0.020101	0.050636	0.102587	0.210721	4.605170	5.991465	7.377759	9.210340	10.596635
3	0.071722	0.114832	0.215795	0.351846	0.584374	6.251388	7.814728	9.348404	11.344867	12.838156
4	0.206989	0.297109	0.484419	0.710723	1.063623	7.779440	9.487729	11.143287	13.276704	14.860259
5	0.411742	0.554298	0.831212	1.145476	1.610308	9.236357	11.070498	12.832502	15.086272	16.749602
6	0.675727	0.872090	1.237344	1.635383	2.204131	10.644641	12.591587	14.449375	16.811894	18.547584
7	0.989256	1.239042	1.689869	2.167350	2.833107	12.017037	14.067140	16.012764	18.475307	20.277740
8	1.344413	1.646497	2.179731	2.732637	3.489539	13.361566	15.507313	17.534546	20.090235	21.954955
9	1.734933	2.087901	2.700390	3.325113	4.168159	14.683657	16.918978	19.022768	21.665994	23.589351
10	2.155856	2.558212	3.246973	3.940299	4.865182	15.987179	18.307038	20.483177	23.209251	25.188180
11	2.603222	3.053484	3.815748	4.574813	5.577785	17.275009	19.675138	21.920049	24.724970	26.756849
12	3.073824	3.570569	4.403789	5.226029	6.303796	18.549348	21.026070	23.336664	26.216967	28.299519
13	3.565035	4.106915	5.008751	5.891864	7.041505	19.811929	22.362032	24.735605	27.688250	29.819471
14	4.074675	4.660425	5.628726	6.570631	7.789534	21.064144	23.684791	26.118948	29.141238	31.319350
15	4.600916	5.229349	6.262138	7.260944	8.546756	22.307130	24.995790	27.488393	30.577914	32.801321
16	5.142205	5.812213	6.907664	7.961646	9.312236	23.541829	26.296228	28.845351	31.999927	34.267187
17	5.697217	6.407760	7.564186	8.671760	10.085186	24.769035	27.587112	30.191009	33.408664	35.718466
18	6.264805	7.014911	8.230746	9.390455	10.864936	25.989423	28.869299	31.526378	34.805306	37.156451
19	6.843971	7.632730	8.906517	10.117013	11.650910	27.203571	30.143527	32.852327	36.190869	38.582257
20	7.433844	8.260398	9.590778	10.850812	12.442609	28.411981	31.410433	34.169607	37.566235	39.996846
21	8.033653	8.897198	10.282898	11.591305	13.239598	29.615089	32.670573	35.478876	38.932173	41.401065
22	8.642716	9.542492	10.982321	12.338015	14.041493	30.813282	33.924439	36.780712	40.289360	42.795655
23	9.260425	10.195716	11.688552	13.090514	14.847956	32.006900	35.172462	38.075627	41.638398	44.181275
24	9.886234	10.856362	12.401150	13.848425	15.658684	33.196244	36.415028	39.364077	42.979820	45.558512
25	10.519652	11.523975	13.119720	14.611408	16.473408	34.381587	37.652484	40.646469	44.314105	46.927890
26	11.160237	12.198147	13.843905	15.379157	17.291885	35.563171	38.885139	41.923170	45.641683	48.289882
27	11.807587	12.878504	14.573383	16.151396	18.113896	36.741217	40.113272	43.194511	46.962942	49.644915
28	12.461336	13.564710	15.307861	16.927875	18.939243	37.915923	41.337138	44.460792	48.278236	50.993376
29	13.121149	14.256455	16.047072	17.708366	19.767744	39.087470	42.556968	45.722286	49.587885	52.335618
30	13.786720	14.953457	16.790772	18.492661	20.599235	40.256024	43.772972	46.979242	50.892181	53.671962

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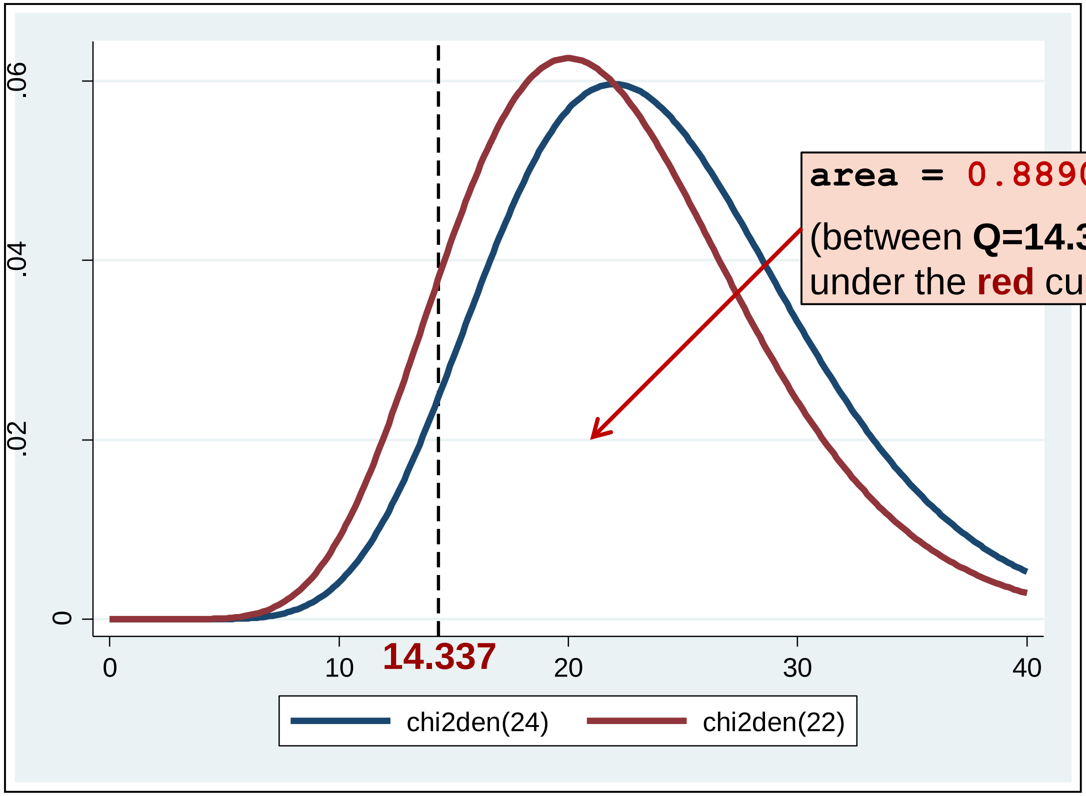
Diagnostic Checking

AR(2) residuals: interpretation of the p-value for the test statistics Q=14.337



area = 0.88908811

(between **Q=14.337** and **+infinity** under the **red** curve)

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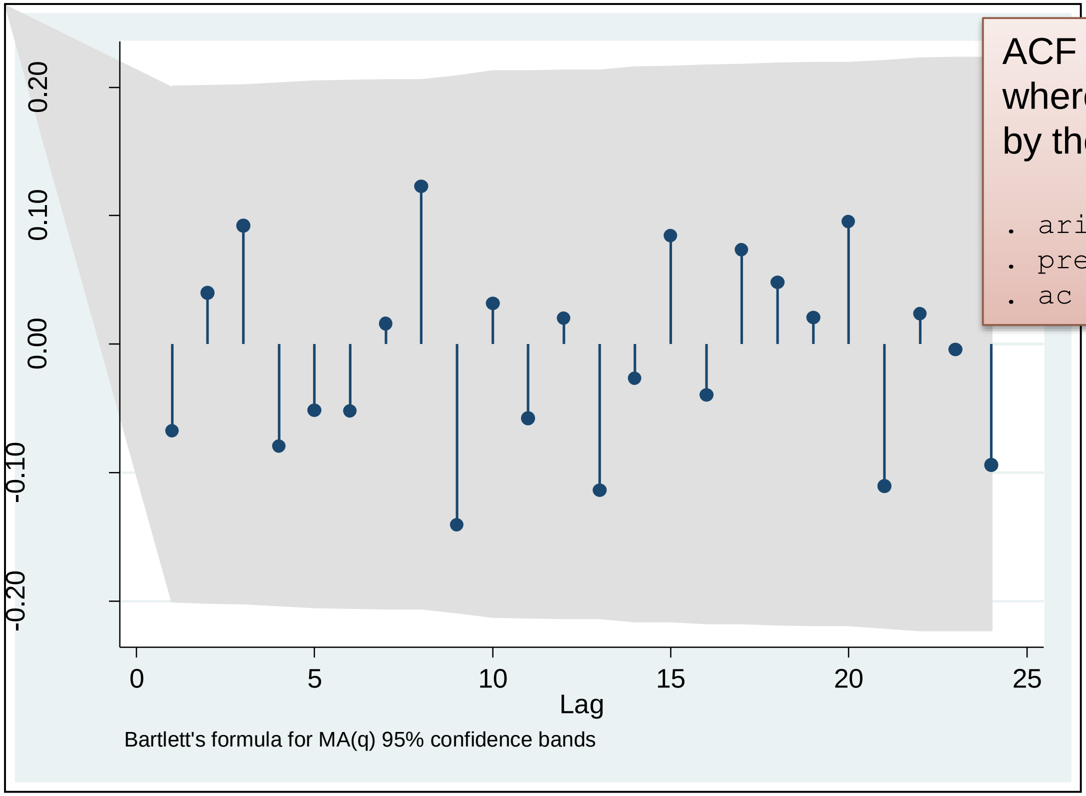
Diagnostic Checking

ACF of the ARMA(1,1) residuals



ACF of the ARMA(1,1) residuals, whereas the ARMA(1,1) was estimated by the ML Method.

```
. arima coal-production, ar(1) ma(1)
. predict res_ARMA11, residuals
. ac res_ARMA11, lags(24)
```

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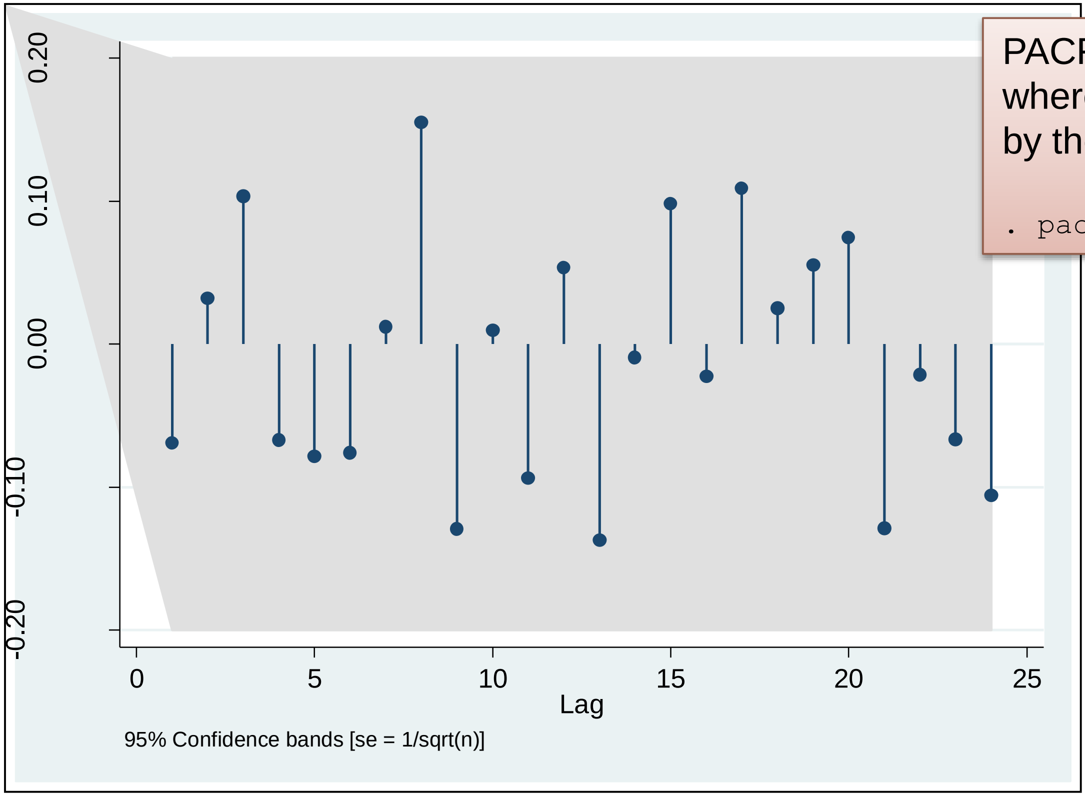
Diagnostic Checking

PACF of the ARMA(1,1) residuals



PACF of the ARMA(1,1) residuals, whereas the ARMA(1,1) was estimated by the ML Method.

```
. pac res_ARMA11, lags(24)
```

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Remember:
H_1: at least one AC-coefficient of the residuals is not equal to zero, i.e. formally

$$\rho_1 \neq 0 \ \textit{or} \ \rho_2 \neq 0 \ \textit{or} \ \ldots \ \textit{or} \ \rho_{24} \neq 0$$

Diagnostic Checking

Box-Ljung test for ARMA(1,1) residuals

H_0: All autocorrelation coefficients up to lag K=24 are zero, i.e. formally

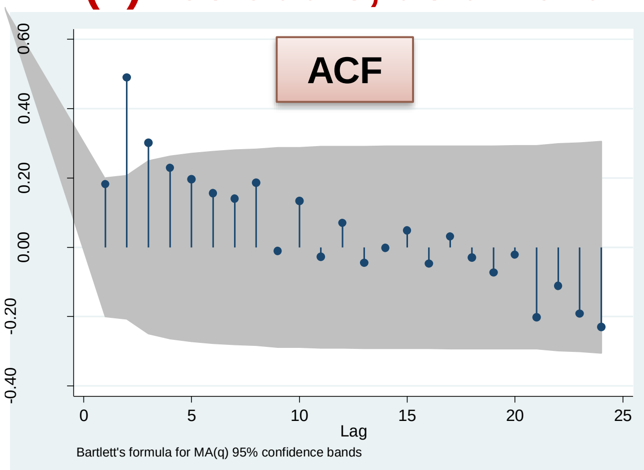
$$\rho_1 = \rho_2 = \ldots = \rho_{24} = 0$$

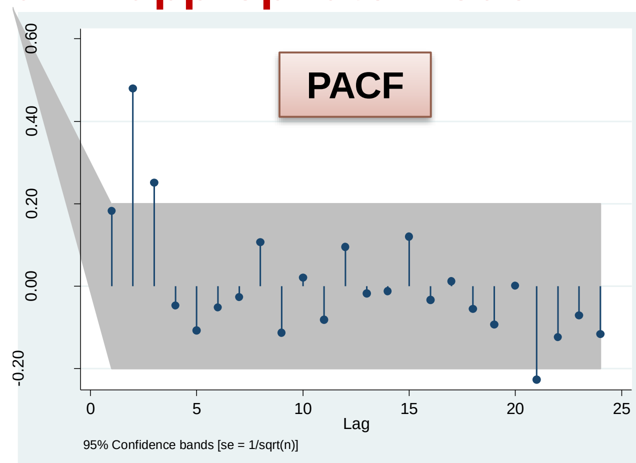
```
. corrgram res_ARMA11, lags(24)
                                                    -1       0       1 -1       0       1
 LAG       AC       PAC       Q      Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1       -0.0678  -0.0692    .4549   0.5000          |                  |
2        0.0398   0.0321   .61312   0.7360          |                  |
[...]
23      -0.0043  -0.0666   13.238   0.9466          |                  |
24      -0.0938  -0.1058   14.387   0.9374          |                  |

. di 1-chi2(22, 14.387)
.88717669
```

On **α=5%(α=0.05)** with the correctly calculated p-value we **can not reject** the null hypothesis as the p-value of **0.88717669** is greater than **0.05**.

Diagnostic Checking

MA(1) residuals, as an example for an inappropriate model






```
. corrgram res_MA1, lags(24)
                                          -1       0       1 -1       0       1
 LAG       AC       PAC       Q     Prob>Q  [Autocorrelation]  [Partial Autocor]
--------------------------------------------------------------------------------
1        0.1822   0.1822   3.2866  0.0698          |-                 |-
2        0.4894   0.4795   27.261  0.0000          |---               |---
[…]
23      -0.1920  -0.0708   69.903  0.0000
24      -0.2299  -0.1163    76.81  0.0000
```

```
. di 1-chi2(23,76.81)
1.035e-07
```

=> reject H_0

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Diagnostic Checking

Comparison of the two candidate models AR(2) and ARMA(1,1)

$$AIC = \log\hat{\sigma}^2 + 2\frac{p+q}{T}$$

AIC of AR(2)

```
. di log(3066.34^2)+2*(2+0)/96
16.098147
```

AIC of ARMA(1,1)

```
. di log(3084.761^2)+2*(1+1)/96
16.110126
```

$$BIC = \log\hat{\sigma}^2 + \frac{p+q}{T}\log T$$

BIC of AR(2)

```
. di log(3066.34^2)+((2+0)/96)*log(96)
16.15157
```

BIC of ARMA(1,1)

```
. di log(3084.761^2)+((1+1)/96)*log(96)
16.16355
```

Recall:

```
. arima coal_production, ar(1/2)
[...]
Sample: 1952m1 to 1959m12   Number of obs = 96
-------------------------------------------------
[...]
-------------------------------------------------
 /sigma |   3066.34
-------------------------------------------------


. arima coal_production, ar(1) ma(1)
[...]
-------------------------------------------------
 /sigma |   3084.761
-------------------------------------------------
```

Exercise 2.16: Forecasting

- Estimate an **AR(2)** model by means of the **ML Method** using only the first **84** observations, i.e. exclude completely the last year.

- Calculate forecasts for all months from **1959m1** to **1961m12**.
  Hint:
  Previously increase the number of observations form 96 to 120 in order to create the additional time periods, i.e. from 1960m1 to 1961m12 (24 additional time periods). To this end you can use the following STATA-code:

```
. set obs 120
. replace time = tm(1952m1)+_n-1
. format time %tm
. tsset time
```

- Plot the original series together with the predicted values. Add to the plot the unconditional mean of the original series.
  What can you conclude?

Dynamic forecasts in Stata

`dynamic(time_constant)` specifies how lags of y_t in the model are to be handled. If `dynamic()` is not specified, actual values are used everywhere that lagged values of y_t appear in the model to produce one-step-ahead forecasts.

`dynamic(time constant)` produces dynamic (also known as recursive) forecasts. `time constant` specifies when the forecast is to switch from one step ahead to dynamic. In dynamic forecasts, references to y_t evaluate to the prediction of y_t for all periods at or after time constant; they evaluate to the actual value of y_t for all prior periods.

For example, **`dynamic(10)`** would calculate predictions in which any reference to y_t with $t < 10$ evaluates to the actual value of y_t and any reference to y_t with $t \geq 10$ evaluates to the prediction of y_t. This means that one-step-ahead predictions are calculated for $t < 10$ and dynamic predictions thereafter.

Stata manual "arima postestimation"

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Exercise (coal_production.dta)

Forecasting: Estimation of AR(2) (without the last year, i.e. without 1959)

```
. arima coal_production if time <= tm(1958m12), ar(1/2)
[...]
Sample:  1952m1 to 1958m12                     Number of obs      =         84
                                               Wald chi2(2)       =      88.28
Log likelihood = -794.6504                     Prob > chi2        =     0.0000
------------------------------------------------------------------------------
             |                 OPG
coal_produ~n |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
coal_produ~n |
       _cons |    38395.6   1560.675    24.60   0.000     35336.73    41454.47
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .4574245   .0965237     4.74   0.000     .2682415    .6466076
         L2. |   .3493262   .0759875     4.60   0.000     .2003935    .4982589
-------------+----------------------------------------------------------------
      /sigma |   3089.115   221.8593    13.92   0.000     2654.279    3523.951
------------------------------------------------------------------------------
```

$\hat{\varphi}_1 = 0.4574 \quad \hat{\varphi}_2 = 0.3493 \quad \hat{\mu} = 38395.6 \quad \hat{\sigma}_\varepsilon = 3089.115 \quad \hat{\delta} = \hat{\mu}\left(1 - \hat{\varphi}_1 - \hat{\varphi}_2\right) = 7419.92$

Pankratz (1983) “Forecasting with univariate Box-Jenkins models”

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Forecasting:

Adjust time periods in order to be able to perform forecasting:

```
. set obs 120
number of observations (_N) was 96, now 120

. replace time = tm(1952m1) + _n - 1
(24 real changes made)

. format time %tm

. tsset time
        time variable:  time, 1952m1 to 1961m12
                delta:  1 month
```

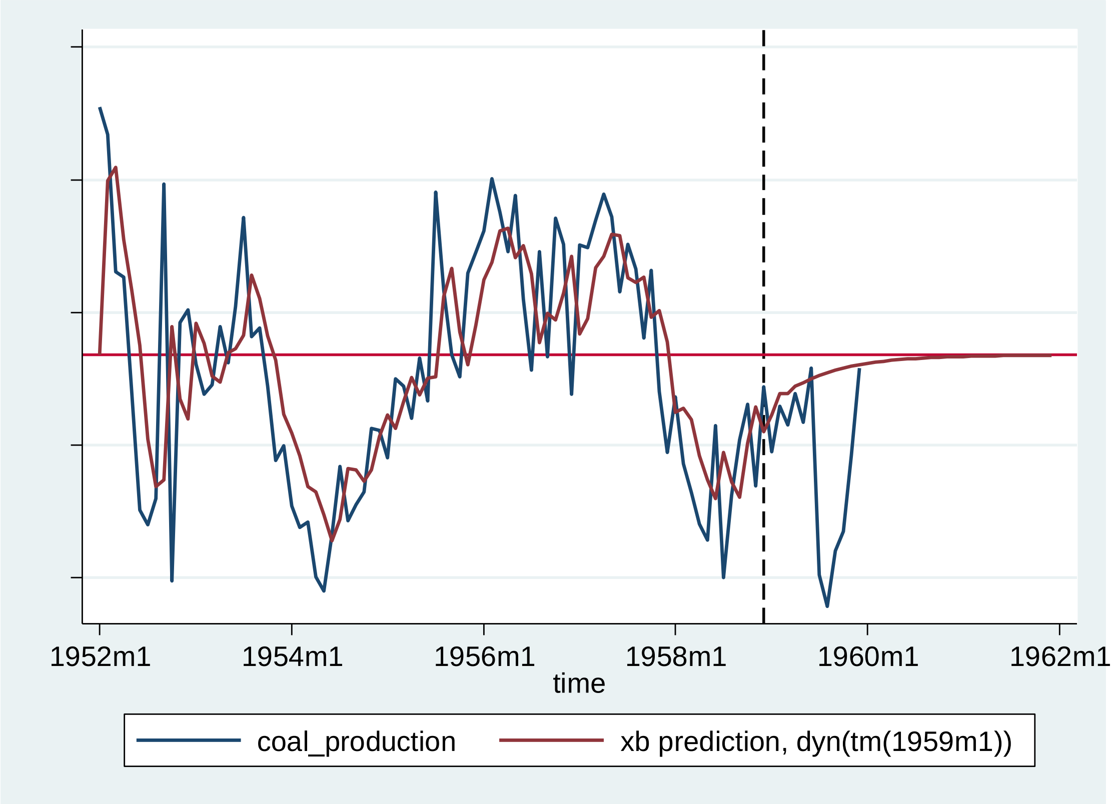
Perform forecasting:

```
. predict forecast_AR2, xb dynamic(tm(1959m1))
```

Stata help "arima postestimation"

Forecasting: Plot the original series and the forecasted values

```
. tsline coal_production forecast_AR2, yline(38395.6) xline(-13)
```



Pankratz (1983) “Forecasting with univariate Box-Jenkins models”

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Exercise 2.17:

Consider the following true model for a time series

$y_t = 0.3 + 0.5\, y_{t\text{-}1} - 0.4\, \varepsilon_{t\text{-}1} + \varepsilon_t$

where ε_t is a zero mean error process.

What is the (unconditional) mean of the series, y_t ?

(1) 0.6

(2) 0.3

(3) 0.0

(4) 0.4

Brooks (2008) "Introductory Econometrics for Finance", Multiple Choice Test Bank Questions

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Solution 2.17

Constant term is the constant of the regression and NOT the (unconditional) mean of the series!

$$E[y_t] = \mu = \frac{\delta}{1 - (\sum_{i=1}^{p} \varphi_i)}$$

$$E[y_t] = \frac{0.3}{1 - 0.5} = 0.6$$

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Exercise 2.18:

Which of the following sets of characteristics would usually best describe an autoregressive process of order 3 (i.e. an AR(3))?

(1) A slowly decaying acf and pacf

(2) An acf and a pacf with 3 significant spikes

(3) A slowly decaying acf, and a pacf with 3 significant spikes

(4) A slowly decaying pacf and an acf with 3 significant spikes

Brooks (2008) “Introductory Econometrics for Finance”, Multiple Choice Test Bank Questions

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Exercise 2.19:

A process, x_t, which has a zero mean, a constant variance, and zero autocovariance for all non-zero lags is best described as

(1) A white noise process

(2) A covariance stationary process

(3) An autocorrelated process

(4) A moving average process

Brooks (2008) “Introductory Econometrics for Finance”, Multiple Choice Test Bank Questions

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Lag-operator notation

Any stationary **ARMA(*p*,*q*)** model

$$y_t = \delta + \varepsilon_t + \underbrace{\varphi_1 y_{t-1} + \ldots + \varphi_p y_{t-p}}_{\text{"AR-part"}} \underbrace{- \Theta_1 \varepsilon_{t-1} + \ldots - \Theta_q \varepsilon_{t-q}}_{\text{"MA-part"}}$$

can be written in **lag-operator** notation, i.e.

$$\underbrace{(1 - \varphi_1 L + \ldots - \varphi_p L^p)}_{:= a_p(L)} \tilde{y}_t = \underbrace{(1 - \Theta_1 L + \ldots - \Theta_q L^q)}_{:= b_q(L)} \varepsilon_t \ , \ \text{with } \tilde{y}_t := y_t - \mu$$

Lag-operator **polynomial** of order **p** with **AR**-coefficients.

Lag-operator **polynomial** of order **q** with **MA**-coefficients.

Time series $\mathbf{y_t}$ written as **deviation form the mean**.

In short: $a_p(L)\tilde{y}_t = b_q(L)\varepsilon_t \ , \text{with } \tilde{y}_t := y_t - \mu$

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Lag-operator notation $a_p(L)\tilde{y}_t = b_q(L)\varepsilon_t$

How to write any stationary **ARMA(*p*,*q*)** in lag operator notation **?**

Proceed as follows:

1. Write $\boldsymbol{y_t}$ in deviations from its mean: $\tilde{y}_t := y_t - \mu$
2. Multiply the result from step 1 by the **appropriate** lag-operator polynomial $\mathbf{a_p(L)}$, whose general form is $\left(1 - \varphi_1 L - \varphi_2 L^2 - \ldots - \varphi_p L^p\right)$.
3. Multiply the random shock, $\boldsymbol{\varepsilon_t}$,by the **appropriate** lag-operator polynomial $\mathbf{b_q(L)}$, whose general form is $\left(1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_q L^q\right)$.
4. Equate the results from steps 2 and 3.

You can **check the correctness** of your result by multiplying out both sides of the equation $a_p(L)\tilde{y}_t = b_q(L)\varepsilon_t$ and comparing it to model written in our usual notation.

Pankratz (1983) “Forecasting with univariate Box-Jenkins models”, p. 99

Exercise 2.20:

Estimated ARMA(p,q) with T = 60 observations:

$\hat{\mu} = 101.26 \quad \hat{\varphi}_1 = 0.62 \quad \hat{\theta}_1 = -0.58 \quad \hat{\sigma}_\varepsilon = 1.6$

a) Determine the order of the estimated ARMA model.

b) Write down the estimated model in our usual notation.

c) Write down the model in lag operator notation.

The last observation in this data series is y_{60} = 96.91 and the predicted value (using the model from above) is $\hat{y}_{60} = 98.28$.

d) With forecast origin T = 60 calculate the first three forecasts from this model.

e) Construct confidence intervals around the three point forecasts.

Hint: $$\left[\tilde{y}_{T+l}/\Omega_T \pm z_{1-\frac{\alpha}{2}}\left(1+\psi_1^2+\ldots+\psi_{l-1}^2\right)^{\frac{1}{2}}\sigma_\varepsilon\right] = \left[\tilde{y}_{T+l}/\Omega_T \pm z_{1-\frac{\alpha}{2}}\left(1+\sum_{j=1}^{l-1}\psi_j^2\right)^{\frac{1}{2}}\sigma_\varepsilon\right]$$

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Solution 2.20-1:

a) Determine the order (i.e. p and q) of the estimated ARMA model.

Given the estimates $\hat{\mu} = 10126$ $\underline{\hat{\varphi}_1 = 0.62}$ $\underline{\hat{\theta}_1 = -0.58}$ $\hat{\sigma}_\varepsilon = 16$

we conclude: p=1, q=1 => ARMA(1,1)

Solution 2.20-2:

b) Write down the estimated model in our usual notation.

ARMA(1,1) in general: $y_t = \delta + \varepsilon_t + \varphi_1 y_{t-1} - \theta_1 \varepsilon_{t-1}$

Given the estimates $\hat{\mu} = 10126 \quad \hat{\varphi}_1 = 0.62 \quad \hat{\theta}_1 = -0.58 \quad \hat{\sigma}_\varepsilon = 16$

=> $\hat{\delta} = \hat{\mu}(1 - \hat{\varphi}_1) = 101.26(1 - 0.62) = 38.4788$

=> $\hat{\varphi}_1 = 0.62, \quad \hat{\theta}_1 = -0.58$

=> Estimated model: $\hat{y}_t = 38.4788 + 0.62 y_{t-1} + 0.58 \varepsilon_{t-1}$

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Exercise

Given the estimated parameters we should write:

$\hat{y}_t = 38.4788 + 0.62 y_{t-1} + 0.58 \varepsilon_{t-1}$

However, here we use the slightly different form to show (and only for that), that our usual notation and the lag-operator notation yield to the same result, **given we know the true model and the true parameters**!

Solution 2.20-3:

c) Write down the model in lag-operator notation.

Our model: $y_t = 38.4788 + 0.62 y_{t-1} + 0.58 \varepsilon_{t-1} + \varepsilon_t$

Lag-operator notation in general: $a_p(L)\tilde{y}_t = b_q(L)\varepsilon_t$

Step 1: y_t as deviation from the mean: $\tilde{y}_t = y_t - 101.26$

Step 2: $a_1(L) = (1 - \varphi_1 L) = (1 - 0.62L)$ => $(1 - 0.62L)\tilde{y}_t$

Step 3: $b_1(L) = (1 - \Theta_1 L) = (1 + 0.58L)$ => $(1 + 0.58L)\varepsilon_t$

Step 4: $(1 - 0.62L)\tilde{y}_t = (1 + 0.58L)\varepsilon_t$ with $\tilde{y}_t = y_t - 10126$

Please, check whether your result is correct.

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Exercise

Solution 2.20-4:

The tilde (~) is usually used for forecasting. Here, however, it is a local definition for the deviation from the mean. You are allowed to use other definitions (e.g. $\dot{y}_t$ or y'_t) if you **clearly state, what those mean**.

Checking the correctness of the result:

$$(1-0.62L)\tilde{y}_t = (1+0.58L)\varepsilon_t \quad \text{with } \tilde{y}_t = y_t - 10126$$

$$\Leftrightarrow (1-0.62L)(y_t - 101.26) = (1+0.58L)\varepsilon_t$$

$$\Leftrightarrow y_t - 0.62y_{t-1} - 101.26 + 0.62 \cdot 101.26 = (1+0.58L)\varepsilon_t$$

$$\Leftrightarrow y_t - 0.62y_{t-1} - 38.4788 = \varepsilon_t + 0.58\varepsilon_{t-1}$$

$$\Leftrightarrow y_t = 38.4788 + 0.62y_{t-1} + \varepsilon_t + 0.58\varepsilon_{t-1}$$

Recall:

Our model: $y_t = 38.4788 + 0.62y_{t-1} + 0.58\varepsilon_{t-1} + \varepsilon_t$

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Solution 2.20-5:

d) With forecast origin T = 60 calculate the first three forecasts from this model.

<u>Recall:</u>

The optimal predictor (minimal *MSE*) is the conditional mean

$$\tilde{y}_{T+l} \,/\, \Omega_T = E\left(Y_{T+l} \,/\, \Omega_T\right), \;\; l = 1, 2, \ldots$$

it minimizes the expected squared forecast error

$$\min_{\tilde{y}_{T+l}} MSE\left(\tilde{y}_{T+l}\right) = E\left[\left(y_{T+l} - \tilde{y}_{T+l}\right)^2 \,/\, \Omega_T\right]$$

Ω_T is the information set at period T, i.e. true model, known parameters, all past observations $y_T, \ldots, y_2, y_1, y_0, y_{-1}, \ldots$

Remember:

$$\tilde{y}_{T+l} / \Omega_T = E\left(Y_{T+l} / \Omega_T\right), \ \ l = 1, 2, \ldots$$

Solution 2.20-6:

d) With forecast origin T = 60 calculate the first three forecasts from this model.

Our model: $y_t = 38.4788 + 0.62 y_{t-1} + 0.58 \varepsilon_{t-1} + \varepsilon_t$

One period ahead forecast: $\tilde{y}_{60+1} / \Omega_{60} = E\left(y_{60+1} / \Omega_{60}\right)$

Given our model: $y_{61} = \boxed{38.4788 + 0.62 y_{60} + 0.58 \varepsilon_{60} + \varepsilon_{61}}$

=> $\tilde{y}_{60+1} / \Omega_{60} = E\left(y_{60+1} / \Omega_{60}\right)$

$$= E\left(\boxed{38.4788 + 0.62 y_{60} + 0.58 \varepsilon_{60} + \varepsilon_{61}} / \Omega_{60}\right)$$

$$= 38.4788 + 0.62 y_{60} + 0.58 \underbrace{E(\varepsilon_{60} / \Omega_{60})}_{=\hat{\varepsilon}_{60}} + \underbrace{E(\varepsilon_{61} / \Omega_{60})}_{=0}$$

with $\hat{\varepsilon}_{60} = (y_{60} - \hat{y}_{60}) = (96.91 - 98.28) = -1.37$

Recall:

y_{60} = 96.91

$\hat{y}_{60} = 98.28$

$$= 38.4788 + 0.62 \cdot 96.91 + 0.58 \cdot (-1.37) = 97.7684 \approx 97.77$$

Remember:

$$\tilde{y}_{T+l} / \Omega_T = E(Y_{T+l} / \Omega_T), \ l = 1, 2, \ldots$$

Solution 2.20-7:

d) With forecast origin T = 60 calculate the first three forecasts from this model.

Our model: $y_t = 38.4788 + 0.62 y_{t-1} + 0.58 \varepsilon_{t-1} + \varepsilon_t$

Two periods ahead forecast: $\tilde{y}_{60+2} / \Omega_{60} = E(y_{60+2} / \Omega_{60})$

Given our model: $y_{62} = \boxed{38.4788 + 0.62 y_{61} + 0.58 \varepsilon_{61} + \varepsilon_{62}}$

Recall:

$$\tilde{y}_{60+1} / \Omega_{60} = E(y_{60+1} / \Omega_{60}) \approx 97.77$$

$$\begin{aligned}
=> \quad \tilde{y}_{60+2} / \Omega_{60} &= E(y_{60+2} / \Omega_{60}) \\
&= E(\boxed{38.4788 + 0.62 y_{61} + 0.58 \varepsilon_{61} + \varepsilon_{62}} / \Omega_{60}) \\
&= 38.4788 + 0.62 E(y_{61} / \Omega_{60}) + 0.58 \underbrace{E(\varepsilon_{61} / \Omega_{60})}_{=0} + \underbrace{E(\varepsilon_{62} / \Omega_{60})}_{=0} \\
&= 38.4788 + 0.62 \cdot \tilde{y}_{61} / \Omega_{60} \\
&= 38.4788 + 0.62 \cdot 97.77 = 99.0962 \approx 99.1
\end{aligned}$$

Remember:

$$\tilde{y}_{T+l} / \Omega_T = E(Y_{T+l} / \Omega_T), \quad l = 1, 2, \ldots$$

Solution 2.20-8:

d) With forecast origin T = 60 calculate the first three forecasts from this model.

Our model: $y_t = 38.4788 + 0.62 y_{t-1} + 0.58\varepsilon_{t-1} + \varepsilon_t$

Three periods ahead forecast: $\tilde{y}_{60+3} / \Omega_{60} = E(y_{60+3} / \Omega_{60})$

Given our model: $y_{63} = \boxed{38.4788 + 0.62 y_{62} + 0.58\varepsilon_{62} + \varepsilon_{63}}$

Recall:

$$\tilde{y}_{60+2} / \Omega_{60} = E(y_{60+2} / \Omega_{60}) \approx 99.1$$

$$\Rightarrow \quad \tilde{y}_{60+3} / \Omega_{60} = E(y_{60+3} / \Omega_{60})$$

$$= E(\boxed{38.4788 + 0.62 y_{62} + 0.58\varepsilon_{62} + \varepsilon_{63}} / \Omega_{60})$$

$$= 38.4788 + 0.62 E(y_{62} / \Omega_{60}) + 0.58 \underbrace{E(\varepsilon_{62} / \Omega_{60})}_{=0} + \underbrace{E(\varepsilon_{63} / \Omega_{60})}_{=0}$$

$$= 38.4788 + 0.62 \cdot \tilde{y}_{62} / \Omega_{60}$$

$$= 38.4788 + 0.62 \cdot 99.1 = 99.9208 \approx 99.92$$

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Solution 2.20-9:

e) Construct confidence intervals around the three point forecasts.

$$\left[\tilde{y}_{T+l}/\Omega_T \pm z_{1-\frac{\alpha}{2}}\left(1+\psi_1^2+\ldots+\psi_{l-1}^2\right)^{\frac{1}{2}}\sigma_\varepsilon\right]=\left[\tilde{y}_{T+l}/\Omega_T \pm z_{1-\frac{\alpha}{2}}\left(1+\sum_{j=1}^{l-1}\psi_j^2\right)^{\frac{1}{2}}\sigma_\varepsilon\right]$$

Recall: Wolds Decomposition

Any stationary ARMA(p,q) process can be written as the linear combination of all shocks, i.e.

$$y_t-\mu=\varepsilon_t+\psi_1\varepsilon_{t-1}+\psi_2\varepsilon_{t-2}+\psi_3\varepsilon_{t-3}+\ldots$$

$$=\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} \quad with \quad \psi_0=1$$

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Solution 2.20-10:

e) Construct confidence intervals around the three point forecasts.

Wolds Decomposition:

$$y_t - \mu = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \ldots = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad \text{with} \quad \psi_0 = 1$$

$\Rightarrow$for any l = 1, 2, …

$$y_{T+l} = \mu + \underbrace{\varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-2} + \ldots + \psi_{l-1} \varepsilon_{T+1}}_{\text{linear combination of future shocks}} + \underbrace{\psi_l \varepsilon_T + \psi_{l+1} \varepsilon_{T-1} + \ldots}_{\text{linear combination of past shocks}}$$

$\Rightarrow$and

$$\tilde{y}_{T+l} / \Omega_T := E(y_{T+l} / \Omega_T) = \mu + \underbrace{\psi_l \varepsilon_T + \psi_{l+1} \varepsilon_{T-1} + \ldots}_{\text{linear combination of past shocks}}$$

Solution 2.20-11:

e) Construct confidence intervals around the three point forecasts.

$\Rightarrow$for any l = 1, 2, …

$$y_{T+l} = \mu + \underbrace{\varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1}\varepsilon_{T+1}}_{linear\ combination\ of\ \underline{future}\ shocks} + \underbrace{\psi_l \varepsilon_T + \psi_{l+1}\varepsilon_{T-1} + \ldots}_{linear\ combination\ of\ \underline{past}\ shocks}$$

$\Rightarrow$and

$$\tilde{y}_{T+l} / \Omega_T := E(y_{T+l} / \Omega_T) = \mu + \underbrace{\psi_l \varepsilon_T + \psi_{l+1}\varepsilon_{T-1} + \ldots}_{linear\ combination\ of\ \underline{past}\ shocks}$$

$\Rightarrow$ forecast error:

$$e_{T+l} := y_{T+l} - \tilde{y}_{T+l} / \Omega_T$$

$$= \underbrace{\varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1}\varepsilon_{T+1}}_{linear\ combination\ of\ \underline{future}\ shocks}$$

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Solution 2.20-12:

e) Construct confidence intervals around the three point forecasts.

$\Rightarrow$ forecast error:

$$e_{T+l} := y_{T+l} - \tilde{y}_{T+l} \,/\, \Omega_T$$
$$= \underbrace{\varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1} \varepsilon_{T+1}}_{\text{linear combination of future shocks}}$$

Hence:

$$E(e_{T+l}) = E(\varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1} \varepsilon_{T+1}) = 0$$

$$Var[e_{T+l}] = E[e_{T+l}^2] - (E[e_{T+l}])^2$$
$$= E\left[(\varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1} \varepsilon_{T+1})^2\right]$$
$$= (1 + \psi_1^2 + \ldots + \psi_{l-1}^2)\sigma_\varepsilon^2$$

Remember:

By definition:

Var(X) = E(X²) – [E(X)]²

By assumption:

$E(\varepsilon_t \varepsilon_{t-k}) = 0$
for any t and any $k \neq 0$

$E(\varepsilon_t^2) = Var(\varepsilon_t) = \sigma_\varepsilon^2$
for any t

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Solution 2.20-13:

e) Construct confidence intervals around the three point forecasts.

$\Rightarrow$ forecast error:

$$e_{T+l} := \underbrace{\varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1} \varepsilon_{T+1}}_{\text{linear combination of future shocks}}$$

with $E(e_{T+l}) = 0$ and $Var\left[e_{T+l}\right] = \left(1 + \psi_1^2 + \ldots + \psi_{l-1}^2\right)\sigma_\varepsilon^2$

Prediction interval:

$$\left[\tilde{y}_{T+l} / \Omega_T \pm z_{1-\frac{\alpha}{2}} \left(1 + \psi_1^2 + \ldots + \psi_{l-1}^2\right)^{\frac{1}{2}} \sigma_\varepsilon\right] = \left[\tilde{y}_{T+l} / \Omega_T \pm z_{1-\frac{\alpha}{2}} \left(1 + \sum_{j=1}^{l-1} \psi_j^2\right)^{\frac{1}{2}} \sigma_\varepsilon\right]$$

Solution 2.20-14:

e) Construct confidence intervals around the three point forecasts.

How do we find ψ_1, ..., ψ_{l-1}?

Lag-operator notation:

$$a_p(L)\tilde{y}_t = b_q(L)\varepsilon_t \ , \ \text{with } \tilde{y}_t := y_t - \mu$$

Wolds Decomposition:

$$\begin{aligned}\tilde{y}_t &= \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots \\ &= (1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \dots)\varepsilon_t \\ &= c(L)\varepsilon_t \ , \text{with } \tilde{y}_t := y_t - \mu\end{aligned}$$

$$a_p(L)c(L)\varepsilon_t = b_q(L)\varepsilon_t$$

The ψ_1, ψ_2, ... coefficients in $c(L)$, can be found by equating coefficients of L^j, j = 1, 2, ... in $a(L)c(L) = b(L)$.

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Recall: $\left[\tilde{y}_{T+l}\,/\Omega_T \pm z_{1-\frac{\alpha}{2}}\left(1+\psi_1^2+\ldots+\psi_{l-1}^2\right)^{\frac{1}{2}}\sigma_\varepsilon\right]$

and: $c(L)=(1+\psi_1 L+\psi_2 L^2+\psi_3 L^3+\ldots)$

Solution 2.20-15:

$$\underbrace{(1-0.62L)}_{a(L)}\tilde{y}_t=\underbrace{(1+0.58L)}_{b(L)}\varepsilon_t$$

$$a(L)c(L)=b(L)$$

$$(1-0.62L)\left(1+\hat{\psi}_1 L+\hat{\psi}_2 L^2+\hat{\psi}_3 L^3+\ldots\right)=1+0.58L$$

$$1-\underline{0.62L}+\underline{\hat{\psi}_1 L}-\underline{0.62\hat{\psi}_1 L^2}+\underline{\hat{\psi}_2 L^2}-\underline{0.62\hat{\psi}_2 L^3}+\underline{\hat{\psi}_3 L^3}+\ldots=1+\underline{0.58L}$$

$$L^1:\ -0.62+\hat{\psi}_1=0.58 \quad\Rightarrow\quad \hat{\psi}_1=0.58+0.62=1.2$$

$$L^2:\ -0.62\hat{\psi}_1+\hat{\psi}_2=0 \quad\Rightarrow\quad \hat{\psi}_2=0.62\cdot 1.2=0.744\approx 0.74$$

$$L^3:\ -0.62\hat{\psi}_2+\hat{\psi}_3=0 \quad\Rightarrow\quad \hat{\psi}_3=0.62\cdot 0.74=0.62\cdot 0.62\cdot 1.2=0.4588\approx 0.46$$

$$L^j:\ -0.62\hat{\psi}_{j-1}+\hat{\psi}_j=0 \quad\Rightarrow\quad \hat{\psi}_j=0.62^{j-1}\cdot 1.2$$

Solution 2.20-16:

$$Estimated\ \ parameters: \hat{\mu} = 101.26, \quad \hat{\varphi}_1 = 0.62, \quad \hat{\theta}_1 = -0.58, \quad \hat{\sigma}_\varepsilon = 1.6$$

$$Predictions: \qquad \hat{y}_{61} = 97.77, \quad \hat{y}_{62} = 99.1, \quad \hat{y}_{63} = 99.92,$$

$$\hat{\psi} - Coefficients: \qquad \hat{\psi}_1 = 1.2, \ \ \hat{\psi}_2 = 0.74, \ \ \hat{\psi}_3 = 0.46$$

$$The\ interval: \quad \left[\tilde{y}_{T+l} / \Omega_T \ \pm \ z_{1-\frac{\alpha}{2}} \left(1 + \psi_1^2 + \ldots + \psi_{l-1}^2\right)^{\frac{1}{2}} \sigma_\varepsilon \right]$$

$$\left[\hat{y}_{61} \pm z_{1-\frac{\alpha}{2}} (1)^{\frac{1}{2}} \hat{\sigma}_\varepsilon \right] \qquad \rightarrow \left[97.77 \pm 1.96 \cdot (1)^{\frac{1}{2}} \cdot 1.6 \right]$$

$$\rightarrow [97.77 \pm 3.14]$$

$$\rightarrow [94.63, 100.91]$$

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Solution 2.20-17:

$$Estimated\ \ parameters: \hat{\mu} = 101.26, \quad \hat{\varphi}_1 = 0.62, \quad \hat{\theta}_1 = -0.58, \quad \hat{\sigma}_\varepsilon = 1.6$$

$$Predictions: \qquad \hat{y}_{61} = 97.77, \quad \hat{y}_{62} = 99.1, \quad \hat{y}_{63} = 99.92,$$

$$\hat{\psi} - Coefficients: \qquad \hat{\psi}_1 = 1.2, \ \ \hat{\psi}_2 = 0.74, \ \ \hat{\psi}_3 = 0.46$$

$$The\ interval: \quad \left[\tilde{y}_{T+l} / \Omega_T \ \pm \ z_{1-\frac{\alpha}{2}} \left(1 + \psi_1^2 + \ldots + \psi_{l-1}^2\right)^{\frac{1}{2}} \sigma_\varepsilon \right]$$

$$\left[\hat{y}_{62} \pm z_{1-\frac{\alpha}{2}} \left(1 + \hat{\psi}_1^2\right)^{\frac{1}{2}} \hat{\sigma}_\varepsilon \right] \qquad \to \left[99.1 \pm 196 \cdot \left(1 + 12^2\right)^{\frac{1}{2}} \cdot 16 \right]$$

$$\to [99.1 \pm 4.9]$$

$$\to [94.2, 104]$$

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Solution 2.20-18:

Estimated parameters : $\hat{\mu} = 101.26, \quad \hat{\varphi}_1 = 0.62, \quad \hat{\theta}_1 = -0.58, \quad \hat{\sigma}_\varepsilon = 1.6$

Predictions : $\hat{y}_{61} = 97.77, \quad \hat{y}_{62} = 99.1, \quad \hat{y}_{63} = 99.92,$

$\hat{\psi}$ *– Coefficients :* $\hat{\psi}_1 = 1.2, \quad \hat{\psi}_2 = 0.74, \quad \hat{\psi}_3 = 0.46$

The interval : $\left[\tilde{y}_{T+l} / \Omega_T \pm z_{1-\frac{\alpha}{2}} \left(1 + \psi_1^2 + \ldots + \psi_{l-1}^2\right)^{\frac{1}{2}} \sigma_\varepsilon \right]$

$$\left[\hat{y}_{63} \pm z_{1-\frac{\alpha}{2}} \left(1 + \hat{\psi}_1^2 + \hat{\psi}_2^2\right)^{\frac{1}{2}} \hat{\sigma}_\varepsilon \right] \rightarrow \left[99.92 \pm 1.96 \cdot \left(1 + 1.2^2 + 0.74^2\right)^{\frac{1}{2}} \cdot 1.6 \right]$$

$$\rightarrow [99.92 \pm 5.42]$$

$$\rightarrow [94.5, 105.34]$$

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Exercise 2.21:

- Write down the following stationary ARMA(2,1) in lag operator notation:

$$y_t = 3 + 0.6y_{t-1} + 0.2y_{t-2} + \varepsilon_t - 0.4\varepsilon_{t-1}$$

- Write down the first difference operator, *Δ*, in lag operator notation.

Hint: $\Delta y_t := y_t - y_{t-1}$

- Write down the simple four-period moving average in lag operator notation.

- Multiply the first difference operator and the simple four-period moving average (both in lag operator notation). Describe the result.

Solution 2.21-1:

$$y_t = 3 + 0.6y_{t-1} + 0.2y_{t-2} + \varepsilon_t - 0.4\varepsilon_{t-1}$$

$$\tilde{y}_t = y_t - \mu$$

$$\mu = \frac{\delta}{(1 - \varphi_1 - \varphi_2)} = \frac{3}{(1 - 0.6 - 0.2)} = 15$$

$$(1 - 0.6L - 0.2L^2)\tilde{y}_t = (1 - 0.4L)\varepsilon_t$$

First difference operator:

$$\Delta = (1 - L)$$

Simple four-period moving average:

$$\frac{1}{4}(1 + L + L^2 + L^3)y_t$$

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Solution 2.21-2:

Multiply the first difference operator and the simple four-period moving average (both in lag operator notation).

$$
\begin{aligned}
(1-L)\cdot\left(1+L+L^2+L^3\right)y_t &= \left[\left(1+L+L^2+L^3\right)-\left(L+L^2+L^3+L^4\right)\right]y_t \\
&= \left(1-L^4\right)y_t \\
&= y_t - y_{t-4}
\end{aligned}
$$

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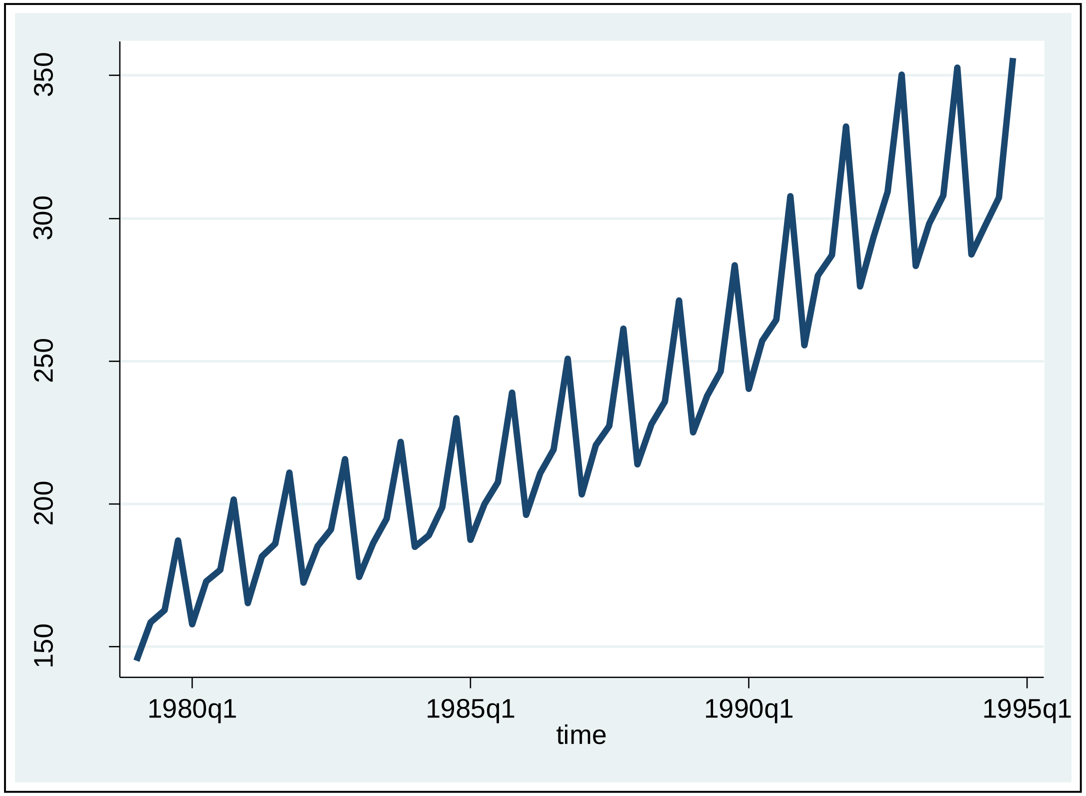
Solution 2.21-3:

Original series (`tsline lohnsumme`)



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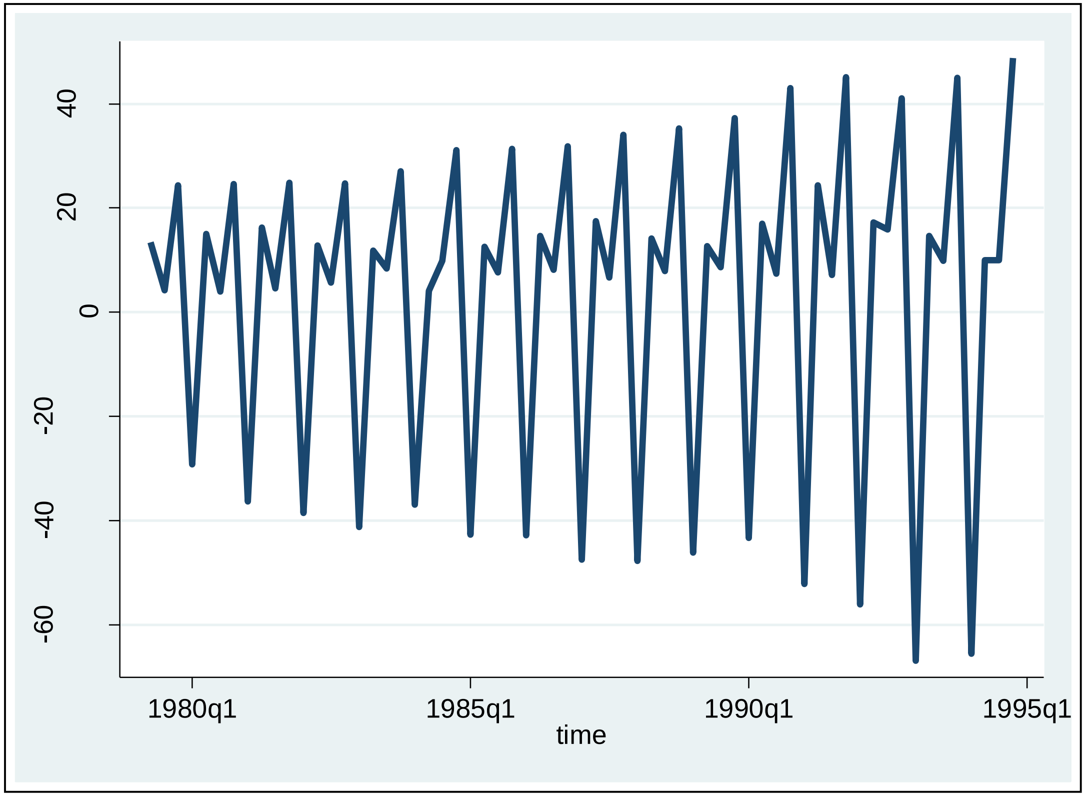
Solution 2.21-4:

Differenced series (`tsline D.lohnsumme`)



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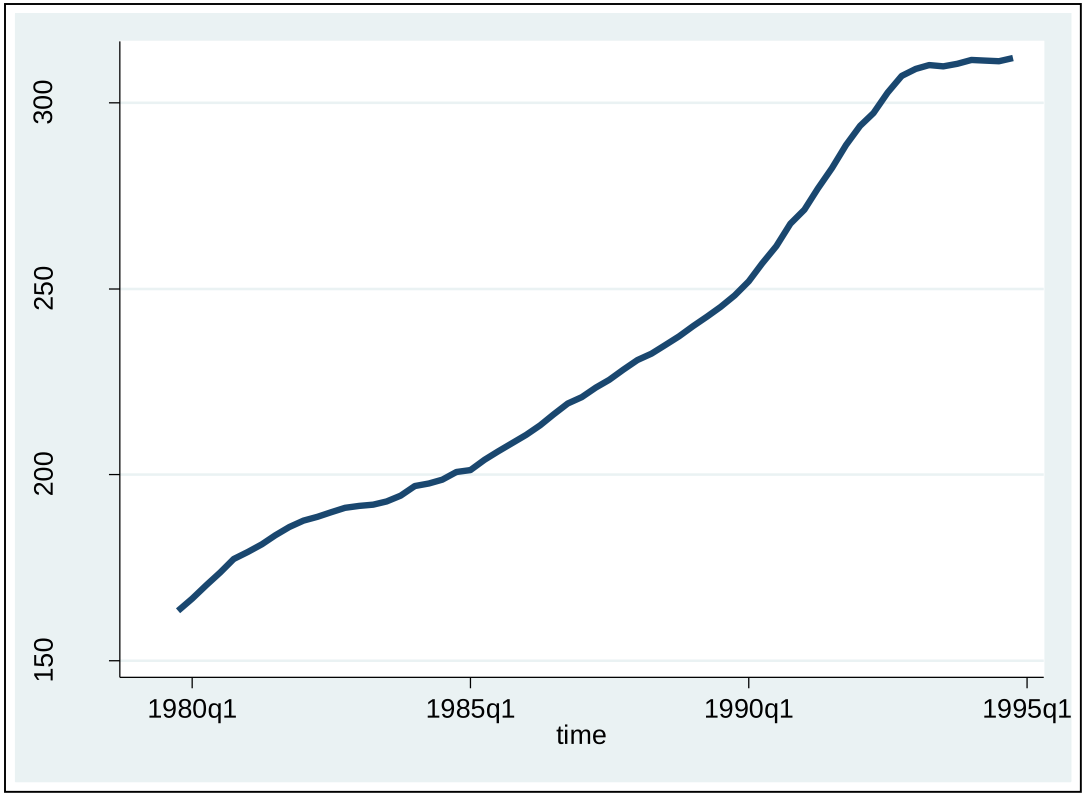
Solution 2.21-5:

Smoothed series (`tsline lohnsumme_ma4`)



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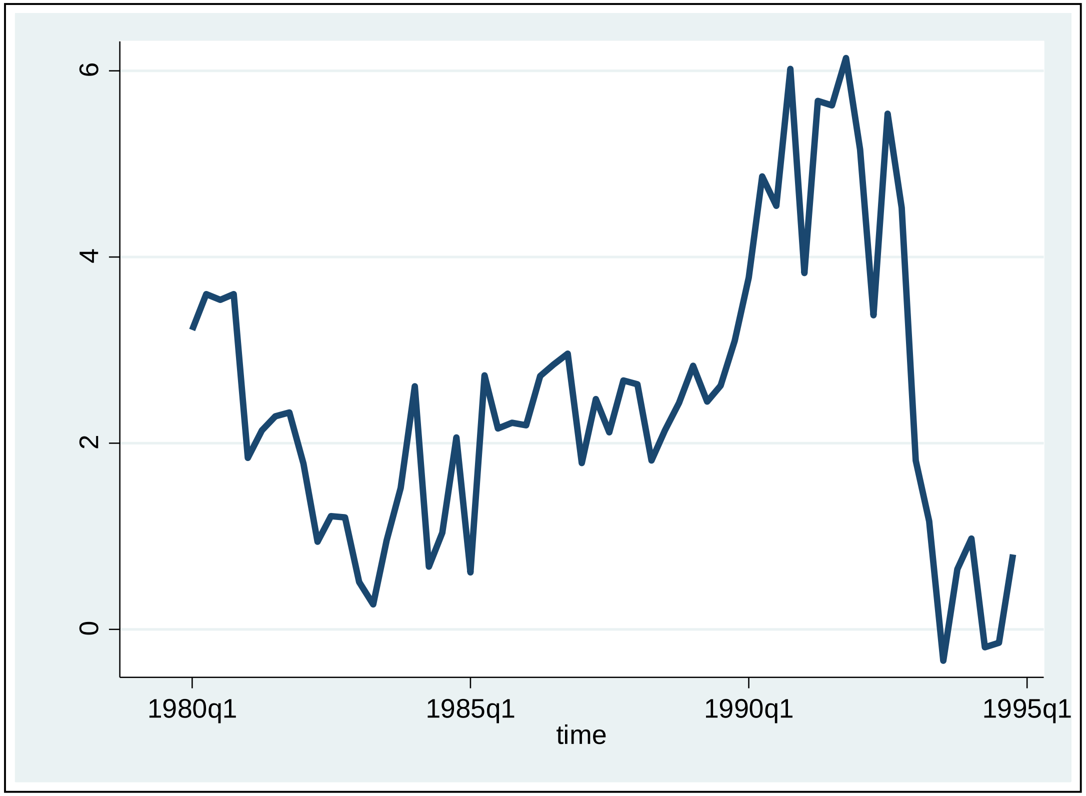
Solution 2.21-6:

Differenced and smoothed series (`tsline D.lohnsumme_ma4`)



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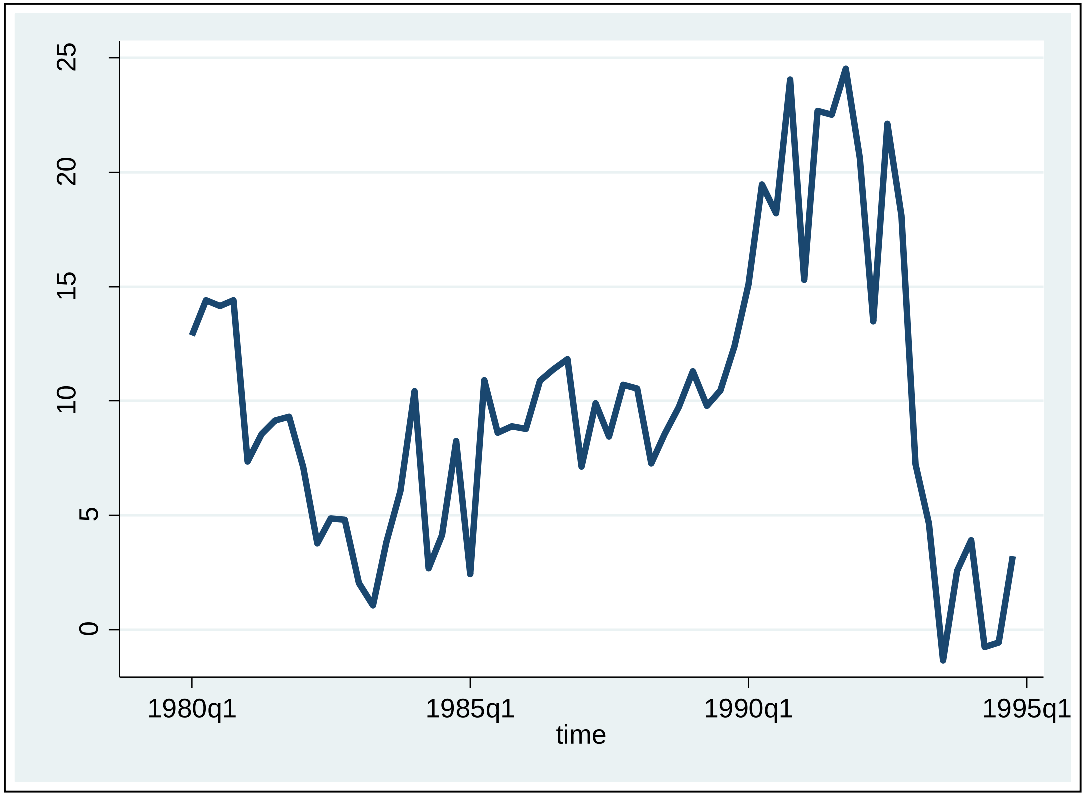
Solution 2.21-7:

Seasonal differenced series (`tsline S4.lohnsumme`)



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Organizational Issues

Please log in to **ISIS** (password: Zeit1718) and **download** the following file:

- ahp.dta

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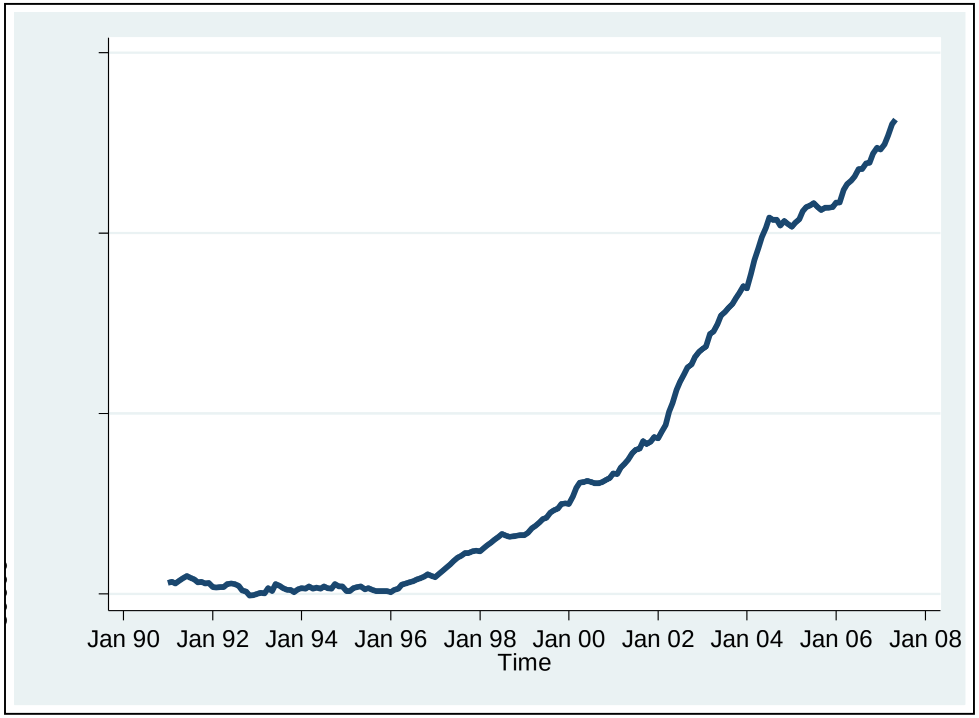
Data set

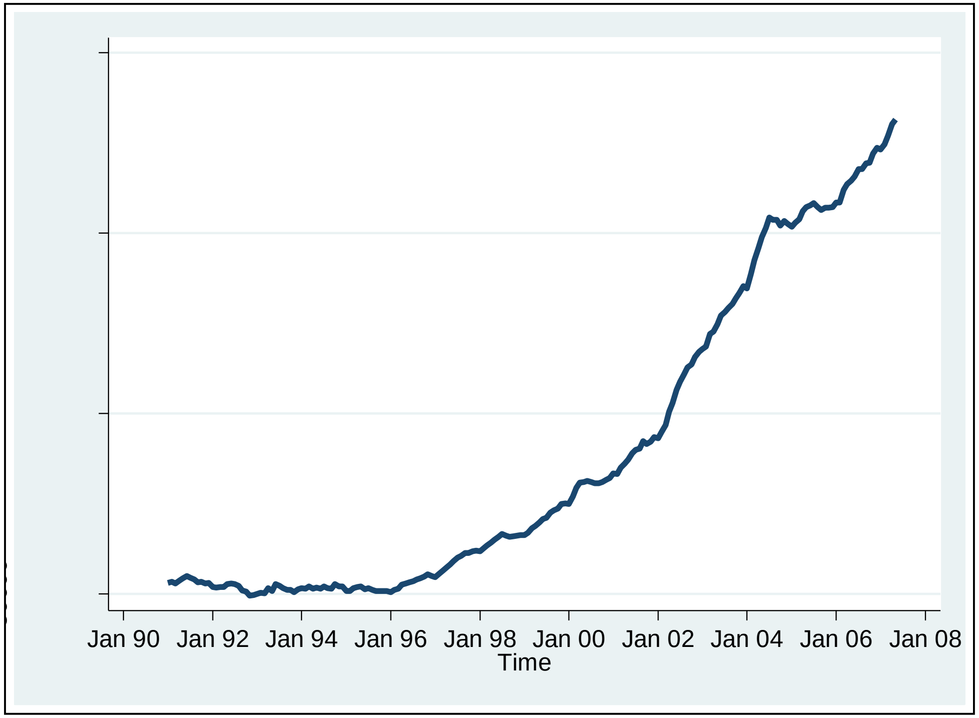
Monthly observations from January 1991 to May 2007 of UK average house prices (`ahp.dta`)




```
.generate time = tm(1991m1)+_n-1
.format time %tm
.tsset time
.tsline ahp
```

Exercise 2.22:

- Does this series look stationary?
- What would the autocorrelation function (ACF) for this series look like?



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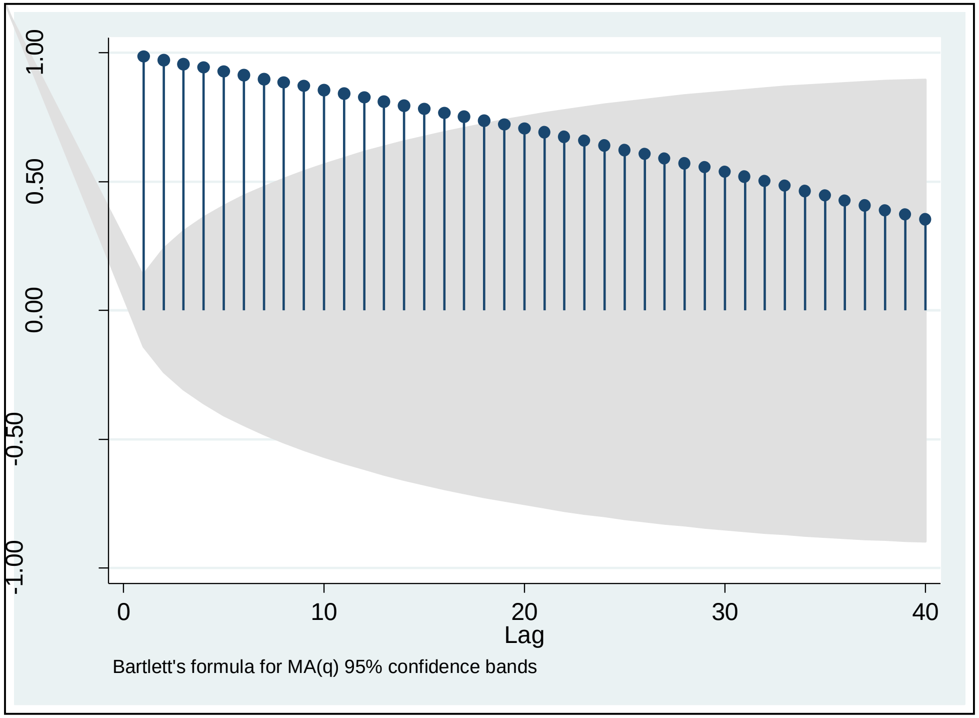
Solution 2.22:

- Does this series look stationary? – **NO** –
- What would the autocorrelation function for this series look like?



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Exercise 2.23:

- Based on the PACF which model seems suitable for the `ahp` series?
- Estimate the model (maximum likelihood).
- Round the estimated coefficients to two decimal places and write down the estimated model in our usual notation and in lag operator notion (both in deviations from the mean).
- Show that the AR polynomial has a unit root.

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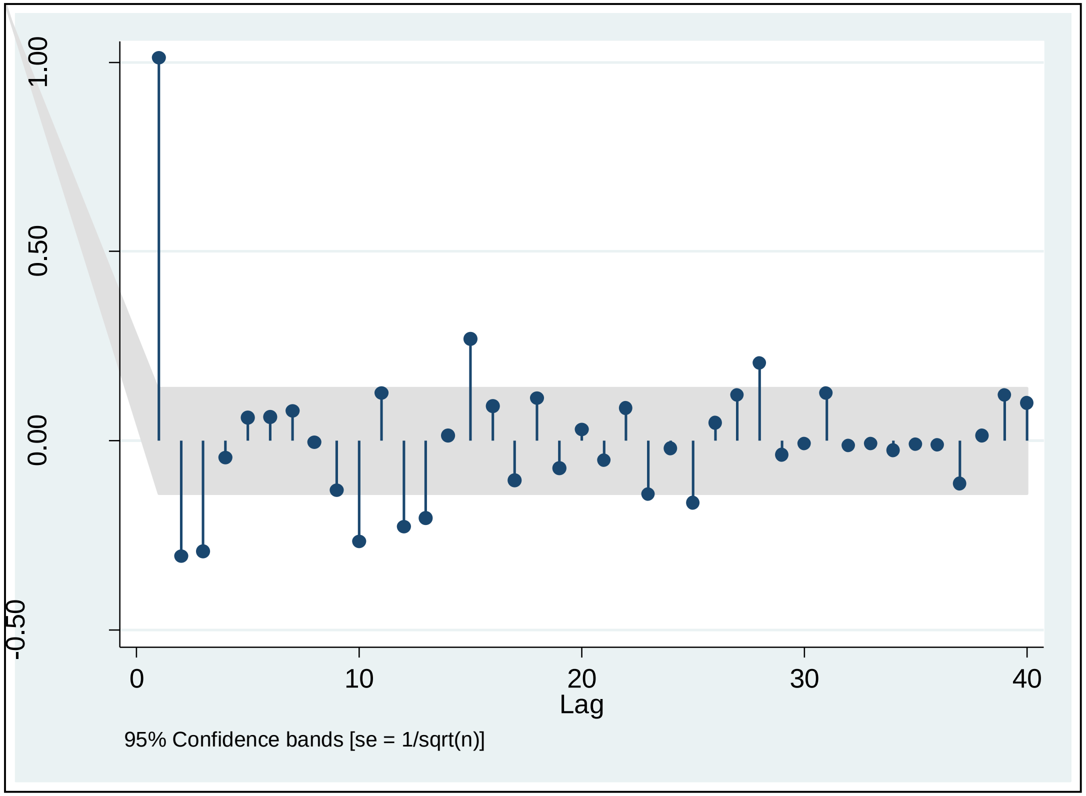
Solution 2.23-1:

PACF



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Solution 2.23-2:

AR(3)

```
ARIMA regression

Sample:  Jan 91 to May 07                       Number of obs      =       197
                                                Wald chi2(3)       = 404294.78
Log likelihood = -1632.543                      Prob > chi2        =    0.0000

------------------------------------------------------------------------------
             |                 OPG
         ahp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ahp          |
       _cons |   117008.8   63528.48     1.84   0.065    -7504.699    241522.4
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   1.341992   .0482209    27.83   0.000      1.24748    1.436503
         L2. |   .0691889    .084424     0.82   0.412     -.096279    .2346568
         L3. |  -.4116066   .0550892    -7.47   0.000    -.5195795   -.3036337
-------------+----------------------------------------------------------------
      /sigma |    938.834   37.79098    24.84   0.000     864.7651    1012.903
------------------------------------------------------------------------------
```

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Solution 2.23-3:

Estimated AR(3) model in our usual notation:

$$\hat{\delta} = \hat{\mu}(1 - \hat{\varphi}_1 - \hat{\varphi}_2 - \hat{\varphi}_3) = 117008.8(1 - 1.34 - 0.07 + 0.41) = 0$$

$$y_t = 1.34 y_{t-1} + 0.07 y_{t-2} - 0.41 y_{t-3} + \hat{\varepsilon}_t$$

Note:
The expression in parentheses is equal to 0, indicating that „there is something wrong" with our model as normally it should hold

$$\mu = \delta / (1 - \varphi_1 - \varphi_2 - \varphi_3)$$

Estimated AR(3) model in lag-operator notation:

$$\tilde{y}_t = (y_t - \hat{\mu}) = (y_t - 117008.8) \text{ - deviation from the mean}$$

$$\left(1 - 1.34L - 0.07L^2 + 0.41L^3\right)\tilde{y}_t = \hat{\varepsilon}_t$$

Checking the correctness of the lag-operator representation:

$$\left(1 - 1.34L - 0.07L^2 + 0.41L^3\right)(y_t - \hat{\mu}) = \hat{\varepsilon}_t$$

$$y_t - 1.34 y_{t\text{-}1} - 0.07 y_{t\text{-}2} + 0.41 y_{t\text{-}3} - \hat{\mu}(1 - 1.34 - 0.07 + 0.41) = \hat{\varepsilon}_t$$

$$y_t = 1.34 y_{t-1} + 0.07 y_{t-2} - 0.41 y_{t-3} + \hat{\varepsilon}_t$$

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Exercise (ahp.dta)

Recall:

$$\underbrace{\left(1-1.34L-0.07L^2+0.41L^3\right)}_{=a_3(L)}\tilde{y}_t=\hat{\varepsilon}_t$$

Solution 2.23-4:

The **lag order polynomial** has a **unit root** if

$$a_p(z)=\left(1-\varphi_1 z-\varphi_2 z^2+\ldots-\varphi_p z^p\right)=0 \quad \text{for} \quad z=1$$

Here, if: $a_3(z)=\left(1-1.34z-0.07z^2+0.41z^3\right)=0 \quad \text{for} \quad z=1$

=> $a_3(1)=\left(1-1.34-0.07+0.41\right)=0$ => the AR polynomial has a **unit root**

Rearrange $a_3(z)=\left(1-\underline{1.34z}-\underline{0.07z^2}+0.41z^3\right)$

$$=\left(\underline{1-1z}\,\underline{-0.34z+0.34z^2}\,\underline{-0.41z^2+0.41z^3}\right)$$

$$=\underline{(1-z)}-0.34z\underline{(1-z)}-0.41z^2\underline{(1-z)}$$

$$=(1-z)(1-0.34z-0.41z^2)$$

=> $a_3(L)=(1-0.34L-0.41L^2)(1-L)$

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Exercise (ahp.dta)

Solution 2.23-5:

Recall:

$$\underbrace{\left(1-1.34L-0.07L^2+0.41L^3\right)}_{=a_3(L)}\tilde{y}_t=\varepsilon_t \quad \& \quad a_3(L)=(1-0.34L-0.41L^2)(1-L)$$

$$(1-0.34L-0.41L^2)\underbrace{(1-L)\tilde{y}_t}_{\substack{=\Delta\tilde{y}_t\\ =\Delta y_t}}=\varepsilon_t$$

Polynomial of order two and such that the roots of the quadratic equation $(1-0.34z-0.41z^2)=0$ are $z_1=1.2012,\quad z_2=-2.031$.

Note: $|z_j|>1$, j = 1, 2

Conclusion: The original process, $\mathbf{y_t}$, was not stationary but the new process, $\mathbf{x_t:=\Delta y_t}$, seems to be stationary.

Exercise 2.24:

- Calculate the monthly percentage change (`dhp`).

$$dhp_t = 100 \cdot \frac{ahp_t - ahp_{t-1}}{ahp_{t-1}}$$

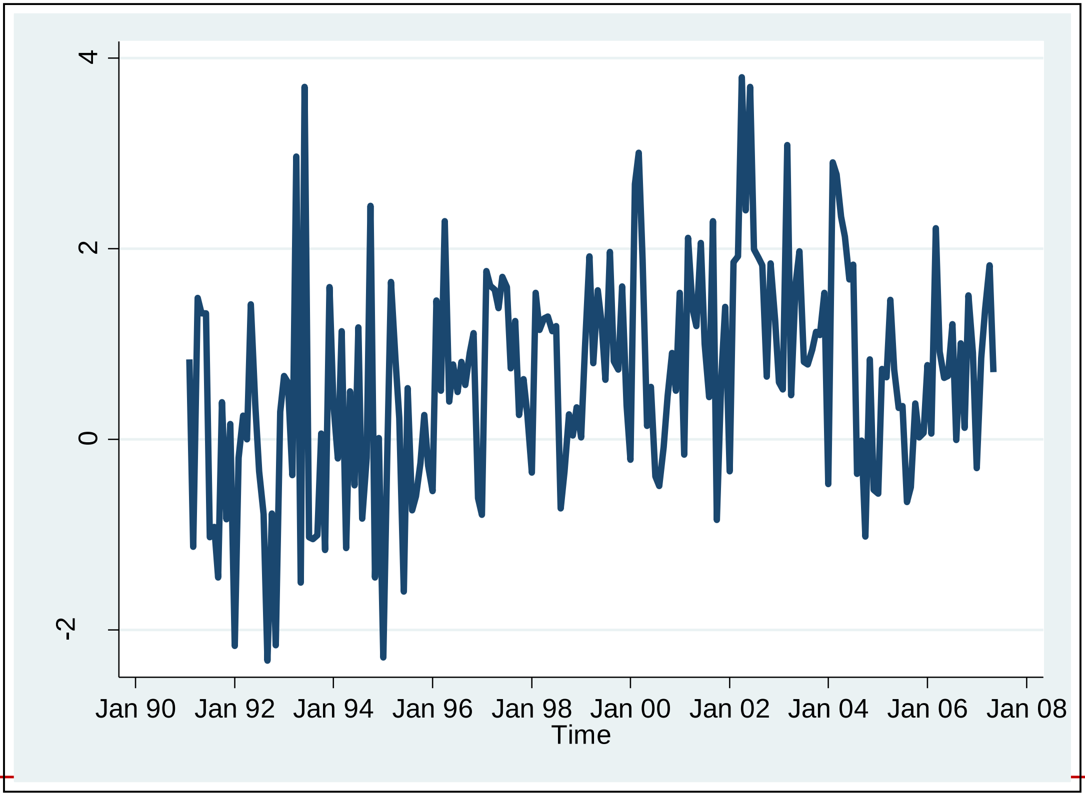
- Plot the `dhp` series.
- Is this series stationary?
- Identify candidate models and estimate them.
- Which of the candidate models is the best in terms of AIC (BIC)?

$$AIC = \log \hat{\sigma}^2 + 2\frac{p+q}{T} \qquad BIC = \log \hat{\sigma}^2 + \frac{p+q}{T}\log T$$

Solution 2.24-1:

Monthly percentage change

```
. gen dhp = 100*(( ahp - L.ahp )/L.ahp)
. tsline dhp, lwidth(thick)
```



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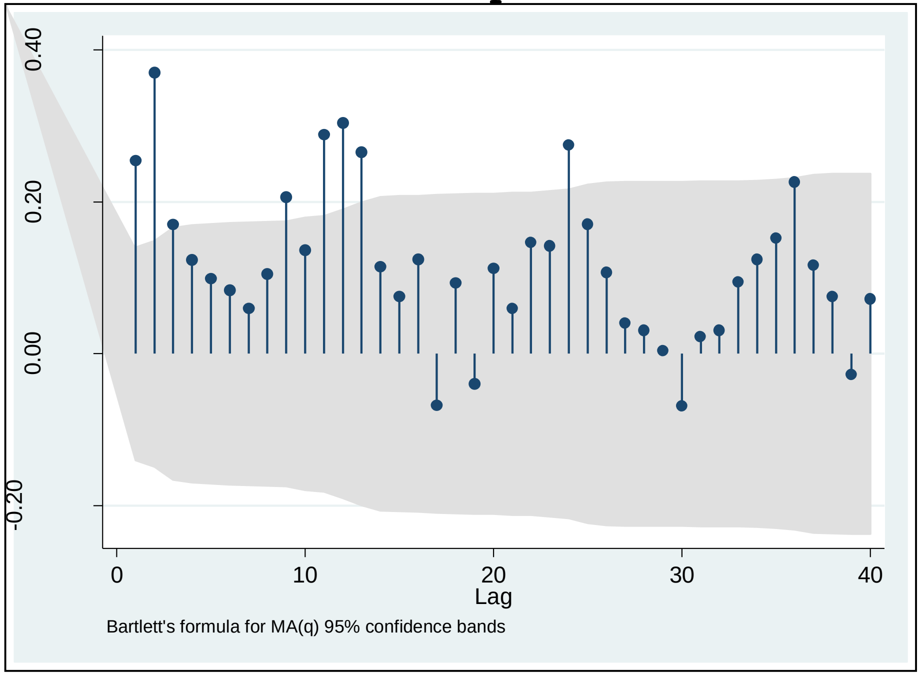
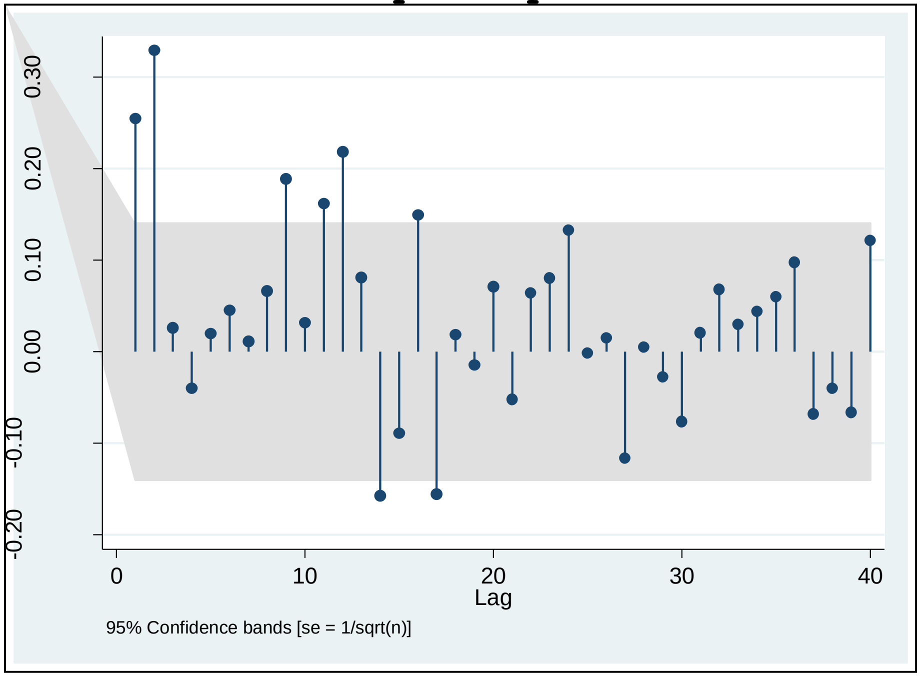
Solution 2.24-2:

ACF and PACF of monthly percentage change

. ac dhp



. pac dhp



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Solution 2.24-3:

Estimation of an **AR(2)** model:

```
Sample:   Feb 91 to May 07                              Number of obs      =        196
                                                        Wald chi2(2)       =      40.65
Log likelihood =    -286.69                             Prob > chi2        =     0.0000
------------------------------------------------------------------------------
             |                 OPG
         dhp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
dhp          |
       _cons |   .6357626   .1490444     4.27   0.000      .343641    .9278843
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .1683468   .0598291     2.81   0.005     .0510839    .2856098
         L2. |    .329707   .0560706     5.88   0.000     .2198106    .4396034
-------------+----------------------------------------------------------------
      /sigma |   1.043935   .0528492    19.75   0.000     .9403523    1.147517
------------------------------------------------------------------------------
```

$$
\begin{aligned}
\hat{y}_t &= \hat{\delta} + \hat{\varphi}_1 y_{t-1} + \hat{\varphi}_2 y_{t-2} \\
&= 0.6358\left(1 - 0.1683 - 0.3297\right) + 0.1683 y_{t-1} + 0.3297 y_{t-2} \\
&= 0.3191 + 0.1683 y_{t-1} + 0.3297 y_{t-2}
\end{aligned}
$$

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Solution 2.24-4:

Estimation of an **ARMA(1,1)** model:

```
Sample:  Feb 91 to May 07                        Number of obs      =       196
                                                 Wald chi2(2)       =    455.46
Log likelihood = -290.2579                       Prob > chi2        =    0.0000
------------------------------------------------------------------------------
             |                 OPG
         dhp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
dhp          |
       _cons |   .6341564   .2159352     2.94   0.003     .2109312    1.057382
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .9181099   .0489344    18.76   0.000     .8222002     1.01402
          ma |
         L1. |  -.7645817    .082139    -9.31   0.000    -.9255711   -.6035923
-------------+----------------------------------------------------------------
      /sigma |   1.063106   .0507664    20.94   0.000     .9636055    1.162606
------------------------------------------------------------------------------
```

$$\hat{y}_t = \hat{\delta} + \hat{\varphi}_1 y_{t-1} - \hat{\theta}_1 \varepsilon_{t-1}$$

$$= 0.6342(1 - 0.9181) + 0.9181 y_{t-1} - 0.7646 \varepsilon_{t-1}$$

$$= 0.0519 + 0.9181 y_{t-1} - 0.7646 \varepsilon_{t-1}$$

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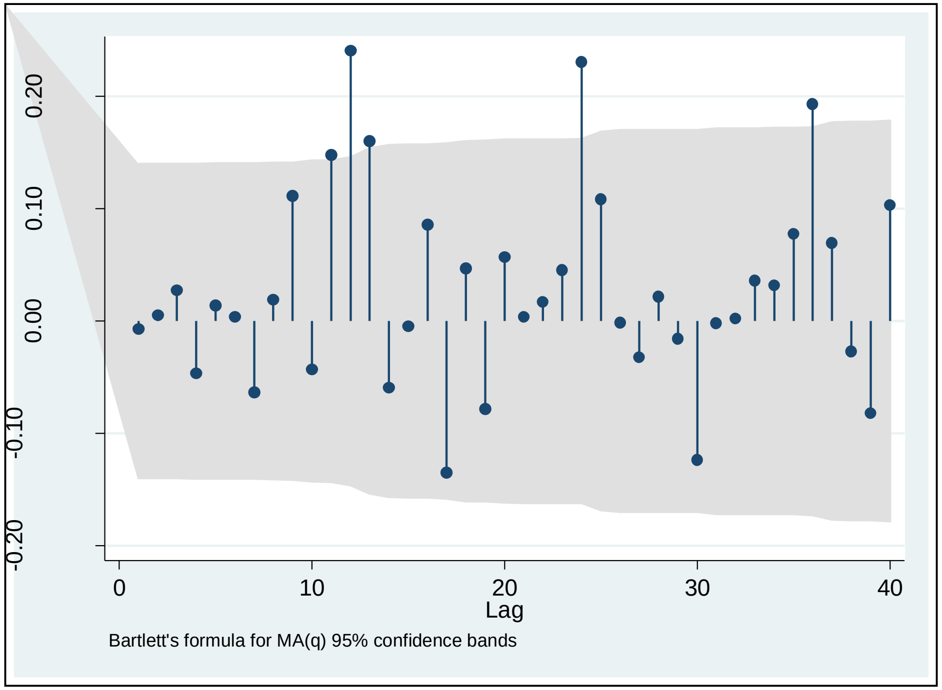
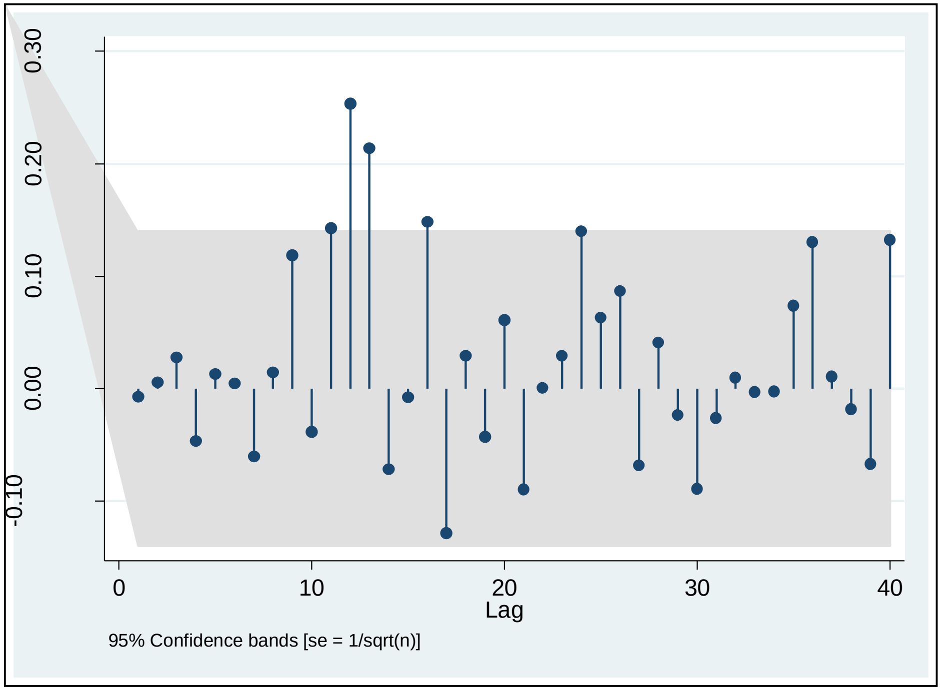
Solution 2.24-5: **ACF** and **PACF** of the **residuals** of the **AR(2)** model:

```
. arima dhp, ar(1/2)
. predict res_AR2, residuals
```

```
. ac res_AR2
```




```
. pac res_AR2
```



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Solution 2.24-6:

```
. corrgram res_AR2
                                          -1       0       1 -1       0       1
 LAG       AC       PAC      Q      Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1        -0.0074  -0.0074   .01091   0.9168          |                  |
2         0.0052   0.0052   .01623   0.9919          |                  |
3         0.0271   0.0274    .1635   0.9833          |                  |
4        -0.0468  -0.0470   .60594   0.9624          |                  |
5         0.0140   0.0128   .64553   0.9858          |                  |
6         0.0034   0.0043   .64794   0.9955          |                  |
7        -0.0634  -0.0606   1.4732   0.9832          |                  |
8         0.0190   0.0142   1.5478   0.9919          |                  |
9         0.1113   0.1184   4.1171   0.9035          |                  |
10       -0.0433  -0.0384   4.5089   0.9215          |                  |
11        0.1478   0.1426   9.0923   0.6134          |-                 |-
12        0.2403   0.2535   21.267   0.0466          |-                 |--
13        0.1601   0.2133   26.705   0.0136          |-                 |-
[...]
40        0.1028   0.1321   71.429   0.0016          |                  |-
```

```
. wntestq res_AR2, lags(12)

Portmanteau test for white noise
---------------------------------------
 Portmanteau (Q) statistic  =    21.2675
 Prob > chi2(12)            =     0.0466
```

$$Q = T(T+2)\sum_{k=1}^{K}\frac{1}{T-k}\hat{\rho}_k^2 \quad \sim \chi^2$$ with $K-p-q$ degrees of freedom

```
. di 1-chi2(12, 21.2675)
.0465979
```

```
. di 1-chi2(10, 21.2675)
.01930359
```

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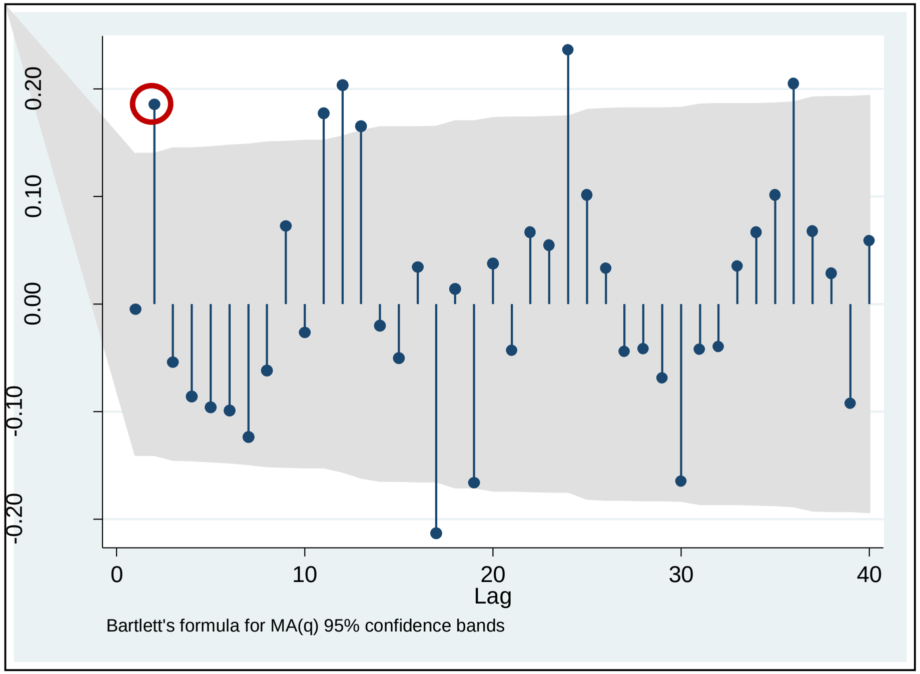
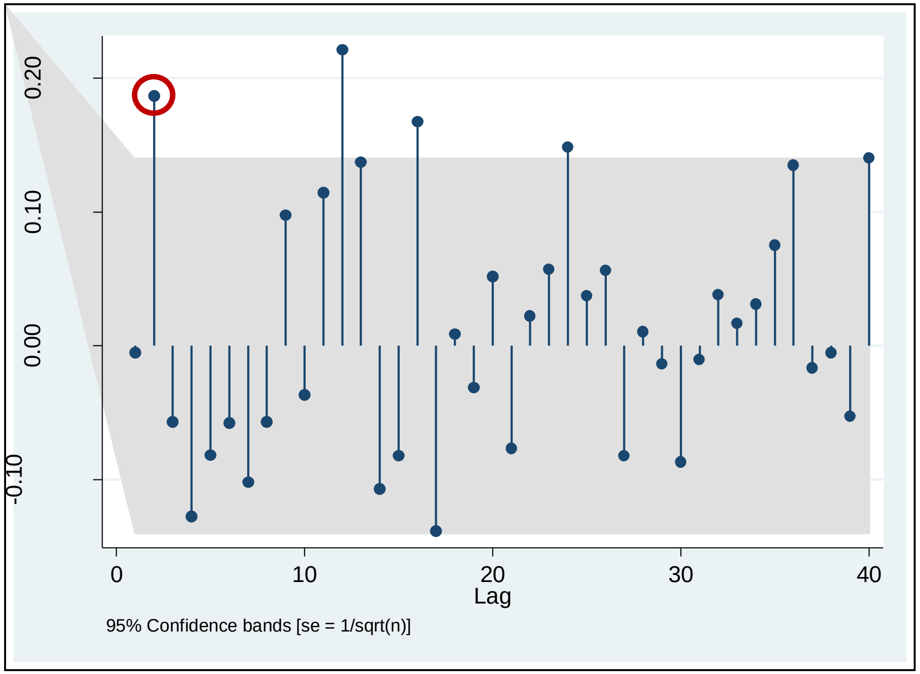
Solution 2.24-7: **ACF** and **PACF** of the **residuals** of **ARMA(1,1)** model:

```
. arima dhp, ar(1) ma(1)
. predict res_ARMA11, residuals
```

```
. ac res_ARMA11
```




```
. pac res_ARMA11
```



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Solution 2.24-8:

Estimation of an **ARMA(2,1)** model:

```
Sample:  Feb 91 to May 07                         Number of obs      =       196
                                                  Wald chi2(3)       =     42.49
Log likelihood = -286.6366                        Prob > chi2        =    0.0000

------------------------------------------------------------------------------
             |                 OPG
         dhp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
dhp          |
       _cons |    .635991   .1524713     4.17   0.000     .3371529    .9348292
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .2253722   .2168956     1.04   0.299    -.1997353    .6504797
         L2. |   .3153073   .0671975     4.69   0.000     .1836026     .447012
          ma |
         L1. |  -.0640492   .2266843    -0.28   0.778    -.5083422    .3802439
-------------+----------------------------------------------------------------
      /sigma |    1.04366   .0528459    19.75   0.000     .9400841    1.147236
------------------------------------------------------------------------------
```

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Solution 2.24-9:

Estimation of the "restricted" **ARMA(2,1)** model: $\varphi_1 = 0$

```
Sample:  Feb 91 to May 07                        Number of obs      =       196
                                                 Wald chi2(2)       =     38.42
Log likelihood = -287.3739                       Prob > chi2        =    0.0000

------------------------------------------------------------------------------
             |                 OPG
         dhp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
dhp          |
       _cons |   .6352416   .1350888     4.70   0.000     .3704725    .9000107
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L2. |   .3555611   .0575959     6.17   0.000     .2426752     .468447
          ma |
         L1. |   .1586386   .0617281     2.57   0.010     .0376538    .2796234
-------------+----------------------------------------------------------------
      /sigma |   1.047677   .0532038    19.69   0.000     .9433994    1.151954
------------------------------------------------------------------------------
```


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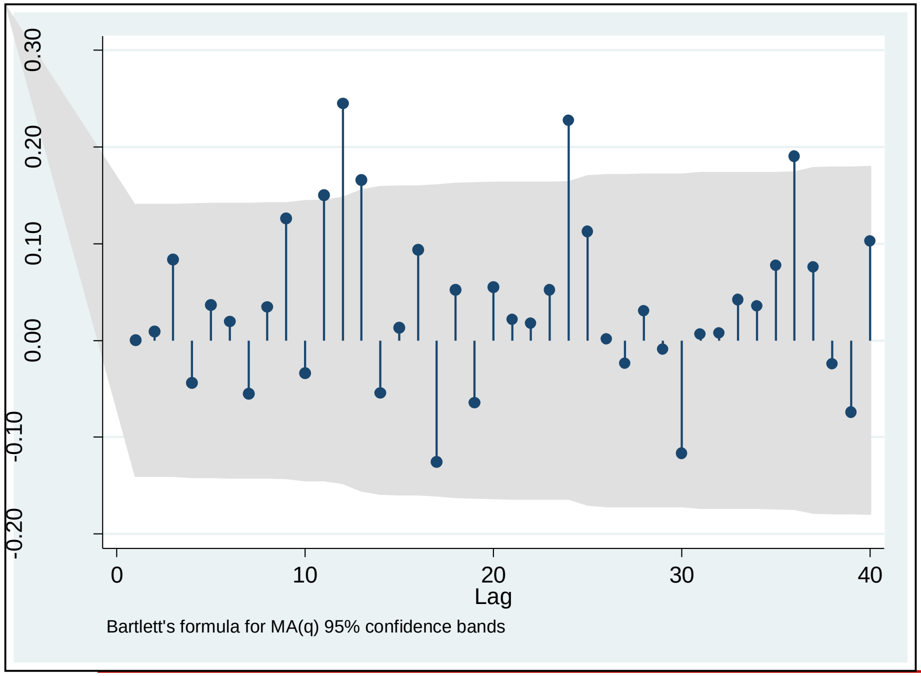
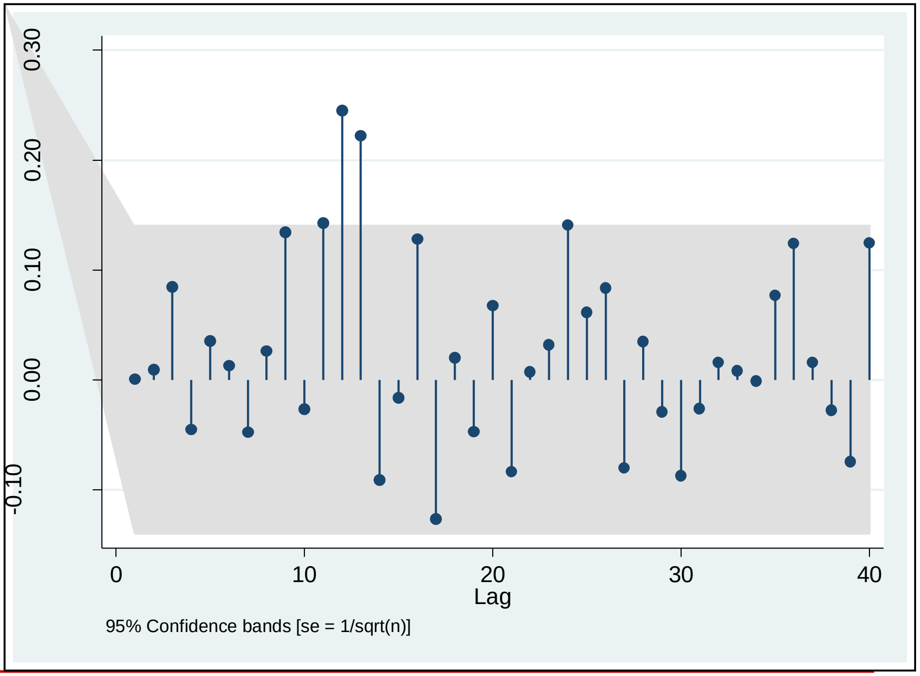
Solution 2.24-10: **ACF** and **PACF** of the **residuals** of the "**restricted**" **ARMA(2,1)** model:

```
. arima dhp, ar(2) ma(1)
. predict res_ARMA21_r, residuals
```

```
. ac res_ARMA21_r
```




```
. pac res_ARMA21_r
```



Solution 2.24-11:

Estimation of an **AR(2)** model:

```
-------------+----------------------------------------------------------------
      /sigma |   1.043935   .0528492    19.75   0.000     .9403523    1.147517
------------------------------------------------------------------------------
```

$$AIC = \log \hat{\sigma}^2 + 2\frac{p+q}{T} = \log\left(1.043935^2\right) + 2\frac{2+0}{196} = 0.10640231$$

$$BIC = \log \hat{\sigma}^2 + \frac{p+q}{T}\log T = \log\left(1.043935^2\right) + \frac{2+0}{196}\log(196) = 0.13985246$$

Estimation of the “restricted” **ARMA(2,1)** model:

```
-------------+----------------------------------------------------------------
      /sigma |   1.047677   .0532038    19.69   0.000     .9433994    1.151954
------------------------------------------------------------------------------
```

$$AIC = \log \hat{\sigma}^2 + 2\frac{p+q}{T} = \log\left(1.047677^2\right) + 2\frac{1+1}{196} = 0.11355855$$

$$BIC = \log \hat{\sigma}^2 + \frac{p+q}{T}\log T = \log\left(1.047677^2\right) + \frac{1+1}{196}\log(196) = 0.1470087$$

Solution 2.24-12:

	AR(2)	AR(1) MA(1)	AR(1/2) MA(1)	AR(2) MA(1)	AR(1) MA(1/2)	AR(1) MA(2)
AIC	.10640231	14279733	.11608015	.11355855	.11804919	.1173732
BIC	.13985246	.17624748	.16625538	.1470087	.16822441	.15082335

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Additional Information

You can use STATA to calculate the general AIC and BIC
After the estimation of the **„restricted“ ARMA(2,1)** you would get:

```
. estat ic
-----------------------------------------------------------------------------
       Model |    Obs    ll(null)   ll(model)     df          AIC         BIC
-------------+---------------------------------------------------------------
           . |    196           .   -287.3739      4     582.7479    595.8603
-----------------------------------------------------------------------------
```

STATA uses the general formulas (k=# of parameters and N=# of obs):

$$BIC = -2 \cdot \ln(likelihood) + \ln(N) \cdot k$$

$$AIC = -2 \cdot \ln(likelihood) + 2 \cdot k$$

For the AIC it would be (k=4, since 4 Parameters were estimated [$\hat{\mu}, \hat{\theta}_1, \hat{\varphi}_2, \hat{\sigma}_\varepsilon$]):

```
. di (-2)*-287.3739 +2*4
582.7478
```

The general AIC by Akaike from 1973 was stated as:

$$AIC = -\frac{2}{T} \cdot \ln(likelihood) + \frac{2}{T} \cdot (\#\,parameters)$$

If we assume Gaussian White Noise for the residuals (what we do, if we estimate our model with the Maximum Likelihood Method), it is shown that the AIC reduces to our formular:

$$\boxed{AIC = \log \hat{\sigma}^2 + 2\frac{p+q}{T}}$$

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Exercise 2.25:

- Estimate an AR(2) model for the `dhp` series (ML-Estimation).
- Estimate an AR(2) model for the `dhp` series (ML-Estimation) without observations from 2007 (i.e. exclude the last five observations from your analysis sample).
- What is the optimal forecast? In which sense is it optimal? What “assumptions” are part of the information set?
- Calculate forecasts for January up to May 2007.
- Compare these forecasts to the actual values and compute the forecast errors.
- What is the forecast for May 2010?

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Solution 2.25-1:

ML-Estimation of an AR(2) model for the `dhp` series:

```
. arima dhp, ar(1/2)
[...]
Sample:  Feb 91 to May 07                        Number of obs      =        196
                                                 Wald chi2(2)       =      40.65
Log likelihood =   -286.69                       Prob > chi2        =     0.0000
------------------------------------------------------------------------------
             |                 OPG
         dhp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
dhp          |
       _cons |   .6357626   .1490444     4.27   0.000      .343641    .9278843
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .1683468   .0598291     2.81   0.005     .0510839    .2856098
         L2. |    .329707   .0560706     5.88   0.000     .2198106    .4396034
-------------+----------------------------------------------------------------
      /sigma |   1.043935   .0528492    19.75   0.000     .9403523    1.147517
------------------------------------------------------------------------------
```

$$\hat{y}_t = 0.3191 + 0.1683 y_{t-1} + 0.3297 y_{t-2}$$

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Solution 2.25-2:

ML-Estimation of an AR(2) model for the `dhp` series (2007 excluded):

```
. arima dhp if time < 564, ar(1/2)   or   . arima dhp if time <= tm(2006m12), ar(1/2)
[...]
Sample:  Feb 91 to Dec 06                        Number of obs      =        191
                                                 Wald chi2(2)       =      41.24
Log likelihood = -280.1191                       Prob > chi2        =     0.0000
------------------------------------------------------------------------------
             |                 OPG
         dhp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
dhp          |
       _cons |   .6291755   .1536857     4.09   0.000      .327957    .9303939
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .1665986   .0604824     2.75   0.006     .0480552     .285142
         L2. |   .3380952     .05664     5.97   0.000     .2270828    .4491076
-------------+----------------------------------------------------------------
      /sigma |   1.047946   .0536005    19.55   0.000     .9428913    1.153001
------------------------------------------------------------------------------
```

$$\hat{y}_t = 0.3116 + 0.1666 y_{t-1} + 0.3381 y_{t-2}$$

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Solution 2.25-3:

Optimal forecast:

$$\tilde{y}_{T+1} \mid \Omega_T = E\left(Y_{T+1} \mid \Omega_T\right)$$

it minimizes the expected squared forecast error

$$\min E\left(e_{T+1}^2\right)$$

$$e_{T+1} = y_{T+1} - \tilde{y}_{T+1} \mid \Omega_T$$

Information set Ω_T:

- true model
- known parameters
- all past observations

Additional assumption:

$$E\left[\varepsilon_{T+k}\right] = 0$$

$$\forall k \geq 1$$

Solution 2.25-4:

One-period ahead forecast of an AR(2):

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \delta + \varepsilon_t \qquad \Rightarrow y_{T+1} = \varphi_1 y_T + \varphi_2 y_{T-1} + \delta + \varepsilon_{T+1}$$

$$\tilde{y}_{T+1} = E(y_{T+1} | \Omega_T)$$

$$\tilde{y}_{T+1} = E(\varphi_1 y_T + \varphi_2 y_{T-1} + \delta + \varepsilon_{T+1} | \Omega_T)$$

$$\tilde{y}_{T+1} = \varphi_1 y_T + \varphi_2 y_{T-1} + \delta$$

$$\Omega_T = \{y_T, \ldots, y_1; y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \delta + \varepsilon\}$$

$$\tilde{y}_{T+1} = 0.3116 + 0.1666 y_T + 0.3381 y_{T-1}$$

$$\tilde{y}_{Jan2007} = 0.3116 + 0.1666 y_{Dec2006} + 0.3381 y_{Nov2006}$$

$$\tilde{y}_{Jan2007} = 0.3116 + 0.1666 \cdot 0.9066 + 0.3381 \cdot 1.5104$$

$$\tilde{y}_{Jan2007} = 0.9733$$

```
. list time  dhp in 188/197

     +---------------------+
     |   time          dhp |
     |---------------------|
188. | Aug 06    -.0071542 |
189. | Sep 06     1.008818 |
190. | Oct 06     .1239574 |
191. | Nov 06     1.510408 |
192. | Dec 06      .906583 |
     |---------------------|
193. | Jan 07     -.299863 |
194. | Feb 07     .8549574 |
195. | Mar 07     1.360572 |
196. | Apr 07     1.824568 |
197. | May 07     .7043269 |
     +---------------------+
```

Solution 2.25-5:

Two-period ahead forecast of an AR(2):

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \delta + \varepsilon_t \quad \Rightarrow y_{T+2} = \varphi_1 y_{T+1} + \varphi_2 y_T + \delta + \varepsilon_{T+2}$$

$$\tilde{y}_{T+2} = E\left(y_{T+2} | \Omega_T\right)$$

$$\tilde{y}_{T+2} = E\left(\varphi_1 y_{T+1} + \varphi_2 y_T + \delta + \varepsilon_{T+2} \mid \Omega_T\right)$$

$$\tilde{y}_{T+2} = \varphi_1 \tilde{y}_{T+1} + \varphi_2 y_T + \delta$$

$$\tilde{y}_{Feb2007} = 0.3116 + 0.1666 \cdot 0.9733 + 0.3381 \cdot 0.9066$$

$$\tilde{y}_{Feb2007} = 0.7803$$

```
. list time  dhp in 188/197

     +---------------------+
     |   time          dhp |
     |---------------------|
188. | Aug 06   -.0071542 |
189. | Sep 06    1.008818 |
190. | Oct 06    .1239574 |
191. | Nov 06    1.510408 |
192. | Dec 06     .906583 |
     |---------------------|
193. | Jan 07    -.299863 |
194. | Feb 07    .8549574 |
195. | Mar 07    1.360572 |
196. | Apr 07    1.824568 |
197. | May 07    .7043269 |
     +---------------------+
```

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Solution 2.25-6:

Three-period ahead forecast of an AR(2):

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \delta + \varepsilon_t \quad \Rightarrow y_{T+3} = \varphi_1 y_{T+2} + \varphi_2 y_{T+1} + \delta + \varepsilon_{T+3}$$

$$\tilde{y}_{T+3} = E\left(y_{T+3} | \Omega_T\right)$$

$$\tilde{y}_{T+3} = E\left(\varphi_1 y_{T+2} + \varphi_2 y_{T+1} + \delta + \varepsilon_{T+3} \,|\, \Omega_T\right)$$

$$\tilde{y}_{T+3} = \varphi_1 \tilde{y}_{T+2} + \varphi_2 \tilde{y}_{T+1} + \delta$$

$$\tilde{y}_{Mar\,2007} = 0.7707$$

Four- and five-period ahead forecast of an AR(2):

$$\tilde{y}_{T+4} = \varphi_1 \tilde{y}_{T+3} + \varphi_2 \tilde{y}_{T+2} + \delta$$

$$\tilde{y}_{Apr\,2007} = 0.7039$$

$$\tilde{y}_{T+5} = \varphi_1 \tilde{y}_{T+4} + \varphi_2 \tilde{y}_{T+3} + \delta$$

$$\tilde{y}_{May\,2007} = 0.6895$$

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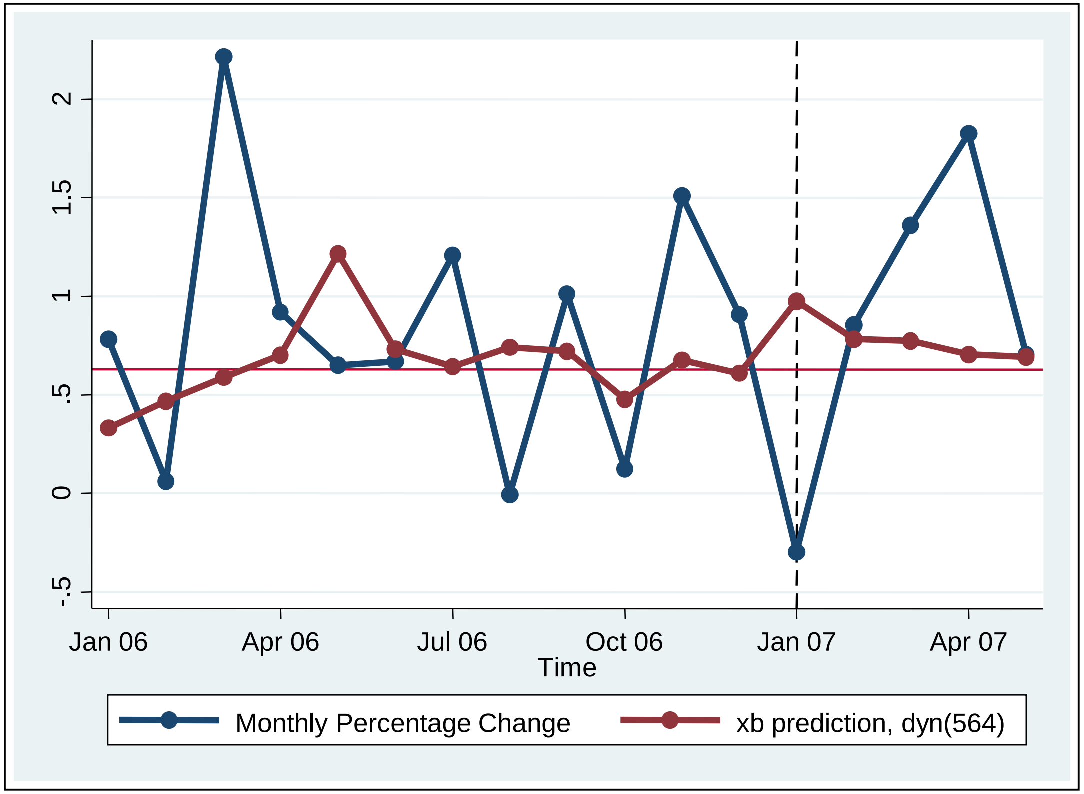
Solution 2.25-7:

l-period ahead forecast of an AR(2): $\hat{y}_t = 0.3116 + 0.1666 y_{t-1} + 0.3381 y_{t-2}$



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Solution 2.25-8:

```
. predict forecast_AR2, xb

. predict forecast_AR2_dyn, xb dynamic(564)

. list time dhp forecast_AR2 forecast_AR2_dyn in 193/197

     +----------------------------------------------+
     |   time          dhp   foreca~2    foreca~n |
     |----------------------------------------------|
193. | Jan 07    -.299863   .9733318    .9733318 |
194. | Feb 07    .8549574   .5681891    .7803016 |
195. | Mar 07    1.360572    .352687    .7707105 |
196. | Apr 07    1.824568   .8273608      .70385 |
197. | May 07    .7043269   1.075608    .6894684 |
     +----------------------------------------------+
```

Recall:

$$\tilde{y}_{Jan\,2007} = 0.9733$$
$$\tilde{y}_{Feb\,2007} = 0.7803$$
$$\tilde{y}_{Mar\,2007} = 0.7707$$
$$\tilde{y}_{Apr\,2007} = 0.7039$$
$$\tilde{y}_{May\,2007} = 0.6895$$

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Solution 2.25-9:

```
. gen forecast_error_AR2 = (dhp-forecast_AR2_dyn) / dhp in 193/197

. list time dhp forecast_AR2_dyn forecast_error_AR2 in 193/197

     +--------------------------------------------+
     |   time        dhp   fo~t_AR2    fo~r_AR2 |
     |--------------------------------------------|
193. | Jan 07   -.299863   .9733318    4.245922 |
194. | Feb 07   .8549574   .7803016     .087321 |
195. | Mar 07   1.360572   .7707105    .4335392 |
196. | Apr 07   1.824568     .70385    .6142375 |
197. | May 07   .7043269   .6894684    .0210959 |
     +--------------------------------------------+
```

Solution 2.25-10:

```
. list time xb_AR2_dyn_long in 193/233

     +--------------------+          |--------------------|
     |   time    xb_AR2~g |     213. | Sep 08    .6292692 |
     |--------------------|     214. | Oct 08    .6292383 |
193. | Jan 07    .9733318 |     215. | Nov 08    .6292176 |
194. | Feb 07    .7803016 |     216. | Dec 08    .6292037 |
195. | Mar 07    .7707105 |     217. | Jan 09    .6291944 |
196. | Apr 07      .70385 |          |--------------------|
197. | May 07    .6894684 |     218. | Feb 09    .6291882 |
     |--------------------|     219. | Mar 09     .629184 |
198. | Jun 07    .6644673 |     220. | Apr 09    .6291812 |
199. | Jul 07    .6554398 |     221. | May 09    .6291793 |
200. | Aug 07    .6454831 |     222. | Jun 09     .629178 |
201. | Sep 07    .6407721 |          |--------------------|
202. | Oct 07     .636621 |     223. | Jul 09    .6291772 |
     |--------------------|     224. | Aug 09    .6291766 |
203. | Nov 07    .6343367 |     225. | Sep 09    .6291763 |
204. | Dec 07    .6325526 |     226. | Oct 09     .629176 |
205. | Jan 08    .6314831 |     227. | Nov 09    .6291758 |
206. | Feb 08    .6307017 |          |--------------------|
207. | Mar 08    .6302099 |     228. | Dec 09    .6291757 |
     |--------------------|     229. | Jan 10    .6291756 |
208. | Apr 08    .6298638 |     230. | Feb 10    .6291755 |
209. | May 08    .6296399 |     231. | Mar 10    .6291755 |
210. | Jun 08    .6294855 |     232. | Apr 10    .6291755 |
211. | Jul 08    .6293842 |          |--------------------|
212. | Aug 08    .6293151 |     233. | May 10    .6291755 |
```

Recall:

$$\hat{\mu} = 0.6291755$$

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