Exercise Sheet 1. Machine Learning.

1. $P(\text{error}) = \int p(\text{error}|x) \cdot p(x)\, dx$

$P(\text{error}|x) = \min(P(w_1|x), P(w_2|x))$

a) the inquality holds: $\min(a,b) \leq \frac{2}{\frac{1}{a}+\frac{1}{b}}$ $\forall a,b>0$.

suppose $a \geq b$: $\min(a,b) = b \leq \frac{2ab}{a+b}$

$\because$ since $\frac{2a}{a+b} \geq 1$

when $0<a<b$, the inquality holds as well.

$$\Rightarrow P(\text{error}|x) \leq \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}}$$

$$\Rightarrow P(\text{error}) \leq \int \frac{2}{P(w_1|x)^{-1} + P(w_2|x)^{-1}} p(x)\, dx. \quad //.$$

b) $P(w_1|x) = \frac{P(w_1) \cdot P(x|w_1)}{P(x)}$, $P(w_2|x) = \frac{P(w_2) \cdot P(x|w_2)}{P(x)}$

by a):
$$P(\text{error}) \leq \int \frac{2}{P(w_1|x)^{-1} + P(w_2|x)^{-1}} p(x)\, dx$$

$$= \int \frac{2}{\frac{1}{P(w_1)P(x|w_1)} + \frac{1}{P(w_2) \cdot P(x|w_2)}} dx$$

$$= \int \frac{2P(w_1)P(w_2)}{\pi[P(w_2)(1+(x-u)^2) + P(w_1)(1+(x+u)^2))} dx$$

$$= \frac{2P(w_1)P(w_2)}{\pi} \int \frac{1}{(P(w_1)+P(w_2))x^2 + 2u[P(w_1)-P(w_2)]x + (1+u^2)P(w_1)+P(w_2)} dx$$

by: $\int \frac{1}{ax^2+bx+c} dx = \frac{2\pi}{\sqrt{4ac-b^2}}$:

$$P(\text{error}) \leq \frac{2P(w_1)P(w_2)}{\pi} \cdot \frac{2\pi}{\sqrt{4(P(w_1)+P(w_2))(1+u^2)^2 - 4u^2(P(w_1)-P(w_2))^2}}$$

$$= \frac{2P(w_1)P(w_2)}{\sqrt{P(w_1)^2 + (4u^2+2)(P(w_1)P(w_2)) + P(w_2)^2}}. \quad //$$

c) If the error has no upper-bounds that are both tight and analytically integrable, we can apply kernel estimation on the training data to estimate the error

2. $p(x|w_1) = \frac{1}{2\sigma}\exp\left(-\frac{|x-u|}{\sigma}\right)$. $p(x|w_2) = \frac{1}{2\sigma}\exp\left(-\frac{|x+u|}{\sigma}\right)$. $\sigma > 0$.

a). $p(w_1|x) = \frac{p(w_1)p(x|w_1)}{p(x)}$, $p(w_2|x) = \frac{p(w_2)\cdot p(x|w_2)}{p(x)}$.

let $p(w_1|x) = p(w_2|x)$,

we get: $p(w_1)\frac{1}{2\sigma}\exp\left(-\frac{|x-u|}{\sigma}\right) = p(w_2)\frac{1}{2\sigma}\exp\left(-\frac{|x+u|}{\sigma}\right)$

$$\Rightarrow \ln(p(w_1)) - \frac{|x-u|}{\sigma} = \ln(p(w_2)) - \frac{|x+u|}{\sigma}$$

$$\sigma\left[\ln(p(w_1)) - \ln(p(w_2))\right] = |x-u| - |x+u|$$

$$\Rightarrow x = \frac{\sigma(\ln(p(w_2)) - \ln(p(w_1)))}{2}$$

b). $\Leftrightarrow p(w_1|x) > p(w_2|x)$ is always true.

$\Leftrightarrow \sigma\left[\ln p(w_1) - \ln(p(w_2))\right] + |x+u| - |x-u| > 0$ is independent of x.

~~so: $u=0$ and~~

$\Leftrightarrow \frac{p(w_1)}{p(w_2)} > \exp\left(\frac{|x-u| - |x+u|}{\sigma}\right)$ is independent of x.

so: $u=0$ and $\frac{p(w_1)}{p(w_2)} > \exp(0)$. $p(w_1) > p(w_2)$

c). ①. $p(w_1|x) = p(w_2|x)$.

i.e.: $\ln(p(w_1)) - \frac{(x-u)^2}{2\sigma^2} = \ln(p(w_2)) - \frac{(x+u)^2}{2\sigma^2}$

$$2\sigma^2\left[\ln p(w_1) - \ln p(w_2)\right] = (x-u)^2 - (x+u)^2$$

$$\Rightarrow x = \frac{\sigma^2}{2u}\ln\frac{p(w_2)}{p(w_1)}$$

② $p(w_1|x) > p(w_2|x)$.

i.e.: $4ux > 2\sigma^2 \ln\frac{p(w_1)}{p(w_2)}$. is independent of x.

so: $u=0$ and $p(w_2) > p(w_1)$