0.1 Chapter 1

Theorem 1. *Let P be a set. An order (or* ***partial order****) on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,*

1. $x \leq x$,
2. $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
3. $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$.

Theorem 2. *Let P be an ordered set. Then P is a* ***chain*** *if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of P are comparable). Alternative names for a chain are* ***linearly ordered set*** *and* ***totally ordered set****. The ordered set is an* ***antichain*** *if $x \leq y$ in P only if $x == y$.*

Theorem 3. *We say that P and Q are (****order-****)****isomorphic****, and write $P \cong Q$, if there exists a map φ from P onto Q such that $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q. Then φ is called* ***order-isomorphism****.*

Theorem 4. *Let P be an ordered set and let $x, y \in P$. We say x is* ***covered by*** *y (or y* ***covers*** *x), and write $x \prec y$, if $x \leq y$ and $x \leq z \leq y$ implies $z = x$. The latter condition is demanding that there be no element z of P with $x \leq z \leq y$.*

Theorem 5. *Let P be a finite ordered set. We can represent P by a configuration of circles (representing the elements of P) and interconnecting lines (indicating the covering relation). The construction goes as follows.*

1. *To each point $x \in P$, associate a point $p(x)$ of the Euclidean plane, depicted by a small circles with centre at $p(x)$.*
2. *For each covering pair $x \prec y$ in P, take a line segment $\ell(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.*
3. *Carry out 1 and 2 in such a way that*
   - *if $x \prec y$, then $p(x)$ is 'lower' than $p(y)$ (that is, in standard Cartesian coordinates, has strictly smaller second coordinate),*

- *the circle at $p(z)$ does not intersect the line segment $\ell(x, y)$ if $z \neq x$ and $z \neq y$.*

Theorem 6. *Let P and Q be finite ordered sets and let $\varphi : P \mapsto Q$ be a bijective map. Then the following are equivalent:*

1. *φ is an order-isomorphism;*
2. *$x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q*
3. *$x \prec y$ in P if and only if $\varphi(x) \prec \varphi(y)$ in Q*

Theorem 7. *Given any ordered set P we can form a new orderer set P^∂ (the **dual** of P) by defining $x \leq y$ to hold in P^∂ if and only if $y \leq x$ hold in P. For P finite, we obtain a diagram for P^∂ simply by 'turning upside down' a diagram for P.*

Theorem 8. *Let P be an ordered set. We say P has a bottom element if there exists $\perp \in P$ (called **bottom**) with the property that $\perp \leq x$ for all $x \in P$. Dually, P has a top element if there exists $\top \in P$ such that $x \leq \top$ for all $x \in P$.*

Theorem 9. *Let P be an ordered set. We say P has a bottom element if there exists $\perp \in P$ (called **bottom**) with the property that $\perp \leq x$ for all $x \in P$. Dually, P has a top element if there exists $\top \in P$ such that $x \leq \top$ for all $x \in P$.*

Theorem 10. *Given an ordered set P (with or without $\perp$), we form $P_\perp$ (called P 'lifted') as follows. Take an element $\boldsymbol{0} \notin P$ and define $\leq$ on $P_\perp := P \cup \{\boldsymbol{0}\}$ by*

$$x \leq y \text{ if and only if } x = \boldsymbol{0} \text{ or } x \leq y \text{ in } P$$

Theorem 11. *Let P be an ordered set and let $Q \subseteq P$. Then $a \in Q$ is a **maximal element** of Q if $a \leq x$ and $x \in Q$ imply $a = x$. We denote the set of maximal elements of Q by $MaxQ$.If Q (with the order inherited from P) has a top element, $\top_Q$, then $MaxQ = \{\top_Q\}$; in this case $\top_Q$ is called the **greatest** (or **maximum**) element of Q, and we write $\top_Q = maxQ$. A **minimal** elment of $Q \subseteq P$ and $minQ$, the **least** (or **minimum**) element of Q (when these exist) are defined dually, that is by reversing the order.*

Theorem 12. *Suppose that P and Q are (disjoint) ordered sets. The **disjoint union** $P \cup Q$ of P and Q is the ordered set formed by defining $x \leq y$ in $P \cup Q$ if and only if* *either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q. A diagram for $P \cup Q$ is formed by placing side by side diagrams for P and Q.*

Theorem 13. *Let P and Q be (disjoint) ordered sets.* ***The linear sum*** *$P \oplus Q$ is defined by taking the following order relation on $P \cup Q : x \leq y$ if and only if*

$$\begin{aligned} &x, y \in P \text{ and } x \leq y \text{ in } P, \\ \text{or } &x, y \in P \text{ and } x \leq y \text{ in } Q, \\ \text{or } &x \in P \text{ and } y \in Q \end{aligned}$$

A diagram for $P \oplus Q$ (when P and Q are finite) is obtained by placing a diagram for P directly below a diagram for Q and then adding a line segment from each each *maximal element of P to* each *minimal element of Q.*

Theorem 14. *Let $P_1, \ldots, P_n$ be ordered sets. The Cartesian product $P_1 \times \ldots \times P_n$ can be made into an ordered set by imposing the coodinatewise order by*

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \iff (\forall i) x_i \leq y_i \text{ in } P_i.$$

Theorem 15. *Let $X = \{1, 2, \ldots, n\}$ and define $\varphi : \wp(X) \to \wp 2^n$ by $\varphi(A) = (\epsilon_1, \ldots, \epsilon_n)$ where*

$$\epsilon_i = \begin{cases} 1 \text{ if } i \in A, \\ 0 \text{ if } i \notin A. \end{cases}$$

Then φ is an order-isomorphism.

Theorem 16. *Let P be an ordered set and $Q \subseteq P$.*

- *Q is a* ***down-set*** *(alternative term include* ***decreasing set*** *and* ***order ideal****) if, whenever $x \in Q$, $y \in P$ and $y \leq x$, we have $y \in Q$.*
- *Dually, Q is an* ***up-set*** *(alternative terms are* ***increasing set*** *and* ***order filter****) if, whenever $x \in Q$, $y \in P$ and $y \geq x$, we have $y \in Q$.*

Theorem 17. *$\downarrow Q := \{y \in P | (\exists x \in Q) y \leq x\}$ and $\uparrow Q := \{y \in P | (\exists x \in Q) y \geq x\}$,*

$$\downarrow x := \{y \in P | y \leq x\} \text{ and } \uparrow x := \{y \in P | y \geq x\}.$$

Theorem 18. *The family of all down-sets of P is denoted by $\mathcal{O}(P)$. It is itself an ordered set, under the inclusion order.*

Theorem 19. *Let P be an ordered set and $x, y \in P$. Then the following are equivalent:*

1. $x \leq y$;

2. $\downarrow x \subseteq \downarrow y$;

3. $(\forall Q \in \mathcal{O}(P)) y \in Q \implies x \in Q$.

Theorem 20.

$$\mathcal{O}(P)^\partial = \mathcal{O}(P^\partial)$$

Theorem 21. *Let P be an ordered set. Then*

1. $\mathcal{O}(P \oplus \mathbf{1}) \cong \mathcal{O}(P) \oplus \mathbf{1}$ *and* $\mathcal{O}(\mathbf{1} \oplus P) \cong \mathbf{1} \oplus \mathcal{O}(P)$;

2. $\mathcal{O}(P_1 \cup P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$.

Theorem 22. *Let P and Q be ordered sets. A map $\varphi : P \to Q$ is said to be*

1. ***order-preserving*** *(or alternatively,* ***monotone****) if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y) in Q$;*

2. *an* ***order-embedding*** *(and we write $\varphi : P \to Q$) if $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q;*

3. *an* ***order-isomorphism*** *if it is an order-embedding which maps P onto Q.*