Technische Universität Berlin
Fakultät IV – Elektrotechnik und Informatik

Probabilistic and Bayesian Modelling in Machine Learning and Artificial Intelligence

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Problem Sheet 1

to be discussed in the tutorial on Monday, May 7

Problem 1 – Random experiments

A dice is thrown repeatedly until it shows a 6. Let T be the number of throws for this to happen. Obviously, T is a random variable. Compute the expectation value $\mathrm{E}[T]$ and the variance $\mathrm{Var}(T)$ of T.

Problem 2 – Addition of variances

Let X and Y be independent random variables. Show that

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y),$$

where the variance is defined as

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mathrm{E}[X])^2].$$

Hint: Use the fact that for independent U and V, $\mathrm{E}[UV] = \mathrm{E}[U]\mathrm{E}[V]$.

Problem 3 – Transformation of probability densities

Let X be uniformly distributed in $(0, 1)$:

$$p(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

A second random variable Y is defined as

$$Y = \tan\left(\pi(X - 1/2)\right).$$

What is the probability density $q(y)$ of Y?

Problem 4 – Gaussian inference

Suppose we have two random variables V_1 and V_2 which are **jointly Gaussian** distributed with zero means $E[V_1] = E[V_2] = 0$ and variances $E[V_1^2] = 16.6$ and $E[V_2^2] = 6.8$. The covariance is $E[V_1 \, V_2] = 6.4$.

Assume that we observe a noisy estimate

$$Y = V_2 + \nu$$

of V_2 where ν is a Gaussian noise variable independent of V_1 and of V_2 with $E[\nu] = 0$ and $E[\nu^2] = 1$.

(a) Calculate the conditional (posterior) densities $p(V_1|Y)$ and $p(V_2|Y)$.

(b) What are the posterior mean predictions of V_1 and V_2 for an observation $Y = 1$ and what are the posterior uncertainties of these predictions.

The following formula could be helpful: The inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is given by

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$