Cayley graphs of given degree and diameter on linear groups

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Overview

Motivation

- In it's simplest form, networks can be modeled by graphs with nodes as vertices and links between them as edges.
- In design of graphs we can take many restrictions into acount such degree, grith, diameter.
- Two important problems concerning degree and diameter and degree and grith of graph

The degree/diameter problem

Degree/diameter problem

Find graph with biggest possible number of vertices with given degree and diameter.

Degree/girth problem

Find graph with smallest possible number of vertices with given degree and diameter.

Moore bound

There is theoretical upper bound for largest order of graph with d-degree and k-diameter.

$$
\begin{aligned}
n_{d,k} \leq M_{d,k} &= 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\
&= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\
&= \begin{cases} 1 + d\frac{(d-1)^k - 1}{d-2}, & \text{if } d > 2 \\ 2k+1, & \text{if } d = 2 \end{cases}
\end{aligned} \tag{1}
$$

Moore bound

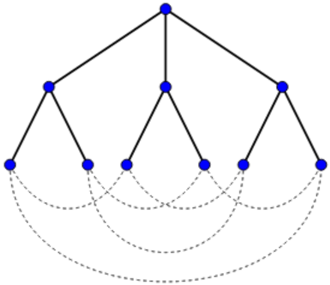

Figure: Peterssen graph is Moore graph with $d = 3$ and $k = 2$

Moore graphs

Graphs with order equal Moore bound are called Moore graphs and are reached only in few cases.

- If $d = 2$ for any $k \geq 1$
- If $k = 1$ for any $d \geq 2$
- For $k = 2$ for $d \in \{3, 7\}$, and possibly 57

For other cases we try to construct graphs with order as close to Moore bound as possible.

Moore graphs

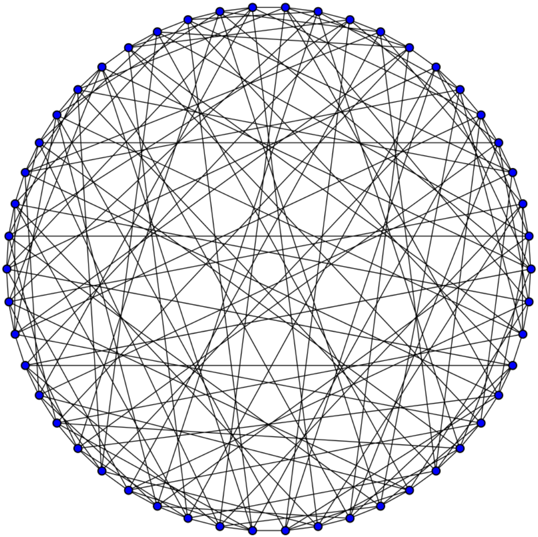

Figure: Hoffman-singleton graph is Moore graph with $d = 7$ and $k = 2$

Graph lifting

Let G be an undirected graph. We will assign direction to every edge of graph and making them *arcs*. *Arc* with *reversed* direction of e is denoted by e^{-1}.

Definition (Graph lifting)

Let G be a graph as above and let Γ be a finite group. The mapping

$$\alpha : D(G) \to \Gamma$$

will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any arc $e \in D(G)$.

Graph lifting example

Obrazok zdvihu na petersenov graf.

Cayley graphs

Let Γ be a group and let $S \subset \Gamma$ be a symmetric unit-free generating set for Γ; that is, we require that $S = S^{-1}$ and $1 \notin S$.

Definition (Cayley graphs)

The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which vertices a, b are adjacent if $a^{-1}b \in S$.

General linear and Special linear groups

Definition (General linear group)

Let q be a power of a prime and let $GF(q)$ be the Galois field of order q. The *general linear group* $GL(m, q)$ consists of all non-singular $m \times m$ matrices over $GF(q)$ under multiplication of matrices. Special linear group is subgroup of $GL(m, q)$ consiting of matrices with determinant equal to 1.

Theorem (Order of $GL(m, q)$)

$|GL(m, q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{n-1})$

Theorem (Order of $SL(m, q)$)

$|SL(m, q)| = |GL(m, q)|/(q - 1)$

Generation of cayley graphs

Computer search of cayley graphs

The End