

Slovak University of Technology in Bratislava
Faculty of Civil Engineering

Cayley graphs of given degree and diameter on linear groups

Bachelor Thesis

Študijný program: Matematicko-počítačové modelovanie
Študijný odbor: Aplikovaná matematika
Školiace pracovisko: Katedra matematiky a deskriptívnej geometrie
Vedúci záverečnej Jana Šiagiová
práce:

Bratislava 2018

Matúš Behun

1 Introduction

In its simplest form, networks can be modeled by graphs in a natural way in which network nodes are represented by vertices of the graph and links between nodes are represented by undirected edges of the graph. Restrictions on the network, such as limits on the number of links attached to a node, or limits on the number of links needed to connect any two nodes, or the length of a shortest circuit, then transform to restrictions on the graph model (for the indicated cases these would be restrictions on vertex degrees, on the diameter, and on the girth of the graph).

In graph theory this leads to two important problems: the *degree/diameter problem* to construct the largest possible graphs of a given maximum degree and a given diameter, and the *degree/girth problem* to construct the smallest possible regular graphs of a given degree and a given girth; in both cases the adjectives ‘large’ and ‘small’ refer to the order (i.e., the number of vertices) of a graph. The survey papers [8] and [4], respectively, contain a large amount of information on both problems. In this work we address both problems restricted to Cayley graphs of certain two-dimensional linear groups, giving details in what follows.

2 The degree/diameter problem

2.1 The Moore bound

There is theoretical upper bound named after E. F. Moore for the largest order of a graph with given diameter $k \geq 1$ and maximum degree $d \geq 2$, which one can derive by building a spanning tree of such a graph. A fixed vertex v of such a graph has at most d neighbours at distance 1. Each of these neighbours has at most $d - 1$ vertices at distance 1 from themselves, giving at most $d(d - 1)$ vertices at distance 2 from v . Iterating this process one obtains at most $d(d - 1)^{i-1}$ vertices at distance i from v . By the diameter requirement, we have at most $d(d - 1)^{k-1}$ vertices at distance k from v . Summing up, the largest order $n_{d,k}$ of a graph of maximum degree $d \geq 2$ and diameter $k \geq 1$ is at most $M_{d,k}$, the Moore bound for the pair (d, k) , where

$$\begin{aligned} n_{d,k} &\leq M_{d,k} = 1 + d + d(d - 1) + \cdots + d(d - 1)^{k-1} \\ &= 1 + d(1 + (d - 1) + \cdots + (d - 1)^{k-1}) \\ &= \begin{cases} 1 + d \frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k + 1, & \text{if } d = 2 \end{cases} \end{aligned} \tag{1}$$

2.2 Moore graphs

It is well known that equality in (1) holds only if $d = 2$ (for any $k \geq 1$), or $k = 1$ (for any $d \geq 2$), or for $k = 2$ and $d \in \{3, 7\}$, and possibly for the pair $(d, k) = (57, 2)$ but for no other $d \geq 2$ and $k \geq 1$. All the graphs for which $n_{d,k} = M_{d,k}$ are called *Moore graphs*; they are necessarily regular and in the order of the listed parameters they are cycles of length $2k + 1$, complete graphs of order $d + 1$, the Petersen graphs and the Hoffman-Singleton graph.

Research into the degree/diameter problem and into Moore graphs in particular was initiated by Hoffman and Singleton in [6] who proved that Moore graphs of diameter $k = 2$ can exist only for $d \in \{2, 3, 7\}$ and possibly 57, proving also their uniqueness except for the last case, and establishing also that the unique Moore graph of diameter 3 is the cycle of length 7. Their proofs exploit eigenvalues and eigenvectors of adjacency matrices (and the corresponding principal submatrices) of graphs.

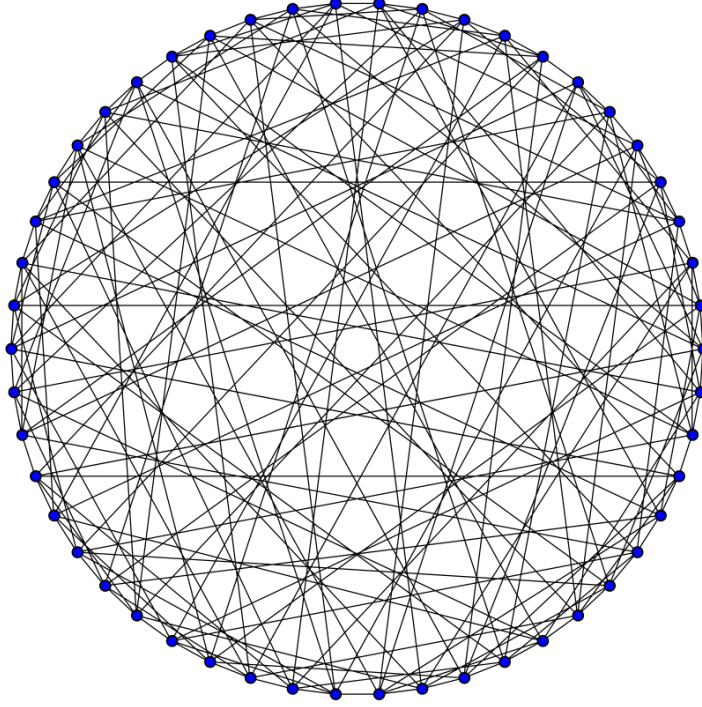


Figure 1: Hoffman-singleton graph is Moore graph with $d = 7$ and $k = 2$

2.3 Graphs of order close to the Moore bound

To facilitate the explanations, any graph of maximum degree d and diameter k will be called a (d, k) -graph. Because for $d \geq 3$ and $k \geq 2$ there are only a few (d, k) -graphs of order equal to the value $M_{d,k}$ of the Moore bound, researchers have tried to construct (d, k) -graphs of order as close to $M_{d,k}$ as possible. For a (d, k) -graph of order $M_{d,k} - \delta$ the quantity δ is known as the *defect*, and then we speak about a $(d, k, -\delta)$ -graph; we note that this terminology is only used for ‘small’ defects.

There are a number of results concerning graphs with small defect. For example, Erdős, Fajtlowitz and Hoffman proved [3] that there is no graph of degree d and diameter 2 with $\delta = 1$ apart from the cycle of length 4. In the case of $\delta = 2$ and $d = 2$ all the $(d, k, -2)$ -graphs are the cycles of length $2k - 1$. For $\delta = 2$ and $d \geq 3$ only five graphs are known at present, namely, two $(3, 2, -2)$ -graphs of order 8, one $(4, 2, -2)$ -graph of order 15, one $(5, 2, -2)$ -graph of order 24, and one $(3, 3, -2)$ -graph of order 20; for details we refer to [8].

2.4 Constructions of large graphs

A different approach to finding (d, k) -graphs of order close to the Moore bound is by constructing appropriate ‘large’ (d, k) -graphs. In most cases the constructions use combinatorics on words or algebraic structures such as groups and fields, so that the resulting graphs turn out to be rich in symmetries, or even vertex-transitive (and sometimes even Cayley).

For a long time one of the asymptotically best families was the one of the *undirected de Bruijn graphs*, which are (d, k) graphs for even d and yield the lower bound

$$n_{d,k} \geq \left(\frac{d}{2}\right)^k.$$

This bound was improved by Baskoro and Miller [1] to

$$n_{d,k} \geq \left(\frac{d}{2}\right)^k + \left(\frac{d}{2}\right)^{k-1} \quad (2)$$

In the special case of diameter $k = 2$, modified Brown graphs can give for sufficiently large d the bound

$$n_{d,2} \geq d^2 - 2d^{1+\varepsilon}$$

where the number $\varepsilon < 1$ depends on results about gaps between consecutive prime numbers; see [2] for details about the current development.

2.5 Graph lifting

Graph lifting is a technique by which one may produce ‘large’ graphs with certain required properties from suitable ‘small’ graphs. The technique is well-known in topological and algebraic graph theory and for historical and mathematical details we refer to the monograph [5]. The technique can be described in the language of the so-called voltage assignments, which we briefly present next.

Let G be a graph. Although our graphs are undirected, we will preassign a direction to every edge. An edge with a preassigned direction is an *arc*. If e is an arc of G , by the symbol e^{-1} we denote the *reverse* of e , obtained simply by changing the preassigned direction on the edge. Let $D(G)$ be the set of all arcs of G ; it follows that the size of $D(G)$ is twice the number of edges of G .

Let G be a graph as above and let Γ be a finite group. The mapping

$$\alpha : D(G) \rightarrow \Gamma$$

will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any arc $e \in D(G)$.

Out of the graph G and the voltage assignment α as introduced above one can construct a ‘larger’ graph, called the *lift* of G (under the assignment α) and denoted G^α . The vertex set and the dart set of the new graph are defined as follows:

$$V(G^\alpha) = V(G) \times \Gamma$$

$$E(G^\alpha) = E(G) \times \Gamma$$

with the condition that if an arc e emanates from a vertex u and terminates in a vertex v in the *base graph* G , then for every $g \in \Gamma$ there is a dart (e, g) emanating from the vertex (u, g) and terminating in the vertex $(v, g\alpha(e))$. It follows that right multiplication by $\alpha(e)$ permutes the terminal vertices of arcs (e, g) emanating from the vertices (u, g) . At the same time this right multiplication induces a group of automorphisms of the lift acting freely on vertices and isomorphic to the *voltage group* Γ . Note that G^α can be considered to be an undirected graph, because the arcs (e, g) and $(e^{-1}, g\alpha(e))$ are reverse of each other.

This procedure can be reversed in the following sense, cf. [5]. Let \tilde{G} be a graph and let Γ be a group of automorphisms of \tilde{G} that acts freely on vertices of \tilde{G} . Then \tilde{G} is a lift of a ‘smaller’ graph, denoted \tilde{G}/Γ and called a *quotient*, which is obtained from \tilde{G} by letting Γ -orbits of the vertex set and the arc set of \tilde{G} to be vertices and arcs of the quotient, with incidence inherited from \tilde{G} .

Of course, degrees of all the vertices (u, g) in the lift are equal to the degree of u in the base graph. An advantage of lifting is that the diameter of the lift can be conveniently controlled by properties of the base graph G and the voltage assignment α as well. A *walk* of length ℓ in G is any sequence $W = e_1 e_2 \dots e_\ell$ of consecutive arcs of G , and the voltage $\alpha(W)$ of the walk is simply the product $\alpha(W) = \alpha(e_1) \alpha(e_2) \dots \alpha(e_\ell)$. Then (cf. e.g. the survey [8]), the lift G^α has diameter at most k if for any two vertices u, v of G and for any element $g \in \Gamma$ there is a walk W of length at most k emanating from u and terminating at v such that $\alpha(W) = g$; in the case when $u = v$ we also require that $g \neq 1$.

A number of the largest currently known d, k -graphs can be described as lifts. For example, all the graphs giving the values of $n_{3,7}$, $n_{3,8}$, $n_{4,4}$, $n_{5,3}$, $n_{5,5}$, $n_{6,3}$, $n_{6,4}$, $n_{7,3}$, $n_{14,3}$ and $n_{16,2}$ obtained by computer search turn out to be lifts [8].

2.6 Cayley graphs

Let Γ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for Γ ; that is, we require that $S = S^{-1}$ and $1 \notin S$. The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which vertices a, b are adjacent if $a^{-1}b \in S$. Observe that now the group Γ acts regularly and hence freely on the vertex set of $C(\Gamma, S)$ as a group of automorphisms, so that the quotient graph $C(\Gamma, S)/\Gamma$ has just one vertex. By the remark in the previous section about groups of automorphisms acting freely in vertices it follows that Cayley graphs are simply lifts of one-vertex graphs (which will in general contain multiple loops and semi-edges but we will not go into the corresponding details).

The diameter testing of lifts now easily translates into testing diameter of Cayley graphs as follows: A Cayley graph $C(\Gamma, S)$ has diameter at most k if and only if every element of Γ can be expressed as a product of at most k elements of the generating set S . Equivalently, the diameter of a Cayley graph $C(\Gamma, S)$ is *equal* to k if and only if k is the smallest number such that every element of Γ can be expressed as a product of at most k elements of S .

By $Cay_{d,k}$ we denote the largest order of a Cayley (d, k) -graph. In [7] it was proved that for every fixed $d \geq 3$ and $c \geq 2$ there is a set A of natural numbers with positive density s.t. $C_{d,k} \leq M_{d,k} - c$ for all $k \in A$.

The currently best available lower bound on $C_{d,2}$ was obtained by Šiagiová and Širáň [9] and reads as follows. Let $D = \{2^{2m+\mu} + (2 + \delta)2^{m+1} - 6, m \geq 1, \mu \in \{0, 1\}\}$. Then, for every $d \in D$ one has $C_{d,2} > d^2 - 6\sqrt{2}d^{3/2}$.

In the special case of Cayley graphs of diameter $k = 2$ on Abelian groups there is a relatively straightforward lower bound of the form

$$n_{d,2} \geq \lfloor \frac{d+2}{2} \rfloor \lceil \frac{d+2}{2} \rceil$$

obtained by considering the product of cyclic groups $Z_{\lfloor (d+2)/2 \rfloor} \times Z_{\lceil (d+2)/2 \rceil}$, with generating set consisting of all pairs (x_1, x_2) with one of the entries equal to zero.

2.7 General linear and special linear groups

Linear groups are usually described in terms of linear transformations of vector spaces over general fields. For our purposes it will be sufficient to work with a more concrete description and restricted to finite fields. Let q be a power of a prime and let $GF(q)$ be the Galois field of order q . The *general linear group* $GL(m, q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication of matrices in the usual sense. By an elementary fact in group theory the order of $GL(m, q)$ is equal to $(q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})$. The *special linear group* $SL(m, q)$ is the subgroup of $GL(m, q)$ consisting of matrices with determinant equal to 1; its order is $|SL(m, q)| = |GL(m, q)| / (q - 1)$.

We will be particularly interested in the case $m = 2$ and $q = p$ for some prime numbers p , and in Cayley graphs of diameter 2 arising from the groups $SL(2, p)$. To the best of our knowledge such a study has not appeared in the available literature.

3 Results

3.1 Example of Cayley graph

For example we generate Cayley graph on group $SL(2, 3)$ with order $|SL(2, 3)| = 24$ consisting of

$$SL(2, 3) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \right\}.$$

For generating set S with three pairs of elements and its inverses

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \right\}$$

Cayley graph $C(SL(2, 3), S)$ has order 24, $d = 6$ and $k = 2$.

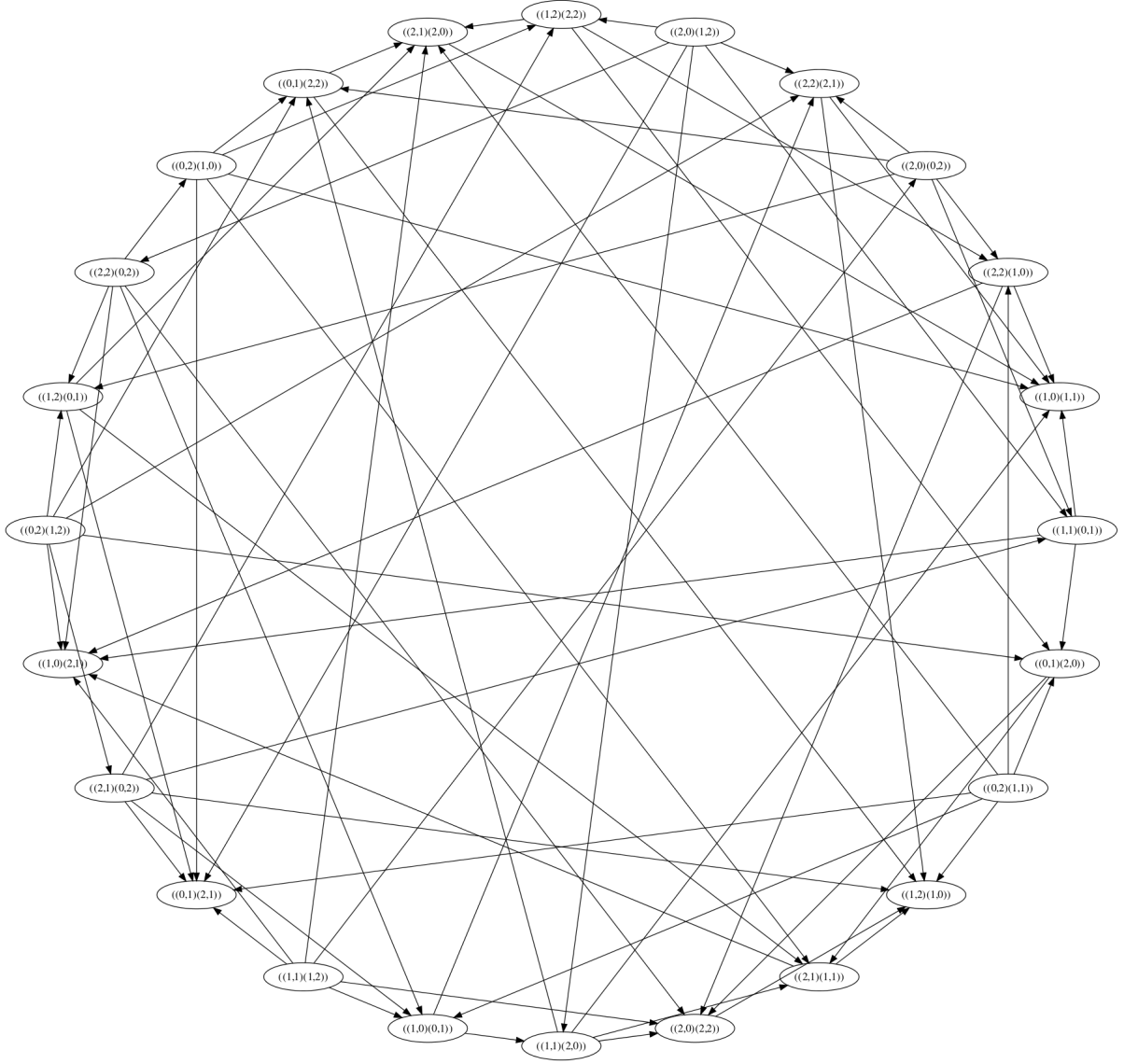


Figure 2: Output of circos [11] of $C(SL(2, 3), S)$

The generating set S of the group $G = SL(2, 3)$ in this example has the form $\{x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}\}$. To verify that the Cayley graph $C(G, S)$ has diameter 2 one would have to show that the set S together with the identity and with all products $x_i^{\pm 1} y_j^{\pm 1}$ gives the entire group G .

3.2 Generation of cayley graphs with given diameter

All of our Cayley graphs which are generated through its symmetric sets were choosed randomly. As mentioned earlier choosing random symmetric sets consits of choosing elements of group with its inverses over finite field arithmetic. In our implementation all elements of group were precomputed. Every element A had its inverse computed

by formula:

$$\frac{1}{\det(A)} \text{adjugate}(A) = A^{-1} \quad (3)$$

We also keep all involutions apart in case of choosing last element for sets with odd size.

We present algorithm for generation of Cayley graph in pseudocode.

```

1 function cayley_graph_generation(generating_set [])
2 {
2     push all from generating_set [] to stack []
3     current_node = first element of stack []
4
5     while stack [] not empty
6         foreach generating_element in generating_set []
7             result = current_node x generating_element % Zp
8             cayley_graph[ current_node ] = result
9
10            if generating_nodes[ result ] is empty
11                push stack [], result
12                generating_nodes[ result ] = result
13
14            shift to the left stack []
15
16            current_node = first element of stack []
17
18    return cayley_graph []
19}

```

On lines 8, 10, 12 we denoted by $[r]$ putting resulting matrix into so called associative array also known as dictionary or hash array. It's worth mentioning that there exists perfect hash function for matrices over finite fields for such purposes. Algorithm uses *stack*[] for storing matrices which weren't multiplied over generating set while *generating_nodes*[] to keep them unique to keep memory overhead low at cost of computation of hash.

Precomputations could be made for finding inverse and creating Cayley table for matrix multiplications, making generation of graph and diameter check matter of accessing memory. While this implementation should give faster performance results, it

was considered at later stages and was not implemented. Precomputations of Cayley table are limited by size of RAM memory and possible only in smaller groups.

To check diameter of graph one has to make only computations on generating set. To check wheter graph has diameter k it's required least if graph has not of diameter $k - 1$. To find unknown k one has to iterate through all $k \in 1 \dots$ diameters. We didn't use algorithm for special case of $k = 1$ which was checked only by checking degree of all nodes in graph.

```

1 function check_diameter(gen_set[], cayley_graph[], diameter)
2 {
3   push reached_nodes[], gen_set[]
4
5   foreach n in [ 2, ... , diameter ]
6     foreach var = variation with repetitions of length len
7       node = gen_set[ var[0] ] x ... x gen_set[ var[len] ]
8       push reached_nodes[], node
9
10    if reached_nodes[] contains all elements of group
11      retrun cayley_graph has given diameter
12
13    retrun cayley_graph has not given diameter
14 }
```

3.3 Computer search of graphs with diameter

Here we present results of computer seach of graphs. We have choosed two approaches in order to find graphs with diameter two. In early stages of program we have tried to find graphs with generating fixed size of graphs on degree d . Later on we used adaptive algorithm to search any graph of diameter two with ability to improve its results if it's feasible.

Search with fixed size of generated graphs was used on two smaller groups $SL(2, 5)$ and $SL(2, 7)$. It's assumed that Cayley graph will yield all elements of groups. Here we present table of results for $SL(2, 5)$ with Moore bound for parameters $k = 2$ and $|SL(2, 5)| = 120$ is $d = 11$, therefore search was done on degree 12. 50000 random graphs were genreated on each degree with following results:

d	12	13	14	15	16	17	18	19	20	21
Found graphs	0	0	0	1	107	345	2451	4120	11669	14926

Same approach was used on graphs in $SL(2, 7)$ with order of 336 with Moore bound on $d = 19$. We have searched 150000 graphs on each degree from 20 to 30 with only 10 graphs found on $d = 30$.

By adaptive algorithm we mean automatized search of graphs in group. Program needs to be set up with starting size of generating set, time limit spent per degree, maximum of generated graphs per degree and number of parallel processes. In order to find some graph of given diameter program's tries to find any graph on degree. In case time limit is up it's continuing on set with bigger size making found graph further from Moore bound with bigger probability to find some. After first successfully found graph search continues on smaller generating size with bigger time limit.

Tu pojdu vysledky z hladania lepsich grafov zo Z_p 11, 13, 19

For comparsion we present latest known results for Cayley graphs with $k = 2$

d	Order	% Moore Bound	d	Order	% Moore Bound
4	13	76.47%	31	648	67.35%
5	18	69.23%	32	648	63.21%
6	32	86.48%	33	648	59.44%
7	36	72%	34	648	56%
8	48	73.48%	35	648	52.85%
9	60	73.17%	36	648	49.96%
10	72	71.28%	37	800	58.39%
11	84	68.85%	38	800	55.36%
12	96	66.20%	39	800	52.56%
13	112	65.88%	40	968	60.46%
14	128	64.97%	41	968	57.55%
15	144	63.71%	42	968	54.84%
16	200	77.82%	43	968	52.32%
17	200	68.96%	44	968	49.97%
18	200	61.53%	45	1058	52.22%
19	200	55.24%	46	1152	54.41%
20	210	55.36%	47	1152	52.12%
21	288	65.15%	48	1152	49.97%
22	288	59.38%	49	1352	56.28%
23	392	73.96%	50	1352	54.05%
24	392	67.93%	51	1352	51.96%
25	392	62.61%	52	1352	49.98%
26	392	57.90%	53	1458	51.88%
27	392	53.69%	54	1568	53.75%
28	512	65.52%	55	1568	51.81%
29	512	60.80%	56	1568	49.98%
30	512	56.82%	57	1682	51.75%

For implementation of our algorithms we choosed *Perl* programming language. Perl is high-level, general-purpose, interpreted programming language with great documentation and online community maintaining easy to use libraries. While Perl being high-level programming language multiplication of matrices in our program were actually implemented in C programming language implementation in choosed language more suitable. For this purpose we have used library called PDL or perl data language available at *CPAN* [12].

4 The degree/girth problem

Finding graphs with smallest possible order denoted $n_{d,g}$ with given degree d and girth $g \geq 3$ is known as *degree/girth problem*. Motivation for findind such graphs could arise from constructing graphs with no cycles with length g or less and its similarity to *degree/diameter problem*. Problem was studied first by Biggs [10] on regular graphs.

4.1 Moore bound

For odd g we have Moore bound denoted by $M_{d,g}$

$$n_{d,g} \geq M(d, g) = 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{\frac{g-3}{2}}$$

and for even g

$$n_{d,g} \geq M(d, g) = 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{\frac{g-2}{2}})$$

For odd g Moore bound constructed same as Moore bound for degree/diameter problem with $k = (g-1)/2$. For even g we start with edge and from incident vertices we continue until we reach depth $(g-2)/2$.

4.2 Moore graphs

Graphs with Moore bound equal to $M(d, g)$ are called Moore graphs or cages. We list all cases with $n_{d,g} = M_{d,g}$ below:

- For $d = 2, g \geq 2$ - circles
- For $g = 3, d \geq 22$ - complete graphs
- For $g = 5, d \geq 2$ - complete bipartite graphs
- For $g = 5, d = 2$ - circle with length 5

$d = 3$ - Peterssen graph

$d = 7$ - Hoffman-Singleton graph

$d = 57$ - possible, graph is unknown

- For $g \in 6, 8, 12$, if $d-1$ is prime factor

4.3 Example of Cayley graph

Here we present example of Cayley graph on group $SL(2, 3)$ with order $|SL(2, 3)| = 24$

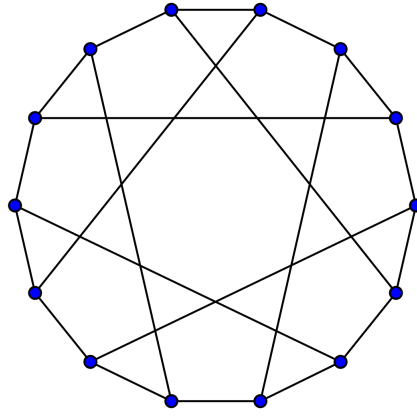


Figure 3: Heawood graph is Moore graph with $d = 3$ and $g = 6$

4.4 Generation of cayley graphs with given girth

Same as for diameter we check girth of graph making computations on generating set. Checking if girth g on graph we have to confirm that graphs doesn't have cycle with length $3 \leq n \leq g - 1$. Here we present algorithm used for computation of girth:

```

1  function find_cycle(generating_set[], girth)
2  {
3    foreach var = variation of length girth
4      node = generating_set[ var[0] ] x ... x generating_set[ var[girth] ]
5      if node equal to eye matrix
6        return found cycle
7
8    return cycle not found
9  }
10
11 function check_given_girth(generating_set[], girth)
12 {
13   if(find_cycle(generating_set[], girth) {
14     foreach i in ( 3..girth - 1 ) {
15       if(find_cycle(generating_set[], i)) {
16         return has not given girth
17       }
18     }
19     return graph has given girth
20   } else {
21     return has not given girth
22   }

```

Tu pojdu vysledky

5 Conclusion

In our work we discussed problems of construction of graphs with restriction on *degree/diameter* and *degree/girth*. We discussed theoretical bounds on both problems called after Moore and special cases of graphs reaching Moore bound. We introduced Cayley graphs as special case with symmetric properties. Symmetric properties allowed making them ideal for computation of our restrictions of random graphs. Considering Cayley graphs generated from special linear group over Galois field we have tried to find best possible results with computer search with some graphs found in smaller groups. We described implementation and our approach in search for such graphs with our results.

6 Abstract

This bachelor thesis discuss two problems concerning construction of graphs with given restriction, namely *degree/diameter* and *degree/girth* problem. In our work we describe both problems, its theoretically possible bounds and some research results with focus on special case of Cayley graphs. We present algorithm for generating Cayley graph and exploitation of symmetry of Cayley graphs with computation of its girth and diameter just on its symmetric generating set. We also tried to find graphs as close to theoretical bound of both problems as possible and we list some of results in our randomized search in Special linear group consisting of linear transformations over Galois field with smaller prime orders.

Bibliography

- [1] E. T. Baskoro and M. Miller, On the construction of networks with minimum diameter, Australian Computer Science Communications C 15 (1993) 739–743.
- [2] D. Bevan, G. Erskine and R. Lewis, Large circulant graphs of fixed diameter and arbitrary degree, Ars Math. contemp. 13 (2017), 275–29.
- [3] P. Erdős, S. Fajtlowicz and A.J. Hoffman, Maximum degree in graphs of diameter 2, Networks 10 (1980), 87–90.
- [4] G. Exoo and R. Jajcay, Dynamic cage survey, Electr. J. Combin. 15 (2008), Dynamic Survey DS16.
- [5] J. L. Gross and T. W. Tucker, Topological Graph Theory. Wiley, 1987 and Dover, 2001.
- [6] A. J. Hoffman and R. R. Singleton, On Moore graphs with diameter 2 and 3, IBM J. Res. Develop. 4 (1960), 497–504.
- [7] R. Jajcay, M. Mačaj and J. Širáň,
- [8] M. Miller and J. Širáň, Moore graphs and beyond: A survey, 2nd Ed., Electr. J. Combin. 2013, Dynamic Survey DS15.
- [9] J. Šiagiová and J. Širáň, Approaching the Moore bound for diameter two by Cayley graphs, J. Combin. Theory Ser. B 102 (2012) 470–473.
- [10] N.I. Biggs, Girth, valency and excess, Linear Algebra Appl. 31 (1980) 55–59
- [11] Circo, <http://circos.ca>
- [12] PDL, <https://metacpan.org/pod/PDL>