eg: $\{a,b,c,b\}$ is not a set!

Sets: an unordered collection of distinct objects.

↳ $\{a,b,c\} = \{c,b,a\}$ is a set of size 3

↳ $\{0,1,2,\ldots\} = \mathbb{N}$ is an infinite set

Set Operations: let A, B be finite sets:

- Cartesian Product: $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$
- Size of Cartesian Product: $|A \times B| = |A| \cdot |B|$

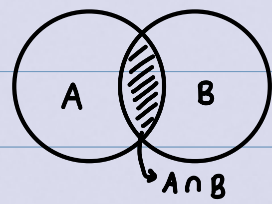
↳ eg: how many binary strings have length 4?

$\underline{0/1}\ \underline{0/1}\ \underline{0/1}\ \underline{0/1} \rightarrow 2\times2\times2\times2 = 2^4 = \underline{16}$.

$\#\text{strings} = |\{0,1\}\times\{0,1\}\times\{0,1\}\times\{0,1\}| = |\{0,1\}|\times|\{0,1\}|\times|\{0,1\}|\times|\{0,1\}| = 2\times2\times2\times2 = 16.$

In general, the number of binary strings of length n is 2^n.

- Union: $A \cup B = \{x : x \in A \text{ or } x \in b\}$.
- Size of Union: $|A \cup B| = |A| + |B| - |A \cap B|$



↳ if $|A \cap B| = 0$, then $|A \cup B| = |A| + |B|$.

↳ $A \cup B$ with $|A \cap B| = 0$ is a disjoint union.

↳ eg: how many binary strings of length 8 are there which begin with 001 or 1011?

let S be the set of all strings.

let A_1 = set of length-8 binary strings that begin with 001.

let A_2 = set of length-8 binary strings that begin with 1011.

then $S = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. So, $|S| = |A_1| + |A_2| = 2^5 + 2^4 = 48$.

Counting Subsets

Permutations: a permutation of a set S is an ordered listing of the elements in S.

↳ eg: the permutations of $\{a, b, c\}$ are abc, acb, bac, bca, cab, cba.

- the number of permutations of a set S of size n is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{2 \text{ choices}} \times \underline{1 \text{ choice}} = n!$$

Partial Permutations: a permutation of a subset of S.

- number of partial permutations of S (with $|S| = n$) of size k is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{n-k+2 \text{ choices}} \times \underline{n-k+1 \text{ choices}}$$

$\therefore$ $n(n-1)(n-2)\dots(n-k+2)(n-k+1)$.

↳ note: this works even if $k > n$.

How many subsets of S (with $|S| = n$) are there of size k?

- number of partial permutations of S of size k is $n(n-1)(n-2)\dots(n-k+1)$.
- each size-k subset of S has $k!$ permutations.

↳ $\therefore$ $$\frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

- **Combinations:** let $0 \le k \le n$. Then "n choose k" is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{(aka, binomial coefficient)}$$

↳ note, if $k > 1$, then $\binom{n}{k} = 0$.

eg: $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = \frac{1}{0!} = 1$.

eg: $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!\,0!} = \frac{1}{0!} = 1$.

eg: $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$.

eg: $\binom{1999}{73} = \dfrac{1999 \times 1998 \times 1997 \times 1927}{73 \times 72 \times \dots \times 1}$ = some positive integer.

Binomial Theorem: for any $n \geq 1$, $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

eg: $(x+1)^3 = (x+1)(x+1)(x+1) = x^3 + 3x^2 + 3x + 1.$

$\binom{3}{3} = 1$ $\quad \binom{3}{2}$ $\quad \binom{3}{1}$ $\quad \binom{3}{0} = 1.$

Combinatorial Proof of the Binomial Theorem:

n copies of $(x+1)$

$(x+1)^n = (x+1)(x+1)\dots(x+1)$

how do we get an x^k term? by selecting x from k binomials and 1 from the other $n-k$ binomials. $\therefore$, the total number of ways is $\binom{n}{k}$.

Therefore, $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. $\square$.

Combinatorial Proofs

- Procedure to prove $M = N$:
  1) "Cleverly" choose a set S
  2) Count the number of elements in S in two ways:
     a) $|S| = M$ and
     b) $|S| = N$
  3) Conclude that $M = N$.

Claim: $2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

↳ note: algebraically, we can prove this by substituting $x=1$ into the binomial theorem.

Combinatorial proof:

- let S be the set of all subsets of $\{1, 2, \dots, n\}$.

a) let S_i be the subsets in S of size i, where $0 \leq i \leq n$.

Then $S = S_0 \cup S_1 \cup \ldots \cup S_n$ (note, a disjoint union).

$\therefore$, $|S| = |S_0| + |S_1| + |S_2| + \ldots + |S_n|$

$|S| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$.

b) to choose a subset X of $\{1, 2, \ldots, n\}$, we do:

- for each i, $1 \le i \le n$, we include i in X, or not.
- Therefore, the total number of choices is 2^n. Hence $|S| = 2^n$.

$\therefore$, $|S| = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. $\square$.

Claim: let $1 \le k \le n$. Then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ [Pascal's Identity].

- Algebraic Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!\,(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} + \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} = \frac{(n-1)! \cdot (n-k)}{k!\,(n-k)!} + \frac{(n-1)! \cdot k}{k!\,(n-k)!}$$

$$= \frac{(n-1)![(n-k)+k]}{k!\,(n-k)!} = \frac{(n-1)! \cdot n}{k!\,(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \quad \square.$$

- Combinatorial Proof:

let S = set of all size-k subsets of $\{1, 2, 3, \ldots, n\}$.

then, $|S| = \binom{n}{k}$ by definition.

let A = size-k subsets that don't include n.

let B = size-k subsets that includes n.

Then $S = A \cup B$ is a disjoint union. So, $|S| = |A| + |B|$.

Now, $|A| = \binom{n-1}{k}$, and $|B| = \binom{n-1}{k-1}$. Thus, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. $\square$.

- Claim: for $k, n \ge 0$, $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n-1}{n-1} + \binom{n}{n-1} + \ldots + \binom{n+k-1}{n-1}$

Combinatorial Proof: $\therefore |S| = \binom{n+k}{n}$

- let S = the size-n subsets of $\{1, 2, \dots, n, n+1, \dots, n+k-1, n+k\}$.
  - ↳ note: the largest number in a size-n subset is between n and $n+k$.

for each $0 \le i \le k$, let S_i = subset whose largest element is $n+i$.
Then, S = disjoint union $S_0 \cup S_1 \cup \dots \cup S_k$. So, $|S| = \sum_{i=0}^{k} |S_i| = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$. $\square$.
Visually:

See that the LHS is size-n subsets of $n+k$ sets. So, we choose subsets of size n (k times):

Subsets: $\{\overbrace{\qquad}^{n \text{ size}}\} \cup \{\overbrace{\qquad}^{n}\} \cup \dots \cup \{\overbrace{\qquad}^{n}\} \cup \{\overbrace{\qquad}^{n}\}$

- $\{\quad\} \cup \{\quad\} \cup \dots \cup \{\quad\} \cup \{\overbrace{\dots, n+k}^{n}\}$
  - ↳ see that the last subset can be counted as $\binom{n+k-1}{n-1}$.

Claim: for $0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$

↳ eg: $\binom{100}{98} = \binom{100}{2} = \frac{100 \times 99}{2}$.

eg: $n=4$, $k=1$: Size-1 subsets of $\{1,2,3,4\} = \{1\}, \{2\}, \{3\}, \{4\}$.

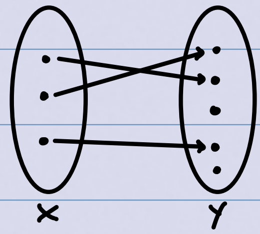
Size-3 subsets of $\{1,2,3,4\} = \{1,2,3\}, \{2,3,4\}, \{1,2,4\}, \{1,3,4\}$.

we can pair these sets by picking the size-3 subset which does not contain the element in the size-1 subset. ie: $\{1\} \leftrightarrow \{2,3,4\}$, $\{2\} \leftrightarrow \{1,3,4\}$...

→ aka, a correspondence or a bijection!

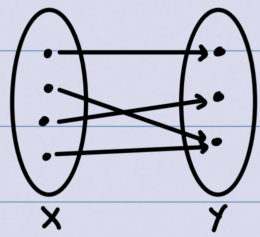
Bijections: let $f: X \to Y$ be a function.

- f is **injective (1-1)** if $\forall x_1, x_2 \in X,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
- Visually:



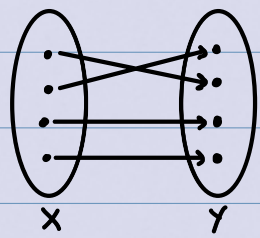
. see that $|X| \leq |Y|$.

- f is **surjective (onto)** if $\forall y \in Y,\ \exists x \in X$ st $f(x) = y$.
- Visually:



. see that $|X| \geq |Y|$

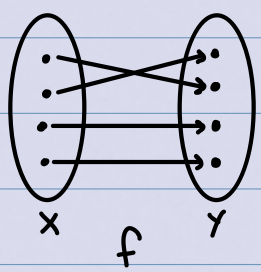
- f is **bijective** if it is injective AND surjective.
- Visually:

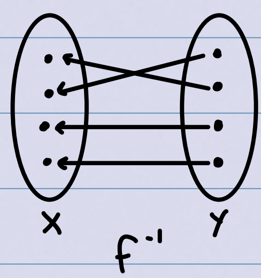


. see that $|X| = |Y|$.

- Theorem: let X, Y be finite sets. Suppose $f: X \to Y$ is a bijection. then, $|X| = |Y|$.

- **Inverse function:** the inverse of a function $f: X \to Y$ is a function $f^{-1}: Y \to X$ such that:
  1) (f^{-1} reverses f) $\Rightarrow \forall x \in X,\ f^{-1}(f(x)) = x$.
  2) (f reverses f^{-1}) $\Rightarrow \forall y \in Y,\ f(f^{-1}(y)) = y$.





- Theorem: let $f: X \to Y$. Then f is a bijection if and only if f has an inverse.

Combinatorial Proofs Using Bijections that $M = N$.

1) Select two sets X, Y, with $|X| = M$ and $|Y| = N$.
2) Define $f: X \to Y$. (f is a bijection).
3) Define $f^{-1}: Y \to X$.
4) Prove that f^{-1} is the inverse function
   ↳ prove the two conditions of the inverse function definition are satisfied.
5) Conclude that $M = N$.

- A function $f: X \to Y$ is well-defined if $\forall x \in X,\ f(x) \in Y$.

Claim: for $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$ (continued)

let X be the set of all size-k subsets of $\{1, 2, \cdots, n\}$.
$\therefore,\ |X| = \binom{n}{k}$.
let Y be the set of all size-$(n-k)$ subsets of $\{1, 2, \ldots, n\}$.
$\therefore,\ |Y| = \binom{n}{n-k}$.

size-k subset
Define $f: X \to Y$ as follows: $\forall S \in X,\ f(S) = S^c$ ($S^c = \{1, 2, \ldots n\} - S$).
- See that f is well-defined, ie, $f(S) = S^c \in Y$ since $|S^c| = n-k$

size-(n-k) subset
Define $f^{-1}: Y \to X$ as follows: $\forall T \in Y,\ f^{-1}(T) = T^c$ ($T^c = \{1, 2, \ldots, n\} - T$).
- see that f^{-1} is well-defined, ie, $f^{-1}(T) = T^c \in X$ since $|T^c| = k$.

now, $\forall S \in X,\ f^{-1}(f(S)) = f^{-1}(S^c) = (S^c)^c = S$.
$\forall T \in Y,\ f(f^{-1}(T)) = f(T^c) = (T^c)^c = T$.

$\therefore$, f is a bijection, and therefore $|X| = |Y|$, so $\binom{n}{k} = \binom{n}{n-k}$.

Generating Series

↳ we'll encode solutions to counting problems as coefficients of a "generating series".

• eg: how many subsets of $\{1,2,3\}$ have size n, $\forall n$ $0 \le n \le 3$?

let S = all size-n subsets of $\{1,2,3\}$ $\forall n$ $0 \le n \le 3$.

define weight function $\omega: S \to \mathbb{N}$, by $\omega(\sigma) = |\sigma|$ $\forall \sigma \in S$.

$\sigma \in S$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2,3\}$
$\omega(\sigma)$	0	1	1	1	2	2	2	3

associate each element $\sigma \in S$ with the term $x^{\omega(\sigma)}$:

$x^{\omega(\sigma)}$	1	x	x	x	x^2	x^2	x^2	x^3

the generating series of X with respect to ω is the sum of the $x^{\omega(\sigma)}$ terms:

3 elements in S have weight 1

$$\Phi_S^\omega(x) = 1 + 3x + 3x^2 + x^3 = (x+1)^3.$$

3 elements in S have weight 2

$\therefore$, the number of size-n subsets of $\{1,2,3\}$ is the coefficient of x^n in $\Phi_S^\omega(x)$, $\forall$ $0 \le n \le 3$.

(of any length!)

•eg: how many binary strings don't have 000 or 00111 as a substring?

·let S = all binary strings with no 000 or 00111 as a substring.

↳ we want to organize these strings by their length, so:

define $w(\sigma)$ be the length of σ (where $\sigma \in S$).

Definition: let S be a set.

- A function $w: S \to \mathbb{N}$ is a weight function if $\forall n \in \mathbb{N}$ there are only finitely many elements in S of weight n.
- The generating series for S with respect to w is $\Phi_w^S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$.
- The coefficient of x^n in a $\Phi_w^S(x)$ is denoted $[x^n]\Phi_w^S(x)$.
  - ↳ eg: $[x^{73}](x+1)^{97} = \binom{97}{73} \to \binom{97}{73}x^{73}$.

↳ result: $[x^n]\Phi_w^S(x)$ is the number of elements in S of weight n.

Eg: S = all subsets from $\{1, 2, \dots, n\}$.

For $\sigma \in S$, $w(\sigma) = |\sigma|$.

Then, $\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n$

So, $[x^i]\Phi_S^w(x) = \binom{n}{i}$.

Eg: S = set of all binary strings.

for $\sigma \in S$, let $w(\sigma)$ = length of σ.

\# binary strings of length 1

Then, $\Phi_S^w(x) = ? + ?x + ?x^2 + ?x^3 + \dots$.

\# binary strings of length 2.

\# binary strings of length 0

$\therefore$, $\Phi_S^w(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$

Formal Power Series (FPS)

- An FPS is an expression of the form $A(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{k \geq 0} a_k x^k$, where $a_i \in \mathbb{R}$.

↳ we don't care about convergence or divergence – we only care about the coefficients.

- So, x is called an "indeterminate".

Operations on Formal Power Series

let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k\geq 0} a_k x^k$
let $B(x) = b_0 + b_1 x + b_2 x^2 + \dots = \sum_{k\geq 0} b_k x^k$.
} FPSs.

1) $(=)$: $A(x) = B(x) \iff a_k = b_k \ \forall k \geq 0$.

2) $(+)$: $A(x) + B(x) = \sum_{k\geq 0} (a_k + b_k) x^k$

2b) $(-)$: $A(x) - B(x) = \sum_{k\geq 0} (a_k - b_k) x^k$

3) $(\times)$: $A(x) \cdot B(x) = C(x) = \sum_{n\geq 0} c_n x^n$, where $c_n = a_0 b_n + a_1 b_{n-1} + \dots a_n b_0 = \sum_{i=0} a_i b_{n-i}$

4) (Inversion): the inverse of a FPS $A(x)$, if it exists, is a new FPS $B(x)$ such that $A(x)B(x) = 1$.

↳ we'll write $B(x) = A(x)^{-1} = 1/A(x)$.

Example: show that $(1-x)^{-1} = 1 + x + x^2 + \dots = \sum_{k\geq 0} x^k$.

let $P(x) = \text{RHS} = \sum_{k\geq 0} x^k$.

Then, $(1-x)P(x) = (1-x)\sum_{k\geq 0} x^k = \sum_{k\geq 0} x^k - x\sum_{k\geq 0} x^k = \sum_{k\geq 0} x^k - \sum_{k\geq 0} x^{k+1}$

re-index the Σ

$= \sum_{k\geq 0} x^k - \sum_{k\geq 1} x^k = 1 + \left(\sum_{k\geq 1} x^k - \sum_{k\geq 1} x^k\right) = 1 + 0 = 1$.

Common Formal Power Series

1) $(1-x)^{-1} = \frac{1}{1-x} = \sum_{k\geq 0} x^k \Rightarrow [x^n](1-x)^{-1} = 1 \quad \forall n \geq 0.$

↳ geometric series!

partial bc not infinite!

2) partial geometric series: $1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}.$

bc if $k>n$, $\binom{n}{k}=0$ anyway.

3) binomial series: $(x+1)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k. \quad \therefore, [x^k](x+1)^n = \binom{n}{k}.$

4) negative binomial series: $\forall n \geq 1, \quad (1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k\geq 0} \binom{n+k-1}{k} x^k$

↳ so $[x^k](1-x)^{-n} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$

Example: determine $[x^n]\left(\frac{x}{1-2x}\right)^5.$

$[x^n]\left(\frac{x}{1-2x}\right)^5 = [x^n]\, x^5(1-2x)^{-5} = [x^{n-5}](1-2x)^{-5}$ for $n \geq 5.$

Then, see that $(1-x)^{-5} = \sum_{k\geq 0} a_k x^k$. So, $(1-2x)^{-5} = \sum_{k\geq 0} a_k x^k 2^k.$

$= 2^{n-5}[x^{n-5}](1-x)^{-5} = 2^{n-5}\binom{n-5+5-1}{5-1} = 2^{n-5}\binom{n-1}{4}.$

Proving the negative binomial theorem $(1-x)^{-n} = \sum_{k\geq 0}\binom{n+k-1}{k}x^k, \; n \geq 1.$

Combinatorial Proof (sketch):

$(1-x)^{-n} = (1-x)^{-1}\cdot(1-x)^{-1}\cdot\ldots\cdot(1-x)^{-1}$ (n times).

$= (1+x+x^2+\dots)\cdot(1+x+x^2+\dots)\cdot\ldots\cdot(1+x+x^2+\dots)$ (n times)

- we get an x^k term by selecting x^{a_1} from the first series, x^{a_2} from the second series, ..., and x^{a_n} from the last series, where $a_1, a_2, \dots, a_k$ are non-negative integers with $a_1+a_2+\dots+a_n = k$. Then, multiplying the x^{a_i} terms gives $x^{a_1}x^{a_2}\cdots x^{a_n} = x^{a_1+a_2+\dots+a_n} = x^k.$

- The number of ways of choosing the a_i's is equal to the coefficient of x^k in $(1-x)^{-n}$.

- Illustration by example: $n=4$, $k=13$:
  - begin with a string of $k=13$ 0s and then insert $n-1=3$ 1s.

    ↳ $\underbrace{000}_{a_1=3}1\underbrace{0000}_{a_2=4}1\underbrace{00}_{a_3=2}1\underbrace{0000}_{a_4=4}.$

    note that $a_1+a_2+a_3+a_4=3+4+2+4=13$!
  - this gives a bijection between binary strings of length $k+n-1$ with exactly $n-1$ ones, and non-negative integers $a_1, a_2, \dots, a_n$ st $\sum_{i=1}^{n} a_i = k$.
  - $\therefore$, the number of non-negative integers $a_1, \dots, a_k$ with $\sum_{i=1}^{n} a_i = k$ is equal to the number of binary strings of length $k+n-1$ with exactly $n-1$ ones, which is $\binom{k+n-1}{n-1} = \binom{n+k-1}{k}$.

- $\therefore$, the coefficient of x^k in $(1-x)^{-n}$ is $\binom{n+k-1}{k}$. $\square$.

Extracting Coefficients from an FPS:

- let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$.

1) $[x^n](A(x) \pm B(x)) = [x^n]A(x) + [x^n]B(x)$.

2) $[x^n](A(x)\cdot B(x)) = \sum_{i=0}^{n} a_i b_{n-1} = \sum_{i=0}^{n} ([x^i]A(x))([x^{n-i}]B(x))$

3) $[x^n]cA(x) = c[x^n]A(x)$, where c is a constant.

4) $[x^n]x^l A(x) = 0$ if $n < l$, and $[x^n]x^l A(x) = [x^{n-l}]A(x)$ if $n \geq l$.

5) $[x^n]A(cx) = c^n [x^n]A(x)$, where c is a constant

6) $[x^n]A(x^\ell)=0$ if $\ell \nmid n$, and $[x^n]A(x^\ell)=[x^{n/\ell}]A(x)$ if $\ell \mid n$.

Example: determine $[x^n](x^3(3x+1)^7)$ where $n \geq 3$.

$$[x^n]\,x^3(3x+1)^7 = [x^{n-3}](3x+1)^7 = 3^{n-3}[x^{n-3}](x+1)^7$$

$$= 3^{n-3}\binom{7}{n-3}.$$

Example: determine $[x^n]\,x^3(3x+1)^7(1-4x^2)^{-m}$, $m \geq 1$, $n \geq 3$.

$$[x^n]\,x^3(3x+1)^7(1-4x^2)^{-m}$$

$$= [x^n]\,x^3\left(\sum_{i\geq 0}\binom{7}{i}(3x)^i\right)\left(\sum_{j\geq 0}\binom{m+j-1}{j}(4x^2)^j\right)$$

$$= [x^n]\sum_{i\geq 0, j\geq 0}\binom{7}{i}\binom{m+j-1}{j}3^i\,4^j\,x^{3+i+2j}$$

$$= \sum_{i\geq 0, j\geq 0}\binom{7}{i}\binom{m+j-1}{j}3^i\,4^j$$, and only care where $3+i+2j=n$.

$\hookrightarrow \therefore$, $i = n-2j-3$, $j = \frac{n-3-i}{2}$

Substituting:

$$= \sum_{j=0}\binom{7}{n-2j-3}\binom{m+j-1}{j}3^{n-2j-3}\,4^j$$

Since $j = \frac{n-3-i}{2}$, $j \leq \frac{n-3}{2}$ (as $i \geq 0$). $\therefore$, $j \leq \lfloor\frac{n-3}{2}\rfloor$

$$= \sum_{j=0}^{\lfloor\frac{n-2}{3}\rfloor}\binom{7}{n-2j-3}\binom{m+j-1}{j}3^{n-2j-3}\,4^j.$$

Generating Series: Counting

example: how many length-64 binary strings have neither 000 nor 00111 as a substring?

- don't know how to do this for specifically length-64. So, let's generalize for length-n and make a generating series.

- Roadmap:
  1) Let S be a set we wish to count by weight
  2) Decompose S into "simpler" sets using disjoint union and cartesian product
  3) Determine the generating series for the simpler sets
  4) Combine the generating series $\Phi_{S_i}(x)$ to get $\Phi_S(x)$.
  5) The number of elements of weight n in S is $[x^n]\Phi_S(x)$.

- Sum Lemma: let $S = A \cup B$ be a disjoint union. Let ω be a weight function $S \to \mathbb{N}$.
  Then, $\Phi_S^\omega(x) = \Phi_A^\omega(x) + \Phi_B^\omega(x)$.

↳ Proof:

$$\Phi_S^\omega(x) = \sum_{\sigma \in S} x^{\omega(\sigma)} = \sum_{\sigma \in A \cup B} x^{\omega(\sigma)} = \sum_{\sigma \in A} x^{\omega(\sigma)} + \sum_{\sigma \in B} x^{\omega(\sigma)} = \Phi_A^\omega(x) + \Phi_B^\omega(x).$$

- Product Lemma: let $S = A \times B = \{(a,b) : a \in A, b \in B\}$. Let $u: A \to \mathbb{N}$ be a weight function on A, and let $v: B \to \mathbb{N}$ be a weight function on B. Define the weight function $\omega: S \to \mathbb{N}$ as follows:
  - $\forall \sigma = (a,b) \in S,\ \omega(\sigma) = u(a) + v(b)$.

  Then, $\Phi_S^\omega(x) = \Phi_A^\omega(x) \cdot \Phi_B^\omega(x)$.

↳ Proof:

$$\Phi_A^\omega(x) \cdot \Phi_B^\omega(x) = \left(\sum_{a \in A} x^{u(a)}\right)\left(\sum_{b \in B} x^{v(b)}\right) = \sum_{a \in A, b \in B} x^{u(a)+v(b)} = \sum_{\sigma \in S} x^{\omega(\sigma)} = \Phi_S^\omega(x).$$

eg: find the generating series of binary strings of length n where the weight of the string is the number of 1s in it.

- Method 1:

let S = all binary strings of length n.

Then, $\Phi_S^w(x) = \sum_{k\geq 0} a_k x^k$, where a_k = number of elements of weight k = number of binary strings of length n with k 1s = $\binom{n}{k}$.

$\therefore$, $\Phi_S^w(x) = \sum_{k\geq 0} \binom{n}{k} x^k = (1+x)^n$

- Method 2:

let $A = \{0, 1\}$. Define $u(0) = 0$ and $u(1) = 1$.

$$\Phi_A^u(x) = x^{u(0)} + x^{u(1)} = x^0 + x^1 = 1 + x.$$

now, we use A to describe the more complicated set S:

$S = A \times A \times A \times \dots \times A$ (n times).

define $w: S \to \mathbb{N}$ as follows:

$\sigma = (a_1, a_2, \dots, a_n)$, then $w(\sigma) = \#$ 1s in σ $= u(a_1) + u(a_2) + \dots + u(a_n) = \sum_{i=0}^{n} u(a_i)$.

→ see that the weight of the complicated set S is the sum of the weight of the less complicated set A.

$\therefore$, by the product lemma, $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_A^u(x) \cdot \dots \cdot \Phi_A^u(x)$ (n times).

$$= [\Phi_A^u(x)]^n = (1+x)^n.$$

↗ note: order matters!!

Compositions: A composition of $n \in \mathbb{N}$ is a sequence of positive integers $(a_1, a_2, \dots, a_k)$ that add to n. k = # of parts.

- example: the compositions of n = 4 are:

$(1,1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(2,1,1)$, $(2,2)$, $(1,3)$, $(3,1)$, (4).

• **note:** $n=0$ has one composition, $()$.

How many k-part compositions of n are there?

let S = the set of all compositions with k parts.
Define $w: S \to \mathbb{N}$ as follows:
if $\sigma = (a_1, a_2, \ldots, a_k)$, then $w(\sigma) = a_1 + a_2 + a_3 + \ldots + a_k$.
Goal: find $\Phi_S^w(x)$, and the answer will be $[x^n]\,\Phi_S^w(x)$.

let $P = \{1, 2, 3, 4, \ldots\}$.
then, $S = P \times P \times P \times \ldots \times P$ (k times) $= P^k$ cartesian product of k copies of P.

define $u: P \to \mathbb{N}$ by $u(a) = a$ $\forall a \in P$.
Then, $w(\sigma) = a_1 + a_2 + \ldots + a_k = u(a_1) + u(a_2) + \ldots + u(a_k)$.

by the product lemma, $\Phi_S^w(x) = \Phi_P^u(x) \times \ldots \times \Phi_P^u(x)$ (k times)
$= [\Phi_P^u(x)]^k$.

Now, $\Phi_P^u(x) = \sum_{a \in P} x^{u(a)} = x + x^2 + x^3 + x^4 + \ldots = x(1 + x + x^2 + \ldots) = x(1-x)^{-1}$.

So, $\Phi_S^w(x) = [\Phi_P^u(x)]^k = x^k (1-x)^{-k}$

$\therefore$, the number of k-part coefficients of n is $[x^n]\, x^k (1-x)^{-k}$
$= [x^{n-k}](1-x)^{-k} = \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$.

String Lemma: let S be a set with weight function w.
Suppose that S has no elements of weight 0. Then,

$$\Phi_{S^*}^{w^*}(x) = \frac{1}{1-\Phi_S^w(x)}.$$

Proof: let S be a set and $w: S \to \mathbb{N}$ it's weight function.
let S^k denote $S \times S \times S \times \ldots \times S$ (k times) $(S^0 = \{\})$.
Define $S^* = S_1 \cup S_2 \cup S_3 \cup \ldots = \bigcup_{k\geq 0} S^k$ is a disjoint union.
Define $w_k: S^k \to \mathbb{N}$: if $(a_1, a_2, \ldots, a_k) \in S^k$, then $w_k = w(a_1) + \ldots + w(a_k)$.
Define $w^*: S^* \to \mathbb{N}$: $w^*(\sigma) = w_k(\sigma)$ where $\sigma \in S^k$.
→ we assume that S has no elements of weight 0!!

Then, $\Phi_{S^*}^{w^*}(x) = \sum_{k\geq 0} \Phi_{S^k}^{w_k}(x)$ by the sum lemma.

Since $\Phi_{S^k}^{w_k}(x) = \prod_{i=1}^{k} \Phi_S^w(x) = [\Phi_S^w(x)]^k$.

We have $\Phi_{S^*}^{w^*}(x) = \sum_{k\geq 0} [\Phi_S^w(x)]^k = \frac{1}{1-\Phi_S^w(x)}$.

How many compositions of n are there where each part is ≥ 2?

let S = all compositions where each part is ≥ 2.
let $P = \{2, 3, 4, \ldots\}$
Then, $S = P^0 \cup P \cup P^2 \cup P^3$. So, $S = P^*$.

By the string lemma, $\Phi_S(x) = \frac{1}{1-\Phi_P(x)}$.

$$\begin{aligned} \text{Now, } \Phi_P(x) &= x^2 + x^3 + x^4 + \ldots \\ &= x^2(1 + x + x^2 + \ldots) \\ &= x^2(1-x)^{-1} \end{aligned}$$