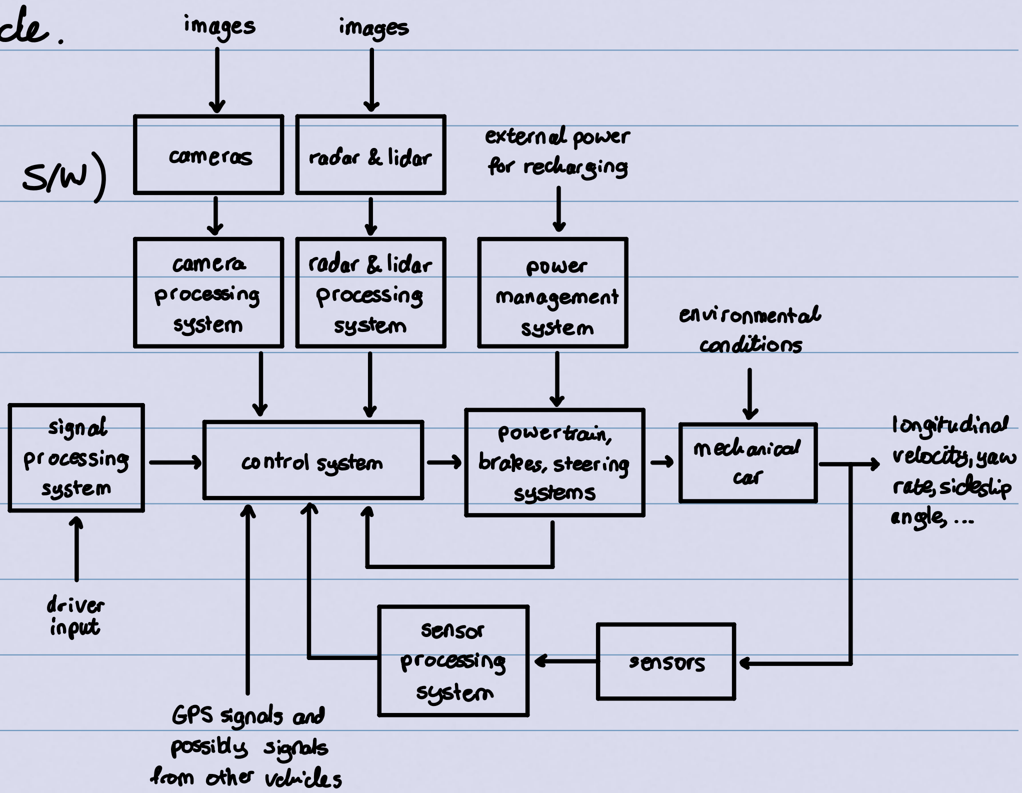
Motivating Example: design of software and hardware used to control an electric vehicle.

boxes = systems (H/W or S/W)

arrows = signals

⇒ this is an example of a block diagram



Can decompose this into 3 types of tasks:

1. Modelling
   - we need some way to describe how systems process to generate outputs
2. Analysis
   - we need tools to determine and study the behavior of the various systems
   - eg. is the control system stable? is it fast or slow? How does it respond in windy conditions?
3. Design
   - we need to have a systematic way to create and tune the various systems (control system, image processing system, etc)

Signal: a function of one or more independent variables, generally containing information about the behavior of some phenomenon of interest.

⇒ Example: a signal may represent a force, a torque, an angle, a speed, a stock price, available SSD memory, etc.

- We will deal with only the situation where there is one independent variable, namely time:
  - If time is varying consistently, it's a continuous-time signal.
    ↳ we denote time by t and continuous-time signals as $x(t), u(t), y(t)$, etc
  - If time jumps from one value to the next, it's a discrete-time signal.
    ↳ we denote time by k and discrete-time signals as $x[k], u[k], y[k]$, etc

System: a device, process, or algorithm that takes one or more input signals and generates one or more output signals.

⇒ Example: each of the blocks in the electric vehicle system, a rocket, a heart, a phone, a planet, etc

It's traditional to denote a generic input signal by u (ie, either $u(t)$ or $u[k]$) and a generic output signal by y (ie, either $y(t)$ or $y[k]$)

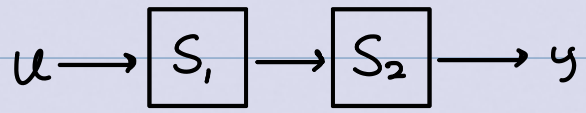
Systems that have one input signal and one output signal are called single-input single-output (SISO). Systems that have multiple inputs and multiple outputs are called multi-input multi-output (MIMO).

The output of the system is also called the response of the system.

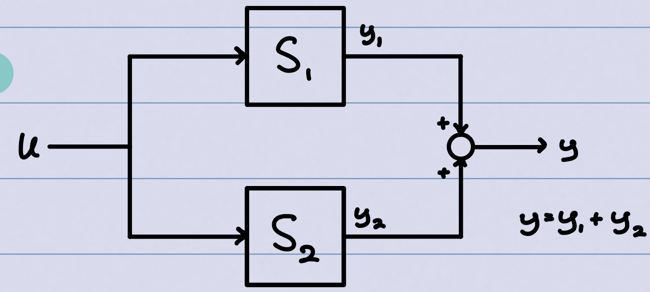
If both the input signal(s) and output signal(s) are continuous-time signals, then we say the system is a continuous-time system. Similarly, if both are discrete-time signals, then we say the system is a discrete-time system. Any other combination results in a hybrid system.

In a block diagram, blocks can be connected in series (aka a cascade connection) or in parallel (with the help of a summer):

series:



parallel:



Differential Equation: any math equation that, in contrast to a purely algebraic equation, includes the derivatives of one or more dependent variable with respect to one or more independent variables.

Ordinary Differential Equation (ODE): a differential equation with only one independent variable.

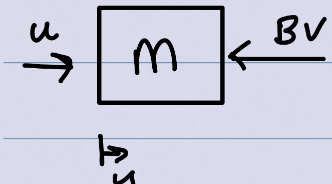
Partial Differential Equation (PDE): a differential equation with more than one independent variable.

Order of a Differential Equation: the order of the highest derivative in the equation.

⇒ Example: are the following algebraic, ODEs, or PDEs?

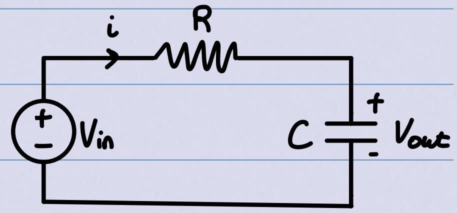
- $\frac{d^3y}{dt^3} + 4y = \frac{du}{dt} + 2u$ — ODE, 3rd order
- $F = ma$ — algebraic
- $F = m\frac{d^2y}{dt^2}$ — ODE, 2nd order
- $\dddot{y} + 2(2 - \dot{y}^3)\dot{y} + 4y = u$ — ODE, 3rd order
- $\frac{\partial y(x,t)}{\partial t} - K\frac{\partial y^2(x,t)}{\partial^2 x} = u(x,t)$ — PDE, 2nd order

Example: consider the dynamics of a vehicle moving in a straight line. The system is affected mainly by the force applied by the engine and air resistance (friction). Let u = input force due to engine and v = output velocity ($=\dot{y}$)



$$F = ma \rightarrow u - Bv = m\ddot{y}$$
$$\rightarrow u - Bv = m\dot{v}$$
$$\rightarrow u = m\dot{v} + Bv$$ $\therefore$, 1st order ODE

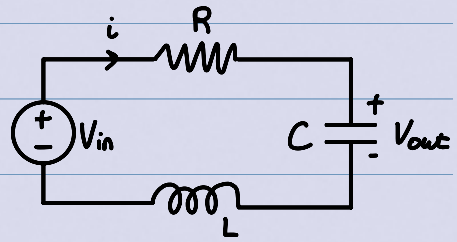
Example: consider this RC circuit. We desire to know the dynamic relationship between the output voltage V_{out} and the input voltage V_{in}. Let i be the current in the loop.



$$Ri - V_{in} + V_{out} = 0, \quad \text{and} \quad i = C\frac{dV_{out}}{dt}.$$

$$\hookrightarrow V_{out} + RC\frac{dV_{out}}{dt} = V_{in}$$ $\therefore$, 1st order ODE

Example: Same as previous example, but now with an inductor included.



$$Ri - V_{in} + V_{out} + L\frac{di}{dt} = 0 \quad \text{and} \quad i = C\frac{dV_{out}}{dt}$$

$$\hookrightarrow V_{out} + RC\frac{dV_{out}}{dt} + LC\frac{d^2V_{out}}{dt^2} = V_{in}$$ $\therefore$, 2nd order ODE

Static / Memoryless System: at each time instant, each possible output doesn't depend on any value of the input except perhaps for the input at the same time instant.

↳ else, the system is said to be Dynamic or to have Memory.

Example : a resistor ($V(t) = i(t)R$) is a static system

Example : a capacitor ($C\frac{dv(t)}{dt} = i(t)$ or $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$) is a dynamic system.

Causal / Non-Anticipative System: at each time instant, each possible output does not depend on the future values of the input.

↳ else, the system is said to be non-causal or acausal.

Example : the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k-1]$ is causal.

Example: the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k+1]$ is noncausal.

Example : the system $y(t) = K\int_{-\infty}^{t} u(\tau)d\tau$ is causal

⇒ A static system is causal!

Linear System: satisfies the superposition property, that is, for any input signals u_1 (w/ output y_1) and u_2 (w/ output y_2) and any constants a_1 and a_2, a response to the input signal $u = a_1u_1 + a_2u_2$ is $y = a_1y_1 + a_2y_2$.

↳ Else, the system is said to be nonlinear.

Homogeneity Property: for any input signal u_1 (w/ output y_1) and any constant a_1, a response to the input signal $u = a_1 u_1$ is $y = a_1 y_1$.

Additivity Property: for any input signals u_1 (w/ output y_1) and u_2 (w/ output y_2), a response to the input signal $u = u_1 + u_2$ is $y = y_1 + y_2$.

A system satisfies the superposition property (ie, the system is linear) if and only if it satisfies both the homogeneity property and the additivity property.

Proof:

($\Rightarrow$) If superposition holds:

- set $a_1 = a_2 = 1$ to conclude additivity holds
- set $a_2 = 0$ to conclude homogeneity holds

($\Leftarrow$) If both homogeneity and additivity hold, then let $y_1 = S(u_1)$ and $y_2 = S(u_2)$. Then a response to input $a_1 u_1 + a_2 u_2$ is:

$S(a_1 u_1 + a_2 u_2)$

$= S(a_1 u_1) + S(a_2 u_2)$ by additivity

$= a_1 S(u_1) + a_2 S(u_2)$ by homogeneity

$= a_1 y_1 + a_2 y_2$.

$\therefore$ superposition property is satisfied.

$\Rightarrow$ it's often faster to check for both homogeneity and additivity instead of superposition directly!

Example: are the following systems linear or nonlinear?

a) $y(t) = K u(t)$

- Satisfies superposition:

Apply input u_1 to get output $y_1 = Ku_1$

Apply input u_2 to get output $y_2 = Ku_2$

Apply input $a_1u_1 + a_2u_2$ to get output $K(a_1u_1 + a_2u_2)$

$= a_1(Ku_1) + a_2(Ku_2) = a_1y_1 + a_2y_2$

$\therefore$, the system is linear.

b) $y(t) = Ku(t) + 1$

- Fails both homogeneity and additivity. Eg:

Apply input u_1 to get output $y_1 = Ku_1 + 1$

Apply input u_2 to get output $y_2 = Ku_2 + 1$

Apply input $u_1 + u_2$ to get output $K(u_1 + u_2) + 1 \neq y_1 + y_2$.

$\therefore$, the system is nonlinear.

c) $y(t) = au(t) + bu^2(t)$

- Fails additivity. Eg:

Apply input u_1 to get output $y_1 = au_1 + bu_1^2$

Apply input u_2 to get output $y_2 = au_2 + bu_2^2$

Apply input $u_1 + u_2$ to get output $a(u_1 + u_2) + b(u_1 + u_2)^2 \neq y_1 + y_2$

$\therefore$, the system is nonlinear (if $b \neq 0$)

d) $y(t) = \sin(u(t))$

- Fails homogeneity. Eg:

Apply input u_1 to get output $y_1 = \sin(u_1)$

Apply input a_1u_1 to get output $y = \sin(a_1u_1) \neq a_1u_1$

$\therefore$, the system is nonlinear

e) $y(t) = \sin(t)u(t)$

· Satisfies superposition!

Apply input u_1 to get output $y_1(t) = \sin(t)\, u_1(t)$

Apply input u_2 to get output $y_2(t) = \sin(t)\, u_2(t)$

Apply input $a_1u_1 + a_2u_2$ to get output

$y(t) = \sin(t)(a_1u_1(t) + a_2u_2(t)) = a_1y_1(t) + a_2y_2(t)$.

$\therefore$, the system is linear!

f) $y(t) = 5u(t) + u(t)y(t)$

Solving for y: $y = 5u + uy \rightarrow y = \dfrac{5u}{1-u}$

Fails homogeneity:

Apply input u_1 to get output $y_1 = \dfrac{5u_1}{1-u_1}$ $(u_1 \neq 1)$

Apply input a_1u_1 to get output $y = \dfrac{5a_1u_1}{1-a_1u_1} \neq a_1y_1$

$\therefore$ the system is nonlinear!

g) $y(t) = K\dfrac{du(t)}{dt}$

· Satisfies superposition:

Apply input u_1 to get output $y_1 = K\dfrac{du_1}{dt}$

Apply input u_2 to get output $y_2 = K\dfrac{du_2}{dt}$

Apply input $a_1u_1 + a_2u_2$ to get output $y = K\dfrac{d}{dt}(a_1u_1 + a_2u_2)$

$\therefore$ the system is linear! $= a_1y_1 + a_2y_2$

h) $y(t) = K\int_{-\infty}^{t} u(\tau)\, d\tau$

· Satisfies superposition!

Apply input u_1 to get output $y_1 = K\int_{-\infty}^{t} u_1(\tau)\, d\tau$

Apply input u_2 to get output $y_2 = K\int_{-\infty}^{t} u_2(\tau)\, d\tau$

Apply input $a_1u_1 + a_2u_2$ to get output $y = K\int_{-\infty}^{t} a_1u_1(\tau) + a_2u_2(\tau)\, d\tau$

$\therefore$, the system is linear! $= a_1y_1 + a_2y_2$

i) $M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + K_s y(t) = u(t)$ (mass-spring damper system)

· Satisfies superposition

Apply input u_1 to get output y_1 satisfying $M\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_s y_1 = u_1$ ①

Apply input u_2 to get output y_2 satisfying $M\frac{d^2y_2}{dt^2} + B\frac{dy_2}{dt} + K_s y_2 = u_2$ ②

Compute $a_1 \cdot$ ① $+ a_2 \cdot$ ②:

$$M\frac{d^2}{dt^2}(a_1y_1 + a_2y_2) + B\frac{d}{dt}(a_1y_1 + a_2y_2) + K_s(a_1y_1 + a_2y_2) = a_1u_1 + a_2u_2$$

$\therefore$, $a_1y_1 + a_2y_2$ is a solution to the ODE when $u = a_1u_1 + a_2u_2$.

So, the system is linear!

j) $\frac{dP(t)}{dt} = aP(t) + bP^2(t) + u(t)$

· Fails homogeneity.

Apply input u_1 to get output satisfying $\frac{dP_1}{dt} = aP_1 + bP_1^2 + u_1$ ①

Suppose the system satisfies homogeneity. Then, $\frac{da_1P_1}{dt} = aa_1P_1 + b(a_1P_1)^2 + u_1$

$\Rightarrow \frac{dP_1}{dt} = aP_1 + ba_1P_1^2 + u_1$ ②

If ① and ② both hold, then so does ①-②:

$0 = bP_1^2 - ba_1P_1^2 \Rightarrow b(a_1 - 1)P_1^2 = 0$ ③

But ③ doesn't hold in general (except if $b=0$ or $a_1 = 1$ or $P_1 = 0$ which are not of interest).

$\therefore$, the system does not satisfy homogeneity by contradiction, and therefore, the system is nonlinear!

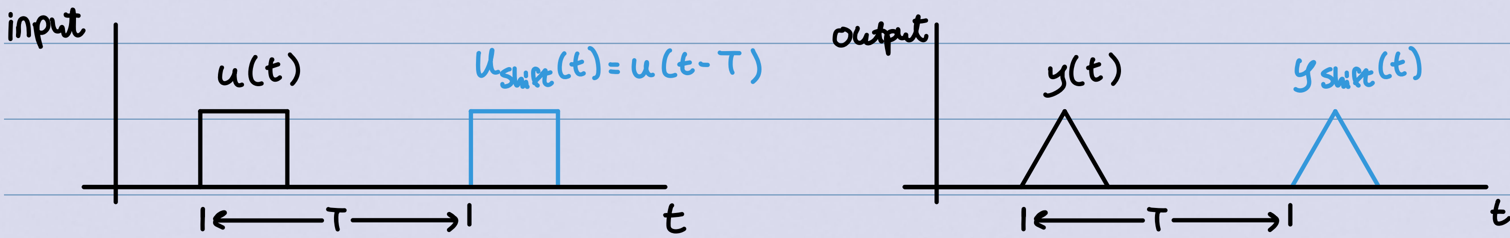
Time-Invariant Systems vs Time-Varying Systems

A system is said to be time-invariant if a time shift in the input signal always causes the same time shift (but no other distortion) in the output signal. More formally:

Assume input u is applied to the system with an associated output y. For a constant T (with $-\infty < T < \infty$), let $u_{shift}(t) = u(t-T)$ denote the shifted output. The system is said to be time-invariant if, for all u and all T, there exists an output associated with the input u_{shift}, denoted y_{shift}, such that:

$$y_{shift}(t) = y(t-T) \text{ for } -\infty < T < \infty.$$

A system that is not time-invariant is said to be time-varying.



↳ system is time-invariant if $y_{shift}(T) = y(t-T)$.

Example: indicate if the following systems are time-invariant or time-varying:

a) $y(t) = Ku(t)$

Apply input u to get output $y(t) = Ku(t)$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t)$

$\therefore$, the system is time-invariant! $\quad = Ku(t-T) = y(t-T)$

b) $y(t) = Ku(t)+1$

Apply input u to get output $y(t) = Ku(t)+1$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t)+1$

$\therefore$, the system is time-invariant! $\quad = K(u(t-T))+1 = y(t-T)$

c) $y(t) = Ku(t) + t$

Apply input u to get output $y(t) = Ku(t) + t$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t) + t$

$\therefore$, the system is time-varying. $= Ku(t-T)+t \neq Ku(t-T)+t-T = y(t-T)$

d) $y(t) = \sin(t)u(t)$

Apply input u to get output $y(t) = \sin(t)u(t)$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(t)u_{shift}(t)$

$\therefore$, the system is time-varying.

$$= \sin(t)u(t-T) \neq \sin(t-T)u(t-T)$$
$$= y(t-T)$$

e) $y(t) = \sin(u(t))$

Apply input u to get output $y(t) = \sin(u(t))$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(u_{shift}(t))$

$\therefore$, the system is time-invariant.

$$= \sin(u(t-T)) = y(t-T).$$

f) $y(t) = K\dfrac{du(t)}{dt}$

Apply input u to get output $y(t) = K\dfrac{du(t)}{dt}$

Apply input u_{shift} to get output $y_{shift}(t) = K\dfrac{du_{shift}(t)}{dt}$

$\therefore$, the system is time-invariant.

$$= K\frac{du(t-T)}{dt} = y(t-T).$$

g) $y(t) = K\int_{-\infty}^{t} u(\tau)\,d\tau$

Apply input u to get output $y(t) = K\int_{-\infty}^{t} u(\tau)\,d\tau$

Apply input u_{shift} to get output $y_{shift}(t) = K\int_{-\infty}^{t} u_{shift}(\tau)\,d\tau$

$\therefore$, the system is time-invariant!

$$= K\int_{-\infty}^{t} u(t-T)\,d\tau = K\int_{-\infty}^{t-T} u(\tau)\,d\tau$$
$$= y(t-T)$$

h) $M\dfrac{d^2y(t)}{dt^2} + B\dfrac{dy(t)}{dt} + K_s y(t) = u(t)$

Apply input u to get output y satisfying:

$$M\frac{d^2y(t)}{dt}\bigg|_t + B\frac{dy}{dt}\bigg|_t + K_s y(t) = u(t) \qquad (1)$$

Apply input $u_{shift}(t)$ to get output y_{shift} satisfying:

$$M\frac{d^2y_{shift}(t)}{dt}\bigg|_t + B\frac{dy_{shift}}{dt}\bigg|_t + K_s y_{shift}(t) = u_{shift}(t) = u(t-T)$$

let $x = t-T \rightarrow dx = dt$

$$\Rightarrow M \left.\frac{d^2 y_{shift}(t)}{dx^2}\right|_{x+T} + B \left.\frac{dy_{shift}}{dx}\right|_{x+T} + K_s\, y_{shift}(t) = u(x) \quad ②$$

① and ② represent the same ODE with the same input.
Therefore, the outputs match: $y_{shift}(x+T) = y(x)$

$$\Leftrightarrow y_{shift}(t) = y(t-T).$$

$\therefore$, the system is time-invariant.

Definition of a solution to an ODE

A general n^{th}-order ODE with input u and output y can be written as: $F(t, y, y', y'', \ldots, y^{(n)}, u, u', u'', \ldots, u^{(m)}) = 0 \quad (n \geq m)$

For a given input u, any output y which satisfies the equation is considered to be a solution to the ODE. More formally:

Assume the input $u(t)$ is defined on some time interval $t_0 < t < t_1$ (possibly with $t_0 = -\infty$ and/or $t_1 = \infty$) and that the first m derivatives of u are defined for $t_0 < t < t_1$. Then a function $y(t) = \phi(t)$ is said to be a solution to the ODE if the first n derivatives of ϕ exist for $t_0 < t < t_1$ and if $F(t, y, y', y'', \ldots, y^{(n)}, u, u', u'', \ldots, u^{(m)}) = 0$ for $t_0 < t < t_1$.

Example: this is the logistic equation which models the dynamics of population growth: $\frac{dP}{dt} = aP - bP^2 + u$. Consider the special case where $u = 0$ and $b = 0$, resulting in $\frac{dP}{dt} = aP$. Show that $P(t) = 3e^{at}$ and $P(t) = 10e^{at}$ (for $-\infty < t < \infty$) are both solutions:

$P(t) = 3e^{at} \rightarrow \frac{dP}{dt} = 3ae^{at} = aP(t)$. $\therefore$, $\frac{dP}{dt} = aP$ ✓

$P(t) = 10e^{at} \rightarrow dP/dt = 10ae^{at} = aP(t)$. $\therefore$, $dP/dt = aP$ ✓

$\therefore$, both $P(t) = 3e^{at}$ and $P(t) = 10e^{at}$ are solutions.

↳ this shows that there need not be only one solution to an ODE. Usually, there exists a whole family of solutions, namely $P(t) = ce^{at}$ (for $-\infty < t < \infty$) for some constant c. Formally:

It's normally (but not always) true that there exists a family of solutions to $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0$, and the family is parameterised by n constants $c_1, c_2, \dots, c_n$. The family of solutions is said to be the general solution to the ODE. If specific values are chosen for the constants $c_1, c_2, \dots, c_n$, we say that the resulting $y(t)$ is a particular solution to the ODE.

Example: imagine a mass-spring damper system modeled by the ODE $M\ddot{y} + B\dot{y} + K_s y = u$. Consider the case where $u=0$, $M=1$, $B=3$, and $K_s = 2$: $\ddot{y} + 3\dot{y} + 2y = 0$.

a) verify that $y(t) = -c_1e^{-2t} + 2c_1e^{-t} - c_2e^{-2t} + c_2e^{-t}$ (for $-\infty < t < \infty$), parameterised by 2 arbitrary constants c_1 & c_2, is a family of solutions.

$$y = -c_1e^{-2t} + 2c_1e^{-t} - c_2e^{-2t} + c_2e^{-t}$$

$$\dot{y} = 2c_1e^{-2t} - 2c_1e^{-t} + 2c_2e^{-2t} - c_2e^{-t}$$

$$\ddot{y} = -4c_1e^{-2t} + 2c_1e^{-t} - 4c_2e^{-2t} + c_2e^{-t}$$

$$\therefore,\ \ddot{y} + 3\dot{y} + 2y = 0 \rightarrow -4c_1e^{-2t} + 2c_1e^{-t} - 4c_2e^{-2t} + c_2e^{-t}$$
$$+ 3(2c_1e^{-2t} - 2c_1e^{-t} + 2c_2e^{-2t} - c_2e^{-t})$$

$$+2(-c_1e^{-2t}+2c_1e^{-t}-c_2e^{-2t}+c_2e^{-t})=0$$

$$\rightarrow -4c_1e^{-2t}+2c_1e^{-t}-4c_2e^{-2t}+c_2e^{-t}+6c_1e^{-2t}-6c_1e^{-t}+6c_2e^{-2t}$$
$$-3c_2e^{-t}-2c_1e^{-2t}+4c_1e^{-t}-2c_2e^{-2t}+2c_2e^{-t}=0$$

→ everything cancels, get $0=0$ ✓

b) find a particular solution to $\ddot{y}+3\dot{y}+2y=0$

arbitrarily, $c_1=c_2=1$: $y(t)=-e^{-2t}+2e^{-t}-e^{-2t}+e^{-t}$
↳ $y(t)=-2e^{-2t}+3e^{-t}$

equally arbitrarily, $c_1=c_2=0$: $y(t)=0+0+0+0$
↳ $y(t)=0$.

We should get used to the idea that there's a family of solutions to an ODE. However, there are exceptions:

example: consider the system described by the followers nonlinear ODE (for $-\infty<t<\infty$): $(dy/dt)^2+y^2=u$

a) if $u=-1$, there are no solutions (why?)

the LHS cannot be negative!

b) if $u=0$, there is only one solution (why?)

only solution is $dy/dt=0$ and $y=0$.

c) if $u=2$, there exist multiple solutions (why?)

One possible solution: $y(t)=\sqrt{2}$, so $dy/dt=0$.

$\therefore \left(\frac{dy}{dt}\right)^2+y^2=\sqrt{2}^2+0=2$.

another possible solution: $y(t)=\sin(t)+\cos(t)$, so $\frac{dy}{dt}=\cos(t)-\sin(t)$.

$\therefore, \left(\frac{dy}{dt}\right)^2 + y^2 = (\cos t - \sin t)^2 + (\sin t + \cos t)^2$

$= \cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\cos t \sin t + \cos^2 t$

$= 2\sin^2 t + 2\cos^2 t = 2(\sin^2 t + \cos^2 t) = 2$

The Initial-Value-Problem (IVP)

Consider the ODE $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0$

Assume the input u is given. The initial-value-problem involves finding a solution (if one exists) of the ODE for $t_0 \leq t \leq t_1$ (or $t_0 \leq t \leq t_1$), possibly with $t_1 = \infty$, subject to the following initial conditions:

$$y(t) = y_0,\ y'(t_0) = y_1,\ \dots,\ y^{(n-1)}(t_0) = y_{n-1}$$

(assuming that $y_0, y_1, \dots, y_{n-1}$ are given constants)

note: initial conditions are all specified at the same time instant, $t = t_0$. If different times are used (eg $y(0) = 2$ and $y'(3) = -4$), the problem is no longer an IVP, but a boundary-value-problem (BVP). We will not deal with BVPs.

note 2: an IVP usually has a unique solution (with exceptions)

Example: consider again the mass-spring-damper system. Find the particular solution to the following IVP:

Solve $\ddot{y} + 3\dot{y} + 2y = 0$ subject to $y(0) = 2$ and $\dot{y}(0) = 3$

We saw previously that $y(t) = -c_1 e^{-2t} + 2c_1 e^{-t} - c_2 e^{-2t} + c_2 e^{-t}$ is a family of solutions to $\ddot{y} + 3\dot{y} + 2y = 0$. at initial $t = 0$:

$y(0) = -c_1 e^{-2(0)} + 2c_1 e^{-(0)} - c_2 e^{-2(0)} + c_2 e^{-(0)} = -c_1 + 2c_1 - c_2 + c_2 = c_1$

$\dot{y}(0) = 2c_1 e^{-2(0)} - 2c_1 e^{-(0)} + 2c_2 e^{-2(0)} - c_2 e^{-(0)} = 2c_1 - 2c_1 + 2c_2 - c_2 = c_2$

$\therefore, y(t) = y(0)(-e^{-2t} + 2e^{-t}) + \dot{y}(0)(-e^{-2t} + e^{-t})$ for $t \geq 0$ is another way of writing the family of solutions.

we wee given now that $y(0)=2$ and $\dot{y}(0)=3$, so:

$$y(t) = 2(-e^{-2t}+2e^{-t}) + 3(-e^{-2t}+e^{-t})$$
$$= -2e^{-2t} + 4e^{-t} - 3e^{-2t} + 3e^{-t} = -5e^{-2t} + 7e^{-t}$$

$\therefore$, $y(t) = 7e^{-t} - 5e^{-2t}$ is the particular solution to the IVP subject to the given initial conditions.

However, it's not always true that an IVP has a unique solution. Non-uniqueness arises in the following example because of the specific form of nonlinearity.

Example: the IVP $\dot{y}(t) - 4t\sqrt{y(t)} = 0$, subject to $y(0)=0$ has 2 solutions.

1. $y(t) = 0$ $(t \geq 0)$
2. $y(t) = t^4$ $(t \geq 0)$ $\rightarrow 4t^3 - 4t\sqrt{t^4} = 4t^3 - 4t^3 = 0$

Factors that affect the solution to an IVP

There are two distinct factors that affect the solution to an IVP: the input signal and the initial conditions.

Example: consider again the mass-spring damper system, but now with $u=2$. As before, we'll have $M=1$, $B=3$, $K_s=2$. Using tools we'll later learn, we can show that the solution to the IVP $\ddot{y} + 3\dot{y} + 2y = 2$ subject to $y(0)=2$ and $\dot{y}(0)=3$ is, for $t \geq 0$:

$$y(t) = \overbrace{y(0)(-e^{-2t}+2e^{-t}) + \dot{y}(0)(-e^{-2t}+e^{-t})}^{\text{response due to initial conditions}} + \overbrace{1 - 2e^{-t} + e^{-2t}}^{\text{response due to input signal}}$$

$= -4e^{-2t} + 5e^{-t} + 1$. $\rightarrow$ same transient but different steady-state response transient response: response for finite t

steady-state response: response as $t \to \infty$

the three approaches that we will consider to solve first-order ODEs are:

1. phase portrait sketch
2. seperation of variables
3. exact differential approach

Approach 1: Phase Portrait Sketch

Simple qualitative approach for determining solutions to an ODE that has the form:

$$\frac{dy}{dt} = f(y, u)$$

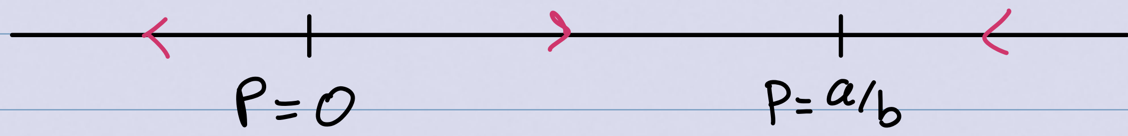
for constant input u. By considering values of y and u where $f(y,u) = 0$, and the sign of $f(y,u)$ between those values, we can get a good idea (qualitatively) of how $y(t)$ behaves, at least for the situation where u is a constant.

Example: consider the logistic equation with $u=0$, $\frac{dP}{dt} = P(a - bP)$

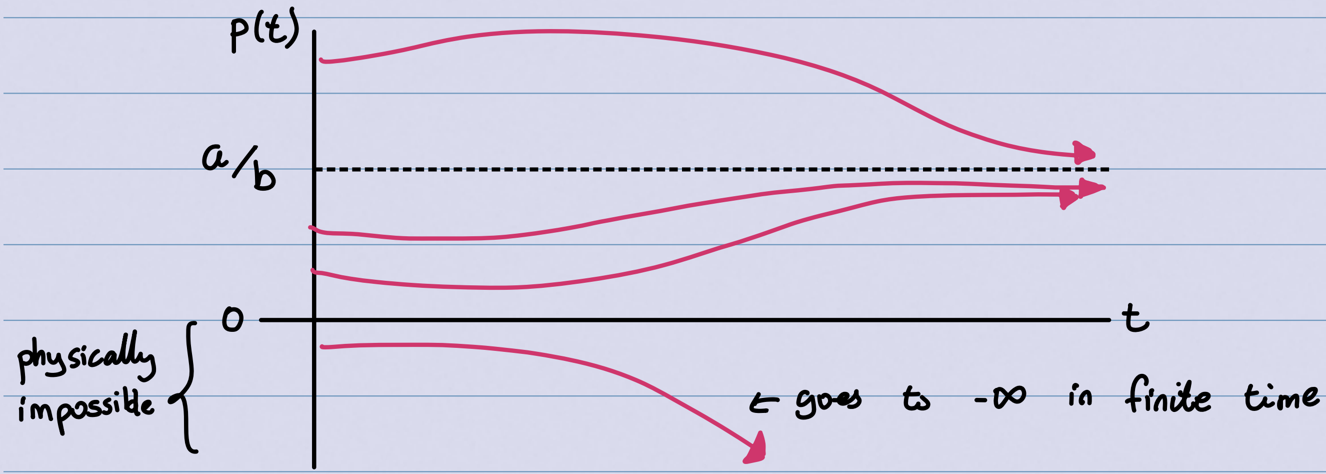
we can determine the sign of the derivative for different values of P:

P value	sign of dP/dt	
$P < 0$	$-$	→ this case isn't possible
$P = 0$	0	
$0 < P < a/b$	$+$	
$P = a/b$	0	
$P > a/b$	$-$	

Using this information, we can sketch on an axis the phase portrait of the ODE:



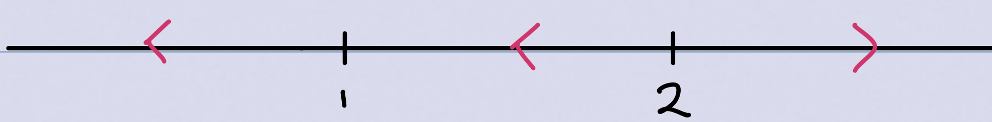
So, we can expect the solution $P(t)$ to look something like:



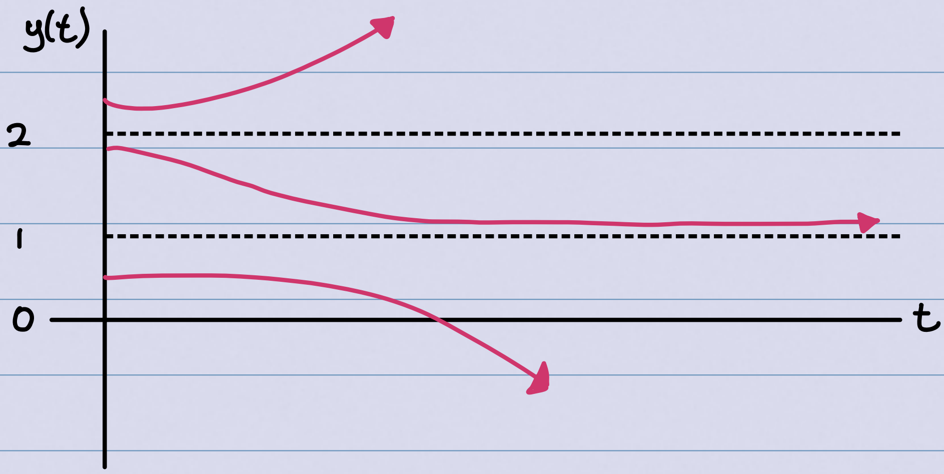
→ we expect it to always converge to a/b in steady-state

Example: consider the following ODE $\frac{dy}{dt} = (y-1)^2(y-2)$ w/ $y(0) = y_0$. Use the phase portrait approach to characterize the solutions.

y value	sign of dy/dt
$y < 1$	$-$
$y = 1$	0
$1 < y < 2$	$-$
$y = 2$	0
$y > 2$	$+$



the expected solution for various values of y_0:



Approach 2: Separation of Variables

Phase portrait method is simple to apply and usually insightful. However, if we want more detail, we need quantitative analysis.

Separation of Variables works for any first-order ODE that can be written (for zero u) in the form: $\frac{dy}{dt} = f(t) \cdot g(y)$.
An ODE in this form is said to be separatable.

↳ for nonzero u, the structure is: $\frac{dy}{dt} = f(t,u) \cdot g(y)$ or $\overbrace{\frac{dy}{dt} = f(t,u) \cdot g(y,u)}^{\text{only if } u \text{ is constant}}$

This form is nice since we can rearrange into a form that can often be solved explicitly via integration.

Example: which of the following can be written in separable form?

a) $\frac{dy}{dt} = e^t y$ yes! already in separate form

b) $t\frac{dy}{dt} = ye^{y+2t} \Rightarrow \frac{dy}{dt} = \left(\frac{e^{2t}}{t}\right)(ye^y)$ yes!

c) $\frac{dy}{dt} = y + e^t$ not separable

Example: a) use separation of variables to determine the general solution to $\frac{dy}{dt} = e^t y$. b) what is the particular solution associated with the initial condition $y(0)=2$?

a) $\frac{dy}{dt} = e^t y$!! can split up dy/dx

$\Rightarrow \frac{dy}{y} = e^t\, dt \Rightarrow \int \frac{1}{y}\, dy = \int e^t\, dt \Rightarrow \ln|y| = e^t + C_1 \Rightarrow |y| = e^{e^t + C_1}$ if $y \neq 0$

$\therefore,\ y(t) = C_2 e^{e^t}$

special case $y(t)=0$: this is a valid solution to the ODE, and the general solution addresses it when $C_2 = 0$. so C_2 can be anything!

$\therefore,\ y(t) = C_2 e^{e^t} \quad (-\infty < t < \infty,\ -\infty < C_2 < \infty)$

b) $y(0)=2 \Rightarrow C_2 e^{e^0} = 2 \Rightarrow C_2 e^1 = 2 \Rightarrow C_2 = 2e^{-1}$

$\therefore,\ y(t) = 2e^{e^t - 1},\quad t \geq 0.$

Example: a) consider the logistic equation, simplified with $b, u = 0$: $\frac{dP(t)}{dt} = aP(t)$. Use the separation of variables method to determine the general solution. b) what is the particular solution associated with the initial condition $P(0) = P_0$?

a) $\frac{dP}{dt} = aP(t)$

$\Rightarrow \int \frac{1}{P}\, dP = \int a\, dt \Rightarrow \ln|P| = at + C_1 \Rightarrow |P| = e^{at + C_1} \Rightarrow P(t) = C_2 e^{at}$

special case: $P(t)=0 \Rightarrow$ a solution to the ODE and the general solution includes this case when $c_2 = 0$.

$\therefore$, $P(t) = c_2 e^{at}$ $(-\infty < t < \infty$, c_2 arbitrary constant$)$

b) $P(0) = P_0$: $P_0 = c_2 e^{a(0)} = c_2$.

$\therefore$, $P(t) = P_0 e^{at}$, $t \geq 0$.

Approach 3: Exact Differential approach

(Review) Schwartz's Theorem: consider a function $\phi(t,y)$ whose second-order (partial) derivatives $\frac{\partial^2\phi}{\partial t^2}$, $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial t\partial y}$, $\frac{\partial^2\phi}{\partial y\partial y}$ all exist and are continuous. Then it's a fact that:

$$\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial^2\phi}{\partial y\partial t}.$$

In addition, the differential : $d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial y}dy$ exists.

Example: consider $\phi(t,y) = t + 2y + e^{ty}$.

All the partial derivatives exist and are continuous:

$\frac{\partial\phi}{\partial t} = 1 + ye^{ty}$

$\hookrightarrow \frac{\partial^2\phi}{\partial y\partial t} = e^{ty} + tye^{ty}$

$\frac{\partial\phi}{\partial y} = 2 + te^{ty}$

$\hookrightarrow \frac{\partial\phi}{\partial t\partial y} = e^{ty} + tye^{ty}$

$$\therefore, \frac{\partial^2\phi}{\partial y\partial t} = \frac{\partial\phi^2}{\partial t\partial y}!$$

& the differential of $\phi(t,y)$ is $d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial y}dy = (1+ye^{ty})dt + (2+te^{ty})dy$

Consider a first-order ODE of the form (for zero u): $\frac{dy}{dt} = -\frac{M(t,y)}{N(t,y)}$

and rewrite it as $M(t,y)\,dt + N(t,y)\,dy = 0$.

If there is a function $\phi(t,y)$ with continuous second-order partial derivatives whose differentials happen to match the above equation, that is, $d\phi(t,y) = \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial y}dy = M(t,y)\,dt + N(t,y)\,dy$, then we say that $M(t,y)\,dt + N(t,y)\,dy$ is an exact differential.

If $M(t,y)\,dt + N(t,y)\,dy$ is an exact differential, then $M(t,y)\,dt + N(t,y)\,dy = 0$ can be written equivalently as $d\phi(t,y) = 0$.

We conclude that the family of solutions is (for arbitrary constant c): $\phi(t,y) = c$.

Example: use the exact differential approach to find all solutions to the ODE $\frac{dy}{dt} = -\left(\frac{1+ye^{ty}}{2+te^{ty}}\right)$

$$(1+ye^{ty})\,dt + (2+te^{ty})\,dy = 0$$

from the previous example, we saw that $(1+ye^{ty})\,dt + (2+te^{ty})\,dy$ is an exact differential with $\phi(t,y) = t + 2y + e^{ty}$.

$\therefore$, general solution of the ODE is $t + 2y + e^{ty} = c$ $\quad(-\infty < t < \infty)$

↳ can't solve for y → an "implicit" solution

Theorem: Assume $M(t,y)$ and $N(t,y)$ are both continuous with continuous first-order partial derivatives. Then $M(t,y)\,dt + N(t,y)\,dy$ is an exact differential if and only if $\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t}$.

|

↳ Proof:

(⇒) if $M(t,y)dt + N(t,y)dy$ is an exact differential, then (by definition) $\exists$ a function $\phi(t,y)$ st. $d\phi(t,y) = M(t,y)dt + N(t,y)dy$ with $\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial^2\phi}{\partial y\partial t}$.

But, $\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial^2\phi}{\partial y\partial t}$ is equivalent to $\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t}$ since:

$$\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial t}(N(t,y)) \quad \text{and} \quad \frac{\partial^2\phi}{\partial y\partial t} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) = \frac{\partial}{\partial y}(M(t,y)).$$

∴, ⇒ proved.

(⇐) suppose that $\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t}$ holds. we must construct a function $\phi(t,y)$ such that $d\phi(t,y) = M(t,y)dt + N(t,y)dy$. This is hard so it'll be illustrated in the following examples:

Example) which of the following are exact differentials?

a) $ydt + tdy$ $\quad \frac{\partial(y)}{\partial y} = \frac{\partial(t)}{\partial t} = 1$ ⇒ yes!

b) $ydt - tdy$ $\quad \frac{\partial}{\partial y}(y) \neq \frac{\partial}{\partial y}(-t)!$ ⇒ no

c) $-y\sin(t)dt + \cos(t)dy$ $\quad \frac{\partial}{\partial y}(-y\sin t) = \frac{\partial}{\partial t}(\cos t)) = -\sin t$ ⇒ yes!

d) $t^2dt + (1+y+y^2)dy$ $\quad \frac{\partial}{\partial y}(t^2) = \frac{\partial}{\partial t}(1+y+y^2) = 0$ ⇒ yes!

Example: use the exact differentials approach to find the solution to $\frac{dy(t)}{dt} + u(t)y(t) = 0$ for the initial condition $y(1) = 4$ with $u(t) = 1/t$ for $t \geq 1$.

$\frac{dy}{dt} + \frac{1}{t}y = 0 \Rightarrow y\,dt + t\,dy = 0$, (so $M = y$ and $N = t$).

See that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} = 1$, so $y\,dt + t\,dy$ is an exact differential.

Therefore, there exists a $\phi(t,y)$ such that $d\phi = y\,dt + t\,dy$:

don't assume $+C$, just do $+f(t)$!!

Step 1: $\frac{\partial \phi}{\partial y} = t \Rightarrow \phi(t,y) = ty + f(t)$, arbitrary function $f(t)$

Step 2: $\frac{\partial \phi}{\partial t} = y \Rightarrow \frac{\partial}{\partial t}(ty + f(t)) = y \Rightarrow y + f'(t) = y \Rightarrow f'(t) = 0.$

Since the derivative of $f(t) = 0$, $f(t)$ must be a constant, say C_1

$\therefore$, $f(t) = C_1$.

So, $\phi(t,y) = ty + C_1$.

$\therefore$, the general solution is $\phi(t,y) = C$, ie $ty = C_2$ or $y(t) = \frac{C_2}{t}$

now force $y(1) = 4$ to get $C_2 = 4$. $\therefore$, $y(t) = 4/t$, $t \geq 1$.

Example: use the exact differential approach for $\frac{dP}{dt} = aP$.

$dP - aP\,dt = 0$ $(M = 1,\ N = aP)$.

but, $\frac{\partial M}{\partial t} = 0 \neq \frac{\partial N}{\partial t} = aP'(t)$.

we can fix this by rearranging to $\underbrace{\frac{1}{P}}_{M}dP - \underbrace{a}_{N}dt = 0$,

so $\frac{\partial M}{\partial t} = 0 = \frac{\partial N}{\partial t}$! Now we must find $\phi(t,y)$ st $d\phi = \frac{1}{P}dP - a\,dt$.

Step 1: $\frac{\partial \phi}{\partial P} = 1/P \Rightarrow \int \partial\phi = \int 1/P\, dP \Rightarrow \phi = \ln|P| + f(t)$

Step 2: $\frac{\partial \phi}{\partial t} = -a \Rightarrow \frac{\partial}{\partial t}(\ln|p| + f(t)) = -a \Rightarrow f'(t) = -a$

so, $f(t) = -at + C_1$. P is function of t?

$\therefore$, $\phi = \ln|P| - at + C_1$. so, general solution is $\phi(t,y) = C$,

ie, $\ln|P| - at = C_1$.

remove absolute value

$\hookrightarrow$ so, $|P(t)| = e^{at+C_1} = C_2 e^{at}$. $\therefore$, $P(t) = C_3 e^{at}$.

now we force $P(0) = P_0$ to get $P_0 = C_3(1) = C_3$.

therefore, $P(t) = P_0 e^{at}$, $t \geq 0$.

Example: use the exact differential approach to find the solution to $\frac{dy}{dt} = \frac{-ty^2}{2+t^2y}$ with initial condition $y(0) = 3$.

rewrite as $ty^2 dt + (2+t^2y)dy = 0$. $(M = ty^2,\ N = 2+t^2y)$

$\frac{\partial}{\partial y}(ty^2) = 2ty = \frac{\partial}{\partial t}(2+t^2y)$! So, we must find $\phi(t,y)$ such that $d\phi(t,y) = ty^2 dt + (2+t^2y)dy$

Step 1: $\frac{\partial \phi}{\partial t} = ty^2 \Rightarrow \int \partial\phi = y^2 \int t\,dt \Rightarrow \phi(t,y) = \frac{y^2t^2}{2} + f(y)$

step 2: $\frac{\partial \phi}{\partial y} = 2+t^2y \Rightarrow \frac{\partial}{\partial y}(\frac{1}{2}y^2t^2 + f(y)) = 2+t^2y \Rightarrow t^2y + f'(y) = 2+t^2y$

$\Rightarrow f'(y) = 2 \Rightarrow f(y) = 2y + C_1$.

$\therefore$, $\phi(t,y) = \frac{1}{2}t^2y^2 + 2y + C_1$.

The family of solutions to the ODE is $\frac{1}{2}t^2y^2 + 2y = C_2$.

force $y(0)=3$ to get $6=C_2$.

$$\therefore, y(t) = \begin{cases} \frac{2}{t^2}\sqrt{1+3t^2} - 1, & t > 0 \\ 3, & t = 0 \end{cases}$$

Use of Integrating Factors

Sometimes a re-arrangement is needed to make an ODE solveable by the exact differential approach.

Suppose that $M(t,y)dt + N(t,y)dy = 0$ is NOT an exact differential. We modify this function by multiplying each side by a yet-to-be-determined $\mu(t,y)$: $\mu(t,y)M(t,y)dt + \mu(t,y)N(t,y)dy = 0$

we know this is an exact differential if and only if:

$$\frac{\partial \mu(t,y)M(t,y)}{\partial y} = \frac{\partial \mu(t,y)N(t,y)}{\partial t}$$

or: $\mu(t,y)\frac{\partial M(t,y)}{\partial y} + \frac{\partial \mu(t,y)}{\partial y}M(t,y) = \mu(t,y)\frac{\partial N(t,y)}{\partial t} + \frac{\partial \mu(t,y)}{\partial t}N(t,y)$

This is super messy, but we can solve for $\mu(t,y)$ under some following special cases:

- suppose that we require that $\mu(t,y)$ depends only on t. then it simplifies to an ODE:

$$\frac{d\mu(t)}{dt} = \Delta_1(t,y)\mu(t) \quad \text{where} \quad \Delta_1(t,y) = \frac{\left(\frac{\partial M(t,y)}{\partial y} - \frac{\partial N(t,y)}{\partial t}\right)}{N(t,y)}$$

if $\Delta_1(t,y)$ also happens to depend only on t, then we can use separation of variables to solve for $\mu(t)$: $\mu(t) = Ce^{\int \Delta_1(t)dt}$

↳ can use $c=1$

• or, suppose $\mu(t,y)$ depends only on y. Then the messy equation simplifies to:

$$\frac{\partial \mu(y)}{\partial y} = \Delta_2(t,y) \quad \text{where} \quad \Delta_2(t,y) = \frac{\left(\frac{\partial N(t,y)}{\partial t} - \frac{\partial M(t,y)}{\partial y}\right)}{M(t,y)}.$$

if $\Delta_2(t,y)$ also happens to depend only on y, we can then use separation of variables to solve for $\mu(y)$: $\mu(y) = Ce^{\int \Delta_2(y)\,dy}$

can use $C=1$

Example: use the exact differential approach to find the solution to $6ty\,dt + (4y + 9t^2)\,dy = 0$ with $y(0) = 1$.

let $ty = M$ & $4y + 9t^2 = N$.

then $\frac{\partial M}{\partial t} = y \neq \frac{\partial N}{\partial y} = 4$. so let's see if the special cases apply:

$$\Delta_1(t,y) = \frac{\left(\frac{\partial M(t,y)}{\partial y} - \frac{\partial N(t,y)}{\partial t}\right)}{N(t,y)} = \frac{6t - 18t}{4y + 9t^2} \times \text{ depends on both } t \,\&\, y$$

$$\Delta_2(t,y) = \frac{\left(\frac{\partial N(t,y)}{\partial t} - \frac{\partial M(t,y)}{\partial y}\right)}{M(t,y)} = \frac{18t - 6t}{6ty} = \frac{12t}{6ty} = \frac{2}{y} \checkmark \text{ depends only on } y!$$

so we can use the following integrating factor: $\mu(y) = e^{\int \Delta_2(y)\,dy}$

$$\mu(y) = e^{\int 2/y\,dy} = e^{2\ln|y|} = e^{\ln y^2} = y^2.$$

the modified ODE is: $6ty^3\,dt + (4y^3 + 9t^2y^2)\,dy = 0$

↳ $\frac{\partial}{\partial y}(6ty^3) = 18ty^2 = \frac{\partial}{\partial t}(4y^3 + 9t^2y^2)$ ✓

Step 1: $\frac{\partial \phi}{\partial t} = 6ty^3 \Rightarrow \phi(t,y) = 6y^3 \int t\,dt \Rightarrow \phi(t,y) = 3t^2y^3 + f(y)$

Step 2: $\frac{\partial \phi}{\partial y} = 4y^3 + 9t^2y^2 \Rightarrow \frac{\partial}{\partial y}(3t^2y^3 + f(y)) = 4y^3 + 9t^2y^2$

$\Rightarrow 9t^2y^2 + f'(y) = 4y^3 + 9t^2y^2 \Rightarrow f'(y) = 4y^3 \Rightarrow f(y) = y^4 + C_1$

$\therefore$, $\phi(t,y) = 3t^2y^3 + y^4 + C_1$.

Solution is $3t^2y^3 + y^4 = C_2$. Apply $y(0)=1$ to get $C_2 = 1$.

$\therefore$, $3t^2y^3 + y^4 = 1$.

Solving Constant-Coefficient Linear ODEs

If we specialize to the case where the ODE is linear and time-invariant, there are 3 methods that can solve any linear constant-coefficient ODE of any order (we will only do the last method)

A general n^{th}-order constant coefficient linear ODE, with input $u(t)$ and output $y(t)$ has the following form (for $a_n \neq 0$):

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \dots + b_1 \frac{du}{dt} + b_0 u$$

We're usually interested in the associated IVP for time $t \geq 0$ with the initial conditions: $y(0) = y_0,\ y'(0) = y_1,\ y''(0) = y_2, \dots, y^{(n-1)}(0) = y_{n-1}$.

Notes:

- we only deal with situations where $n \geq m$
- there is no loss of generality in stating the initial conditions

at time $t_0 = 0$ (because it's time-invariant!)

The presence of initial conditions somewhat complicates the type of linearity and time-invariance analysis that we did previously.

Linearity: the IVP is linear if and only if the initial conditions $y_0, \dots, y_{n-1}$ are all zero. However, we will now allow for the possibility of nonzero initial conditions throughout.

Time-invariance: the presence of initial conditions slightly affects how we analyse time-invariance - the time at which the ICs are assessed must be shifted in the same manner that the input signal is shifted.

Theorem: Consider the IVP above. There exists a solution defined for all $t \geq 0$, and the solution is unique!

Also, for constant-coefficient linear ODEs, the system can be decomposed into the sum of two responses, one due to the initial conditions and the other due to the input signal:

$$y(t) = (\text{response due to initial conditions}) + (\text{response due to input signal})$$

Laplace Transform Method

Fundamental Idea: ODEs and IVPs are challenging because of the presence of derivatives, so we seek a method of transforming the original time-domain IVP into a new domain where the IVP is represented purely by algebraic equations. We call the new domain the Laplace domain or the s-domain. $\therefore$, Calculations with derivatives now become routine algebraic calculations!

Major Benefits of the Laplace Method

- Systematic and Straightforward!
- Can readily handle inputs that are discontinuous or that contain impulses
- Clearly distinguishes between the effect of the initial conditions and that of the input signal.

Example: use the Laplace Transform Methods to find the solution to:

solve $\ddot{y} + 3\dot{y} + 2y = a$ subject to $y(0) = y_0$ and $\dot{y}(0) = 0$.

Step 1: apply the Laplace transform of each side of the ODE to map the problem to the s-domain:

$$\left(s^2 Y(s) - s y_0\right) + 3\left(sY(s) - y_0\right) + 2Y(s) = \frac{a}{s}$$

↳ $Y(s)$ denotes the Laplace transform of $y(t)$.

Step 2: solve algebraically for $Y(s)$:

$$Y(s) = \left[\frac{1}{s(s^2+3s+2)}\right]a + \left[\frac{s+3}{s^2+3s+2}\right]y_0$$

$$= \left[\frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2}\right]a + \left[\frac{2}{s+1} - \frac{1}{s+2}\right]y_0$$

Step 3: apply the inverse Laplace transform to map back to time:

$$y(t) = (0.5 - e^{-t} + 0.5e^{-2t})a + (2e^{-t} - e^{-2t})y_0 \quad \text{for} \quad t \geq 0.$$

The Laplace transform of a signal $f(t)$ is denoted equivalently by

$\mathcal{L}\{f(t)\}$, or $\mathcal{L}\{f\}$, or $F(s)$

The inverse Laplace transform of $F(s)$ recovers $f(t)$. We write

$\mathcal{L}^{-1}\{F(s)\}$

time domain $\xrightarrow[\text{}]{\text{Laplace Transform } F(s) = \mathcal{L}\{f(t)\}}$ Laplace domain

Laplace domain $\xrightarrow[\text{inverse Laplace transform } f(t) = \mathcal{L}^{-1}\{F(s)\}]{}$ time domain

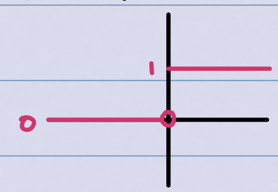
the Laplace transform of a signal $f(t)$ defined for $t \geq 0$ is the following function of $s \in \mathbb{C}$:

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st} \, dt \qquad \left(= \lim_{\substack{T \to \infty \\ \varepsilon \to 0^+}} \int_{-\varepsilon}^{T} f(t) e^{-st} \, dt \right)$$

Region of Convergence (ROC) of the Laplace transform: the set of all s values for which the above integral converges

↳ Depending on the signal $f(t)$, the ROC may be the empty set, the entire complex plane, or something in between.

Example: the unit step (aka Heaviside function) is defined to be the signal: $u_{step}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$. Determine it's Laplace transform.



$$\mathcal{L}\{u_{step}(t)\} = \int_{0^-}^{\infty} u_{step}(t) e^{-st} \, dt$$

$$= \lim_{T \to \infty} \int_{0}^{T} 1 \cdot e^{-st} \, dt \;=\; \lim_{T \to \infty} -\frac{1}{s} e^{-st} \Big|_{0}^{T}$$

$$= \lim_{T \to \infty} -\frac{1}{s} \left(e^{-sT} - 1 \right) = \frac{1}{s} - \frac{1}{s} \lim_{T \to \infty} e^{-sT}$$

Analysing $\lim_{T \to \infty} e^{-sT}$: let $s = \sigma + j\omega$.

$$\therefore \lim_{T \to \infty} e^{-st} = \lim_{T \to \infty} e^{-(\sigma + j\omega)T} = \lim_{T \to \infty} e^{-\sigma T} e^{-j\omega T}.$$

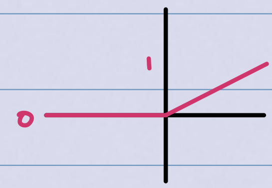
if $\sigma < 0$: $|e^{-\sigma T} e^{-j\omega t}| = e^{-\sigma T} \to \infty$ $\therefore$, limit DNE

if $\sigma = 0$: $e^{-\sigma T} e^{-j\omega t} = e^{-j\omega T}$ a complex # that rotates as $T \uparrow$ $\therefore$, limit DNE

if $\sigma > 0$: $|e^{-\sigma T} e^{-j\omega T}| = e^{-\sigma T} \to 0$ as $T \to \infty$ $\therefore$, limit converges

so, $\mathcal{L}\{u_{step}(t)\} = \frac{1}{s}$ and the ROC is the half-plane $\{s : Re(s) > 0\}$.

Example: determine the Laplace transform of the unit ramp, defined to be $u_{ramp}(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$. Note that $u_{ramp}(t) = t\,u_{step}(t)$.



$$\mathcal{L}\{u_{ramp}(t)\} = \int_{0^-}^{\infty} u_{ramp}(t) e^{-st}\,dt = \int_{0^-}^{\infty} t e^{-st}\,dt$$

$$= \lim_{T\to\infty} \int_{0}^{T} t e^{-st}\,dt = \lim_{T\to\infty} \left(-\frac{te^{-st}}{s} \Big|_{0^-}^{T} + \frac{1}{s} \int_{0^-}^{T} e^{-st}\,dt \right)$$

by integration by parts!
$u = t$ $\quad dv = e^{-st}$
$du = 1$ $\quad v = -\frac{1}{s}e^{-st}$

$$= \lim_{T\to\infty} \left(-\frac{te^{-st}}{s} \Big|_{0}^{T} - \frac{e^{-st}}{s^2} \Big|_{0^-}^{T} \right) = \lim_{T\to\infty} \left(-\frac{Te^{-sT}}{s} - 0 - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right)$$

$$= \frac{1}{s^2} - \frac{1}{s^2} \lim_{T\to\infty} (sT+1)e^{-sT}$$

analysing $\lim_{T\to\infty} (sT+1)e^{-sT}$: set $s = \sigma + j\omega$

$$\lim_{T\to\infty} (sT+1)e^{-sT} = \lim_{T\to\infty} (\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}$$

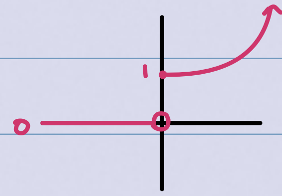
if $\sigma < 0$, the limit DNE since $|(\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}| = |\sigma T + j\omega T + 1| e^{-\sigma T} \geq e^{-\sigma T} \to \infty$ as $T \to \infty$

if $\sigma = 0$, the limit DNE since $|(\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}| = |j\omega T + 1| \to \infty$ as $T \to \infty$

if $\sigma > 0$, the limit is 0 since $|(\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}| = |\sigma T + j\omega T + 1| e^{-\sigma T}$
$\leq (|\sigma T + j\omega T + 1|) e^{-\sigma T} = T|\sigma + j\omega| e^{-\sigma T} + e^{-\sigma T} \to 0$ as $T \to \infty$

so, $\mathcal{L}\{u_{ramp}(t)\} = \frac{1}{s^2}$ and the ROC is the half-plane $\{s : Re(s) > 0\}$

Example: determine the Laplace transform of the exponential function $f(t) = e^{at} u_{step}(t)$. Allow for the possibility that a is a complex number.



$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st}\, dt = \lim_{T\to\infty} \int_0^T e^{(a-s)t}\, dt$$

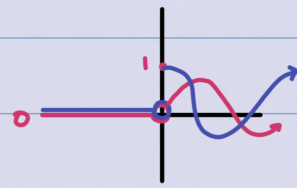
$$= \lim_{T\to\infty} \frac{1}{a-s} e^{(a-s)t} \Big|_0^T = \lim_{T\to\infty} \left(\frac{e^{(a-s)T}}{a-s} - \frac{1}{a-s} \right)$$

$$= \frac{1}{s-a} - \frac{1}{s-a} \lim_{T\to\infty} e^{(a-s)T}$$

similar analysis to the unit step example (unit step is identical, but with $a=0$)

so, $\mathcal{L}\{f(t)\} = \frac{1}{s-a}$ and the ROC is $\{s : Re(s) > Re(a)\}$.

Example: determine the LT of the signals $f(t) = \sin(at) u_{step}(t)$ and $f(t) = \cos(at) u_{step}(t)$.



$$\mathcal{L}\{\cos(at) u_{step}(at)\} = \int_{0^-}^{\infty} \cos(at) e^{-st}\, dt$$

$$= \lim_{T\to\infty} \int_0^T \cos(at) e^{-st}\, dt \xrightarrow{\text{from table of integrals}} \lim_{T\to\infty} \left(\frac{a\sin(at) - s\cos(at)}{s^2 + a^2} e^{-st} \right) \Big|_{0^-}^{T}$$

$$= \lim_{T\to\infty} \left(\frac{a\sin(aT) - s\cos(aT)}{s^2 + a^2} e^{-sT} \right) - \left(\frac{-s}{s^2+a^2} \right)$$

$$= \frac{s}{s^2+a^2} + \frac{1}{s^2+a^2} \lim_{T\to\infty} [\underbrace{a\sin(aT) - s\cos(aT)}_{\text{let this be } z}] e^{-sT}$$

let $s = \sigma + j\omega$. $\therefore$, $\lim_{T\to\infty} z \cdot e^{-(\sigma + j\omega)T} = \lim_{T\to\infty} z \cdot e^{-\sigma T} e^{-j\omega T}$

$z \cdot e^{-\sigma T} e^{-j\omega T}$ is a complex number with magnitude $|z| \cdot e^{-\sigma T}$.

if $\sigma < 0$: limit DNE since $|z|$ varies with T but doesn't converge to 0 and $e^{-\sigma T} \to \infty$ as $T \to \infty$.

if $\sigma = 0$: limit DNE since $|z|$ varies with T but doesn't converge to 0 and $e^{-\sigma T} = 1$ and $e^{-j\omega T}$ has magnitude 1 with rotating phase.

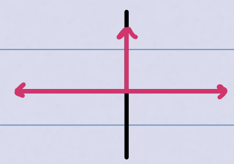
if $\sigma > 0$: limit is 0 since $|z|$ is a bounded function of T (in fact, $|z| \le a + |s|$) and $e^{-\sigma T} \to 0$ as $T \to \infty$.

So, $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2 + a^2}$ and the ROC is the half-plane $\{s : Re(s) > 0\}$

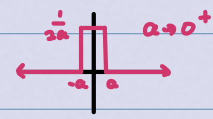
Similarly, $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2 + a^2}$ and the ROC is also the half-plane $\{s : Re(s) > 0\}$.

The Impulse Function $\delta(t)$:

loosely: $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$



better: $\delta_a(t) = \begin{cases} 1/2a, & -a \le t \le a \\ 0, & \text{otherwise} \end{cases}$



note: the area under the curve of $\delta_a(t)$ is 1 for all $a > 0$.

we rely heavily on the following facts:

1. $\int_{-\infty}^{\infty} \delta_a(t)\,dt = 1$
2. $\int_{-\infty}^{\infty} \delta_a(t)\, g(t)\,dt = g(0)$
3. $\int_{-\infty}^{\infty} \delta(t - t_0)\, g(t)\,dt = g(t_0)$
4. $\int_{-\infty}^{\infty} \delta(t)\, g(t - t_0)\,dt = g(-t_0)$

} the sifting property of the impulse function!

Example: $\int_{-\infty}^{\infty} \delta(t)(t+2)\,dt = 2$.

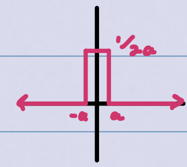
Example: $\int_{-\infty}^{\infty} \delta(t-5)(t^2)\,dt = 5^2 = 25$.

Also, $\delta(t)$ is (kind of) the derivative of $u_{step}(t)$: $\frac{d}{dt} u_{step}(t) = \delta(t)$.

Example: find the Laplace transform of the impulse function.

we need to handle the lower limit carefully. Let's use $t = -\varepsilon$ (for small $\varepsilon > 0$) instead of the vague "0^-" notation, and consider only a small enough that $0 < a < \varepsilon$.

$$\mathcal{L}\{f_a(t)\} = \int_{-\varepsilon}^{\infty} f_a(t) e^{-st}\,dt = \int_{-a}^{a} \frac{1}{2a} e^{-st}\,dt$$



$$= \frac{1}{2a}\left(-\frac{e^{-st}}{s}\right)\Big|_{-a}^{a} = \frac{e^{as} - e^{-as}}{2as}$$

let's now take the limit $a \to 0^+$ as follows:

$$\mathcal{L}\{\delta(t)\} = \lim_{a\to 0^+} \mathcal{L}\{f_a(t)\} = \lim_{a \to 0^+} \frac{e^{as} - e^{-as}}{2as} \overset{L'H}{=} \lim_{a\to 0^+} \frac{se^{as} + se^{-as}}{2s}$$

$$= \frac{2s}{2s} = 1$$

Alternatively, the result quickly follows from the sifting property of the impulse function:

$$\text{LT of } \delta(t): \int_{0^-}^{\infty} \delta(t) e^{-st}\,dt = e^{-st}\Big|_{t=0} = 1 \,/\!/.$$

$\therefore$ $\mathcal{L}(\delta_a(t)) = 1$ and the ROC is the entire S-plane!

Laplace Transform Pairs Table

Time Domain	S-Domain, $F(s) = \mathcal{L}\{f(t)\}$
1. $u_{step}(t)$	$\frac{1}{s}$

2. $u_{ramp}(t)$ (ie, $t u_{step}(t)$)	$\frac{1}{s^2}$	
3. $e^{at} u_{step}(t)$	$\frac{1}{s-a}$	memorise these!
4. $\sin(at) u_{step}(t)$	$\frac{a}{s^2+a^2}$	
5. $\cos(at) u_{step}(t)$	$\frac{s}{s^2+a^2}$	
6. $\delta(t)$	1	
7. $t^n u_{step}(t)$ for $n \geq 1$	$\frac{n!}{s^{n+1}}$	
8. $\sin^2(at) u_{step}(t)$	$\frac{2a^2}{s(s^2+4a^2)}$	
9. $\cos^2(at) u_{step}(t)$	$\frac{s^2+2a^2}{s(s^2+4a^2)}$	
10. $t\sin(at) u_{step}(t)$	$\frac{2as}{(s^2+a^2)^2}$	
11. $t\cos(at) u_{step}(t)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	
12. $[\sin(at) - at\cos(at)] u_{step}(t)$	$\frac{2a^3}{(s^2+a^2)^2}$	
13. $e^{bt}\sin(at) u_{step}(t)$	$\frac{a}{(s-b)^2+a^2}$	
14. $e^{bt}\cos(at) u_{step}(t)$	$\frac{s-b}{(s-b)^2+a^2}$	
15. $te^{at} u_{step}(t)$	$\frac{1}{(s-a)^2}$	
16. $t^2e^{at} u_{step}(t)$	$\frac{2}{(s-a)^3}$	
17. $t^ne^{at} u_{step}(t)$ for $n \geq 1$	$\frac{n!}{(s-a)^{n+1}}$	