ML CS10: Ex. 1

$$k_\ell(x,x') = \phi_\ell(x)^T \phi_\ell(x')$$

$$k(x,x') = \sum_{\ell=1}^{L} \beta_\ell k_\ell(x,x')$$

a) $\sum_i \sum_j c_i c_j k(x,x') \Leftrightarrow \sum_i \sum_j c_i c_j \sum_{\ell=1}^{L} \beta_\ell k_\ell(x,x')$

$\Leftrightarrow \sum_i \sum_j \sum_\ell c_i c_j \beta_\ell k_\ell(x,x') \Leftrightarrow \sum_i \sum_j \sum_\ell (c_i \sqrt{\beta_\ell} \sqrt{k_\ell(x,x')})(c_j \sqrt{\beta_\ell} \sqrt{k(x,x')})$

$\Leftrightarrow \sum_{i,j} \sum_\ell (c_i \sqrt{\beta_\ell} \sqrt{k_\ell(x,x')})^2$

$\sqrt{\beta_\ell}$: ≥ 0; k_ℓ: is kernel $\to \geq 0$

$\underbrace{\qquad\qquad}_{\geq 0}$

b) $\phi(x)$ such that $k(x,x') = \phi(x)^T \phi(x')$

$\Leftrightarrow \sum_{\ell=1}^{L} \beta_\ell k_\ell(x,x') \Leftrightarrow \sum_{\ell=1}^{L} \beta_\ell \phi_\ell(x)^T \phi_\ell(x')$

with $\phi_\ell(x)^T \phi_\ell(x') = \phi(x)^T \phi(x') \Leftrightarrow \sum_{\ell=1} \beta_\ell \phi(x)^T \phi(x')$

$\Leftrightarrow \sum_{\ell=1}^{L} \frac{1}{L} \phi(x)^T \phi(x') \Leftrightarrow \phi(x)^T \phi(x')$

$\Leftrightarrow \sum_{\ell=1}^{L} \frac{1}{\sqrt{L}} \phi(x)^T \cdot \frac{1}{\sqrt{L}} \phi(x')$

Ex. 2 $k_{struct}((x,y),(x',y')) = k(x,x') \cdot 1_{[y=y']}$

a) $\sum_i \sum_j c_i c_j k(x,x') \cdot 1_{[y=y']}$

$\Leftrightarrow \sum_i \sum_j (c_i \sqrt{k(x,x')} \sqrt{1_{[y=y']}})(c_j \sqrt{k(x,x')} \sqrt{1_{[y=y']}})$

$\Leftrightarrow \sum_{i,j} (c_i \sqrt{k(x,x')}\, 1_{[y=y']})^2$

$\sqrt{k}$: is kernel $\to \geq 0$; $1_{[y=y']}$: 1 or 0 $\to \geq 0$

$\underbrace{\qquad\qquad}_{\geq 0}$

b) $\phi_{struct}(x,y)$ such that $k_{struct}((x,y),(x',y')) = \phi_{struct}(x,y)^T \phi_{struct}(x',y')$

$k_{struct}(\cdot) = k(x,x') \cdot 1_{[y=y']} \Leftrightarrow (\sqrt{k(x,x')} \cdot 1_{[y=y']})^T (\sqrt{k(x,x')}\, 1_{[y=y']})$

$\phi_{struct}(x,y)^T \cdot \phi_{struct}(x',y')$