Cayley graphs of given degree and diameter on linear groups

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Overview

1. Degree/diameter
2. Graph lifting and Cayley graphs
3. Computer search of graphs

Motivation

- In it's simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In design of graphs we can take many restrictions into acount such as degree, grith, diameter.

The degree/diameter problem

Degree/diameter problem

Find graph with biggest possible number of vertices with given degree and diameter.

Edward Forrest Moore

Edward Forrest Moore was first who proposed problem of describing and classifying these graphs.

Moore bound

There is theoretical upper bound for largest order of graph with d-degree and k-diameter.

$$
\begin{aligned}
n_{d,k} \leq M_{d,k} &= 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\
&= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\
&= \begin{cases} 1 + d\frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k+1, & \text{if } d = 2 \end{cases}
\end{aligned} \tag{1}
$$

Moore bound

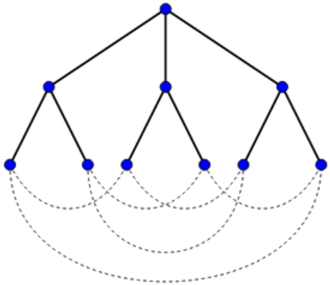

Figure: Peterssen graph is Moore graph with $d = 3$ and $k = 2$

Moore graphs

Graphs with order equal Moore bound are called Moore graphs and are reached only in few cases:

- If $d = 2$ for any $k \geq 1$
- If $k = 1$ for any $d \geq 2$
- For $k = 2$ for $d \in \{3, 7\}$, and possibly 57

For other cases we try to construct graphs with order as close to Moore bound as possible.

Moore graphs

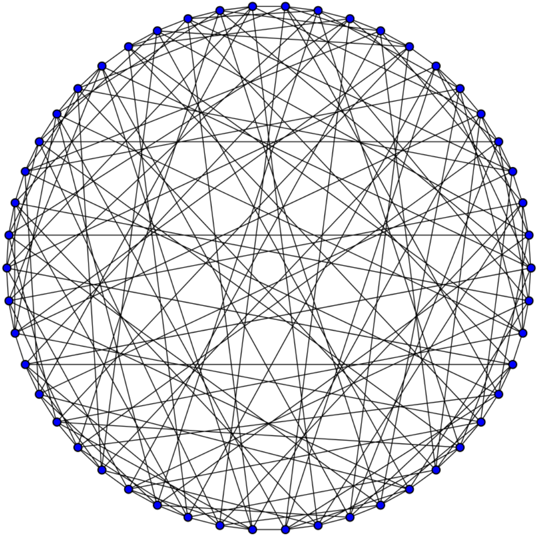

Figure: Hoffman-singleton graph is Moore graph with $d = 7$ and $k = 2$

Construction of large graphs

Finding large graphs close to Moore bound is approached by many techniques in most cases by combinatorics on words algebraic structures.

Here we present two best bounds on large graphs:

$$n_{d,k} \geq \left(\frac{d}{2}\right)^k + \left(\frac{d}{2}\right)^{k-1} \text{ [?]}$$

In the special case of diameter $k = 2$, modified Brown graphs can give for sufficiently large d the bound

$$n_{d,2} \geq d^2 - 2d^{1+\varepsilon}$$

Graph lifting

Let G be an undirected graph. We will assign direction to every edge of graph and making them *arcs*. *Arc* with *reversed* direction of e is denoted by e^{-1}.

Definition (Graph lifting)

Let G be a graph as above and let Γ be a finite group. The mapping

$$\alpha : D(G) \to \Gamma$$

will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any arc $e \in D(G)$.

Graph lifting example

- A *walk* of length ℓ in G is any sequence $W = e_1 e_2 \dots e_\ell$ of consecutive arcs of G, and the voltage $\alpha(W)$ of the walk is simply the product $\alpha(W) = \alpha(e_1)\alpha(e_2)\cdots\alpha(e_\ell)$.
- Lift G^α has diameter at most k if for any two vertices u, v of G and for any element $g \in \Gamma$ there is a walk W of length at most k emanating from u and terminating at v such that $\alpha(W) = g$; in the case when $u = v$ we also require that $g \neq 1$.

Obrazok zdvihu na petersenov graf.

Cayley graphs

Let Γ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for Γ; that is, we require that $S = S^{-1}$ and $1 \notin S$.

Definition (Cayley graphs)

The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which vertices a, b are adjacent if $a^{-1}b \in S$.

Best lower bound on Cayley graphs obtained by Šiagiová and Širáň [?]:

Theorem

Let $D = \{2^{2m+\mu} + (2+\delta)2^{m+1} - 6, m \geq 1, \mu \in \{0, 1\}\}$. *Then, for every* $d \in D$ *one has* $C_{d,2} > d^2 - 6\sqrt{2}d^{3/2}$.

General linear and Special linear groups

Definition (General linear group)

Let q be a power of a prime and let $GF(q)$ be the Galois field of order q. The *general linear group* $GL(m, q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication of matrices. Special linear group is subgroup of $GL(m, q)$ consiting of matrices with determinant equal to 1.

Theorem (Order of $GL(m, q)$)

$|GL(m, q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{n-1})$

Theorem (Order of $SL(m, q)$)

$|SL(m, q)| = |GL(m, q)|/(q - 1)$

Computer search of Cayley graphs

In order to find graphs with given d and k we generted Cayley graphs from random symmetric unit-free sets S of $SL(2,5)$. With assumptions that:

- Generating sets yield all elements of $|SL(2,5)| = 120$
- Cayley graphs are regular with with $d = |S|$

we know by Moore bound that size of S must be at least 12 and all we have to check is diameter of graph.

Computer search of Cayley graphs

With one involution in $SL(2,5)$ and with unit-free property of symmetry we have 59 pairs of elements with its inverses. To check all graphs for $d = 12$ we have to generate and check diameter on $\binom{59}{6} = 45057474$ graphs.

In our search we generated 150000 graphs for all listed degrees.

d	13	14	15	16	17	18	19	20	21
Found graphs	0	0	1	107	345	2451	4120	11669	14926

Algorithm for generation of cayley graph

```
 while(@stack) {
  foreach my $gen_element (@gen_set) {
   my $mul_result = ($current_node x $gen_element) % $zp;
   insert_result(\@mul_results, $current_node, $mul_result, $zp)

   if(find_hash(\%gen_nodes, $mul_result, $zp)) {
    insert_hash(\%gen_nodes, $mul_result, $zp);
    push @keys, compute_hash($mul_result, $zp);
    push @stack, $mul_result;
   }
  }

  shift @stack;

  if(@stack) {
   $current_node = $stack[0];
  }
 }
}
```

Algorithm for diameter

```
foreach my $i ( 2..$diameter ) {
 my $var = variations_with_repetition([0...$#gen_set], $i);
 while (my $var = $variations->next) {
  my $m = $gen_set[$variation ->[0]];
   foreach my $j ( 1..($i-1) ) {
    $m = ($m x $gen_set[$var->[$j]]) % $zp;
   }
   push @hash_storage, compute_hash($m, $zp);
 }
}

foreach my $m ( @{ $gen_set_ref } ) {
 push @hash_storage, compute_hash($m, $zp);
}

foreach my $key ( @{ $keys_ref } ) {
 return 0 unless(List::Util::any {$_ eq $key} @hash_storage);
}

return 1;
```

Output of program

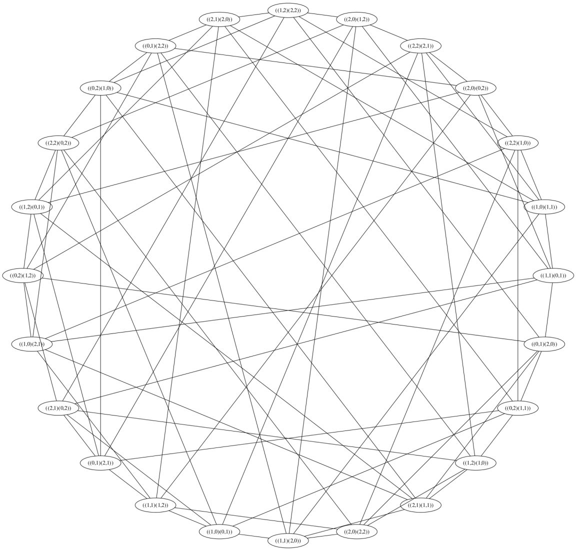

Figure: One of the generated graphs from $SL(2,3)$

Further research

- Finding smallest graph with given girth and degree is konwn as Degree/girth problem. With small changes to program we can systematically or randomly look for Cayley graphs and check its girth.
- Computations to find smallest graphs with $k = 2$ in bigger Galois fields .

The End