

Cayley graphs of given degree and diameter on linear groups

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Overview

- 1 Degree/diameter
- 2 Graph lifting and Cayley graphs
- 3 Computer search of graphs

- In it's simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In design of graphs we can take many restrictions into account such degree, grith, diameter.
- Two important problems concerning degree and diameter and degree and grith of graph

The degree/diameter problem

Degree/diameter problem

Find graph with biggest possible number of vertices with given degree and diameter.

There is theoretical upper bound for largest order of graph with d -degree and k -diameter.

$$\begin{aligned} n_{d,k} &\leq M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\ &= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\ &= \begin{cases} 1 + d \frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k + 1, & \text{if } d = 2 \end{cases} \end{aligned} \tag{1}$$

Moore bound

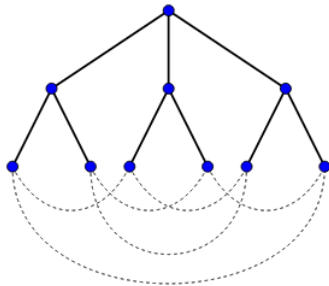


Figure: Petersen graph is Moore graph with $d = 3$ and $k = 2$

Graphs with order equal Moore bound are called Moore graphs and are reached only in few cases.

- If $d = 2$ for any $k \geq 1$
- If $k = 1$ for any $d \geq 2$
- For $k = 2$ for $d \in \{3, 7\}$, and possibly 57

For other cases we try to construct graphs with order as close to Moore bound as possible.

Moore graphs

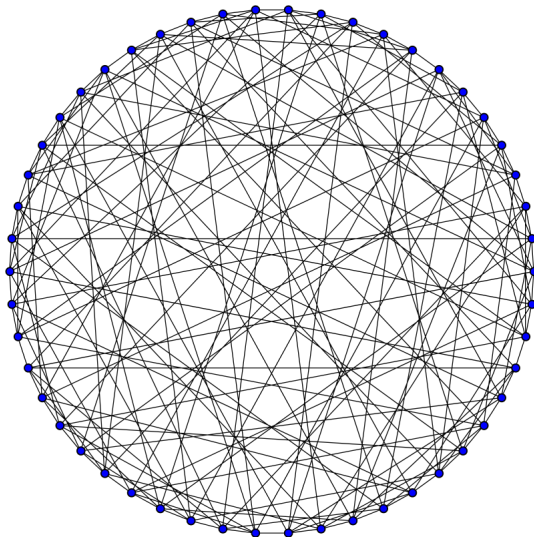


Figure: Hoffman-singleton graph is Moore graph with $d = 7$ and $k = 2$

Let G be an undirected graph. We will assign direction to every edge of graph and making them *arcs*. Arc with *reversed* direction of e is denoted by e^{-1} .

Definition (Graph lifting)

Let G be a graph as above and let Γ be a finite group. The mapping

$$\alpha : D(G) \rightarrow \Gamma$$

will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any arc $e \in D(G)$.

Graph lifting example

Obrazok zdvihu na petersenov graf.

Cayley graphs

Let Γ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for Γ ; that is, we require that $S = S^{-1}$ and $1 \notin S$.

Definition (Cayley graphs)

The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which vertices a, b are adjacent if $a^{-1}b \in S$.

General linear and Special linear groups

Definition (General linear group)

Let q be a power of a prime and let $GF(q)$ be the Galois field of order q . The *general linear group* $GL(m, q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication of matrices. Special linear group is subgroup of $GL(m, q)$ consisting of matrices with determinant equal to 1.

Theorem (Order of $GL(m, q)$)

$$|GL(m, q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{n-1})$$

Theorem (Order of $SL(m, q)$)

$$|SL(m, q)| = |GL(m, q)| / (q - 1)$$

Computer search of Cayley graphs

In order to find graphs with given d and k we generated Cayley graphs from random symmetric unit-free sets S of $SL(2, 5)$. With assumptions that:

- Generating sets yield all elements of $|SL(2, 5)| = 120$
- Cayley graphs are regular with $d = |S|$

we know by Moore bound that size of S must be at least 12 and all we have to check is diameter of graph.

Computer search of Cayley graphs

With one involution in $SL(2, 5)$ and with unit-free property of symmetry we have 59 pairs of elements with its inverses. To check all graphs for $d = 12$ we have to generate and check diameter on $\binom{59}{6} = 45057474$ graphs. In our computation we generated 150000 graphs for all listed degrees.

d	13	14	15	16	17	18	19	20	21
Found graphs	0	0	1	107	345	2451	4120	11669	14926

Algorithm for generation of cayley graph

Algoritmus na generovanie grafu

Algorithm for diameter

Algoritmus na kontrolu priemeru grafu

Output of program

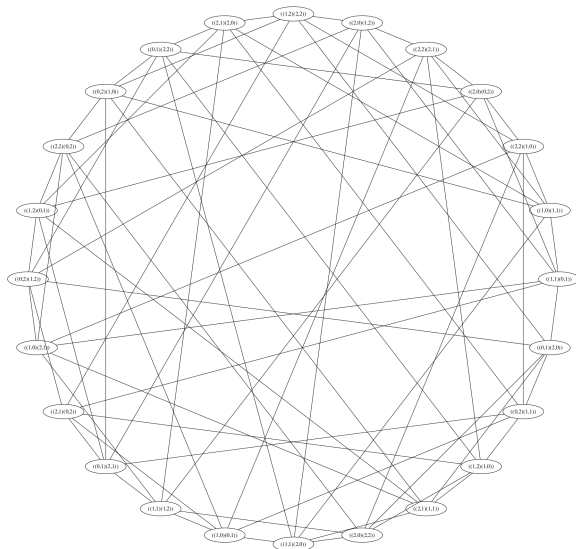


Figure: One of the generated graphs from $SL(2, 3)$

- Finding smallest graph with given girth and degree is known as Degree/girth problem. With small changes to program we can systematically or randomly look for Cayley graphs and check its girth.
- Computations to find smallest graphs with $k = 2$ in bigger Galois fields .

The End