Cayley graphs of given degree and diameter on linear groups

Matúš Behun

Slovak University of Technology in Bratislava

June 10, 2019

Overview

1. Degree/diameter problem
2. Cayley graphs
3. Computer search

Motivation

- In its simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In modeling interconnection networks by graphs one often considers two restrictions:
  - limitation on the number of physical links leaving a node
  - limitation on accessibility, that is, any two nodes should be accessible using at most a certain number of physical links.

  In terms of a graph model this translates into restriction on the maximum degree and on the diameter.

The degree/diameter problem

Degree/diameter problem

Find graphs with the largest possible number of vertices, with given degree and diameter.

Edward Forrest Moore

Edward Forrest Moore was first who proposed problem of describing and classifying such graphs.

Moore bound

There is a theoretical upper bound on the largest order of a graph of maximum degree $d \geq 2$ and diameter $k \geq 1$.

$$
\begin{aligned}
n_{d,k} \leq M_{d,k} &= 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\
&= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\
&= \begin{cases} 1 + d\frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k+1, & \text{if } d = 2 \end{cases}
\end{aligned}
\tag{1}
$$

Graphs of maximum degree d and diameter k with $M_{d,k}$ vertices are called *Moore graphs*.

Moore graphs are necessarily regular.

An example of Moore graph

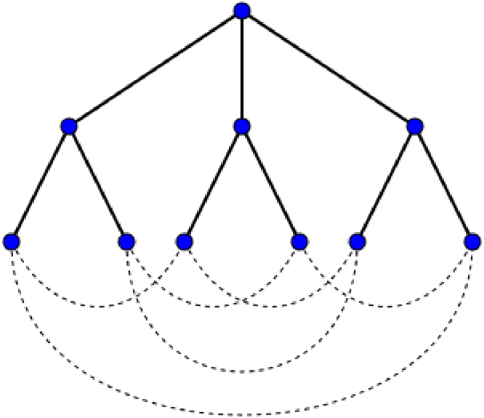

Figure: The Petersen graph is a Moore graph with $d = 3$ and $k = 2$

Moore graphs

Moore graphs are rare; they exist only for the following degrees and diameters:

- If $d = 2$ for any $k \geq 1$
- If $k = 1$ for any $d \geq 2$
- If $k = 2$ and $d \in \{3, 7\}$, and possibly 57

In the remaining cases one tries to construct graphs of maximum degree d and diameter k of order as close to the Moore bound $M_{d,k}$ as possible.

Moore graphs

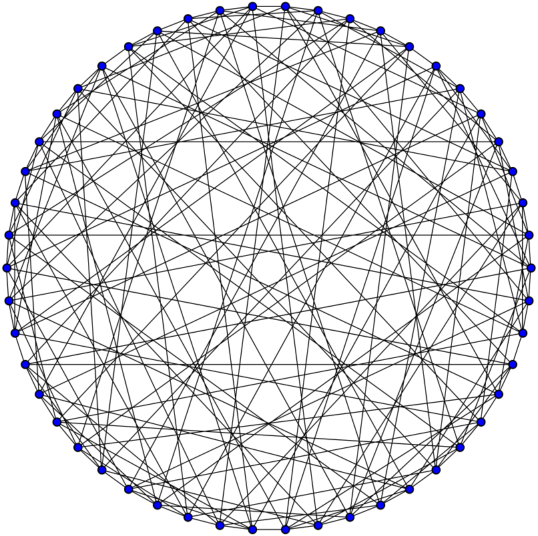

Figure: The Hoffman-Singleton graph is Moore graph with $(d, k) = (7, 2)$

Construction of large (d, k)-graphs

Finding (d, k)-graph of large order is approached by many techniques, mostly using combinatorics on words or various algebraic structures.

We present two examples of bounds arising from such constructions:
Using combinatorics on words, Baskoro and Miller(1993) [1] proved that

$$n_{d,k} \geq \left(\frac{d}{2}\right)^k + \left(\frac{d}{2}\right)^{k-1}$$

With the help of finite fields Bevan, Erskine and Lewis(2017) [2] observed that modified Brown graphs give the bound

$$n_{d,2} \geq d^2 - 2d^{1+\varepsilon}$$

where ε depends on results about graphs between consecutive primes.

Cayley graphs

Let Γ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for Γ; that is, we require that $S = S^{-1}$ and $1 \notin S$.

Definition (Cayley graphs)

The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which vertices a, b are adjacent if $a^{-1}b \in S$.

By $C_{d,k}$ we denote the largest order of Cayley graph of degree d and diameter k. The best currently known lower bound on diameter 2 for an infinite family of degrees is due to Šiagiová and Širáň(2012) [3]:

Theorem

Let $D = \{2^{2m+\mu} + (2+\delta)2^{m+1} - 6, m \geq 1, \mu \in \{0,1\}\}$. Then, for every $d \in D$ one has $C_{d,2} > d^2 - 6\sqrt{2}d^{3/2}$.

General and special linear groups

Definition (General and special linear group)

Let q be a power of a prime and let $GF(q)$ be the Galois field of order q. The *general linear group* $GL(m, q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication.

The special linear group is the subgroup of $GL(m, q)$ consiting of matrices with determinant 1.

Theorem (Order of $GL(m, q)$)

$$|GL(m, q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})$$

Theorem (Order of $SL(m, q)$)

$$|SL(m, q)| = |GL(m, q)|/(q - 1)$$

Computer search

We created a program generating Cayley graphs of given degree and diameter based on $SL(2, q)$.

As an example, consider the problem of generating a Cayley graph for the group $SL(2, 5)$ of diameter 2 and minimum possible degree:

In order to find graphs with given d and k we generated Cayley graphs from random symmetric unit-free sets S of $SL(2, 5)$. The order of $SL(2, 5)$ is 120, so that by the Moore bound the first feasible degree (equal to sthe size of a generating set) would be 12.

Computer search

The group $SL(2,5)$ has only one involution; all the remaining 118 elements form 59 pairs of the form $\{x, x^{-1}\}$.

To check all Cayley graphs $C(G,S)$ for $G = SL(2,5)$ and $|S| = 12$ we would have to generate all the $\binom{59}{6} = 45057474$ possibilities for S and then check for the diameter of the resulting Cayley graphs.

For the group $G = SL(2,5)$ and generating sets S such that $|S| = d$ by our randomized algorithm we found the following number of generating sets giving Cayley graphs $C(G,S)$ of diameter 2:

d	13	14	15	16	17	18	19	20	21
Found graphs	0	0	1	107	345	2451	4120	11669	14926

Algorithm for generation of cayley graph

```
push all from generating_set[] to stack[]
current_node = first element of stack[]

while stack[] not empty
    foreach generating_element in generating_set[]
        r = current_node x generating_element % zp
        cayley_graph[ current_node ] = r

        if generating_nodes[r] is empty
            push stack[], r
            generating_nodes[r] = r

    shift to the left stack[]

    if stack[] not empty
        current_node = first element of stack[]
```

Algorithm for diameter

```
push reached_nodes[], generating_set[]

foreach len in [ 2, ... , diameter ]
    foreach variation with repetitions of length len
        node = gen_set[ var[0] ] x ... x gen_set[ var[len] ]
        push reached_nodes[], node

foreach node in cayley_graph[]
    if any in reached_nodes[] not equal to node
        return cayley_graph has not given diameter

return cayley_graph has given diameter
```

An example of the output for $GL(2,3)$

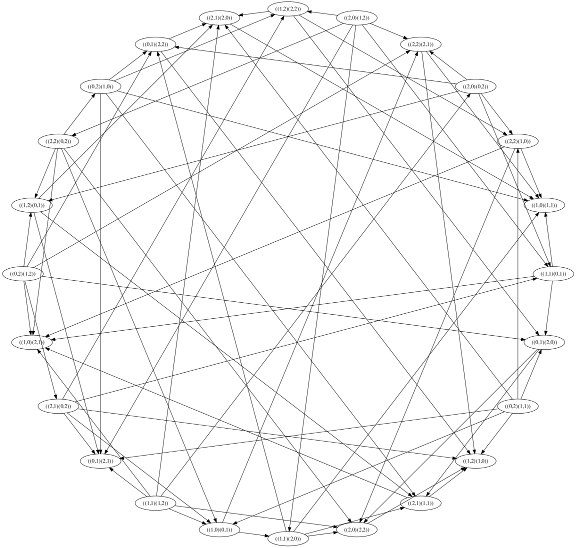

Figure: One of the generated graphs from $GL(2,3)$

The degree/girth problem

Definition (Degree/girth problem)

Finding regular graphs with smallest possible order denoted $M(d, g)$ with given degree d and girth g ≥ 3 is known as degree/girth problem.

Moore bound

For odd g

$$n_{d,g} \geq M(d,g) = 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{\frac{g-3}{2}}$$

and for even g

$$n_{d,g} \geq M(d,g) = 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{\frac{g-2}{2}})$$

The Heawood graph

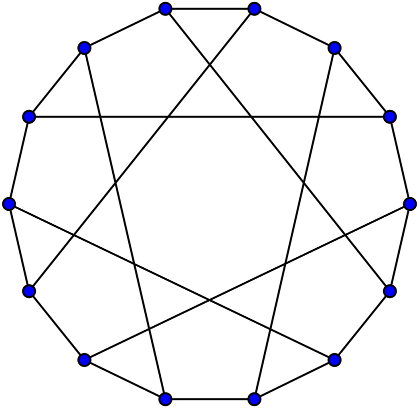

Figure: The Heawood graph is a Moore graph with $d = 3$ and $g = 6$

Algorithm for checking girth

```
function find_cycle(generating_set[], girth)
    foreach variation with len girth
        node = gen_set[ var[0] x ... x gen_set[ var[len] ]
        if node equal to eye matrix
            return found cycle

if find_cycle(generating_set, grith)
    foreach i in [ 3, ... , girth - 1 ]
        if(find_cycle(generating_set[], i))
            return girth i

    return graph has given girth
```

References

E. T. Baskoro and M. Miller, On the construction of networks with minimum diameter, Australian Computer Science Communications C 15 (1993) 739?743.

D. Bevan, G. Erskine and R. Lewis, Large circulant graphs of fized diameter and arbitrary degree, Ars Math. ontemp. 13 (2017), 275–29.

J. Šiagiová and J. Širáň, Approaching the Moore bound for diameter two by Cayley graphs, J. Combin. Theory Ser. B 102 (2012) 470–4 73.

The End