Cayley graphs of given diameter or girth on linear groups

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Motivation

- In its simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In modeling interconnection networks by graphs one often considers these restrictions:
  - limitation on the number of physical links leaving a node
  - limitation on accessibility, that is, any two nodes should be accessible using at most a certain number of physical links.
  - limitation on length of smallest cycle in graph.

In graph theory this leads to two important problems: the *degree/diameter problem* to construct the largest possible graphs of a given maximum degree and a given diameter, and the *degree/girth problem* to construct the smallest possible regular graphs of a given degree and a given girth; in both cases the adjectives 'large' and 'small' refer to the order (i.e., the number of vertices) of a graph.

Degree/diameter problem

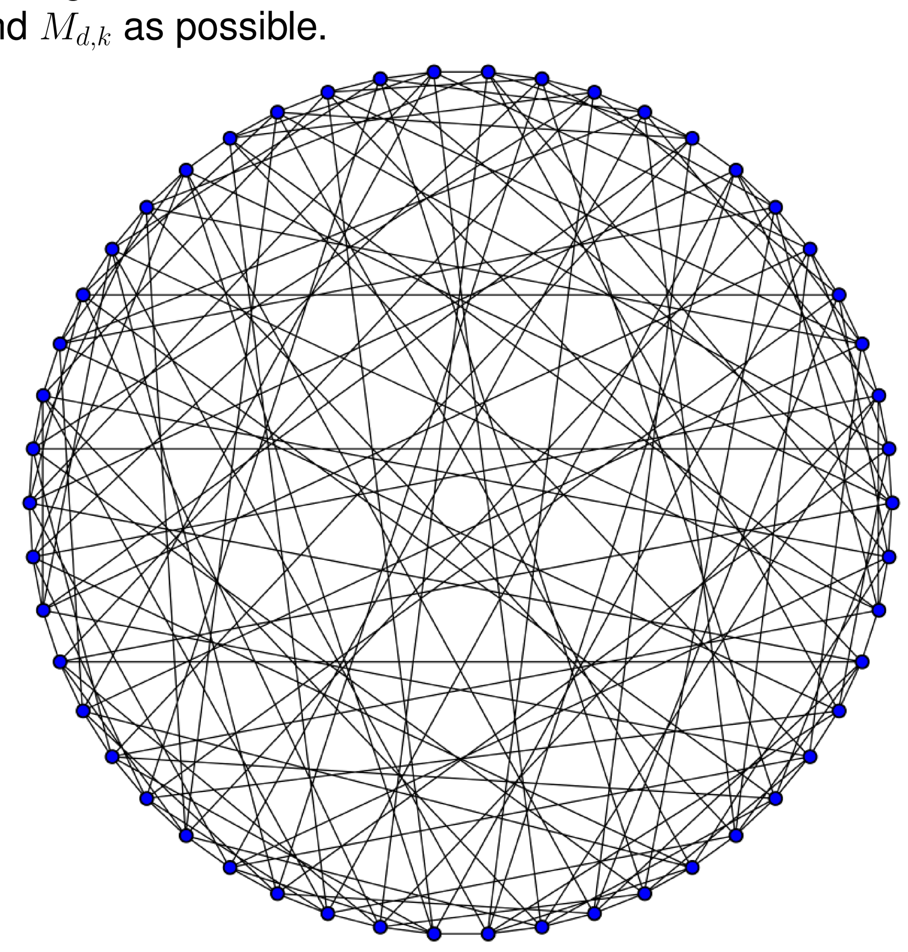
There is a theoretical upper bound on the largest order of a graph of maximum degree $d \geq 2$ and diameter $k \geq 1$.

$$\begin{aligned} n_{d,k} \leq M_{d,k} &= 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\ &= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\ &= \begin{cases} 1 + d\frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k+1, & \text{if } d = 2 \end{cases} \end{aligned} \tag{1}$$

Graphs of maximum degree d and diameter k with $M_{d,k}$ vertices are called *Moore graphs*. Moore graphs are rare; they exist only for the following degrees and diameters:

- If $d = 2$ for any $k \geq 1$
- If $k = 1$ for any $d \geq 2$
- If $k = 2$ and $d \in \{3, 7\}$, and possibly 57

In the remaining cases one tries to construct graphs of maximum degree d and diameter k of order as close to the Moore bound $M_{d,k}$ as possible.



Hoffman-Singleton graph with $k = 2$ and $d = 7$

Construction of large graphs

Finding (d,k)-graph of large order is approached by many techniques, mostly using combinatorics on words or various algebraic structures.

We present two examples of bounds arising from such constructions:
Using combinatorics on words, Baskoro and Miller(1993) [?] proved that

$$n_{d,k} \geq \left(\frac{d}{2}\right)^k + \left(\frac{d}{2}\right)^{k-1}$$

With the help of finite fields Bevan, Erskine and Lewis(2017) [?] observed that modified Brown graphs give the bound

$$n_{d,2} \geq d^2 - 2d^{1+\varepsilon}$$

where ε depends on results about graphs between consecutive primes.

Cayley graphs

Let Γ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for Γ; that is, we require that $S = S^{-1}$ and $1 \notin S$. The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which vertices a, b are adjacent if $a^{-1}b \in S$.

By $C_{d,k}$ we denote the largest order of Cayley graph of degree d and diameter k.
The best currently known lower bound on diameter 2 for an infinite family of degrees is due to Šiagiová and Širáň(2012):
Let $D = \{2^{2m+\mu} + (2+\delta)2^{m+1} - 6, m \geq 1, \mu \in \{0,1\}\}$. Then, for every $d \in D$ one has $C_{d,2} > d^2 - 6\sqrt{2}d^{3/2}$.

Special linear group

Let q be a power of a prime and let $GF(q)$ be the Galois field of order q. The *general linear group* $GL(m,q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication.

The special linear group is the subgroup of $GL(m,q)$ consiting of matrices with determinant 1.
$|GL(m,q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})$
$|SL(m,q)| = |GL(m,q)|/(q-1)$

Computer search

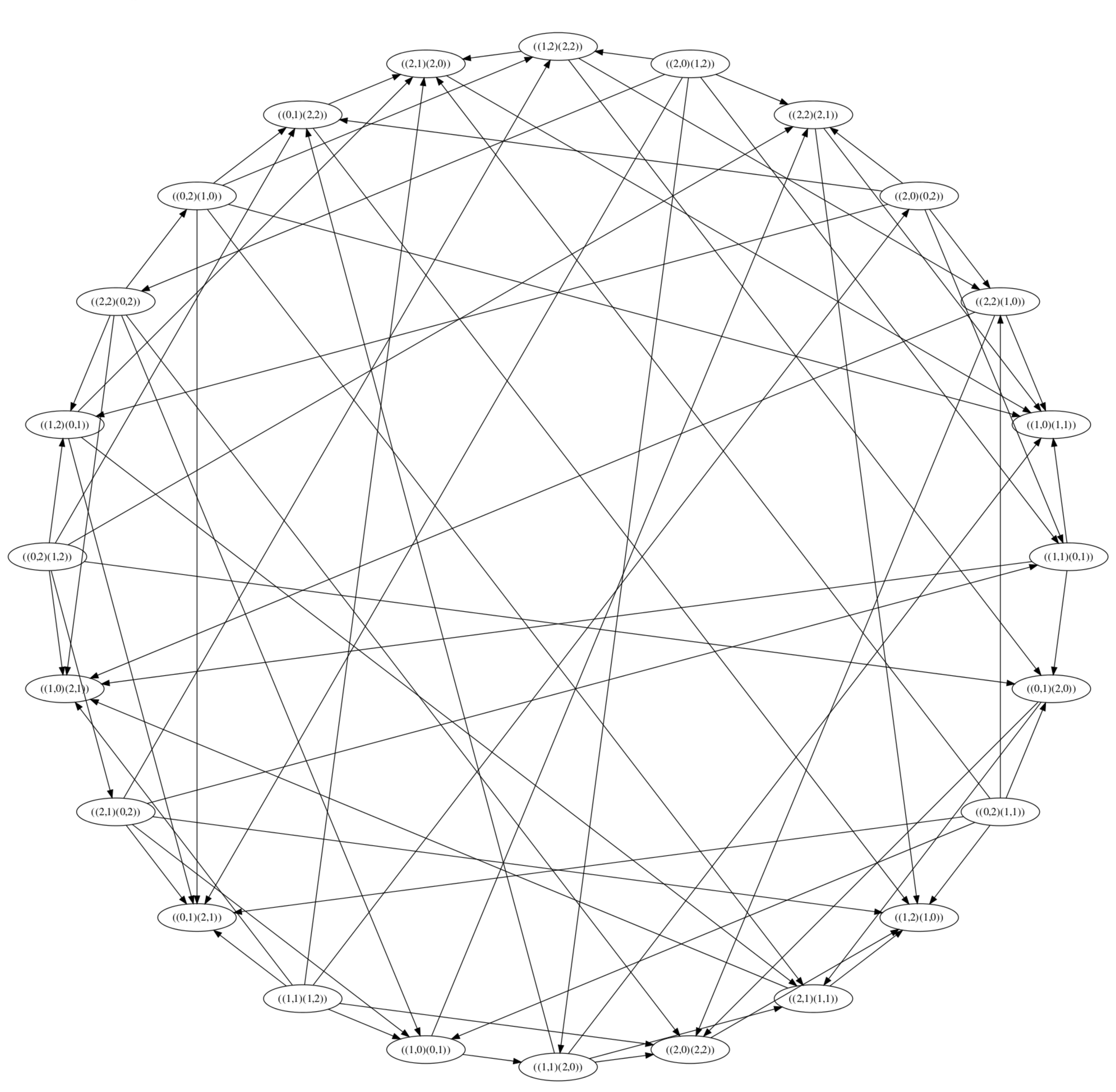
We created a program generating Cayley graphs of given degree and diameter based on $SL(2,q)$.

As an example, consider the problem of generating a Cayley graph for the group $SL(2,5)$ of diameter 2 and minimum possible degree.
In order to find graphs with given d and k we generted Cayley graphs from random symmetric unit-free sets S of $SL(2,5)$. The order of $SL(2,5)$ is 120, so that by the Moore bound the first feasible degree (equal to sthe size of a generating set) would be 12.
The group $SL(2,5)$ has only one involution; all the remaining 118 elements form 59 pairs of the form $\{x, x^{-1}\}$.

To check all Cayley graphs $C(G,S)$ for $G = SL(2,5)$ and $|S| = 12$ we would have to generate all the $\binom{59}{6} = 45057474$ possibilities for S and then check for the diameter of the resulting Cayley graphs.

For the group $G = SL(2,5)$ and generating sets S such that $|S| = d$ by our randomized algorithm we found the following number of generating sets giving Cayley graphs $C(G,S)$ of diameter 2:

d	13	14	15	16	17	18	19	20	21
Found graphs	0	0	1	107	345	2451	4120	11669	14926



Example of graph generated on $SL(2,3)$

Genearation of Cayley graph

```
function cayley_graph_generation(generating_set[])
{
    push all from generating_set[] to stack[]

    foreach element in generating_set[]
        generating_nodes[ element ] = element

    current_node = first element of stack[]

    while stack[] not empty
        foreach generating_element in generating_set[]
            result = current_node x generating_element % Zp
            cayley_graph[ current_node ] = result

            if generating_nodes[ result ] is empty
                push stack[], result
                generating_nodes[ result ] = result

        shift to the left stack[]
        current_node = first element of stack[]

    return cayley_graph[]
}
```

Check of diameter of Cayley graph

```
function check_diameter(generating_set[], diameter)
{
 push reached_nodes[], generating_set[]

 foreach n in [ 2, ... ,diameter ]
    foreach var = variation of indeces of generating_set with length n
        node = generating_set[ var[0] ] x ... x generating_set[ var[len
        push reached_nodes[], node
       if reached_nodes[] contains all elements of group
         retrun cayley_graph has diameter n

retrun cayley_graph has bigger diameter
}
```

Degree/girth problem

Finding regular graphs with smallest possible order denoted $n_{d,g}$ with given degree d and girth $g \geq 3$ is known as *degree/girth problem.* Motivation for finding such graphs could arise from constructing graphs with no cycles of length less than g and its similarity to *degree/diameter problem.*

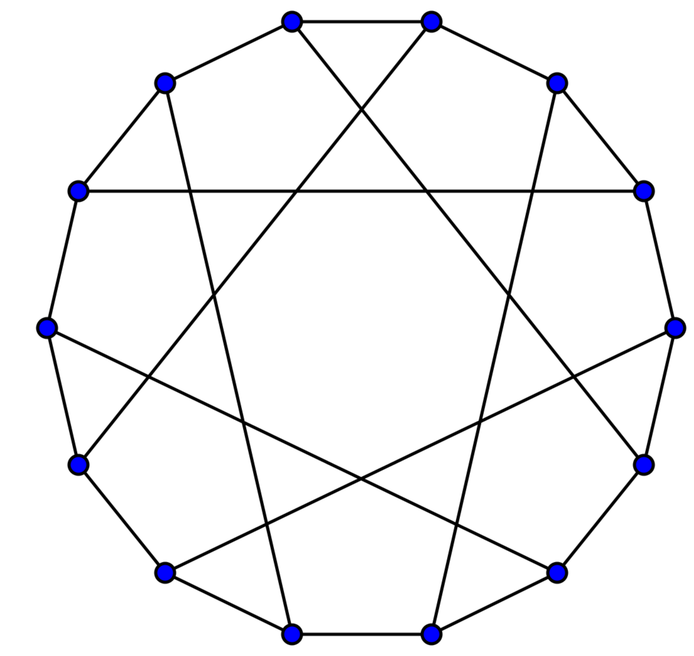
$$n_{d,g} \geq M(d,g) = 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{\frac{g-3}{2}}$$

and for even g

$$n_{d,g} \geq M(d,g) = 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{\frac{g-2}{2}})$$

For odd g the Moore bound is obtained in the same way as the Moore bound for the degree/diameter problem for the diameter $k = (g-1)/2$. For even g we start with an edge and from its incident vertices we continue growing two spanning trees until we reach depth $(g-2)/2$.
Graphs of degree d and girth g with order equal to $M(d,g)$ are called Moore graphs or cages. We list all cases with $n_{d,g} = M_{d,g}$ below:

- For $d = 2$, $g \geq 2$ - circles
- For $g = 3$, $d \geq 2$ - complete graphs
- For $g = 4$, $d \geq 2$ - complete bipartite graphs
- For $g = 5$, $d = 2$ - circle of length 5
  $d = 3$ - Peterssen graph
  $d = 7$ - Hoffman-Singleton graph
  $d = 57$ - this value has not been excluded but no such graph has been found yet
- For $g \in 6, 8, 12$, if $d - 1$ is a prime power



Heawood graph with $d = 3$ and $g = 6$

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