eg: $\{a, b, c, b\}$ is not a set!

Sets: an unordered collection of distinct objects.

↳ $\{a, b, c\} = \{c, b, a\}$ is a set of size 3

↳ $\{0, 1, 2, \ldots\} = \mathbb{N}$ is an infinite set

Set operations: let A, B be finite sets:

- Cartesion Product: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$
- Size of Cartesian Product: $|A \times B| = |A| \cdot |B|$

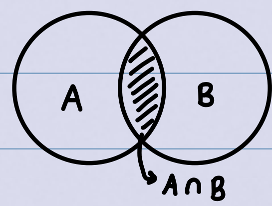
↳ eg: how many binary strings have length 4?

$\underline{0/1}\ \underline{0/1}\ \underline{0/1}\ \underline{0/1} \rightarrow 2 \times 2 \times 2 \times 2 = 2^4 = \underline{16}$.

$\#\text{strings} = |\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}| = |\{0,1\}| \times |\{0,1\}| \times |\{0,1\}| \times |\{0,1\}| = 2 \times 2 \times 2 \times 2 = 16$.

In general, the number of binary strings of length n is 2^n.

- Union: $A \cup B = \{x : x \in A \text{ or } x \in b\}$.
- Size of Union: $|A \cup B| = |A| + |B| - |A \cap B|$



↳ if $|A \cap B| = 0$, then $|A \cup B| = |A| + |B|$.

↳ $A \cup B$ with $|A \cap B| = 0$ is a disjoint union.

↳ eg: how many binary strings of length 8 are there which begin with 001 or 1011?

let S be the set of all strings.

let A_1 = set of length-8 binary strings that begin with 001.

let A_2 = set of length-8 binary strings that begin with 1011.

then $S = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. So, $|S| = |A_1| + |A_2| = 2^5 + 2^4 = 48$.

Counting Subsets

Permutations: a permutation of a set S is an ordered listing of the elements in S.

↳ eg: the permutations of $\{a, b, c\}$ are abc, acb, bac, bca, cab, cba.

- the number of permutations of a set S of size n is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{2 \text{ choices}} \times \underline{1 \text{ choice}} = n!$$

Partial Permutations: a permutation of a subset of S.

- number of partial permutations of S (with $|S| = n$) of size k is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{n-k+2 \text{ choices}} \times \underline{n-k+1 \text{ choices}}$$

$\therefore\ n(n-1)(n-2)\dots(n-k+2)(n-k+1).$

↳ note: this works even if $k > n$.

How many subsets of S (with $|S| = n$) are there of size k?

- number of partial permutations of S of size k is $n(n-1)(n-2)\dots(n-k+1)$.
- each size-k subset of S has $k!$ permutations.

↳ $\therefore\ \dfrac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot \dfrac{(n-k)!}{(n-k)!} = \dfrac{n!}{k!(n-k)!} = \dbinom{n}{k}$

- **Combinations:** let $0 \le k \le n$. Then "n choose k" is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ (aka, binomial coefficient)

↳ note, if $k > 1$, then $\binom{n}{k} = 0$.

eg: $\dbinom{n}{0} = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{0!n!} = \dfrac{1}{0!} = 1.$

eg: $\dbinom{n}{n} = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!\,0!} = \dfrac{1}{0!} = 1.$

eg: $\dbinom{10}{3} = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$

eg: $\binom{1999}{73} = \dfrac{1999 \times 1998 \times 1997 \times 1927}{73 \times 72 \times \dots \times 1}$ = Some positive integer.

Binomial Theorem: for any $n \geq 1$, $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

eg: $(x+1)^3 = (x+1)(x+1)(x+1) = x^3 + 3x^2 + 3x + 1.$

$\binom{3}{3}=1 \quad \binom{3}{2} \quad \binom{3}{1} \quad \binom{3}{0}=1.$

Combinatorial Proof of the Binomial Theorem:

$(x+1)^n = (x+1)(x+1)\dots(x+1)$ ← n copies of $(x+1)$

how do we get an x^k term? by selecting x from k binomials and 1 from the other $n-k$ binomials. ∴, the total number of ways is $\binom{n}{k}$.

Therefore, $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. $\square$.

Combinatorial Proofs

- **Procedure to prove $M = N$:**

1) "Cleverly" choose a set S
2) Count the number of elements in S in two ways:
   a) $|S| = M$ and
   b) $|S| = N$
3) Conclude that $M = N$.

Claim: $2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

↳ note: algebraically, we can prove this by substituting $x=1$ into the binomial theorem.

Combinatorial proof:

- let S be the set of all subsets of $\{1, 2, \dots, n\}$.

a) let S_i be the subsets in S of size i, where $0 \leq i \leq n$.

Then $S = S_0 \cup S_1 \cup \ldots \cup S_n$ (note, a disjoint union).

$\therefore$, $|S| = |S_0| + |S_1| + |S_2| + \ldots + |S_n|$

$$|S| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}.$$

b) to choose a subset X of $\{1, 2, \ldots, n\}$, we do:

- for each i, $1 \leq i \leq n$, we include i in X, or not.
- Therefore, the total number of choices is 2^n. Hence $|S| = 2^n$.

$\therefore$, $|S| = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. $\square$.

Claim: let $1 \leq k \leq n$. Then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ [Pascal's Identity].

- Algebraic Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!\,(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} + \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} = \frac{(n-1)! \cdot (n-k)}{k!\,(n-k)!} + \frac{(n-1)! \cdot k}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,[(n-k)+k]}{k!\,(n-k)!} = \frac{(n-1)! \cdot n}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}. \quad \square.$$

- Combinatorial Proof:

let S = set of all size-k subsets of $\{1, 2, 3, \ldots, n\}$.

then, $|S| = \binom{n}{k}$ by definition.

let A = size-k subsets that don't include n.

let B = size-k subsets that includes n.

Then $S = A \cup B$ is a disjoint union. So, $|S| = |A| + |B|$.

Now, $|A| = \binom{n-1}{k}$, and $|B| = \binom{n-1}{k-1}$. Thus, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. $\square$.

- Claim: for $k, n \geq 0$, $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n-1}{n-1} + \binom{n}{n-1} + \ldots + \binom{n+k-1}{n-1}$

Combinatorial Proof:

$\therefore |S| = \binom{n+k}{n}$

- let S = the size-n subsets of $\{1, 2, \dots, n, n+1, \dots, n+k-1, n+k\}$.
  - ↳ note: the largest number in a size-n subset is between n and $n+k$.

for each $0 \le i \le k$, let S_i = subset whose largest element is $n+i$.

Then, S = disjoint union $S_0 \cup S_1 \cup \dots \cup S_k$. So, $|S| = \sum_{i=0}^{k} |S_i| = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$. □.

Visually:

See that the LHS is size-n subsets of $n+k$ sets. So, we choose subsets of size n (k times):

Subsets: $\{\overbrace{\qquad\qquad}^{n\text{ size}}\} \cup \{\overbrace{\qquad\qquad}^{n}\} \cup \dots \cup \{\overbrace{\qquad\qquad}^{n}\} \cup \{\overbrace{\qquad\qquad}^{n}\}$

$\cdot\ \{\qquad\qquad\} \cup \{\qquad\qquad\} \cup \dots \cup \{\qquad\qquad\} \cup \{\overbrace{\dots, n+k}^{n}\}$

↳ see that the last subset can be counted as $\binom{n+k-1}{n-1}$.

Claim: for $0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$

↳ eg: $\binom{100}{98} = \binom{100}{2} = \frac{100 \times 99}{2}$.

eg: $n = 4$, $k = 1$: size-1 subsets of $\{1, 2, 3, 4\} = \{1\}, \{2\}, \{3\}, \{4\}$.

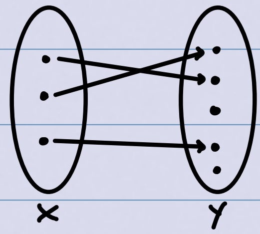
size-3 subsets of $\{1, 2, 3, 4\} = \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}$.

we can pair these sets by picking the size-3 subset which does not contain the element in the size-1 subset. ie: $\{1\} \leftrightarrow \{2, 3, 4\}$, $\{2\} \leftrightarrow \{1, 3, 4\}$...

→ aka, a correspondence or a bijection!

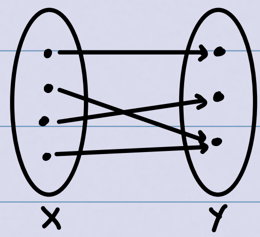
Bijections: let $f: X \to Y$ be a function.

- f is **injective (1-1)** if $\forall x_1, x_2 \in X,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
- Visually:



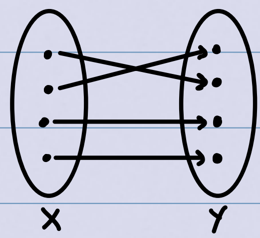
. see that $|X| \leq |Y|$.

- f is **surjective (onto)** if $\forall y \in Y$, $\exists x \in X$ st $f(x) = y$.
- Visually:



. see that $|X| \geq |Y|$

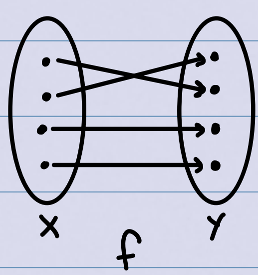
- f is **bijective** if it is injective AND surjective.
- Visually:

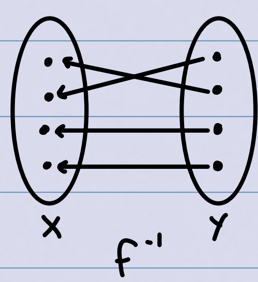


. see that $|X| = |Y|$.

- Theorem: let X, Y be finite sets. Suppose $f: X \to Y$ is a bijection. then, $|X| = |Y|$.

- **Inverse function:** the inverse of a function $f: X \to Y$ is a function $f^{-1}: Y \to X$ such that:
  1) (f^{-1} reverses f) $\Rightarrow \forall x \in X,\ f^{-1}(f(x)) = x$.
  2) (f reverses f^{-1}) $\Rightarrow \forall y \in Y,\ f(f^{-1}(y)) = y$.





• Theorem: let $f: X \to Y$. Then f is a bijection if and only if f has an inverse.

Combinatorial Proofs Using Bijections that $M = N$.

1) Select two sets X, Y, with $|X| = M$ and $|Y| = N$.
2) Define $f: X \to Y$. (f is a bijection).
3) Define $f^{-1}: Y \to X$.
4) Prove that f^{-1} is the inverse function
   ↳ prove the two conditions of the inverse function definition are satisfied.
5) Conclude that $M = N$.

• A function $f: X \to Y$ is well-defined if $\forall x \in X$, $f(x) \in Y$.

Claim: for $0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$ (continued)

let X be the set of all size-k subsets of $\{1, 2, \cdots, n\}$.
$\therefore$, $|X| = \binom{n}{k}$.

let Y be the set of all size-$(n-k)$ subsets of $\{1, 2, \ldots, n\}$.
$\therefore$, $|Y| = \binom{n}{n-k}$.

size-k subset

Define $f: X \to Y$ as follows: $\forall S \in X$, $f(S) = S^c$ ($S^c = \{1, 2, \ldots n\} - S$).
• See that f is well-defined, ie, $f(S) = S^c \in Y$ since $|S^c| = n-k$

size-(n-k) subset

Define $f^{-1}: Y \to X$ as follows: $\forall T \in Y$, $f^{-1}(T) = T^c$ ($T^c = \{1, 2, \ldots, n\} - T$).
• see that f^{-1} is well-defined, ie, $f^{-1}(T) = T^c \in X$ since $|T^c| = k$.

now, $\forall S \in X$, $f^{-1}(f(S)) = f^{-1}(S^c) = (S^c)^c = S$.
$\forall T \in Y$, $f(f^{-1}(T)) = f(T^c) = (T^c)^c = T$.

$\therefore$, f is a bijection, and therefore $|X| = |Y|$, so $\binom{n}{k} = \binom{n}{n-k}$.

Generating Series

↳ we'll encode solutions to counting problems as coefficients of a "generating series".

• eg: how many subsets of $\{1,2,3\}$ have size n, $\forall n \; 0 \le n \le 3$?

let S = all size-n subsets of $\{1,2,3\}$ $\forall n \; 0 \le n \le 3$.

define weight function $\omega: S \to \mathbb{N}$, by $\omega(\sigma) = |\sigma| \; \forall \sigma \in S$.

$\sigma \in S$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2,3\}$
$\omega(\sigma)$	0	1	1	1	2	2	2	3

associate each element $\sigma \in S$ with the term $x^{\omega(\sigma)}$:

$x^{\omega(\sigma)}$	1	x	x	x	x^2	x^2	x^2	x^3

the generating series of X with respect to ω is the sum of the $x^{\omega(\sigma)}$ terms:

3 elements in S have weight 1

$$\Phi_S^{\omega}(x) = 1 + 3x + 3x^2 + x^3 = (x+1)^3.$$

3 elements in S have weight 2

$\therefore$, the number of size-n subsets of $\{1,2,3\}$ is the coefficient of x^n in $\Phi_S^{\omega}(x)$, $\forall \, 0 \le n \le 3$.

(of any length!)

•eg: how many binary strings don't have 000 or 00111 as a substring?

· let S = all binary strings with no 000 or 00111 as a substring.

↳ we want to organize these strings by their length, so:

define $w(\sigma)$ be the length of σ (where $\sigma \in S$).

Definition: let S be a set.

- A function $w: S \to \mathbb{N}$ is a weight function if $\forall n \in \mathbb{N}$ there are only finitely many elements in S of weight n.
- The generating series for S with respect to w is $\Phi_w^S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$
- The coefficient of x^n in a $\Phi_w^S(x)$ is denoted $[x^n]\Phi_w^S(x)$.

↳ eg: $[x^{73}](x+1)^{97} = \binom{97}{73} \rightarrow \binom{97}{73}x^{73}$.

↳ result: $[x^n]\Phi_w^S(x)$ is the number of elements in S of weight n.

Eg: S = all subsets from $\{1, 2, \ldots, n\}$.

For $\sigma \in S$, $w(\sigma) = |\sigma|$.

Then, $\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n$

So, $[x^i]\Phi_S^w(x) = \binom{n}{i}$.

Eg: S = set of all binary strings.

for $\sigma \in S$, let $w(\sigma)$ = length of σ.

\# binary strings of length 1

Then, $\Phi_S^w(x) = ? + ?x + ?x^2 + ?x^3 + \ldots$

\# binary strings of length 2.

\# binary strings of length 0

$\therefore$, $\Phi_S^w(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$

Formal Power Series (FPS)

- An FPS is an expression of the form $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k \geq 0} a_k x^k$, where $a_i \in \mathbb{R}$.

↳ we don't care about convergence or divergence - we only care about the coefficients.

- So, x is called an "indeterminate".

Operations on Formal Power Series

let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k\geq 0} a_k x^k$
let $B(x) = b_0 + b_1 x + b_2 x^2 + \ldots = \sum_{k\geq 0} b_k x^k$.
$\Big\}$ FPSs.

1) $(=)$: $A(x) = B(x) \iff a_k = b_k \;\; \forall k \geq 0$.

2) $(+)$: $A(x) + B(x) = \sum_{k\geq 0} (a_k + b_k) x^k$

2b) $(-)$: $A(x) - B(x) = \sum_{k\geq 0} (a_k - b_k) x^k$

3) $(\times)$: $A(x) \cdot B(x) = C(x) = \sum_{n\geq 0} c_n x^n$, where $c_n = a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0 = \sum_{i=0} a_i b_{n-i}$

4) (Inversion): the inverse of a FPS $A(x)$, if it exists, is a new FPS $B(x)$ such that $A(x)B(x) = 1$.

↳ we'll write $B(x) = A(x)^{-1} = 1/A(x)$.

Example: show that $(1-x)^{-1} = 1 + x + x^2 + \ldots = \sum_{k\geq 0} x^k$.

let $P(x) = \text{RHS} = \sum_{k\geq 0} x^k$.

Then, $(1-x)P(x) = (1-x)\sum_{k\geq 0} x^k = \sum_{k\geq 0} x^k - x\sum_{k\geq 0} x^k = \sum_{k\geq 0} x^k - \sum_{k\geq 0} x^{k+1}$

re-index the Σ

$= \sum_{k\geq 0} x^k - \sum_{k\geq 1} x^k = 1 + \left(\sum_{k\geq 1} x^k - \sum_{k\geq 1} x^k\right) = 1 + 0 = 1$.

Common Formal Power Series

1) $(1-x)^{-1} = \frac{1}{1-x} = \sum_{k\geq 0} x^k \rightarrow [x^n](1-x)^{-1} = 1 \;\forall n \geq 0.$

↳ geometric series!

partial bc not infinite!

2) partial geometric series: $1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$.

bc if $k>n$, $\binom{n}{k}=0$ anyway.

3) binomial series: $(x+1)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}x^k$. $\therefore, [x^k](x+1)^n = \binom{n}{k}$.

4) negative binomial series: $\forall n \geq 1$, $(1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k\geq 0}\binom{n+k-1}{k}x^k$

↳ so $[x^k](1-x)^{-n} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

Example: determine $[x^n]\left(\frac{x}{1-2x}\right)^5$.

$[x^n]\left(\frac{x}{1-2x}\right)^5 = [x^n]\, x^5(1-2x)^{-5} = [x^{n-5}](1-2x)^{-5}$ for $n \geq 5$.

Then, see that $(1-x)^{-5} = \sum_{k\geq 0} a_k x^k$. So, $(1-2x)^{-5} = \sum_{k\geq 0} a_k x^k 2^k$.

$= 2^{n-5}[x^{n-5}](1-x)^{-5} = 2^{n-5}\binom{n-5+5-1}{5-1} = 2^{n-5}\binom{n-1}{4}$.

Proving the negative binomial theorem $(1-x)^{-n} = \sum_{k\geq 0}\binom{n+k-1}{k}x^k$, $n \geq 1$.

Combinatorial Proof (sketch):

$(1-x)^{-n} = (1-x)^{-1}\cdot(1-x)^{-1}\cdot\ldots\cdot(1-x)^{-1}$ (n times).

$= (1+x+x^2+\dots)\cdot(1+x+x^2+\dots)\cdot\ldots\cdot(1+x+x^2+\dots)$ (n times)

· we get an x^k term by selecting x^{a_1} from the first series, x^{a_2} from the second series, ..., and x^{a_n} from the last series, where $a_1, a_2, \dots, a_k$ are non-negative integers with $a_1+a_2+\dots+a_n = k$. Then, multiplying the x^{a_i} terms gives $x^{a_1}x^{a_2}\dots x^{a_n} = x^{a_1+a_2+\dots+a_n} = x^k$.

- The number of ways of choosing the a_i's is equal to the coefficient of x^k in $(1-x)^{-n}$.

- Illustration by example: $n=4$, $k=13$:
  - begin with a string of $k=13$ 0s and then insert $n-1=3$ 1s.

    ↳ $\underbrace{000}_{a_1=3}1\underbrace{0000}_{a_2=4}1\underbrace{00}_{a_3=2}1\underbrace{0000}_{a_4=4}.$

    note that $a_1+a_2+a_3+a_4=3+4+2+4=13$!
  - this gives a bijection between binary strings of length $k+n-1$ with exactly $n-1$ ones, and non-negative integers $a_1, a_2, \dots, a_n$ st $\sum_{i=1}^{n} a_i = k$.
  - ∴, the number of non-negative integers $a_1, \dots, a_k$ with $\sum_{i=1}^{n} a_i = k$ is equal to the number of binary strings of length $k+n-1$ with exactly $n-1$ ones, which is $\binom{k+n-1}{n-1} = \binom{n+k-1}{k}$.

  - ∴, the coefficient of x^k in $(1-x)^{-n}$ is $\binom{n+k-1}{k}$. □.

Extracting Coefficients from an FPS:

- let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$.

1) $[x^n](A(x) \pm B(x)) = [x^n]A(x) + [x^n]B(x)$.

2) $[x^n](A(x)\cdot B(x)) = \sum_{i=0}^{n} a_i b_{n-1} = \sum_{i=0}^{n} ([x^i]A(x))([x^{n-i}]B(x))$

3) $[x^n]cA(x) = c[x^n]A(x)$, where c is a constant.

4) $[x^n]x^l A(x) = 0$ if $n < l$, and $[x^n]x^l A(x) = [x^{n-l}]A(x)$ if $n \geq l$.

5) $[x^n]A(cx) = c^n [x^n]A(x)$, where c is a constant

6) $[x^n]A(x^\ell)=0$ if $\ell \nmid n$, and $[x^n]A(x^\ell)=[x^{n/\ell}]A(x)$ if $\ell \mid n$.

Example: determine $[x^n](x^3(3x+1)^7)$ where $n \geq 3$.

$$[x^n]\,x^3(3x+1)^7 = [x^{n-3}](3x+1)^7 = 3^{n-3}[x^{n-3}](x+1)^7$$

$$= 3^{n-3}\binom{7}{n-3}.$$

Example: determine $[x^n]\,x^3(3x+1)^7(1-4x^2)^{-m}$, $m \geq 1$, $n \geq 3$.

$$[x^n]\,x^3(3x+1)^7(1-4x^2)^{-m}$$

$$= [x^n]\,x^3\Big(\sum_{i\geq 0}\binom{7}{i}(3x)^i\Big)\Big(\sum_{j\geq 0}\binom{m+j-1}{j}(4x^2)^j\Big)$$

$$= [x^n]\sum_{i\geq 0,\, j\geq 0}\binom{7}{i}\binom{m+j-1}{j}3^i\,4^j\,x^{3+i+2j}$$

$$= \sum_{i\geq 0,\, j\geq 0}\binom{7}{i}\binom{m+j-1}{j}3^i\,4^j$$, and only care where $3+i+2j=n$.

↳ $\therefore$, $i = n-2j-3$, $j = \frac{n-3-i}{2}$

Substituting:

$$= \sum_{j=0}\binom{7}{n-2j-3}\binom{m+j-1}{j}3^{n-2j-3}\,4^j$$

Since $j = \frac{n-3-i}{2}$, $j \leq \frac{n-3}{2}$ (as $i \geq 0$). $\therefore$, $j \leq \lfloor \frac{n-3}{2} \rfloor$

$$= \sum_{j=0}^{\lfloor \frac{n-2}{3} \rfloor}\binom{7}{n-2j-3}\binom{m+j-1}{j}3^{n-2j-3}\,4^j.$$

Generating Series: Counting

example: how many length-64 binary strings have neither 000 nor 00111 as a substring?

- don't know how to do this for specifically length-64. So, let's generalize for length-n and make a generating series.

- Roadmap:
  1) Let S be a set we wish to count by weight
  2) Decompose S into "simpler" sets using disjoint union and cartesian product
  3) Determine the generating series for the simpler sets
  4) Combine the generating series $\Phi_{S_i}(x)$ to get $\Phi_S(x)$.
  5) The number of elements of weight n in S is $[x^n]\Phi_S(x)$.

- Sum Lemma: let $S = A \cup B$ be a disjoint union. Let ω be a weight function $S \to \mathbb{N}$.

Then, $\Phi_S^\omega(x) = \Phi_A^\omega(x) + \Phi_B^\omega(x)$.

↳ Proof:

$$\Phi_S^\omega(x) = \sum_{\sigma \in S} x^{\omega(\sigma)} = \sum_{\sigma \in A \cup B} x^{\omega(\sigma)} = \sum_{\sigma \in A} x^{\omega(\sigma)} + \sum_{\sigma \in B} x^{\omega(\sigma)} = \Phi_A^\omega(x) + \Phi_B^\omega(x).$$

- Product Lemma: let $S = A \times B = \{(a,b) : a \in A, b \in B\}$. Let $u : A \to \mathbb{N}$ be a weight function on A, and let $v : B \to \mathbb{N}$ be a weight function on B. Define the weight function $\omega : S \to \mathbb{N}$ as follows:
  - $\forall \sigma = (a,b) \in S,\ \omega(\sigma) = u(a) + v(b)$.

Then, $\Phi_S^\omega(x) = \Phi_A^\omega(x) \cdot \Phi_B^\omega(x)$.

↳ Proof:

$$\Phi_A^\omega(x) \cdot \Phi_B^\omega(x) = \left(\sum_{a \in A} x^{u(a)}\right)\left(\sum_{b \in B} x^{v(b)}\right) = \sum_{a \in A, b \in B} x^{u(a)+v(b)} = \sum_{\sigma \in S} x^{\omega(\sigma)} = \Phi_S^\omega(x).$$

eg: find the generating series of binary strings of length n where the weight of the string is the number of 1s in it.

- **Method 1:**

let S = all binary strings of length n.

Then, $\Phi_S^w(x) = \sum_{k\geq 0} a_k x^k$, where a_k = number of elements of weight k = number of binary strings of length n with k 1s = $\binom{n}{k}$.

$\therefore$, $\Phi_S^w(x) = \sum_{k\geq 0} \binom{n}{k} x^k = (1+x)^n$

- **Method 2:**

let $A = \{0, 1\}$. Define $u(0) = 0$ and $u(1) = 1$.

$\Phi_A^u(x) = x^{u(0)} + x^{u(1)} = x^0 + x^1 = 1 + x$.

now, we use A to describe the more complicated set S:

$S = A \times A \times A \times \dots \times A$ (n times).

define $w: S \to \mathbb{N}$ as follows:

$\sigma = (a_1, a_2, \dots, a_n)$, then $w(\sigma) = \#$ 1s in σ $= u(a_1) + u(a_2) + \dots + u(a_n) = \sum_{i=0}^{n} u(a_i)$.

→ see that the weight of the complicated set S is the sum of the weight of the less complicated set A.

$\therefore$, by the product lemma, $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_A^u(x) \cdot \ldots \cdot \Phi_A^u(x)$ (n times).

$= [\Phi_A^u(x)]^n = (1+x)^n$.

note: order matters!!

Compositions: A composition of $n \in \mathbb{N}$ is a sequence of positive integers $(a_1, a_2, \dots, a_k)$ that add to n. k = # of parts.

- example: the compositions of n = 4 are:

(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2), (1,3), (3,1), (4).

• note: $n=0$ has one composition, ().

How many k-part compositions of n are there?

let S = the set of all compositions with k parts.
Define $w: S \to \mathbb{N}$ as follows:
if $\sigma = (a_1, a_2, \dots, a_k)$, then $w(\sigma) = a_1 + a_2 + a_3 + \dots + a_k$.
Goal: find $\Phi_S^w(x)$, and the answer will be $[x^n]\, \Phi_S^w(x)$.

let $P = \{1, 2, 3, 4, \dots\}$.
then, $S = P \times P \times P \times \dots \times P$ (k times) $= P^k$ cartesian product of k copies of P.

define $u: P \to \mathbb{N}$ by $u(a) = a$ $\forall a \in P$.
Then, $w(\sigma) = a_1 + a_2 + \dots + a_k = u(a_1) + u(a_2) + \dots + u(a_k)$.

by the product lemma, $\Phi_S^w(x) = \Phi_P^u(x) \times \dots \times \Phi_P^u(x)$ (k times)
$= [\Phi_P^u(x)]^k$.

Now, $\Phi_P^u(x) = \sum_{a \in P} x^{u(a)} = x + x^2 + x^3 + x^4 + \dots = x(1 + x + x^2 + \dots) = x(1-x)^{-1}$.

So, $\Phi_S^w(x) = [\Phi_P^u(x)]^k = x^k (1-x)^{-k}$

$\therefore$, the number of k-part coefficients of n is $[x^n]\, x^k (1-x)^{-k}$
$= [x^{n-k}](1-x)^{-k} = \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$.

String Lemma: let S be a set with weight function w.
Suppose that S has no elements of weight 0. Then,

$$\Phi_{S^*}^{w^*}(x) = \frac{1}{1-\Phi_S^w(x)}.$$

Proof: let S be a set and $w: S \to \mathbb{N}$ it's weight function.
let S^k denote $S \times S \times S \times \ldots \times S$ (k times) $(S^0 = \{\})$.
Define $S^* = S^1 \cup S^2 \cup S^3 \cup \ldots = \bigcup_{k \geq 0} S^k$ is a disjoint union.
Define $w_k: S^k \to \mathbb{N}$: if $(a_1, a_2, \ldots, a_k) \in S^k$, then $w_k = w(a_1) + \ldots + w(a_k)$.
Define $w^*: S^* \to \mathbb{N}$: $w^*(\sigma) = w_k(\sigma)$ where $\sigma \in S^k$.
→ we assume that S has no elements of weight 0!!

Then, $\Phi_{S^*}^{w^*}(x) = \sum_{k \geq 0} \Phi_{S^k}^{w_k}(x)$ by the sum lemma.

Since $\Phi_{S^k}^{w_k}(x) = \prod_{i=1}^{k} \Phi_S^w(x) = [\Phi_S^w(x)]^k$.

We have $\Phi_{S^*}^{w^*}(x) = \sum_{k \geq 0} [\Phi_S^w(x)]^k = \frac{1}{1-\Phi_S^w(x)}$.

How many compositions of n are there where each part is ≥ 2 ?

let S = all compositions where each part is ≥ 2.
let $P = \{2, 3, 4, \ldots\}$
Then, $S = P^0 \cup P \cup P^2 \cup P^3$. So, $S = \bigcup_{k \geq 0} P^k$, $\therefore$ $S = P^*$.

By the string lemma, $\Phi_S(x) = \frac{1}{1-\Phi_P(x)}$.

$$\begin{aligned} \text{Now, } \Phi_P(x) &= x^2 + x^3 + x^4 + \ldots \\ &= x^2(1 + x + x^2 + \ldots) \\ &= x^2(1-x)^{-1} \end{aligned}$$

$\therefore$, $\Phi_S(x) = \frac{1}{1-\frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}$. So, $[x^n] \frac{1-x}{1-x-x^2}$

A formal power series $A(x)$ is rational if there exist polynomials $P(x)$, $Q(x)$ such that $A(x) = P(x)/Q(x)$.

- we extract coefficients from ration power series using recurrences and partial fractions.

How many compositions have an odd number of parts, where each part $\equiv 1 \pmod 3$.

Let S = the set of all compositions with an odd number of parts, with each part $\equiv 1 \pmod 3$.

Let $P = \{1, 4, 7, 10, 13, \ldots\}$ (ie, $P = \{x : x \equiv 1 \pmod 3\}$).

Then, $S = P^1 \cup P^3 \cup P^5 \cup P^7 \cup \ldots$ since $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$$= P \times (P^0 \cup P^2 \cup P^4 \cup P^6 \cup \ldots)$$
$$= P \times (P^2)^*$$

$\therefore$, $\Phi_S(x) = \Phi_P(x) \cdot \Phi_{(P^2)^*}(x)$ by the product lemma

$$= \Phi_P(x) \cdot \frac{1}{1 - \Phi_{P^2}(x)}$$ by the string lemma

$\therefore$, $\Phi_P(x) = x + x^4 + x^7 + x^{10} + \ldots = \dfrac{x}{1 - x^3}$

$\Phi_{P^2}(x) = \Phi_P(x) \cdot \Phi_P(x)$ by the product lemma

$$= \frac{x^2}{(1-x^3)^2}$$

So, $\Phi_S(x) = \dfrac{x}{1-x^3} \cdot \dfrac{1}{1 - \frac{x^2}{(1-x^3)^2}} = \dfrac{x}{1-x^3} \cdot \dfrac{(1-x^3)^2}{(1-x^3)^2 - x^2} = \dfrac{x - x^4}{1 - x^2 - 2x^3 + x^6}$.

$\therefore$, $[x^n] \dfrac{x - x^4}{1 - x^2 - 2x^3 + x^6}$

Binary Strings

A binary string σ is a sequence of bits $b_1 b_2 \dots b_n$, where $b_i = 0$ or 1 for $1 \leq i \leq n$, and n is the length of σ.

- There's only one binary string of length 0: the empty string ε.
- We will (almost) always use the length of a binary string as it's weight!

If $a = a_1 a_2 \dots a_m$, $b = b_1 b_2 \dots b_n$ are binary strings, their concatenation is $ab = a_1 a_2 \dots a_m b_1 b_2 \dots b_n$.

↳ note the length of ab is $m+n$.

↳ also, $\varepsilon\sigma = \sigma$ $\forall$ strings σ.

a is a substring of b if $b = cad$ for some strings c and d.

$\overbrace{00}\underbrace{111}\overbrace{0}\underbrace{1}\overbrace{000000}\underbrace{1}$ → blocks

A block of a binary string is a non-empty maximal substring of all 0s or all 1s.

Regular Expressions: a method for generating/producing a set of binary strings.

Let A, B be binary strings.

1) $A \cup B$ is the union of A and B

↳ eg: $A = \{00\}$, $B = \{1, 11, 111\}$. $A \cup B = \{00, 1, 11, 111\}$ or $00 \cup (1 \cup 11 \cup 111)$.

2) $AB = \{ab : a \in A, b \in B\}$ is the concatenation of A and B

↳ eg: $A = \{1, 11\}$, $B = \{001\}$. $AB = \{1001, 11001\}$, or $(1 \cup 11)001$.

3) $A^* = \bigcup_{k \geq 0} A^k$, where $A^k = \underbrace{AAA \dots A}_{k \text{ times}}$ and $A^0 = \{\varepsilon\}$

↳ eg: 0^* → all 0-strings of any length (including 0).

Examples: $(00)^* = \{\varepsilon, 00, 0000, 000000, \ldots\}$.

$0(00)^*$ $(= (00)^*0)$ = all odd-length strings of 0s

$(0 \cup 1)^*$ = all binary strings!

$(0 \cup 111)^*$ = all binary strings where each block of 1s has length a multiple of 3.

- Ideally, we'd like to use the Sum, Product, and String lemmas to find the generating series for a set of strings produced by a regular expression. But you have to be careful!

eg: let $A = \{1, 0\}$, $B = \{1, 01\}$

Then $AB = \{11, 101, 01, 001\}$

$A \times B = \{(1,1), (1,01), (0,1), (0,01)\}$.

same(ish)

eg: let $A = \{1, 10\}$, $B = \{1, 01\}$

→ sets can't have duplicates!

Then $AB = \{11, 101, \cancel{101}, 1001\}$

$A \times B = \{(1,1), (1,01), (10,1), (10,01)\}$.

not same!

So, $|AB| \neq |A \times B|$!! Issue when we're trying to count. Removing commas from $A \times B$ might cause ambiguity.

A regular expression is unambiguous if every string generated by the expression can be generated in exactly one way.

↳ eg: $(1 \cup 0)(1 \cup 01)$ is unambiguous.

$(1 \cup 10)(1 \cup 01)$ is ambiguous, since 101 can be generated in two ways: $\underbracket{1}\underbracket{01}$ and $\underbracket{10}\underbracket{1}$.

Example: Is $0^* \cup 1^*$ ambiguous or not?

↳ ambiguous, because the empty string ε can be generated in two ways. (So, $\Phi_{0^* \cup 1^*} \neq \Phi_{0^*} + \Phi_{1^*}$).

Example: $S = (0^*11 \cup 001^*)^*$. Is this an ambiguous expression?
↳ ambiguous, since 0011 can be generated in two ways.

Let A, B be sets of strings.
1) Then $A \cup B$ is unambiguous if the union is disjoint (ie $A \cap B = \emptyset$).
2) Then AB is unambiguous if $\forall \sigma \in AB$, there's exactly one pair of strings $a, b \in A \times B$ such that $\sigma = ab$.
3) Then A^* is unambiguous if A^k is unambiguous $\forall k$, and $A^0, A^1, A^2, \ldots$ are disjoint.

- If $A \cup B$ is unambiguous, then $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$. SUM LEMMA
- If AB is unambiguous, then $\Phi_{AB}(x) = \Phi_A(x) \cdot \Phi_B(x)$. PRODUCT LEMMA
- If A^* is unambiguous, then $\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$ STRING LEMMA

Counting Binary Strings

eg: find the generating series for all binary strings where weight = length.

1) $\Phi_S(x) = \sum_{n \geq 0} a_n x^n$, where $a_n = \#$ strings of length $n = 2^n$.

$$= \sum_{n \geq 0} 2^n x^n = \frac{1}{1 - 2x}.$$

2) $(0 \cup 1)^*$ is an unambiguous expression for S.
So, $\Phi_S(x) = \Phi_{(0 \cup 1)^*}(x) = \frac{1}{1 - \Phi_{(0 \cup 1)}(x)}$ by string lemma

$$= \frac{1}{1 - (x^1 + x^1)} = \frac{1}{1 - 2x}$$

3) Claim: $1^*(00^*11^*)^*0^*$ is an unambiguous expression for S.

eg: $\underbrace{}_{1^*}\underbrace{000011}_{00^*11^*}\underbrace{011}_{00^*11^*}\underbrace{001}_{00^*11^*}\underbrace{011}_{00^*11^*}\underbrace{00}_{0^*}$

justification: the decomposition of a string into it's blocks is unique.
So, $\Phi_S(x) = \Phi_{1^*(00^*11^*)0^*}(x)$

- $\Phi_{1^*}(x) = \frac{1}{1-\Phi_1(x)} = \frac{1}{1-x} = \Phi_{0^*}(x)$.

- $\Phi_{00^*11^*}(x) = \Phi_0(x)\cdot\Phi_{0^*}(x)\cdot\Phi_1(x)\cdot\Phi_{1^*}(x)$
$= x\cdot\frac{1}{1-x}\cdot x\cdot\frac{1}{1-x} = \frac{x^2}{(1-x)^2}$

So, $\Phi_{(00^*11^*)^*}(x) = \frac{1}{1-\Phi_{00^*11^*}(x)} = \frac{1}{1-\frac{x^2}{(1-x)^2}} = \frac{(1-x)^2}{(1-x)^2-x^2}$

$\therefore$, $\Phi_{1^*(00^*11^*)^*0^*}(x) = \Phi_{1^*}(x)\cdot\Phi_{(00^*11^*)^*}(x)\cdot\Phi_{0^*}(x)$
$= \frac{1}{1-x}\cdot\frac{(1-x)^2}{(1-x)^2-x^2}\cdot\frac{1}{1-x} = \frac{1}{(1-x)^2-x^2} = \frac{1}{1-2x}$.

eg: find the generating series for all strings where every block of 0s has even length and must be followed by a block of at least 3 1s.

let S = set of all binary strings where every block of 0s has even length and is followed by a block of at least 3 1s.

idea: begin with $1^*(00^*11^*)^*0^*$. If we replace any portion of this block decomposition with an unambiguous expression for a non-empty subset of that portion, then the new expression is also unambiguous.

$$1^*(00^*11^*)^*0^*$$
$$1^* \quad 00(00)^* \quad 1111^* \quad \varepsilon$$

$\therefore$, $1^*(00(00)^*1111^*)^*$ is an unambiguous expression for S.

$$\Phi_S(x) = \frac{1}{1-x} \cdot \frac{1}{1-\left(x^2 \cdot \frac{1}{1-x^2} \cdot x^3 \cdot \frac{1}{1-x}\right)} = \frac{1}{(1-x)\left(1 - \frac{x^5}{(1-x^2)(1-x)}\right)}$$

$$= \frac{(1-x)(1-x^2)}{(1-x)((1-x^2)(1-x)-x^5)} = \frac{1-x^2}{1-x-x^2+x^3-x^5}$$

Let A, B be sets such that $B \subseteq A$. Then, $\Phi_{A \setminus B}(x) = \Phi_A(x) - \Phi_B(x)$, since $|A \setminus B| = |A| - |B|$.

Eg: how many length-n binary strings don't have 111 as a substring?

$(\varepsilon \cup 1 \cup 11)(00^*(1 \cup 11 \cup \varepsilon))^* 0^*$

Eg: how many length-n binary strings don't have 110100 as a substring?
↳ we must use recursive decomposition here!!

Recursive Decompositions

- express a set in terms of itself

Eg: find a recursive decomposition for A = all binary strings.

every string σ either begins with a 0 or a 1 followed by a binary string, except if $\sigma = \varepsilon$.

So, $A = \underbrace{\varepsilon}_{\text{base case}} \underbrace{(0 \cup 1)A}_{\text{recursive step}}$ is an unambiguous expression for A.

$\therefore$, $\Phi_A(x) = 1 + (2x \cdot \Phi_A(x))$. Solving, we get $\Phi_A(x) = \frac{1}{1-2x}$.

Eg: find a recursive decomposition which don't have 111 as a substring.

let A be the set of all strings which don't have 111 as a substring.
let $\sigma \in A$. Either σ has a 0 or it doesn't.

- If it doesn't, then $\sigma \in \{\varepsilon, 1, 11\}$.
- If it does, then either σ begins with a 0, 10, or 110, followed by a string in A.

So, $A = \underbrace{(\varepsilon \cup 1 \cup 11)}_{\text{base case}} \cup \underbrace{(0 \cup 10 \cup 110) A}_{\text{recursive step}}$ is unambiguous.

$\Phi_A(x) = (1+x+x^2) + (x+x^2+x^3)\Phi_A(x)$, so $\Phi_A(x) = \dfrac{1+x+x^2}{1-x-x^2-x^3}$.

Eg: how many length-4 binary strings have neither 000 nor 00111 as a substring?

i) find a block decomposition for the set of all such strings.
ii) find $\Phi_S(x) = \dfrac{1+x+x^2}{1-x-x^2-x^3+x^5}$
iii) answer is $[x^4]\Phi_S(x)$.

we'll see that the number of such strings of length n is given by the recurrence relation $a_n - a_{n-1} - a_{n-2} - a_{n-3} + a_{n-5} = 0 \ \forall n \geq 5$, which comes from the denominator of $\Phi_S(x)$.
So, $a_n = a_{n-1} + a_{n-2} + a_{n-3} - a_{n-5}$, but we need initial conditions s.t. $n \leq 4$.
By inspection, $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 7$, and $a_4 = 13$.
Then, $a_5 = 13 + 7 + 4 - 1 = 23$, $a_6 = 41$, $a_7 = 73$, $a_{64} = 13,076,512,262,747,676$.

Rational Power Series

(I) Rational series → recurrence relation:

eg: consider $A(x) = \dfrac{P(x)}{Q(x)} = \dfrac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \sum_{n\geq 0} a_n x^n$. Find a recurrence relation for a_n.

↳ Strategy: multiply both sides by $Q(x)$ and equate coefficients of x^n on both sides:

$$(1-8x+21x^2-18x^3)\sum_{n\geq 0} a_n x^n = 1-2x+3x^2$$

$$\hookrightarrow \sum_{n\geq 0} a_n x^n - 8\sum_{n\geq 0} a_n x^{n+1} + 21\sum_{n\geq 0} a_n x^{n+2} - 18\sum_{n\geq 0} a_n x^{n+3} = 1-2x+3x^2$$

$$\hookrightarrow \sum_{n\geq 0} a_n x^n - 8\sum_{n\geq 1} a_{n-1} x^n + 21\sum_{n\geq 2} a_{n-2} x^n - 18\sum_{n\geq 3} a_{n-3} x^n = 1-2x+3x^2$$

extracting $[x^n]$ of both sides:

$n=0$: $a_0 = 1 \Rightarrow a_0 = 1$

$n=1$: $a_1 - 8a_0 = -2 \Rightarrow a_1 = 6$

$n=2$: $a_2 - 8a_1 + 21a_0 = 3 \Rightarrow a_2 = 30$

$n\geq 3$: $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$.

$\therefore$, for $n\geq 3$, we have that $a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n+3}$.

↳ NOTE!! : $Q(x)$ was $1-8x+21x^2-18x^3$, so these coefficients will always match!

Let $a_0, a_1, a_2, \ldots$ be a sequence of complex numbers.
Let $c_1, c_2, \ldots, c_d$ be constants.
Then, a_n satisfies a (linear homogeneous) recurrence relation if
$a_n + c_1 a_{n+1} + c_2 a_{n+2} + \ldots + c_d a_{n+d} = 0 \quad \forall n \geq d$.

↳ note: $a_0, a_1, \ldots, a_{d-1}$ are the initial conditions

Theorem: let $A(x) = P(x)/Q(x)$ be a rational power series with $Q(x) = 1 + c_1 x + c_2 x^2 + \dots + c_d x^d$ AND degree $(P) <$ degree (Q)

Then the a_n satisfy a recurrence relation:

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d} \quad \forall n \geq d$$

with initial conditions $a_0, a_1, a_2, \dots, a_{d-1}$.

(II) reccurrence relation → rational series:

eg: consider $a_n - 7a_{n-1} - 16a_{n-2} + 12a_{n-3} = 0$, with $a_0 = 1$, $a_1 = 0$, and $a_2 = 2$.
find $P(x)$, $Q(x)$ such that $A(x) = \sum_{n \geq 0} a_n x^n = \frac{P(x)}{Q(x)}$.

Strategy: multiply both sides of the reccurrence relation by $A(x)$ and sum for $n \geq 3$:

$$\hookrightarrow \sum_{n\geq 3} a_n x^n - 7\sum_{n \geq 3} a_{n-1} x^n - 16 \sum_{n\geq 3} a_{n-2} x^n + 12 \sum_{n \geq 3} a_{n-3} x^n = 0$$

$$\sum_{n\geq 3} a_n x^n - 7x \sum_{n\geq 3} a_{n-1} x^{n-1} - 16x^2 \sum_{n\geq 3} a_{n-2} x^{n-2} + 12x^3 \sum_{n\geq 3} a_{n-3} x^{n-3} = 0$$

$$\sum_{n\geq 3} a_n x^n - 7x \sum_{n\geq 2} a_n x^n - 16x^2 \sum_{n\geq 1} a_n x^n + 12x^3 \sum_{n\geq 0} a_n x^n = 0$$

$$\therefore (A(x) - a_0 - a_1 x - a_2 x^2) - 7x(A(x) - a_0 - a_1 x) - 16x^2 (A(x) - a_0) + 12x^3 A(x) = 0$$

$$\hookrightarrow A(x)\left[1 - 7x - 16x^2 + 12x^3\right] = a_0 + a_1 x + a_2 x^2 - 7x(a_0 + a_1 x) - 16x^2 a_0$$

since $a_0 = 1$, $a_1 = 0$, and $a_2 = 2$, we get:

$$A(x) = [1 - 7x - 16x^2 + 12x^3] = 1 - 7x - 14x^2, \quad \text{so:}$$

$$A(x) = \frac{1 - 7x - 14x^2}{1 - 7x - 16x^2 + 12x^3}.$$

Theorem: let $a_0, a_1, a_2, \dots$ be a sequence that satisfies a reccurrence relation $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d} = 0 \quad \forall n \geq d$ with initial conditions $a_0, a_1, \dots, a_{d-1}$. Let $A(x) = \sum_{n \geq 0} a_n x^n$. Then, $A(x) = \frac{P(x)}{Q(x)}$ where:

$$Q(x) = 1 + c_1 x + c_2 x^2 + \dots + c_d x^d \text{ and } \deg(P) < d.$$

(III) Partial Fractions (rational series → "explicit formula")

eg: let $A(x) = \frac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \frac{P(x)}{Q(x)} = \sum_{n \geq 0} a_n x^n$. Find an "explicit formula" for a_n.

1) Factor $Q(x)$ into a product of linear factors:

$$Q(x) = (1-\lambda_1 x)^{d_1} (1-\lambda_2 x)^{d_2} \dots (1-\lambda_m x)^{d_m}$$

we have $Q(x) = (1-2x)(1-3x)^2$.

2) Partial Fractions : $A(x) = \frac{c_1}{1-2x} + \frac{c_2}{1-3x} + \frac{c_3}{(1-3x)^2}$

where c_1, c_2, c_3 are unique constants.

3) Find the constants (c_1, c_2, c_3):

multiply of sides of the partial series expression by $Q(x) = (1-2x)(1-3x)^2$

$\hookrightarrow c_1(1-3x)^2 + c_2(1-2x)(1-3x) + c_3(1-2x) = 1-2x+3x^2$

$c_1(1-6x+9x^2) + c_2(1-5x+6x^2) + c_3(1-2x) = 1-2x+3x^2$

equating coefficients of x^0, x^1, x^2 on both sides:

$x^0 \to c_1 + c_2 + c_3 = 1$

$x^1 \to -6c_1 - 5c_2 - 2c_3 = -2$

$x^2 \to 9c_1 + 6c_2 = 3$

solving the systems of linear equations, we get that $c_1 = 3$, $c_2 = -4$, $c_3 = 2$.

So, $A(x) = \frac{3}{1-2x} - \frac{4}{1-3x} + \frac{2}{(1-3x)^2}$

4) Extract coefficients

geometric series!!

negative binomial

$$a_n = [x^n]A(x) = [x^n]\frac{3}{1-2x} - [x^n]\frac{4}{1-3x} + [x^n]\frac{2}{(1-3x)^2}$$

$$= 3 \cdot 2^n - 4 \cdot 3^n + 2[x^n]\frac{1}{(1-3x)^2}$$

$$= 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n [x^n]\frac{1}{(1-x)^2} = 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \binom{n+2-1}{2-1}$$

$$= 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \binom{n+1}{1} = 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \cdot (n+1)$$

$$= 3 \cdot 2^n + (2n-2)3^n$$

$\therefore$, $a_n = 3 \cdot 2^n + (2n-2)3^n \quad \forall n \geq 0$.

Note: such a factorisation always exists if you allow λ_i to be complex, by the fundamental theorem of algebra!

↳ if you cannot factor $Q(x)$, then the method fails.

Notation: $Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2} \dots (1-\lambda_m x)^{d_m}$, where λ_i is an "inverse root" of $Q(x)$, and d_i is its multiplicity.

eg: suppose $A(x) = \frac{1-3x^2+7x^3}{(1+4x)^2(1-3x)^4}$

↳ $Q(x)$ has two inverse roots, -4 with multiplicity 2, and 3 with multiplicity 4.

$$A(x) = \frac{C_1}{1+4x} + \frac{C_2}{(1+4x)^2} + \frac{C_3}{1-3x} + \frac{C_4}{(1-3x)^2} + \frac{C_5}{(1-3x)^3} + \frac{C_6}{(1-3x)^4}$$

$$\text{Then } a_n = [x^n]A(x) = \underbrace{(B+Cn)}_{\text{polynomial of } n \text{ in degree } <2,\ \lambda_1}(-4)^n + \underbrace{(D+En+Fn^2+Gn^3)}_{\text{polynomial in } n \text{ of degree } <4,\ \lambda_2}\cdot 3^n$$

Theorem: let $A(x) = \frac{P(x)}{Q(x)}$, where $\deg(P) < \deg(Q)$ and the constant term of $Q(x)$ is 1. Factor $Q(x)$ into a product of linear polynomials $Q(x) = (1-\lambda_1 x)^{d_1} \dots (1-\lambda_m x)^{d_m}$ where λ_i are the inverse roots of $Q(x)$, with multiplicities d_i. Then, there exist unique constants C_{ij}, where $1 \le i \le m$, $1 \le j \le d_i$ such that:

$$A(x) = \left(\frac{C_{1,1}}{1-\lambda_1 x} + \frac{C_{1,2}}{(1-\lambda_1 x)^2} + \frac{C_{1,d_1}}{(1-\lambda_1 x)^{d_1}}\right) + \dots + \left(\frac{C_{m,1}}{1-\lambda_m x} + \frac{C_{m,2}}{(1-\lambda_m x)^2} + \dots + \frac{C_{m,d_m}}{(1-\lambda_m x)^{d_m}}\right).$$

Moreover, $[x^n]A(x) = p_1(n)\lambda_1^n + \dots + p_m(n)\lambda_m^n \quad \forall n \ge 0$, where $p_i(n)$ is a polynomial of degree $< d_i$.

(IV) recurrence → explicit formula

eg: given $a_n - 6a_{n-1} + 12a_{n-2} - 10a_{n-3} + 3a_{n-4} = 0 \quad \forall n \ge 4$, with $a_0 = 5$, $a_1 = 11$, $a_2 = 23$, $a_3 = 49$. Find an explicit formula for a_n.

let $A(x) = \sum_{k \ge 0} a_k x^k = P(x)/Q(x)$,

where $Q(x) = 1 - 6x + 12x^2 - 10x^3 + 3x^4$ ← directly from recurrence!

$= (1-3x)(1-x)^3$ ← given!

the inverse roots of $Q(x)$ are $\lambda_1 = 3$ and $\lambda_2 = 1$ with multiplicities $d_1 = 1$ and $d_2 = 3$, respectively.

$\therefore$, an explicit formula for a_n is: $a_n = \binom{\text{Polynomial in } n}{\text{of degree} < d_1 = 1} 3^n + \binom{\text{Polynomial in } n}{\text{of degree} < d_2 = 3} 1^n$

So, $a_n = C \cdot 3^n + (D + En + Fn^2)$ $\forall n \geq 0$.

Now, we use the initial conditions to find the constants C, D, E, and F:

$$\left.\begin{array}{l} \cdot\, n=0:\ a_0 = 5 = C + D \\ \cdot\, n=1:\ a_1 = 11 = 3C + D + E + F \\ \cdot\, n=2:\ a_2 = 23 = 9C + D + 2E + 4F \\ \cdot\, n=3:\ a_3 = 49 = 27C + D + 3E + 9F \end{array}\right\}$$

Solving, we get:
$C = 1$, $D = 4$, $E = 3$, $F = 1$

$\therefore$, $a_n = 3^n + (4 + 3n + n^2)$ $\forall n \geq 0$.

General Method: given a recurrence relation $a_n + C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_d n_{n-d}$ $\forall n \geq d$ with initial conditions $a_0, a_1, \ldots, a_{d-1}$, we find the explicit formula by:

1) Defining the denominator $Q(x) = 1 + C_1 x + C_2 x^2 + \ldots + C_d x^d$
2) Find the inverse roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ and their multiplicities $d_1, d_2, \ldots, d_m$ of $Q(x)$ by factoring $Q(x)$ into a product of linear factors.
3) Set up the general form $a_n = p_1(n)\lambda_1^n + \ldots + p_m(n)\lambda_m^n$ for all $n \geq 0$, where $p_i(n)$ has degree $< d_i$ with unknown coefficients.
4) Use the initial conditions to determine these coefficients.

(V) explicit formula → recurrence

eg: given $a_n = (2n-3)(-1)^n + (2n^2+n-5)4^n$ $\forall n \geq 0$. Find the recurrence for a_n.

let $A(x) = \sum_{k \geq 0} a_k x^k = P(x)/Q(x)$.

Then, $Q(x)$ must be a polynomial with inverse roots $\lambda_1 = -1$ and $\lambda_2 = 4$, with multiplicities $d_1 = 2$ and $d_2 = 3$. $2n-3$ and $2n^2+n-5$

↳ note, we get multiplicities by seeing degree of respective polynomials, plus one!

So, $Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2)^{d_2} = (1+x)^2(1-4x)^3$.

expanding, we get: $Q(x) = 1 - 10x + 25x^2 + 20x^3 - 80x^4 - 64x^5$, so:

$a_n - 10a_{n-1} + 25a_{n-2} + 20a_{n-3} - 80a_{n-4} - 64a_{n-5} = 0 \quad \forall n \geq 5$.

let's find initial conditions by plugging $n \in \{0,1,2,3,4\}$ into the given explicit formula: $a_0 = -8$, $a_1 = -7$, $a_2 = 81$, $a_3 = 1021$, $a_4 = 7941$.

General Method: given an explicit formula of the form $a_n = p_1(n)\lambda_1^n + \ldots + p_m(n)\lambda_m^n$ for all $n \geq 0$, where $p_i(n)$ has degree $< d_i$, we get the recurrence relation by:

1) Define the denominator polynomial $Q(x) = (1-\lambda_1 x)^{d_1} \ldots (1-\lambda_m x)^{d_m}$
2) Compute $Q(x) = 1 + c_1 x + c_2 x^2 + \ldots + c_d x^d$
3) A recurrence relation for the a_n is then:

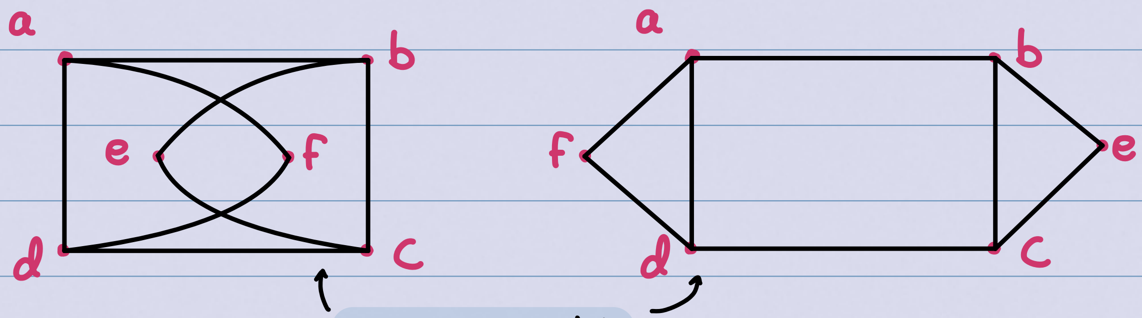
   $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d} = 0 \quad \forall n \geq d$.
4) The initial conditions $a_0, a_1, \ldots, a_{d-1}$ can be determined from the explicit formula.

Graph Theory

A graph has a finite set of vertices $V(G)$.

Vertices that are connected are paired in the Edge Set $E(G)$.

Example: $V(G) = \{a, b, c, d, e, f\}$, $E(G) = \{(a,b), (b,c), (c,d), (d,a), (a,f), (d,f), (b,e), (c,e)\}$



Same graph!

A graph G consists of a pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of objects called vertices, and $E(G)$ is a set of unordered pairs of distinct vertices called edges.

Terminology: for an edge $(u,v) \in E(G)$, we say:

- u is adjacent to v
- u is a neighbor of v
- e is incident with u and v
- u and v are the ends of e,
- e joins u and v

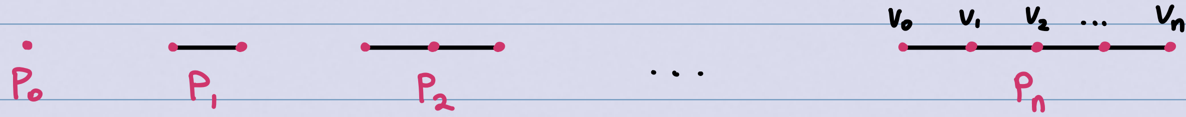
→ we use the shorthand $e = uv$ to represent the edge $\{u, v\}$.

→ note, if $e = uv$, then $e = vu$.

The degree of a vertex v in a graph G is the number of edges incident with v in G, denoted $\deg(v)$.

Handshaking Lemma: for any graph G, $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

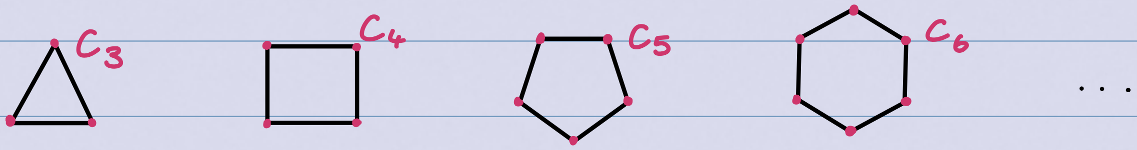
Paths:



A path of length n:

- $P_n = (\{v_0, v_1, \dots, v_n\}, \{v_0v_1, v_1v_2, v_2v_3, \dots, v_{n-1}, v_n\})$
- number of vertices is $n+1$
- number of edges is n

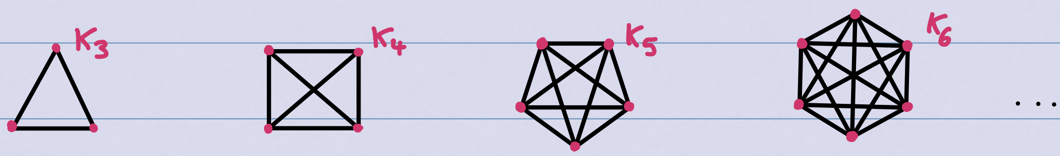
Cycles:



A cycle of length n:

- $C_n = (\{v_1, v_2, \dots, v_n\}, \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\})$
- number of vertices is n
- number of edges is n

Complete Graphs:



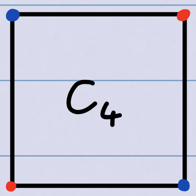
A complete graph on n vertices:

- Every pair of vertices is adjacent
- $K_n = (\{v_1, v_2, \dots, v_n\}, \{v_iv_j : 1 \leq i \leq j \leq n\})$
- number of vertices is n
- number of edges is $\binom{n}{2} = \frac{n(n-1)}{2}$

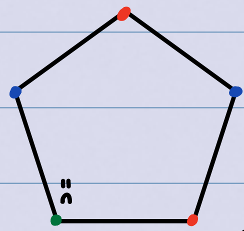
A graph is bipartite if there is a partition (A, B) of it's vertices such that every edge joins a vertex in A with a vertex in B.

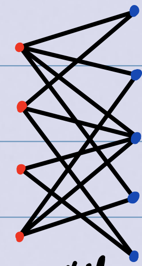
- we can (informally) check if a graph is bipartite by coloring it's vertices. If two adjacent vertices have the same color, then the graph is not bipartite!

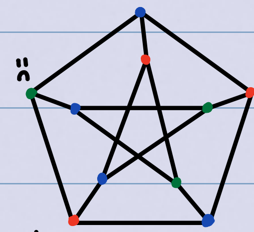
C_n is bipartite if and only if n is even.
Moreover, a graph is bipartite if and only if it does not contain an odd cycle.



bipartite!


not bipartite!


bipartite!

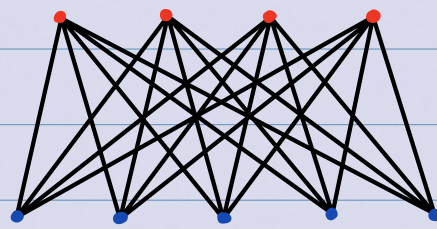

not bipartite!

Complete Bipartite Graph: a bipartite graph with bipartition (A, B), where $|A| = m$, $|B| = n$, and every vertex in A is adjacent to every vertex in B

↳ $K_{m,n} = \left(\{u_1, u_2, \dots, u_m\} \cup \{v_1, v_2, \dots, v_n\},\ \{u_i v_j : 1 \leq i \leq m \leq j \leq n\} \right)$.

- number of vertices is $m+n$
- number of edges is mn

$K_{4,5}$:



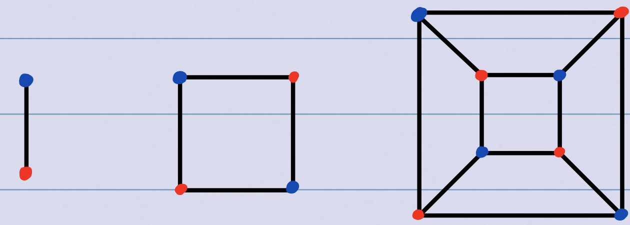
A graph is regular if every vertex has the same degree.
↳ more precisely, a graph is k-regular if every vertex has degree k.

- Paths → P_0 is 0-regular, P_1 is 1-regular, P_n is not regular for $n \geq 2$
- Cycles → cycles are 2-regular
- The complete graph K_n → $(n-1)$-regular
- The complete bipartite graph $K_{m,n}$ → m-regular and n-regular
- Number of edges in a k-regular graph with n vertices → $kn/2$.

n-Cubes (Hypercubes): graphs whose vertices are the length-n binary strings, and two vertices are adjacent if and only if their binary strings differ in exactly one bit.

Properties of the n-cube:

- number of vertices is 2^n
- n-cubes are n-regular
- number of edges is $n2^{n-1}$
- n-cubes are bipartite

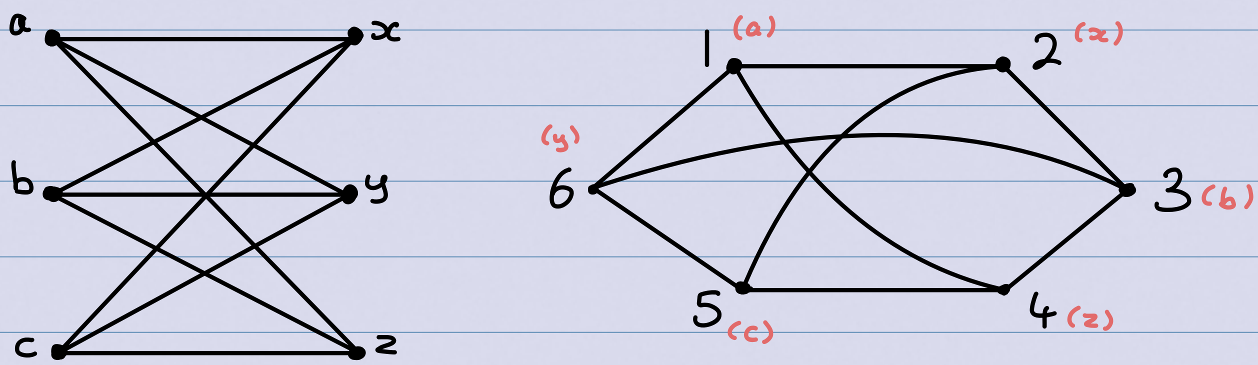


1-cube 2-cube 3-cube

Isomorphic Graphs

Informally: two graphs are isomorphic (ie, essentially the same) if one can be obtained from the other by "relabelling" vertices.

Eg: Are the following graphs isomorphic?

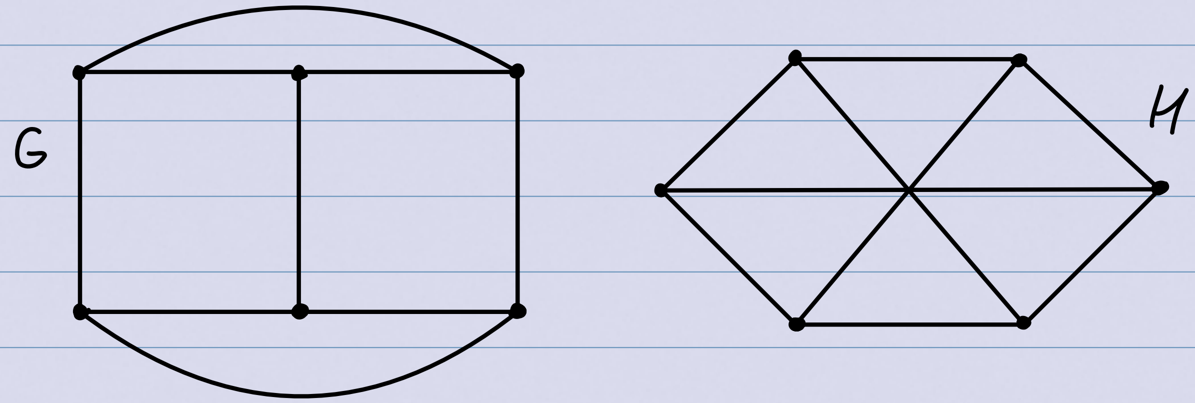


Yes! we rename vertices as follows:

1)	1	2	3	4	5	6
2)	a	x	b	z	c	y

Formally, two graphs G and H are isomorphic if there is a bijection $f: V(G) \to V(H)$ which preserves adjacency, ie, $uv \in E(G)$ if and only if $f(u)f(v) \in H$.

Eg: are the following graphs isomorphic?



No! G has 3 mutually adjacent vertices (a triangle), but H doesn't. So, there doesn't exist a bijection from V(G) to V(H) that preserves adjacency.

Graph Invariants

- A graph invariant is a property of a graph that does not change over time.
  - ↳ examples of possible graph invariants:
    1) number of vertices (isomorphic graphs have the same number of vertices)
    2) number of edges (isomorphic graphs have the same number of edges)
    3) the degree sequence: a list of all vertex degrees in non-decreasing sequence (isomorphic graphs have the same degree sequence)
    4) bipartiteness (isomorphic graphs are either both bipartite or both non-bipartite.
    5) containment of a cycle of a certain length
    6) etc

Determining if two graphs are isomorphic

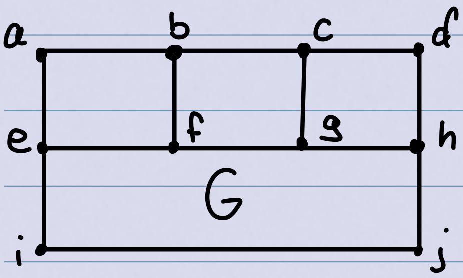
• to show that two graphs G and H are isomorphic, give a bijection from $V(G)$ to $V(H)$ that preserves adjacency.

• to show that two graphs G and H are not isomorphic, find a graph invariant that is different for G and H (eg, # vertices, # edges, degree sequence, etc), or find an adjacency structure that is in G but not in H.

• note: no efficient algorithm is known for determining whether two graphs are isomorphic or not!

Fundemental Concepts in Graph Theory

Walks and Paths



• A walk in G of length 5 b, f, g, f, e, i is called a b, i walk

• a, b, c, b, a is a closed a, a walk

• b is a path (and a closed b, b walk)
↳ length 0 walk

Definition: let $u, v \in V(G)$. A u,v-walk in G is a sequence of vertices $v_0, v_1, \dots, v_k$, where $v_0 = u$, $v_k = v$, and $v_i v_{i+1} \in E(G)$
↳ $\forall\, 0 \le i \le k+1$

• The length of the walk is k

• The walk is closed if $v_0 = v_k$ (end where we start)

• A u,v-path in G is a u,v-walk with no repeated vertices (and no repeated edges).

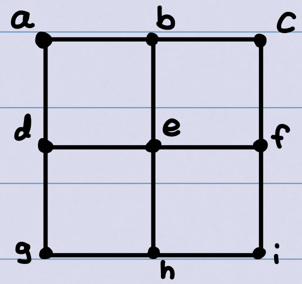
Theorem: let $u, v \in V(G)$. If there exists a u,v-walk in G, then there exists a u,v-path in G.

↳ Proof: let $w = v_0, v_1, v_2, \dots, v_k$ be a u-v walk in G (so

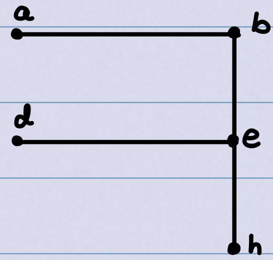
$V_0 = u$ and $V_k = V$) of shortest length. Suppose there exists indices i and j with $i<j$ and $V_i = V_j$. Then, $\omega' = V_0, V_1, \ldots, V_{j+1}, \ldots, V_k$ is a u,v-walk in G that is shorter than ω. So, this contradicts the choice of ω. ∴, the vertices in ω are distinct, and so ω is a u,v-path in G.

↳ Corolary: let $u, v, w \in V(G)$. If there exists a u,v-path and a v,w-path in G, then there exists a u,w-path in G.

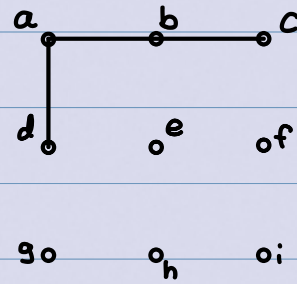
Subgraphs



h o subgraph



subgraph of G



spanning subgraph

A graph H is a subgraph of G if:

1) $V(H) \subseteq V(G)$

2) $E(H) \subseteq E(G)$

3) each edge in $E(H)$ has both ends in $V(H)$

→ if $V(H) = V(G)$ then H is a spanning subgraph of G

note: a u,v-path in G is a subgraph of G.

Cycles: A cycle in a graph G is a closed walk $V_0, V_1, V_2, \ldots, V_k$ and where $V_1, V_2, \ldots, V_k$ are distinct and $k \geq 2$ and $V_0 = V_k$ (the length of the cycle is k).

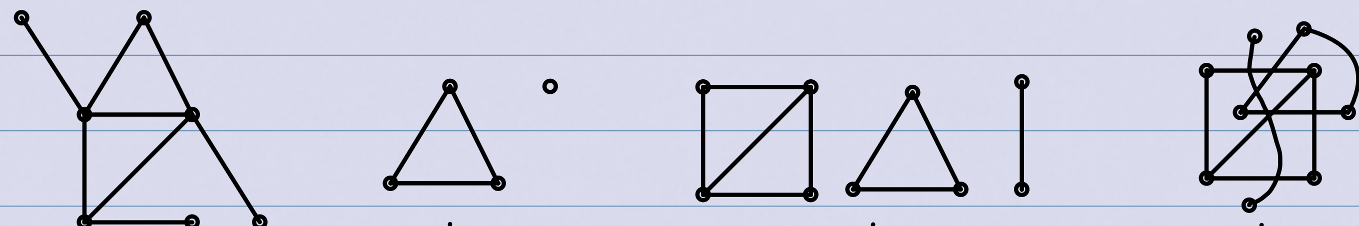
Theorem: let G be a graph where each vertex has degree ≥ 2. Then G contains a cycle!

· **proof:** let $P = v_0, v_1, v_2, \ldots, v_k$ be the longest path in G. note that $k \geq 2$, since every vertex has degree at least 2, so any vertex v, together with it's two neighbors, gives a path of at least 2.
now, v_0 is adjacent to v_1 since $\deg(v_0) \geq 2$, v_0 is adjacent to at least one other vertex, say $x \neq v_1$. If x is not on path P, then $x, v_0, \ldots, v_k$ is a path that is longer than P, contradicting the choice of P. $\therefore$, x is on P, so $x = v_i$ for some $2 \leq i \leq k$. Then $v_0, v_1, \ldots, v_i = x, v_0$ is a cycle in G (of length ≥ 3).

Connectedness + Components

A graph G is connected if $\forall x, y \in V(G)$, there exists an x,y-path in G.

eg:

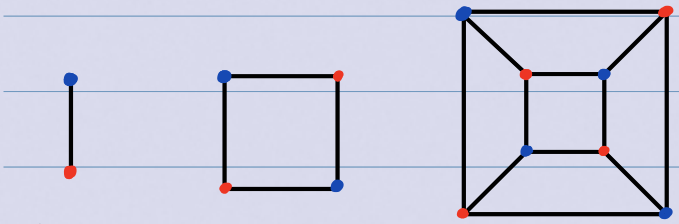


connected | not connected | not connected | not connected

Theorem: let $v \in V(G)$. If there exists a v,w-path in G $\forall w \in V(G)$, then G is connected.

proof: let $x, y \in V(G)$. By assumption, there is a v,x-path in G, so there's also a x,v-path in G. Also by assumption, there's a v,y-path in G. By the previous corollary, there exists an x,y-path in G.
$\therefore$, G is connected.

eg: show that the n-cube is connected.

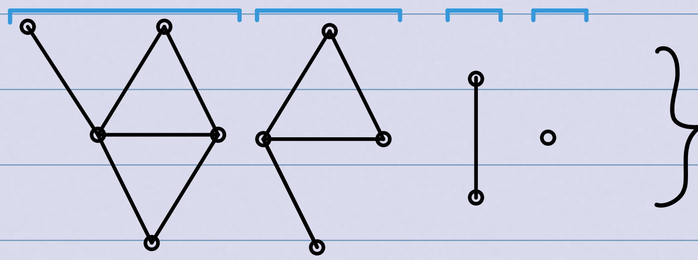


1-cube 2-cube 3-cube

consider $v = 000000$, $w = 010110$. then, 000000, 010000, 010100, 010110 is a v,w-walk in the 6-cube.

formally, let $v = \overbrace{000\ldots00}^{n}$ and $w = \{0,1\}^n$. Suppose w has exactly k 1s $(0 \le k \le n)$ with 1s in positions $1 \le i_1 < i_2 < \ldots < i_3 \le n$. For each $1 \le j \le k$, define v_i to be a length-n binary string with 1s in positions $i_1, i_2, \ldots, i_j$ and 0s elsewhere. Then $v, v_1, v_2, \ldots, \overset{=v_k}{w}$ is a v,w-path in the n-cube. $\therefore$, by the previous theorem, the n-cube is connected.

Components



4 components

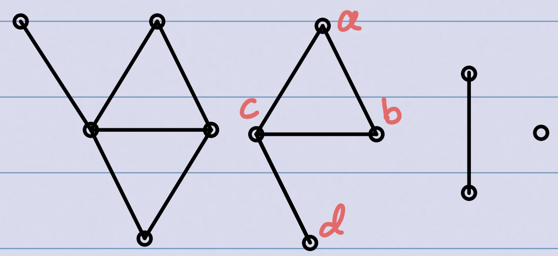
A component of a graph G is a maximal connected subgraph of G.

Observation: a graph is connected if and only if it has only one component!

Cuts

Let $X \subseteq V(G)$. The cut, induced by X, is the set of all edges in $E(G)$ with exactly one end in X.

Eg:



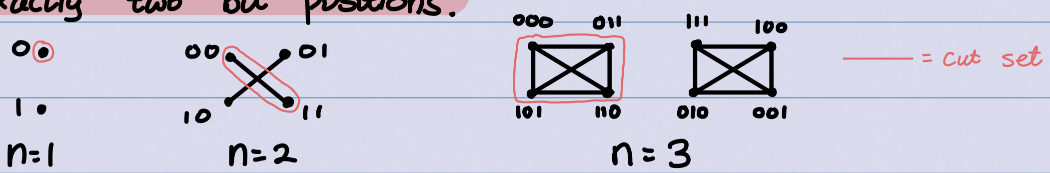
if $X = \{a,b,c,d\}$, then the cut induced by X is empty.

Theorem: a graph is connected if and only if <u>for all</u> non-empty proper subset $X \subseteq V(G)$, the cut induced by X is non-empty.

↳ subset that cannot be equal

Eg: let $n \geq 1$. Define G_n to be a graph whose vertex set is the of length-n binary strings, with two vertices being adjacent iff they differ in exactly two bit positions.

examples:



Claim: G_n is not connected!

let X = vertices in G_n labelled with an even number of 1s. Then, X is <u>non-empty</u> (since the string $000...000 \in X$) and is a <u>proper subset</u> of $V(G_n)$ (since $1000...000 \notin X$). We'll argue that the cut induced by X is empty:

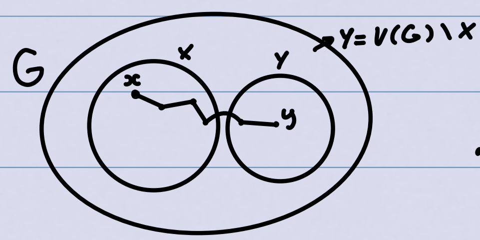
ie, $y \in V(G) \setminus X$

let $x \in X$ and $y \notin X$ (but $y \in V(G)$). Now x has an even number of 1s (by defn. of X). Flipping one bit in X gives a vertex in $V(G) \setminus X$ (not in X). Flipping a second bit of this vertex gives a vertex with an even number of 1s, so it's in X. $\therefore$, there's no edge from a vertex in X to a vertex outside of X.

Therefore, G_n is not connected!

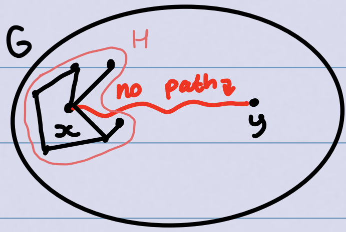
Proof w/ a sketch:

($\Rightarrow$)



- x exists bc X non-empty
- y exists bc X is proper subset

($\Leftarrow$)
contrapositive



• if the cut induced by X is empty, the graph is not connected.

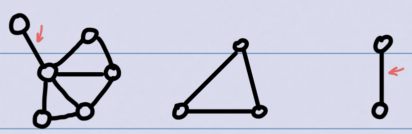
(Formal) Proof:

($\Rightarrow$) Suppose that G is connected. Let X be any non-empty proper subset of $V(G)$, and let $Y = V(G) \setminus X$. Let $x \in X$ (which exists since X is non-empty) and $y \in Y$ (which exists since X is proper). Since G is connected, there exists an x,y-path in G: $x = x_0, x_1, x_2, \dots, x_k = y$ $(k \geq 1)$.

Let i be the largest index such that $x_i \in X$ and $x_{i+1} \notin X$. Such an index exists since $x_0 \in X$ (so $i \geq 0$) and $x_k \notin X$ (so $i \leq k-1$). Then, $x_i x_{i+1}$ is an edge with one end in X and the other end not in X. $\therefore$, the cut induced by X is non-empty.

contrapositive
($\Leftarrow$) Suppose G is not connected. So, there exists $x, y \in V(G)$ such that there's no x,y-path. Let H be the component of the graph that contains x, and let $X = V(H)$. Then, X is non-empty (since $x \in X$), and is proper (since $y \notin X$). Since H is a maximal connected subgraph of G, there's no edge in the graph with one end in X and the other end not in X. $\therefore$, the cut induced by X is empty!

Bridges



$G - e$ is the graph obtained by deleting the edge e from G.

An edge $e = uv$ is a bridge if $G - e$ has more components than G.

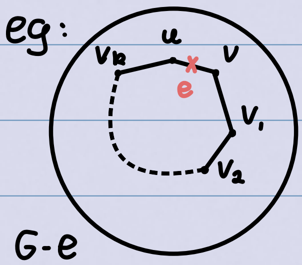
Theorem: let e be a bridge of a connected graph G. Then, $G-e$ must have 2 components.

↳ Equivalently, e is contained in a cycle G if and only if G is not a bridge.

• **Proof:**

($\Rightarrow$) Assume that $e=uv$ is contained in a cycle C of G. Say $C = u, v, v_1, v_2, \ldots, v_k, u$.

eg:



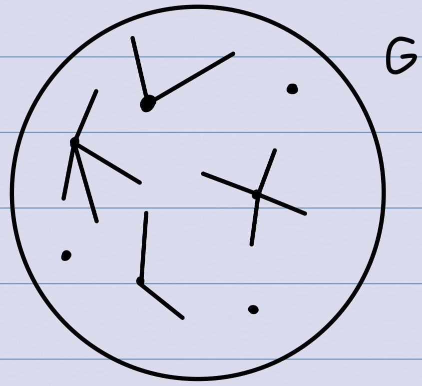
There is a path $\sim u$ & v in $G-e$, namely $v, v_1, \ldots, v_k, v$. Let H be the component of G that contains e. Now, u & v are still components of $G-e$. So, $G-e$ has the same number of components, and therefore, e is not a bridge of G.

($\Leftarrow$) Assume $e=uv$ is not a bridge of G.
Let H be the component of G that contains e.
Since e is not a bridge of G, $G-e$ has the same number of components as G.
In particular, u and v are in the same component of $G-e$. So, there is a path p $\sim u$ & v in $G-e$. Then, p together with e gives a cycle in G. So, e is contained in a cycle of G.

Recall the theorem that states that if every vertex of G has degree ≥ 2, then G has a cycle.

↳ **Corollary:** suppose G has no vertex of degree one, then G has a cycle, except if G has no edges (ie, $E(G) = \emptyset$ or $V(G) = 0$).

Example: let G be a graph where each vertex has an even degree. Prove that G has no bridges.



If G has no edges, then G has no bridges.
Suppose that G has at least one edge. By the corollary, G has a cycle C.
Remove the edges of C from G to get a graph G_1. Every vertex of G still has vertices of even degree. So, if G_1 is not empty, it must contain a cycle C_1.
Then, we remove the edges of C_1 from G_1 to yield G_2. Again, G_2 must have vertices of even degree, and therefore a cycle C_2.
... continue the process ...
The process will terminate, bc the sequence $|E(G)|$, $|E(G_2)|$, $|E(G_3)|$, ... is a strictly decreasing sequence of non-negative integers, and therefore must eventually reach 0.
This shows that every edge of G is on at least 1 cycle, and therefore, G has no bridges.

Trees

- A tree is a connected graph with no cycles.
- root and leaves ⇒ a leaf is a vertex of degree 1.

Properties of Trees

1) Every edge of a tree is a bridge

↳ Proof: trees have no cycles

2) A tree is a minimally connected graph. ie, if any edge is deleted, the resulting graph is disconnected (since every edge is a bridge).

3) if a tree T has n vertices, it has $m=n-1$ edges.

- Proof: by induction on # vertices n:
  - Base Case: if T has $n=1$ vertex, it has 0 edges. $\therefore$, $m=n-1=0$, so BC holds.
  - Induction Hypothesis: let $n \geq 2$. Assume the statement is true $\forall$ trees with fewer than n vertices.
  - Inductive Step: let T be a tree with n vertices. We will show that it has $m=n-1$ edges.

  Let e be any edge in T.

  Since e is a bridge, $T-e$ has exactly 2 components T_1 and T_2. However, T_1 and T_2 have no cycles, so T_1 and T_2 are trees.

  Let $n_1 = |T_1|$ and $n_2 = |T_2|$. Note, $n = n_1 + n_2$, $n_1 \geq 1$, $n_2 \geq 1$. So, $n_1 < n$ and $n_2 < n$.

  By the inductive hypothesis, T_1 has $n_1 - 1$ edges and T_2 has $n_2 - 1$ edges.

  $\therefore$, the total number of edges in T is $\underbrace{n_1 - 1}_{|T_1|} + \underbrace{n_2 - 1}_{|T_2|} + \underbrace{1}_{\text{the edge e we deleted}}$

  $= n_1 + n_2 - 1 = n - 1$.

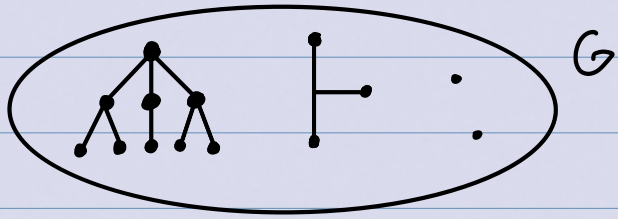
  So, the statement is true for trees with n vertices.

  Therefore, by induction, the statement is true $\forall$ trees.

4) if a tree T be a tree with at least 2 vertices, then T has at least 2 leaves.

- Proof: let P be a longest-path in T, say $v_1, v_2, \ldots, v_k$ where $k \geq 2$. Now, v_1 cannot be adjacent to any vertex in P other than v_2 as otherwise, we'd have a cycle in T. So, v_1 and similarly v_k have degree 1, so they are leaves.

Forests



A forest is a graph with no cycles.

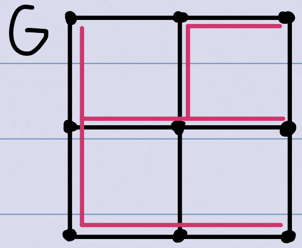
Properties:

1) Each component of a forest is a tree
2) Every edge of a forest is a bridge
3) If a forest has n vertices and c components, then it has $m = n - c$ edges.
   - ↳ Proof: each of the c components of the forest has one less edge than vertices, since each component is a tree. So, in total, the number of edges in the forest is c less than the # of vertices.

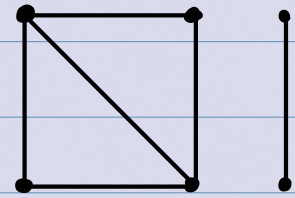
Spanning Trees

A spanning tree of a graph G is a spanning subgraph of G that is also a tree.

↳ Recall: a subgraph H of G is a spanning subgraph if $V(H) = V(G)$.



• = spanning subgraph of G



doesn't have a spanning subgraph!

Theorem: a graph is connected if and only if it has a spanning tree.

Proof:

($\Rightarrow$) let G be a connected graph.

- if G has no cycles, then G is a tree. So, G itself is a spanning tree of G.
- Suppose now that G has at least one cycle C. Let e be any edge in C, then G-e is connected (since e is not a bridge). Then, G-e is a spanning subgraph of G. note that G-e has at least one less cycle than G. Repeat the process, which must eventually terminate with a graph T. Then T is a spanning subgraph of G with no cycles, and is connected. So, T is a spanning subgraph of G.

($\Leftarrow$) Suppose a graph G has a spanning subtree T. Let $x, y \in V(G)$. Since T is a spanning subgraph of G, we have that $x, y \in V(T)$. Since T is connected, there is an x,y-path in T. Since T is a subgraph of G, this path is also a path in G. So, G is connected.

Corollary: let G be a graph with n vertices and $n-1$ edges. Then G is a tree.

Proof: Since G is connected, it has a spanning tree T. Since T is a spanning subgraph of G, T also has n vertices. Since T is a tree, it has $n-1$ edges. These are all the vertices and edges in G, so $G = T$. Therefore, G is a tree.

Summary: Let G be a graph with n vertices. Then, G is a tree if and only if any two of the following conditions are true:

1) G is connected
2) G has no cycles
3) G has $n-1$ edges.

Justification:

- (1) and (2): definition of a tree
- (1) and (3): by previous corollary
- (2) and (3): G is a forest, say with c components and hence $n-c$ edges. Since G has $n-c$ components, c must be 1. So, G is connected and so it's a tree.