# Cayley graphs of given degree and diameter on linear groups

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# Motivation

- In it's simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In design of graphs we can take many restrictions into acount such degree, grith, diameter.
- Two important problems concerning degree and diameter and degree and grith of graph

# The degree/diameter problem

## Degree/diameter problem

Find graph with biggest possible number of vertices with given degree and diameter.

## Moore bound

There is theoretical upper bound for largest order of graph with $d$-degree and $k$-diameter.

$$\begin{aligned} n_{d,k} \leq M_{d,k} &= 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\ &= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\ &= \begin{cases} 1 + d\frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k + 1, & \text{if } d = 2 \end{cases} \end{aligned} \tag{1}$$
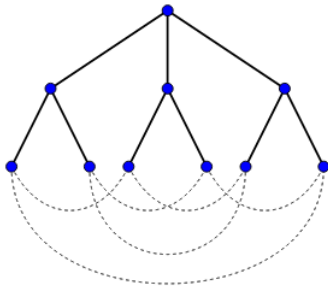
Figure: Peterssen graph is Moore graph with $d = 3$ and $k = 2$

# Moore graphs

Graphs with order equal Moore bound are called Moore graphs and are reached only in few cases.

- If $d = 2$ for any $k \geq 1$
- If $k = 1$ for any $d \geq 2$
- For $k = 2$ for $d \in \{3, 7\}$, and possibly 57

For other cases we try to construct graphs with order as close to Moore bound as possible.
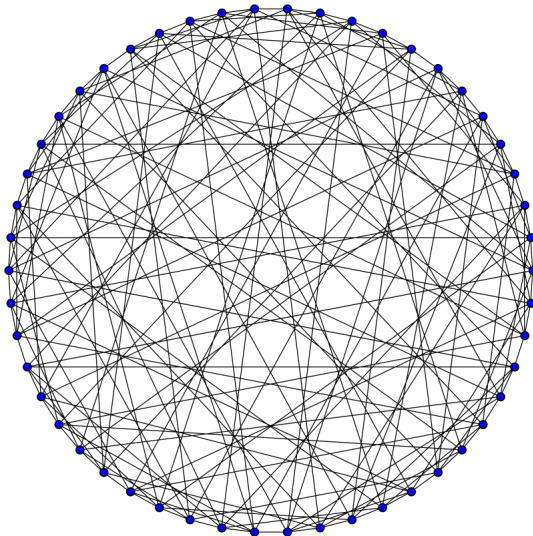
# Moore graphs



Figure: Hoffman-singleton graph is Moore graph with $d = 7$ and $k = 2$

# Graph lifting

Let $G$ be an undirected graph. We will assign direction to every edge of graph and making them *arcs*. Arc with *reversed* direction of $e$ is denoted by $e^{-1}$.

## Definition (Graph lifting)

Let $G$ be a graph as above and let $\Gamma$ be a finite group. The mapping

$$\alpha : D(G) \to \Gamma$$

will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any arc $e \in D(G)$.

# Graph lifting example

Obrazok zdvihu na petersenov graf.

# Cayley graphs

Let $\Gamma$ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for $\Gamma$; that is, we require that $S = S^{-1}$ and $1 \notin S$.

## Definition (Cayley graphs)

The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set $\Gamma$ in which vertices $a, b$ are adjacent if $a^{-1}b \in S$.

# General linear and Special linear groups

## Definition (General linear group)

Let $q$ be a power of a prime and let $GF(q)$ be the Galois field of order $q$. The *general linear group* $GL(m, q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication of matrices. Special linear group is subgroup of $GL(m, q)$ consiting of matrices with determinant equal to 1.

## Theorem (Order of $GL(m, q)$)

$|GL(m, q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{n-1})$

## Theorem (Order of $SL(m, q)$)

$|SL(m, q)| = |GL(m, q)|/(q - 1)$

# Computer search of Cayley graphs

In order to find graphs with given $d$ and $k$ we genertet Cayley graphs from random symmetric unit-free sets $S$ of $SL(2,5)$. With assumptions that:

- Generating sets yield all elements of $|SL(2,5)| = 120$
- Cayley graphs are regular with with $d = |S|$

we know by Moore bound that size of $S$ must be at least 12 and all we have to check is diameter of graph.

# The End