Cayley graphs of given degree and diameter on linear groups

Matúš Behun

Slovak University of Technology in Bratislava

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Overview

Motivation

- In it's simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In design of graphs we can take many restrictions into acount such degree, grith, diameter.
- Two important problems concerning degree and diameter and degree and grith of graph

The degree/diameter problem

Degree/diameter problem

Find graph with biggest possible number of vertices with given degree and diameter.

Moore bound

There is theoretical upper bound for largest order of graph with d-degree and k-diameter.

$$
\begin{aligned}
n_{d,k} \leq M_{d,k} &= 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\
&= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\
&= \begin{cases} 1 + d\frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k+1, & \text{if } d = 2 \end{cases}
\end{aligned} \tag{1}
$$

Moore bound

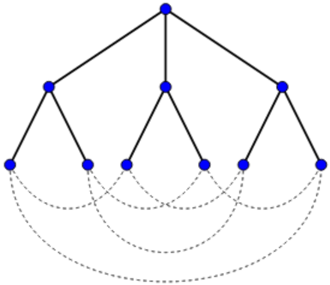

Figure: Peterssen graph is Moore graph with $d = 3$ and $k = 2$

Moore graphs

Graphs with order equal Moore bound are called Moore graphs and are reached only in few cases.

- If $d = 2$ for any $k \geq 1$
- If $k = 1$ for any $d \geq 2$
- For $k = 2$ for $d \in \{3, 7\}$, and possibly 57

For other cases we try to construct graphs with order as close to Moore bound as possible.

Moore graphs

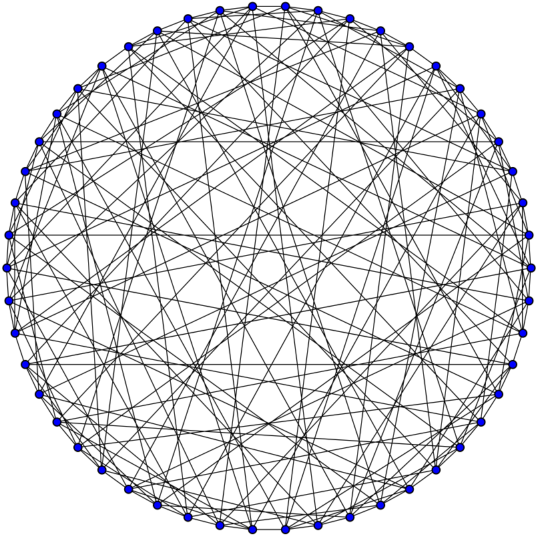

Figure: Hoffman-singleton graph is Moore graph with $d = 7$ and $k = 2$

Graph lifting

Let G be an undirected graph. We will assign direction to every edge of graph and making them *arcs*. *Arc* with *reversed* direction of e is denoted by e^{-1}.

Definition (Graph lifting)

Let G be a graph as above and let Γ be a finite group. The mapping

$$\alpha : D(G) \to \Gamma$$

will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any arc $e \in D(G)$.

Graph lifting example

Obrazok zdvihu na petersenov graf.

Cayley graphs

Let Γ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for Γ; that is, we require that $S = S^{-1}$ and $1 \notin S$.

Definition (Cayley graphs)

The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which vertices a, b are adjacent if $a^{-1}b \in S$.

General linear and Special linear groups

Definition (General linear group)

Let q be a power of a prime and let $GF(q)$ be the Galois field of order q. The *general linear group* $GL(m, q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication of matrices. Special linear group is subgroup of $GL(m, q)$ consiting of matrices with determinant equal to 1.

Theorem (Order of $GL(m, q)$)

$|GL(m, q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{n-1})$

Theorem (Order of $SL(m, q)$)

$|SL(m, q)| = |GL(m, q)|/(q - 1)$

Computer search of Cayley graphs

In order to find graphs with given d and k we generted Cayley graphs from random symmetric unit-free sets S of $SL(2,5)$. With assumptions that:

- Generating sets yield all elements of $|SL(2,5)| = 120$
- Cayley graphs are regular with with $d = |S|$

we know by Moore bound that size of S must be at least 12 and all we have to check is diameter of graph.

Computer search of Cayley graphs

With one involution in $SL(2,5)$ and with unit-free property of symmetry we have 59 pairs of elements with its inverses. To check all graphs for $d = 12$ we have to generate and check diameter on $\binom{59}{6} = 45057474$ graphs.
In our computation we generated 150000 graphs for all listed degrees.

d	13	14	15	16	17	18	19	20	21
Found graphs	0	0	1	107	345	2451	4120	11669	14926

Algorithm for generation of cayley graph

Algoritmus na generovanie grafu

Algorithm for diameter

Algoritmus na kontrolu priemeru grafu

Algorithm for diameter

Algoritmus na kontrolu priemeru grafu

Output of program

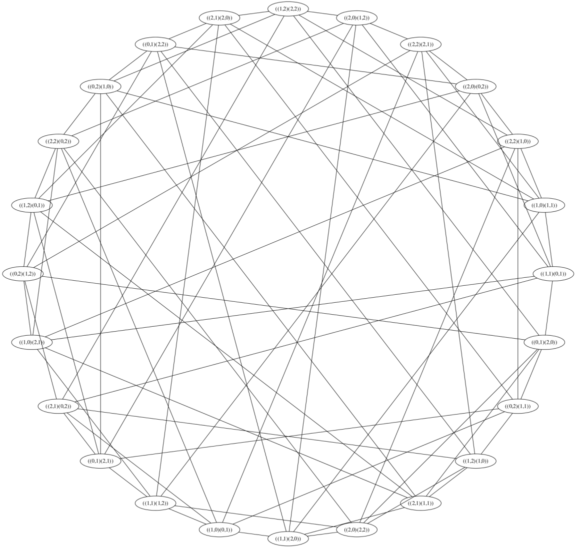

Figure: One of the generated graphs from $SL(2,3)$

The End