Solution of the Transmission-Line Equations for Three-Conductor Lines in Homogeneous Media

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Abstract—**Formulas are obtained for the terminal voltages of a transmission line consisting of three conductors immersed in a homogeneous medium. No simultaneous equations need be solved and the result does not require that the line be electrically short. Typical applications are lines composed of three wires, two wires above a ground plane, and two wires within a cylindrical shield. Certain interesting properties of the solution are also obtained.**

Key Words: **Transmission lines, three conductors, coupling, formulas.**

I. INTRODUCTION

WIRE-COUPLED interference (crosstalk) results from the unintentional coupling of signals from one circuit into another by virtue of the electromagnetic interactions among wires (cylindrical conductors) which are in close proximity in a cable bundle. In this paper, explicit formulas are obtained for the solution of the transmission-line equations which characterize this type of coupling for the special case of three-conductor transmission lines in homogeneous media. This specific situation is often used to model coupling in cable bundles to provide estimates of crosstalk.

Consider the three-conductor line shown in Fig. 1. The line is of total length L and consists of a generator circuit and a receptor circuit. The generator circuit consists of a wire, the "generator wire," and a reference conductor for the line voltages. The receptor circuit consists of another wire, the "receptor wire," and the reference conductor. One generally excites one end of the generator circuit and is then interested in determining the induced signals, the voltages V_R (0) and V_R (L), at each end of the receptor circuit. Some examples of typical physical configurations to which these results apply are shown as cross-sectional views in Fig. 2 where the reference conductor is either another wire, an infinite ground plane, or an overall cylindrical shield. The results in this paper are not restricted to these configurations, but only require that the medium surrounding the three conductors be homogeneous.

Techniques for characterizing these structures as multiconductor transmission lines are, with the aid of matrix analysis, straightforward extensions of the familiar results for two-conductor lines [1], [2]. However, for lines consisting of more than two conductors, the end result requires the solution of complex, simultaneous equations at each frequency for which the response is desired. For example, if the line consists of n wires and a reference conductor, then n simultaneous equations must be solved at each frequency.

For three-conductor lines as are considered here, the two required simultaneous equations will be solved explicity, resulting in general formulas for the receptor voltages V_R (0)

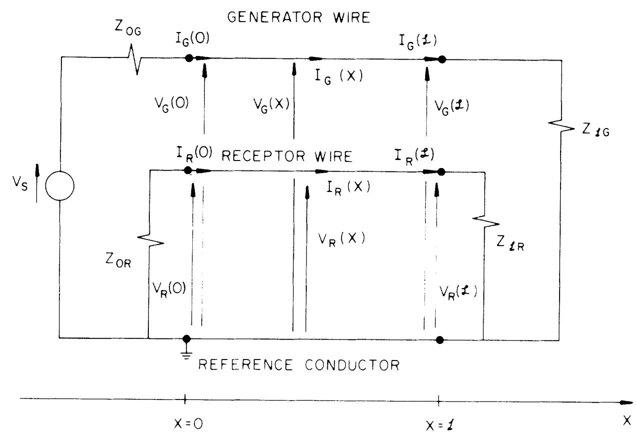

Fig. 1. The three-conductor transmission line.

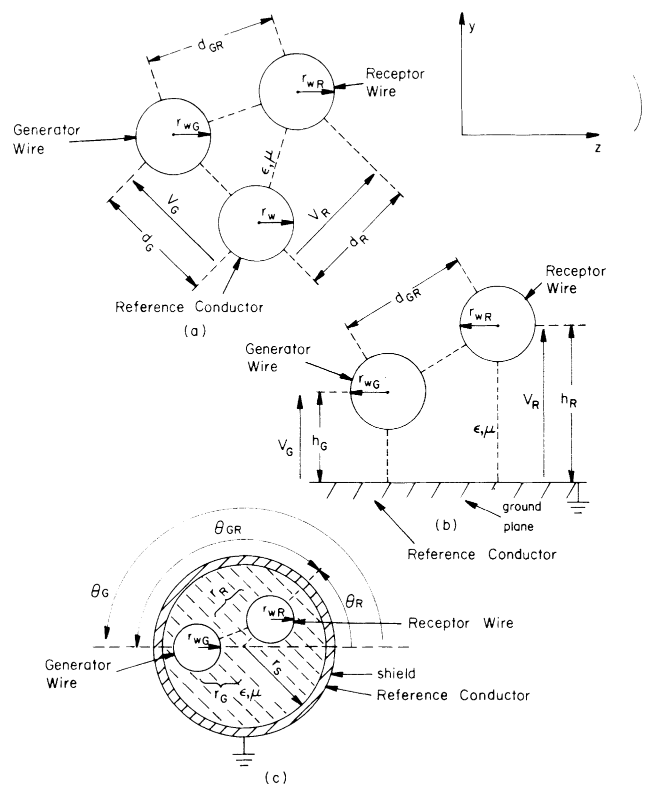

Fig. 2. Typical cross-sectional geometries to which the results apply.

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and V_R (L). This approach bypasses the requirement for the repeated solution of simultaneous equations (two of them in this case). Although the numerical solution of the two simultaneous equations is straightforward, there are two advantages to solving them *a priori* in literal form. First, there will be no need to obtain a subroutine to solve the required complex simultaneous equations when one implements the general solution on a digital computer. Thus the general solution could be obtained when using a digital computer having limited facilities, e.g., minicomputers or even programmable hand-held calculators. Secondly, the explicit formulas provide considerable insight into the behavior of the coupling phenomenon which is obscured in the simultaneous-equation formulation.

Perhaps the most important benefit of these results is the following. Quite often, only a low-frequency approximation is used to obtain estimates of crosstalk [4]. Perhaps the main reason for using this low-frequency approximation lies in the simplicity of the resulting equations for the receptor terminal voltages, V_R (0) and V_R (L). In this approximate model, the crosstalk is considered to be the superposition of the portions of the coupling due to mutual capacitance ("capacitive coupling") and mutual inductance ("inductive coupling") between the generator and receptor circuits (see Matick [4, pp. 287–291]). The region of validity for this low-frequency approximation, however, is a complicated function of, not only frequency, but also the values of the termination impedances, as will be shown in this paper. The exact formulas (34) for the crosstalk which are derived in this paper from the transmission-line equations are placed in a form which is quite similar to, and almost as simple as, this conventional low-frequency approximation. Therefore, these exact expressions may be used for determining crosstalk instead of the low-frequency approximations with no significant increase in computational complexity.

Other attempts at simplifying the crosstalk equations assume weak coupling between the generator and receptor circuits [5]. In this model, the effect of the receptor circuit on the generator circuit is disregarded. The exact formulas derived in this paper are almost as simple as the weak-coupling approximations and are not restricted to the weak-coupling case. The specific solution will be obtained under certain assumptions:

1) The medium surrounding the transmission line is homogeneous, linear, isotropic, and lossless, and is characterized by permittivity ϵ and permeability μ.

2) The line is uniform in the sense that there is no variation in the conductor cross sections along the line length and all conductors are parallel to each other.

3) All conductors are perfect (lossless).

These assumptions simplify the solution considerably and are generally implicit in most other formulations.

II. DERIVATION OF THE TERMINAL-VOLTAGE EQUATIONS

The interest in this paper is in the sinusoidal steady-state behavior of the line, i.e., V_s in Fig. 1 is a complex, phasor voltage which we could assume to be $V_s = 1\angle 0^\circ$ since the equations describing the line will be linear and superposition may be applied for any other V_s. In this case, the voltage transfer functions, $V_R(0)/V_s$ and $V_R(L)/V_s$ would be obtained. The axis of the line is denoted by X. We characterize an electrically short section ΔX of the line as shown in Fig. 3 with lumped per-unit-length parameters of self inductance l_G and l_R, mutual inductance l_m, self capacitance c_G and c_R, and mutual capacitance c_m [2]. The transmission-line equations can be derived from the circuit of Fig. 3 and become, in the limit as $\Delta X \to 0$ [2],

$$\frac{dV_G(X)}{dX} = -j\omega l_G I_G(X) - j\omega l_m I_R(X) \tag{1a}$$

$$\frac{dV_R(X)}{dX} = -j\omega l_m I_G(X) - j\omega l_R I_R(X) \tag{1b}$$

$$\frac{dI_G(X)}{dX} = -j\omega(c_G + c_m) V_G(X) + j\omega c_m V_R(X) \tag{1c}$$

$$\frac{dI_R(X)}{dX} = j\omega c_m V_G(X) - j\omega(c_R + c_m) V_R(X). \tag{1d}$$

The matrix chain parameters provide a solution to these transmission-line equations by relating the voltages and currents at one end of the line, $V_G(L)$, $V_R(L)$, $I_G(L)$, and $I_R(L)$, to the voltages and currents at the other end of the line, $V_G(0)$, $V_R(0)$, $I_G(0)$, and $I_R(0)$, as [1], [2]

$$\begin{bmatrix} \bar{V}(L) \\ \bar{I}(L) \end{bmatrix} = \begin{bmatrix} \tilde{\phi}_{11}(L) & \tilde{\phi}_{12}(L) \\ \tilde{\phi}_{21}(L) & \tilde{\phi}_{22}(L) \end{bmatrix} \begin{bmatrix} \bar{V}(0) \\ \bar{I}(0) \end{bmatrix} \tag{2}$$

where

$$\bar{V}(L) = \begin{bmatrix} V_G(L) \\ V_R(L) \end{bmatrix} \qquad \bar{V}(0) = \begin{bmatrix} V_G(0) \\ V_R(0) \end{bmatrix}$$

$$\bar{I}(L) = \begin{bmatrix} I_G(L) \\ I_R(L) \end{bmatrix} \qquad \bar{I}(0) = \begin{bmatrix} I_G(0) \\ I_R(0) \end{bmatrix} \tag{3}$$

The matrix chain parameters in (2) become, for the above-stated assumptions, [2]

$$\tilde{\phi}_{11}(L) = \cos(\beta L)\tilde{1}_2 \tag{4a}$$

$$\begin{aligned} \tilde{\phi}_{12}(L) &= -jv\sin(\beta L)\tilde{L} \\ &= -j\omega L\left\{\frac{\sin(\beta L)}{\beta L}\right\}\tilde{L} \end{aligned} \tag{4b}$$

$$\begin{aligned} \tilde{\phi}_{21}(L) &= -jv\sin(\beta L)\tilde{C} \\ &= -j\omega L\left\{\frac{\sin(\beta L)}{\beta L}\right\}\tilde{C} \end{aligned} \tag{4c}$$

$$\tilde{\phi}_{22}(L) = \cos(\beta L)\tilde{1}_2 \tag{4d}$$

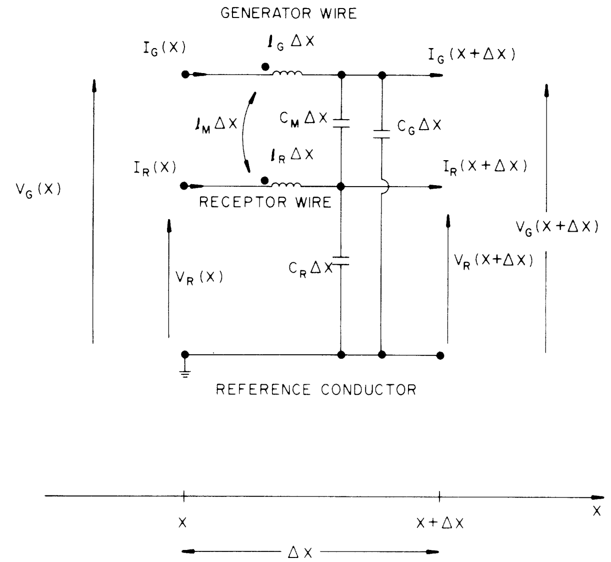

Fig. 3. The per-unit-length model.

where β is the phase constant given by

$$\beta = \frac{\omega}{v} = \frac{2\pi}{\lambda} \tag{5}$$

ω is the radian frequency of excitation, λ is a wavelength at this frequency,

$$\lambda = v/f \tag{6}$$

and v is the velocity of propagation in the surrounding (homogeneous) medium,

$$v = \frac{1}{\sqrt{\mu\epsilon}}. \tag{7}$$

The per-unit-length inductance and capacitance matrices, $\tilde{L}$ and $\tilde{C}$, respectively, are given by [2]

$$\tilde{L} = \begin{bmatrix} l_G & l_m \\ l_m & l_R \end{bmatrix} \tag{8a}$$

$$\tilde{C} = \begin{bmatrix} (c_G + c_m) & -c_m \\ -c_m & (c_R + c_m) \end{bmatrix} \tag{8b}$$

and

$$\tilde{1}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{9}$$

Matrices $\tilde{L}$ and $\tilde{C}$ satisfy (for this special case of a homogeneous medium) [2]

$$\tilde{L}\tilde{C} = \mu\epsilon \tilde{1}_2 = \frac{1}{v^2}\tilde{1}_2. \tag{10}$$

Values for the entries in $\tilde{L}$ for the three structures in Fig. 2 are given in [3, eqs. (20), (23), and (24)].

The termination networks are described by Generalized Thevenin Equivalents [2]:

$$\bar{V}(0) = \bar{V}_0 - \tilde{Z}_0 \bar{I}(0) \tag{11a}$$

$$\bar{V}(L) = \bar{V}_L + \tilde{Z}_L \bar{I}(L) \tag{11b}$$

where, from Fig. 1,

$$\bar{V}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_s \qquad \bar{V}_L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\tilde{Z}_0 = \begin{bmatrix} Z_{0G} & 0 \\ 0 & Z_{0R} \end{bmatrix} \quad \tilde{Z}_L = \begin{bmatrix} Z_{LG} & 0 \\ 0 & Z_{LR} \end{bmatrix}. \tag{12}$$

Substituting (11) and (4) into (2) yields [2]

$$\left[\cos(\beta L)\{\tilde{Z}_L + \tilde{Z}_0\} + \left(\frac{\sin(\beta L)}{\beta L}\right) \cdot \{\tilde{Z}_L(j\omega\tilde{C}L)\tilde{Z}_0 + j\omega\tilde{L}L\}\right]\bar{I}(0) = \left[\cos(\beta L)\tilde{1}_2 + \left(\frac{\sin(\beta L)}{\beta L}\right)\tilde{Z}_L(j\omega\tilde{C}L)\right]\bar{V}_0 - \bar{V}_L. \tag{13}$$

A similar equation for determining $\bar{I}(L)$ can be obtained by replacing $\bar{I}(0)$, $\tilde{Z}_L$, $\tilde{Z}_0$, $\bar{V}_0$, and $\bar{V}_L$ in (13) with $-\bar{I}(L)$, $\tilde{Z}_0$, $\tilde{Z}_L$, $\bar{V}_L$, and $\bar{V}_0$, respectively, to obtain

$$\left[\cos(\beta L)\{\tilde{Z}_0 + \tilde{Z}_L\} + \left(\frac{\sin(\beta L)}{\beta L}\right) \cdot \{\tilde{Z}_0(j\omega\tilde{C}L)\tilde{Z}_L + j\omega\tilde{L}L\}\right]\bar{I}(L) = -\left[\cos(\beta L)\tilde{1}_2 + \left(\frac{\sin(\beta L)}{\beta L}\right)\tilde{Z}_0(j\omega\tilde{C}L)\right]\bar{V}_L + \bar{V}_0. \tag{14}$$

This step should be clear since the transmission line is reciprocal.

The solution for the terminal currents of the receptor circuit can be obtained by expanding (13) and (14) and using Cramer's rule. To simplify the notation somewhat, we will employ the symbols C and S to denote

$$C = \cos(\beta L) \tag{15a}$$

$$S = \frac{\sin(\beta L)}{\beta L} \tag{15b}$$

Equation (14) is expanded as

$$[C\{Z_{0G} + Z_{LG}\} + j\omega LS\{Z_{LG}Z_{0G}(c_G + c_m) + l_G\}]I_G(L) + [j\omega LS\{l_m - c_m Z_{0G}Z_{LR}\}]I_R(L) = V_s \tag{16a}$$

$$[j\omega LS\{l_m - c_m Z_{LG}Z_{0R}\}]I_G(L) + [C\{Z_{0R} + Z_{LR}\} + j\omega LS\{Z_{0R}Z_{LR}(c_R + c_m) + l_R\}]I_R(L) = 0. \tag{16b}$$

Solving for $I_R(L)$ with Cramer's rule and utilizing $V_R(L) = Z_{LR}\, I_R(L)$, one obtains

$$V_R(L) = \frac{j\omega LSZ_{LR}\{c_m Z_{LG}Z_{0R} - l_m\}}{\Delta} V_s \tag{17}$$

where Δ is the determinant of the coefficients in (16) and can be written as

$$\begin{aligned}\Delta = {} & (Z_{0G} + Z_{LG})(Z_{0R} + Z_{LR})C^2 \\ & - \omega^2 L^2 S^2 [\{Z_{LG}Z_{0G}(c_G + c_m) + l_G\} \\ & \cdot \{Z_{0R}Z_{LR}(c_R + c_m) + l_R\} \\ & - \{l_m - c_m Z_{0G}Z_{LR}\}\{l_m - c_m Z_{LG}Z_{0R}\}] \\ & + j\omega LSC[(Z_{0R} + Z_{LR})\{Z_{LG}Z_{0G}(c_G + c_m) + l_G\} \\ & + (Z_{0G} + Z_{LG})\{Z_{0R}Z_{LR}(c_R + c_m) + l_R\}].\end{aligned} \tag{18}$$

Similarly, from (13) one can obtain, using $V_R(0) = -Z_{0R}\, I_R(0)$,

$$V_R(0) = \frac{j\omega LSZ_{0R}}{\Delta} [C\{Z_{LR}Z_{LG}c_m + l_m\} + j\omega LS\{Z_{LR}c_m l_G + Z_{LG}l_m(c_G + c_m)\}]\, V_s \tag{19}$$

and Δ is given by (18). These equations will be simplified in the following section.

III. PROPERTIES OF THE SOLUTION

Several observations immediately arise. Note in (17) that there is the possibility that $V_R(L)$ may be identically zero for all frequencies whereas $V_R(0)$ in (19) may not be zero. Coupled transmission lines are sometimes purposely designed for this condition and are called directional couplers [4]. The condition for the directional coupler effect is that the numerator of (17) be identically zero. This results in the condition

$$Z_{0R}Z_{LG} = l_m/c_m. \tag{20}$$

From (8) and (10) one can obtain

$$l_G(c_G + c_m) = l_R(c_R + c_m) \tag{21a}$$

$$l_m(c_G + c_m) = l_R c_m \tag{21b}$$

$$l_m(c_R + c_m) = l_G c_m \tag{21c}$$

and (20) can be written, by employing (21b) and (21c), as

$$\begin{aligned} Z_{0R}Z_{LG} &= \sqrt{\frac{l_m^2}{c_m^2}} \\ &= \sqrt{\frac{l_G}{(c_R + c_m)}\frac{l_R}{(c_G + c_m)}} \\ &= Z_{CG}Z_{CR} \end{aligned} \tag{22}$$

where

$$Z_{CG} = \sqrt{\frac{l_G}{(c_G + c_m)}} \tag{23a}$$

$$Z_{CR} = \sqrt{\frac{l_R}{(c_R + c_m)}}. \tag{23b}$$

The item $Z_{CG}(Z_{CR})$ is generally referred to as the characteristic impedance of the generator (receptor) circuit in the presence of the receptor (generator) circuit [4].

Certain low-frequency approximations of the solution may also be obtained. If the line is electrically short, i.e., $\beta L \ll 1$, then the following approximations may be used in the terminal-voltage equations (17), (18), (19):

$$\begin{aligned} C &= \cos(\beta L) \\ &\cong 1 \end{aligned} \tag{24a}$$

$$\begin{aligned} S &= \frac{\sin(\beta L)}{\beta L} \\ &\cong 1. \end{aligned} \tag{24b}$$

In addition, if the termination impedances, $Z_{0G}, Z_{LG}, Z_{0R}, Z_{LR}$, are frequency-independent, i.e., purely resistive, then, for a sufficiently small frequency, Δ may be approximated by $\Delta \cong (Z_{0R} + Z_{LR})(Z_{0G} + Z_{LG})$ and (17) and (19) become[1]

$$V_R(L) \cong -\left(\frac{Z_{LR}}{Z_{0R} + Z_{LR}}\right)(j\omega l_m L)I_{G_{\mathrm{DC}}} + \left(\frac{Z_{0R}Z_{LR}}{Z_{0R} + Z_{LR}}\right)(j\omega c_m L)V_{G_{\mathrm{DC}}} \tag{25a}$$

[1] If the termination impedances are not frequency-independent, they may increase with decreasing frequency, e.g., $Z_{LG} = 1/j\omega C_{LG}$, and it may not be possible to find a sufficiently small frequency so that these approximations hold.

$$V_R(0) \cong \left(\frac{Z_{0R}}{Z_{0R}+Z_{LR}}\right)(j\omega l_m L)\, I_{G_{DC}} + \left(\frac{Z_{0R}Z_{LR}}{Z_{0R}+Z_{LR}}\right)(j\omega c_m L)\, V_{G_{DC}} \tag{25b}$$

where the zero-frequency (dc) values of the voltage and current of the generator circuit are

$$V_{G_{DC}} = \frac{Z_{LG}}{Z_{0G}+Z_{LG}} V_s \tag{26a}$$

$$I_{G_{DC}} = \frac{1}{Z_{0G}+Z_{LG}} V_s. \tag{26b}$$

One can obtain the same result from the lumped-circuit representation of the receptor circuit in Fig. 4.

From the low-frequency approximation in (25), it is clear that there are two contributions to each receptor voltage; a term due to the mutual inductance l_m, which will be classified as an inductive-coupling contribution, and a term due to the mutual capacitance c_m, which will be classified as a capacitive-coupling contribution. This resolution of the receptor terminal voltages into inductive- and capacitive-coupling contributions has been referred to in numerous other places and is clearly valid only for a sufficiently small frequency and resistive terminations. The justification for this resolution of the total coupling into inductive and capacitive contributions is generally argued on intuitive grounds.

Another property which is generally argued on intuitive grounds is the dominance of one portion of the coupling over another. For example, it is generally argued that for "high-impedance loads" the capacitive coupling dominates the inductive coupling and vice versa for "low-impedance loads." The low-frequency approximation in (25) clearly demonstrates what is meant by the terms "low-impedance loads" and "high-impedance loads." Clearly in (25a) the inductive-coupling contribution dominates the capacitive-coupling contribution in $V_R(L)$ if

$$l_m \gg c_m Z_{0R} Z_{LG} \tag{27}$$

which becomes (see (20) and (22))

$$\frac{Z_{LG}}{Z_{CG}}\frac{Z_{0R}}{Z_{CR}} \ll 1. \tag{28}$$

Similarly, in (25b), the inductive-coupling contribution dominates the capacitive-coupling contribution in $V_R(0)$ if

$$\frac{Z_{LG}}{Z_{CG}}\frac{Z_{LR}}{Z_{CR}} \ll 1. \tag{29}$$

Capacitive coupling dominates the inductive coupling when the above inequalities are reversed.

Clearly, the dominance of one form of coupling over another can only be discussed when the frequency is sufficiently small,

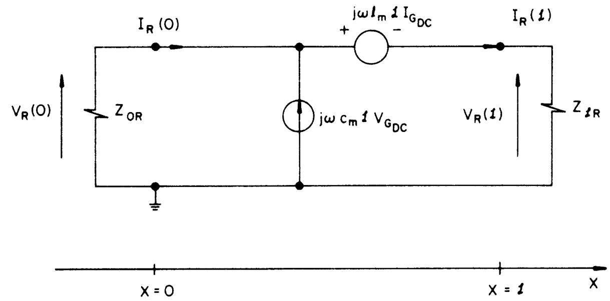

Fig. 4. The low-frequency approximation.

the line is electrically short, and the termination impedances are purely resistive. When this is the case, the dominance of one form of coupling over the other depends upon the ratio of the resistance terminating the generator circuit to the characteristic impedance of the generator circuit in the presence of the receptor circuit and the ratio of the resistance at the end of the receptor circuit opposite the voltage in question to the characteristic impedance of the receptor circuit in the presence of the generator circuit. Obviously, it is possible to have a "low-impedance" on the generator circuit, i.e., $Z_{LG} < Z_{CG}$, and have a "high impedance" on the receptor circuit, i.e., $Z_{0R} > Z_{CR}$, and have either the capacitive or the inductive contribution dominate in $V_R(L)$ depending upon the ratios Z_{CG}/Z_{LG} and Z_{CR}/Z_{0R}. Similar observations apply to $V_R(0)$.

Although the solutions for the terminal voltages are obtained in (17), (18), and (19), it is helpful to rewrite them in an equivalent form. This will illustrate certain facets of the coupling when the low-frequency approximation is not valid and will be simpler to remember. To do this, define the following quantities:

$$\tau_R = \frac{l_R L}{Z_{0R}+Z_{LR}} + (c_R + c_m)L\left(\frac{Z_{0R}Z_{LR}}{Z_{0R}+Z_{LR}}\right) \tag{30a}$$

$$\tau_G = \frac{l_G L}{Z_{0G}+Z_{LG}} + (c_G + c_m)L\left(\frac{Z_{0G}Z_{LG}}{Z_{0G}+Z_{LG}}\right) \tag{30b}$$

$$k = \frac{l_m}{\sqrt{l_G l_R}} = \frac{c_m}{\sqrt{(c_G+c_m)(c_R+c_m)}} \qquad (k \leq 1). \tag{30c}$$

The items τ_R and τ_G have the dimensions of seconds and are logically defined as time constants of the appropriate circuits if the termination impedances are purely resistive. The quantity k is logically defined as the coupling coefficient between the circuits [4]. The characteristic impedances in (23) can be written in terms of the coupling coefficient k. From (8) and (10), note that $(c_R + c_m) = l_G/v^2(l_G l_R - l_m^2) = 1/v^2 l_R(1-k^2)$ and $(c_G + c_m) = 1/v^2 l_G(1-k^2)$. Therefore, one may write

$$Z_{CG} = v l_G \sqrt{1-k^2} \tag{31a}$$

$$Z_{CR} = v l_R \sqrt{1-k^2} \tag{31b}$$

and for weakly coupled lines (small k), these become the characteristic impedances of the isolated circuits. Also, in view of the importance of the ratios of a terminating impedance to its circuit characteristic impedance in our previous discussions, define

$$\alpha_{0R} = \frac{Z_{0R}}{Z_{CR}} \quad \alpha_{LR} = \frac{Z_{LR}}{Z_{CR}}$$

$$\alpha_{0G} = \frac{Z_{0G}}{Z_{CG}} \quad \alpha_{LG} = \frac{Z_{LG}}{Z_{CG}}. \tag{32}$$

One can write, using the relations in (23) and (32),

$$\tau_R = \frac{l_R L}{Z_{0R} + Z_{LR}} (1 + \alpha_{0R}\alpha_{LR}) = (c_R + c_m)L \left(\frac{Z_{0R}Z_{LR}}{Z_{0R} + Z_{LR}}\right)\left(1 + \frac{1}{\alpha_{0R}\alpha_{LR}}\right) \tag{33a}$$

$$\tau_G = \frac{l_G L}{Z_{0G} + Z_{LG}} (1 + \alpha_{0G}\alpha_{LG}) = (c_G + c_m)L \left(\frac{Z_{0G}Z_{LG}}{Z_{0G} + Z_{LG}}\right)\left(1 + \frac{1}{\alpha_{0G}\alpha_{LG}}\right). \tag{33b}$$

Note, for example, if $\alpha_{0R}\alpha_{LR} \ll 1$, i.e., $Z_{0R}Z_{LR} \ll Z_{CR}Z_{CR}$, then the time constant of the receptor circuit is determined by the self-inductance of the receptor circuit l_R, which makes sense, intuitively, especially when both Z_{0R} and Z_{LR} are "low impedances." If $\alpha_{0R}\alpha_{LR} \gg 1$, the time constant of the receptor circuit is determined by the sum of the self capacitance and mutual capacitance $(c_R + c_m)$ of the receptor circuit.

Using the above terms, one may write the exact solution in (17), (18), and (19) in an alternate form as

$$V_R(L) = \frac{S}{\text{Den}}\left[-\left(\frac{Z_{LR}}{Z_{0R} + Z_{LR}}\right)(j\omega l_m L)I_{G_{DC}} + \left(\frac{Z_{0R}Z_{LR}}{Z_{0R} + Z_{LR}}\right)(j\omega c_m L)V_{G_{DC}}\right] \tag{34a}$$

$$V_R(0) = \frac{S}{\text{Den}}\left[\left(\frac{Z_{0R}}{Z_{0R} + Z_{LR}}\right)(j\omega l_m L) \cdot \left[C + \frac{j2\pi(L/\lambda)}{\sqrt{1-k^2}}\alpha_{LG}S\right] I_{G_{DC}} + \left(\frac{Z_{0R}Z_{LR}}{Z_{0R} + Z_{LR}}\right) \cdot (j\omega c_m L)\left[C + \frac{j2\pi(L/\lambda)}{\sqrt{1-k^2}}\frac{1}{\alpha_{LG}}S\right]V_{G_{DC}}\right] \tag{34b}$$

$$\text{Den} = C^2 - S^2\omega^2\tau_R\tau_G \cdot \left\{1 - k^2\frac{(1 - \alpha_{0G}\alpha_{LR})(1 - \alpha_{LG}\alpha_{0R})}{(1 + \alpha_{0R}\alpha_{LR})(1 + \alpha_{0G}\alpha_{LG})}\right\} + j\omega CS\{\tau_R + \tau_G\} \tag{34c}$$

where C and S are given in (15) as $C = \cos(\beta L)$, $S = \sin(\beta L)/(\beta L)$. Note that the exact value of $V_R(L)$ in (34a) is the low-frequency approximation given in (25a) and (26) multiplied by the factor S/Den. The exact value of $V_R(0)$ in (34b) yields virtually the same conclusion, except that $I_{G_{DC}}$ and $V_{G_{DC}}$ in the low-frequency approximation in (25b) and (26) are to be multiplied by the terms

$$\left(C + \frac{j2\pi(L/\lambda)}{\sqrt{1-k^2}}\alpha_{LG}S\right) \quad \text{and} \quad \left(C + \frac{j2\pi(L/\lambda)}{\sqrt{1-k^2}}\frac{1}{\alpha_{LG}}S\right)$$

respectively.

The denominator of both expressions, Den in (34c), can be simplified for weakly coupled lines, i.e., for k sufficiently small, to

$$\text{Den}_{(\text{small } k)} \cong [C^2 - S^2\omega^2\tau_R\tau_G + j\omega CS\{\tau_R + \tau_G\}] = (C + j\omega S\tau_R)(C + j\omega S\tau_G). \tag{35}$$

In this case, the denominator of the response is a simple function of the time constants of the individual circuits.

A careful study of the general equations in (34) in terms of the relationship of each termination impedance to its appropriate characteristic impedance for a particular situation will be considerably more helpful in understanding the factors affecting the response than would (17), (18), and (19).

Quite often, only the low-frequency approximations in (25) are used to obtain estimates of crosstalk, perhaps because of the simplicity of these equations. The exact equations in (34) may now be used to obtain more accurate results without any significant increase in computational complexity.

IV. SUMMARY

Explicit formulas (34) for the electromagnetic coupling within three-conductor transmission lines immersed in homogeneous media are derived. The result is exact for the TEM mode of propagation and does not require that the line be electrically short. The formulas may be used to determine crosstalk for three-conductor lines and are suitable for hand calculations as well as programmable calculators and minicomputers with limited facilities. Certain interesting properties, such as directional coupling and the separation of the response into capacitive- and inductive-coupling contributions, are easily seen from this result.

REFERENCES

[1] C. R. Paul, "Useful matrix chain parameter identities for the analysis of multiconductor transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, no. 9, pp. 756–760, Sept. 1975.

[2] C. R. Paul, "Applications of multiconductor transmission line theory to the prediction of cable coupling, Vol. I., Multiconductor transmission line theory," Tech. Rep. RADC-TR-76-101, Rome Air Development Center, Griffiss AFB, Rome, NY, Apr. 1976.

[3] C. R. Paul and A. E. Feather, "Computation of the transmission line inductance and capacitance matrices from the generalized capacitance matrix," *IEEE Trans. Electromag. Compat.*, vol. EMC-18, no. 4, pp. 175–183, Nov. 1976.

[4] R. E. Matick, *Transmission Lines for Digital and Communication Networks.* New York: McGraw-Hill, 1969.
[5] T. K. Liu, "Electromagnetic coupling between multiconductor transmission lines in a homogeneous medium," Tech. Rep. AFWL-TR-76-333, Air Force Weapons Laboratory, Kirtland AFB, NM, May 1977.

EDITORIAL SUMMARY

The problem of mutual coupling among wires has long plagued EMC engineers (see editorial summary to [3]); it normally involves the unintentional coupling of signals from one circuit into another. Often, only a low-frequency approximation is used to obtain estimates of crosstalk [4] because of simplicity of the equations for the receptor terminal voltages. In this approximate model, the crosstalk is considered to be a superposition of the portions of the coupling due to mutual capacitance ("capacitive coupling") and mutual inductance ("inductive coupling") between the generator and receptor circuits. The region of validity for this low-frequency approximation, however, is a complicated function of, not only frequency, but also the values of the termination impedances, as shown in this paper. The exact formulas (34) for the crosstalk which are derived in this paper from the transmission-line equations are placed in a form which is quite similar to, and almost as simple as, this conventional low-frequency approximation. *Therefore, these exact expressions may be used for determining crosstalk instead of the low-frequency approximations, with no significant increase in computational complexity.*

Other attempts at simplifying the crosstalk equations assume weak coupling between the generator and receptor circuits [5]; the effect of the receptor circuit on the generator circuit is disregarded. The exact formulas derived in this paper are almost as simple as the weak-coupling approximations and are not restricted to the weak-coupling case. The specific solution is obtained under certain assumptions:

1) The medium surrounding the transmission line is homogeneous, linear, isotropic, and lossless.

2) The line is uniform in the sense that there is no variation in the conductor cross sections along the line length and all conductors are parallel to each other.

3) All conductors are perfect (lossless).

These explicit assumptions simplify the solution considerably and are generally implicit in most other formulations.

Derivation of the terminal-voltage equations is for sinusoidal steady-state behavior and makes use of matrix chain parameters developed earlier [1]. A number of properties of the solution (17)-(19) are noted:

1) A condition for directional coupling is apparent.

2) Low-frequency approximations show two contributions to each receptor voltage, due to 1) mutual inductance and 2) mutual capacitance; valid only for resistive terminations.

3) They show dominance of capacitive coupling for high-impedance loads, and inductive coupling for low-impedance loads. Dominance depends upon the ratios of 1) resistance terminating the generator circuit to its characteristic impedance in the presence of the receptor circuit and (2) resistance at the end of the receptor circuit to its characteristic impedance in the presence of the generator circuit.

The last two properties have often been accepted on intuitive grounds in the past.

By defining three new quantities (30a, b, c), two of which are basically time constants of the appropriate circuits and the other is the coupling coefficient between the circuits, the basic equations (17)-(19) are put in an alternate form (34a, b, c) to display additional properties. Equation (34a) shows that the receptor voltage at the end of the receptor line is simply the low-frequency approximation multiplied by a factor which depends upon frequency, as well as other parameters. For weakly coupled lines, these parameters are simply the time constants of the individual circuits. Not only do (34) readily display these properties, but they are convenient to use for obtaining accurate results without significant increase in computational complexity over the low-frequency approximations (25). The formulas are suitable for hand calculations as well as programmable calculators and minicomputers.

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