a)

$$H = -\int \exp(s) \log \exp(s)\, dx = -\int \exp(s)\, s\, dx$$

$$\Lambda = -\int \exp(s)\, s\, dx - \lambda_1 \left(\int \exp(s)\, dx - 1 \right) - \lambda_2 \int x \exp(s)\, dx - \lambda_3 \left[\int x^2 \exp(s)\, dx - \int x \exp(s)\, dx - \sigma^2 \right]$$

b)

$$\frac{\partial \Lambda}{\partial s} = -\exp(s)\, s - \exp(s) - \lambda_1 \exp(s) - \lambda_2\, x \exp(s) - \lambda_3 \left[x^2 \exp(s) - x \exp(s) \right] = 0$$

quadratic in x

(c) $\Lambda = -\int p(x) \ln p(x) . dx + \lambda_1 \left(\int p(x) . dx - 1 \right) + \lambda_2 \int p(x) . x . dx$
$+ \lambda_3 \left(\int p(x) . x^2 . dx - \sigma^2 \right)$

$$\frac{\partial \Lambda}{\partial p(x)} = \int \ln p(x) + \frac{p(x)}{p(x)} + \lambda_1 + \lambda_2 x + \lambda_3 x^2 . dx = 0$$

$$\ln P(x) + 1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 . \cancel{dx} = 0$$

$$P(x) = e^{-1 - \lambda_1 - \lambda_2 x - \lambda_3 x^2}$$

To show $P(x)$ is Gaussian distributed

Choose $\lambda_1, \lambda_2, \lambda_3$ as

$$\lambda_1 = -1 - \ln \frac{1}{\sqrt{2\pi}(\sigma)}$$

$$\lambda_2 = 0$$

$$\lambda_3 = \frac{1}{2\sigma^2}$$

plugging these values in $p(x)$ we get

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{x^2/2\sigma^2}$$

then

$$\frac{\partial \Lambda}{\partial p(x)} = \ln (P(x)) + 1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2$$

$$= \ln \left(e^{-1 - \lambda_1 - \lambda_2 x - \lambda_3 x^2} \right) + 1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2$$

$$= \ln \left(e^{-1 + 1 + \ln \frac{1}{\sqrt{\sigma^2 2\pi}} + 0 - \frac{x^2}{2\sigma^2}} \right) + 1 \left(-1 - \ln \frac{1}{\sqrt{\sigma^2 2\pi}} \right.$$
$$\left. + 0 + \frac{x^2}{2\sigma^2} \right.$$

$$= \ln \left(\frac{1}{\sqrt{\sigma^2 2\pi}} \right) - \frac{x^2}{2\sigma^2} - \frac{\ln}{\sqrt{\sigma^2 2\pi}} + \frac{x^2}{2\sigma^2} = 0$$

$\therefore$ we can show $P(x)$ is gaussian distributed

If ~~f is~~ a random variable with pdf $p(x)$ has zero mean and variance σ^2, then

$$\left[\text{and } \varphi(x) \sim N(0, \sigma^2)\right]$$

$$-\int p(x) \log \varphi(x)\, dx$$

$$= -\int p(x) \left(\log \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{x^2}{2\sigma^2} \right) dx$$

$$= \frac{1}{2} \log (2\pi\sigma^2) + \frac{E_p[x^2]}{2\sigma^2} = \frac{1}{2} \log (2\pi e \sigma^2)$$

Now it is also true that if X has pdf $p(x)$ with zero mean & variance σ^2, then

$$\int p(x) \log \varphi(x)\, dx - \underbrace{\int p(x) \log p(x)\, dx}_{H(X)}$$

$$= \int p(x) \log \frac{\varphi(x)}{p(x)}\, dx \geq \log \int p(x) \frac{\varphi(x)}{p(x)}\, dx \quad \left[\begin{array}{l}\text{Jensen's inequality,}\\ \log \text{ is convex}\end{array}\right]$$

$$= \log \int \varphi(x)\, dx = 0$$

$$\Rightarrow H(X) \leq \int p(x) \log \varphi(x)\, dx = \frac{1}{2} \log(2\pi e \sigma^2)$$

We just showed that if $x^* \sim N(0,\sigma^2)$

then $H(x^*) = \frac{1}{2}\log(2\pi e\sigma^2)$.

We also showed that for a random var

with zero mean and variance σ^2 it holds that

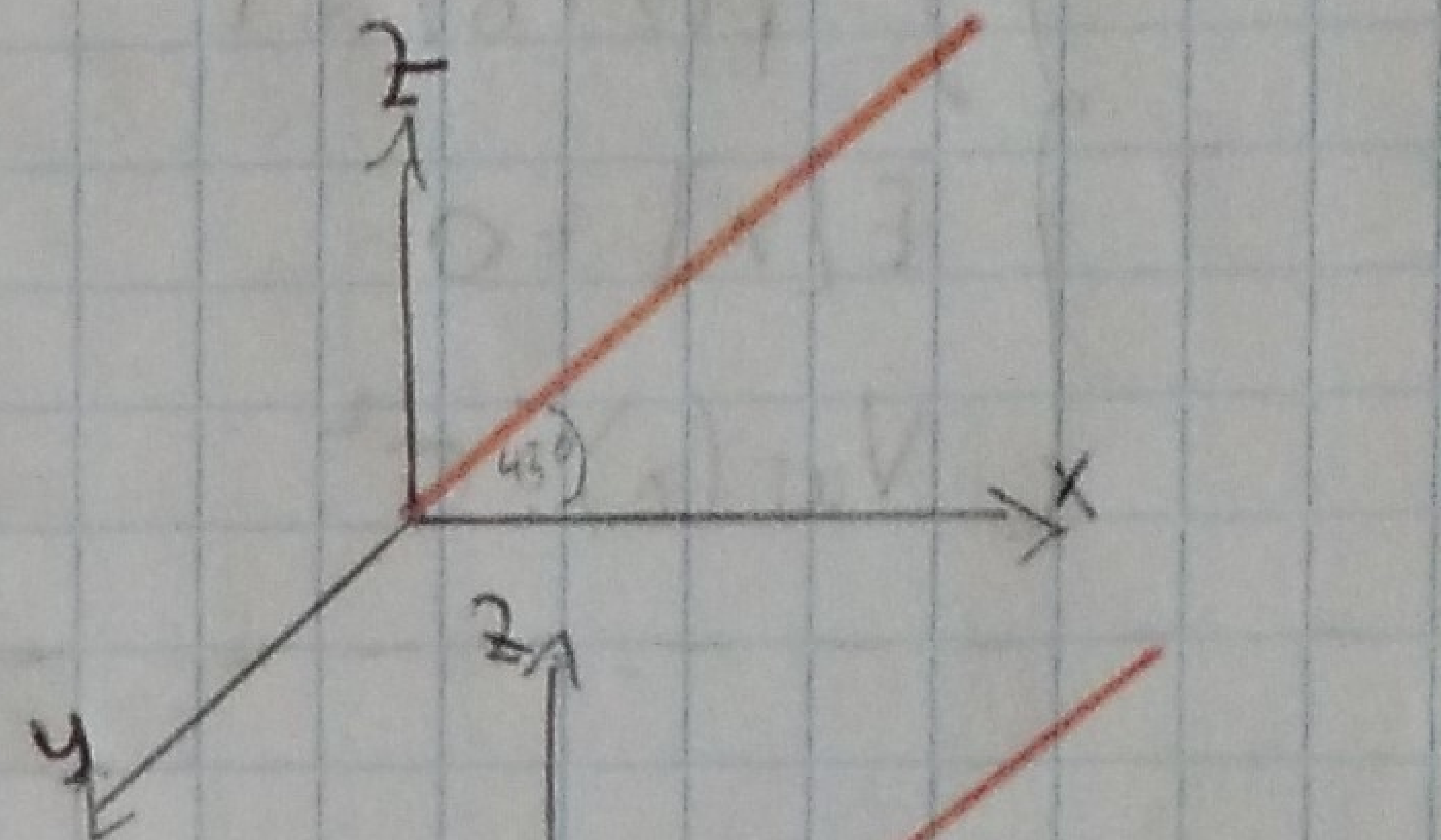
$$H(x) \leq \frac{1}{2}\log(2\pi e\sigma^2).$$

Therefore $J(x) = H(x^*) - H(x) \geq 0$

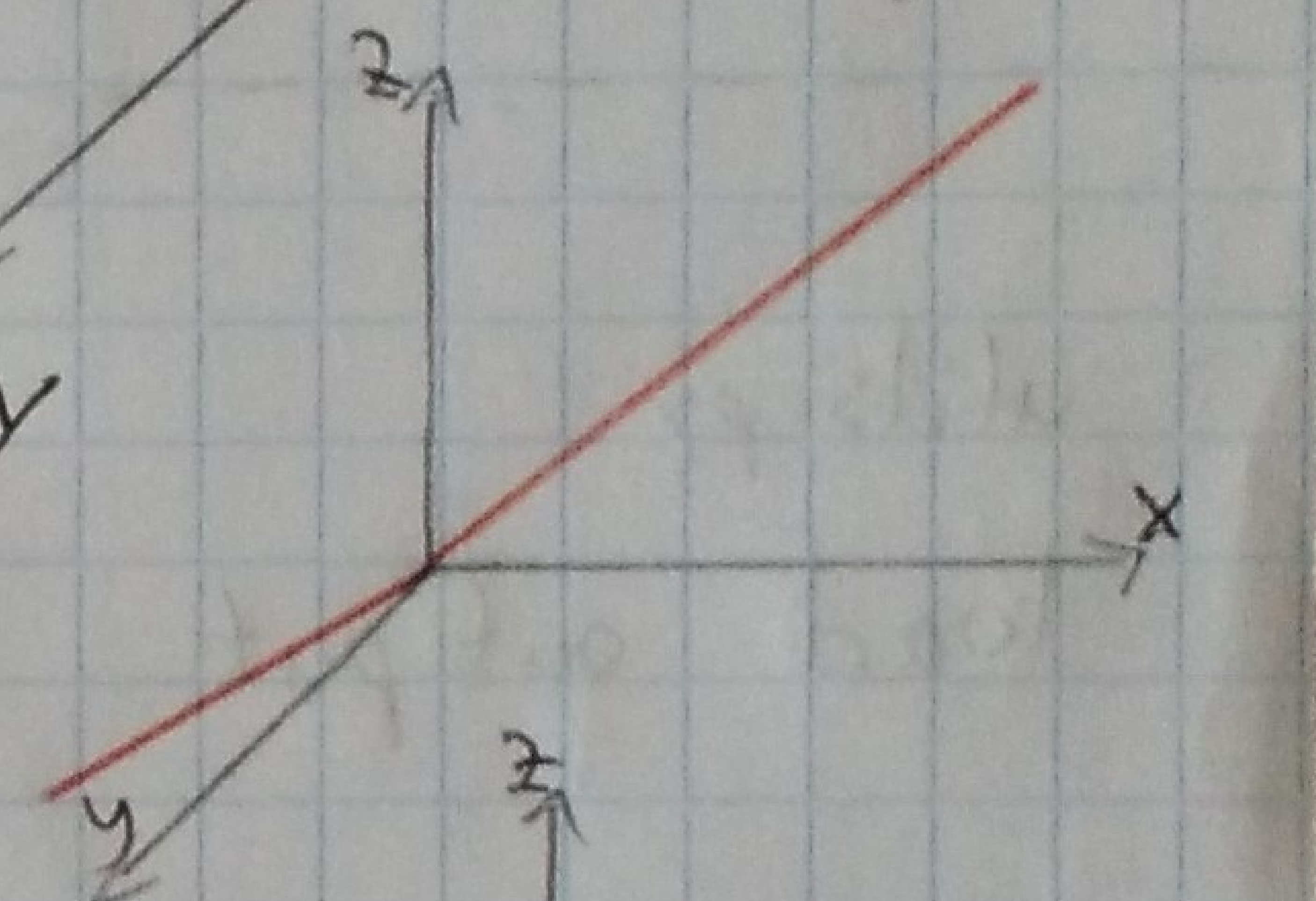
and equal if x Gaussian.

Exercise 2

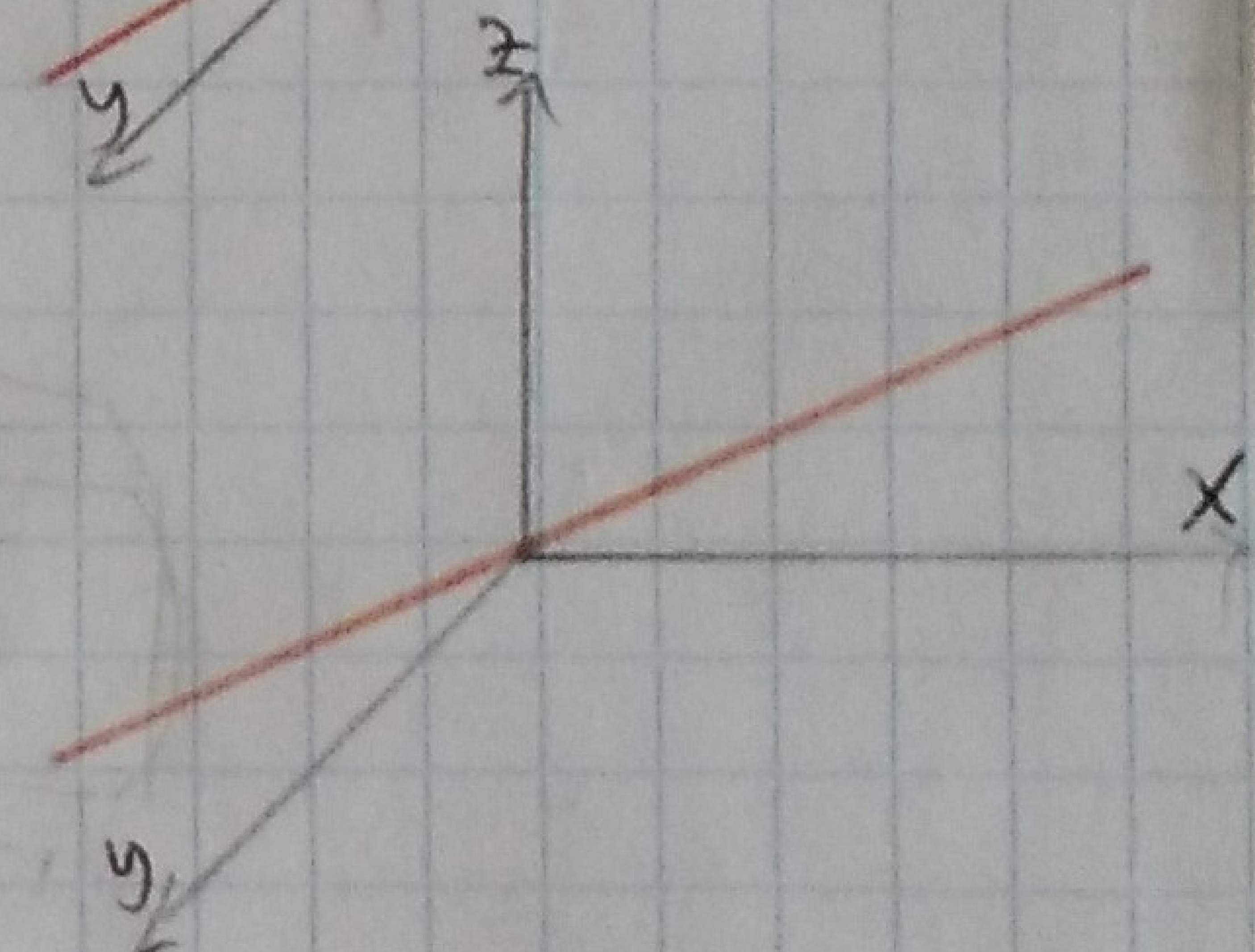
a) $\theta = 0 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad z(0) = x$



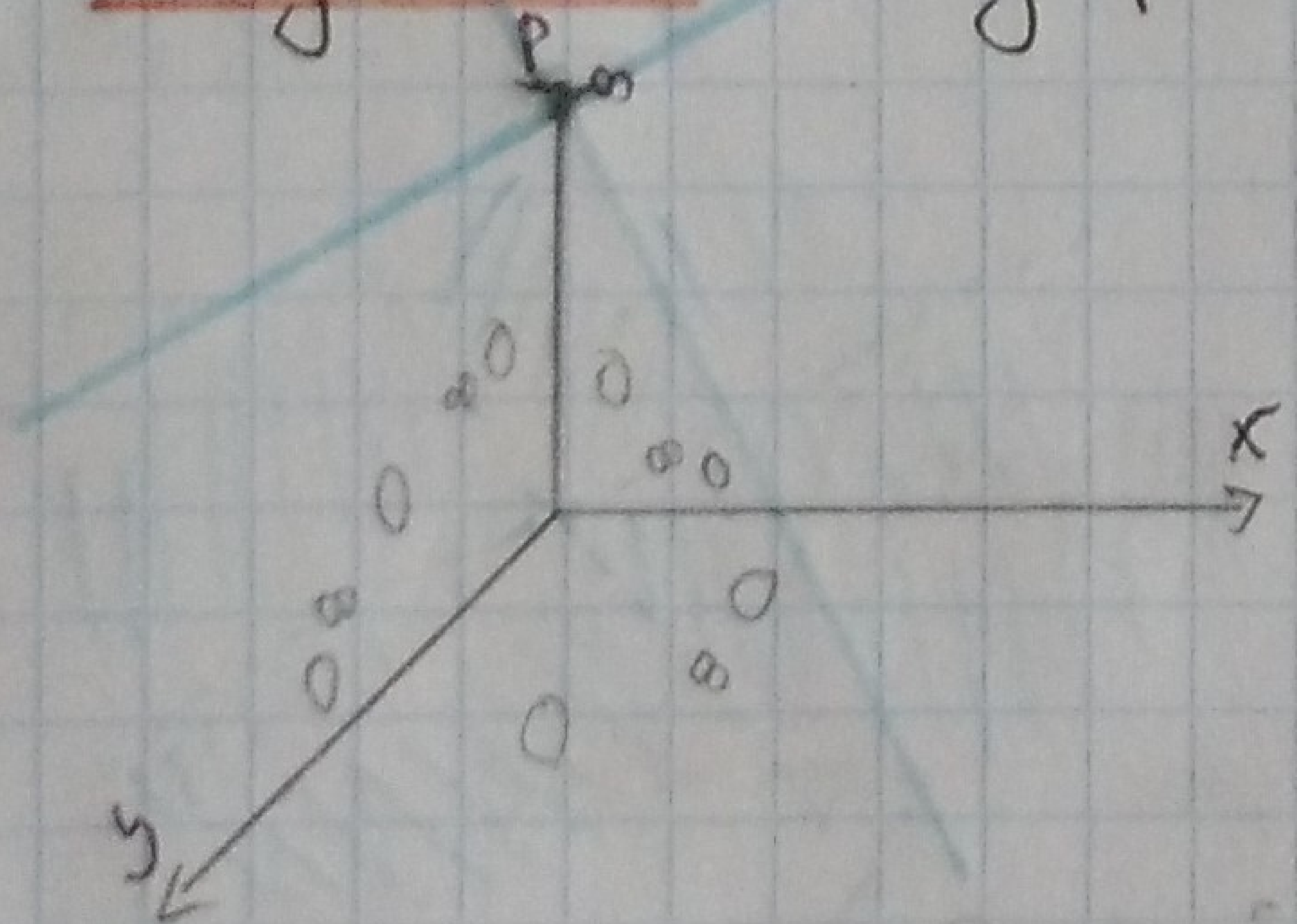
$\theta = \frac{\pi}{8} \rightarrow \begin{pmatrix} 0,924 \\ 0,383 \end{pmatrix} \qquad z\left(\frac{\pi}{8}\right) = 0,924x + 0,383y$



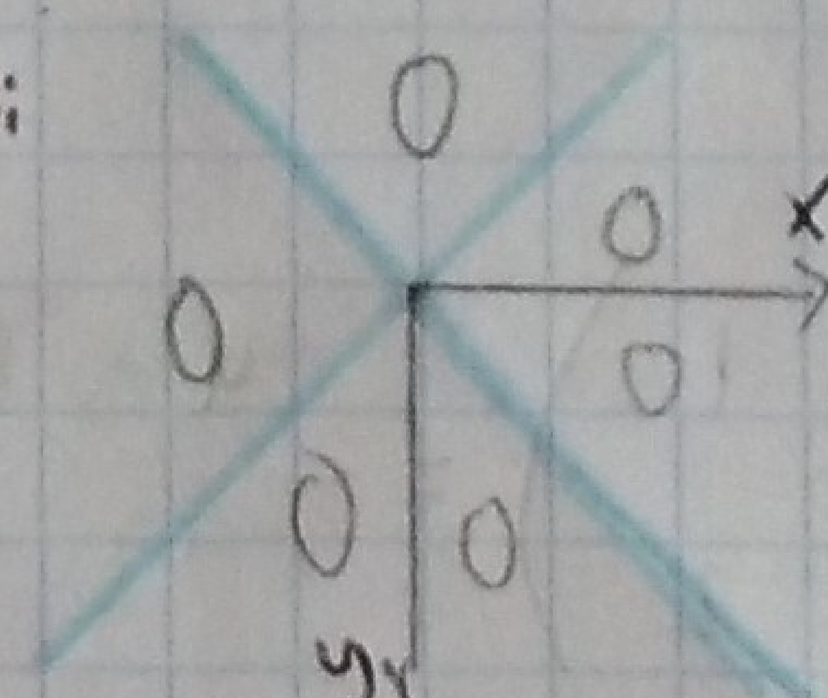
$\theta = \frac{\pi}{4} \rightarrow \begin{pmatrix} 0,707 \\ 0,707 \end{pmatrix} \qquad z\left(\frac{\pi}{4}\right) = 0,707(x+y)$



The orange lines on graphs denote planes



From above:



The blue lines denote infinity. On the z axis joint probability $p(x,y)$

b) $Var(z(\theta)) = E[z^2(\theta)] - E[z(\theta)]^2$

$$Var(z(\theta)) = E[(x\cos\theta + y\sin\theta)^2] - E[x\cos\theta + y\sin\theta]^2 =$$

$$= \underbrace{E[x^2\cos^2\theta]}_{(*)} + E[2xy\cos\theta\sin\theta] + E[y^2\sin^2\theta] - (\underbrace{E[x\cos\theta]}_{(*)} + E[y\sin\theta])^2 =$$

$$= 2\cos\theta\sin\theta E[xy] + \sin^2\theta E[y^2] - (E[y\sin\theta])^2 =$$

$$= 2\cos\theta\sin\theta E[xy] + \sin^2\theta Var(y)$$

let $\cos^2\theta$, $\cos\theta$ be const, then $(*)$ is 0 as $x \sim N(0,1)$

$$\frac{\partial Var(z(\theta))}{\partial\theta} = -2E[xy]\sin^2\theta + 2E[xy]\cos^2\theta + \sin 2\theta \, Var(y) = 0$$

$$\sin 2\theta \, Var(y) = 2E[xy](\sin^2\theta - \cos^2\theta)$$

$$\frac{\sin 2\theta}{4\sin^2\theta - 2} = \frac{E[xy]}{Var(y)}$$

$$\operatorname{kurt}[z(\theta)] = \frac{E[(z(\theta)-E[z(\theta)])^4]}{(Var[z(\theta)])^2} - 3$$

$Var[z(\theta)]$ from part b: $Var[z(\theta)] = 2\cos(\theta)\sin(\theta)E[xy] + \sin^2(\theta)\,Var(y)$

$$(Var[z(\theta)])^2 = 4\cos^2(\theta)\sin^2(\theta)E[xy]^2 + 4\cos(\theta)\sin^3(\theta)E[xy]Var(y) + \sin^4(\theta)Var(y)^2$$

$$E[(z(\theta)-E[z(\theta)])^4] = E[(x\cos(\theta)+y\sin(\theta) - E[x\cos(\theta)+y\sin(\theta)])^4] \quad \text{with } E[x\cos(\theta)]=0$$

$$= E[(x\cos(\theta)+y\sin(\theta)-\sin(\theta)E[y])^4] \quad \text{with } E[y]=y$$

$$= E[(x\cos(\theta))^4]$$

$$= \cos^4(\theta)E[x^4]$$

$$\frac{\partial}{\partial\theta}\; \frac{\cos^4(\theta)E[x^4]}{4\cos^2(\theta)\sin^2(\theta)E[xy]^2 + 4\cos(\theta)\sin^3(\theta)E[xy]Var(y) + \sin^4(\theta)Var(y)^2} - 3$$

s.t. $\theta \in [0, 2\pi]$

$$= \frac{-4E[x^4]\sin(\theta)\cos^3(\theta)\cdot(2\cos(\theta)\sin(\theta)E[xy]+\sin^2(\theta)Var(y))^2 - \cos^4(\theta)E[x^4]\cdot\big(2(2\cos(\theta)\sin(\theta)E[xy] + \sin^2(\theta)Var(y))\cdot(-2\sin^2(\theta)E[xy]+2\cos^2(\theta)E[xy]+\sin(2\theta)Var(y))\big)}{(2\cos(\theta)\sin(\theta)E[xy]+\sin^2(\theta)Var(y))^3}$$

$$\overset{!}{=} 0$$