Time Series Analysis

Discussion Section 03

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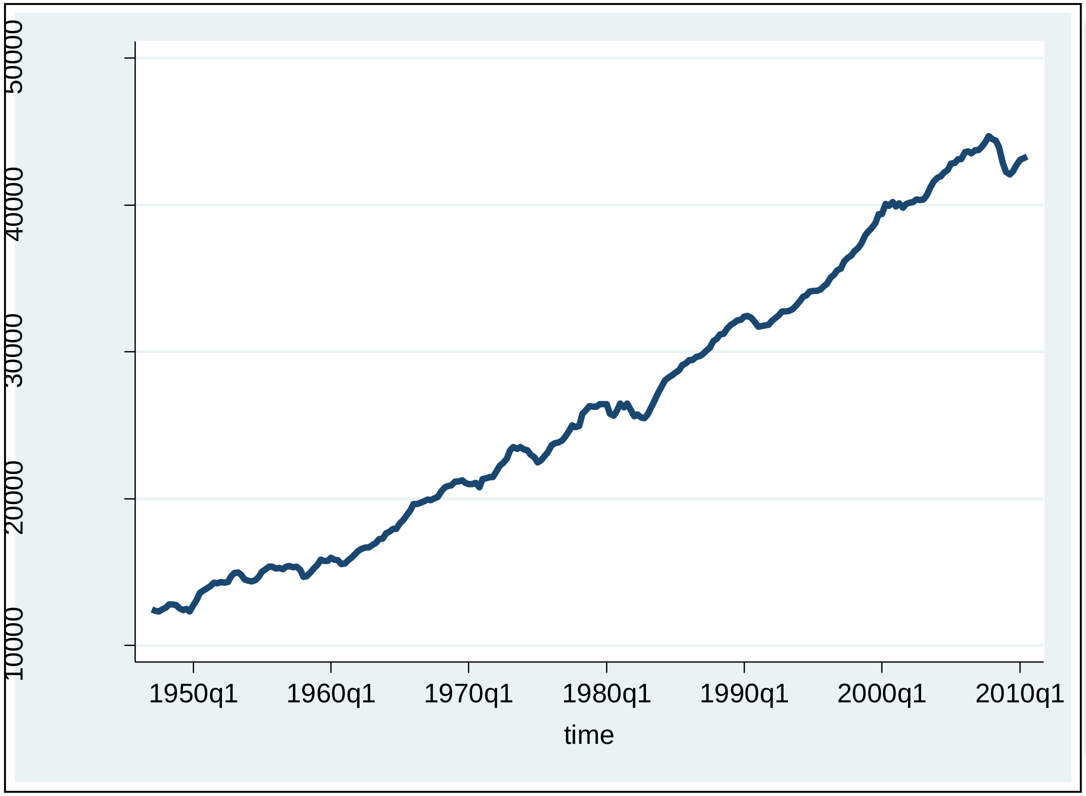
Nonstationary Stochastic Processes

- **Introduction**
- **Nonstationarity and Trends**
- **ARIMA Models**
- Unit Root Tests
- Seasonal ARIMA

Original Time Series (1947q1 to 2010q3)

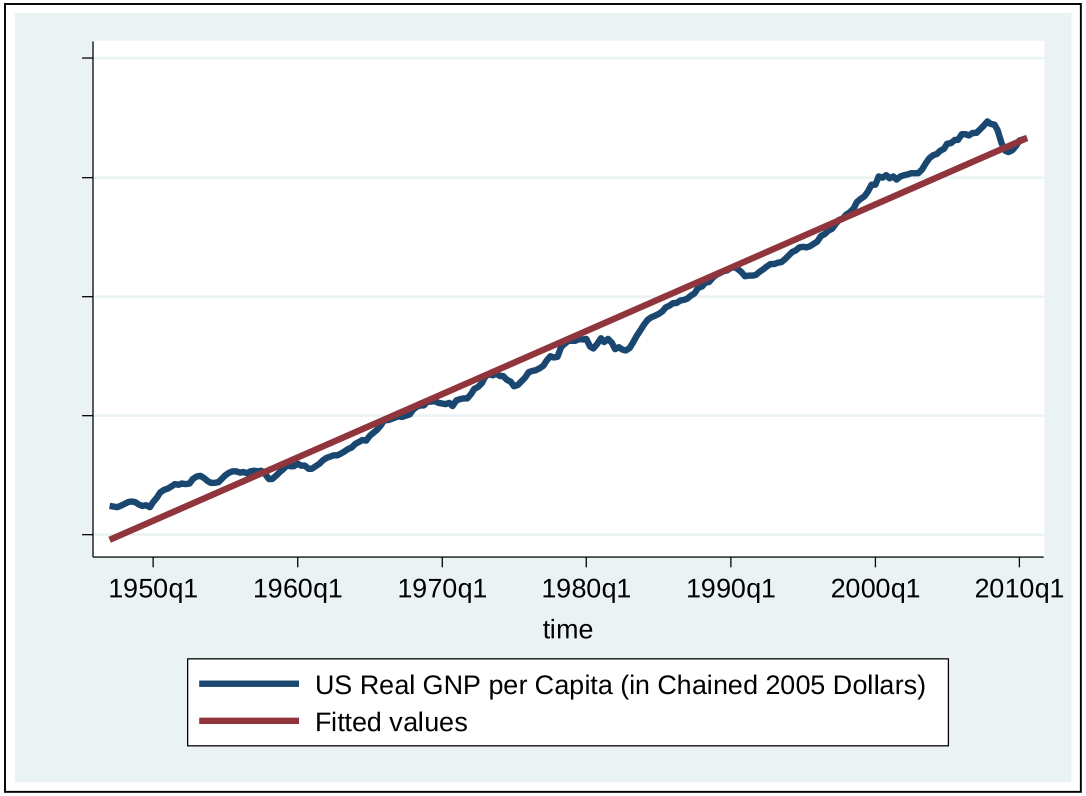
U.S. postwar real GNP per capita (in chained 2005 dollars)



Chained dollars -- A measure used to express real prices. Real prices are those that have been adjusted to remove the effect of changes in the purchasing power of the dollar; they usually reflect buying power relative to a reference year. Prior to 1996, real prices were expressed in constant dollars, a measure based on the weights of goods and services in a single year, usually a recent year. In 1996, the U.S. Department of Commerce introduced the chained-dollar measure. The new measure is based on the average weights of goods and services in successive pairs of years. It is "chained" because the second year in each pair, with its weights, becomes the first year of the next pair. The advantage of using the chained-dollar measure is that it is more closely related to any given period covered and is therefore subject to less distortion over time.

Original Time Series (1947q1 to 2010q3)

U.S. postwar real GNP per capita (in chained 2005 dollars)



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Excursus: Logarithmic Transformation

Before we even remove a deterministic trend in the trend stationary model (TS-model) or difference in the difference stationary model (DS-model), it is often useful to first lake logs of the original series.

This will linearize an exponential trend,

i.e. constant proportional growth.

$$ln(e^{\delta t}) = \delta t$$

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Excursus: Logarithmic Transformation

Moreover, 1st differences of log-series are approximately growth rates (percentage changes) which can be expected to be stationary even if the original series is not.

$$\Delta ln(y_t) = (1-L)ln(y_t) = ln(y_t) - ln(y_{t-1})$$

$$= ln\left(\frac{y_t}{y_{t-1}}\right) = ln\left(\frac{y_{t-1} + y_t - y_{t-1}}{y_{t-1}}\right)$$

$$= ln\left(1 + \frac{y_t - y_{t-1}}{y_{t-1}}\right) \approx \frac{y_t - y_{t-1}}{y_{t-1}}$$

Recall:

$$ln(1+x) \approx x$$

$$for\ \ x - small$$

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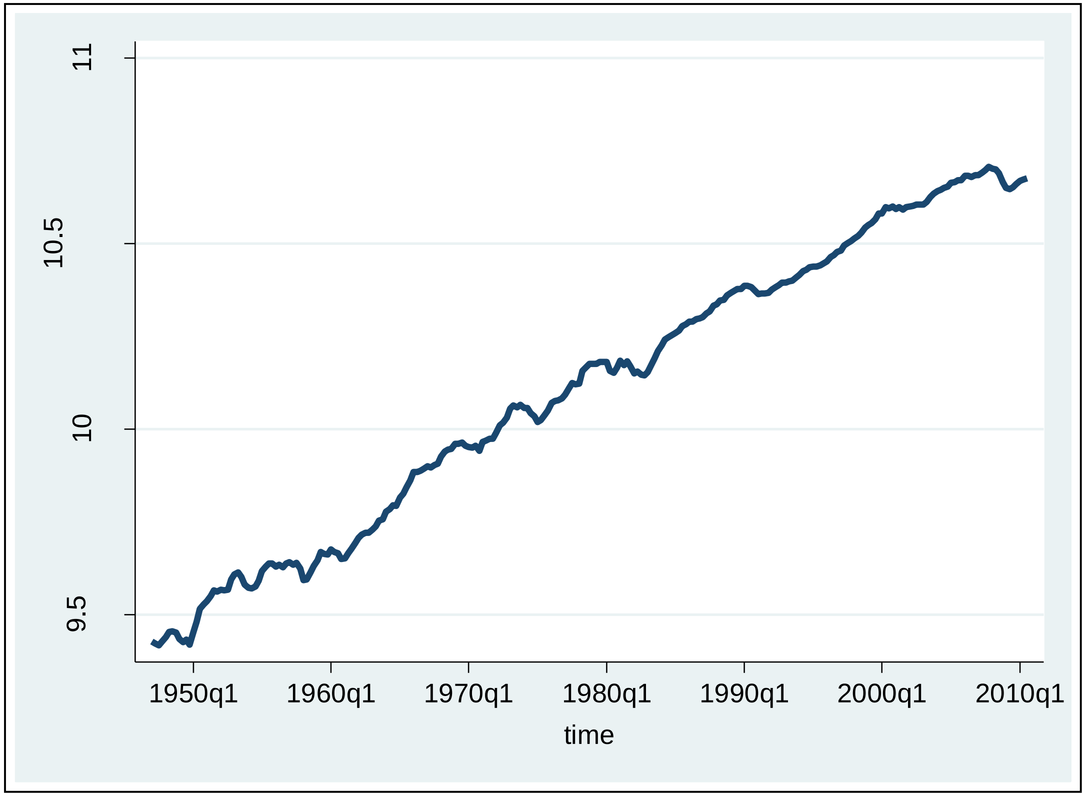
Log Time Series (1947q1 to 2010q3)

y_t = U.S. postwar log real GNP per capita (in chained 2005 dollars)



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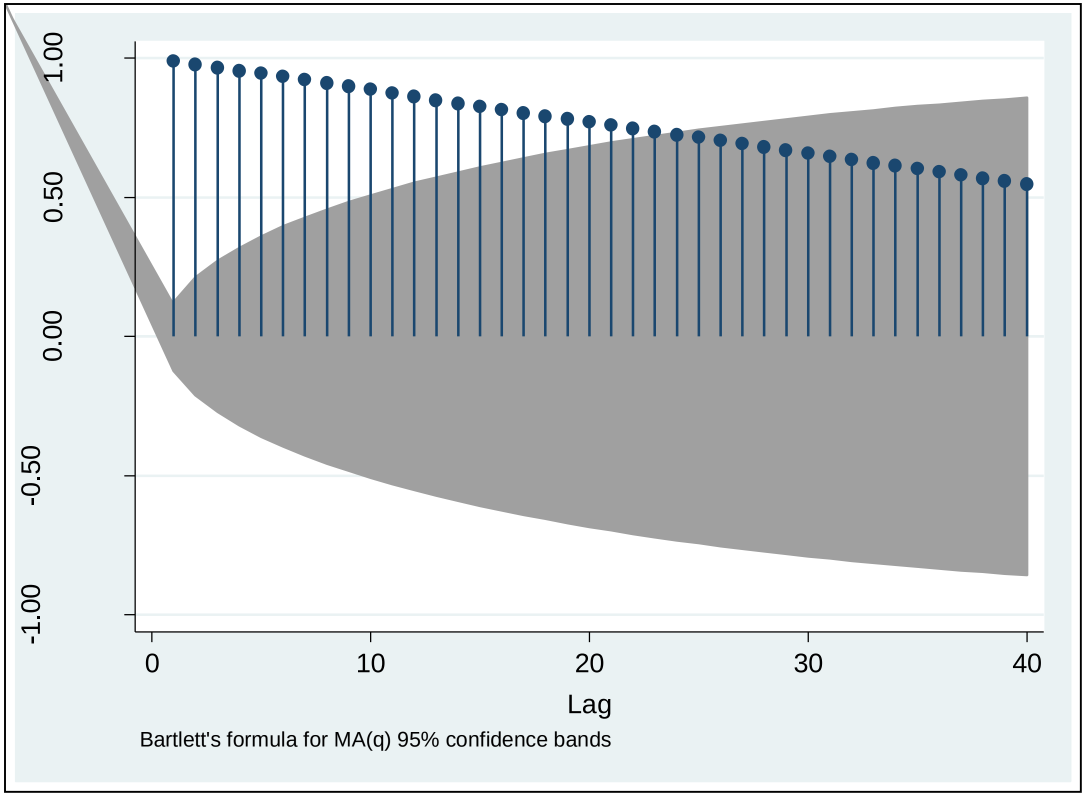
Autocorrelation Function

U.S. postwar log real GNP per capita (in chained 2005 dollars)



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Exercise 3.1:

- Write down the **general formulas** for trend-stationary (TS) and the difference-stationary (DS) models.
- Describe the **difference** between the trend-stationary (TS) and the difference-stationary (DS) model with respect to the persistence of their dynamic response to a random shock to real GNP per capita.

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Solution 3.1:

Trend-stationary (TS) model

$$y_t = \sum_{j=0}^{m} \delta_j \cdot t^j + u_t \text{ with } a(L)u_t = b(L)\varepsilon_t$$

Difference-stationary (DS) model

$$a(L)(1-L)y_t = \delta + b(L)\varepsilon_t$$

Examples of TS-models:

$y_t = \delta_0 + \delta_1 t + u_t$ - *linear trend*

$y_t = \delta_0 + \delta_1 t + \delta_2 t^2 + u_t$ - *quadratic trend*

Examples of DS-models:

$(1-L)y_t = \varepsilon_t$ - *randomwalk*

$(1-L)y_t = \delta + \varepsilon_t$ - *randomwalkwithdrift*

"In the DS model of output, the effect of a shock persists forever because the disturbance changes the trend component and thus affects the level of output in all future periods. In contrast, the impact of a shock in the TS model is transitory and is eliminated quite quickly as output reverts to its steady trend."

Rudebusch (1993) "The Uncertain Unit Root in Real GNP", p. 264

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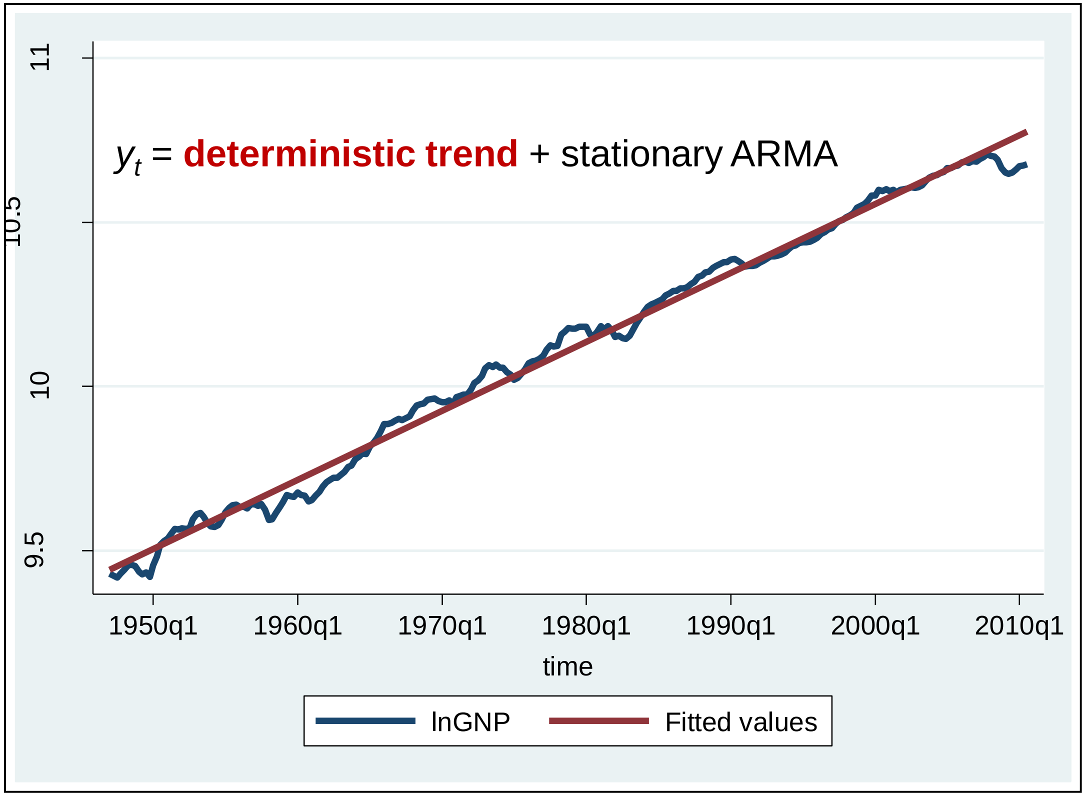
Trend-stationary (TS) Model

U.S. postwar log real GNP per capita (in chained 2005 dollars)



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Trend-stationary (TS) Model

OLS estimate of the deterministic trend

```
. regress lnGNP time

      Source |       SS       df       MS              Number of obs =     255
-------------+------------------------------           F(  1,   253) =28663.44
       Model |  38.2339928     1  38.2339928           Prob > F      =  0.0000
    Residual |  .337475217   253  .001333894           R-squared     =  0.9913
-------------+------------------------------           Adj R-squared =  0.9912
       Total |   38.571468   254  .151856173           Root MSE      =  .03652

------------------------------------------------------------------------------
       lnGNP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        time |   .0052603   .0000311   169.30   0.000     .0051991    .0053215
       _cons |   9.714362   .0032651  2975.18   0.000     9.707932    9.720793
------------------------------------------------------------------------------
```

$$y_t = \sum_{j=0}^{m} \delta_j \cdot t^j + u_t \text{ with } u_t \sim ARMA(p,q) \longrightarrow \hat{y}_t = 9.714 + 0.005 \cdot t + \hat{u}_t$$

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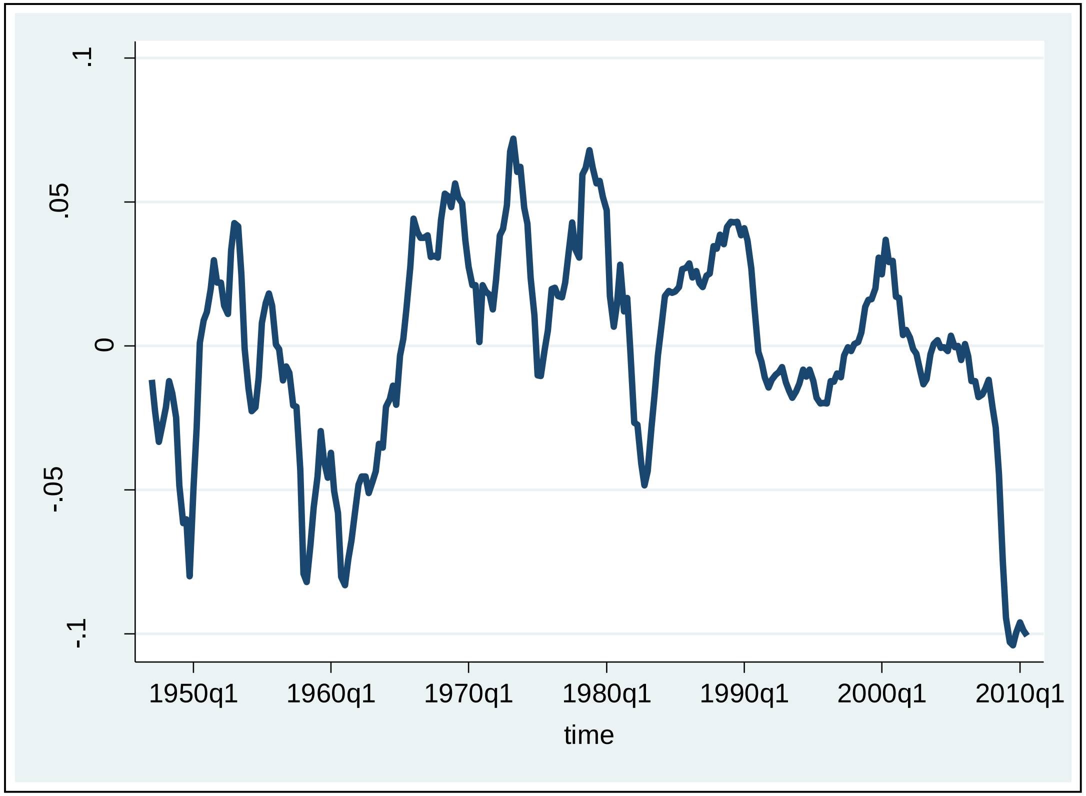
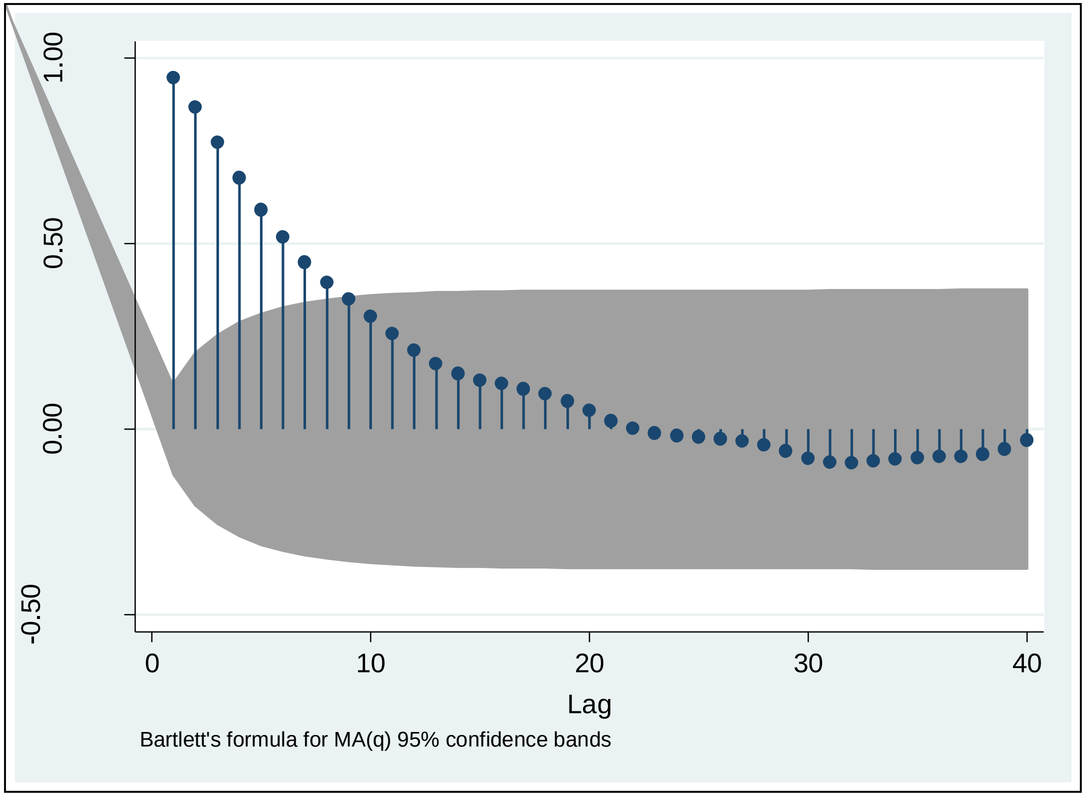
Trend-stationary (TS) Model

OLS Residuals



Trend-stationary (TS) Model

ACF of OLS Residuals



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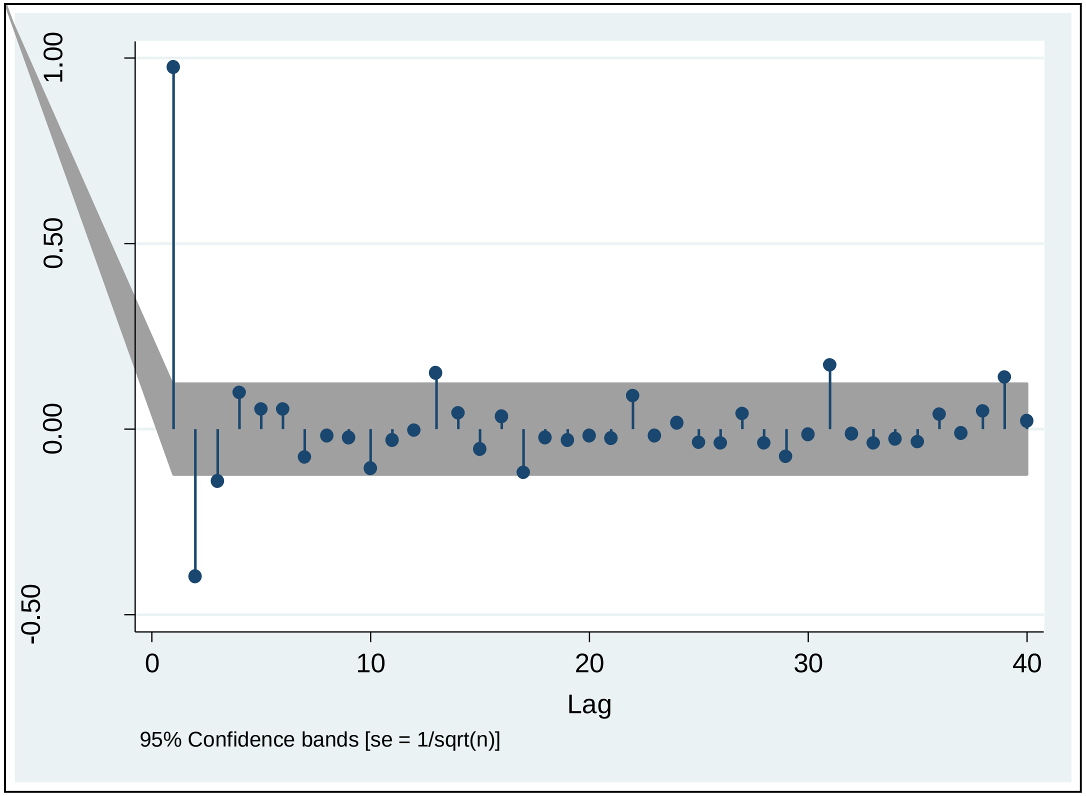
Trend-stationary (TS) Model

PACF of OLS Residuals



Trend-stationary (TS) Model

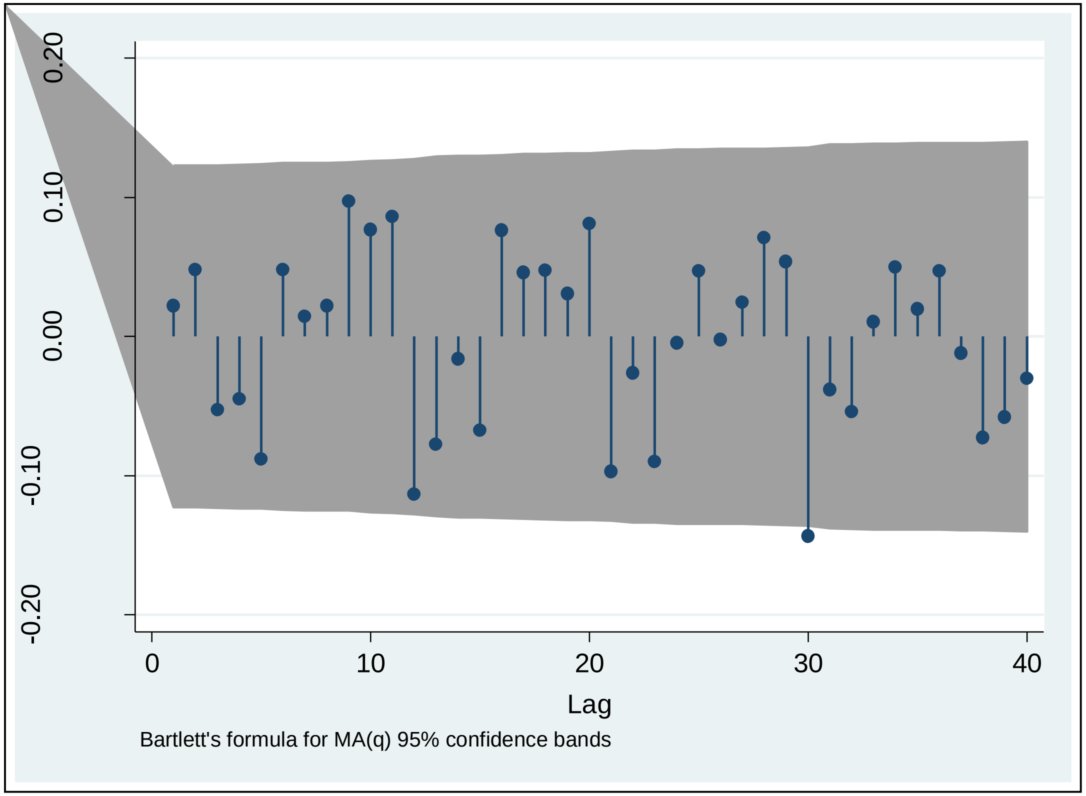
ML estimate of the stationary fluctuations

```
. arima res_OLS, ar(1/3) noconstant
[...]
------------------------------------------------------------------------------
             |                 OPG
     res_OLS |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   1.296006   .0497034    26.07   0.000     1.198589    1.393423
         L2. |  -.2122003   .0861312    -2.46   0.014    -.3810144   -.0433861
         L3. |  -.1392797   .0561655    -2.48   0.013     -.249362   -.0291974
-------------+----------------------------------------------------------------
      /sigma |   .0090287    .000296    30.50   0.000     .0084486    .0096089
------------------------------------------------------------------------------
```

$$\longrightarrow \quad \hat{u}_t = 1.296u_{t-1} - 0.212u_{t-2} - 0.139u_{t-3}$$

Trend-stationary (TS) Model

ACF of AR(3) Residuals



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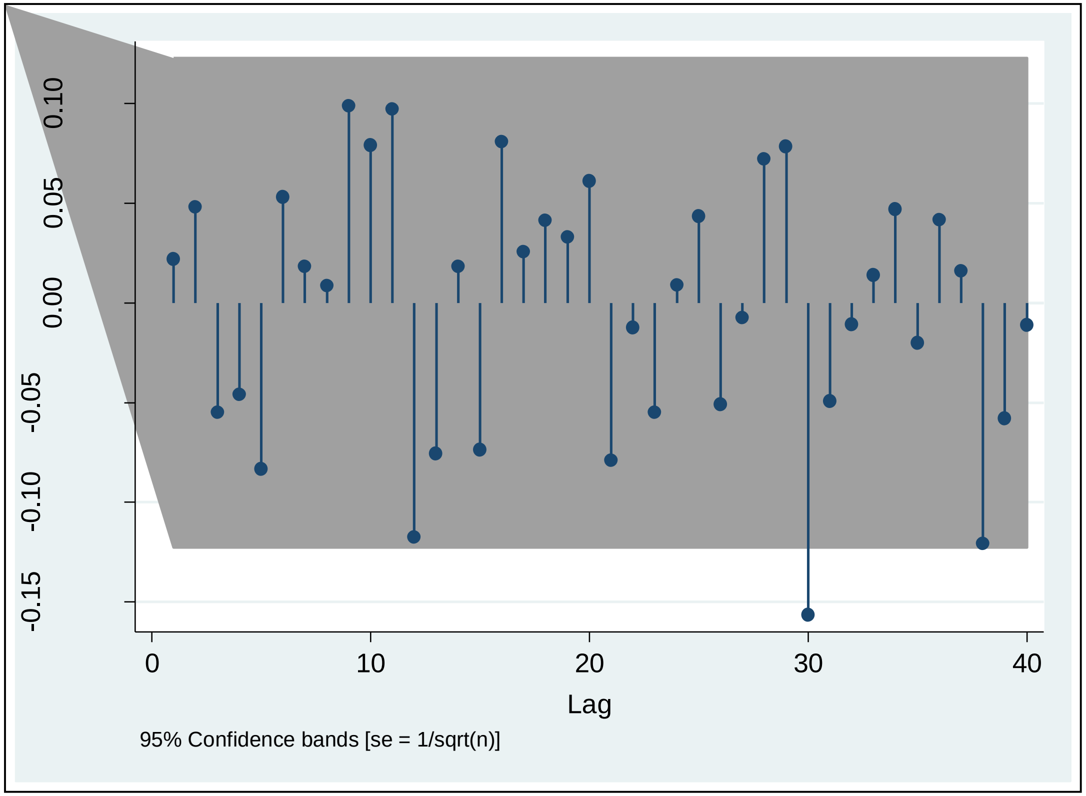
Trend-stationary (TS) Model

PACF of AR(3) Residuals



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Trend-stationary (TS) Model

Q statistics computed from AR(3) Residuals

```
. corrgram res_AR3
                                                 -1       0       1 -1       0       1
 LAG       AC       PAC        Q      Prob>Q  [Autocorrelation]  [Partial Autocor]
---------------------------------------------------------------------------------------
1        0.0220   0.0220   .12491   0.7238           |                  |
2        0.0480   0.0479   .72119   0.6973           |                  |
3       -0.0529  -0.0548   1.4485   0.6942           |                  |
4       -0.0447  -0.0460   1.9709   0.7411           |                  |
5       -0.0883  -0.0832   4.0146   0.5473           |                  |
6        0.0481   0.0533   4.6233   0.5930           |                  |
7        0.0144   0.0182   4.6777   0.6992           |                  |
8        0.0224   0.0087   4.8107   0.7776           |                  |
9        0.0972   0.0986   7.3293   0.6029           |                  |
10       0.0769   0.0792   8.9114   0.5405           |                  |
11       0.0862   0.0971   10.906   0.4512           |                  |
12      -0.1136  -0.1175   14.384   0.276
[...]
38      -0.0727  -0.1209   40.921   0.343
39      -0.0579  -0.0579   41.939   0.344
40      -0.0304  -0.0109   42.221   0.375
```

$$Q = T(T+2)\sum_{k=1}^{K} \frac{1}{T-k}\hat{\rho}_k^2 \quad \sim \chi^2 \quad \text{with } K-p-q \text{ degrees of freedom}$$

```
. di 1-chi2(1, 1.9709)
.16035236
```

Exercise 3.2:

Trend-stationary (TS) Model

$$y_t = \sum_{j=0}^{m} \delta_j \cdot t^j + u_t \text{ with } u_t \sim ARMA(p,q)$$

$$\longrightarrow \quad \hat{y}_t = 9.714 + 0.005 \cdot t + \hat{u}_t$$

$$\longrightarrow \quad \hat{u}_t = 1.296 u_{t-1} - 0.212 u_{t-2} - 0.139 u_{t-3}$$

- Show that the series y_t is not stationary if the estimated TS model is the right model.

  Hint: Consider $E[y_t]$.
- Calculate the average percentage annual growth rate of the log GNP per capita.

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Solution 3.2-1:

Trend-stationary (TS) Model

$$y_t = \sum_{j=0}^{m} \delta_j \cdot t^j + u_t \text{ with } u_t \sim ARMA(p,q)$$

$$\longrightarrow \quad \hat{y}_t = 9.714 + 0.005 \cdot t + \hat{u}_t$$

$$\longrightarrow \quad \hat{u}_t = 1.296 u_{t-1} - 0.212 u_{t-2} - 0.139 u_{t-3}$$

Show that the series y_t is not stationary if the estimated TS model is the right model.

$$E[y_t] = E[\hat{y}_t] = E[9.714 + 0.005 \cdot t + \hat{u}_t] = 9.714 + 0.005 \cdot t = \mu_t$$

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Solution 3.2-2:

Trend-stationary (TS) Model

$$y_t = \sum_{j=0}^{m} \delta_j \cdot t^j + u_t \text{ with } u_t \sim ARMA(p,q)$$

$$\longrightarrow \quad \hat{y}_t = 9.714 + 0.005 \cdot t + \hat{u}_t$$

$$\longrightarrow \quad \hat{u}_t = 1.296 u_{t-1} - 0.212 u_{t-2} - 0.139 u_{t-3}$$

Calculate the average percentage annual growth rate of the log GNP per capita.

- Average percentage **quarterly** growth rate of the log GNP per capita = 0.5%
- Average percentage **annual** growth rate of the log GNP per capita:

$$1.005 \cdot 1.005 \cdot 1.005 \cdot 1.005 - 1 = 0.0201505 \approx 2\%$$

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Exercise 3.3:

Trend-stationary (TS) Model

$$y_t = \sum_{j=0}^{m} \delta_j \cdot t^j + u_t \text{ with } u_t \sim ARMA(p,q)$$

$$\longrightarrow \quad \hat{y}_t = 9.714 + 0.005 \cdot t + \hat{u}_t$$

$$\longrightarrow \quad \hat{u}_t = 1.296 u_{t-1} - 0.212 u_{t-2} - 0.139 u_{t-3}$$

How does a shock today affect the level of y_t one year hence and infinitely far in the future?

Hint: MA representation of y_t

$$y_t = \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots = \mu + \psi(L)\varepsilon_t$$

Solution 3.3-1:

$$\hat{y}_t = 9.714 + 0.005 \cdot t + \hat{u}_t \text{ with } \hat{u}_t = 1.296 u_{t-1} - 0.212 u_{t-2} - 0.139 u_{t-3}$$

MA representation of y_t

$$y_t = \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots = \mu + \psi(L)\varepsilon_t$$

$$y_t = c(L)\varepsilon_t$$

$$\boxed{y_t = \varphi_1 y_{t-1} + \ldots + \varphi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \ldots - \theta_q \varepsilon_{t-q}}$$

$$\left(1 - \varphi_1 L - \varphi_2 L^2 - \ldots - \varphi_p L^p\right) y_t = \left(1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_q L^q\right)\varepsilon_t$$

$$a(L) y_t = b(L)\varepsilon_t$$

$$a(L)c(L) = b(L)$$

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Solution 3.3-2:

$$\hat{y}_t = 9.714 + 0.005 \cdot t + \hat{u}_t \text{ with } \hat{u}_t = 1.296u_{t-1} - 0.212u_{t-2} - 0.139u_{t-3}$$

$$\boxed{y_t = \varphi_1 y_{t-1} + \ldots + \varphi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \ldots - \theta_q \varepsilon_{t-q}}$$

$$\hat{y}_t = 9.714 + 0.005 \cdot t + 1.296u_{t-1} - 0.212u_{t-2} - 0.139u_{t-3}$$

$$y_{t-1} = \delta_0 + \delta_1(t-1) + u_{t-1} \Rightarrow u_{t-1} = y_{t-1} - \delta_0 - \delta_1(t-1)$$

$$y_{t-2} = \delta_0 + \delta_1(t-2) + u_{t-2} \Rightarrow u_{t-2} = y_{t-2} - \delta_0 - \delta_1(t-2)$$

$$y_{t-3} = \delta_0 + \delta_1(t-3) + u_{t-3} \Rightarrow u_{t-3} = y_{t-3} - \delta_0 - \delta_1(t-3)$$

$$\begin{aligned}\hat{y}_t = {} & 9.714 + 0.005 \cdot t \\ & + 1.296(y_{t-1} - 9.714 - 0.005(t-1)) \\ & - 0.212(y_{t-2} - 9.714 - 0.005(t-2)) \\ & - 0.139(y_{t-3} - 9.714 - 0.005(t-3)) \\ = {} & 0.537 + 0.000275t + 1.296y_{t-1} - 0.212y_{t-2} - 0.139y_{t-3}\end{aligned}$$

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Solution 3.3-3:

$$\hat{y}_t = 9.714 + 0.005 \cdot t + 1.296 u_{t-1} - 0.212 u_{t-2} - 0.139 u_{t-3}$$

```
. regress lnGNP time L.lnGNP L2.lnGNP L3.lnGNP

      Source |       SS       df       MS              Number of obs =     252
-------------+------------------------------           F(  4,   247) =       .
       Model |  37.1251786     4  9.28129465           Prob > F      =  0.0000
    Residual |  .020552824   247   .00008321           R-squared     =  0.9994
-------------+------------------------------           Adj R-squared =  0.9994
       Total |  37.1457314   251  .147990962           Root MSE      =  .00912

------------------------------------------------------------------------------
       lnGNP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        time |   .0002734   .0000889     3.08   0.002     .0000983    .0004486
       lnGNP |
         L1. |   1.291802   .0629791    20.51   0.000     1.167757    1.415846
         L2. |  -.2091396   .1027163    -2.04   0.043    -.4114512    -.006828
         L3. |  -.1359682   .0638015    -2.13   0.034    -.2616327   -.0103038
       _cons |   .5207294   .1627163     3.20   0.002     .2002411    .8412177
------------------------------------------------------------------------------
```


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General Solution: Use “MA representation”

Any stationary ARMA(p,q) process can be written as an infinite MA:

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad \text{with } \psi_0 = 1$$

$$y_{T+l} = \varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1}\varepsilon_{T+1} + \psi_l \varepsilon_T + \psi_{l+1}\varepsilon_{T-1} + \ldots$$

$$\tilde{y}_{T+l} = \psi_l \varepsilon_T + \psi_{l+1}\varepsilon_{T-1} + \ldots$$

Forecast Error

$$e_{T+l} = y_{T+l} - \tilde{y}_{T+l|T} = \varepsilon_{T+l} + \psi_1 \varepsilon_{T+l-1} + \ldots + \psi_{l-1}\varepsilon_{T+1}$$

$$E\left[e_{T+l}^2\right] = Var\left[e_{T+l}\right] = \left(1 + \psi_1^2 + \ldots + \psi_{l-1}^2\right)\sigma_\varepsilon^2$$

Prediction Interval

$$\left[\tilde{y}_{T+l|T} \pm z_{1-\frac{\alpha}{2}} \left(1 + \psi_1^2 + \ldots + \psi_{l-1}^2\right)^{\frac{1}{2}} \sigma_\varepsilon\right]$$

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How do we find $\psi_1, \ldots, \psi_{l-1}$?

$$y_t = \varphi_1 y_{t-1} + \ldots + \varphi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \ldots - \theta_q \varepsilon_{t-q}$$

$$\left(1 - \varphi_1 L - \varphi_2 L^2 - \ldots - \varphi_p L^p\right) y_t = \left(1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_q L^q\right) \varepsilon_t$$

$$a(L) y_t = b(L) \varepsilon_t$$

$$y_t = c(L) \varepsilon_t$$

$$= \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad \text{with} \quad \psi_0 = 1$$

So the ψ_1, ψ_2, ... coefficients in $c(L)$, can be obtained by equating coefficients of L^j, j = 1, 2, ... in $a(L)c(L) = b(L)$.

Exercises

Solution 3.3-4:

$$\hat{y}_t = 9.714 + 0.005 \cdot t + 1.296 u_{t-1} - 0.212 u_{t-2} - 0.139 u_{t-3}$$

$$y_t = \varphi_1 y_{t-1} + \ldots + \varphi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \ldots - \theta_q \varepsilon_{t-q}$$

$$\left(1 - \varphi_1 L - \varphi_2 L^2 - \ldots - \varphi_p L^p\right) y_t = \left(1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_q L^q\right) \varepsilon_t$$

$$a(L) y_t = b(L) \varepsilon_t$$

$$\left(1 - 1.296L + 0.212L^2 + 0.139L^3\right) y_t = 0.537 + 0.000275t + \varepsilon_t$$

Attention: Here Lag-Operator Notation ➔ ε_t

$$y_t = \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots = \mu + \psi(L) \varepsilon_t$$

$$y_t = c(L) \varepsilon_t$$

$$a(L) c(L) = b(L)$$

b(L) ≡ all information from ε at t-i, with i={0,….,p}, with the corresponding Lag-Operator. Since we only have ε_t without a factor it´s equal to 1.

$$\underbrace{\left(1 - 1.296L + 0.212L^2 + 0.139L^3\right)}_{a(L)} \underbrace{\left(1 + \psi_1 L + \psi_2 L^2 + +\psi_3 L^3 + \psi_4 L^4 \ldots\right)}_{c(L)} = \overbrace{1}^{b(L)}$$

a(L) ≡ all information from y at t-i, with i={0,….,p}, with the corresponding Lag-Operator

$$c(L) = \tilde{y}_t = \left(1 + \hat{\psi}_1 L + \hat{\psi}_2 L^2 + \hat{\psi}_3 L^3 + \ldots\right)$$

Solution 3.3-5:

$$(1-1.296L+0.212L^2+0.139L^3)(1+\psi_1L+\psi_2L^2++\psi_3L^3+\psi_4L^4\ldots)=1$$

$$1-1.296L+0.212L^2+0.139L^3$$

$$+\psi_1L-1.296\psi_1L^2+0.212\psi_1L^3+0.139\psi_1L^4$$

$$+\psi_2L^2-1.296\psi_2L^3+0.212\psi_2L^4+0.139\psi_2L^5$$

$$+\psi_3L^3-1.296\psi_3L^4+0.212\psi_3L^5+0.139\psi_3L^6$$

$$+\psi_4L^4-1.296\psi_4L^5+0.212\psi_4L^6+0.139\psi_4L^7$$

$$+\psi_5L^5-1.296\psi_5L^6+0.212\psi_5L^7+0.139\psi_5L^8+\ldots=1$$

$$-1.296+\psi_1=0\Rightarrow\psi_1=1.296$$

$$0.212-1.296\psi_1+\psi_2=0\Rightarrow\psi_2=-0.212+1.296\psi_1=-0.212+1.296^2=1.467$$

$$0.139+0.212\psi_1-1.296\psi_2+\psi_3=0\Rightarrow\psi_3=1.487$$

$$0.139\psi_1+0.212\psi_2-1.296\psi_3+\psi_4=0\Rightarrow\psi_4=1.436$$

$$0.139\psi_2+0.212\psi_3-1.296\psi_4+\psi_5=0\Rightarrow\psi_5=1.342$$

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Solution 3.3-5:

Quarter	1	2	3	4	8	16	32	64
ψ	1.296	1.467	1.487	1.436	0.988	0.339	0.034	0.000

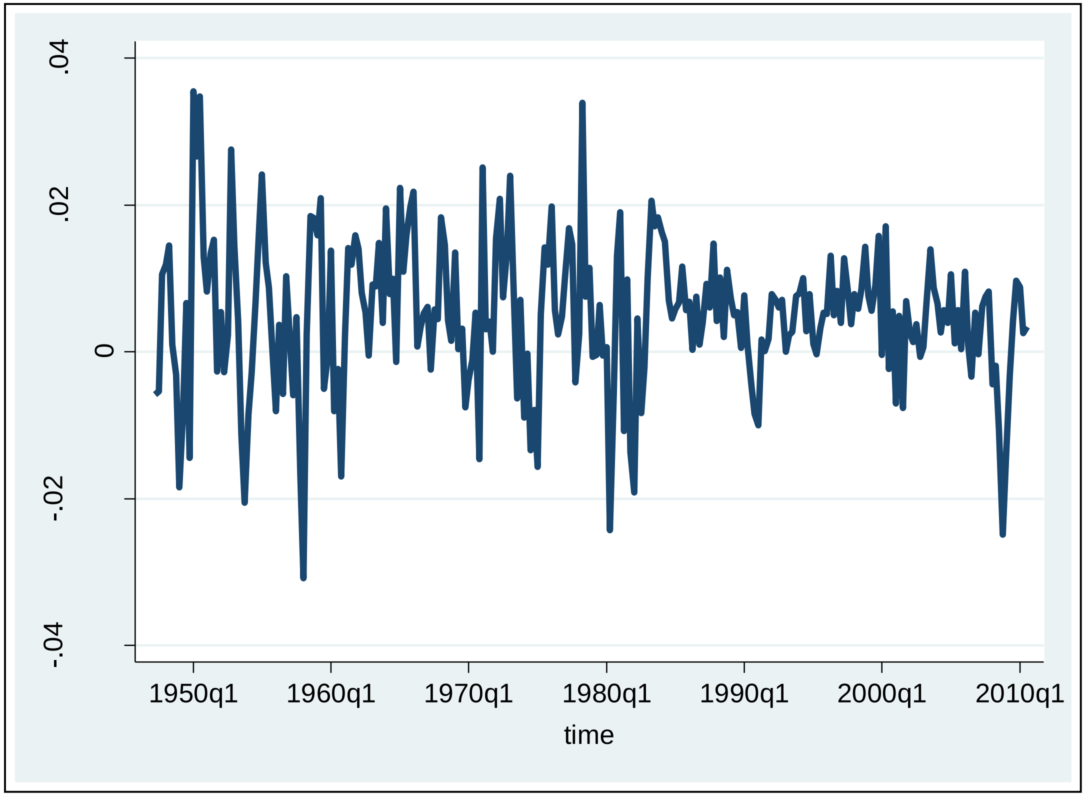
The TS model exhibits fairly rapid reversion to trend, with about two-thirds of a shock dissipated after four years.

For any TS series, $\psi_\infty = 0$, because the effect of any shock is eliminated as reversion to the deterministic trend eventually dominates.

Rudebusch (1993) "The Uncertain Unit Root in Real GNP", p. 266

Difference-stationary (DS) Model

U.S. postwar log real GNP per capita (in chained 2005 dollars), D.



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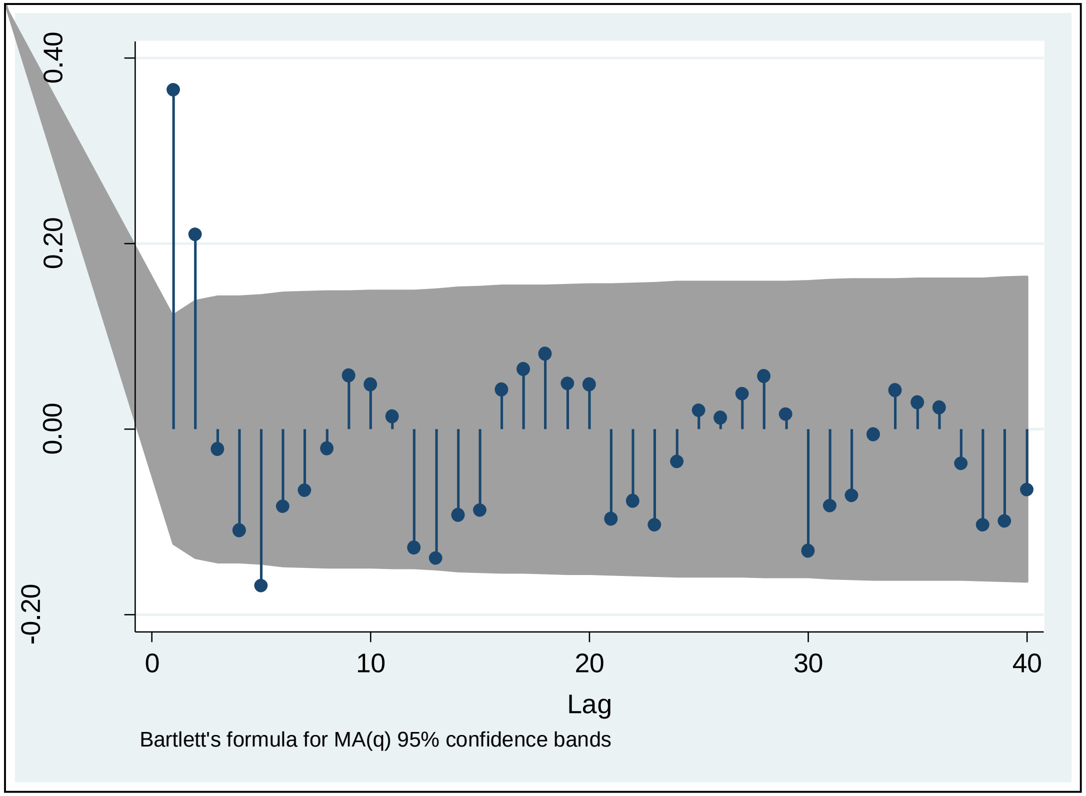
Difference-stationary (DS) Model

ACF of D.lnGNP



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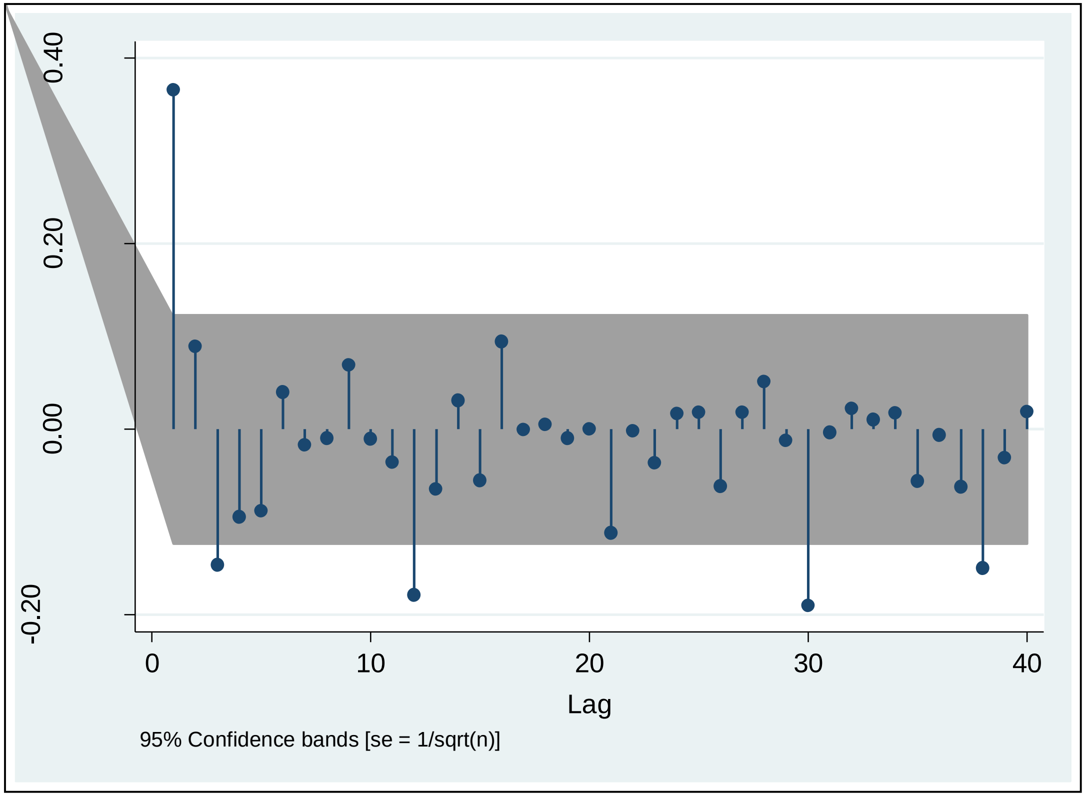
Difference-stationary (DS) Model

PACF of D.lnGNP



Difference-stationary (DS) Model

arima(#p,#d,#q) is an alternative, shorthand notation for specifying models with ARMA disturbances.
The dependent variable and any independent variables are differenced #d times, and 1 through #p lags of autocorrelations and 1 through #q lags of moving averages are included in the model. For example, the specification

Estimated ARIMA(3,1,0)

`. arima D.lnGNP, ar(1/3)` **or** `. arima lnGNP, arima(3,1,0)`

```
[...]
------------------------------------------------------------------------------
             |                 OPG
     D.lnGNP |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
lnGNP        |
       _cons |   .0048788   .0009102     5.36   0.000     .0030949    .0066627
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .3468049   .0515245     6.73   0.000     .2458187    .4477911
         L2. |   .1381909   .0561872     2.46   0.014      .028066    .2483158
         L3. |  -.1459299   .0568711    -2.57   0.010    -.2573953   -.0344646
-------------+----------------------------------------------------------------
      /sigma |   .0091259   .0003021    30.21   0.000     .0085338     .009718
------------------------------------------------------------------------------
```

$$\left(1-0.347L-0.138L^2+0.146L^3\right)(1-L)y_t = 0.0032+\varepsilon_t$$

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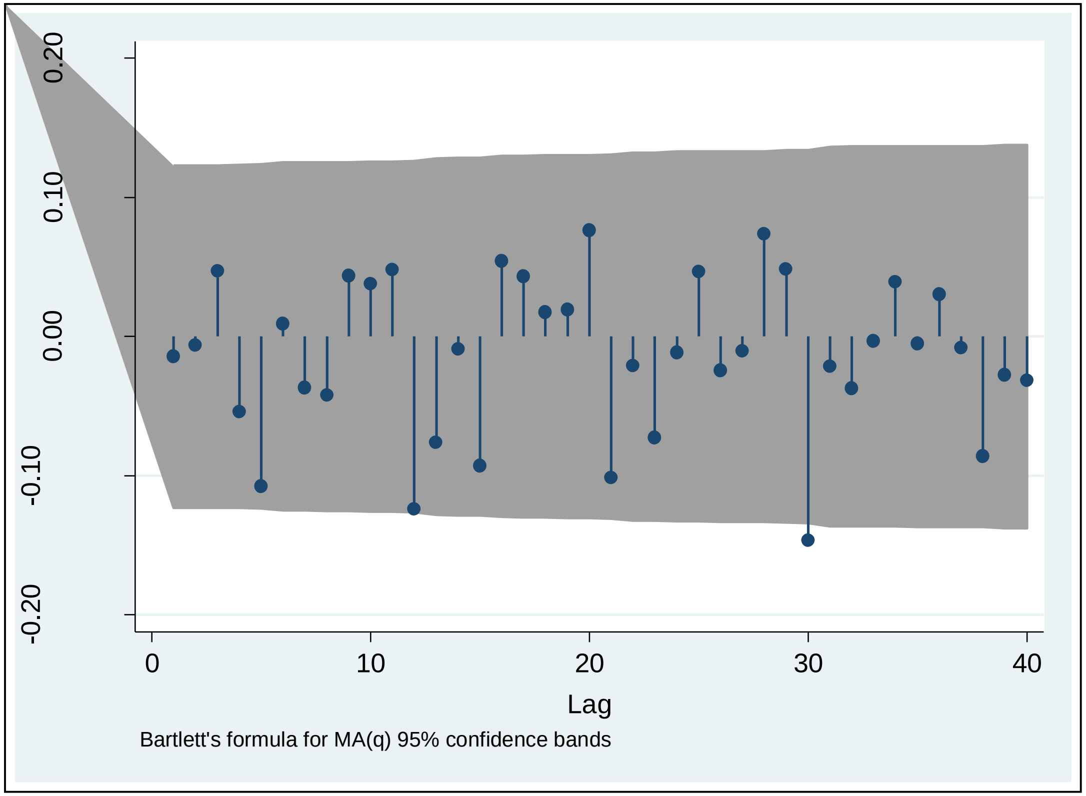
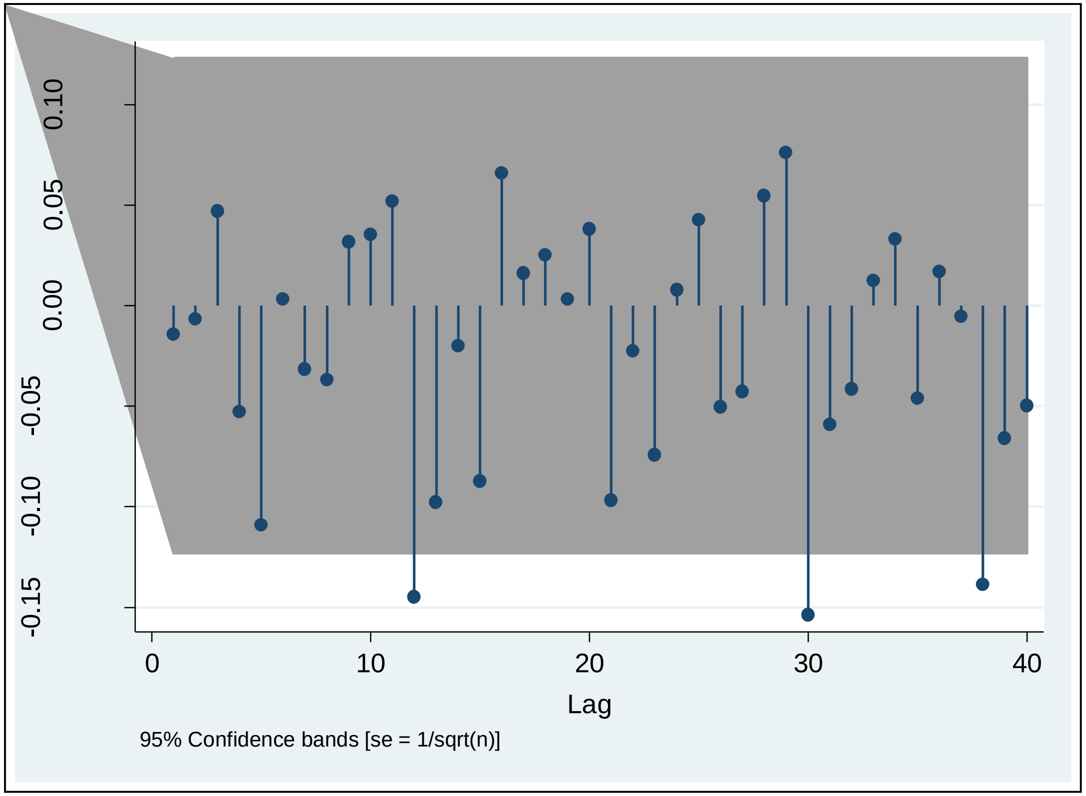
Difference-stationary (DS) Model

ACF of the residuals of the estimated ARIMA(3,1,0)



Difference-stationary (DS) Model

PACF of the residuals of the estimated ARIMA(3,1,0)



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Exercise 3.4:

Difference-stationary (DS) Model

$$\left(1-0.347L-0.138L^2+0.146L^3\right)\left(1-L\right)y_t=0.0032+\varepsilon_t$$

How does a shock today affect the level of y_t one year hence and infinitely far in the future?

Hint: MA representation of Δy_t

$$\Delta y_t=\mu+\varepsilon_t+\psi_1\varepsilon_{t-1}+\psi_2\varepsilon_{t-1}+\ldots=\mu+\psi\left(L\right)\varepsilon_t$$

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Solution 3.4-1:

$$\left(1-0.347L-0.138L^2+0.146L^3\right)\left(1+\psi_1L+\psi_2L^2++\psi_3L^3+\psi_4L^4...\right)=1$$
$$1-0.347L-0.138L^2+0.146L^3$$
$$+\psi_1L-0.347\psi_1L^2-0.138\psi_1L^3+0.146\psi_1L^4$$
$$+\psi_2L^2-0.347\psi_2L^3-0.138\psi_2L^4+0.146\psi_2L^5$$
$$+\psi_3L^3-0.347\psi_3L^4-0.138\psi_3L^5+0.146\psi_3L^6$$
$$+\psi_4L^4-0.347\psi_4L^5-0.138\psi_4L^6+0.146\psi_4L^7$$
$$+\psi_5L^5-0.347\psi_5L^6-0.138\psi_5L^7+0.146\psi_5L^8+...=1$$
$$-0.347+\psi_1=0\Rightarrow\psi_1=0.347$$
$$-0.138-0.347\psi_1+\psi_2=0\Rightarrow\psi_2=0.258$$
$$0.146-0.138\psi_1-0.347\psi_2+\psi_3=0\Rightarrow\psi_3=-0.009$$
$$0.146\psi_1-0.138\psi_2-0.347\psi_3+\psi_4=0\Rightarrow\psi_4=-0.018$$
$$0.146\psi_2-0.138\psi_3-0.347\psi_4+\psi_5=0\Rightarrow\psi_5=-0.045$$

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Solution 3.4-2:

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
ψ	0.347	0.258	-0.009	-0.018	-0.045	-0.017	-0.009	0.001	0.002	0.002	0.001	0.000

$$\Delta y_t = \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-1} + \ldots = \mu + \psi(L)\varepsilon_t$$

A unit shock in period t affects ΔY_{t+h} by ψ_h and Y_{t+h} by $c_h = 1 + \psi_1 + \ldots + \psi_h$.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
c	1.347	1.605	1.596	1.578	1.533	1.516	1.506	1.507	1.509	1.511	1.512	1.512

The impulse response of the DS model implies not only shock persistence but shock magnification. The effect of an innovation is not reversed through time, and it eventually increases the level of real GNP by more than one and a half times the size of the innovation.

For a DS series, $c_\infty \neq 0$, that is, each shock has some permanent effect.

Rudebusch (1993) “The Uncertain Unit Root in Real GNP”, p. 266

Part Availability

“The data for this case are adapted from a series provided by a large U.S. corporation. There are 90 weekly observations showing the percent of the time that parts for an industrial product are available when needed.”

Pankratz (1983) “Forecasting with univariate Box-Jenkins models”

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Exercise 3.5:

- **Identification**: Which model would you chose and why?
- **Estimation**: Estimate your model!
- **Diagnostic checking**: Is the selected model a statistically adequate representation of the available data?

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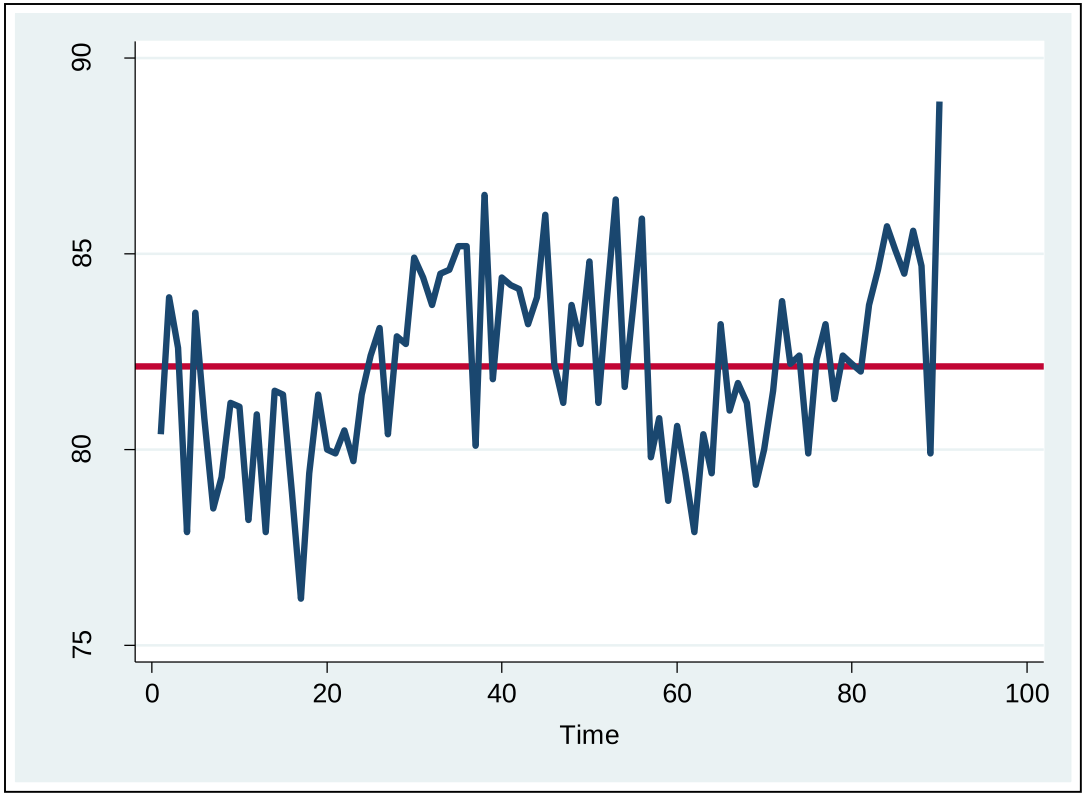
Solution 3.5-1:

Original Time Series



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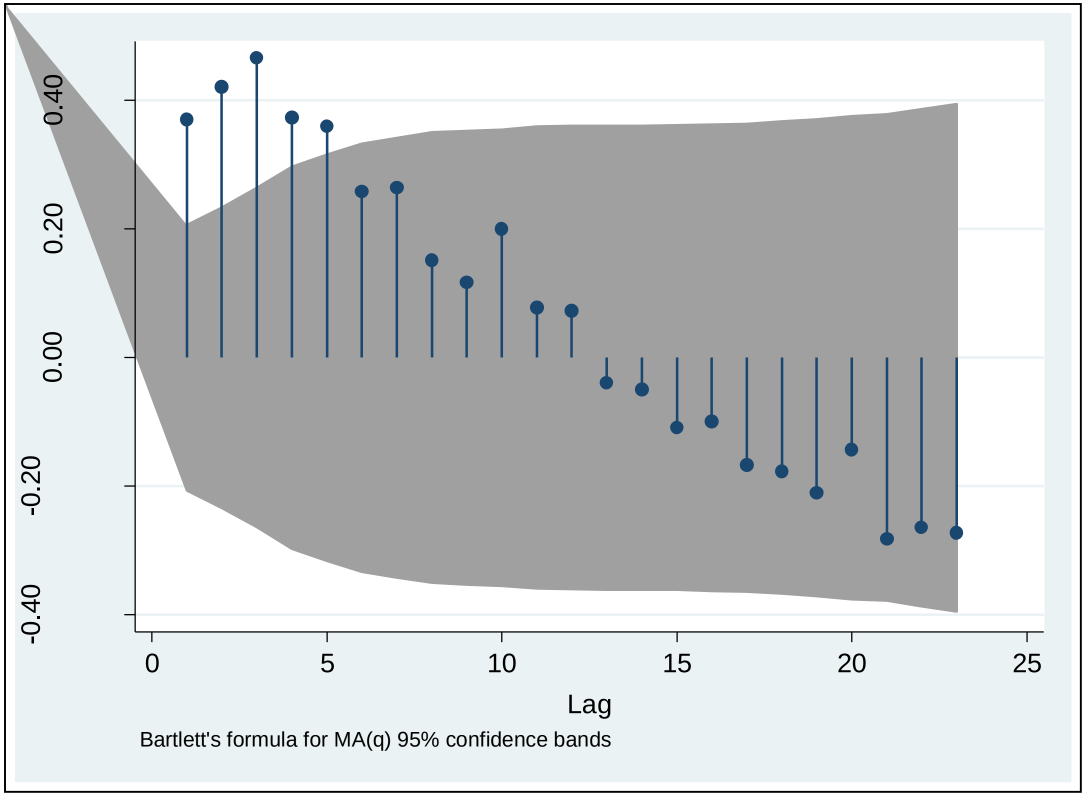
Solution 3.5-2:

ACF → maybe not stationary



$$Var(\hat{\rho}_k) = \begin{cases} \frac{1}{T} & k = 1 \\ \frac{1}{T}\left\{1 + 2\sum_{i=1}^{k-1}\hat{\rho}_i^2\right\} & k > 1 \end{cases}$$

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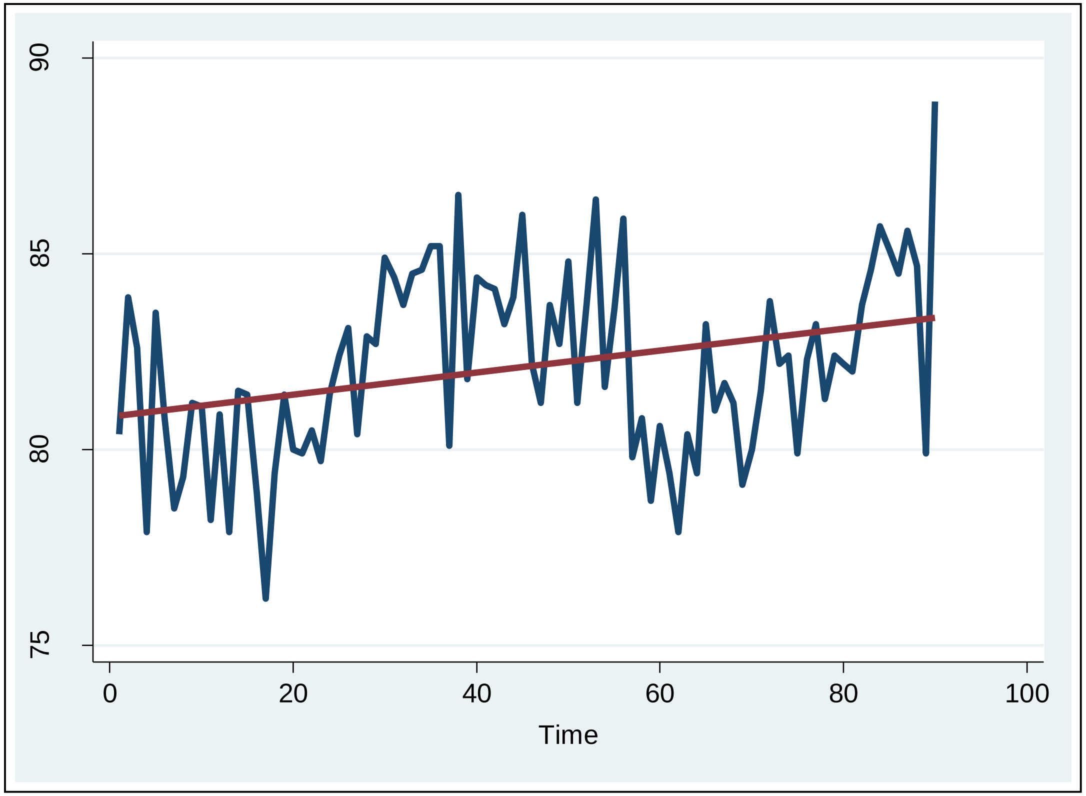
Solution 3.5-3:

Deterministic Trend



Solution 3.5-4:

Deterministic Trend

```
. regress parts_availability time

      Source |       SS           df       MS      Number of obs   =        90
-------------+----------------------------------   F(1, 88)        =      9.31
       Model |  48.2660077         1  48.2660077   Prob > F        =    0.0030
    Residual |  456.383779        88  5.18617931   R-squared       =    0.0956
-------------+----------------------------------   Adj R-squared   =    0.0854
       Total |  504.649787        89  5.67022232   Root MSE        =    2.2773

------------------------------------------------------------------------------
parts_avai~y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        time |   .0281887   .0092401     3.05   0.003     .0098259    .0465514
       _cons |   80.83853   .4841298   166.98   0.000     79.87642    81.80063
------------------------------------------------------------------------------
```

$$\hat{y}_t = 80.84 + 0.028 \cdot t + \hat{u}_t \quad with \quad u_t \approx ARMA(p,q) \qquad (?)$$

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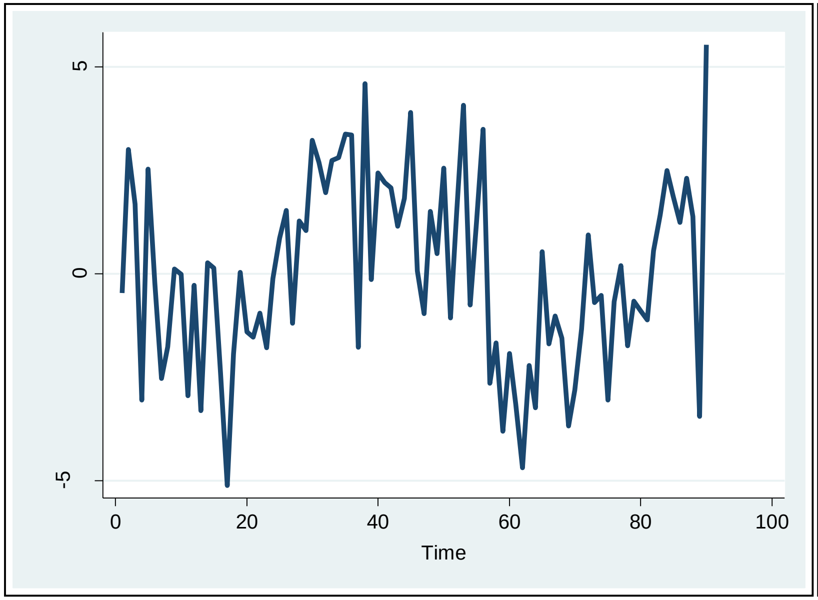
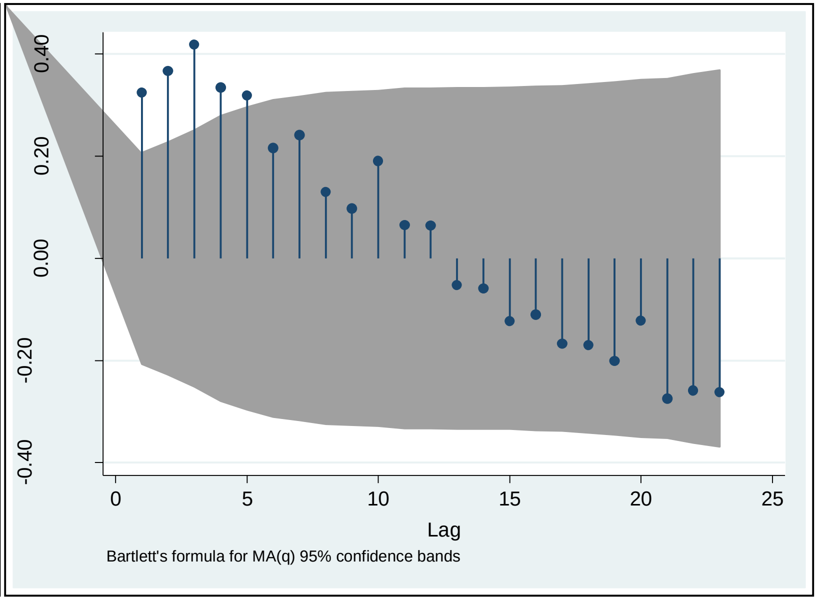
Solution 3.5-5:

OLS Residuals



ACF OLS Residuals



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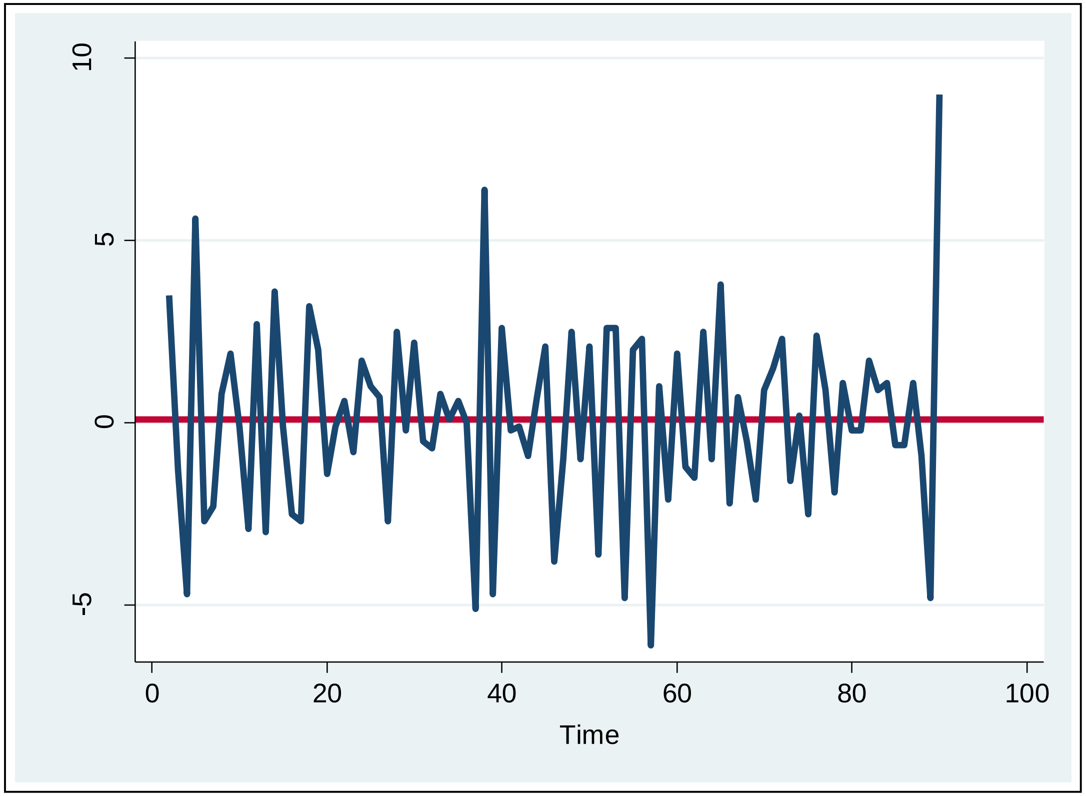
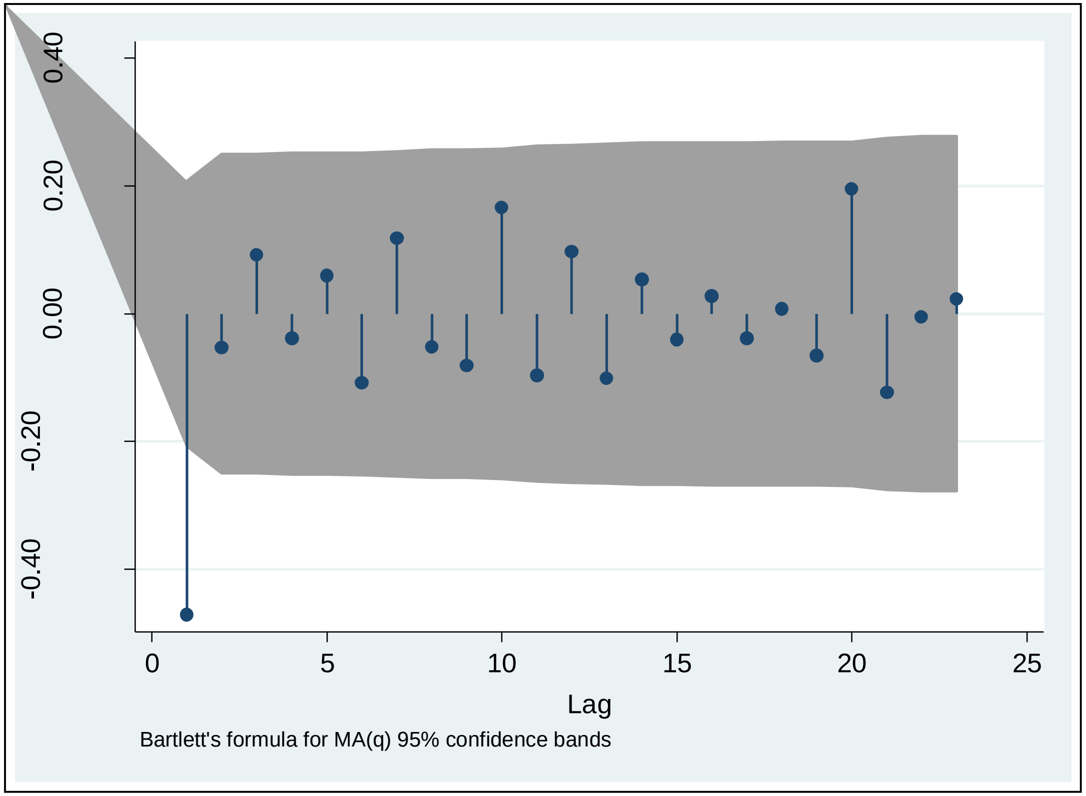
Solution 3.5-6:

Differenced Series



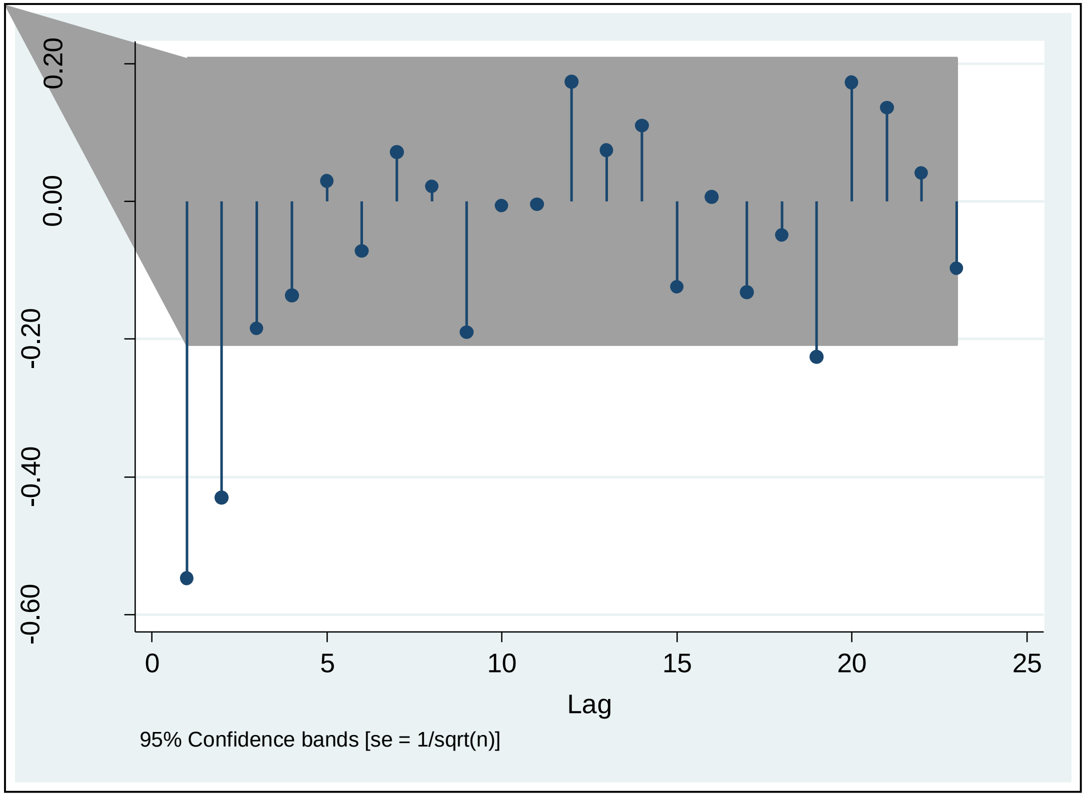
Solution 3.5-7:

ACF of the Differenced Series



Solution 3.5-8:

PACF of the Differenced Series



Solution 3.5-8:

```
. arima parts_availability, arima(0 1 1)
[...]

Sample:   2 to 90                               Number of obs      =         89
                                                Wald chi2(1)       =      69.20
Log likelihood = -188.7081                      Prob > chi2        =     0.0000

------------------------------------------------------------------------------
             |                 OPG
D.           |
parts_avai~y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
parts_avai~y |
       _cons |   .0426991   .0657141     0.65   0.516    -.0860981    .1714963
-------------+----------------------------------------------------------------
ARMA         |
          ma |
         L1. |  -.7242702   .0870639    -8.32   0.000    -.8949124    -.553628
-------------+----------------------------------------------------------------
      /sigma |   2.008118   .1710866    11.74   0.000     1.672794    2.343441
------------------------------------------------------------------------------
```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Solution 3.5-9: Alternative

```
. arima D.parts_availability, ma(1)
[...]

Sample:   2 to 90                               Number of obs      =        89
                                                Wald chi2(1)       =     69.20
Log likelihood = -188.7081                      Prob > chi2        =    0.0000

------------------------------------------------------------------------------
             |                 OPG
D.           |
parts_avai~y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
parts_avai~y |
       _cons |   .0426991   .0657141     0.65   0.516    -.0860981    .1714963
-------------+----------------------------------------------------------------
ARMA         |
          ma |
         L1. |  -.7242702   .0870639    -8.32   0.000    -.8949124    -.553628
-------------+----------------------------------------------------------------
      /sigma |   2.008118   .1710866    11.74   0.000     1.672794    2.343441
------------------------------------------------------------------------------
```

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Solution 3.5-10:

Stata's arima command

$$y_t = \sum_{i=1}^{p} \varphi_i y_{t-i} \boxed{+} \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$$

```
. arima parts_availability, arima(0 1 1) noconstant
[...]

Sample:   2 to 90                              Number of obs      =         89
                                               Wald chi2(1)       =      63.33
Log likelihood = -188.9507                     Prob > chi2        =     0.0000

------------------------------------------------------------------------------
             |                 OPG
D.           |
parts_avai~y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ARMA         |
          ma |
         L1. |  -.7175448   .0901645    -7.96   0.000    -.8942639   -.5408257
-------------+----------------------------------------------------------------
      /sigma |    2.01387   .1709397    11.78   0.000     1.678834    2.348906
------------------------------------------------------------------------------
```

$x_t \sim \text{ARMA}(0,1) \qquad x_t = (1 - 0.7175448L)\varepsilon_t$

$y_t \sim \text{ARIMA}(0,1,1) \qquad (1-L)y_t = (1 - 0.7175448L)\varepsilon_t$

Lag-Operator-Notation!

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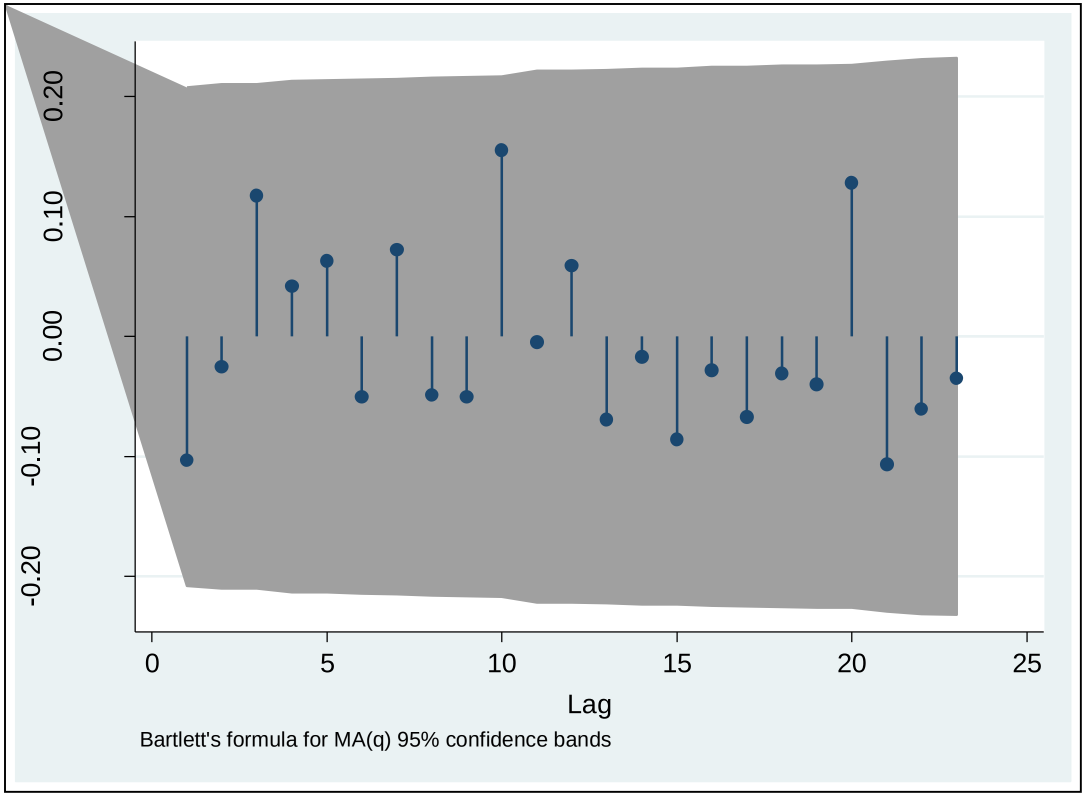
Solution 3.5-11:

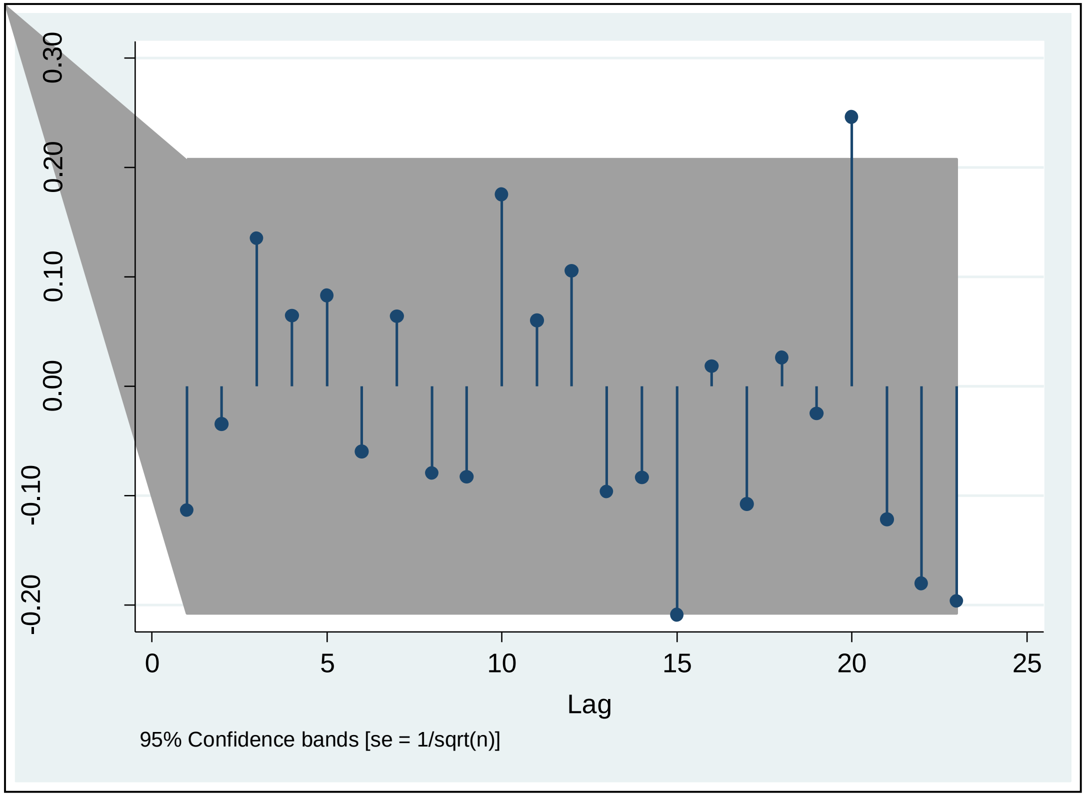
ACF of the Residuals of the estimated ARIMA(0,1,1) Model



Raum: H 5103 D, **Tel.:** 314-78734, **E-Mail:** franziska.plitzko@tu-berlin.de

Solution 3.5-12:

PACF of the Residuals of the estimated ARIMA(0,1,1) Model



Raum: H 5103 D, **Tel.:** 314-78734, **E-Mail:** franziska.plitzko@tu-berlin.de

Solution 3.5-13:

```
. corrgram residuals, lags(22)
                                                    -1       0       1 -1       0       1
 LAG       AC       PAC        Q     Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1       -0.1034  -0.1132   .98422  0.3212          |                  |
2       -0.0256  -0.0343    1.045  0.5930          |                  |
3        0.1172   0.1351   2.3394  0.5050          |                  |-
4        0.0415   0.0648   2.5036  0.6440          |                  |
5        0.0629   0.0833   2.8846  0.7178          |                  |
6       -0.0504  -0.0593   3.1327  0.7920          |                  |
[...]
12       0.0590   0.1052   6.9822  0.8588          |                  |
13      -0.0689  -0.0963   7.4876  0.8753          |                  |
14      -0.0168  -0.0833   7.5181  0.9129          |                  |
15      -0.0856  -0.2088   8.3196  0.9103          |                 -|
16      -0.0279   0.0180   8.4058  0.9359          |                  |
17      -0.0672  -0.1072   8.9144  0.9429          |                  |
18      -0.0307   0.0264   9.0221  0.959
19      -0.0397  -0.0253   9.2042  0.969
20       0.1282   0.2464   11.133  0.942
21      -0.1067  -0.1214   12.489  0.925
22      -0.0606  -0.1801   12.933
```

$$Q = T(T+2)\sum_{k=1}^{K}\frac{1}{T-k}\hat{\rho}_k^2 \sim \chi^2 \quad \text{with } K-p-q \text{ degrees of freedom}$$

```
. di 1-chi2(21, 12.933)
. 91095258
```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Percentiles of the chi-squared distribution

Percentiles of the χ^2 Distribution

	Percent									
df	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
1	0.000039	0.000157	0.000982	0.003932	0.015791	2.705544	3.841459	5.023886	6.634897	7.879439
2	0.010025	0.020101	0.050636	0.102587	0.210721	4.605170	5.991465	7.377759	9.210340	10.596635
3	0.071722	0.114832	0.215795	0.351846	0.584374	6.251388	7.814728	9.348404	11.344867	12.838156
4	0.206989	0.297109	0.484419	0.710723	1.063623	7.779440	9.487729	11.143287	13.276704	14.860259
5	0.411742	0.554298	0.831212	1.145476	1.610308	9.236357	11.070498	12.832502	15.086272	16.749602
6	0.675727	0.872090	1.237344	1.635383	2.204131	10.644641	12.591587	14.449375	16.811894	18.547584
7	0.989256	1.239042	1.689869	2.167350	2.833107	12.017037	14.067140	16.012764	18.475307	20.277740
8	1.344413	1.646497	2.179731	2.732637	3.489539	13.361566	15.507313	17.534546	20.090235	21.954955
9	1.734933	2.087901	2.700390	3.325113	4.168159	14.683657	16.918978	19.022768	21.665994	23.589351
10	2.155856	2.558212	3.246973	3.940299	4.865182	15.987179	18.307038	20.483177	23.209251	25.188180
11	2.603222	3.053484	3.815748	4.574813	5.577785	17.275009	19.675138	21.920049	24.724970	26.756849
12	3.073824	3.570569	4.403789	5.226029	6.303796	18.549348	21.026070	23.336664	26.216967	28.299519
13	3.565035	4.106915	5.008751	5.891864	7.041505	19.811929	22.362032	24.735605	27.688250	29.819471
14	4.074675	4.660425	5.628726	6.570631	7.789534	21.064144	23.684791	26.118948	29.141238	31.319350
15	4.600916	5.229349	6.262138	7.260944	8.546756	22.307130	24.995790	27.488393	30.577914	32.801321
16	5.142205	5.812213	6.907664	7.961646	9.312236	23.541829	26.296228	28.845351	31.999927	34.267187
17	5.697217	6.407760	7.564186	8.671760	10.085186	24.769035	27.587112	30.191009	33.408664	35.718466
18	6.264805	7.014911	8.230746	9.390455	10.864936	25.989423	28.869299	31.526378	34.805306	37.156451
19	6.843971	7.632730	8.906517	10.117013	11.650910	27.203571	30.143527	32.852327	36.190869	38.582257
20	7.433844	8.260398	9.590778	10.850812	12.442609	28.411981	31.410433	34.169607	37.566235	39.996846
21	8.033653	8.897198	10.282898	11.591305	13.239598	29.615089	32.670573	35.478876	38.932173	41.401065
22	8.642716	9.542492	10.982321	12.338015	14.041493	30.813282	33.924439	36.780712	40.289360	42.795655
23	9.260425	10.195716	11.688552	13.090514	14.847956	32.006900	35.172462	38.075627	41.638398	44.181275
24	9.886234	10.856362	12.401150	13.848425	15.658684	33.196244	36.415028	39.364077	42.979820	45.558512
25	10.519652	11.523975	13.119720	14.611408	16.473408	34.381587	37.652484	40.646469	44.314105	46.927890
26	11.160237	12.198147	13.843905	15.379157	17.291885	35.563171	38.885139	41.923170	45.641683	48.289882
27	11.807587	12.878504	14.573383	16.151396	18.113896	36.741217	40.113272	43.194511	46.962942	49.644915
28	12.461336	13.564710	15.307861	16.927875	18.939243	37.915923	41.337138	44.460792	48.278236	50.993376
29	13.121149	14.256455	16.047072	17.708366	19.767744	39.087470	42.556968	45.722286	49.587885	52.335618
30	13.786720	14.953457	16.790772	18.492661	20.599235	40.256024	43.772972	46.979242	50.892181	53.671962

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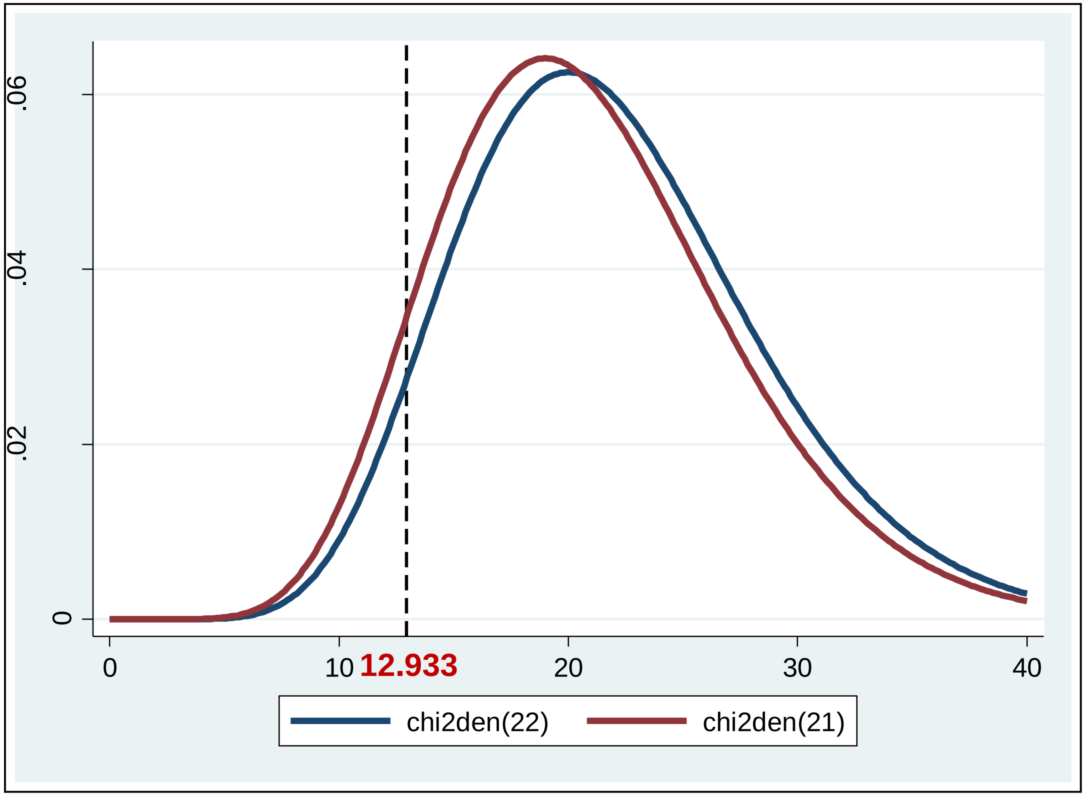
Solution 3.5-14:

χ^2 Distribution



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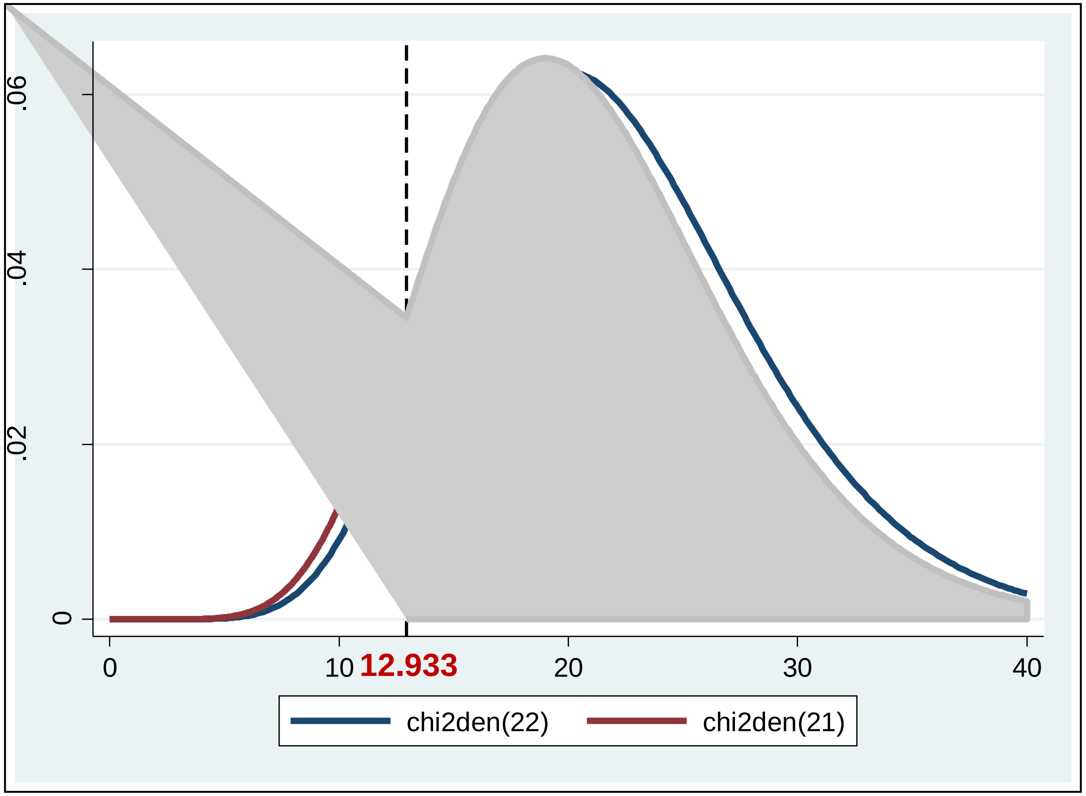
Solution 3.5-15:

χ^2 Distribution




```
. di 1-chi2(21, 12.933)
.91095258
```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Exercise 3.6:

Forecasting

- Forecast x_t from one to four weeks ahead!
- Forecast y_t from one to four weeks ahead!
- Forecast x_t and y_t from one to four weeks ahead using the information that we know at the end of week 91 that y_{91} = 87 and that we know at the end of week 92 that y_{92} = 86.5.

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Forecasting

Optimal forecast:

$$\tilde{y}_{T+1} \mid \Omega_T = E\left(Y_{T+1} \mid \Omega_T\right)$$

it minimizes the expected squared forecast error

$$\min E\left(e_{T+1}^2\right)$$

$$e_{T+1} = y_{T+1} - \tilde{y}_{T+1} \mid \Omega_T$$

Information set Ω_T:

- true model
- known parameters
- all past observations

Additional assumption:

$$E\left[\varepsilon_{T+k}\right] = 0$$

$$\forall k \geq 1$$

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Forecasting an ARIMA (*p*,1,*q*)

$$x_t = y_t - y_{t-1} \qquad => \qquad \boxed{y_t = y_{t-1} + x_t}$$

$$In\ period\ (T+1): \quad y_{T+1} = y_T + x_{T+1} \quad \to \tilde{y}_{T+1/\Omega_T} = E(y_T + x_{T+1} / \Omega_T) = y_T + \tilde{x}_{T+1/\Omega_T}$$

$$In\ period\ (T+2): \quad y_{T+2} = y_{T+1} + x_{T+2} = (y_T + x_{T+1}) + x_{T+2}$$

$$\to \tilde{y}_{T+2/\Omega_T} = E(y_T + x_{T+1} + x_{T+2} / \Omega_T) = \underbrace{y_T + \tilde{x}_{T+1/\Omega_T}}_{\tilde{y}_{T+1/\Omega_T}} + \tilde{x}_{T+2/\Omega_T}$$

$$In\ period\ (T+3): \quad y_{T+3} = y_{T+2} + x_{T+3} = (y_T + x_{T+1} + x_{T+2}) + x_{T+3}$$

$$\to \tilde{y}_{T+3/\Omega_T} = E(y_T + x_{T+1} + x_{T+2} + x_{T+3} / \Omega_T)$$

$$= \underbrace{y_T + \tilde{x}_{T+1/\Omega_T} + \tilde{x}_{T+2/\Omega_T}}_{\tilde{y}_{T+2/\Omega_T}} + \tilde{x}_{T+3/\Omega_T}$$

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ARMA(*p*,*q*) process at time *T* + *l*:

$$\tilde{x}_{T+l|\Omega_T} = \varphi_1 \tilde{x}_{T+l-1|\Omega_T} + \ldots + \varphi_p \tilde{x}_{T+l-p|\Omega_T} + \tilde{\varepsilon}_{T+l|\Omega_T} - \theta_1 \tilde{\varepsilon}_{T+l-1|\Omega_T} - \ldots - \theta_q \tilde{\varepsilon}_{T+l-q|\Omega_T}$$

Recursive forecasting recipe:

1. replace unknown x_{T+l} by their forecasts for $l > 0$;
2. "forecasts" of x_{T+l} , $l \leq 0$, are simply the known values x_{T+l}
3. since ε_t is white noise, the optimal forecast of ε_{T+l}, $l > 0$, is simply zero
4. "forecasts" of ε_{T+l}, $l \leq 0$, are just the known values ε_{T+l}

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Solution 3.6-1:

MA(1) without constant: $x_t = \varepsilon_t - \hat{\theta}_1 \varepsilon_{t-1} = \varepsilon_t - 0.7175448\varepsilon_{t-1}$

```
. list time parts_availability x_tilde in 88/90
```

	time	y_t :parts_~y	$\tilde{x}_t$:x_tilde
88.	88	84.7	-.947457
89.	89	79.9	-.0340514
90.	90	88.9	3.419778

$$\tilde{x}_{T+1|\Omega_T} = \tilde{\varepsilon}_{T+1|\Omega_T} - \theta_1 \tilde{\varepsilon}_{T|\Omega_T}$$

$$\tilde{x}_{T+1|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+1|\Omega_T}}_{0} - \theta_1 \tilde{\varepsilon}_{T|\Omega_T} \quad \text{with} \quad \tilde{\varepsilon}_{T|\Omega_T} = x_T - \tilde{x}_T = (y_T - y_{T-1}) - \tilde{x}_T$$

$$= 88.9 - 79.9 - 3.419778 = 5.580222$$

$$= -0.7175448 \cdot 5.580222 = -4.004059$$

$$\tilde{x}_{T+2|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+2|\Omega_T}}_{0} - \theta_1 \underbrace{\tilde{\varepsilon}_{T+1|\Omega_T}}_{0} = 0$$

$$\vdots = \vdots$$

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Solution 3.6-2:

MA(1) without constant: $x_t = (1 - \hat{\theta}_1 L)\varepsilon_t = (1 - 0.7175448L)\varepsilon_t$

```
. list time parts_availability x_tilde in 88/90

     +------------------------------+
     | time   parts_~y     x_tilde |
     |------------------------------|
 88. |   88       84.7    -.947457 |
 89. |   89       79.9   -.0340514 |
 90. |   90       88.9    3.419778 |
     +------------------------------+
```

$$\tilde{x}_{T+1|\Omega_T} = -4.004059$$

$$\tilde{x}_{T+2|\Omega_T} = 0$$

$$\mathrm{M}$$

$$\tilde{y}_{T+1|\Omega_T} = y_T + \tilde{x}_{T+1|\Omega_T}$$

$$= 88.9 - 4.004059 = 84.89594$$

$$\tilde{y}_{T+2|\Omega_T} = \tilde{y}_{T+1|\Omega_T} + \tilde{x}_{T+2|\Omega_T}$$

$$= 84.89594 + 0 = 84.89594$$

$$\mathrm{M} = \mathrm{M}$$

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Solution 3.6-3:

MA(1) without constant: $x_t = (1-\hat{\theta}_1 L)\varepsilon_t = (1-0.7175448L)\varepsilon_t$

Forecast x_t and y_t from one to four weeks ahead using the information that we know at the end of week 91 that y_{91} = 87 and that we know at the end of week 92 that y_{92} = 86.5.

```
. list time parts_availability x_tilde in 88/90

     +-------------------------------+
     | time   parts_~y     x_tilde |
     |-------------------------------|
 88. |   88       84.7    -.947457 |
 89. |   89       79.9   -.0340514 |
 90. |   90       88.9    3.419778 |
     +-------------------------------+
```

$$\tilde{x}_{T+1|\Omega_T} = -0.7175448 \cdot 5.580222 = -4.004059$$

$$\tilde{\varepsilon}_{T+1|\Omega_T} = x_{T+1} - \tilde{x}_{T+1|\Omega_T} = \left(y_{T+1} - y_T\right) - \tilde{x}_{T+1|\Omega_T} = -19 - \left(-4.004059\right) = 2.104057$$

$$\tilde{x}_{T+2|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+2|\Omega_T}}_{0} - \theta_1 \tilde{\varepsilon}_{T+1|\Omega_T} = -0.7175448 \cdot 2.104057 = -1509755$$

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Solution 3.6-4:

MA(1) without constant: $\mathrm{x}_t = (1-\hat{\theta}_1 L)\varepsilon_t = (1-0.7175448L)\varepsilon_t$

Forecast x_t and y_t from one to four weeks ahead using the information that we know at the end of week 91 that y_{91} = 87 and that we know at the end of week 92 that y_{92} = 86.5.

$$\tilde{x}_{T+1|\Omega_T} = -0.7175448 \cdot 5.580222 = -4.004059$$

$$\tilde{\varepsilon}_{T+1|\Omega_T} = x_{T+1} - \tilde{x}_{T+1|\Omega_T} = \left(y_{T+1} - y_T\right) - \tilde{x}_{T+1|\Omega_T} = -19 - \left(-4.004059\right) = 2.104057$$

$$\tilde{x}_{T+2|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+2|\Omega_T}}_{0} - \theta_1 \tilde{\varepsilon}_{T+1|\Omega_T} = -0.7175448 \cdot 2.104057 = -1509755$$

$$\tilde{\varepsilon}_{T+2|\Omega_T} = x_{T+2} - \tilde{x}_{T+2|\Omega_T} = -0.5 - \left(-1509755\right) = 1009755$$

$$\tilde{x}_{T+3|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+3|\Omega_T}}_{0} - \theta_1 \tilde{\varepsilon}_{T+2|\Omega_T} = -0.7175448 \cdot 1009755 = -0.7245447$$

$$\tilde{x}_{T+4|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+4|\Omega_T}}_{0} - \theta_1 \underbrace{\tilde{\varepsilon}_{T+3|\Omega_T}}_{0} = 0$$

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Solution 3.6-5:

MA(1) without constant: $x_t = (1 - \hat{\theta}_1 L)\varepsilon_t = (1 - 0.7175448L)\varepsilon_t$

Forecast x_t and y_t from one to four weeks ahead using the information that we know at the end of week 91 that y_{91} = 87 and that we know at the end of week 92 that y_{92} = 86.5.

```
. list time parts_availability x_tilde in 88/90

     +-----------------------------+
     | time   parts_~y     x_tilde |
     |-----------------------------|
 88. |   88       84.7    -.947457 |
 89. |   89       79.9   -.0340514 |
 90. |   90       88.9    3.419778 |
     +-----------------------------+
```

$$\tilde{x}_{T+1|\Omega_T} = -4.004059$$

$$\tilde{x}_{T+2|\Omega_T} = -1509755$$

$$\tilde{x}_{T+3|\Omega_T} = -0.7245447$$

$$\tilde{x}_{T+4|\Omega_T} = 0$$

$$\tilde{y}_{T+1|\Omega_T} = y_T + \tilde{x}_{T+1|\Omega_T} = 88.9 - 4.004059 = 84.89594$$

$$\tilde{y}_{T+2|\Omega_T} = y_{T+1} + \tilde{x}_{T+2|\Omega_T} = 87 - 1509755 = 85.49024$$

$$\tilde{y}_{T+3|\Omega_T} = y_{T+2} + \tilde{x}_{T+3|\Omega_T} = 86.5 - 0.7245447 = 85.77545$$

$$\tilde{y}_{T+4|\Omega_T} = \tilde{y}_{T+3|\Omega_T} + \tilde{x}_{T+4|\Omega_T} = 85.77545 + 0 = 85.77545$$

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Dynamic forecasts in Stata

For example, `dynamic(10)` would calculate predictions in which any reference to y_t with $t < 10$ evaluates to the actual value of y_t and any reference of y_t with $t > 10$ evaluates to the prediction of y_t. This means that one-step-ahead predictions are calculated for $t < 10$ and dynamic predictions thereafter.

```
. set obs 94
. replace time = _n
. replace parts_availability = 87   in 91
. replace parts_availability = 86.5 in 92
. tsset time
. arima parts_availability in 1/90, arima(0,1,1)
  noconstant
. predict x_tilde_dyn, xb dynamic(91)
. predict y_tilde_dyn, y dynamic(91)
. predict x_tilde, xb
. predict y_tilde, y
```

Stata help “arima postestimation”

Solution 3.6-6:

without information about $\mathbf{y}_{91}$ and $\mathbf{y}_{92}$

$$\tilde{x}_{T+1/\Omega_T} = -4.004059 \qquad \tilde{y}_{T+1/\Omega_T} = y_T + \tilde{x}_{T+1/\Omega_T} = 88.9 - 4.004059 = 84.89594$$

$$\tilde{x}_{T+j/\Omega_T} = 0,\ \ j = 2,3,4 \qquad \tilde{y}_{T+j/\Omega_T} = 84.89594,\ \ j = 2,3,4$$

with information about $\mathbf{y}_{91}$ and $\mathbf{y}_{92}$

$$\tilde{x}_{T+1|\Omega_T} = -4.004059 \qquad \tilde{y}_{T+1|\Omega_T} = y_T + \tilde{x}_{T+1|\Omega_T} = 88.9 - 4.004059 = 84.89594$$

$$\tilde{x}_{T+2|\Omega_T} = -1509755 \qquad \tilde{y}_{T+2|\Omega_T} = y_{T+1} + \tilde{x}_{T+2|\Omega_T} = 87 - 1509755 = 85.49024$$

$$\tilde{x}_{T+3|\Omega_T} = -0.7245447 \qquad \tilde{y}_{T+3|\Omega_T} = y_{T+2} + \tilde{x}_{T+3|\Omega_T} = 86.5 - 0.7245447 = 85.77545$$

$$\tilde{x}_{T+4|\Omega_T} = 0 \qquad \tilde{y}_{T+4|\Omega_T} = \tilde{y}_{T+3|\Omega_T} + \tilde{x}_{T+4|\Omega_T} = 85.77545 + 0 = 85.77545$$

```
. list time parts_availability x_tilde_dyn y_tilde_dyn x_tilde y_tilde
in 91/94

     +--------------------------------------------------------------+
     | time   parts_~y   x_tilde~n   y_tild~n     x_tilde    y_tilde |
     |--------------------------------------------------------------|
 91. |   91         87   -4.004059   84.89594   -4.004059   84.89594 |
 92. |   92       86.5           0   84.89594   -1.509755   85.49024 |
 93. |   93          .           0   84.89594   -.7245447   85.77545 |
 94. |   94          .           0   84.89594           0          . |
     +--------------------------------------------------------------+
```

Exercise 3.7:

Forecasting

- Calculate the forecast error and then the MSE for the (true) model

  $x_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$ and $y_t = y_{t-1} + x_t = y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$
- Calculate confidence intervals for $\tilde{x}_{91}, \tilde{x}_{92}, \tilde{x}_{93}, \tilde{y}_{91}, \tilde{y}_{92}$, and $\tilde{y}_{93}$.

Hint:

$$MSE\left(\tilde{y}_{T+s|\Omega_T}\right) = E\left[\left(y_{T+s} - \tilde{y}_{T+s|\Omega_T}\right)^2\right]$$

$$\left[\tilde{y}_{T+s|\Omega_T} \pm 1.96 \cdot \sqrt{MSE\left(\tilde{y}_{T+s|\Omega_T}\right)}\right]$$

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Solution 3.7-1:

Forecast errors for x_t

$$x_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$x_{T+1} = \varepsilon_{T+1} - \theta_1 \varepsilon_T \qquad \tilde{x}_{T+1|\Omega_T} = -\theta_1 \varepsilon_T$$

$$x_{T+2} = \varepsilon_{T+2} - \theta_1 \varepsilon_{T+1} \qquad \tilde{x}_{T+2|\Omega_T} = 0$$

$$x_{T+3} = \varepsilon_{T+3} - \theta_1 \varepsilon_{T+2} \qquad \tilde{x}_{T+3|\Omega_T} = 0$$

$$e_{T+1} = x_{T+1} - \tilde{x}_{T+1|\Omega_T} = \varepsilon_{T+1} - \theta_1 \varepsilon_T + \theta_1 \varepsilon_T = \varepsilon_{T+1}$$

$$e_{T+2} = x_{T+2} - \tilde{x}_{T+2|\Omega_T} = \varepsilon_{T+2} - \theta_1 \varepsilon_{T+1} + 0 = \varepsilon_{T+2} - \theta_1 \varepsilon_{T+1}$$

$$e_{T+3} = x_{T+3} - \tilde{x}_{T+3|\Omega_T} = \varepsilon_{T+3} - \theta_1 \varepsilon_{T+2} + 0 = \varepsilon_{T+3} - \theta_1 \varepsilon_{T+2}$$

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Solution 3.7-2:

MSE for x_t

$$e_{T+1} = x_{T+1} - \tilde{x}_{T+1|\Omega_T} = \varepsilon_{T+1} - \theta_1\varepsilon_T + \theta_1\varepsilon_T = \varepsilon_{T+1}$$

$$e_{T+2} = x_{T+2} - \tilde{x}_{T+2|\Omega_T} = \varepsilon_{T+2} - \theta_1\varepsilon_{T+1} + 0 = \varepsilon_{T+2} - \theta_1\varepsilon_{T+1}$$

$$e_{T+3} = x_{T+3} - \tilde{x}_{T+3|\Omega_T} = \varepsilon_{T+3} - \theta_1\varepsilon_{T+2} + 0 = \varepsilon_{T+3} - \theta_1\varepsilon_{T+2}$$

$$E\left(e_{T+1}^2\right) = E\left(\varepsilon_{T+1}^2\right) = Var\left(\varepsilon_{T+1}\right) = \sigma_\varepsilon^2$$

$$\begin{aligned} E\left(e_{T+2}^2\right) &= E\left(\varepsilon_{T+2} - \theta_1\varepsilon_{T+1}\right)^2 = E\left(\varepsilon_{T+2}^2 - 2\theta_1\varepsilon_{T+1}\varepsilon_{T+2} + \theta_1^2\varepsilon_{T+1}^2\right) \\ &= E\left(\varepsilon_{T+2}^2\right) - 2\theta_1 E\left(\varepsilon_{T+1}\varepsilon_{T+2}\right) + \theta_1^2 E\left(\varepsilon_{T+1}^2\right) \\ &= \sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2 = \left(1 + \theta_1^2\right)\sigma_\varepsilon^2 \end{aligned}$$

$$E\left(e_{T+3}^2\right) = E\left(\varepsilon_{T+3} - \theta_1\varepsilon_{T+2}\right)^2 = \ldots = \left(1 + \theta_1^2\right)\sigma_\varepsilon^2$$

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Solution 3.7-3:

Forecast error for y_t: $y_t = y_{t-1} + x_t = y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$

$$\begin{aligned} y_{T+1} &= y_T + x_{T+1} \\ &= y_T + \varepsilon_{T+1} - \theta_1 \varepsilon_T \end{aligned}$$

$$\begin{aligned} y_{T+2} &= y_{T+1} + x_{T+2} \\ &= y_{T+1} + \varepsilon_{T+2} - \theta_1 \varepsilon_{T+1} \\ &= y_T + \varepsilon_{T+1} - \theta_1 \varepsilon_T + \varepsilon_{T+2} - \theta_1 \varepsilon_{T+1} \end{aligned}$$

$$\begin{aligned} y_{T+3} &= y_{T+2} + x_{T+3} \\ &= y_{T+2} + \varepsilon_{T+3} - \theta_1 \varepsilon_{T+2} \\ &= y_T + \varepsilon_{T+1} - \theta_1 \varepsilon_T + \varepsilon_{T+2} - \theta_1 \varepsilon_{T+1} + \varepsilon_{T+3} - \theta_1 \varepsilon_{T+2} \end{aligned}$$

Recall :

$$\tilde{x}_{T+1/\Omega_T} = -\theta_1 \varepsilon_T$$

$$\tilde{x}_{T+2/\Omega_T} = 0$$

$$\tilde{x}_{T+3/\Omega_T} = 0$$

$$\tilde{y}_{T+1|\Omega_T} = y_T + \tilde{x}_{T+1|\Omega_T} = y_T - \theta_1 \varepsilon_T$$

$$\tilde{y}_{T+2|\Omega_T} = \tilde{y}_{T+1|\Omega_T} + \tilde{x}_{T+2|\Omega_T} = y_T - \theta_1 \varepsilon_T + 0 = y_T - \theta_1 \varepsilon_T$$

$$\tilde{y}_{T+3|\Omega_T} = \tilde{y}_{T+2|\Omega_T} + \tilde{x}_{T+3|\Omega_T} = y_T - \theta_1 \varepsilon_T + 0 = y_T - \theta_1 \varepsilon_T$$

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Solution 3.7-4:

Forecast error for y_t

$$e_{T+1} = y_{T+1} - \tilde{y}_{T+1|\Omega_T} = y_T + \varepsilon_{T+1} - \theta_1\varepsilon_T - (y_T - \theta_1\varepsilon_T)$$
$$= \varepsilon_{T+1}$$

$$e_{T+2} = y_{T+2} - \tilde{y}_{T+2|\Omega_T} = y_T + \varepsilon_{T+1} - \theta_1\varepsilon_T + \varepsilon_{T+2} - \theta_1\varepsilon_{T+1} - (y_T - \theta_1\varepsilon_T)$$
$$= \varepsilon_{T+1} + \varepsilon_{T+2} - \theta_1\varepsilon_{T+1}$$

$$e_{T+3} = y_{T+3} - \tilde{y}_{T+3|\Omega_T} = y_T + \varepsilon_{T+1} - \theta_1\varepsilon_T + \varepsilon_{T+2} - \theta_1\varepsilon_{T+1}$$
$$+ \varepsilon_{T+3} - \theta_1\varepsilon_{T+2} - (y_T - \theta_1\varepsilon_T)$$
$$= \varepsilon_{T+1} + \varepsilon_{T+2} - \theta_1\varepsilon_{T+1} + \varepsilon_{T+3} - \theta_1\varepsilon_{T+2}$$

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Solution 3.7-5:

MSE for y_t

$$E\left(e_{T+1}^2\right) = E\left(\varepsilon_{T+1}^2\right) = Var\left(\varepsilon_{T+1}\right) = \sigma_\varepsilon^2$$

$$E\left(e_{T+2}^2\right) = E\left(\varepsilon_{T+1} + \varepsilon_{T+2} - \theta_1\varepsilon_{T+1}\right)^2$$

$$= E\begin{pmatrix} \varepsilon_{T+1}^2 + \varepsilon_{T+1}\varepsilon_{T+2} - \theta_1\varepsilon_{T+1}^2 \\ + \varepsilon_{T+1}\varepsilon_{T+2} + \varepsilon_{T+2}^2 - \theta_1\varepsilon_{T+1}\varepsilon_{T+2} \\ - \theta_1\varepsilon_{T+1}^2 - \theta_1\varepsilon_{T+1}\varepsilon_{T+2} + \theta_1^2\varepsilon_{T+1}^2 \end{pmatrix}$$

$$= E\left(\varepsilon_{T+1}^2\right) + E\left(\varepsilon_{T+2}^2\right) + \left(2 - 2\theta_1\right)E\left(\varepsilon_{T+1}\varepsilon_{T+2}\right) - 2\theta_1 E\left(\varepsilon_{T+1}^2\right) + \theta_1^2 E\left(\varepsilon_{T+1}^2\right)$$

$$= \sigma_\varepsilon^2 + \sigma_\varepsilon^2 - 2\theta_1\sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2 = \left[1 + \left(1 - \theta_1\right)^2\right]\sigma_\varepsilon^2$$

$$E\left(e_{T+3}^2\right) = \ldots = \left[1 + \left(1 - \theta_1\right)^2 + \left(1 - \theta_1\right)^2\right]\sigma_\varepsilon^2$$

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Comparison of Forecast Errors

DS model

The *s*-period-ahead forecast error is:

$$y_{T+s|T} - \tilde{y}_{T+s|\Omega_T} = \ldots = \varepsilon_{T+s} + \{1+\psi_1\}\varepsilon_{T+s-1} + \{1+\psi_1+\psi_2\}\varepsilon_{T+s-2} + \ldots$$
$$+ \{1+\psi_1+\psi_2+\ldots+\psi_{s-1}\}\varepsilon_{T+1}$$

MSE of this forecast is:

$$E\left(y_{T+s} - \tilde{y}_{T+s|\Omega_T}\right)^2 = \left\{1+(1+\psi_1)^2+(1+\psi_1+\psi_2)^2+\ldots+(1+\psi_1+\psi_2+\ldots+\psi_{s-1})^2\right\}\sigma_\varepsilon^2$$

for s = 1, 2, 3:

$$E\left(y_{T+1} - \tilde{y}_{T+1|\Omega_T}\right)^2 = \sigma_\varepsilon^2$$

$$E\left(y_{T+2} - \tilde{y}_{T+2|\Omega_T}\right)^2 = \left[1+(1+\psi_1)^2\right]\sigma_\varepsilon^2$$

$$E\left(y_{T+3} - \tilde{y}_{T+3|\Omega_T}\right)^2 = \left[1+(1+\psi_1)^2+(1+\psi_1+\psi_2)^2\right]\sigma_\varepsilon^2$$

Hamilton (1994) “Time Series Analysis”, p. 435-442

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Solution 3.7-6:

$$E\left(y_{T+1} - \tilde{y}_{T+1|\Omega_T}\right)^2 = \sigma_\varepsilon^2$$

$$E\left(y_{T+2} - \tilde{y}_{T+2|\Omega_T}\right)^2 = \left[1 + (1+\psi_1)^2\right] \sigma_\varepsilon^2$$

$$E\left(y_{T+3} - \tilde{y}_{T+3|\Omega_T}\right)^2 = \left[1 + (1+\psi_1)^2 + (1+\psi_1+\psi_2)^2\right] \sigma_\varepsilon^2$$

$$E\left(e_{T+1}^2\right) = \sigma_\varepsilon^2$$

$$E\left(e_{T+2}^2\right) = \left[1 + (1-\theta_1)^2\right] \sigma_\varepsilon^2$$

$$E\left(e_{T+3}^2\right) = \ldots = \left[1 + (1-\theta_1)^2 + (1-\theta_1)^2\right] \sigma_\varepsilon^2$$

$$\left(1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \ldots\right) = (1 - \theta_1 L)$$

$$L^1 : \psi_1 = -\theta_1$$

$$L^2 : \psi_2 = 0$$

$$L^s : \psi_s = 0$$

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Solution 3.7-7:

Confidence Intervals for x_t forecasts

$$\left[\tilde{x}_{T+s|\Omega_T} \pm 1.96 \cdot \sqrt{MSE\left(\tilde{x}_{T+s|\Omega_T}\right)}\,\right]$$ with $MSE\left(\tilde{x}_{T+s|\Omega_T}\right) = E\left[\left(x_{T+l} - \tilde{x}_{T+s|\Omega_T}\right)^2\right]$

$$MSE\left(\tilde{x}_{T+1|\Omega_T}\right) = E\left(e_{T+1}^2\right) = \sigma_\varepsilon^2 \qquad \tilde{x}_{T+1|\Omega_T} = -4.004059 \qquad [-0.0568738, -7.9512442]$$

$$MSE\left(\tilde{x}_{T+2|\Omega_T}\right) = E\left(e_{T+2}^2\right) = \left(1+\theta_1^2\right)\sigma_\varepsilon^2 \qquad \tilde{x}_{T+2|\Omega_T} = 0 \qquad [-4.85819859, 4.85819859]$$

$$MSE\left(\tilde{x}_{T+3|\Omega_T}\right) = E\left(e_{T+3}^2\right) = \left(1+\theta_1^2\right)\sigma_\varepsilon^2 \qquad \tilde{x}_{T+3|\Omega_T} = 0 \qquad [-4.85819859, 4.85819859]$$

```
. arima parts_availability, arima(0 1 1) noconstant
[...]
             |                 OPG
D.           |
parts_avai~y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ARMA         |
          ma |
         L1. |  -.7175448   .0901645    -7.96   0.000    -.8942639   -.5408257
-------------+----------------------------------------------------------------
      /sigma |    2.01387   .1709397    11.78   0.000     1.678834    2.348906
------------------------------------------------------------------------------
```

Solution 3.7-8:

Confidence Intervals for y_t forecasts

$$\left[\tilde{y}_{T+s|\Omega_T} \pm 1.96 \cdot \sqrt{MSE\left(\tilde{y}_{T+s|\Omega_T}\right)}\right] \text{ with } MSE\left(\tilde{y}_{T+s|\Omega_T}\right) = E\left[\left(y_{T+l} - \tilde{y}_{T+s|\Omega_T}\right)^2\right]$$

$$MSE\left(\tilde{y}_{T+1|\Omega_T}\right) = \sigma_\varepsilon^2 \qquad \tilde{y}_{T+1|\Omega_T} = 84.89594 \qquad [80.949, 88.843]$$

$$MSE\left(\tilde{y}_{T+2|\Omega_T}\right) = \left[1 + (1-\theta_1)^2\right]\sigma_\varepsilon^2 \qquad \tilde{y}_{T+2|\Omega_T} = 84.89594 \qquad [80.794, 88.997]$$

$$MSE\left(\tilde{y}_{T+3|\Omega_T}\right) = \left[1 + (1-\theta_1)^2 + (1-\theta_1)^2\right]\sigma_\varepsilon^2 \qquad \tilde{y}_{T+3|\Omega_T} = 84.89594 \qquad [80.645, 89.146]$$

```
. arima parts_availability, arima(0 1 1) noconstant
[...]
             |                 OPG
D.           |
parts_avai~y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ARMA         |
          ma |
         L1. |  -.7175448   .0901645    -7.96   0.000    -.8942639   -.5408257
-------------+----------------------------------------------------------------
      /sigma |    2.01387   .1709397    11.78   0.000     1.678834    2.348906
------------------------------------------------------------------------------
```

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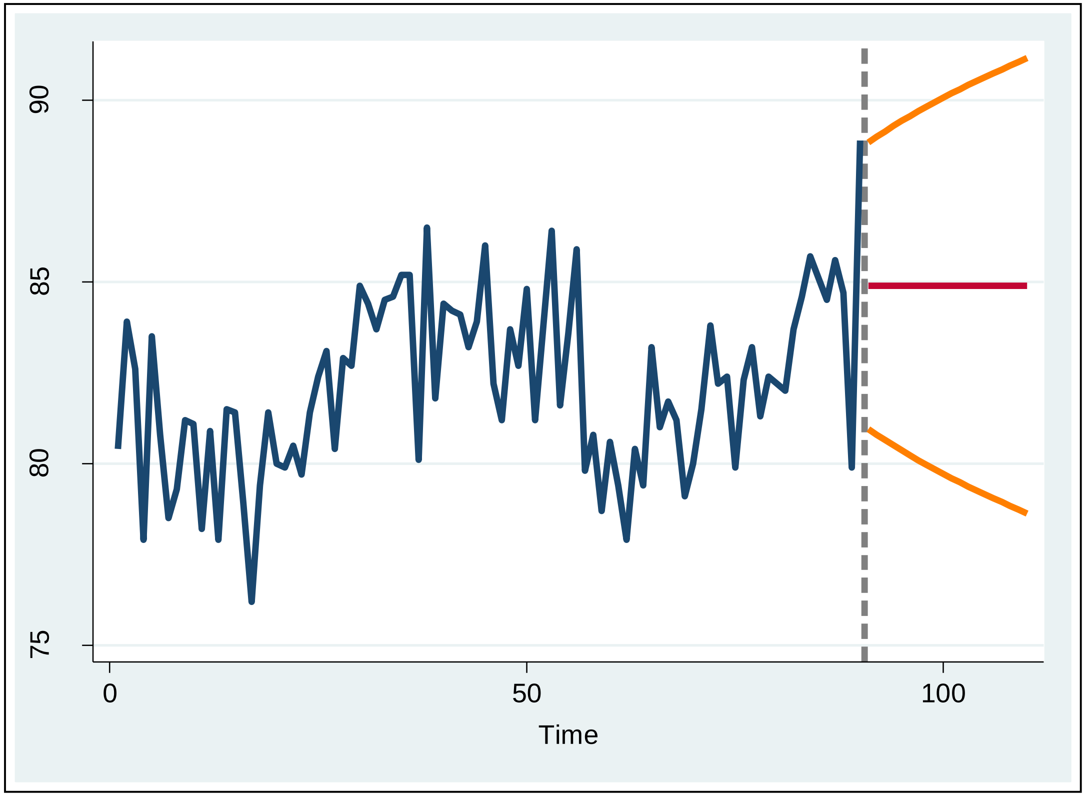
Solution 3.7-9:

Confidence Intervals for y_t forecasts



Nonstationary Stochastic Processes

- Introduction
- Nonstationarity and Trends
- ARIMA Models
- **Unit Root Tests**
- Seasonal ARIMA

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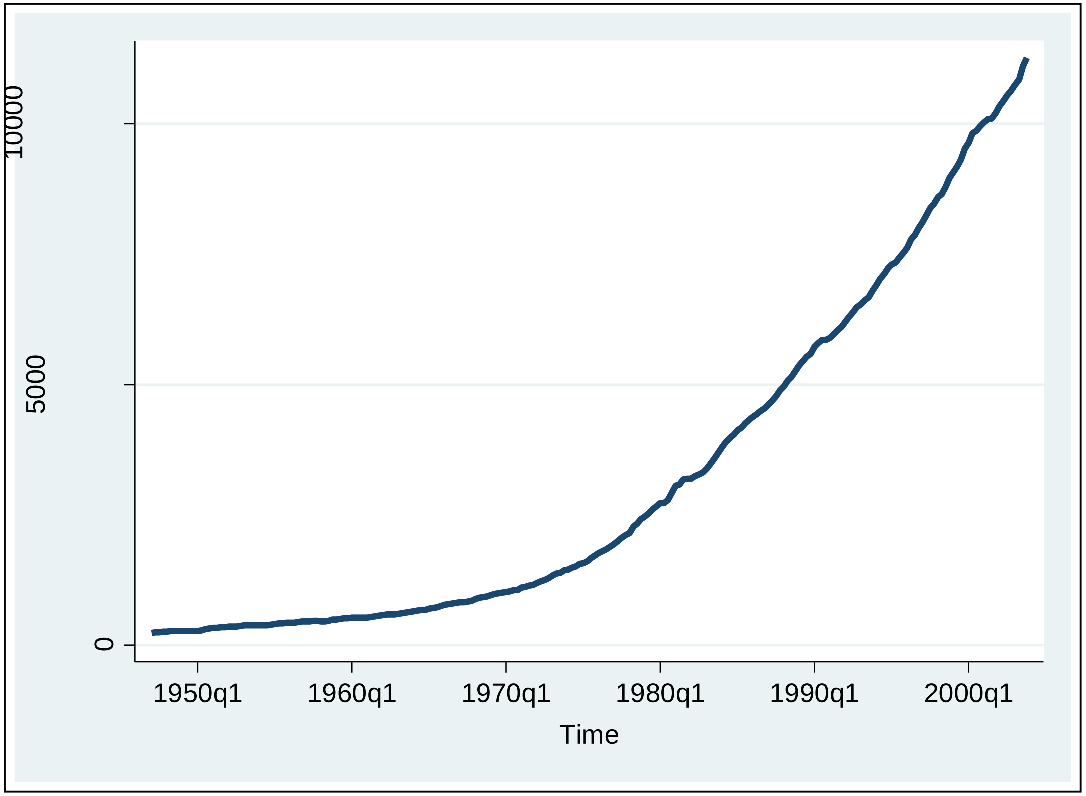
Original Time Series

U.S. quarterly GDP (1947q1 – 2003q4)



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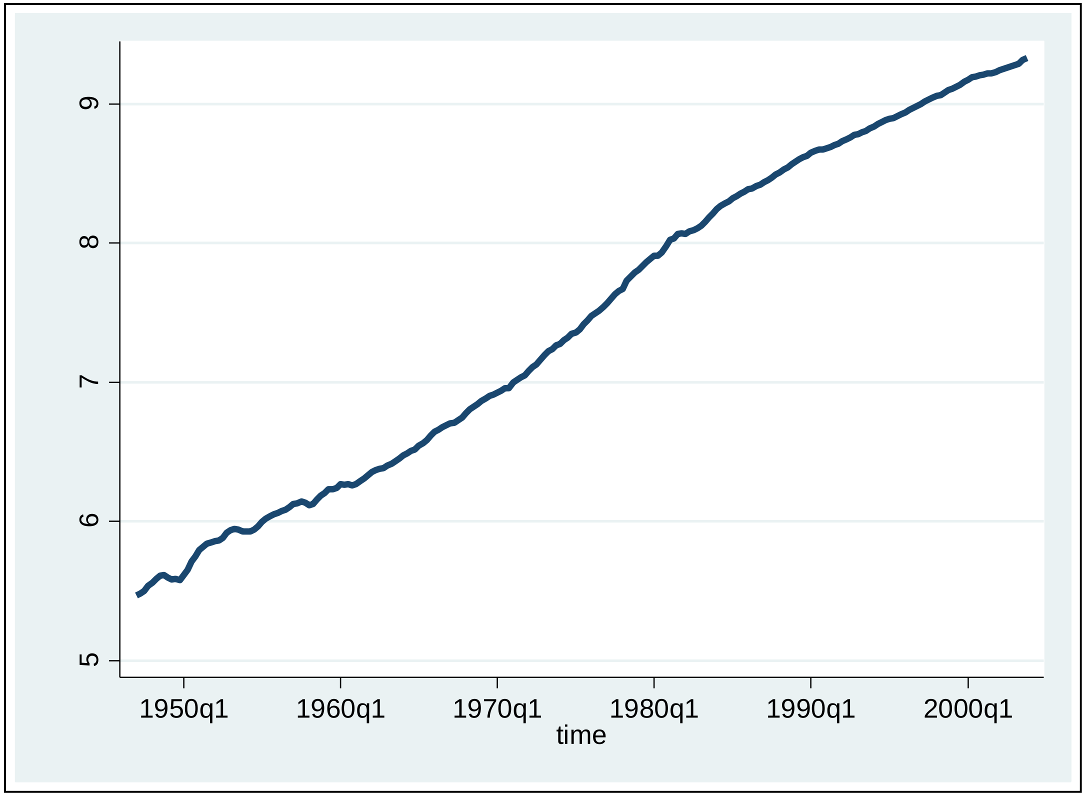
Logarithm of GDP

U.S. quarterly GDP (1947q1 – 2003q4), log



Which unit root test is adequate?

„Fit a specification that is a **plausible description of the data** under both the null and the alternative hypothesis.“

→ **“constant and trend”**

$$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$$

$$H_0: \varphi_1 = 1, (\gamma = 0)$$
$$y_t = y_{t-1} + \delta + \varepsilon_t$$

H_0: random walk with drift

$$H_1: \varphi_1 < 1, (\gamma \neq 0)$$
$$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$$

H_1: trend stationary model with AR(1) errors

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Testing for Unit Roots

no constant, no trend	constant, no trend	constant and trend
$y_t = \varphi_1 y_{t-1} + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$
$H_0{:}\ \varphi_1 = 1$	$H_0{:}\ \varphi_1 = 1,\ (\delta = 0)$	$H_0{:}\ \varphi_1 = 1,\ (\gamma = 0)$
$y_t = y_{t-1} + \varepsilon_t$	$y_t = y_{t-1} + \varepsilon_t$	$y_t = y_{t-1} + \delta + \varepsilon_t$
$H_1{:}\ \varphi_1 < 1$	$H_1{:}\ \varphi_1 < 1,\ (\delta \neq 0)$	$H_1{:}\ \varphi_1 < 1,\ (\gamma \neq 0)$
$y_t = \varphi_1 y_{t-1} + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$
• H_0: pure random walk (no drift) • H_1: stationary AR(1) with mean zero (i.e. strictly speaking $0 \le \varphi_1 < 1$) • simplest case, mostly educational value • “Testing with zero intercept is extremely restrictive, so much that it is hard to imagine ever using it with economic time series”*	• H_0: pure random walk (no drift) • H_1: stationary AR(1) with arbitrary mean • applies to non-growing series • typical examples: “rates” (interest rates, inflation rates, unemployment rates)	• H_0: random walk with drift • H_1: trend stationary model with AR(1) errors • applies to growing series (but not explosive) • typical examples: GDP, consumption, investment

* Davidson, MacKinnon (1993) “Estimation and inference in econometrics ”, p.702

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Testing for Unit Roots

no constant, no trend	constant, no trend	constant and trend
$y_t = \varphi_1 y_{t-1} + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$
$H_0 : \varphi_1 = 1$	$H_0 : \varphi_1 = 1, (\delta = 0)$	$H_0 : \varphi_1 = 1, (\gamma = 0)$
$y_t = y_{t-1} + \varepsilon_t$	$y_t = y_{t-1} + \varepsilon_t$	$y_t = y_{t-1} + \delta + \varepsilon_t$
$H_1 : \varphi_1 < 1$	$H_1 : \varphi_1 < 1, (\delta \neq 0)$	$H_1 : \varphi_1 < 1, (\gamma \neq 0)$
$y_t = \varphi_1 y_{t-1} + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$	$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$
Estimating equations	Estimating equations	Estimating equations
$y_t = \varphi_1 y_{t-1} + \varepsilon_t$ or	$y_t = \varphi_1 y_{t-1} + \delta + \varepsilon_t$ or	$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$ or
$\Delta y_t = \theta y_{t-1} + \varepsilon_t$	$\Delta y_t = \theta y_{t-1} + \delta + \varepsilon_t$	$\Delta y_t = \theta y_{t-1} + \delta + \gamma t + \varepsilon_t$
$\theta = (\varphi_1 - 1)$	$\theta = (\varphi_1 - 1)$	$\theta = (\varphi_1 - 1)$
Test statistics	Test statistics	Test statistics
$t = \frac{(\hat{\varphi}_1 - 1)}{\sigma_{\hat{\varphi}_1}}$ or $t = \frac{\hat{\theta}}{\sigma_{\hat{\theta}}}$	$t = \frac{(\hat{\varphi}_1 - 1)}{\sigma_{\hat{\varphi}_1}}$ or $t = \frac{\hat{\theta}}{\sigma_{\hat{\theta}}}$	$t = \frac{(\hat{\varphi}_1 - 1)}{\sigma_{\hat{\varphi}_1}}$ or $t = \frac{\hat{\theta}}{\sigma_{\hat{\theta}}}$

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Dickey-Fuller Unit Root Test with constant and trend

$$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t$$

$$H_0 : \varphi_1 = 1, (\gamma = 0)$$

$$y_t - y_{t-1} = \varphi_1 y_{t-1} - y_{t-1} + \delta + \gamma t + \varepsilon_t$$

$$\Delta y_t = (\varphi_1 - 1) y_{t-1} + \delta + \gamma t + \varepsilon_t \qquad \theta = (\varphi_1 - 1)$$

$$H_0 : \theta = 0, (\gamma = 0)$$

$$\Delta y_t = \theta y_{t-1} + \delta + \gamma t + \varepsilon_t$$

```
. regress D.lnGDP L.lnGDP time
[...]
```

```
. di -4.58e-06/.0070557
-.00064912
```

$$\boxed{t = \frac{\hat{\theta}}{\hat{\sigma}_{\hat{\theta}}} \nless t_{1-\frac{\alpha}{2},T-1}}$$

```
------------------------------------------------------------------------------
     D.lnGDP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lnGDP |
         L1. |  -4.58e-06   .0070557    -0.00   0.999    -.0139087    .0138995
        time |  -.0000116   .0001295    -0.09   0.929    -.0002668    .0002436
       _cons |   .0177608   .0445152     0.40   0.690    -.0699614     .105483
------------------------------------------------------------------------------
```

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Dickey-Fuller Unit Root Test with constant and trend

$$y_t = \varphi_1 y_{t-1} + \delta + \gamma t + \varepsilon_t \quad H_0 : \varphi_1 = 1, (\gamma = 0)$$

```
. regress lnGDP L.lnGDP time
[...]
------------------------------------------------------------------------------
       lnGDP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lnGDP |
         L1. |   .9999954   .0070557   141.73   0.000     .9860913      1.0139
        time |  -.0000116   .0001295    -0.09   0.929    -.0002668    .0002436
       _cons |   .0177608   .0445152     0.40   0.690    -.0699614     .105483
------------------------------------------------------------------------------
. di  (.9999954-1)/.0070557
-.00065196
```

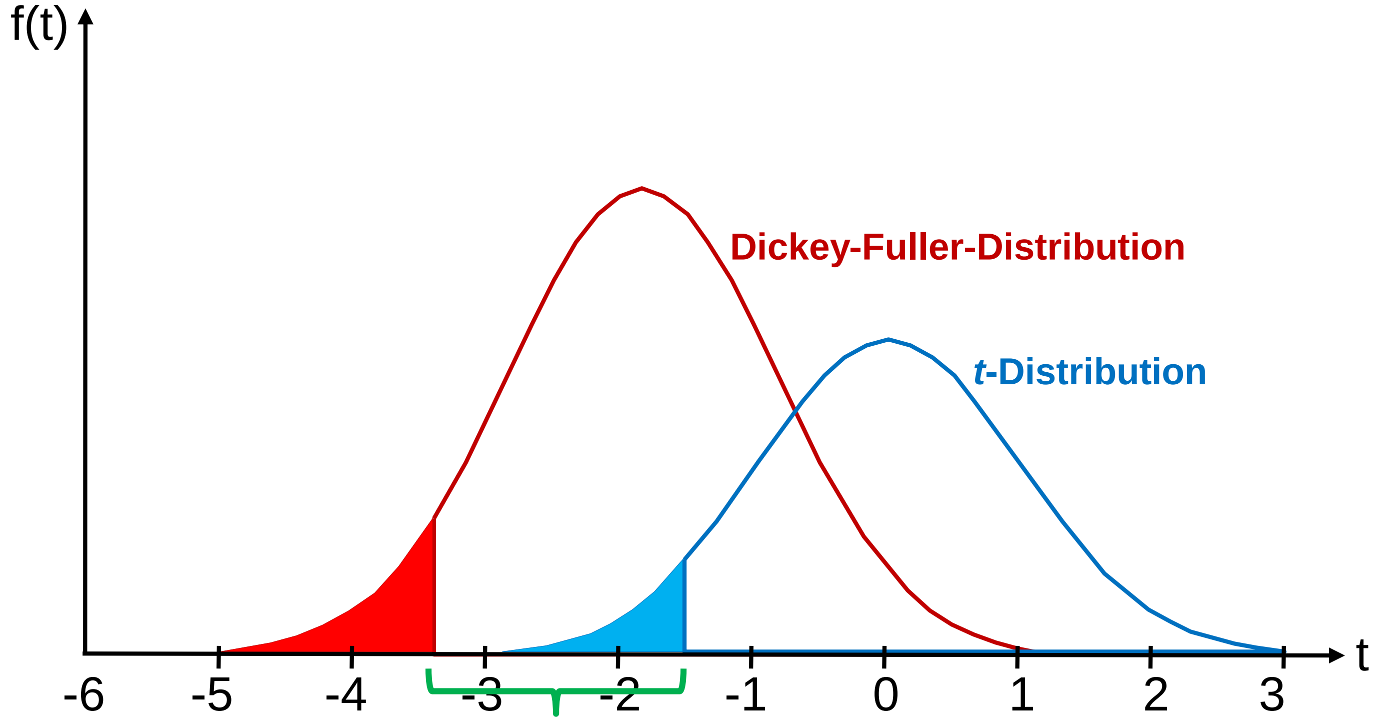
```
. regress D.lnGDP L.lnGDP time
[...]
```

```
. di -4.58e-06/.0070557
-.00064912
```

$$t = \frac{\hat{\theta}}{\hat{\sigma}_{\hat{\theta}}} \nsim t_{1-\frac{\alpha}{2}, T-1}$$

```
------------------------------------------------------------------------------
     D.lnGDP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lnGDP |
         L1. |  -4.58e-06   .0070557    -0.00   0.999    -.0139087    .0138995
        time |  -.0000116   .0001295    -0.09   0.929    -.0002668    .0002436
       _cons |   .0177608   .0445152     0.40   0.690    -.0699614     .105483
------------------------------------------------------------------------------
```

Dickey-Fuller-Distribution vs. *t*-Distribution



If we use the *t*-distribution instead of the Dickey-Fuller-distribution we would reject the null hypothesis too often.

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Critical values for Dickey-Fuller tests

U.S. quarterly GDP (1947q1 – 2003q4), log

Sample Size T	No constant, no trend		Constant, no trend		Constant, trend	
	1%	5%	1%	5%	1%	5%
25	-2.66	-1.95	-3.75	-3.00	-4.38	-3.60
50	-2.62	-1.95	-3.58	-2.93	-4.15	-3.50
100	-2.60	-1.95	-3.51	-2.89	-4.04	-3.45
250	-2.58	-1.95	-3.46	-2.88	-3.99	$\boxed{-3.43}$
500	-2.58	-1.95	-3.44	-2.87	-3.98	-3.42
∞	-2.58	-1.95	-3.43	-2.86	-3.96	-3.41

Verbeek (2000) “A Guide to Modern Econometrics ”

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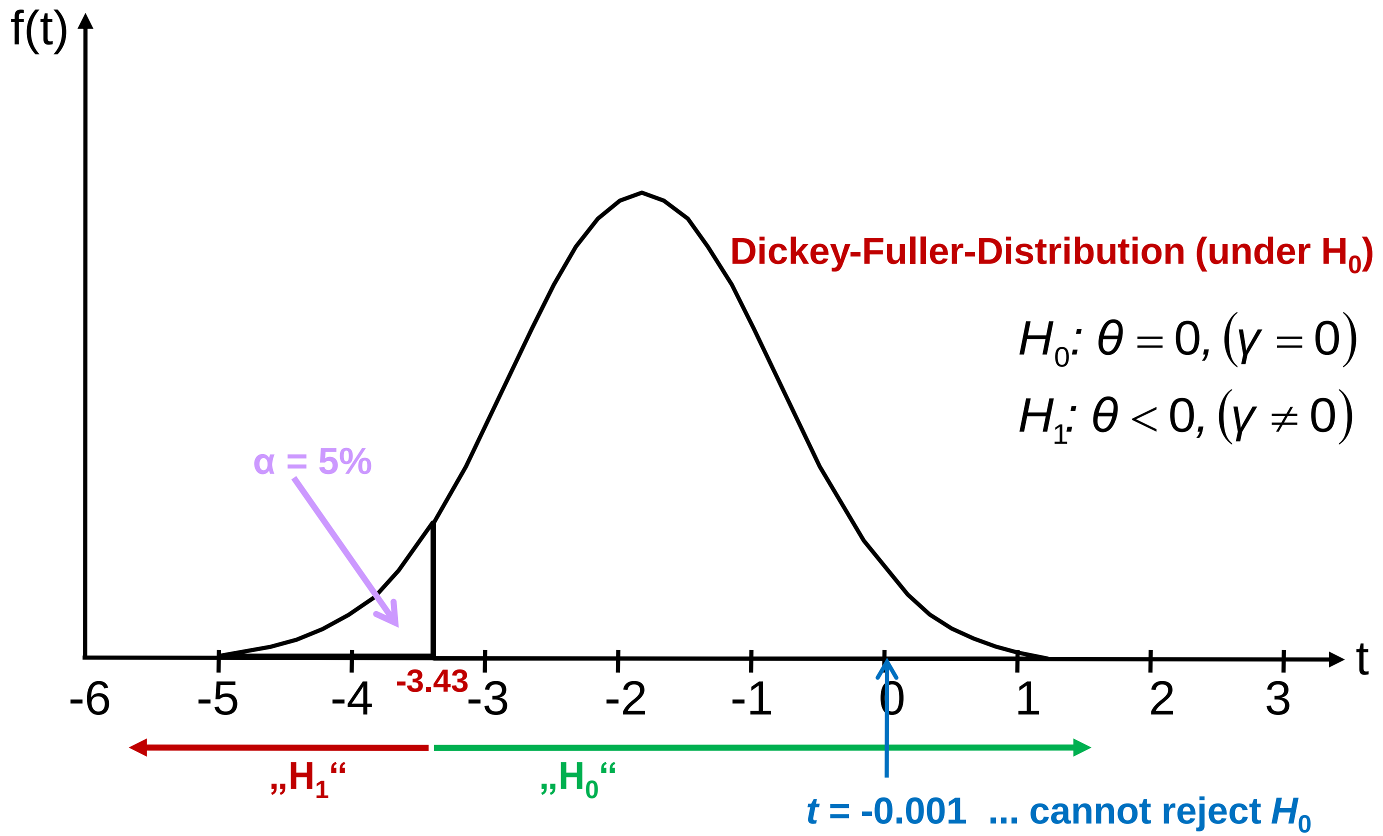
Dickey-Fuller Unit Root Test



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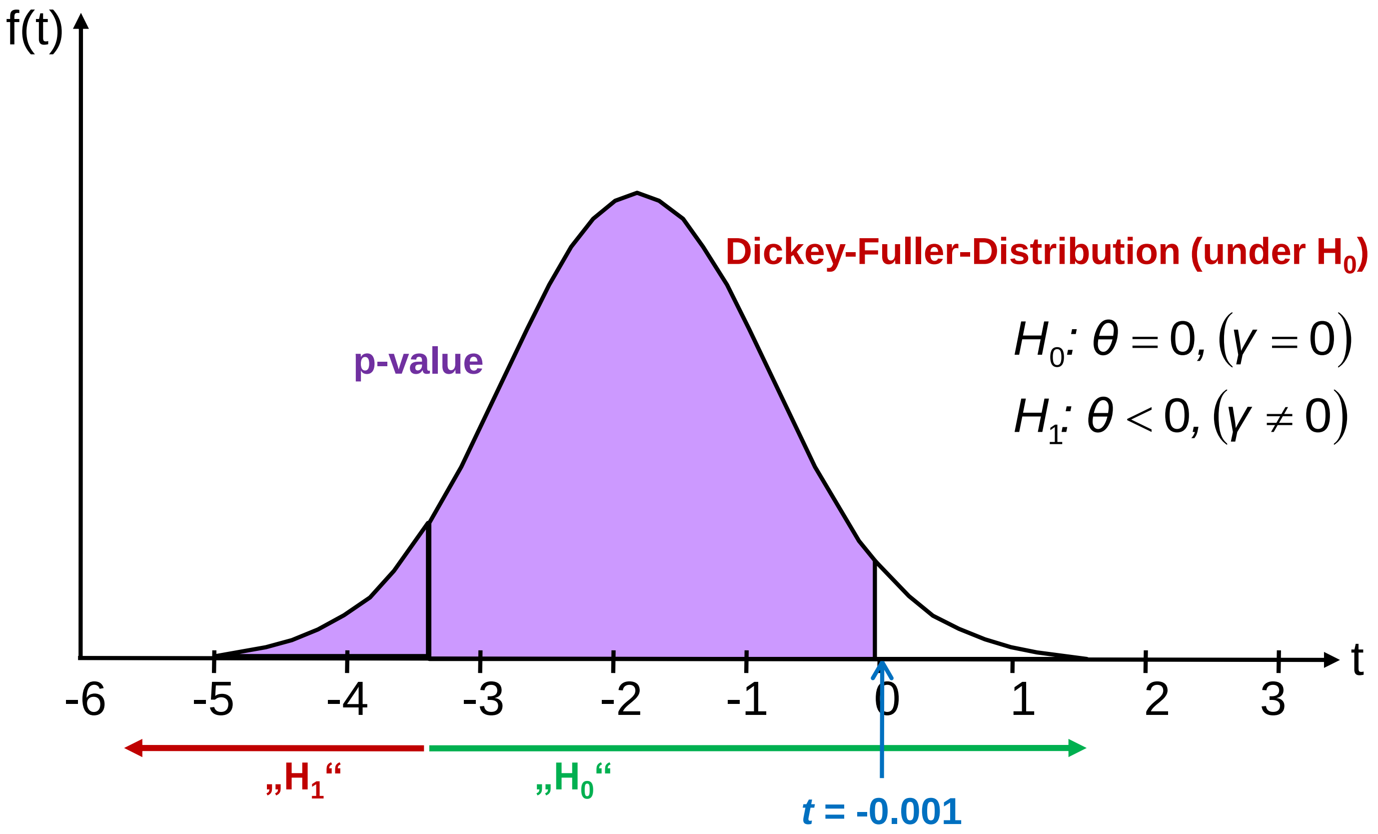
Dickey-Fuller Unit Root Test



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Dickey-Fuller Unit Root Test

```
. regress D.lnGDP L.lnGDP time
[...]
------------------------------------------------------------------------------
     D.lnGDP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lnGDP |
         L1. |  -4.58e-06   .0070557    -0.00   0.999    -.0139087    .0138995
        time |  -.0000116   .0001295    -0.09   0.929    -.0002668    .0002436
       _cons |   .0177608   .0445152     0.40   0.690    -.0699614     .105483
------------------------------------------------------------------------------
. di -4.58e-06/.0070557
-.00064912

. dfuller lnGDP, trend
Dickey-Fuller test for unit root                   Number of obs   =       227

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -0.001            -3.998            -3.433            -3.133
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.9942
```

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Augmented Dickey-Fuller Unit Root Test

Not all time-series processes can be well represented by an AR(1) process. It is possible to use Dickey-Fuller tests in higher-order equations.

Example: AR(2) without constant, no trend

$H_0: \varphi_1 + \varphi_2 = 1$ given $|\varphi_2| < 1$

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \varepsilon_t \qquad \Big| + \varphi_2 y_{t-1} - \varphi_2 y_{t-1}$$

$$= (\varphi_1 + \varphi_2) y_{t-1} - \varphi_2 (y_{t-1} - y_{t-2}) + \varepsilon_t \qquad \Big| - y_{t-1}$$

$$\Delta y_t = (\varphi_1 + \varphi_2 - 1) y_{t-1} - \varphi_2 \Delta y_{t-1} + \varepsilon_t \qquad \Big| \text{ with } \pi_1 = \varphi_1 + \varphi_2 - 1 \text{ and } \pi_2 = -\varphi_2$$

$$= \pi_1 y_{t-1} + \pi_2 \Delta y_{t-1} + \varepsilon_t$$

$H_0: \pi_1 = \varphi_1 + \varphi_2 - 1 = 0$

In general, for an AR(*p*): $\Delta y_t = \pi_1 y_{t-1} + \pi_2 \Delta y_{t-1} + \ldots + \pi_p \Delta y_{t-p+1} + \varepsilon_t$

Any **ARMA model** (with an invertible MA polynomial) can be written as an infinite autoregressive process.

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Augmented Dickey-Fuller Unit Root Test

Any unknown **ARIMA(p, d, q)** process can be well approximated by an **ARIMA(p*, d, 0)** of order no more than **$T^{1/3}$.** (Said and Dickey(1984), Enders(1995), p.226)

=> The above **augmented regression** can also be used to **test for a unit root** in any **ARMA model**.

$$\Delta y_t = \pi_1 y_{t-1} + \pi_2 \boxed{\Delta y_{t-1}} + \ldots + \pi_p \boxed{\Delta y_{t-p+1}} + \delta + \gamma t + \varepsilon_t$$

"augmentation terms"

Note: order **p** => **(p-1)** augmentation terms

$$\begin{aligned}
\pi_1 &= \varphi_1 + \varphi_2 + \varphi_3 + \ldots + \varphi_p - 1 && \text{corresponds to } y_{t-1} \\
\pi_2 &= -(\varphi_2 + \varphi_3 + \ldots + \varphi_p) && \text{corresponds to } \Delta y_{t-1} \\
\pi_3 &= -(\varphi_3 + \ldots + \varphi_p) && \text{corresponds to } \Delta y_{t-2} \\
&\ldots \\
\pi_p &= -\varphi_p && \text{corresponds to } \Delta y_{t-p+1}
\end{aligned}$$

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Augmented Dickey-Fuller Unit Root Test

Why is it important to select the appropriate lag length?

Including **too many** lags:

reduces power of the test to reject the null of a unit root:

- because the number of parameters estimated has increased and
- because the number of usable observations has decreased.

Including **too few** lags:

will not appropriately capture the actual error process and φ_1 and its standard error will not be properly estimated.

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Augmented Dickey-Fuller Unit Root Test

How to select the appropriate lag length?

- Start with a relatively long lag length (p^*) and **pare down the model by the usual *t*-test**.

  $$\Delta y_t = \pi_1 y_{t-1} + \pi_2 \Delta y_{t-1} + \ldots + \pi_p \Delta y_{t-p+1} + \varepsilon_t$$

  If the null hypothesis $\pi_{p^*} = 0$ is accepted, reestimate the regression using a lag length of p^*-1. Repeat the process until the p^*-ℓ is significantly different from zero. If no value of ℓ leads to rejection, the simple Dickey-Fuller test is used.

- Use a model selection criterion to determine the order of the regression, e.g. the **Hannan-Quinn criterion**:

  $$HQ(p) = \log \hat{\sigma}^2(p) + (1+p)\frac{2\ln(\ln(T))}{T} \qquad \hat{\sigma}_\varepsilon^2 = \frac{1}{(T-p)}\sum_{t=1}^{T-p}\hat{\varepsilon}_t^2$$

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Augmented Dickey-Fuller Unit Root Test

Any unknown ARIMA(ρ, d, q) process can be well approximated by an ARIMA(p*, d, 0) of order no more than $T^{1/3}$

```
. di 228^(1/3)
6.1091147
```

Pare down the model by the usual t-test **starting with an AR(p^* = 7)**

```
. regress D.lnGDP L.lnGDP D.L1.lnGDP D.L2.lnGDP D.L3.lnGDP D.L4.lnGDP D.L5.lnGDP
          D.L6.lnGDP time
[...]
------------------------------------------------------------------------------
     D.lnGDP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lnGDP |
         L1. |  -.0052131   .0064806    -0.80   0.422    -.0179877    .0075615
         LD. |   .4261161   .0683021     6.24   0.000     .2914779    .5607543
[...]
        L6D. |   .1092259   .0680106     1.61   0.110    -.0248378    .2432897
        time |   .0000917   .0001197     0.77   0.445    -.0001443    .0003277
       _cons |   .0424386   .0403873     1.05   0.295    -.0371736    .1220507
------------------------------------------------------------------------------
```

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Augmented Dickey-Fuller Unit Root Test

Hannan-Quinn criterion for AR(7)

```
. regress D.lnGDP L.lnGDP D.L1.lnGDP D.L2.lnGDP D.L3.lnGDP D.L4.lnGDP
          D.L5.lnGDP D.L6.lnGDP time
[...]
```

$$\hat{\sigma}_{\varepsilon}^{2} = \frac{1}{(T-p)} \sum_{t=1}^{T-p} \hat{\varepsilon}_t^2$$

```
. predict res_AR7, res
. gen res_AR7_squared = res_AR7^2
. sum res_AR7_squared
    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
res_AR7_sq~d |        221    .0000944    .0001899   6.19e-11   .0017038
```

$$HQ(p) = \log \hat{\sigma}^2(p) + (1+p)\frac{2\ln(\ln(T))}{T}$$

```
. local HQ_AR7 = log(`r(mean)')+((1+7)*(2*log(log(228))/228))
. di `HQ_AR7'
-9.1494445
```

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Augmented Dickey-Fuller Unit Root Test

Any unknown ARIMA(p, d, q) process can be well approximated by an ARIMA(p, d, 0) of order no more than $T^{1/3}$

```
. di 228^(1/3)
6.1091147
```

Pare down the model by the usual t-test **starting with an AR(p^* = 7)**

```
. regress D.lnGDP L.lnGDP D.L1.lnGDP D.L2.lnGDP D.L3.lnGDP D.L4.lnGDP D.L5.lnGDP
         time
[...]
------------------------------------------------------------------------------
      D.lnGDP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        lnGDP |
          L1. |  -.0039244   .0064403    -0.61   0.543    -.0166188    .0087701
          LD. |   .4132228   .0679335     6.08   0.000     .2793183    .5471273
[...]
         L5D. |  -.1101527   .0677209    -1.63   0.105    -.2436381    .0233326
         time |   .0000668   .0001189     0.56   0.574    -.0001674    .0003011
        _cons |    .035684   .0402132     0.89   0.376    -.0435806    .1149486
------------------------------------------------------------------------------
```

Augmented Dickey-Fuller Unit Root Test

Any unknown ARIMA(p, d, q) process can be well approximated by an ARIMA(p, d, 0) of order no more than $T^{1/3}$

```
. di 228^(1/3)
6.1091147
```

Pare down the model by the usual t-test **starting with an AR(p^* = 7)**

```
. regress D.lnGDP L.lnGDP D.L1.lnGDP D.L2.lnGDP D.L3.lnGDP D.L4.lnGDP time
[...]
------------------------------------------------------------------------------
     D.lnGDP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lnGDP |
         L1. |  -.0049456   .0063957    -0.77   0.440    -.0175516    .0076603
         LD. |   .4192889   .0678941     6.18   0.000     .2854692    .5531085
        L2D. |   .1968177   .0725389     2.71   0.007      .053843    .3397924
        L3D. |  -.1302062   .0726467    -1.79   0.074    -.2733933    .0129809
        L4D. |  -.0479274   .0677443    -0.71   0.480    -.1814519     .085597
        time |   .0000859   .0001179     0.73   0.467    -.0001465    .0003183
       _cons |   .0409559   .0400094     1.02   0.307     -.037903    .1198147
------------------------------------------------------------------------------
```

Augmented Dickey-Fuller Unit Root Test

Any unknown ARIMA(p, d, q) process can be well approximated by an ARIMA(p, d, 0) of order no more than $T^{1/3}$

```
. di 228^(1/3)
6.1091147
```

Pare down the model by the usual t-test **starting with an AR(p^* = 7)**

```
. regress D.lnGDP L.lnGDP D.L1.lnGDP D.L2.lnGDP D.L3.lnGDP time
[...]
```

D.lnGDP	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
lnGDP						
L1.	-.005618	.0063167	$\boxed{-0.89}$	~~0.375~~	-.0180677	.0068316
LD.	.4236657	.0660744	6.41	0.000	.2934392	.5538922
L2D.	.1899376	.0712332	2.67	0.008	.0495436	.3303316
L3D.	**-.1487725**	**.0667618**	**-2.23**	**0.027**	**-.2803537**	**-.0171914**
time	.0000989	.0001163	0.85	0.396	-.0001304	.0003281
_cons	.0446789	.0395887	1.13	0.260	-.0333466	.1227045

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Augmented Dickey-Fuller Unit Root Test

Hannan-Quinn criterion for AR(1) to AR(7)

$$HQ(p) = \log \hat{\sigma}^2(p) + (1+p)\frac{2\ln(\ln(T))}{T}$$

p	1	2	3	4	5	6	7
HQ(p)	-8.9346092	-9.155521	-9.1521224	$\boxed{-9.1789607}$	-9.1624933	-9.1556618	-9.1494445

```
. dfuller lnGDP, trend lags(3) regress
Augmented Dickey-Fuller test for unit root         Number of obs   =       224

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -0.889            -3.999            -3.433            -3.133
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.9573
[...]
```

... cannot reject H_0

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Additional Information

For trending series we have to discriminate between deterministic and stochastic trends.

In a Trend-Stationary-Model we have a deterministic trend with *stationary stochastic* fluctuations around this deterministic trend. A Difference-Stationary-Model can have a stochastic trend (e.g. a Random Walk) or a combination of a stochastic and deterministic trend (e.g. a Random Walk with drift). To check if the time series contains a stochastic trend (with a unit root) we can use the (Augmented) Dickey-Fuller-Test. This test is not that powerful, but it gives us a hint, that we can use a Difference-Stationary-Model to deal with the non-stationary time series. If we can reject the hypothesis of a unit root in the data, it means we can still try to get rid of the non-stationarity by using a Trend-Stationary-Model.

For the augmented Dickey-Fuller-Test we consider the general formula (with constant and trend) to test for one unit root (if there is more then one unit root we would have to consider differencing the series more then just once):

$$\Delta y_t = \pi_1 y_{t-1} + \pi_2 \Delta y_{t-1} + \ldots + \pi_p \Delta y_{t-p+1} + \delta + \gamma t + \varepsilon_t$$

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Additional Information

The null hypothesis of the test is:

$$\pi_1 = \sum_k \varphi_k - 1 = 0$$

We estimate the equation:

$$\Delta y_t = \pi_1 y_{t-1} + \pi_2 \Delta y_{t-1} + \ldots + \pi_p \Delta y_{t-p+1} + \delta + \gamma t + \varepsilon_t$$

By ordinary least squares and focus on the estimate of π_1.

For the coefficients π_j with j>1 of the OLS-Regression we can use the standard t-statistic, but for π_1 , we have to consider the Dickey-Fuller-Distribution to get correct critical values.

You can either test for various p or you use information criteria, such as AIC, BIC and the Hannan-Quinn-Criteria (HQIC), to get the correct number of lags. All three criteria tend to come to the same conclusion. AIC sometimes overestimates the lag length, because it is the least strict one to penalize an high order of lags. BIC is the strictest in penalizing loss of degree of freedom by having more parameters in the fitted model. The HQIC holds the middle ranking in penalizing and is therefore often used.

Additional Information

After we have found evidence for a unit root and a DS-Model we have to identify the correct ARMA-Model for the differenced series. For this purpose we use our well-known Box-Jenkins-Approach and apply it to the differenced series.

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Difference Stationary Model

- **Identification**
  Which model would you chose and why?
- **Estimation**
  Estimate your model.
- **Diagnostic checking**
  Is the selected model a statistically adequate representation of the available data?

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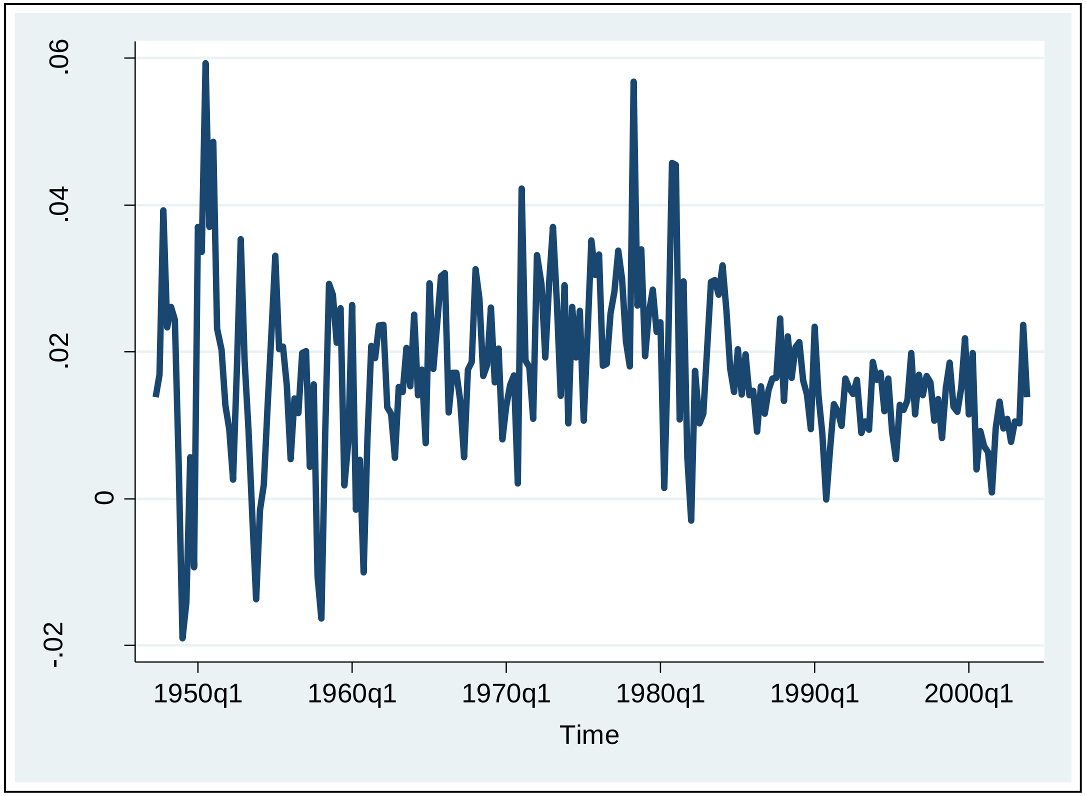
Identification

First difference of logarithm of U.S. quarterly GDP (1947q1 – 2003q4)



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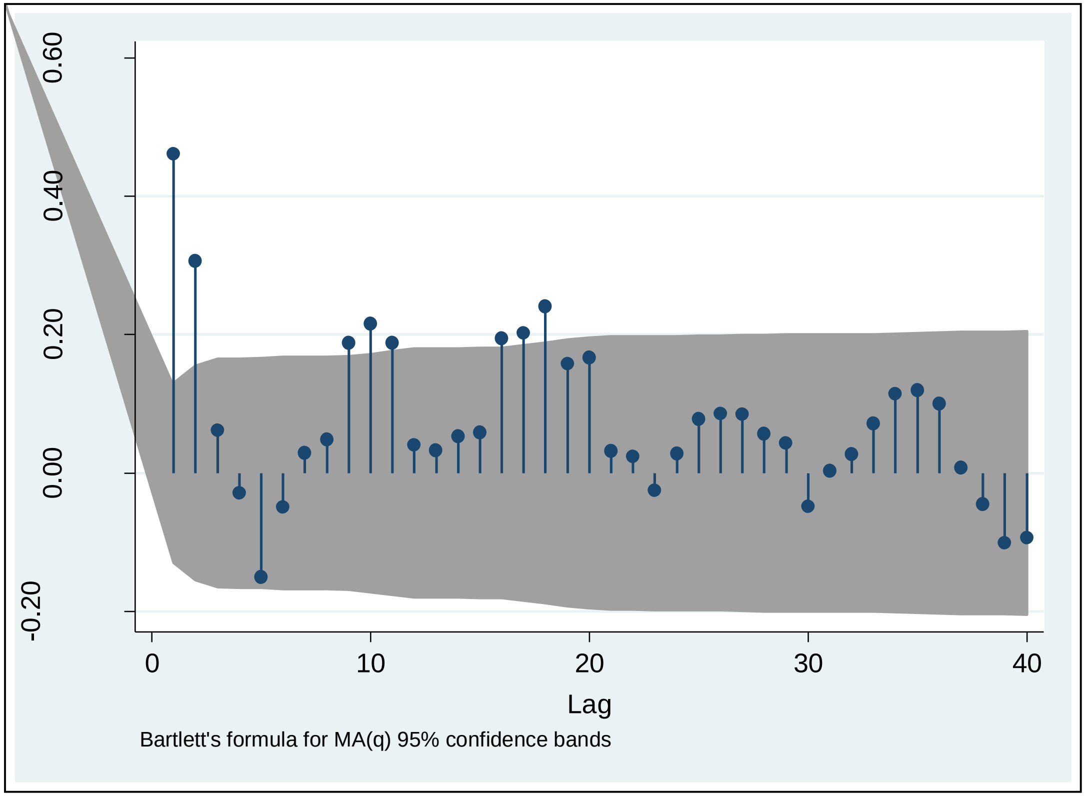
Identification

First difference of logarithm of U.S. quarterly GDP (1947q1 – 2003q4)



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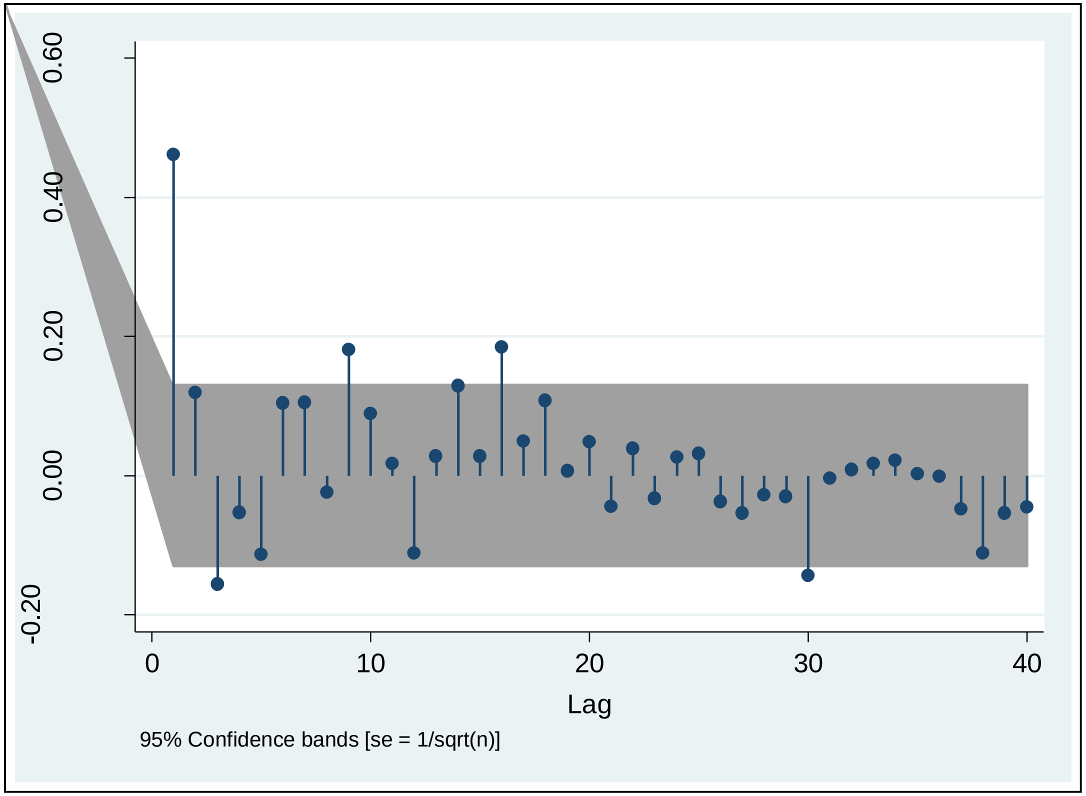
Identification

First difference of logarithm of U.S. quarterly GDP (1947q1 – 2003q4)



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Estimation

Stata's arima command

$$y_t = \sum_{i=1}^{p} \varphi_i y_{t-i} \boxed{+} \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$$

```
. arima D.lnGDP, ma(1/2)
[...]
ARIMA regression
Sample:  1947q4 to 2004q2                       Number of obs      =       227
                                                Wald chi2(2)       =     73.01
Log likelihood =  723.7383                      Prob > chi2        =    0.0000
------------------------------------------------------------------------------
             |                 OPG
     D.lnGDP |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
lnGDP        |
       _cons |   .0169737   .0011325    14.99   0.000     .0147541    .0191934
-------------+----------------------------------------------------------------
ARMA         |
          ma |
         L1. |   .3981594   .0504263     7.90   0.000     .2993257     .496993
         L2. |   .2694164   .0490873     5.49   0.000     .1732071    .3656257
-------------+----------------------------------------------------------------
      /sigma |   .0099742    .000341    29.25   0.000     .0093059    .0106425
------------------------------------------------------------------------------
```

$$\hat{x}_t = 0.0169737 + 0.3981594\varepsilon_{t-1} + 0.2694164\varepsilon_{t-2}$$

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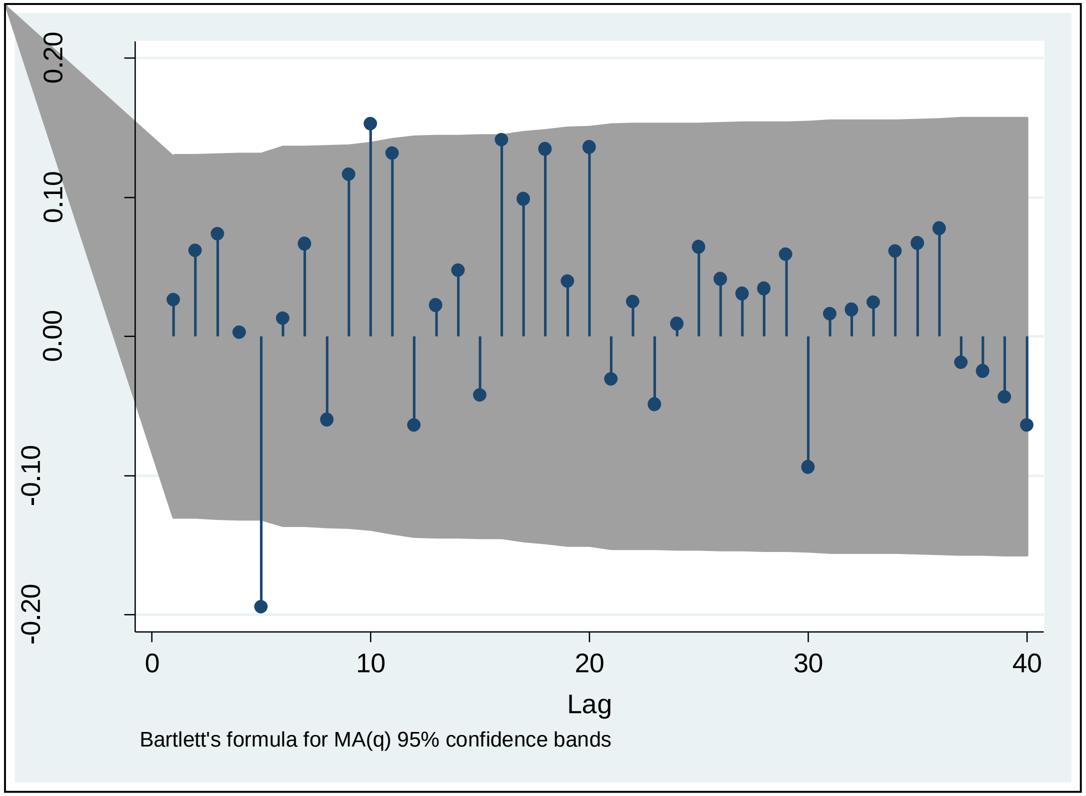
Diagnostic Checking

ACF of the residuals of an ARIMA(0,1,2)



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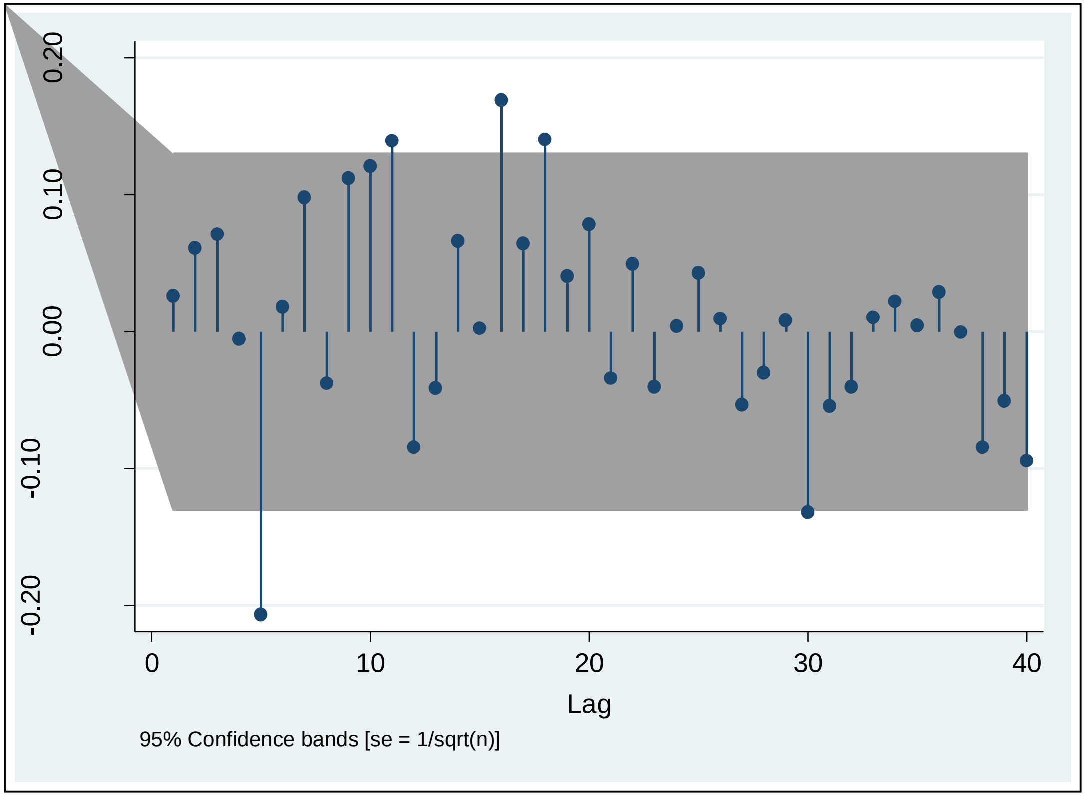
Diagnostic Checking

PACF of the residuals of an ARIMA(0,1,2)



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Diagnostic Checking

$$AIC = \log\hat{\sigma}^2 + 2\frac{p+q}{T} \qquad BIC = \log\hat{\sigma}^2 + \frac{p+q}{T}\log T$$

AIC and BIC for ARMA(p ,1 ,q)

p \ q	0	1	2	3	4	5
0	-8.9685045 -8.9835924	-9.1125014 -9.0974135	-9.1978922 -9.1677164	-9.2028873 -9.1576237	-9.2126863 -9.1523348	-9.2270162 -9.1515768
1	-9.190383 -9.1752951	-9.1899704 -9.1597946	-9.213996 -9.1838202	-9.2070468 -9.1617831	-9.2249688 -9.1646173	-9.2274977 -9.1520582
2	-9.1959458 -9.16577	-9.2037504 -9.1584868	$\boxed{\text{-9.2932324}}$ $\boxed{\text{-9.2479687}}$	-9.2013771 -9.1410256	-9.2728801 -9.1974407	-9.2270026 -9.1364753
3	-9.2121577 -9.1668941	-9.2079973 -9.1476458	-9.2648833 -9.2045317	-9.1936841 -9.1182447	-9.1839081 -9.0933808	-9.2258877 -9.1202725
4	-9.2062074 -9.1458558	-9.1995611 -9.1241217	-9.2577521 -9.1823127	-9.2707273 -9.1802	-9.2335015 -9.1278864	-9.2097806 -9.0890775
5	-9.2105199 -9.1350805	-9.2105449 -9.1200176	-9.2291609* -9.1386336*	-9.2009942 -9.095379	-9.2166735 -9.0959704	-9.2490434 -9.1132524

* conditional ML estimation

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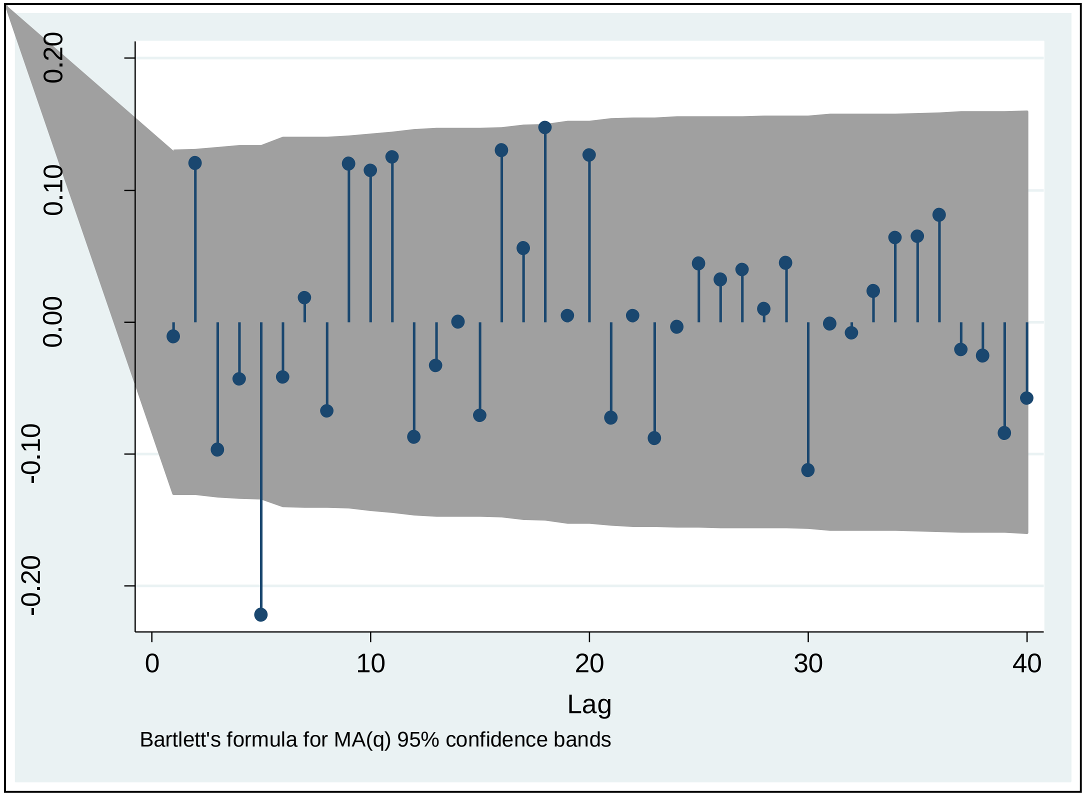
Diagnostic Checking

ACF of the residuals of an ARIMA(2,1,2)



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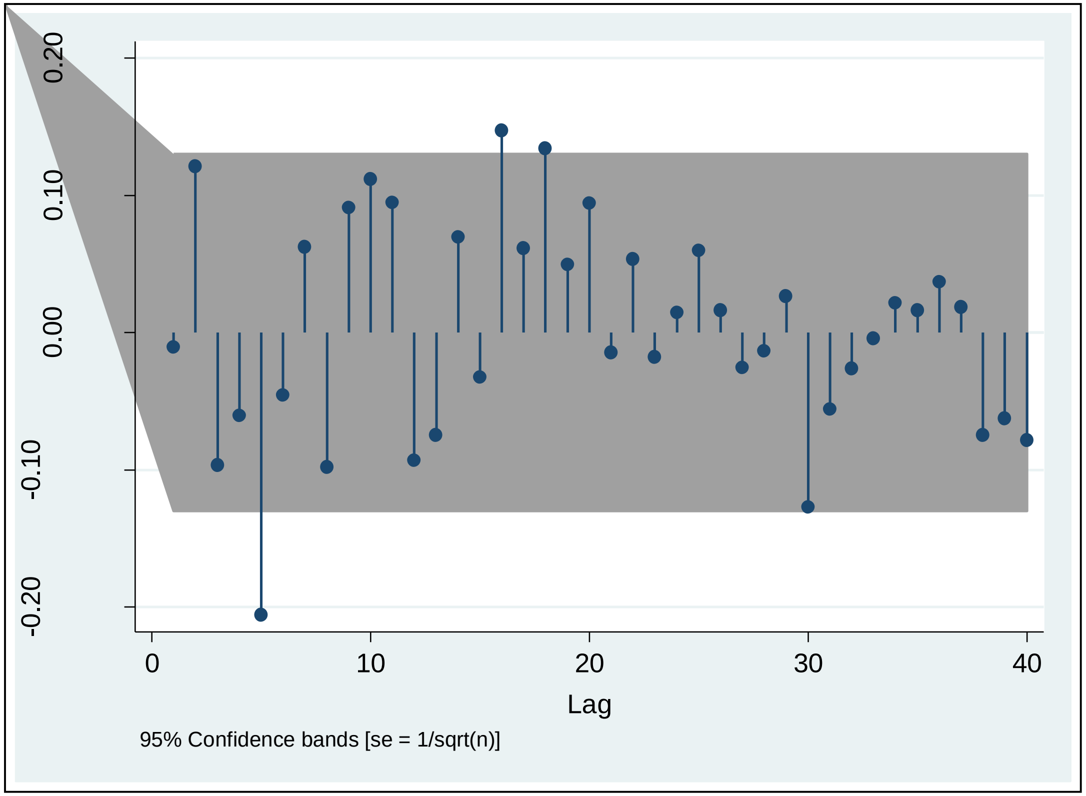
Diagnostic Checking

PACF of the residuals of an ARIMA(2,1,2)



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Exercise 3.8:

Forecasting (without using Stata's forecast commands)

- Forecast x_t from one to four quarters ahead.
- Forecast y_t from one to four quarters ahead.
- Forecast x_t and y_t from one to four quarters ahead using the information that we know at the end of 2004q1 that $y_{04q1} = 9.36$ and that we know at the end of 2004q2 that $y_{04q2} = 9.38$.

$$\hat{x}_t = 0.0169737 + 0.3981594\varepsilon_{t-1} + 0.2694164\varepsilon_{t-2}$$

```
. list time lnGDP res_MA2 in 226/228

     +---------------------------------+
     |   time       lnGDP      res_MA2 |
     |---------------------------------|
226. | 2003q2   9.2916164   -.0037882 |
227. | 2003q3   9.3153305    .0091118 |
228. | 2003q4   9.3291893   -.0057223 |
     +---------------------------------+
```

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Solution 3.8-1:

MA(2) with constant $x_t = \mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$

$$\hat{x}_t = 0.0169737 + 0.3981594\varepsilon_{t-1} + 0.2694164\varepsilon_{t-2}$$

$$\tilde{x}_{T+1|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+1|\Omega_T}}_{0} + \mu - \theta_1\tilde{\varepsilon}_{T|\Omega_T} - \theta_2\tilde{\varepsilon}_{T-1|\Omega_T}$$

with $\tilde{\varepsilon}_{T|\Omega_T} = x_T - \tilde{x}_{T|\Omega_T}$ and $\tilde{\varepsilon}_{T-1|\Omega_T} = x_{T-1} - \tilde{x}_{T-1|\Omega_T}$

```
. list time res_MA2 in 226/228

      +--------------------+
      |   time     res_MA2 |
      |--------------------|
226.  | 2003q2   -.0037882 |
227.  | 2003q3    .0091118 |
228.  | 2003q4   -.0057223 |
      +--------------------+
```

```
. di .0169737+(.3981594*(-.0057223))+(.2694164*.0091118)
.0171502
```

$$\tilde{x}_{T+2|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+2|\Omega_T}}_{0} + \mu - \theta_1\underbrace{\tilde{\varepsilon}_{T+1|\Omega_T}}_{0} - \theta_2\tilde{\varepsilon}_{T|\Omega_T}$$

```
. di .0169737+(.2694164*-.00572227)
.01543203
```

$$\tilde{x}_{T+3|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+3|\Omega_T}}_{0} + \mu - \theta_1\underbrace{\tilde{\varepsilon}_{T+2|\Omega_T}}_{0} - \theta_2\underbrace{\tilde{\varepsilon}_{T+1|\Omega_T}}_{0}$$

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Solution 3.8-2:

ARIMA(0,1,2) with constant

$$\tilde{x}_{T+1|\Omega_T} = .0171502 \; \tilde{x}_{T+2|\Omega_T} = .01543203 \quad \tilde{x}_{T+3|\Omega_T} = \tilde{x}_{T+4|\Omega_T} = .0169737$$

$$\tilde{y}_{T+1|\Omega_T} = y_{T|\Omega_T} + \tilde{x}_{T+1|\Omega_T}$$
$$= 9.3291893 + 0.0171502 = 9.3463395$$

$$\tilde{y}_{T+2|\Omega_T} = \tilde{y}_{T+1|\Omega_T} + \tilde{x}_{T+2|\Omega_T}$$
$$= 9.3463395 + .01543203 = 9.3617715$$

$$\tilde{y}_{T+3|\Omega_T} = \tilde{y}_{T+2|\Omega_T} + \tilde{x}_{T+3|\Omega_T}$$
$$= 9.3617715 + 0.0169737 = 9.3787452$$

$$\tilde{y}_{T+4|\Omega_T} = \tilde{y}_{T+3|\Omega_T} + \tilde{x}_{T+4|\Omega_T}$$
$$= 9.3787452 + 0.0169737 = 9.3957189$$

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Solution 3.8-3:

MA(2) with constant $x_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$

$$\hat{x}_t = 0.0169737 + 0.3981594\varepsilon_{t-1} + 0.2694164\varepsilon_{t-2}$$

$$\tilde{x}_{T+1|\Omega_T} = .0171502$$

$$\tilde{x}_{T+2|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+2|\Omega_T}}_{0} + \mu - \theta_1 \tilde{\varepsilon}_{T+1|\Omega_T} - \theta_2 \tilde{\varepsilon}_{T|\Omega_T}$$

$$\tilde{\varepsilon}_{T+1|\Omega_T} = x_{T+1} - \tilde{x}_{T+1|\Omega_T} = \left(y_{T+1} - y_T\right) - \tilde{x}_{T+1|\Omega_T}$$

$$= \left(9.36 - 9.3291893\right) - 0.0171502 = 0.0136605$$

$$\tilde{x}_{T+2|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+2|\Omega_T}}_{0} + \mu - \theta_1 \tilde{\varepsilon}_{T+1|\Omega_T} - \theta_2 \tilde{\varepsilon}_{T|\Omega_T}$$

$$= 0.0169737 + \left(0.3981594 \cdot 0.0136605\right) + \left(0.2694164 \cdot \left(\text{-}0.00572227\right)\right)$$

$$= 0.02087108$$

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Solution 3.8-4:

$$\tilde{x}_{T+3|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+3|\Omega_T}}_{0} + \mu - \theta_1\tilde{\varepsilon}_{T+2|\Omega_T} - \theta_2\tilde{\varepsilon}_{T+1|\Omega_T}$$

$$\tilde{\varepsilon}_{T+2|\Omega_T} = x_{T+2} - \tilde{x}_{T+2|\Omega_T} = \left(y_{T+2} - y_{T+1}\right) - \tilde{x}_{T+2|\Omega_T}$$

$$= \left(9.38 - 9.36\right) - 0.02087108 = -0.00087108$$

$$\tilde{x}_{T+3|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+3|\Omega_T}}_{0} + \mu - \theta_1\tilde{\varepsilon}_{T+2|\Omega_T} - \theta_2\tilde{\varepsilon}_{T+1|\Omega_T}$$

$$= 0.0169737 + \left(0.3981594 \cdot \left(-0.00087108\right)\right) + \left(0.2694164 \cdot 0.0136605\right)$$

$$= 0.02030723$$

$$\tilde{x}_{T+4|\Omega_T} = \underbrace{\tilde{\varepsilon}_{T+4|\Omega_T}}_{0} + \mu - \theta_1\underbrace{\tilde{\varepsilon}_{T+3|\Omega_T}}_{0} - \theta_2\tilde{\varepsilon}_{T+2|\Omega_T}$$

$$= 0.0169737 + 0.2694164 \cdot \left(-0.00087108\right) = 0.01673902$$

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Solution 3.8-5:

$$\tilde{x}_{T+1|\Omega_T} = 0.0171502 \qquad \tilde{x}_{T+2|\Omega_T} = 0.02087108 \qquad \tilde{x}_{T+3|\Omega_T} = 0.02030723$$

$$\tilde{x}_{T+4|\Omega_T} = 0.01673902$$

$$\begin{aligned}\tilde{y}_{T+1|\Omega_T} &= y_{T|\Omega_T} + \tilde{x}_{T+1|\Omega_T}\\ &= 9.3291893 + 0.0171502 = 9.3463395\end{aligned}$$

$$\begin{aligned}\tilde{y}_{T+2|\Omega_T} &= y_{T+1|\Omega_T} + \tilde{x}_{T+2|\Omega_T}\\ &= 9.36 + 0.02087108 = 9.3808711\end{aligned}$$

$$\begin{aligned}\tilde{y}_{T+3|\Omega_T} &= y_{T+2|\Omega_T} + \tilde{x}_{T+3|\Omega_T}\\ &= 9.38 + 0.02030723 = 9.4003072\end{aligned}$$

$$\begin{aligned}\tilde{y}_{T+4|\Omega_T} &= \tilde{y}_{T+3|\Omega_T} + \tilde{x}_{T+4|\Omega_T}\\ &= 9.4003072 + 0.01673902 = 9.4170462\end{aligned}$$

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Solution 3.8-6:

- Forecast x_t and y_t from one to four quarters ahead.

$$\tilde{x}_{T+1|\Omega_T} = .0171502 \quad \tilde{x}_{T+2|\Omega_T} = .01543203 \quad \tilde{x}_{T+3|\Omega_T} = \tilde{x}_{T+4|\Omega_T} = .0169737$$

$$\tilde{y}_{T+1|\Omega_T} = 9.3463395 \quad \tilde{y}_{T+2|\Omega_T} = 9.3617715 \quad \tilde{y}_{T+3|\Omega_T} = 9.3787452 \quad \tilde{y}_{T+4|\Omega_T} = 9.3957189$$

- Forecast x_t and y_t from one to four quarters ahead using the information that we know at the end of 2004q1 that y_{04q1} = 9.36 and that we know at the end of 2004q2 that y_{04q2} = 9.38.

$$\tilde{x}_{T+1|\Omega_T} = 0.0171502 \quad \tilde{x}_{T+2|\Omega_T} = 0.02087108 \quad \tilde{x}_{T+3|\Omega_T} = 0.02030723 \quad \tilde{x}_{T+4|\Omega_T} = 0.01673902$$

$$\tilde{y}_{T+1|\Omega_T} = 9.3463395 \quad \tilde{y}_{T+2|\Omega_T} = 9.3808711 \quad \tilde{y}_{T+3|\Omega_T} = 9.4003072 \quad \tilde{y}_{T+4|\Omega_T} = 9.4170462$$

```
. list time lnGDP x_tilde_dyn y_tilde_dyn x_tilde y_tilde in 229/232

     +-------------------------------------------------------------+
     |   time   lnGDP    x_tild~n    y_tild~n     x_tilde    y_tilde |
     |-------------------------------------------------------------|
229. | 2004q1    9.36    .0171502    9.346339    .0171502   9.346339 |
230. | 2004q2    9.38     .015432    9.361772     .020871   9.380871 |
231. | 2004q3       .    .0169737    9.378745    .0203074   9.400308 |
232. | 2004q4       .    .0169737    9.395719    .0167392          . |
     +-------------------------------------------------------------+
```