Schedule

Deter-ministic Models	• Components of a Time Series • Additive and Multiplicative Models • Parametric (global) and flexible (local) approaches

Stationary Stochastic Processes	• Introduction
	• Identification
	  • Autocorrelation Function
	  • Moving Average and Autoregressive Models
	  • Partial Autocorrelation Function
	  • ARMA Models
	• Estimation
	• Diagnostic Checking
	• Forecasting

Non-stationary Stochastic Processes	• Introduction • Nonstationarity and Trends • ARIMA Models • Unit Root Tests • Seasonal ARIMA

Deterministic Models

- Deterministic = non-stochastic
- No explicit probabilistic model for time series is used
- Time series is assumed to consist of several components (e.g. a trend)
- “Estimate” these components
  - to describe(“understand”) the series
  - to remove unwanted components (“seasonal adjustment”)
  - to analyze components of interest (“trend analysis”)
  - to extrapolate into the future (“forecast”)

Additive Model

Series y_t is assumed to be a **sum of four components:**

$$y_t = \underbrace{L_t + C_t}_{G_t \text{ long term behavior}} + S_t + I_t \qquad t = 1, \ldots, T$$

L_t long-term change in the mean level, the 'trend'

C_t long-term cyclical component (due to business cycle; in macroeconomics over ≈ 2 - 7 years)

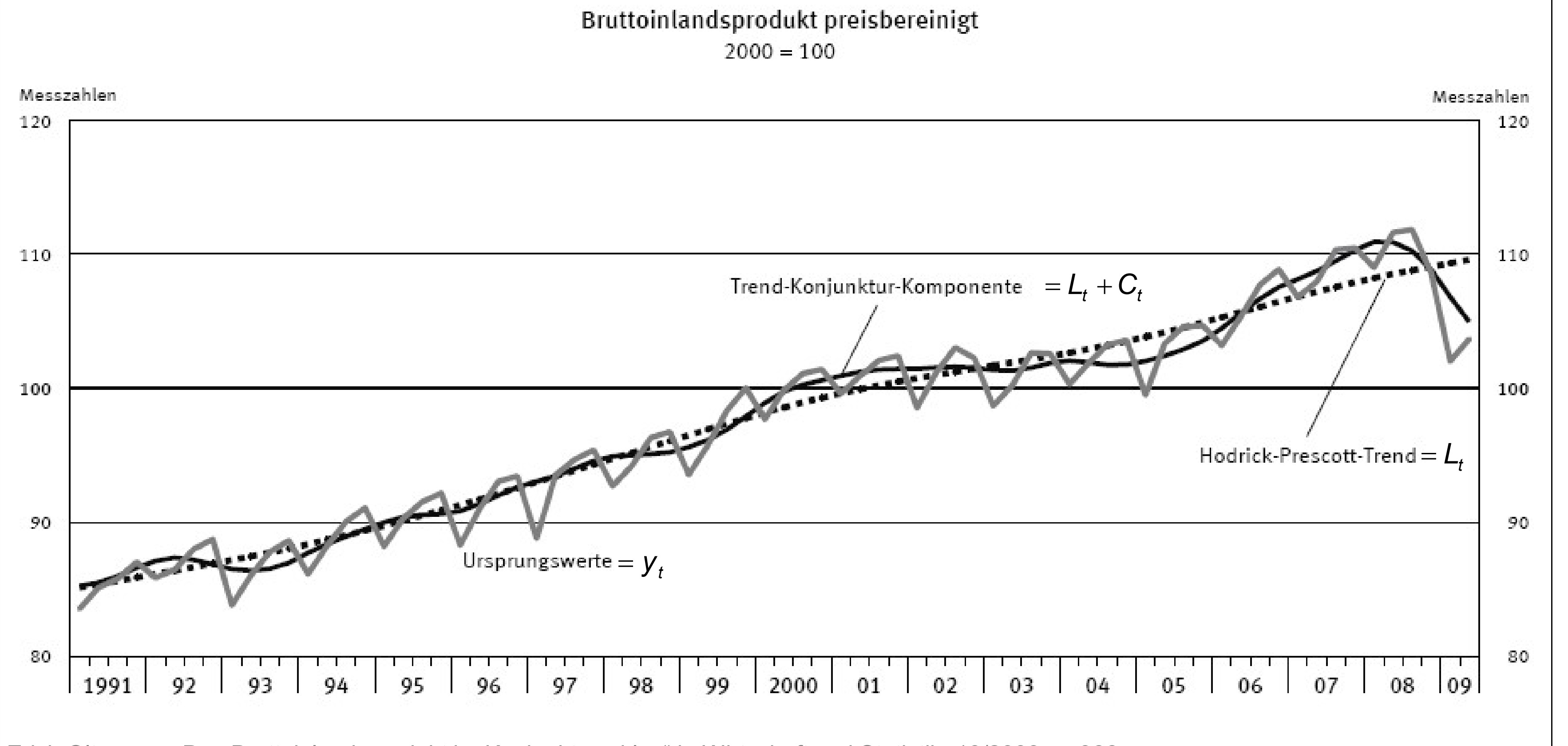
S_t short-term cyclic influence: the 'seasonal' component

I_t irregular component (random deviations from the non-stochastic components)

Additive Model for a Time Series

Additive Model $$y_t = \underbrace{L_t + C_t}_{G_t \text{ long term behavior}} + S_t + I_t \qquad t = 1, \ldots, T$$

Example: German real Gross Domestic Product



Erich Oltmanns „Das Bruttoinlandsprodukt im Konjunkturzyklus“ in Wirtschaft und Statistik, 10/2009, p. 966

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Additive Model for a Time Series

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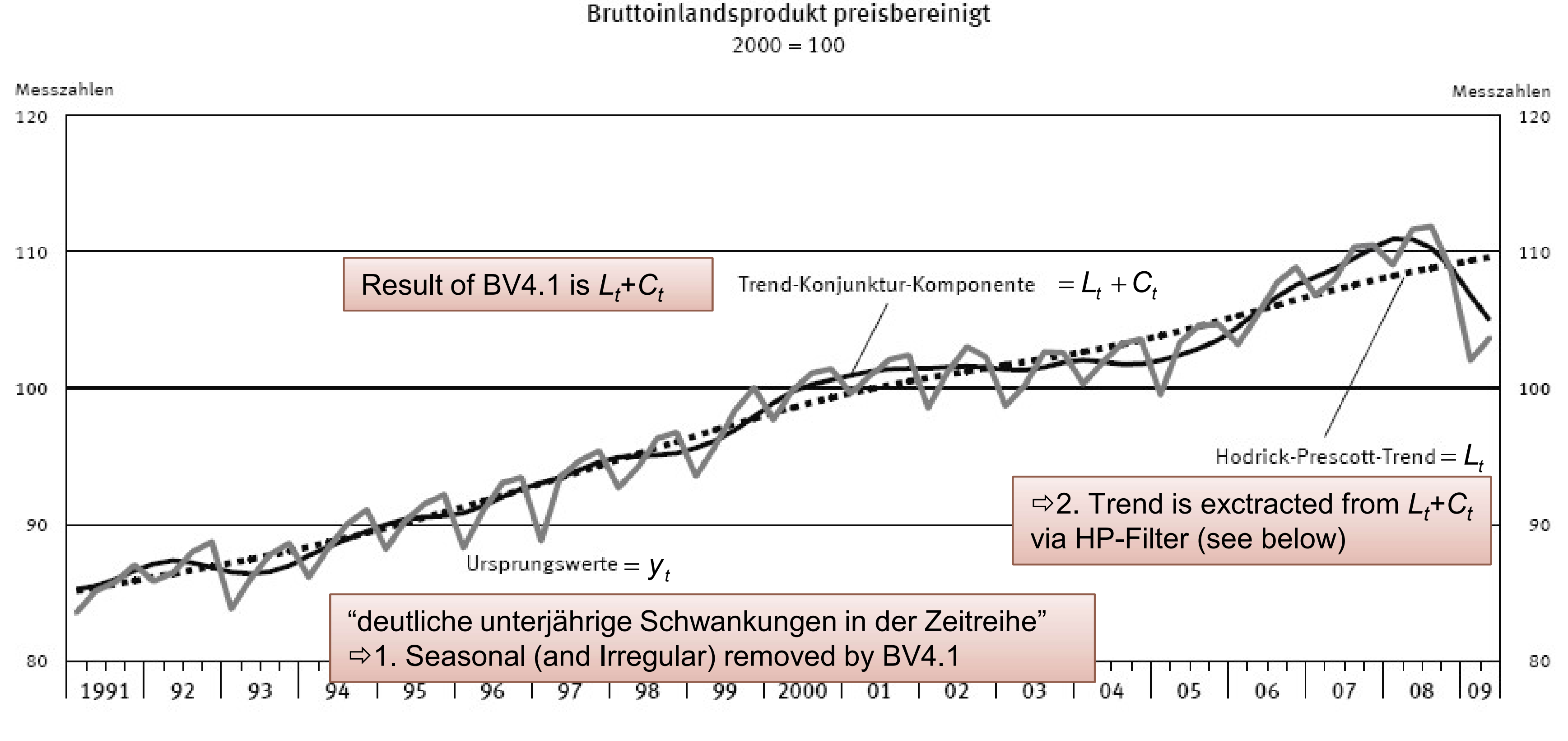


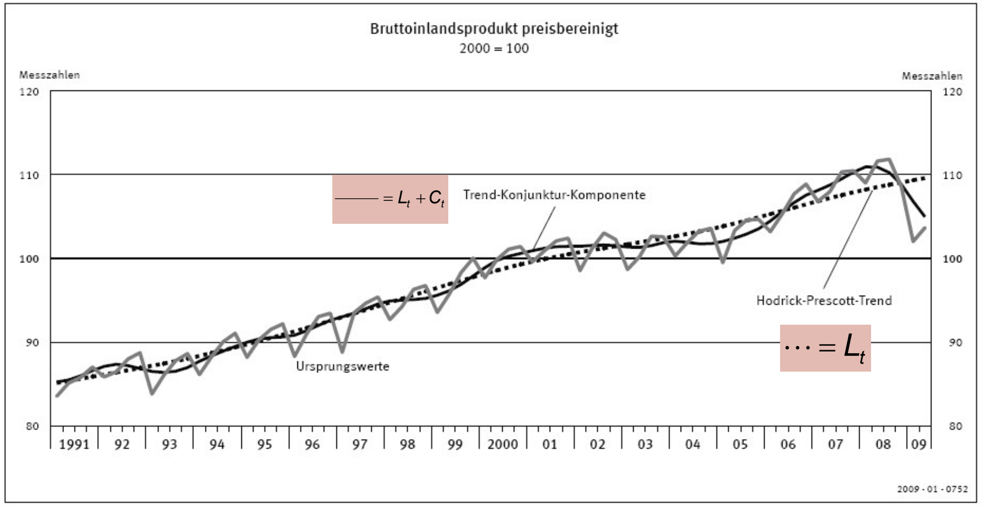
Erich Oltmanns „Das Bruttoinlandsprodukt im Konjunkturzyklus" in Wirtschaft und Statistik, 10/2009, p. 966

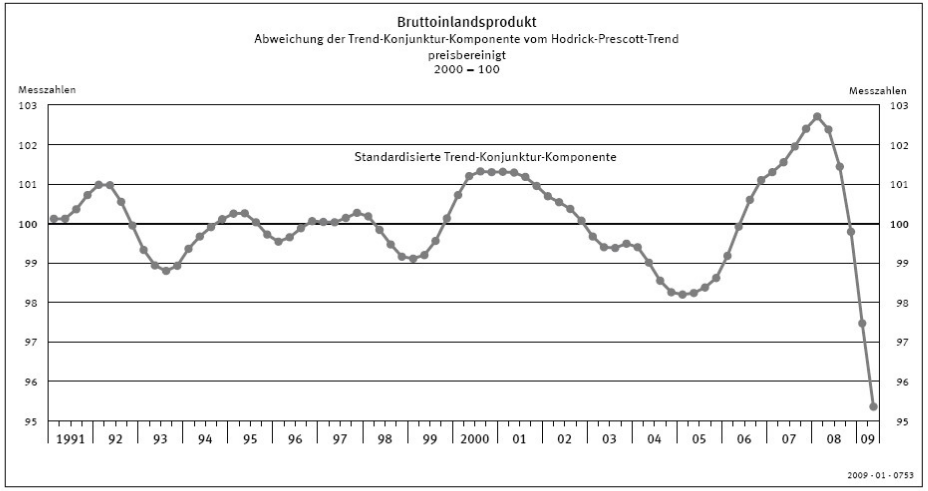
Additive Model for a Time Series

Additive Model $y_t = L_t + C_t + S_t + I_t \quad t = 1,\ldots,T$

For extraction/removal: use simple algebra

Example: German real Gross Domestic Product





⇨ This is C_t, the difference between the solid line L_t+C_t and the dotted trend L_t

Erich Oltmanns „Das Bruttoinlandsprodukt im Konjunkturzyklus“ in Wirtschaft und Statistik, 10/2009, p. 966

Multiplicative Model

y_t is assumed to be the **product of four components:**

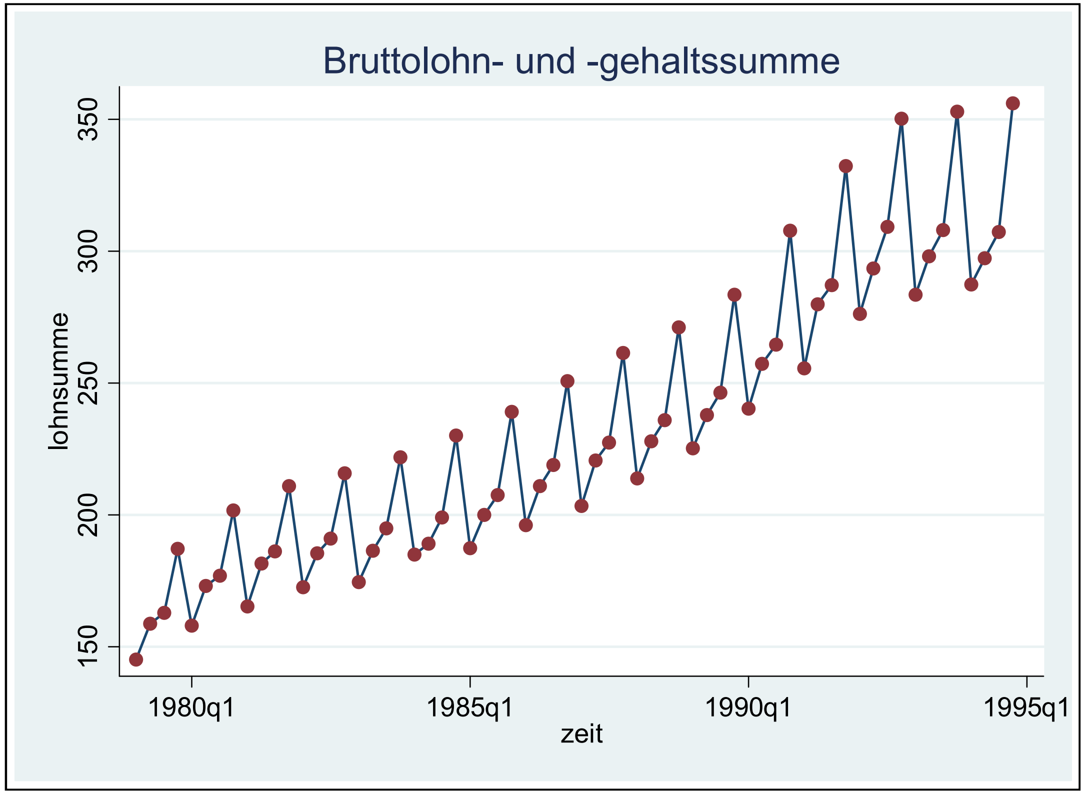
$$y_t = \underbrace{L_t \cdot C_t}_{G_t \text{ long term behavior}} \cdot S_t \cdot I_t \qquad t = 1,\dots,T$$

- S_t, say, is proportional to trend (level)
- Example: 1st quarter of each year, y_t may be raised by a certain factor

Note that: $\ln y_t = \ln L_t + \ln C_t + \ln S_t + \ln I_t$ ("log-additive decomposition"); will not be covered.

Example:

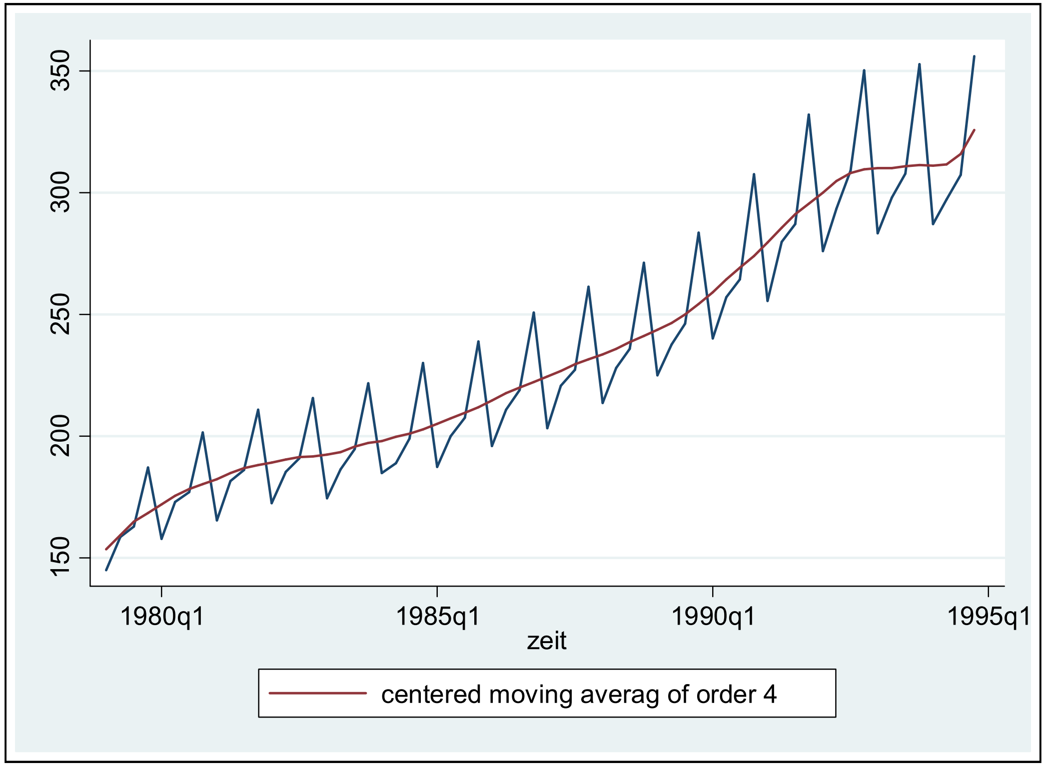
Seasonal oscillations appear to increase with the level of the series



$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t$$

Example:

Seasonal oscillations appear to increase with the level of the series



$$y_t = \underbrace{L_t \cdot C_t}_{} \cdot S_t \cdot I_t$$

G_t long term behavior
= "Level"
= red line

S_t is then a multiplicative factor that is >1 during "high season" and <1 during low season

Components

L_t trend

C_t cyclical component

S_t seasonal component

“... the relative importance of the three components depends largely on how far ahead one is forecasting. In the very short run, the trend and cycle components may have changed very little, so forecasting of the third component are the most important. However, when long-run forecasting, the trend term is usually dominant ...”

C.W.J. Granger (1989) “Forecasting in Business and Economics”, p. 26/27

Long-term Components: L_t and C_t

Additive Model $$y_t = \underbrace{L_t + C_t}_{G_t \text{ long term behavior}} + S_t + I_t \quad t = 1,\ldots,T$$

Multiplicative Model $$y_t = \underbrace{L_t \cdot C_t}_{G_t \text{ long term behavior}} \cdot S_t \cdot I_t \quad t = 1,\ldots,T$$

Distinction between L_t and C_t often not clear and they are considered as one long-term component G_t

"In this article, the trend is defined as the cyclical movements in the time series with periods longer than the business cycle (that is, longer than 8 years)."

Mark W. Watson (2007), Economic Quarterly—Vol. 93 (2), p. 144

Seasonal Component: S_t

$$y_t = L_t + C_t + S_t + I_t \quad t = 1,\ldots,T$$

$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t \quad t = 1,\ldots,T$$

Bundesbank:

„Die **Saisonbereinigung** von Zeitreihen durch die Deutsche Bundesbank zielt darauf ab, aus den Bewegungen der betrachteten Zeitreihe die **üblichen** Saisonausschläge **herauszufiltern**. Als übliche Saisonausschläge werden die Jahr für Jahr **zur gleichen Jahreszeit mit ähnlicher Intensität wiederkehrenden** Bewegungen verstanden, die aufgrund von Schwankungen der jeweiligen Zeitreihe in der Vergangenheit unter normalen Umständen zu erwarten sind.“

Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 85

Components of a Time Series and Forecasting

Additive Model

$$y_t = \underbrace{L_t + C_t}_{G_t \text{ long term behavior}} + S_t + I_t \qquad t = 1,\ldots,T$$

Multiplicative Model

$$y_t = \underbrace{L_t \cdot C_t}_{G_t \text{ long term behavior}} \cdot S_t \cdot I_t \qquad t = 1,\ldots,T$$

We will cover **two approaches** for estimating/removing these components :

- Parametric, global approach
  - based on parametric functional forms, e.g. $L_t = c_1 + c_2 t$
  - uses all observations at once (“global”)
- Flexible, local approach
  - no pre-specified functional forms
  - uses data in a neighborhood (“local”)

Components of a Time Series and Forecasting

Additive Model

$$y_t = \underbrace{L_t + C_t}_{G_t \text{ long term behavior}} + S_t + I_t \quad t = 1,\ldots,T$$

Multiplicative Model

$$y_t = \underbrace{L_t \cdot C_t}_{G_t \text{ long term behavior}} \cdot S_t \cdot I_t \quad t = 1,\ldots,T$$

Which of the **components** should be tackled first?

- No clear rule (but it matters: what one component "grabs" isn't available for subsequently calculated components)
- Often, researchers work with seasonally adjusted data, so extracting/removing S_t, seems to be first job
- However, in the process of extracting/removing S_t, the trend component is often initially removed…
- We will start with the trend … assuming S_t has been removed (which we will show last)

- Parametric, global approach can treat components simultaneously

Trend component L_t

(or trend + cycle)

Trend and Long-Term Forecasting/Extrapolation

L_t long-term change in the mean level, the ‘trend’

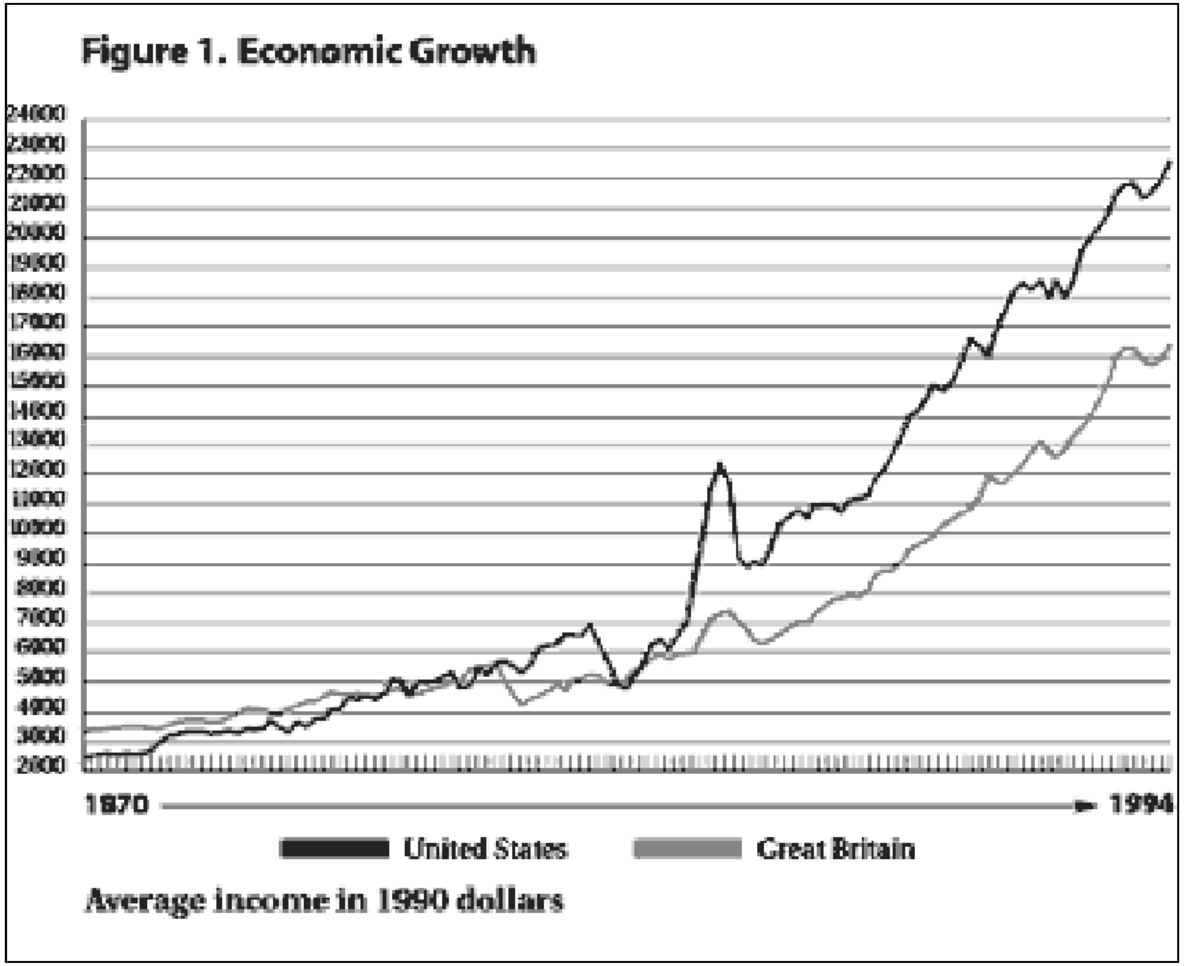
“A time series that appears to contain a smoothly increasing (or decreasing) component is said to contain a trend term.”

C.W.J. Granger (1989) “Forecasting in Business and Economics”, p. 23

„…die **längerfristige Grundtendenz**…“

Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 87

The Macroeconomic Trend - Economic Growth



Trend and Long-Term Forecasting/Extrapolation

$$y_t = L_t + C_t + S_t + I_t \quad t = 1, \ldots, T$$

$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t \quad t = 1, \ldots, T$$

Often, S_t is removed first before L_t is estimated

“It is usually worthwhile removing or at least reducing the importance of the other components before analyzing the trend. This is done by *smoothing* the series…”

C.W.J. Granger (1989) “Forecasting in Business and Economics”, p. 27

➔ see below how to do "seasonal adjustment"
(here we assume that y_t is seasonally adjusted or has no seasonal component)

Trend and Long-Term Forecasting/Extrapolation

$y_t = L_t + C_t + S_t + I_t$

$y_t = L_t \cdot C_t \cdot S_t \cdot I_t$

Approaches to estimating L_t:

1. Parametric, global approach
   - Trend as a deterministic function of time via OLS fit
2. Flexible, local approach ("Smoothing")
   - Moving averages
   - Exponential Smoothing
   - Hodrick-Prescott Filter

Trend as a **deterministic function of time**

Linear Trend $$L_t = c_1 + c_2 t$$

Quadratic Trend $$L_t = c_1 + c_2 t + c_3 t^2$$

Cubic Trend $$L_t = c_1 + c_2 t + c_3 t^2 + c_4 t^3$$

Exponential Trend $$L_t = A \cdot e^{rt}$$

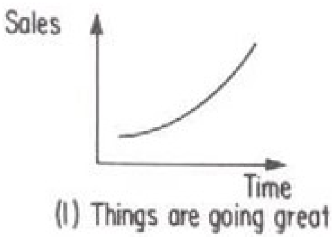
Logistic Trend* $$L_t = \frac{1}{k + ab^t} \qquad b > 0$$

C.W.J. Granger (1989) "Forecasting in Business and Economics", p. 28/29

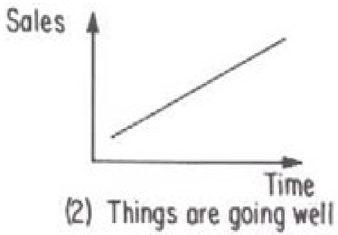
* : We will use a slightly more complicated and flexible version of the Logistic trend model below

Simple Trend Models - Overview

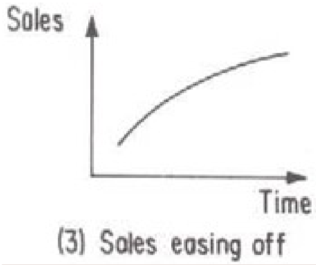
„Suppose that a sales manager of some manufacturing company looks at the chart of sales over the last few years and sees a smooth curve of one of the shapes ... An appropriate curve might be ...”



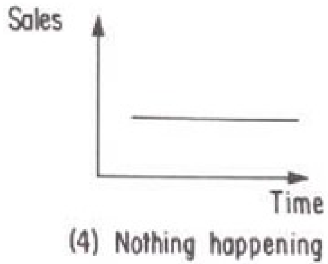
Exponential curve
($A > 0$, $r > 0$)



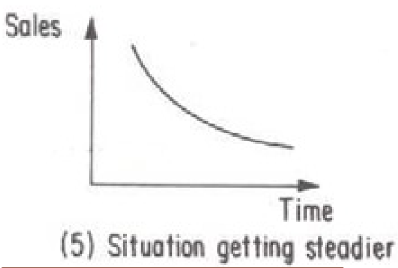
Linear trend
($c_2 > 0$)



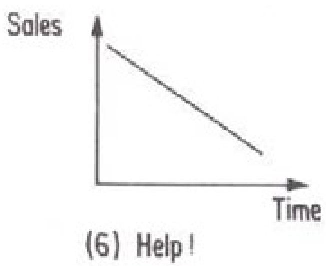
Logistic curve
($a < 0$)



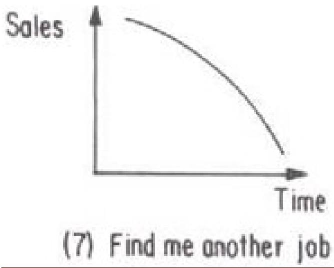
Linear trend
($c_2 = 0$)



Logistic curve
($a > 0$)



Linear trend
($c_2 < 0$)



Exponential curve
($A < 0$, $r > 0$)

C.W.J. Granger (1989) “Forecasting in Business and Economics”, p. 28/29

Deterministic Trend Models

- First, pick and estimate (by Least Squares) a model

  Linear example: $\hat{L}_t = \hat{c}_1 + \hat{c}_2 t$

- Estimated Model can be used to **detrend** y_t:

  (i.e. calculate residuals from trend regression)

  Linear example: $\hat{u}_t = y_t - \hat{L}_t = y_t - \left(\hat{c}_1 + \hat{c}_2 t\right)$

- or as a **Simple Extrapolation Model**

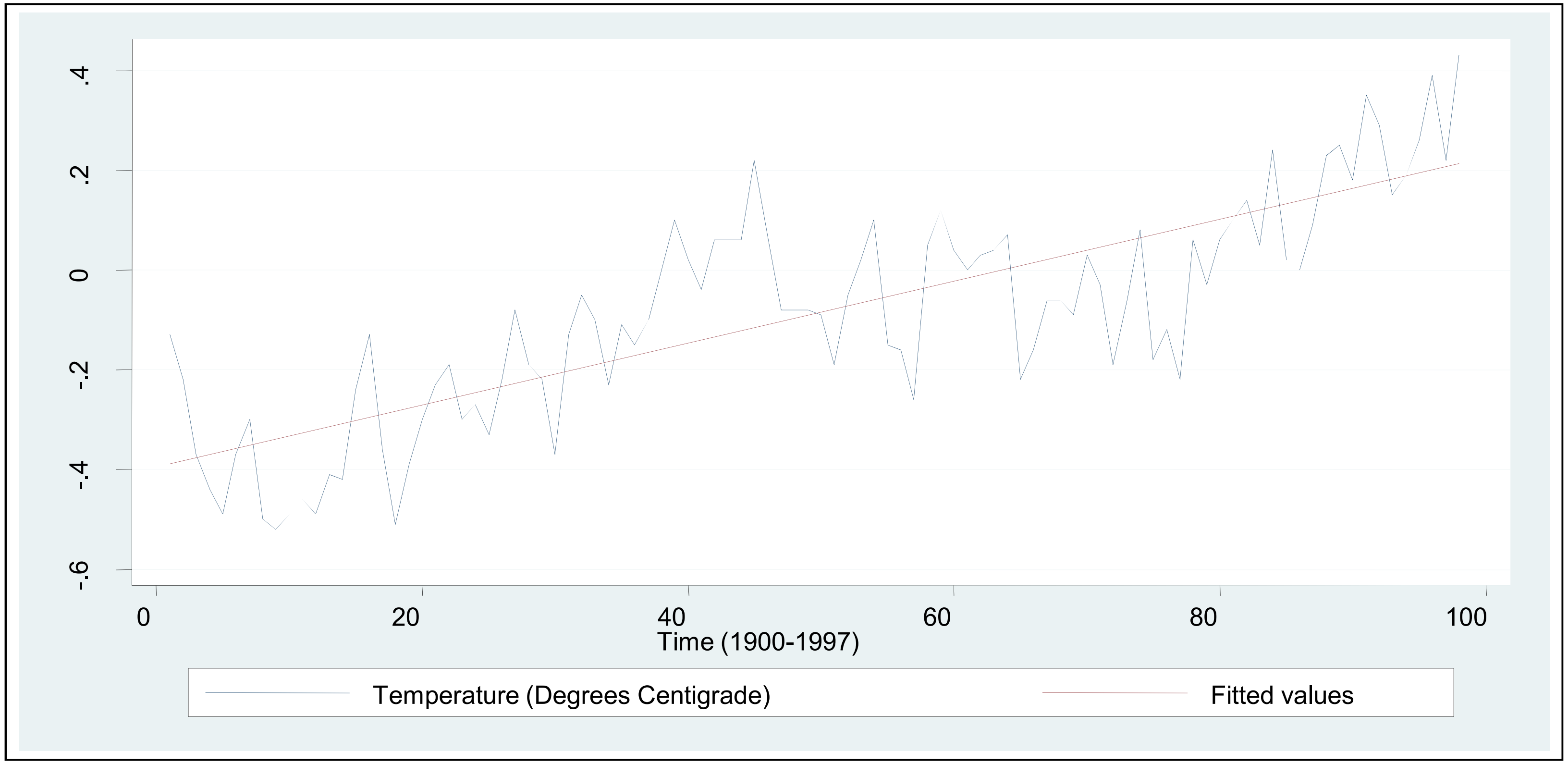
  (particularly for long term forecasting)

Linear Trend Model for the global temperature data

```
. regress temperature time

      Source |       SS       df       MS              Number of obs =      98
-------------+------------------------------           F(  1,    96) =  179.49
       Model |  3.02385927     1  3.02385927           Prob > F      =  0.0000
    Residual |  1.61729685    96  .016846842           R-squared     =  0.6515
-------------+------------------------------           Adj R-squared =  0.6479
       Total |  4.64115612    97  .04784697            Root MSE      =   .1298

------------------------------------------------------------------------------
 temperature |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        time |   .0062095   .0004635    13.40   0.000     .0052895    .0071295
       _cons |  -.3946139   .0264246   -14.93   0.000    -.4470664   -.3421615
------------------------------------------------------------------------------
```

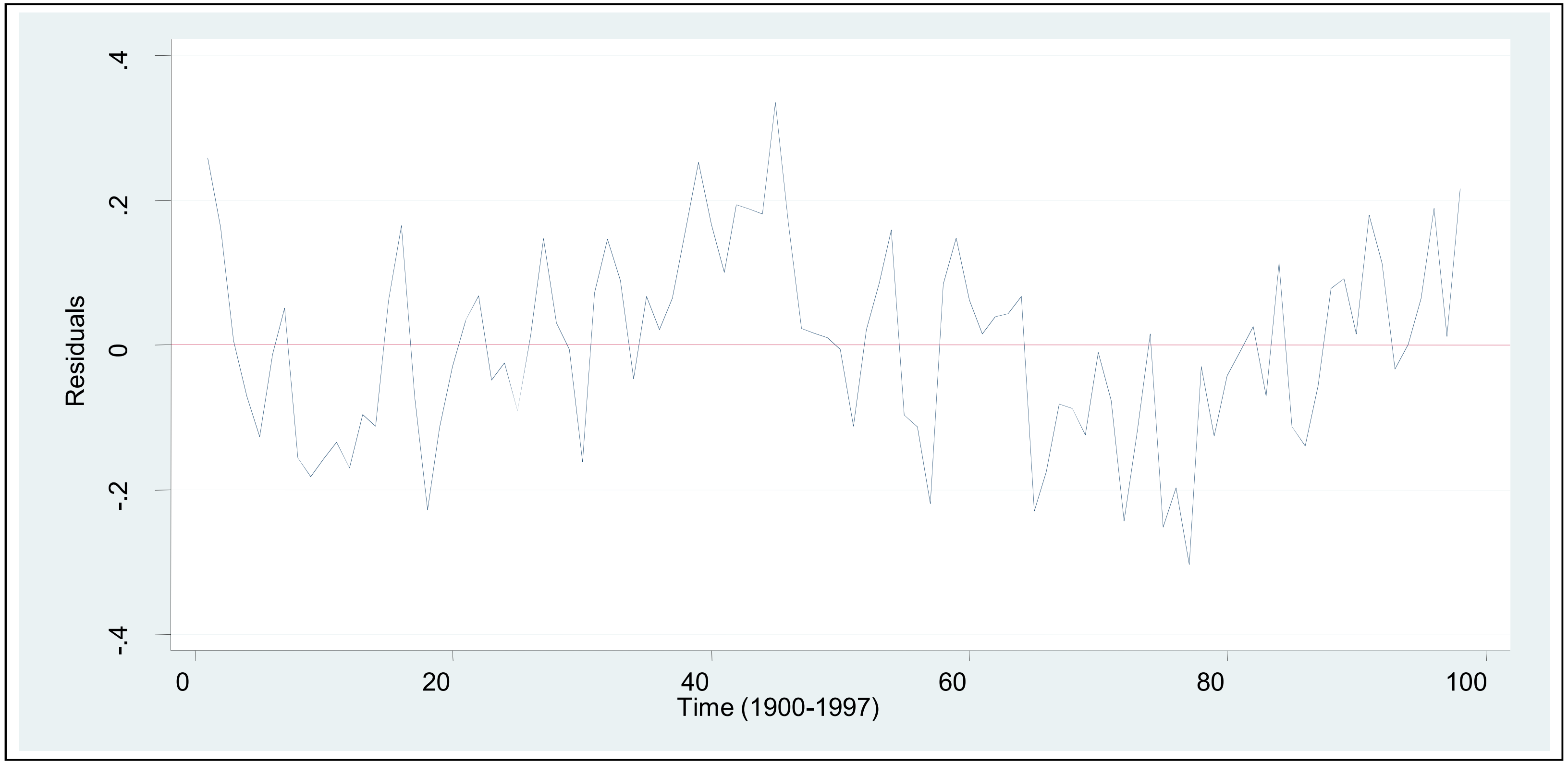
We can predict that the temperature in 1998 will be 0.0062 degrees Centigrade higher than in 1997.

Linear Trend Model for the global temperature data



Brockwell/Davis (1996) “Introduction to Time Series and Forecasts”

Detrended global warming data



Brockwell/Davis (1996) "Introduction to Time Series and Forecasts"

Estimation

If models are (or can be made) linear in the parameters: estimation by OLS regression of (seasonally adjusted) y_t on t according to L_t

Example: $L_t = c_1 + c_2 t \Rightarrow$ OLS of $y_t = c_1 + c_2 t + u_t$

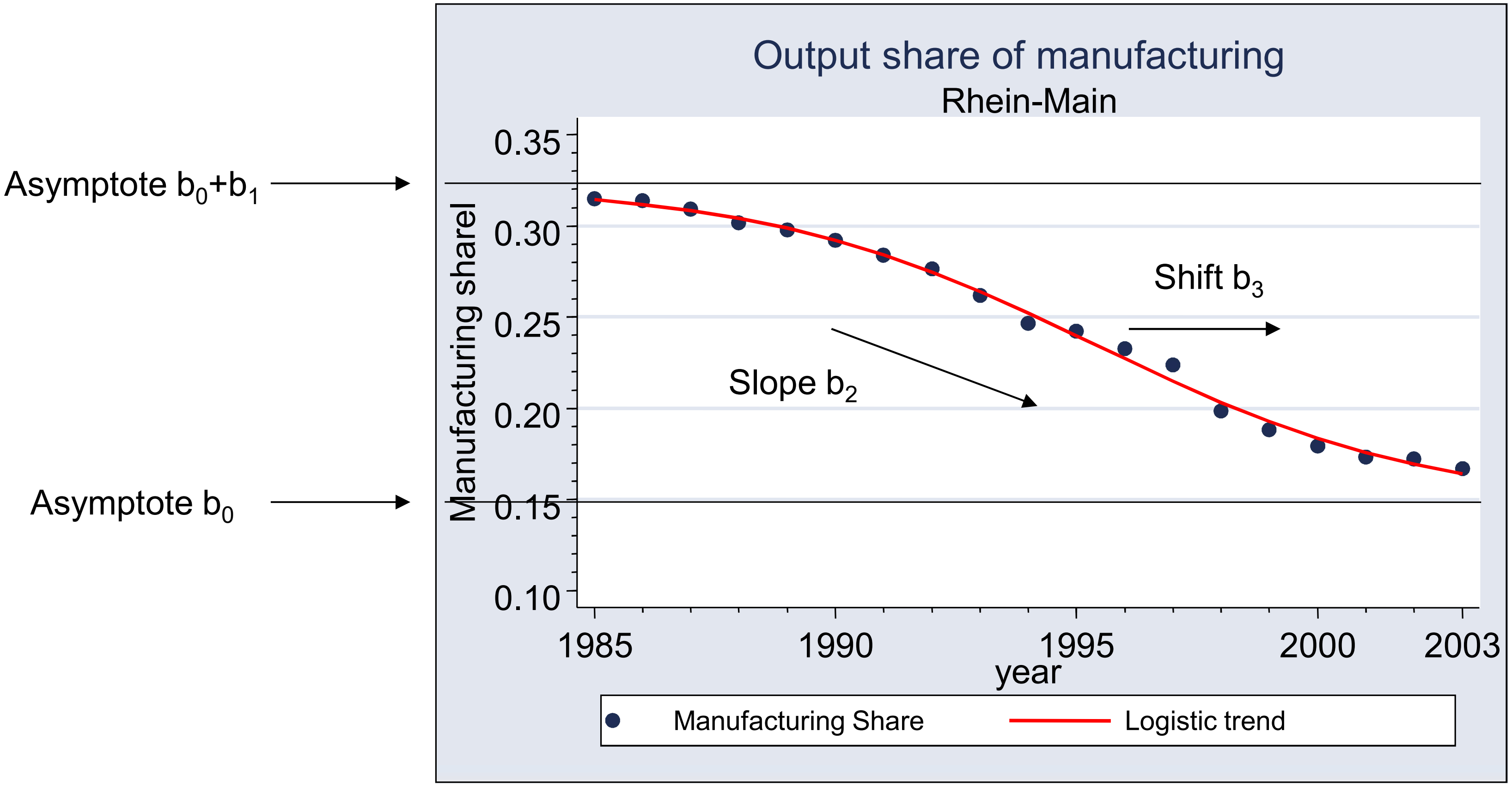
Note that for the exponential trend model: $L_t = A \cdot e^{rt} \Rightarrow \ln(L_t) = \ln(A) + rt$

It can be made linear in the parameters (ln(A) and r) and still be estimated by simple least squares.

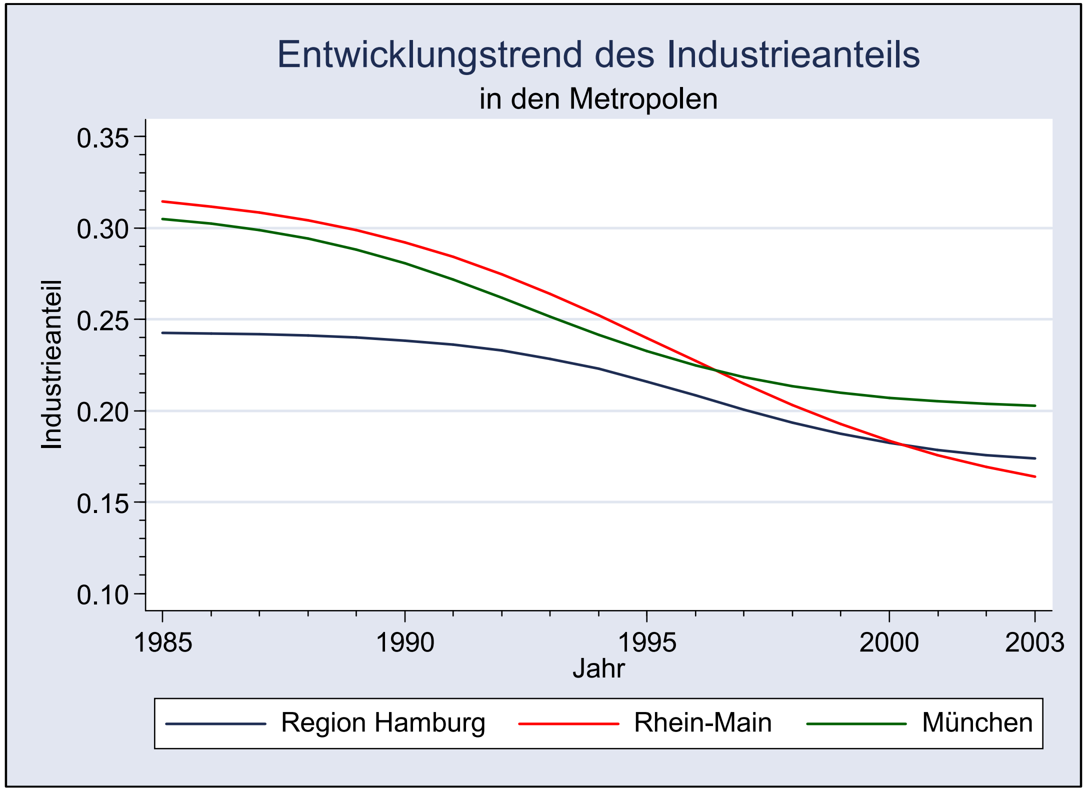
However, some non-linear trend functions can't be made linear in the parameters by simple transformations (see next slide)

➔Use nonlinear least squares (not covered).

Nonlinear Example: $L_t = b_0 + \frac{b_1}{\{1 + \exp[-b_2(t - b_3)]\}}$



Example: Rhein-Main compared with other regions



Linear Trend Model for industrial production

```
. reg  Produktiosindex t

      Source |       SS       df       MS              Number of obs =     236
-------------+------------------------------           F(  1,   234) =  203.63
       Model |  10360.419     1   10360.419           Prob > F      =  0.0000
    Residual |  11905.552   234  50.8784272           R-squared     =  0.4653
-------------+------------------------------           Adj R-squared =  0.4630
       Total |  22265.971   235  94.7488127           Root MSE      =  7.1329

------------------------------------------------------------------------------
Produktios~x |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |   .0972557   .0068154    14.27   0.000     .0838283    .1106832
       _cons |   82.78113    .931585    88.86   0.000     80.94576    84.61649
------------------------------------------------------------------------------
```

$$\hat{L}_t = 82.7 + 0.097\, t$$

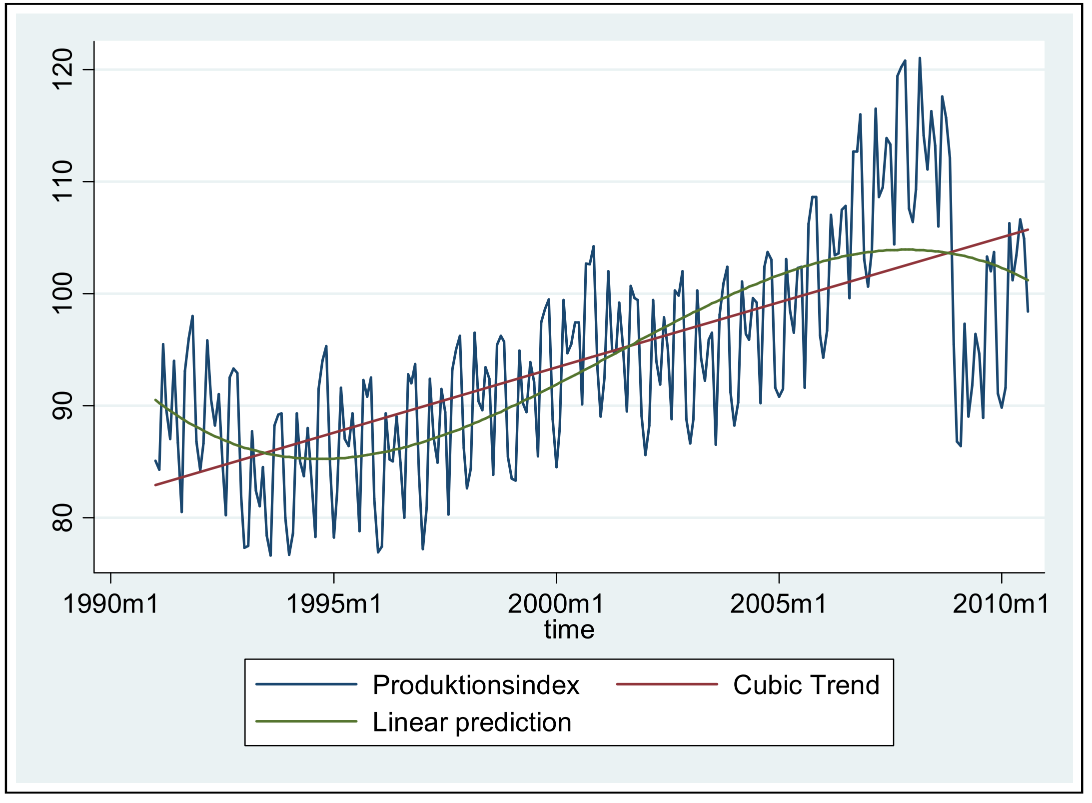
adopted from Winker (2007) “Empir. Wirtschaftsfor”

Cubic Trend Model for industrial production

```
. reg  Produktiosindex t t2 t3

      Source |       SS       df       MS              Number of obs =     236
-------------+------------------------------           F(  3,   232) =   86.87
       Model |  11779.7907     3  3926.5969            Prob > F      =  0.0000
    Residual |  10486.1803   232  45.1990529           R-squared     =  0.5290
-------------+------------------------------           Adj R-squared =  0.5230
       Total |   22265.971   235  94.7488127           Root MSE      =   6.723

------------------------------------------------------------------------------
Produktion~x |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |  -.2622828    .064859    -4.04   0.000    -.3900707   -.1344948
          t2 |    .003533   .0006352     5.56   0.000     .0022815    .0047845
          t3 |  -9.46e-06   1.76e-06    -5.37   0.000    -.0000129   -5.99e-06
       _cons |   90.74309   1.778714    51.02   0.000     87.23859    94.24758
------------------------------------------------------------------------------
```

$$\hat{L}_t = 90.74 + -0.262t + 0.0035t^2 + -0.0000095t^3$$

Linear vs. Cubic Trend Model for industrial production



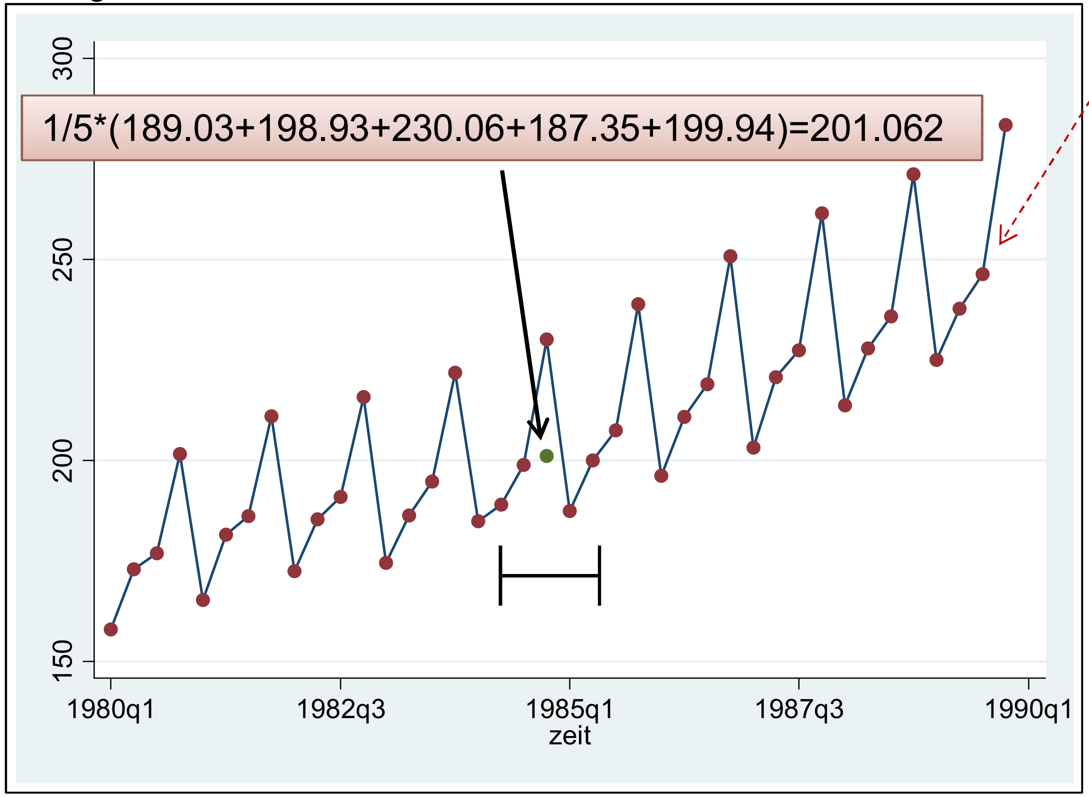
Moving Averages

Averaging over several consecutive values of the series in order to

- determine L_t, C_t or their combination ($L_t + C_t$ or $L_t \cdot C_t$)
  ➔ "local" trends by "**smoothing**" the series
- eliminate S_t (and I_t)
  ➔ "**filtering**" the series

Example: Centered moving average of order 5

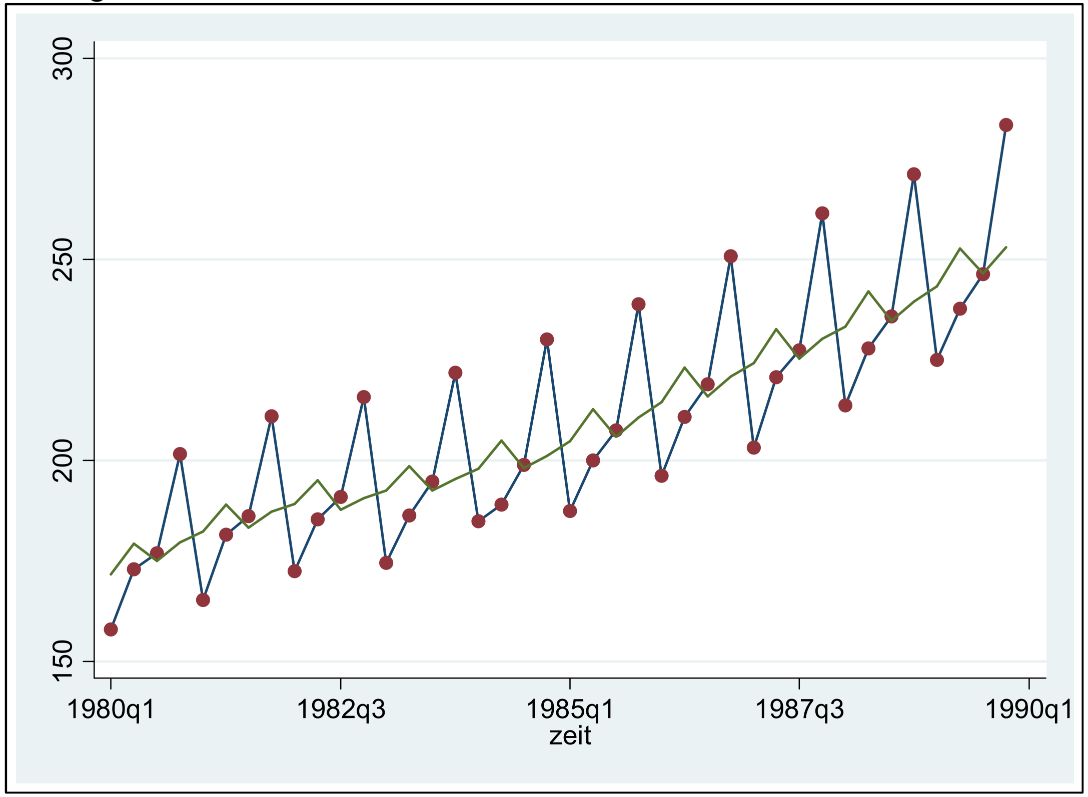
$$\tilde{y}_t = \frac{1}{5}(y_{t+2} + y_{t+1} + y_t + y_{t-1} + y_{t-2})$$



this series is NOT seasonally adjusted

Example: Centered moving average of order 5

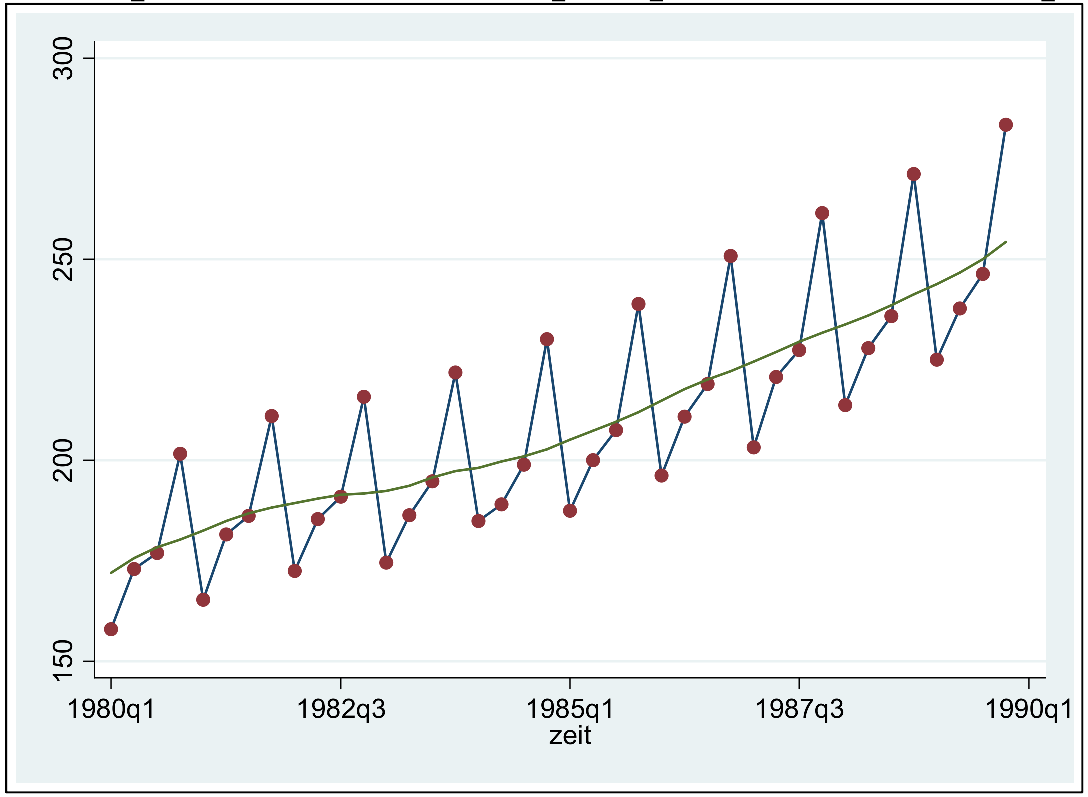
$$\tilde{y}_t = \frac{1}{5}\left(y_{t+2} + y_{t+1} + y_t + y_{t-1} + y_{t-2}\right)$$



STATA: `tssmooth ma lohnsumme_ma5=lohnsumme, window(2 1 2)`

Example: Centered moving average of order 4

$$\tilde{y}_t = \frac{1}{2}\cdot\left[\frac{1}{4}\left(y_{t+1}+y_t+y_{t-1}+y_{t-2}\right)+\right]+\frac{1}{2}\cdot\left[\frac{1}{4}\left(y_{t+2}+y_{t+1}+y_t+y_{t-1}\right)\right]$$



Note: order of moving average is exactly equal seasonal periodicity of this quarterly data!

Centered Moving Averages for **Filtering/Smoothing**

$$\tilde{y}_t = \frac{1}{2k+1} \sum_{j=-k}^{k} y_{t+j}$$

(of odd oder **2k+1**)

Example: order 5 → k=2

$$\tilde{y}_t = \frac{1}{(2 \cdot 2 + 1)} \left(y_{t-2} + y_{t-1} + y_t + y_{t+1} + y_{t+2} \right)$$

Centered Moving Averages for **Filtering/Smoothing**

(of even order **2k**)

$$\begin{aligned}
\tilde{y}_t &= \frac{1}{2}\tilde{y}_t' + \frac{1}{2}\tilde{y}_t'' \\
&= \frac{1}{2}\frac{1}{2k}\sum_{j=-k}^{(k-1)} y_{t+j} + \frac{1}{2}\frac{1}{2k}\sum_{j=-(k-1)}^{k} y_{t+j} \\
&= \frac{1}{2k}\left[\frac{1}{2}y_{t-k} + \sum_{j=-(k-1)}^{(k-1)} y_{t+j} + \frac{1}{2}y_{t+k}\right]
\end{aligned}$$

Example: order 4 → k=2

$$\begin{aligned}
\tilde{y}_t &= \frac{1}{2}\cdot\left[\frac{1}{4}\left(y_{t+1} + y_t + y_{t-1} + y_{t-2}\right) +\right] + \frac{1}{2}\cdot\left[\frac{1}{4}\left(y_{t+2} + y_{t+1} + y_t + y_{t-1}\right)\right] \\
&= \frac{1}{4}\cdot\left[\frac{1}{2}y_{t-2} + \left(y_{t+1} + y_t + y_{t-1}\right) + \frac{1}{2}y_{t+2}\right]
\end{aligned}$$

What happens at the boundaries?

From the STATA manual:

❑ Technical note

`tssmooth ma` gives any missing observations a coefficient of zero in both the uniformly weighted and weighted moving-average filters. This simply means that missing values or missing periods are excluded from the moving average.

Sample restrictions, via `if` and `in`, cause the expression smoothed by `tssmooth ma` to be missing for the excluded observations. Thus sample restrictions have the same effect as missing values in a variable that is filtered in the expression. Also, gaps in the data that are longer than the span of the filter will generate missing values in the filtered series.

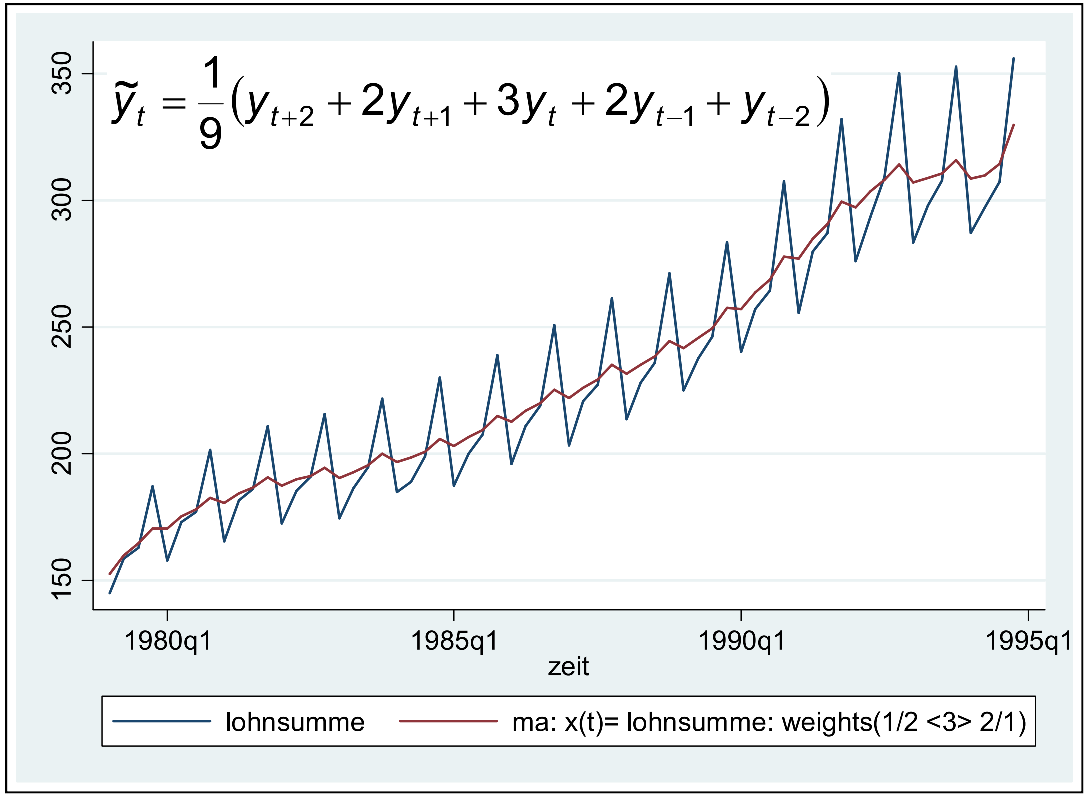
Because the first l observations and the last f observations will be outside the span of the filter, those observations will be set to missing in the moving-average series.

❑

Smoothing Techniques - Moving Averages

Lots of possibilities: weighted ma, unsymmetric ma, uncentered ma…

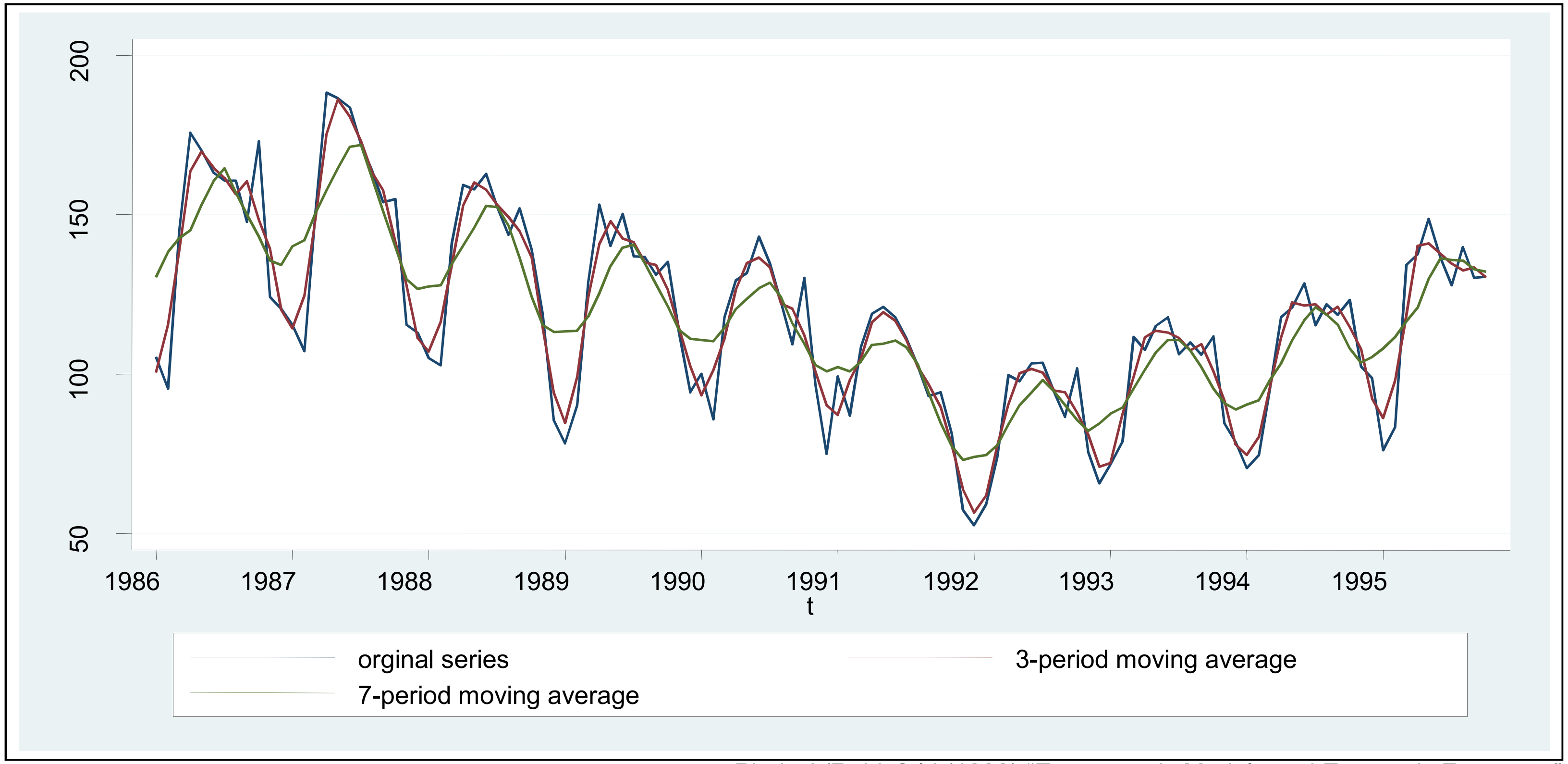
Example: Centered **weighted** moving average of order 5



STATA: use weights option of `tssmooth ma`

Choice of order strongly effects results

Centered Moving Average of Monthly Housing Starts



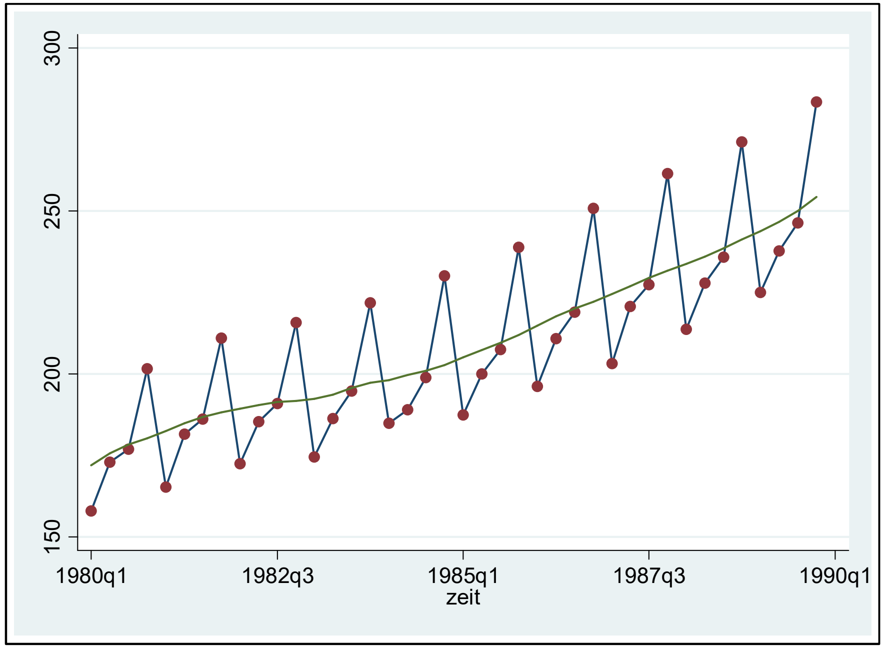
Pindyck/Rubinfeld (1998) “Econometric Models and Economic Forecasts”

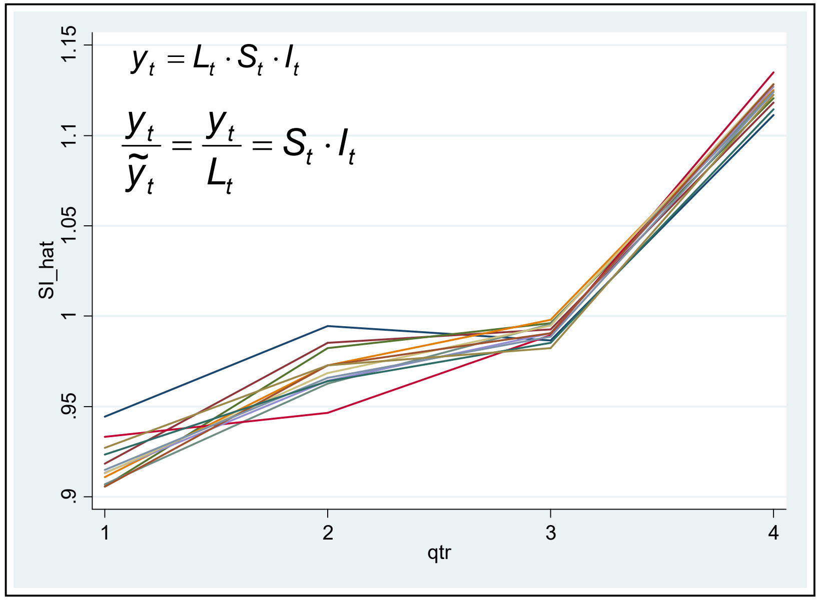
How to choose order? Depends on purpose!

- For **filtering** out S_t choose the order equal to (a multiple of) the seasonal periodicity:
  e.g., 4 for quarterly data, 12 for monthly data
- For **smoothing** the series in order to obtain L_t, choose large(r) values of the order.
- In any case, it is ad hoc.

Filtering out seasonal fluctuations with a moving average

Example: quarterly date and order 4





Seasonal pattern after removing trend

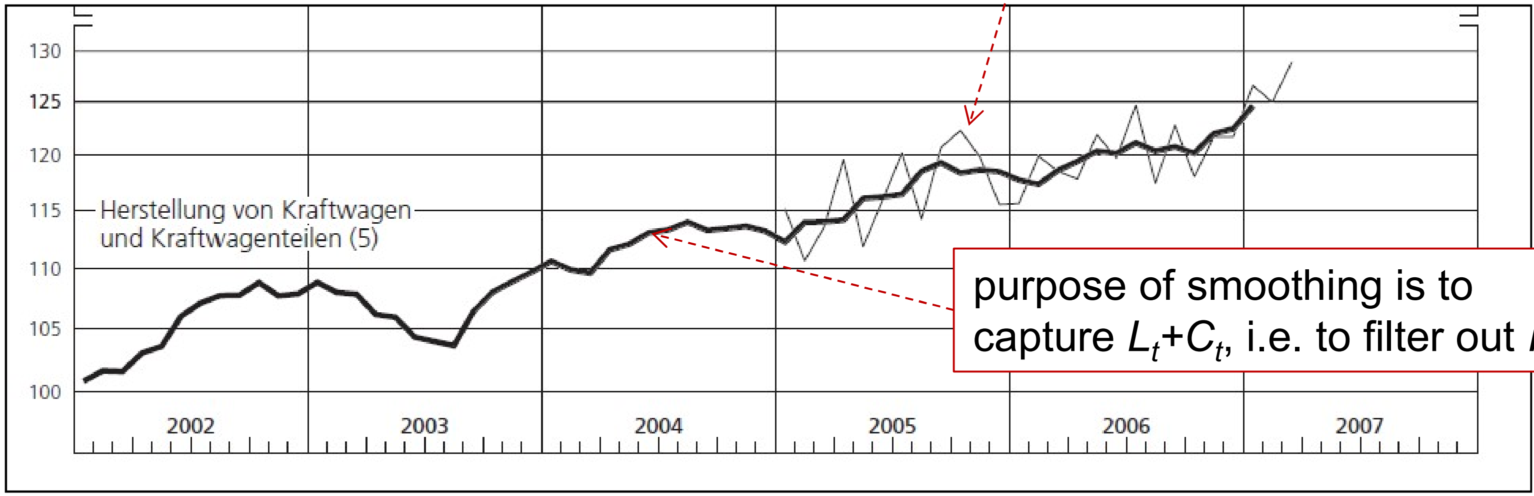
$$\tilde{y}_t = \frac{1}{2} \cdot \left[\frac{1}{4} \left(y_{t+1} + y_t + y_{t-1} + y_{t-2} \right) + \right] + \frac{1}{2} \cdot \left[\frac{1}{4} \left(y_{t+2} + y_{t+1} + y_t + y_{t-1} \right) \right]$$

Moving Averages

this series is already seasonally adjusted

„Zur deutlicheren Kennzeichnung der **konjunkturellen** Entwicklung sind in den Schaubildern in der Regel neben saisonbereinigten Monatswerten daraus errechnete **gleitende Durchschnitte** dargestellt; die Zahl der in die Berechnung einbezogenen Werte ist an der jeweiligen Kurve (in Klammern) angegeben.“

Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 86



purpose of smoothing is to capture L_t+C_t, i.e. to filter out I_t

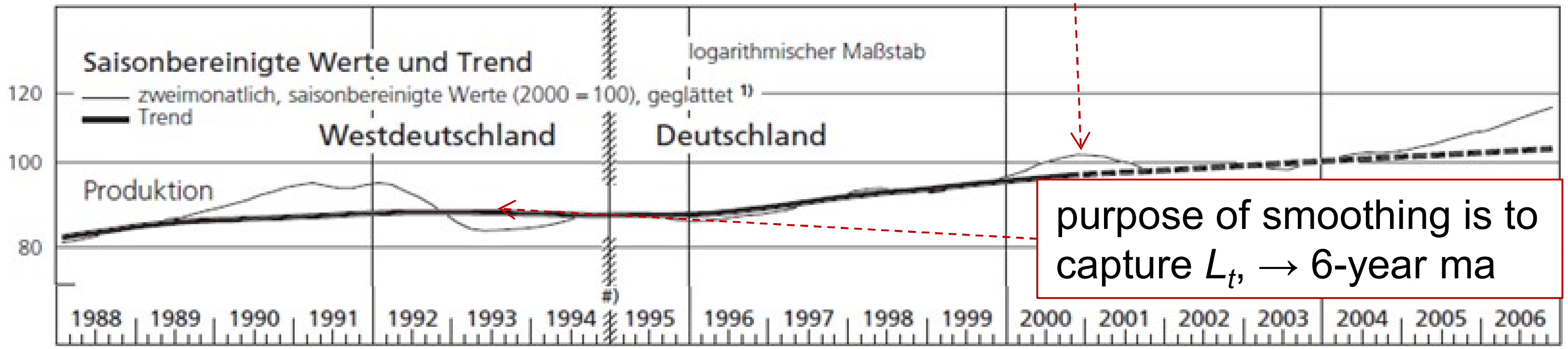
Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 35

original series is a 2-month average; this version seasonally adjusted and smoothed (3-quarter ma)

Moving Averages

„.. ein als **gleitender Durchschnitt** über mehrere Jahre (**in der Regel sechs Jahre**) ermittelter **Trend**… . Der Trend soll kurz- und mittelfristige Schwankungen im Verlauf der saisonbereinigten Reihe ausgleichen und die **längerfristige Grundtendenz** darstellen. Die Trendwerte für die Jahre am Reihenende, für die keine gleitenden Durchschnitte ermittelt werden können, werden durch Extrapolation des Trendverlaufs der letzten Jahre geschätzt, sie sind daher vorläufig..“

Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 87



purpose of smoothing is to capture L_t, → 6-year ma

* Sowie Ergebnisse für den Kohlenbergbau und die Mineralölverarbeitung. Indizes für Westdeutschland und für Deutschland über Jahresdurchschnitt 1995 verkettet. Trend bzw. Trendabweichungen im besonders markierten Bereich am Reihenende wegen der erforderlichen Trendextrapolation unsicher. — 1 Zweimonatsdurchschnitte (Kapazitätsauslastung: Vierteljahreswerte), mit einem gleitenden Dreiperiodendurchschnitt geglättet. — 2 Kapazitätsauslastung in % der betriebsüblichen Vollauslastung, vierteljährliche Angaben, ohne Bergbau, bis 1994 ohne Nahrungs- und Genussmittelgewerbe sowie ohne Chemische Industrie (Quelle der Ursprungswerte: ifo Institut). — # Vergleichbarkeit wegen Umstellung der Erhebungen auf EU-einheitliche Systematiken gestört.

Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 81

Moving Averages and Forecasting

Use uncentered “backward-looking” (“nachlaufend”) version:

$$\hat{y}_{T+1} = \frac{1}{K}\left(y_T + y_{T-1} + \ldots + y_{T-(K-1)}\right)$$

Moving Averages and Forecasting

The moving average forecast are **adaptive forecasts**: They automatically adjust themselves to the most recently available data.

Example: Simple four-period moving average

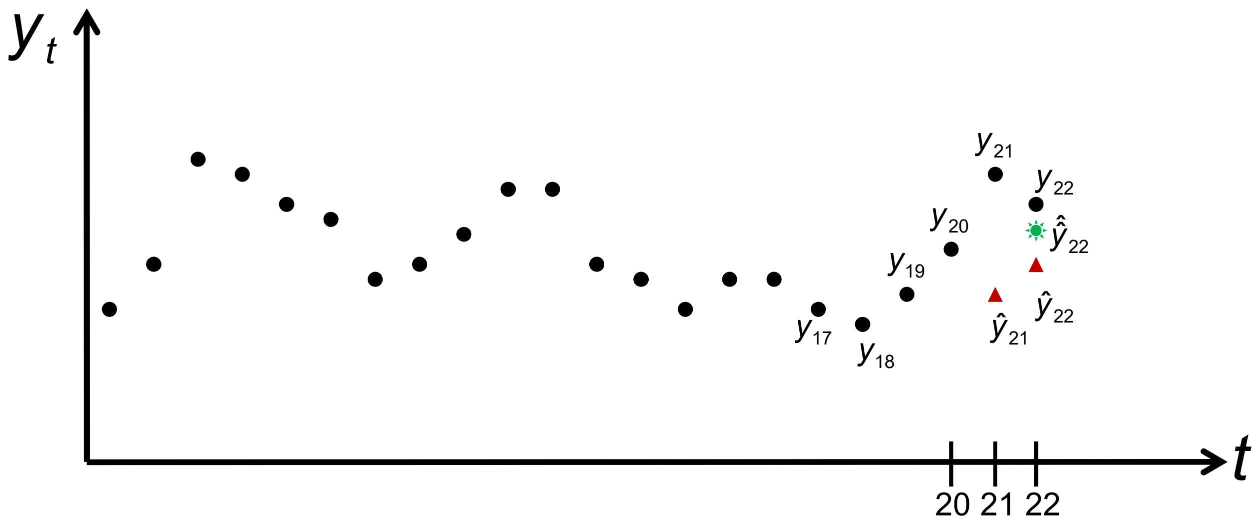
$$\hat{y}_{21} = \frac{1}{4}\left(y_{20} + y_{19} + y_{18} + y_{17}\right)$$

$$\hat{y}_{22} = \frac{1}{4}\left(\hat{y}_{21} + y_{20} + y_{19} + y_{18}\right)$$

$$= \frac{5}{16} y_{20} + \frac{5}{16} y_{19} + \frac{5}{16} y_{18} + \frac{1}{16} y_{17}$$

But shouldn't more recent values play a greater role...?

Example: Simple four-period moving average



If y_{21} were known, we would forecast y_{22} one period ahead as

$$\hat{\hat{y}}_{22} = \frac{1}{4}\left(y_{21} + y_{20} + y_{19} + y_{18}\right)$$

Moving Averages

Uncentered moving average as **adaptive forecasts**:

Example: Simple four-period moving average

$$\hat{y}_{21} == \frac{1}{4} y_{20} + \frac{1}{4} y_{19} + \frac{1}{4} y_{18} + \frac{1}{4} y_{17}$$

$$\hat{y}_{22} = \frac{5}{16} y_{20} + \frac{5}{16} y_{19} + \frac{5}{16} y_{18} + \frac{1}{16} y_{17}$$

An observation is given full weight and suddenly the next period (almost) none

And shouldn’t more recent values play a greater role...?

But shouldn't more recent values play a greater role...?

Exponential Smoothing (Exponentially Weighted Moving Average)

$$\tilde{y}_t = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \alpha(1-\alpha)^3 y_{t-2} + \ldots$$

for $\alpha = 0.8$: $\tilde{y}_t = 0.8 y_t + 0.16 y_{t-1} + 0.032 y_{t-2} + 0.0064 y_{t-2} + 0.00128 y_{t-2} + 0.000256 y_{t-2} \ldots$

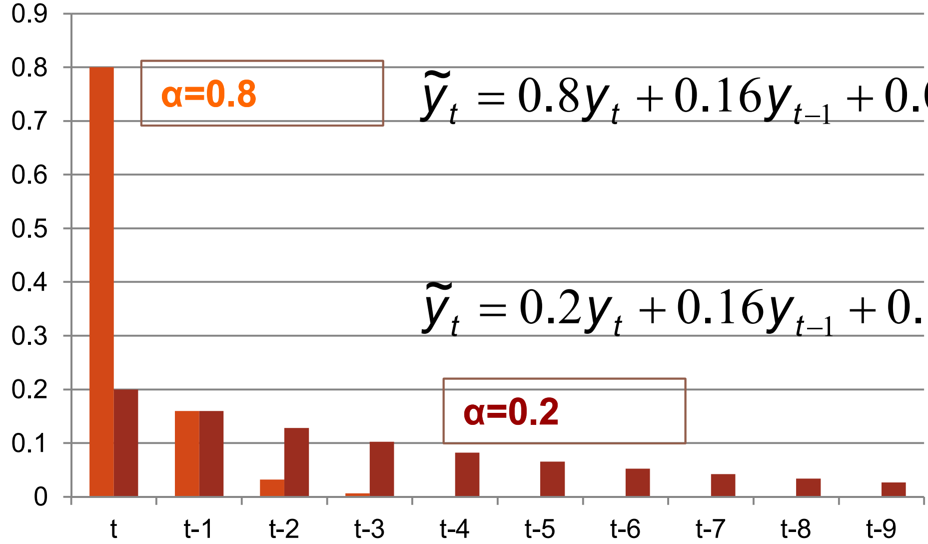
Note that there are different weights for different periods but just one tuning parameter.

Can be used for two purposes:

"Exponential smoothing can be viewed either as an adaptive-forecasting algorithm or, equivalently as a geometrically weighted moving-average filter." STATA TS Manual p. 336

Exponential Smoothing (Exponentially Weighted Moving Average)

$$\tilde{y}_t = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \alpha(1-\alpha)^3 y_{t-2} + \ldots$$



$$\tilde{y}_t = 0.8y_t + 0.16y_{t-1} + 0.032y_{t-2} + 0.0064y_{t-2} + \ldots$$

$$\tilde{y}_t = 0.2y_t + 0.16y_{t-1} + 0.128y_{t-2} + 0.1024y_{t-2} + \ldots$$

- The smaller α (where $0 \leq \alpha \leq 1$), the more heavy the smoothing
- Can be shown that $\alpha \sum_{\tau=0}^{\infty} (1-\alpha)^{\tau} = \frac{\alpha}{1-(1-\alpha)} = 1$

Smoothing Techniques – Exponential Smoothing

Exponential Smoothing (Exponentially Weighted Moving Average)

$$\tilde{y}_t = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \dots \qquad \text{1}$$

$$\tilde{y}_{t-1} = \alpha y_{t-1} + \alpha(1-\alpha)y_{t-2} + \alpha(1-\alpha)^2 y_{t-3} + \dots \qquad \text{L.1}$$

$$(1-\alpha)\tilde{y}_{t-1} = \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \dots \qquad (1\text{-}\alpha)\cdot\text{L.1}$$

$$\tilde{y}_t = \alpha y_t + (1-\alpha)\tilde{y}_{t-1}$$

- For forecasting:

$$\hat{y}_{T+1} = \alpha y_T + \alpha(1-\alpha)y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \dots$$

$$\hat{y}_{T+1} = \alpha y_T + (1-\alpha)\hat{y}_T$$

$$\hat{y}_{T+3} = \hat{y}_{T+2} = \hat{y}_{T+1}$$

Exponential Smoothing (Exponentially Weighted Moving Average)

$$\begin{aligned}\tilde{y}_t &= \alpha y_t + (1-\alpha)\tilde{y}_{t-1} \\ &= \tilde{y}_{t-1} + \alpha\left(y_t - \tilde{y}_{t-1}\right)\end{aligned}$$

The latest observation is compared with the previous weighted average and an adjustment is made.Smoothing parameter α governs how much of an adjustment is made.

This will work reasonably well unless the series has an upward or downward trend (see below).

Exponential Smoothing (Exponentially Weighted Moving Average)

$$\tilde{y}_t = \alpha y_t + \alpha(1-\alpha) y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \alpha(1-\alpha)^3 y_{t-2} + \ldots$$

In practice, we can't go back to the infinite past..

$$\tilde{y}_t = \alpha y_t + (1-\alpha)\tilde{y}_{t-1}$$

$$\tilde{y}_{t-1} = \alpha y_{t-1} + (1-\alpha)\tilde{y}_{t-1}$$

$$\tilde{y}_t = \alpha y_t + (1-\alpha)\left[\alpha y_{t-1} + (1-\alpha)\tilde{y}_{t-1}\right]$$

$$\vdots$$

$$\tilde{y}_t = \alpha y_t + \alpha(1-\alpha) y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \ldots + \alpha(1-\alpha)^{t-1} y_1 + (1-\alpha)^t \tilde{y}_0$$

Typically weight of starting value is very small.

STATA: "the default is to use the mean calculated over the first half of the sample."

Exponential Smoothing

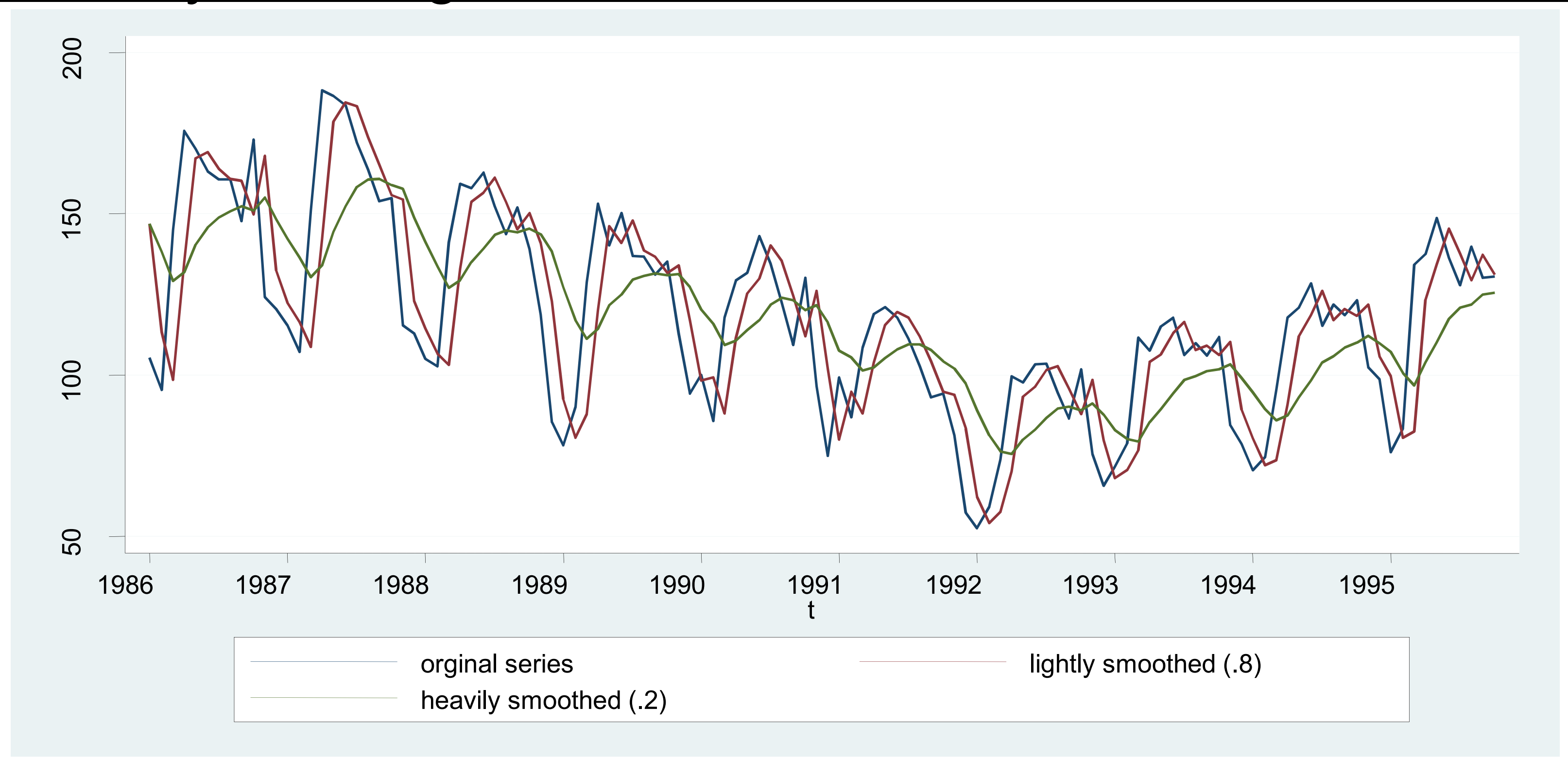
```
. regress hs6fr t

      Source |       SS       df       MS              Number of obs =     118
-------------+------------------------------           F(  1,   116) =   34.04
       Model |  24063.3943     1  24063.3943           Prob > F      =  0.0000
    Residual |  81990.1647   116  706.811764           R-squared     =  0.2269
-------------+------------------------------           Adj R-squared =  0.2202
       Total |  106053.559   117  906.440675           Root MSE      =  26.586

------------------------------------------------------------------------------
       hs6fr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |   -.419239   .0718514    -5.83   0.000    -.5615497   -.2769283
       _cons |   142.9204   4.863919    29.38   0.000     133.2868     152.554
------------------------------------------------------------------------------
```

1. Detrend the original series (assume a linear trend and use the above regression)
2. Smooth the detrended series (the residuals from the above regression)
3. Add the trend back (fit the smoothed residuals in the regression function)

Pindyck/Rubinfeld (1998) “Econometric Models and Economic Forecasts”

Exponential Smoothing

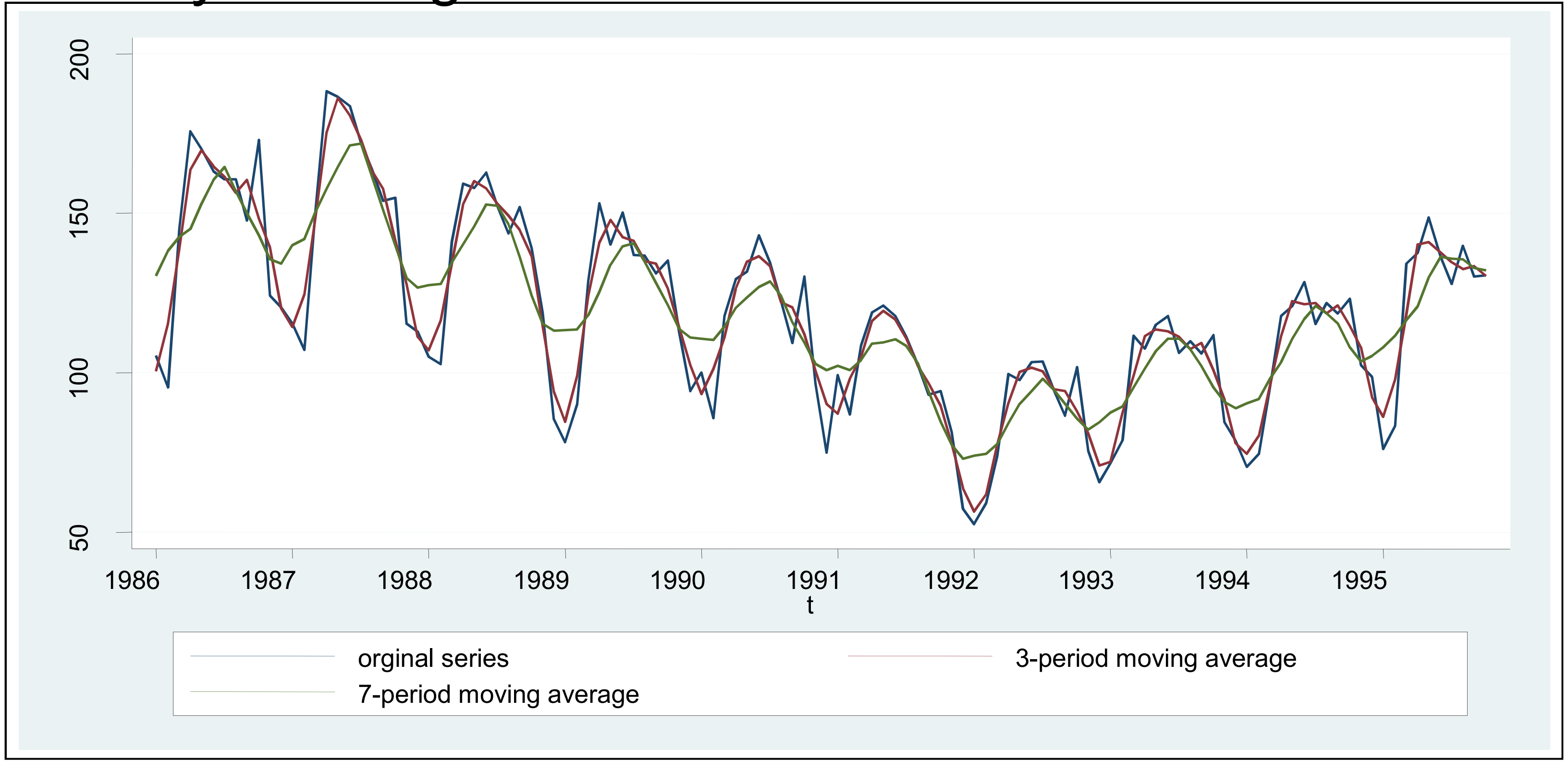
Monthly Housing Starts data



Pindyck/Rubinfeld (1998) “Econometric Models and Economic Forecasts”

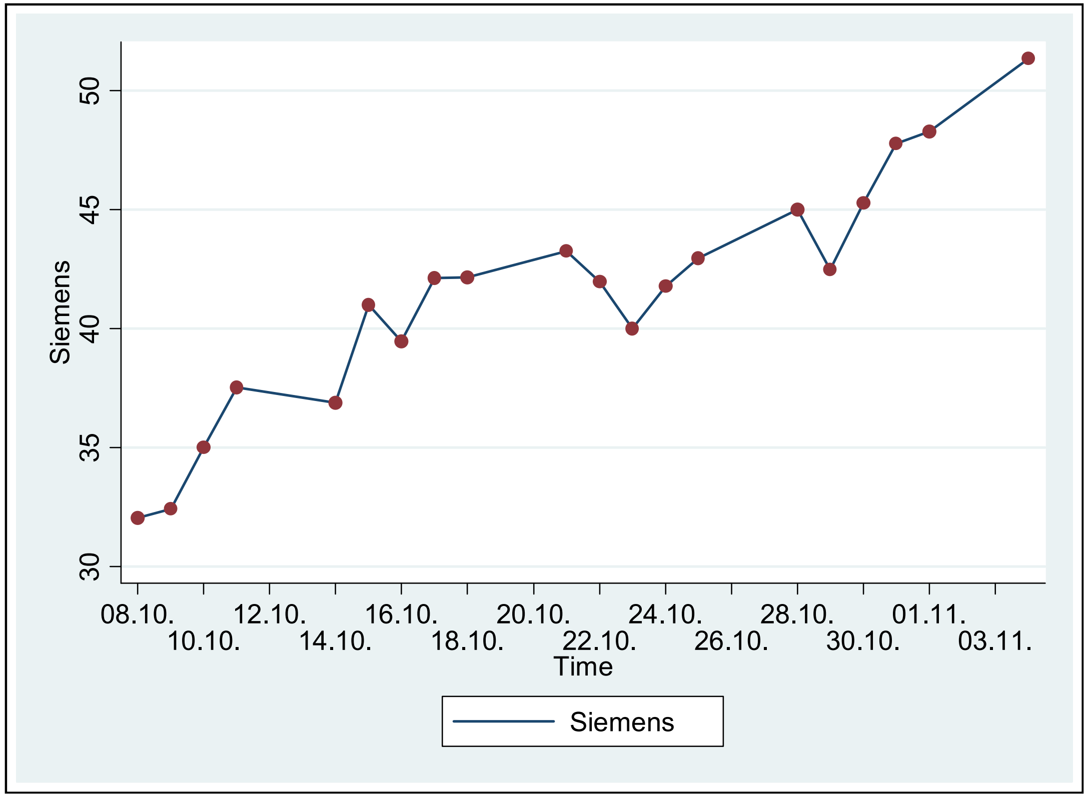
Centered moving average: $\tilde{y}_t = \frac{1}{n}\sum_{i=0}^{n-1} y_{t+\frac{1}{2}(n-1)-i}$

Monthly Housing Starts data



Pindyck/Rubinfeld (1998) "Econometric Models and Economic Forecasts"

Example:


Siemens
30
35
40
45
50
08.10.
10.10.
12.10.
14.10.
16.10.
18.10.
20.10.
22.10.
24.10.
26.10.
28.10.
30.10.
01.11.
03.11.
Time
Siemens

Example:

Time	SIE	Time	SIE
08.10.02	32.05	22.10.02	41.98
09.10.02	32.42	23.10.02	40.00
10.10.02	35.00	24.10.02	41.77
11.10.02	37.53	25.10.02	42.98
14.10.02	36.88	28.10.02	45.02
15.10.02	41.00	29.10.02	42.48
16.10.02	39.48	30.10.02	45.29
17.10.02	42.13	31.10.02	47.79
18.10.02	42.15	01.11.02	48.30
21.10.02	43.25	04.11.02	51.35

Exponential Smoothing

$$\tilde{y}_t = \alpha y_t + (1-\alpha)\tilde{y}_{t-1}$$

$$\alpha = 0.3 \quad \tilde{y}_1 = 32.05$$

$$\tilde{y}_2 = 0.3 \cdot 32.42 + 0.7 \cdot 32.05 = 32.161$$

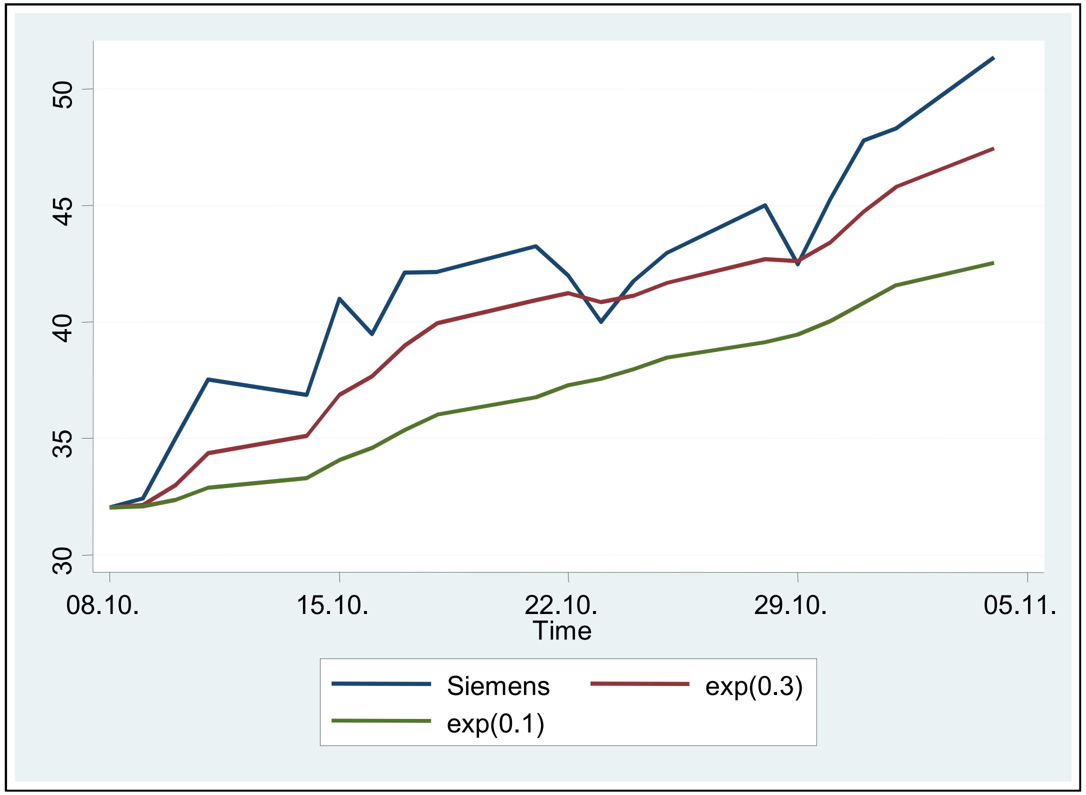
$$\tilde{y}_3 = 0.3 \cdot 35 + 0.7 \cdot 32.161 = 33.013$$

$$\alpha = 0.1 \quad \tilde{y}_1 = 32.05$$

$$\tilde{y}_2 = 0.1 \cdot 32.42 + 0.9 \cdot 32.05 = 32.087$$

$$\tilde{y}_3 = 0.1 \cdot 35 + 0.9 \cdot 32.087 = 32.378$$

Exponential Smoothing



Exponential Smoothing

If the time series has an upward (downward) trend, the **EWMA model** will underpredict (overpredict) future values of y_t.

→ remove any trend from the data before using EWMA

→ the trend term can be added to the untrended initial forecast to obtain the final forecast

Alternatively: use Holt-Winter method (see next slides)

Holt-Winter's two parameter exponential smoothing

$$\tilde{y}_t = \alpha y_t + (1-\alpha)(\tilde{y}_{t-1} + r_{t-1})$$
$$r_t = \gamma(\tilde{y}_t - \tilde{y}_{t-1}) + (1-\gamma)r_{t-1}$$

Here r_t is a smoothed series representing the trend, for example the average rate of increase.

Equivalently:

$$\tilde{y}_t = \tilde{y}_{t-1} + r_{t-1} + \alpha\left[(y_t - \tilde{y}_{t-1}) - r_{t-1}\right]$$
$$r_t = r_{t-1} + \gamma\left[(\tilde{y}_t - \tilde{y}_{t-1}) - r_{t-1}\right]$$

Holt-Winter's two parameter exponential smoothing

$$\tilde{y}_t = \alpha y_t + (1-\alpha)(\tilde{y}_{t-1} + r_{t-1})$$
$$r_t = \gamma(\tilde{y}_t - \tilde{y}_{t-1}) + (1-\gamma)r_{t-1}$$

Here r_t is a smoothed series representing the trend, for example the average rate of increase.

Holt-Winter's two parameter exponential smoothing

$$\tilde{y}_t = \alpha y_t + (1-\alpha)(\tilde{y}_{t-1} + r_{t-1})$$

$$r_t = \gamma(\tilde{y}_t - \tilde{y}_{t-1}) + (1-\gamma) r_{t-1}$$

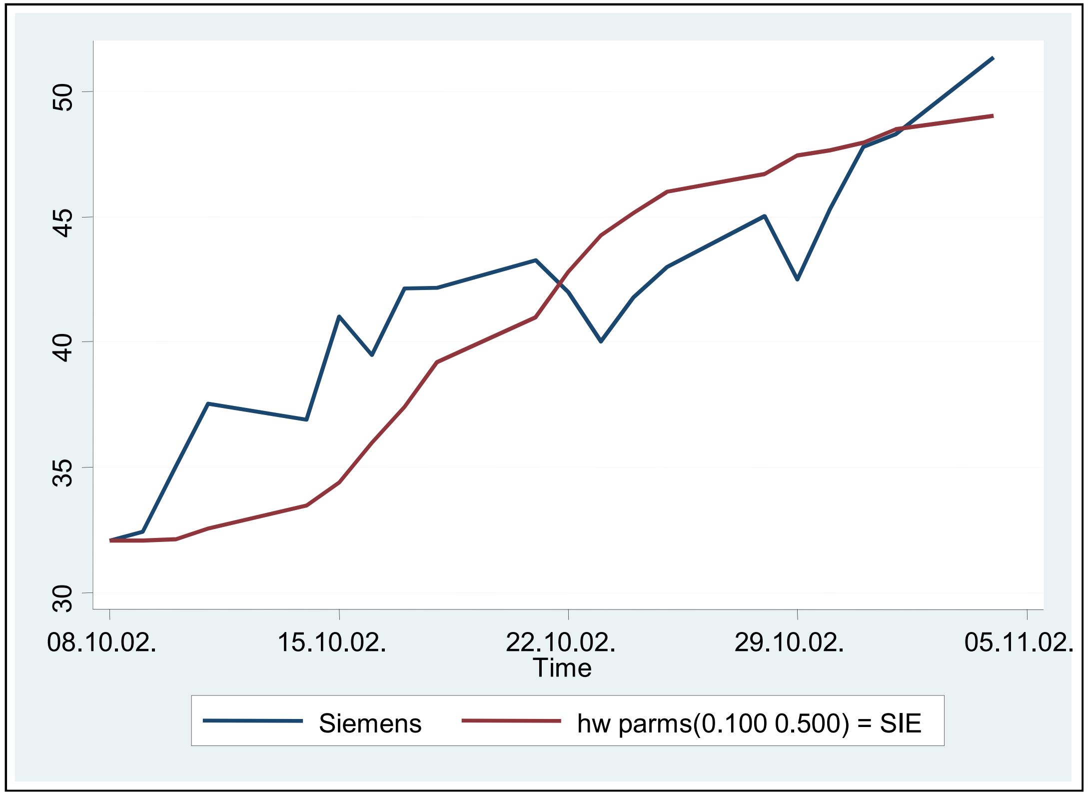
$$\alpha = 0.1 \quad \gamma = 0.5 \quad \tilde{y}_1 = 32.05 \quad r_1 = 0$$

$$\tilde{y}_2 = 0.1 \cdot y_2 + (1-0.1)(\tilde{y}_1 + r_1) = 0.1 \cdot 32.42 + 0.9 \cdot 32.05 = 32.087$$

$$r_2 = 0.5 \cdot (\tilde{y}_2 - \tilde{y}_1) + (1-0.5) r_1 = 0.5 \cdot (32.087 - 32.05) + 0.5 \cdot 0 = 0.018$$

$$\tilde{y}_3 = 0.1 \cdot y_3 + (1-0.1)(\tilde{y}_2 + r_2) = 0.1 \cdot 35 + 0.9 \cdot (32.05 + 0.018) = 32.395$$

Holt-Winter's two parameter exponential smoothing



Hodrick-Prescott Filter

The HP filter is often used (by academics, IMF, OECD, ECB,..) for detrending macroeconomic time series.

$$\underset{\{T_t\}_{t=1}^{T}}{\text{Min}} \sum_{t=1}^{T} \underbrace{\left(y_t - T_t\right)^2}_{\text{Fit}} + \lambda \underbrace{\left[\left(T_{t+1} - T_t\right) - \left(T_t - T_{t-1}\right)\right]^2}_{\text{Smoothness}}$$

Fit is minimized if $y_t = T_t$

Squared second difference of trend is minimized (=0 if trend is linear

The “penalty” parameter λ, controls the smoothness of the resulting trend. The larger λ ,the smoother the trend. HP recommend to set λ equal to 100, 1600, 14400 for annual, quarterly, monthly data.

Hodrick-Prescott Filter

$$\underset{\{T_t\}_{t=1}^{T}}{\text{Min}} \sum_{t=1}^{T} \underbrace{(y_t - T_t)}_{c_t}^2 + \lambda \sum_{t=2}^{T-1} \left[(T_{t+1} - T_t) - (T_t - T_{t-1})\right]^2$$

$$= (y_1 - T_1)^2 + \ldots + (y_T - T_T)^2 + \lambda\left[(T_3 - T_2) - (T_2 - T_1)\right]^2 + \ldots + \lambda\left[(T_T - T_{T-1}) - (T_{T-1} - T_{T-2})\right]^2$$

FOC with respect to T_1:

$$-2(y_1 - T_1) + 2\lambda\left[(T_3 - T_2) - (T_2 - T_1)\right] = 0$$
$$\Rightarrow c_1 = \lambda\left[T_1 - 2T_2 + T_3\right]$$

Hodrick-Prescott Filter

$$\operatorname*{Min}_{\{T_t\}_{t=1}^{T}} \sum_{t=1}^{T} \underbrace{(y_t - T_t)^2}_{c_t} + \lambda \sum_{t=2}^{T-1} \left[(T_{t+1} - T_t) - (T_t - T_{t-1})\right]^2$$

FOC:

$$c_1 = \lambda\left[T_1 - 2T_2 + T_3\right]$$

$$c_2 = \lambda\left[-2T_1 + 5T_2 - 4T_3 + T_4\right]$$

$$c_t = \lambda\left[T_{t-2} - 4T_{t-1} + 6T_t - 4T_{t+1} + T_{t+2}\right] \qquad t = 3,4,\ldots,T-2$$

$$c_{T-1} = \lambda\left[T_{T-3} - 4T_{T-2} + 5T_{T-1} - 2T_T\right]$$

$$c_T = \lambda\left[T_{T-2} - 2T_{T-1} + T_T\right]$$

$$\mathbf{c} = \lambda \mathbf{F}\mathbf{T}$$

Hodrick-Prescott Filter

FOC: $$\mathbf{c} = \lambda \mathbf{F}\mathbf{T}$$

$$
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \\ \vdots \\ \vdots \\ c_{T-2} \\ c_{T-2} \\ c_{T-1} \\ c_T \end{bmatrix}
=
\begin{bmatrix} (y_1 - T_1) \\ (y_2 - T_2) \\ (y_3 - T_3) \\ (y_4 - T_4) \\ (y_5 - T_5) \\ \vdots \\ \vdots \\ \vdots \\ (y_{T-3} - T_{T-3}) \\ (y_{T-2} - T_{T-2}) \\ (y_{T-1} - T_{T-1}) \\ (y_T - T_T) \end{bmatrix}
= \lambda
\begin{bmatrix}
1 & 2 & 1 & 0 & \cdots & & & & & 0 \\
-2 & 5 & -4 & 1 & 0 & \cdots & & & & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & 0 \\
0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
0 & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\
0 & & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\
0 & & & & \cdots & 0 & 1 & -4 & 5 & -2 \\
0 & & & & & \cdots & 0 & 1 & -2 & 1
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ \vdots \\ \vdots \\ \vdots \\ T_{T-3} \\ T_{T-2} \\ T_{T-1} \\ T_T \end{bmatrix}
$$

Hodrick-Prescott Filter

FOC:

$$\mathbf{c} = \lambda \mathbf{F} \mathbf{T}$$

$$\mathbf{y} - \mathbf{T} = \lambda \mathbf{F} \mathbf{T}$$

$$\mathbf{y} = (\lambda \mathbf{F} + \mathbf{I}) \mathbf{T}$$

$$\mathbf{T} = (\lambda \mathbf{F} + \mathbf{I})^{-1} \mathbf{y}$$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ \vdots \\ \vdots \\ \vdots \\ T_{T-3} \\ T_{T-2} \\ T_{T-1} \\ T_T \end{bmatrix} = \left\{ \lambda \begin{bmatrix} 1 & 2 & 1 & 0 & \cdots & & & & & 0 \\ -2 & 5 & -4 & 1 & 0 & \cdots & & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & \vdots \\ 0 & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & & & & \cdots & 0 & 1 & -4 & 5 & -2 \\ 0 & & & & & \cdots & 0 & 1 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & 0 \\ \vdots & \vdots & & & \vdots \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & 1 & 0 \\ 0 & & & 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \\ \vdots \\ \vdots \\ y_{T-3} \\ y_{T-2} \\ y_{T-1} \\ y_T \end{bmatrix}$$

Hodrick-Prescott Filter

FOC: $$\mathbf{c} = \lambda \mathbf{F} \mathbf{T}$$

$$\mathbf{y} - \mathbf{T} = \lambda \mathbf{F} \mathbf{T}$$

$$\mathbf{y} = \left(\lambda \mathbf{F} + \mathbf{I} \right) \mathbf{T}$$

$$\mathbf{T} = \left(\lambda \mathbf{F} + \mathbf{I} \right)^{-1} \mathbf{y}$$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ \vdots \\ \vdots \\ \vdots \\ T_{T-3} \\ T_{T-2} \\ T_{T-1} \\ T_T \end{bmatrix} = \left\{ \lambda \begin{bmatrix} 1 & 2 & 1 & 0 & \cdots & & & & & 0 \\ -2 & 5 & -4 & 1 & 0 & \cdots & & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & \vdots \\ 0 & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & & & & \cdots & 0 & 1 & -4 & 5 & -2 \\ 0 & & & & & \cdots & 0 & 1 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & 0 \\ \vdots & \vdots & & & \vdots \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & 1 & 0 \\ 0 & & & 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \\ \vdots \\ \vdots \\ y_{T-3} \\ y_{T-2} \\ y_{T-1} \\ y_T \end{bmatrix}$$

Turns out that solution is a two-sided linear filter. Weights depend on λ. They can be negative. For time periods in the middle of the window they look like this:

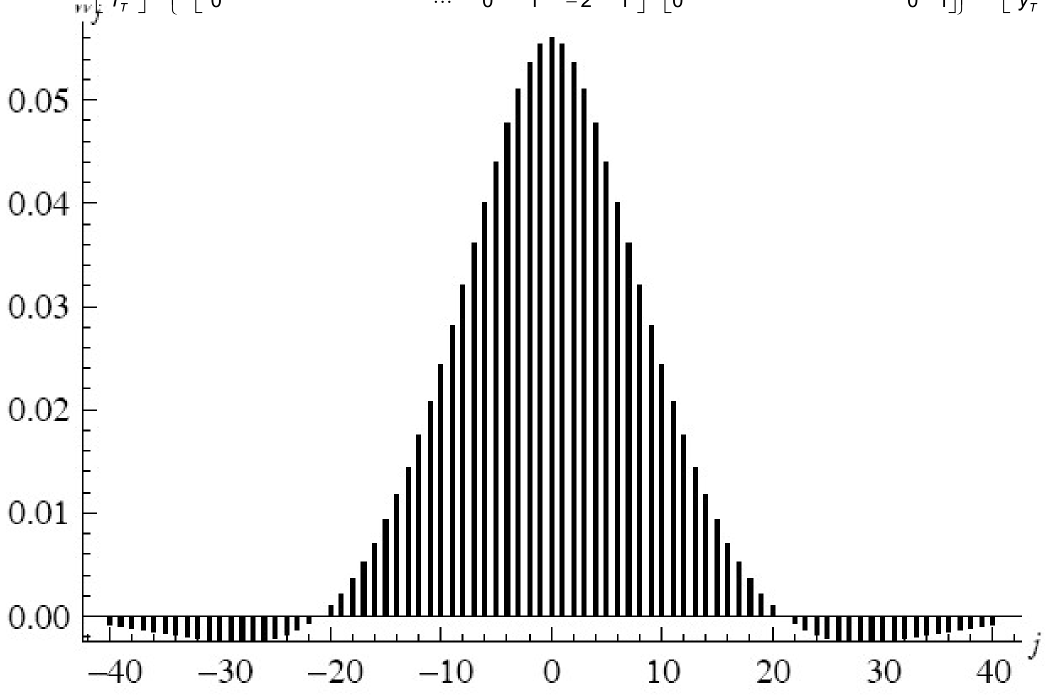
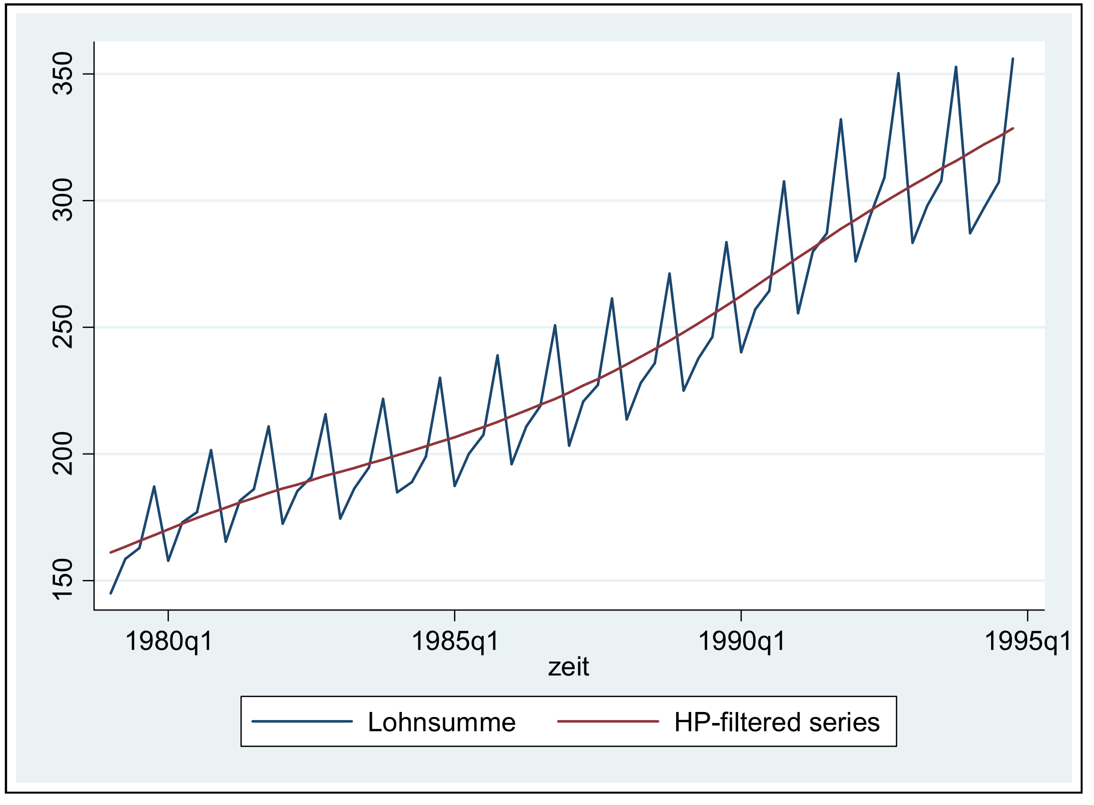


Figure 1. Weights for $HP(1600)$ filter.

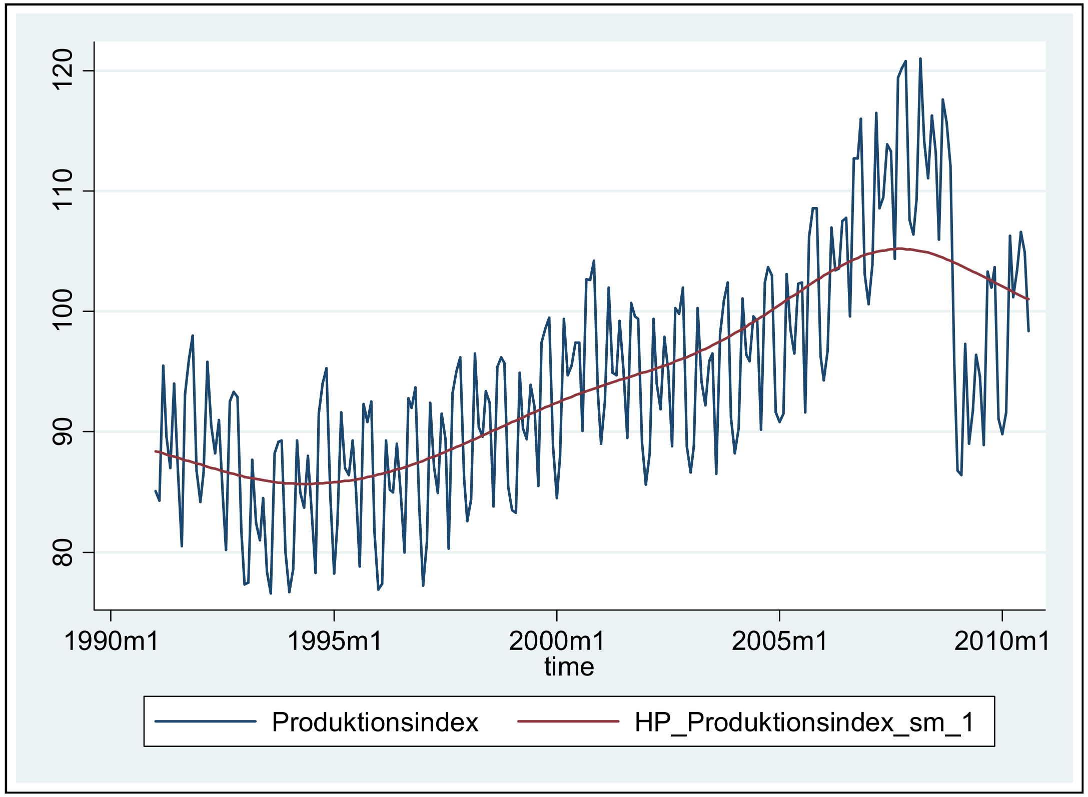
Hodrick-Prescott Filter

Example 1



Hodrick-Prescott Filter

Example 2



Cons:

-deterministic (i.e., no account of stochastics):

- not well-suited for forecasting (trend extrapolation)

- “no way of determining the ‘correct’ value of the smoothing parameters”

Pindyck/Rubinfeld (1998) “Econometric Models and Economic Forecasts”

Pros:

- “The simplicity and underlying philosophy of this approach is appealing and it is much used in industry, possibly also because of its low cost of operation. In practice, the forecasts produced are usually far from optimal but are generally by no means valueless. In terms of cost effectiveness, such methods have much to recommend them”

C.W.J. Granger (1989) “Forecasting in Business and Economics”, p. 103

- valuable if we simply want to smooth the series to interpret it

Cyclical component C_t

Cyclical Component: C_t

Additive Model

$$y_t = \underbrace{L_t + C_t}_{G_t \text{ long term behavior}} + S_t + I_t \quad t = 1,\ldots,T$$

Multiplicative Model

$$y_t = \underbrace{L_t \cdot C_t}_{G_t \text{ long term behavior}} \cdot S_t \cdot I_t \quad t = 1,\ldots,T$$

An estimate of C_t can be obtained by

1. first estimating G_t (using, say a suitable MA)
2. and L_t (using a deterministic trend model or HP) and
3. obtaining C_t by eliminating the trend from G_t:

$$\hat{C}_t = \hat{G}_t - \hat{L}_t \quad \text{or} \quad \hat{C}_t = \frac{\hat{G}_t}{\hat{L}_t}$$

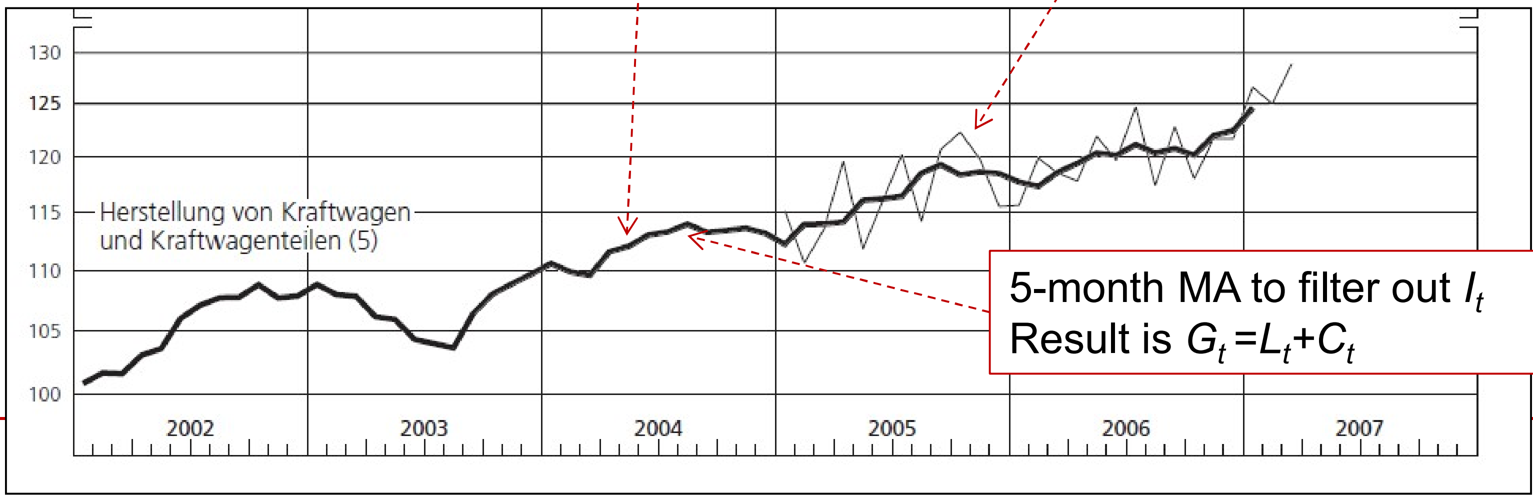
Cyclical Component: L_t and C_t

1. first estimating G_t (using, say a suitable MA)
2. and L_t (using a deterministic trend model or HP) and
3. obtaining C_t by eliminating the trend from G_t:

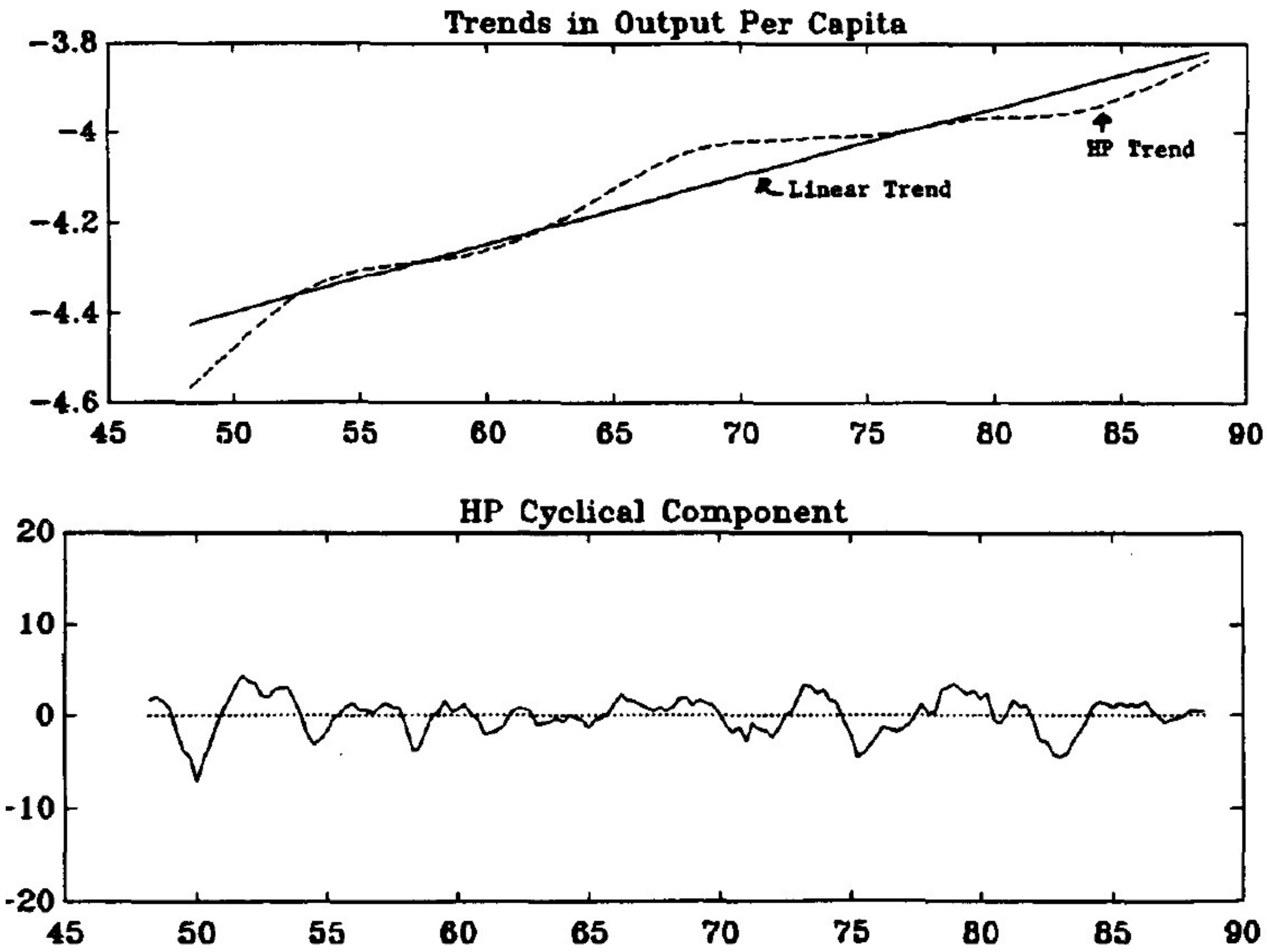
Subtract from this series an estimate of the trend to obtain cyclical component

this series is already seasonally adjusted

5-month MA to filter out I_t
Result is $G_t = L_t + C_t$



Cyclical Component:



Second Example

- Seasonal adjusted series is starting point (not shown)
- L_t is estimated via HP filter
- C_t is simply difference between seasonally adjusted y_t and HP trend:

Seasonal component S_t

Seasonal adjustment

"Seasonal adjustment removes from the time series the average effect of variations that normally occur at about the same time and in about the same magnitude each year—for example, weather and holidays. After seasonal adjustment, cyclical and other short-term changes in the economy stand out more clearly."

Seskin and Parker; A Guide to the NIPA's ; http://www.bea.gov/scb/account_articles/national/0398niw/maintext.htm

Seasonal Component: S_t

$$y_t = L_t + C_t + S_t + I_t \qquad t = 1,\ldots,T$$

$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t \qquad t = 1,\ldots,T$$

Bundesbank:

„Die **Saisonbereinigung** von Zeitreihen durch die Deutsche Bundesbank zielt darauf ab, aus den Bewegungen der betrachteten Zeitreihe die **üblichen** Saisonausschläge **herauszufiltern**. Als übliche Saisonausschläge werden die Jahr für Jahr **zur gleichen Jahreszeit mit ähnlicher Intensität wiederkehrenden** Bewegungen verstanden, die aufgrund von Schwankungen der jeweiligen Zeitreihe in der Vergangenheit unter normalen Umständen zu erwarten sind.“

Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 85

Seasonal Component: S_t

$$y_t = L_t + C_t + S_t + I_t \quad t = 1,\ldots,T$$

$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t \quad t = 1,\ldots,T$$

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Deutsche Bundesbank (Mai 1997), Saisonbereinigte Wirtschaftszahlen, S. 85

Seasonal Adjustment

Additive Model

$$y_t = \underbrace{L_t + C_t}_{G_t \text{ long term behavior}} + S_t + I_t \qquad t = 1, \ldots, T$$

Multiplicative Model

$$y_t = \underbrace{L_t \cdot C_t}_{G_t \text{ long term behavior}} \cdot S_t \cdot I_t \qquad t = 1, \ldots, T$$

To obtain **Seasonal Component:** S_t

1. first estimate G_t (using MA, EWMA or HP)
2. remove G_t to obtain combined seasonal and irregular components

$$y_t - G_t = S_t + I_t \qquad\qquad \frac{y_t}{G_t} = S_t \cdot I_t$$

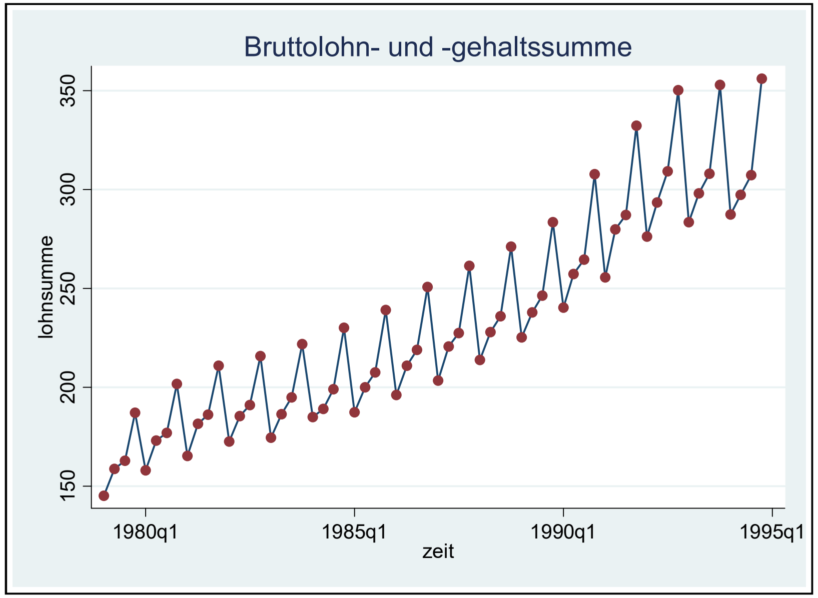
3. Obtain S_t by eliminating I_t from either $S_t + I_t$ or $S_t \cdot I_t$, e.g. by averaging over observations from same seasonal periods
4. Eliminate S_t (if desired) to obtain seasonally adjusted series

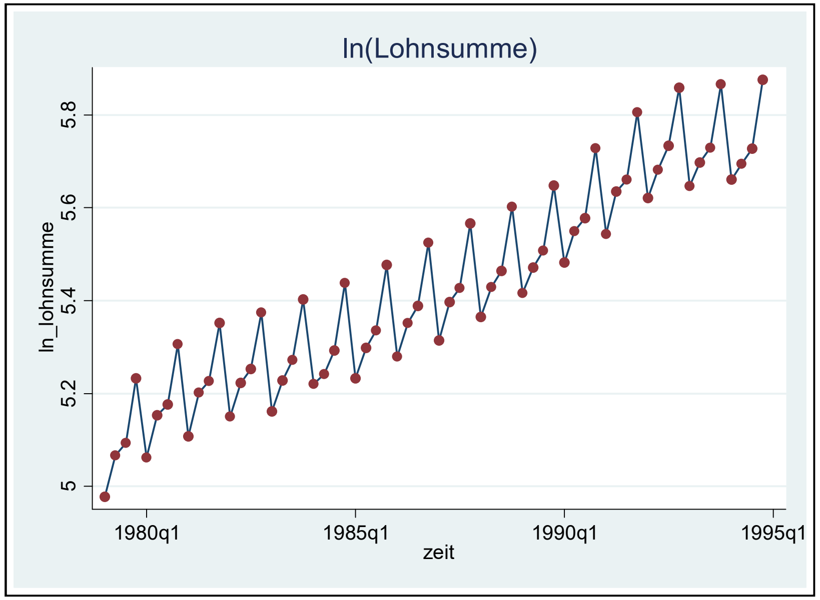
$$y_t^{SA} = y_t - S_t \qquad\qquad y_t^{SA} = \frac{y_t}{S_t}$$

Example:

Seasonal oscillations appear to increase with the level of the series

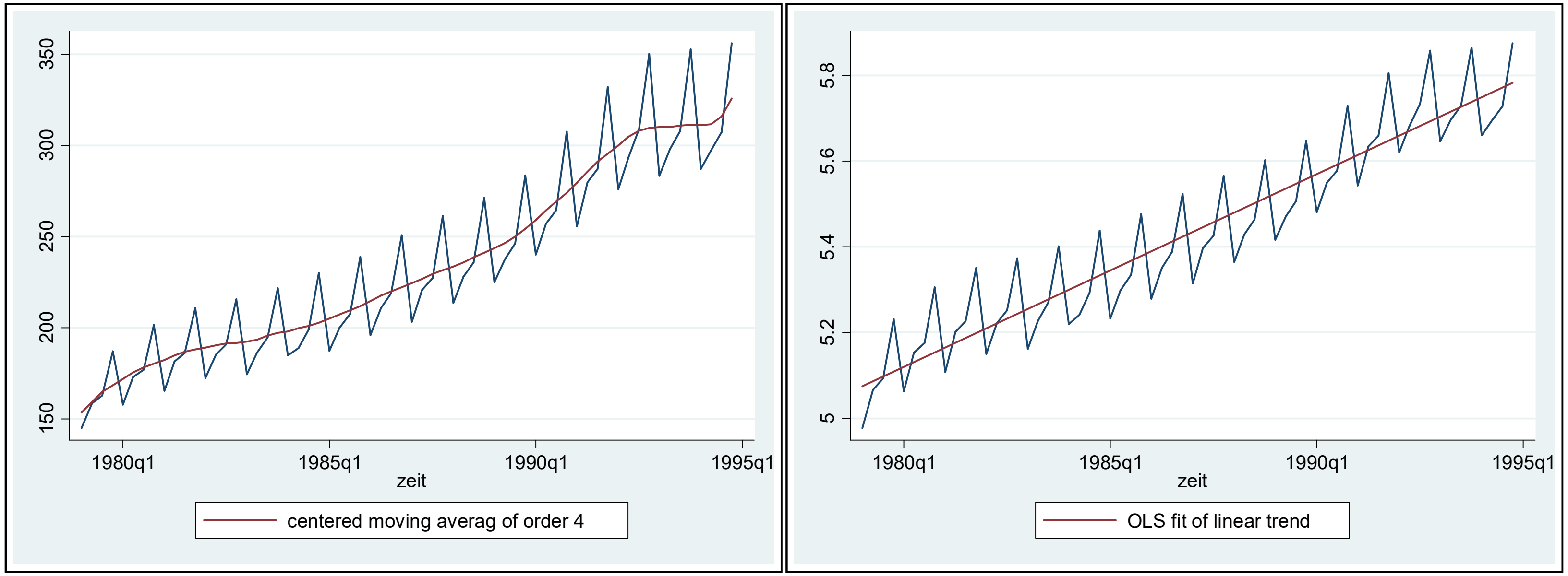
$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t \qquad t = 1,\ldots,T \qquad\qquad \ln y_t = \ln L_t + \ln C_t + \ln S_t + \ln I_t$$





Step1: Isolate G_t, the combined long-term trend and cyclical components

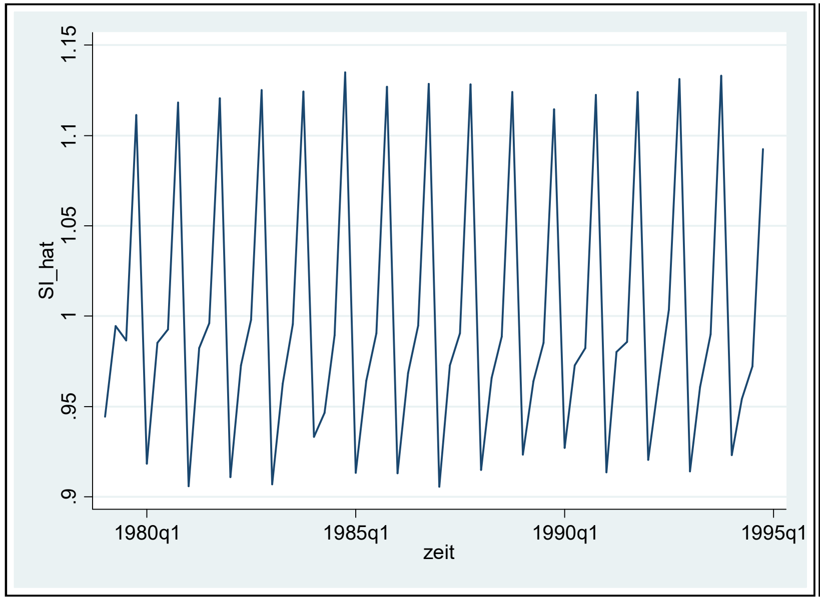
$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t \quad t = 1,\ldots,T \qquad \ln y_t = \ln L_t + \ln C_t + \ln S_t + \ln I_t$$

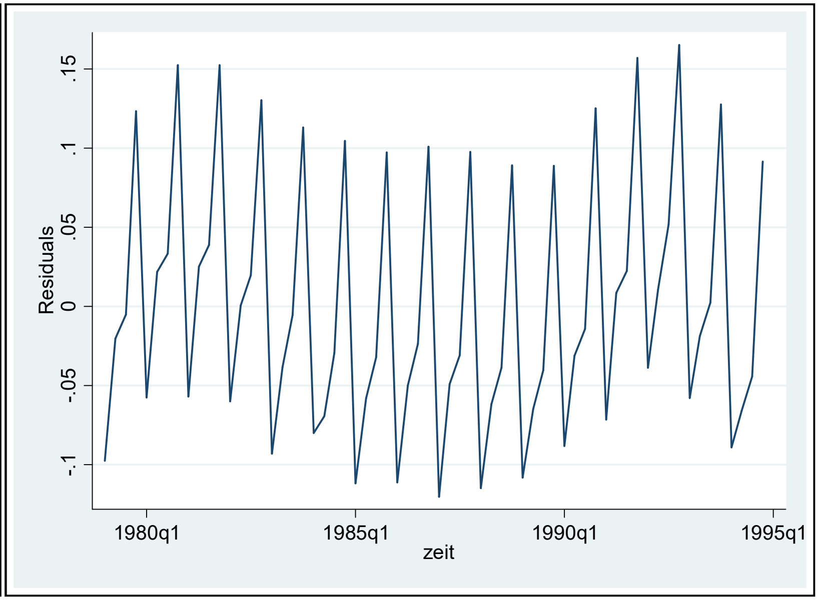


Seasonal Adjustment – Multiplicative Model

Step2: Remove G_t

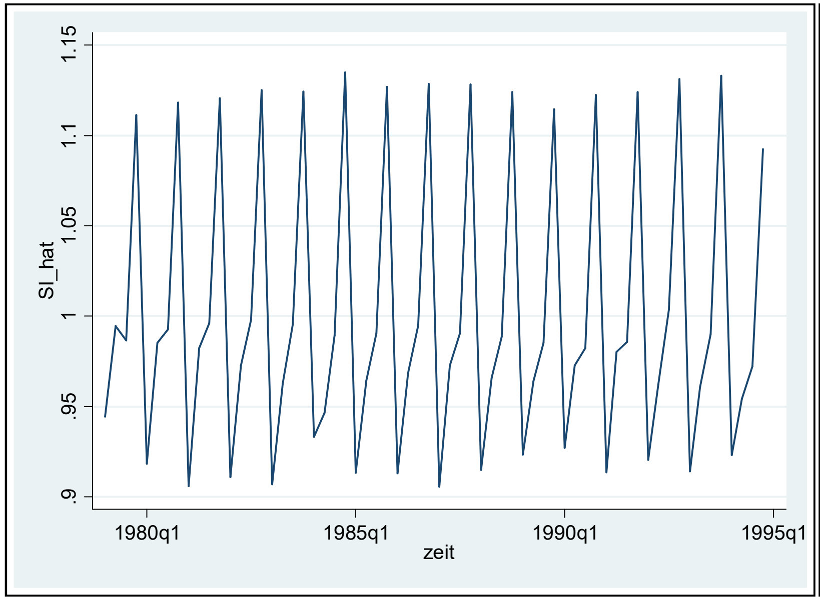
$$\frac{L_t \cdot C_t \cdot S_t \cdot I_t}{L_t \cdot C_t} = S_t \cdot I_t = \frac{y_t}{\tilde{y}_t} = z_t \qquad \ln y_t - (\ln L_t + \ln C_t) = \ln S_t + \ln I_t$$

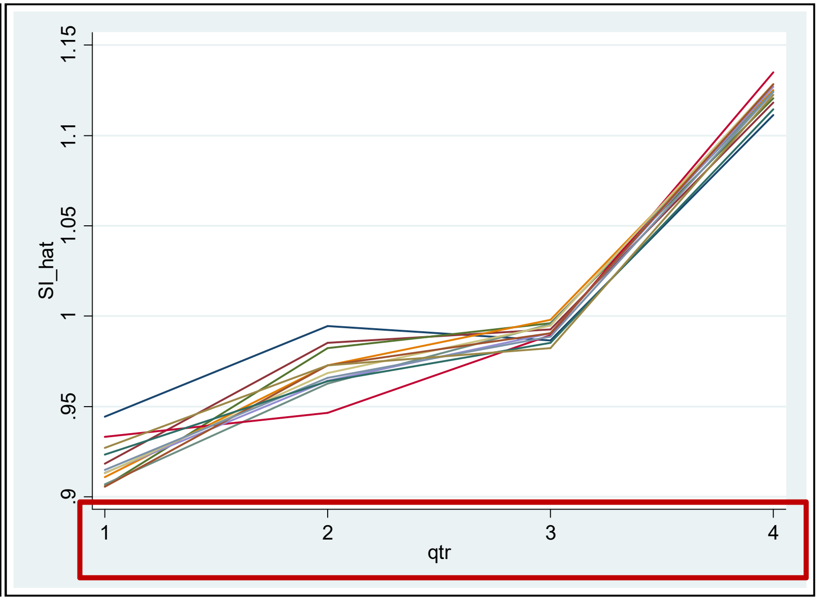




Step 3: Eliminate I_t to obtain S_t

$$\frac{L_t \cdot C_t \cdot S_t \cdot I_t}{L_t \cdot C_t} = S_t \cdot I_t = \frac{y_t}{\tilde{y}_t} = z_t$$





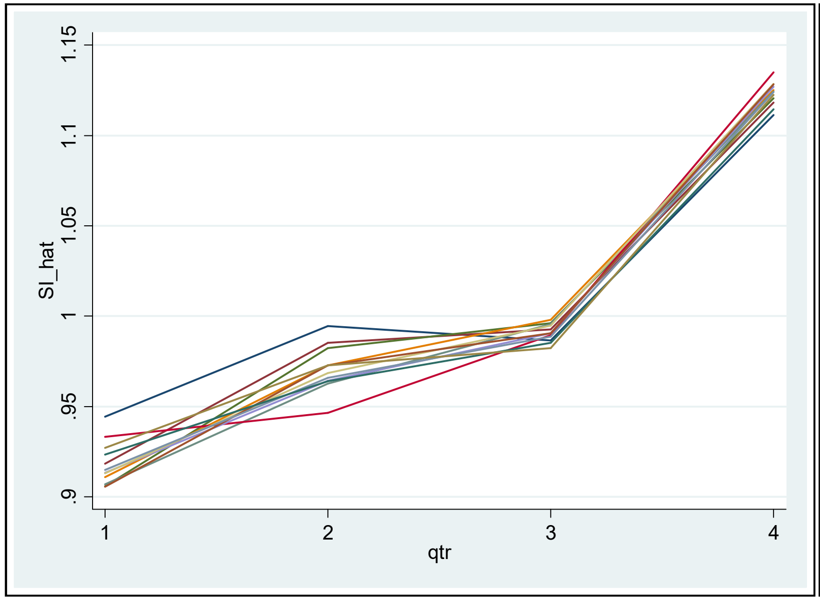
Seasonal Adjustment – Multiplicative Model

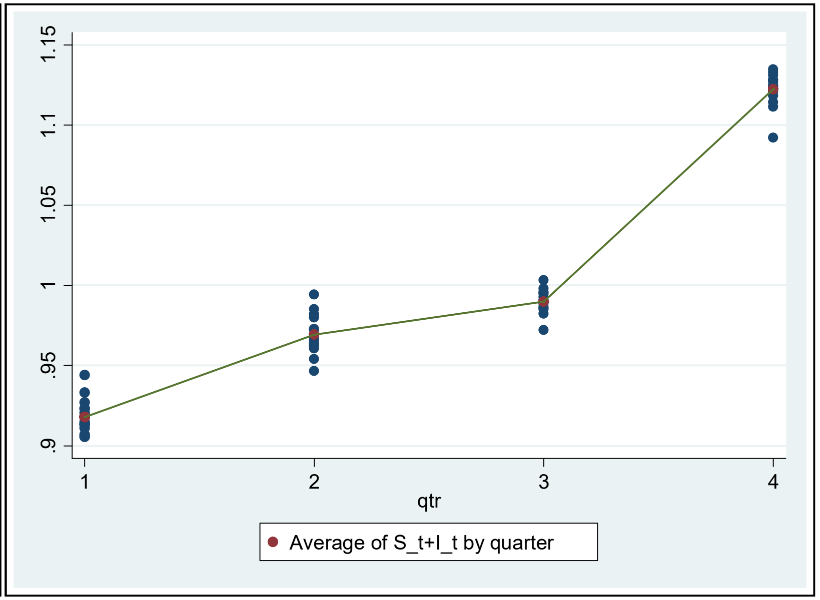
Step 3: Eliminate I_t to obtain S_t

“Phasendurchschnitt”

$$S_t = \overline{S}_q = \frac{1}{\#\text{of years}} \sum_{\text{same quarter obs}} (S_t + I_t)$$

qtr	mean(Q_avg)
1	0.918
2	0.969
3	0.990
4	1.123
Total	1.000



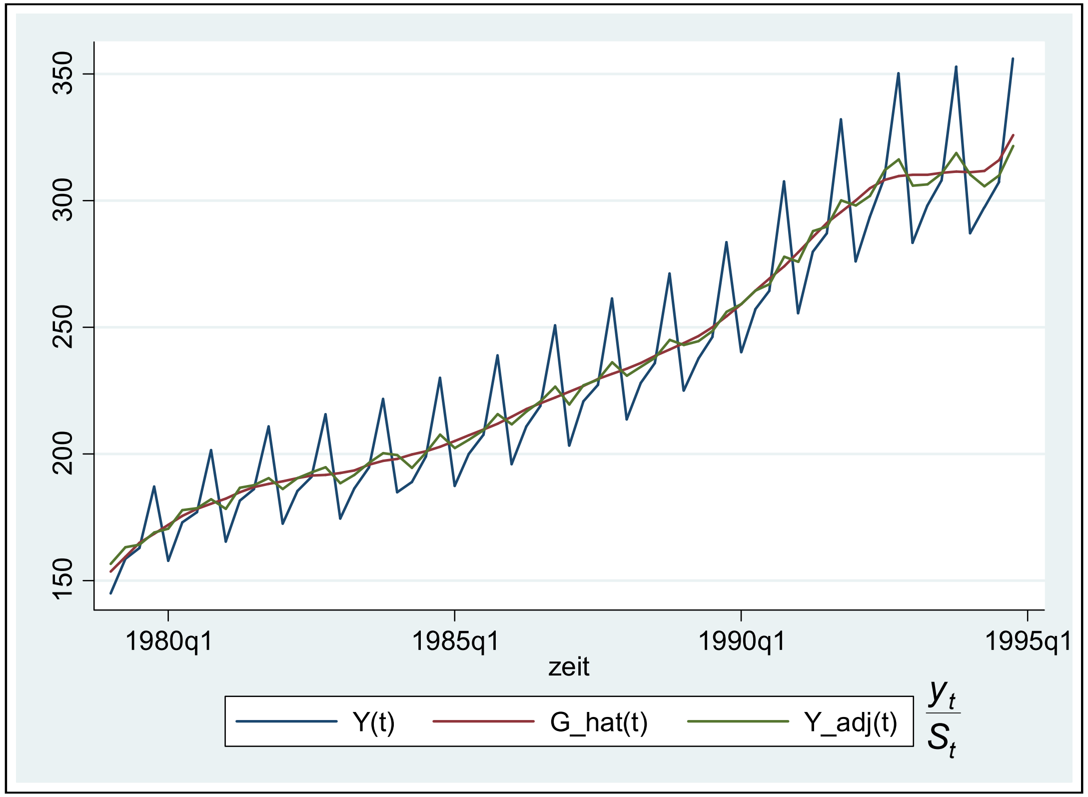


Step 3: Eliminate I_t to obtain S_t

In general, average the values of the combined seasonal and irregular components corresponding to the same seasonal period (month, quarter). These averages will then be estimates of the seasonal indices. Final seasonal indices are computed by dividing these averages by their overall average.

```
. summm Q_avg

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       Q_avg |        64    .9999733    .0760634   .9179683   1.122522
```

Step 3: Eliminate S_t to obtain seasonally adjusted series



Alternative: **Regression with season dummy variables**

$$y_t = L_t \cdot C_t \cdot S_t \cdot I_t \qquad t = 1,\ldots,T$$

$$\ln y_t = \ln L_t + \ln C_t + \ln S_t + \ln I_t$$

Suppose we knew (had already estimated) trend and cycle.

$$\ln y_t - (\ln L_t + \ln C_t) = \ln S_t + \ln I_t$$

$$\underbrace{\ln y_t - (\ln L_t + \ln C_t)}_{\ln \ddot{y}_t} = \sum_{k=1} \beta_k \cdot I[t \text{ is in season } k] + \ln I_t$$

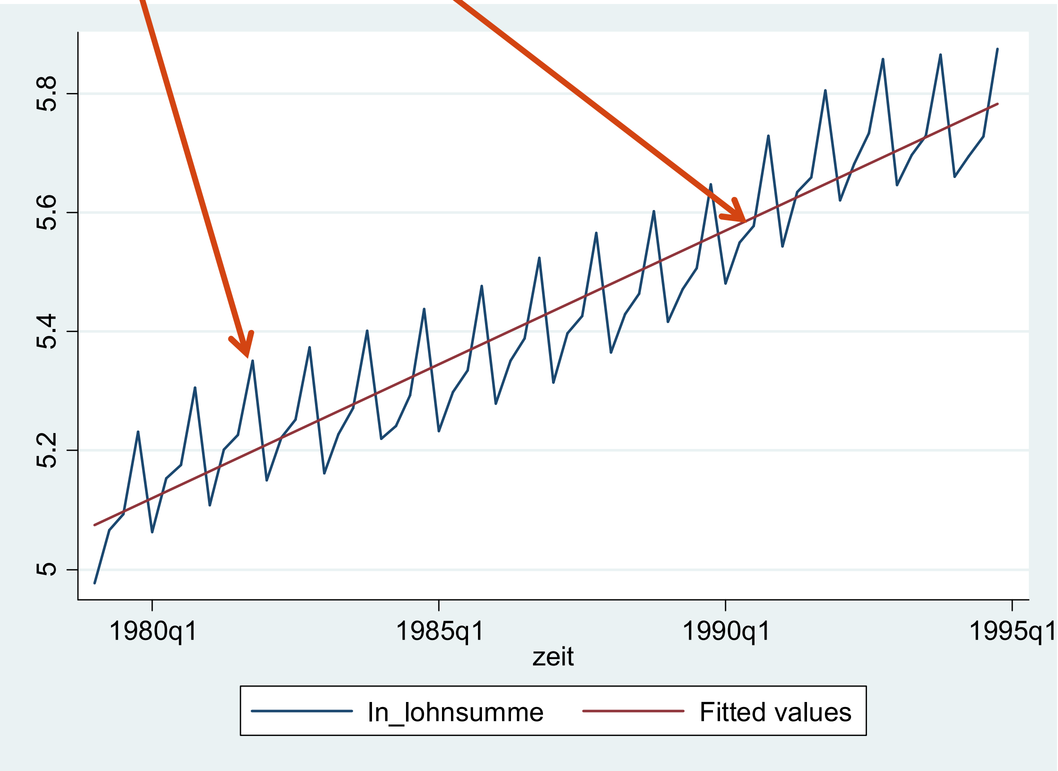
Example: Linear trend fit to log-earnings time series

$$\underbrace{\ln y_t - \underbrace{(\ln L_t)}_{\alpha_0+\alpha_1 \cdot t}}_{\ln \ddot{y}_t} = \sum_{k=1} \beta_k \cdot I\left[t \text{ is in season } k\right] + \ln I_t$$

```
reg   ln_lohnsumme zeit

      Source |       SS       df       MS              Number of obs =      64
-------------+------------------------------           F(  1,     62) =  417.49
       Model |  2.76061369     1  2.76061369           Prob > F      =  0.0000
    Residual |  .409968538    62  .006612396           R-squared     =  0.8707
-------------+------------------------------           Adj R-squared =  0.8686
       Total |  3.17058223    63  .050326702           Root MSE      =  .08132

------------------------------------------------------------------------------
ln_lohnsumme |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        zeit |   .0112429   .0005502    20.43   0.000     .0101429    .0123428
       _cons |   4.220691   .0600179    70.32   0.000     4.100717    4.340665

predict ln_deterended, res
```

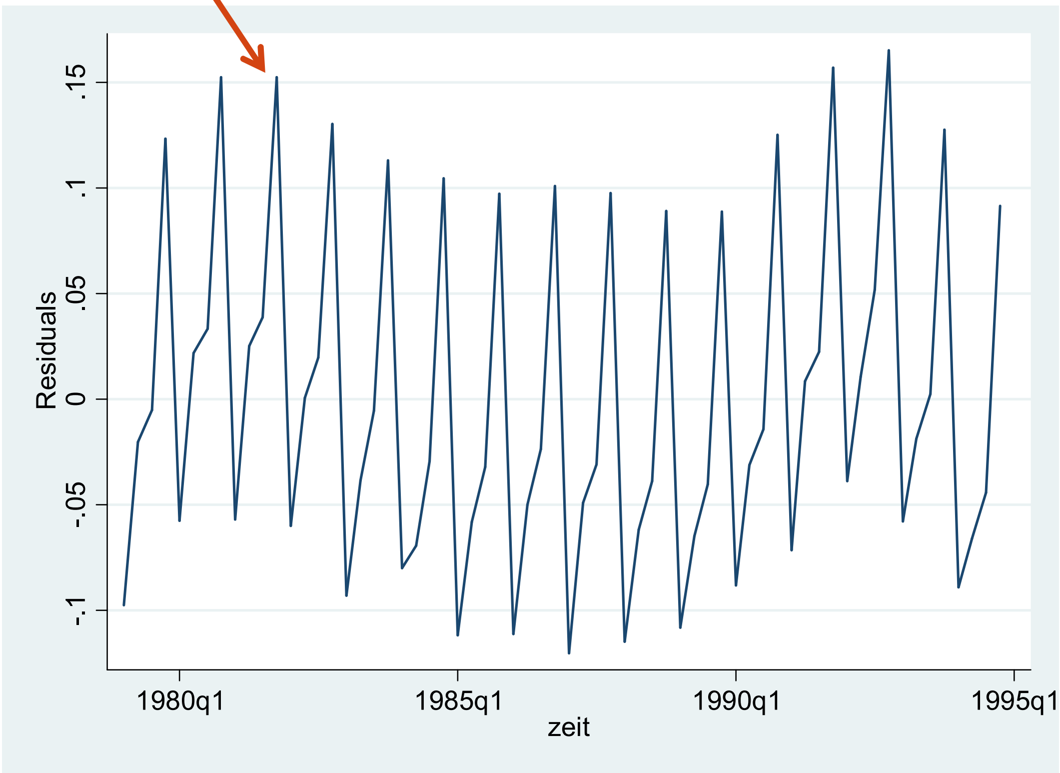
Seasonal Adjustment – Multiplicative Model

$$\ln y_t - \underbrace{(\ln L_t)}_{\alpha_0 + \alpha_1 \cdot t} = \sum_{k=1} \beta_k \cdot I[t \text{ is in season } k] + \ln I_t$$



Seasonal Adjustment – Multiplicative Model

$$\underbrace{\ln y_t - (\underbrace{\ln L_t}_{\alpha_0 + \alpha_1 \cdot t})}_{\ln \ddot{y}_t} = \sum_{k=1} \beta_k \cdot I[t \text{ is in season } k] + \ln I_t$$



Seasonal Adjustment – Multiplicative Model

Example: Fitting time dummies to deviation from trend

$$\underbrace{\ln y_t - (\underbrace{\ln L_t}_{\alpha_0+\alpha_1 \cdot t})}_{\ln \ddot{y}_t} = \sum_{k=1} \beta_k \cdot I[t \text{ is in season } k] + \ln I_t$$

```
. reg ln_d  qtr_1 qtr_2 qtr_3 qtr_4, noc

      Source |       SS       df       MS              Number of obs =      64
-------------+------------------------------           F(  4,    60) =  105.98
       Model |   .35913857     4  .089784643           Prob > F      =  0.0000
    Residual |  .050829968    60  .000847166           R-squared     =  0.8760
-------------+------------------------------           Adj R-squared =  0.8677
       Total |  .409968538    64  .006405758           Root MSE      =  .02911

------------------------------------------------------------------------------
ln_deteren~d |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       qtr_1 |  -.0849292   .0072765   -11.67   0.000    -.0994844    -.070374
       qtr_2 |  -.0288997   .0072765    -3.97   0.000    -.0434549   -.0143444
       qtr_3 |  -.0060121   .0072765    -0.83   0.412    -.0205673    .0085431
       qtr_4 |    .119841   .0072765    16.47   0.000     .1052857    .1343962
------------------------------------------------------------------------------
```

Seasonal Adjustment – Multiplicative Model

Example: Finally converting dummy coefficients from log-scale to natural scale

```
. reg ln_d  qtr_1 qtr_2 qtr_3 qtr_4, noc

      Source |       SS       df       MS              Number of obs =      64
-------------+------------------------------           F(  4,    60) =  105.98
       Model |   .35913857     4  .089784643           Prob > F      =  0.0000
    Residual |  .050829968    60  .000847166           R-squared     =  0.8760
-------------+------------------------------           Adj R-squared =  0.8677
       Total |  .409968538    64  .006405758           Root MSE      =  .02911

------------------------------------------------------------------------------
ln_deteren~d |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       qtr_1 |  -.0849292   .0072765   -11.67   0.000    -.0994844    -.070374
       qtr_2 |  -.0288997   .0072765    -3.97   0.000    -.0434549   -.0143444
       qtr_3 |  -.0060121   .0072765    -0.83   0.412    -.0205673    .0085431
       qtr_4 |    .119841   .0072765    16.47   0.000     .1052857    .1343962
------------------------------------------------------------------------------
```

```
. di exp(_coef[qtr_1])
.91857732

. di exp(_coef[qtr_2])
.97151395

. di exp(_coef[qtr_3])
.99400592

. di exp(_coef[qtr_4])
1.1273176
```

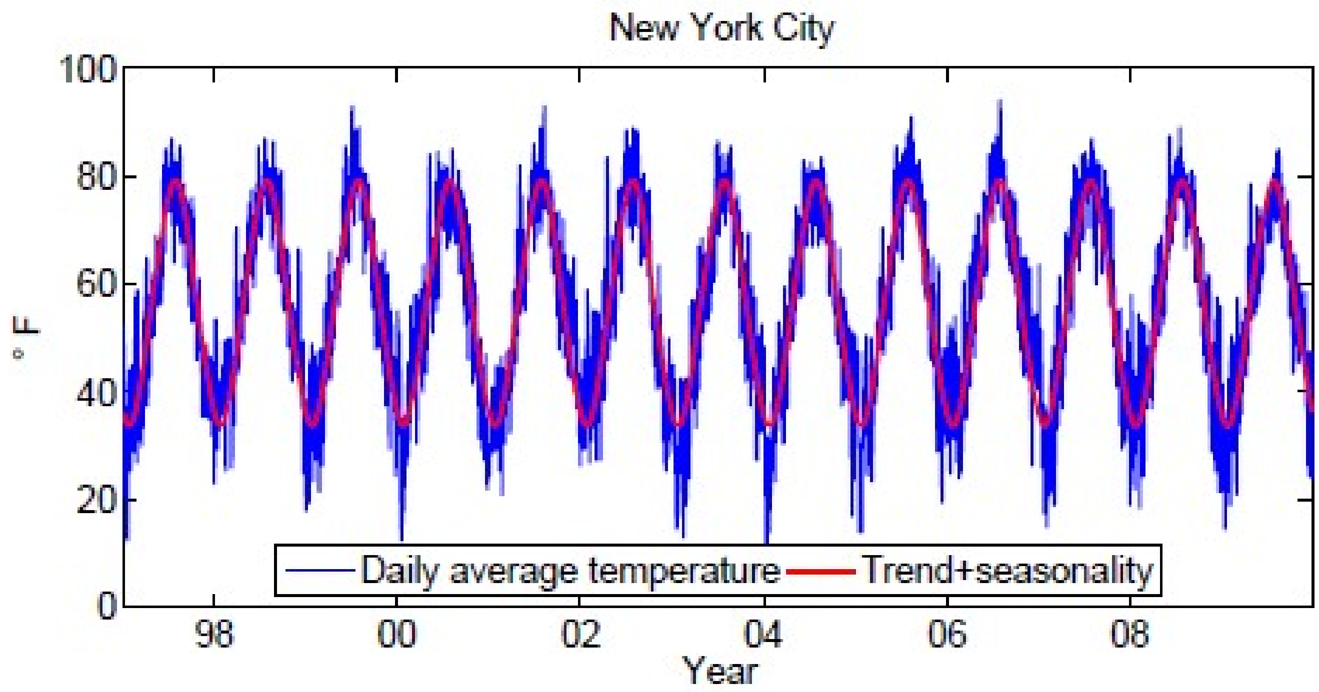
Comparison with earlier result

```
----------------------
      qtr | mean(Q_avg)
----------+-----------
        1 |     0.918
        2 |     0.969
        3 |     0.990
        4 |     1.123
          |
    Total |     1.000
----------------------
```

Seasonal Modelling via Trigonometric Functions

Model for the average daily temperature in New York City

$$T_t = \Lambda_t + X_t,$$



$$\text{with} \qquad \Lambda_t = \underbrace{a + bt}_{\text{trend}} + \underbrace{\sum_{p=1}^{P} \left[a_p \cos\left(\frac{2\pi pt}{365}\right) + b_p \sin\left(\frac{2\pi pt}{365}\right)\right]}_{\text{seasonality}},$$

$$X_t = \underbrace{\sum_{l=1}^{L} \rho_l X_{t-l}}_{\text{autoregression}} + \underbrace{\sigma_t \epsilon_t}_{\text{stochastic}},$$

Meteorological Forecasts and the Pricing of Temperature Futures
Matthias Ritter, Oliver Musshoff, and Martin Odening
The Journal of Derivatives
Winter 2011, Vol. 19, No. 2: pp. 45-60