Solution for Sheet 3, Exercise 1(c) We would like to show that the distribution of mean 0 and variance σ^2 that maximizes the entropy is Gaussian. As shown in the previous questions, the maximum entropy probability function is given by a saddle node of the Lagrange function found at the intersection of the following equations:

$$\forall_{x\in\mathbb{R}}: \; q(x) = \exp(ax^2 + bx + c) \quad \text{with} \quad a, b, c \in \mathbb{R} \tag{1}$$

$$\int_{-\infty}^{\infty} q(x)dx = 1 \tag{2}$$

$$\int_{-\infty}^{\infty} xq(x)dx = 0 \tag{3}$$

$$\int_{-\infty}^{\infty} x^2q(x)dx = \sigma^2 \tag{4}$$

We rewrite $q(x)$ as follows:

$$\begin{aligned} q(x) &= \exp(ax^2 + bx + c) \\ &= \exp\left(-\frac{1}{2}\frac{x^2 + \frac{b}{a}x + \frac{c}{a}}{-\frac{1}{2a}}\right) \\ &= \exp\left(-\frac{1}{2}\frac{x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}}{-\frac{1}{2a}}\right) \\ &= \exp\left(-\frac{1}{2}\left(\frac{x - (-\frac{b}{2a})}{\sqrt{-\frac{1}{2a}}}\right)^2\right)\exp\left(-\frac{1}{2}\frac{(\frac{b}{2a})^2}{\frac{1}{2a}}\right)\cdot\exp(c) \end{aligned}$$

To satisfy Eq. 2, one needs to set $a < 0$ and choose an appropriate parameter c. By doing so, $q(x)$ can be identified as a Gaussian distribution of mean $-\frac{b}{2a}$ and variance $-\frac{1}{2a}$. To satisfy Eq. 3 (i.e. for the Gaussian distribution to have mean 0), one needs to choose $b = 0$. Finally, to satisfy Eq. 4 (i.e. for the Gaussian distribution to have variance σ^2), one needs to choose $a = -\frac{1}{2\sigma^2}$. Note that at each step, there was only one possible solution for the parameters c, b, a. The Gaussian distribution of mean 0 and variance σ^2 is therefore also the only one that simultaneously satisfies all the equations.