

# Cayley graphs of given degree and diameter on linear groups

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# Overview

- 1 Degree/diameter
- 2 Graph lifting and Cayley graphs
- 3 Computer search of graphs

- In it's simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In design of graphs we can take many restrictions into account such as degree, girth, diameter.
- Two important problems concerning degree and diameter and degree and girth of graph

# The degree/diameter problem

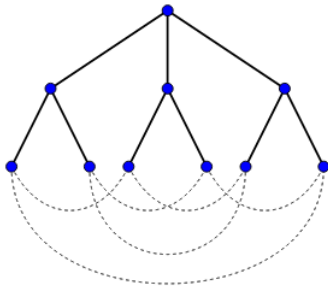
## Degree/diameter problem

Find graph with biggest possible number of vertices with given degree and diameter.

There is theoretical upper bound for largest order of graph with  $d$ -degree and  $k$ -diameter.

$$\begin{aligned} n_{d,k} \leq M_{d,k} &= 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\ &= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\ &= \begin{cases} 1 + d \frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k + 1, & \text{if } d = 2 \end{cases} \end{aligned} \tag{1}$$

# Moore bound



**Figure:** Peterssen graph is Moore graph with  $d = 3$  and  $k = 2$

Graphs with order equal Moore bound are called Moore graphs and are reached only in few cases.

- If  $d = 2$  for any  $k \geq 1$
- If  $k = 1$  for any  $d \geq 2$
- For  $k = 2$  for  $d \in \{3, 7\}$ , and possibly 57

For other cases we try to construct graphs with order as close to Moore bound as possible.

# Moore graphs

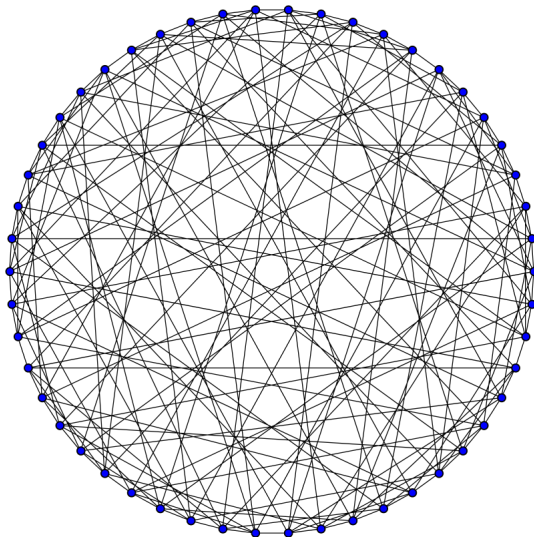


Figure: Hoffman-singleton graph is Moore graph with  $d = 7$  and  $k = 2$



Let  $G$  be an undirected graph. We will assign direction to every edge of graph and making them *arcs*. Arc with *reversed* direction of  $e$  is denoted by  $e^{-1}$ .

## Definition (Graph lifting)

Let  $G$  be a graph as above and let  $\Gamma$  be a finite group. The mapping

$$\alpha : D(G) \rightarrow \Gamma$$

will be called a *voltage assignment* if  $\alpha(e^{-1}) = (\alpha(e))^{-1}$ , for any arc  $e \in D(G)$ .

# Graph lifting example

- A *walk* of length  $\ell$  in  $G$  is any sequence  $W = e_1 e_2 \dots e_\ell$  of consecutive arcs of  $G$ , and the voltage  $\alpha(W)$  of the walk is simply the product  $\alpha(W) = \alpha(e_1)\alpha(e_2) \cdots \alpha(e_\ell)$ .
- Lift  $G^\alpha$  has diameter at most  $k$  if for any two vertices  $u, v$  of  $G$  and for any element  $g \in \Gamma$  there is a walk  $W$  of length at most  $k$  emanating from  $u$  and terminating at  $v$  such that  $\alpha(W) = g$ ; in the case when  $u = v$  we also require that  $g \neq 1$ .

Obrazok zdvihu na petersenov graf.

# Cayley graphs

Let  $\Gamma$  be a group and let  $S \subset \Gamma$  be a symmetric unit-free generating set for  $\Gamma$ ; that is, we require that  $S = S^{-1}$  and  $1 \notin S$ .

## Definition (Cayley graphs)

The *Cayley graph*  $C(\Gamma, S)$  is the graph with vertex set  $\Gamma$  in which vertices  $a, b$  are adjacent if  $a^{-1}b \in S$ .

# General linear and Special linear groups

## Definition (General linear group)

Let  $q$  be a power of a prime and let  $GF(q)$  be the Galois field of order  $q$ . The *general linear group*  $GL(m, q)$  consists of all non-singular  $m \times m$  matrices over  $GF(q)$  under multiplication of matrices. Special linear group is subgroup of  $GL(m, q)$  consisting of matrices with determinant equal to 1.

## Theorem (Order of $GL(m, q)$ )

$$|GL(m, q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})$$

## Theorem (Order of $SL(m, q)$ )

$$|SL(m, q)| = |GL(m, q)| / (q - 1)$$

# Computer search of Cayley graphs

In order to find graphs with given  $d$  and  $k$  we generated Cayley graphs from random symmetric unit-free sets  $S$  of  $SL(2, 5)$ . With assumptions that:

- Generating sets yield all elements of  $|SL(2, 5)| = 120$
- Cayley graphs are regular with  $d = |S|$

we know by Moore bound that size of  $S$  must be at least 12 and all we have to check is diameter of graph.

# Computer search of Cayley graphs

With one involution in  $SL(2, 5)$  and with unit-free property of symmetry we have 59 pairs of elements with its inverses. To check all graphs for  $d = 12$  we have to generate and check diameter on  $\binom{59}{6} = 45057474$  graphs. In our computation we generated 150000 graphs for all listed degrees.

$d$	13	14	15	16	17	18	19	20	21
Found graphs	0	0	1	107	345	2451	4120	11669	14926

# Algorithm for generation of cayley graph

```
while (@stack) {  
    foreach my $gen_element (@gen_set) {  
        my $mul_result = ($current_node x $gen_element) % $zp;  
        insert_result(\@mul_results, $current_node, $mul_result  
  
        if (find_hash(\%gen_nodes, $mul_result, $zp)) {  
            insert_hash(\%gen_nodes, $mul_result, $zp);  
            push @keys, compute_hash($mul_result, $zp);  
            push @stack, $mul_result;  
        }  
    }  
  
    shift @stack;  
  
    if (@stack) {  
        $current_node = $stack[0];  
    }  
}
```

# Algorithm for diameter

```
foreach my $i ( 2..$diameter ) {
    my $var = variations_with_repetition([0...$#gen_set], $i);
    while (my $var = $variations->next) {
        my $m = $gen_set[$variation->[0]];
        foreach my $j ( 1..($i-1) ) {
            $m = ($m x $gen_set[$var->[$j]]) % $zp;
        }
        push @hash_storage, compute_hash($m, $zp);
    }
}

foreach my $m ( @{ $gen_set_ref } ) {
    push @hash_storage, compute_hash($m, $zp);
}

foreach my $key ( @{ $keys_ref } ) {
    return 0 unless(List::Util::any {$_ eq $key} @hash_storage);
}

return 1;
```



# Output of program

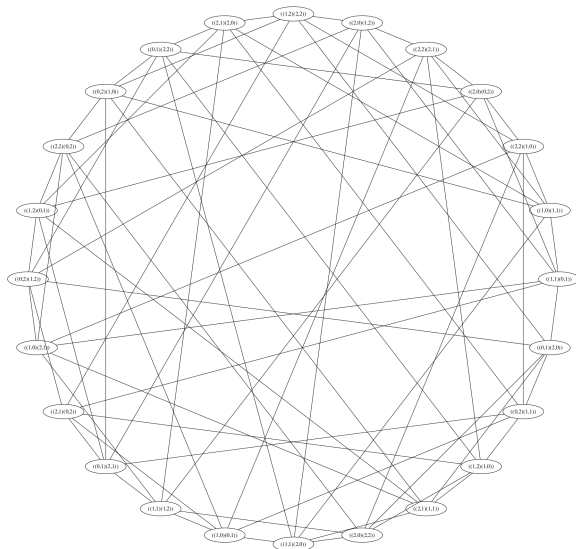


Figure: One of the generated graphs from  $SL(2, 3)$

- Finding smallest graph with given girth and degree is known as Degree/girth problem. With small changes to program we can systematically or randomly look for Cayley graphs and check its girth.
- Computations to find smallest graphs with  $k = 2$  in bigger Galois fields .

The End