0.1 Chapter 2

Theorem 1. *Let P be an ordered set and let $S \subseteq P$. An element $x \in P$ is an* ***upper bound*** *of S if $s \leq x$ for all $s \in S$. A* ***lower bound*** *is defined dually. The set of all upper bounds of S is denoted by S^u (read as 'S upper') and the set of all lower bounds by S^l (read as 'S* ***lower****'):*

$$S^u := \{x \in P | (\forall s \in S) s \leq x\} \text{ and } S^l := \{x \in P | (\forall s \in S) s \geq x\}$$

Since $\leq$ is transitive, S^u is always an up-set and S^l is a down-set. If S^u has a least element x, then x is called the ***least upper bound*** *of S. Equivalently, x is the least upper bound of S if*

1. *x is an upper bound of S and*

2. *$x \leq y$ for all upper bounds y of S.*

The least upper bounds of S exists if and only if there exists $x \in P$ such that

$$(\forall y \in P)[((\forall s \in S) s \leq y) \iff x \leq y]$$

Dually, if S^l has a greatest element, x, then x is called the ***greatest lower bound*** *of S. The least upper bound of S is also called the* ***supremum*** *of S and is denoted by* $\sup S$*; the greatest lower bound of S if also called the* ***infinum*** *of S and is denoted by* $\inf S$.

Theorem 2. *We write $x \vee y$(read as 'x* ***join*** *y') in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ (read as 'x* ***meet*** *y' in place of $\inf\{x, y\}$ when it exists. Similiarly we write $\bigvee S$ (the '****join of*** *S') and $\bigwedge S$ (the '****meet of*** *S') instead of $\sup S$ and $\inf S$ when these exist. It is sometimes necessary to indicate that the join or meet is being found in a particular ordered set P, in which case we write $\bigvee_P S$ or $\bigwedge_P S$.*

Theorem 3. *Let P be a non-empty ordered set.*

1. *If $x \vee y$ and $x \wedge y$ exists for all $x, y \in P$, then P is called* ***latttice***

2. *If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a* ***complete lattice****.*

Theorem 4. *Let L be a lattice and let $a, b \in L$. Then the following are equivalent:*

1. $a \leq b$*;*

2. $a \vee b = b$*;*

3. $a \wedge b = a$*.*

Theorem 5. *Let L be a lattice. Then $\vee$ and $\wedge$ satisfy, for all $a, b, c, \in L$,*

(L1) $(a \vee b) \vee c = a \vee (b \vee c)$ *(associative law)*

(L2) $a \vee = b \vee a$ *(commutative law)*

(L3) $a \vee a = a$ *(idempotency law)*

(L4) $a \vee (a \wedge b) = a$ *(absorption law)*

Theorem 6. *Let $\langle L, \vee, \wedge \rangle$ be a non-empty set equipped with two binary operations which satisfy* ***(L1)-(L4)***

(i) For all $a, b \in L$, we hae $a \vee b = b$ if and only if $a \wedge b = a$.

(ii) Defined $\leq$ on L by $a \leq b$ if $a \vee b = b$. Then $\leq$ is an order relation.

(iii) With $\leq$ as in (ii), $\langle L; \leq \rangle$ is a lattice in which the original operations agree with the induced operations, that is, for all $a, b \in L$,

$$a \vee b = \sup\{a, b\} \text{ and } a \wedge b = \inf\{a, b\}.$$

Theorem 7. *Let L be a lattice and $\emptyset \neq M \subseteq L$. Then M is a* ***sublattice*** *of L if*

$$a, b \in M \text{ implies } a \vee b \in M \text{ and } a \wedge b \in M$$

We denote the collection of all sublattices of L by $subL$ and let $Sub_0 L = SubL \cup \{\emptyset\}$; both are ordered by inclusion.

Theorem 8. *Let L and K be lattices. Define $\vee$ and $\wedge$ coordinatewise on $L \times K$, as follows:*

$$(\ell_1, k_1) \vee (\ell_2, k_2) = (\ell_1 \vee \ell_2, k_1 \vee k_2)$$

$$(\ell_1, k_1) \wedge (\ell_2, k_2) = (\ell_1 \wedge \ell_2, k_1 \wedge k_2)$$

Theorem 9. *Let L and K be a lattices. A map $f : L \to K$ is said to be a **homomorphism** (or, for emphasis, **lattice homomorphism**) if f is a **join-preserving** and **meet-preserving**, that is, for all $a, b \in L$,*

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b).$$

*A bijective homomorphism is a **(lattice) isomorphism**. If $f : L \to K$ is a one-to-one homomorphism, then the sublattice $f(L)$ of K is isomorphic to L and we refer to f as an **embeding (of** L **into** K).*

Theorem 10. *Let L and K be lattices and $f : L \to K$ a map.*

(i) The following are equivalent:

(a) f is order-preserving;

(b) $(\forall a, b \in L) f(a \vee b) \geq f(a) \vee f(b)$;

(c) $(\forall a, b \in L) f(a \wedge b) \leq f(a) \wedge f(b)$.
In particular, if f is homomorphism, then f is order-perserving.

(ii) f is a lattice isomorphism if and only if it is an order-isomorphism.

Theorem 11. *Let L be a lattice. A non-empty subset J of L is call an **ideal** if*

(i) $a, b \in J$ implies $a \vee b \in J$,

(ii) $a \in L, b \in J$ and $a \leq b$ imply $a \in J$.

Theorem 12. *Non-empty sdubset G of L is called a **filter** if*

(i) $a, b \in G$ implies $a \wedge b \in G$,

(ii) $a \in L, b \in G$ and $a \geq b$ imply $a \in G$

The set of all ideals (filters) of L is denoted by $\mathcal{I}(L)$