

eg:  $\{a, b, c, b\}$  is not a set!

Sets: an unordered collection of distinct objects.

↳  $\{a, b, c\} = \{c, b, a\}$  is a set of size 3

↳  $\{0, 1, 2, \dots\} = \mathbb{N}$  is an infinite set

Set operations: let  $A, B$  be finite sets:

• Cartesian Product:  $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

• Size of Cartesian Product:  $|A \times B| = |A| \cdot |B|$

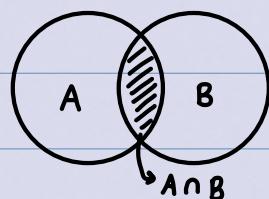
↳ eg: how many binary strings have length 4?

$$\underline{0/1} \quad \underline{0/1} \quad \underline{0/1} \quad \underline{0/1} \rightarrow 2 \times 2 \times 2 \times 2 = 2^4 = 16.$$

$$\# \text{ strings} = |\{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\}| = |\{0, 1\}| \times |\{0, 1\}| \times |\{0, 1\}| \times |\{0, 1\}| = 2 \times 2 \times 2 \times 2 = 16.$$

In general, the number of binary strings of length  $n$  is  $2^n$ .

• Union:  $A \cup B = \{x : x \in A \text{ or } x \in B\}$ .



• Size of Union:  $|A \cup B| = |A| + |B| - |A \cap B|$

↳ if  $|A \cap B| = 0$ , then  $|A \cup B| = |A| + |B|$ .

↳  $A \cup B$  with  $|A \cap B| = 0$  is a disjoint union.

↳ eg: how many binary strings of length 8 are there which begin with 001 or 1011?

let  $S$  be the set of all strings.

let  $A_1$  = set of length-8 binary strings that begin with 001.

let  $A_2$  = set of length-8 binary strings that begin with 1011.

then  $S = A_1 \cup A_2$  and  $A_1 \cap A_2 = \emptyset$ . So,  $|S| = |A_1| + |A_2| = 2^5 + 2^4 = 48$ .

## Counting Subsets

Permutations: a permutation of a set  $S$  is an ordered listing of the elements in  $S$ .

↳ eg: the permutations of  $\{a, b, c\}$  are abc, acb, bac, bca, cab, cba.

• the number of permutations of a set  $S$  of size  $n$  is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{2 \text{ choices}} \times \underline{1 \text{ choice}} = n!$$

Partial Permutations: a permutation of a subset of  $S$ .

• number of partial permutations of  $S$  (with  $|S|=n$ ) of size  $k$  is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{n-k+2 \text{ choices}} \times \underline{n-k+1 \text{ choices}}$$

$$\therefore n(n-1)(n-2)\dots(n-k+2)(n-k+1).$$

↳ note: this works even if  $k > n$ .

How many subsets of  $S$  (with  $|S|=n$ ) are there of size  $k$ ?

- number of partial permutations of  $S$  of size  $k$  is  $n(n-1)(n-2)\dots(n-k+1)$ .
- each size- $k$  subset of  $S$  has  $k!$  permutations.

↳  $\therefore \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

• Combinations: let  $0 \leq k \leq n$ . Then "n choose  $k$ " is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (\text{aka, binomial coefficient})$$

↳ note, if  $k > 1$ , then  $\binom{n}{k} = 0$ .

eg:  $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = \frac{1}{0!} = 1$ .

eg:  $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{1}{0!} = 1$ .

eg:  $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ .

eg:  $\binom{1999}{73} = \frac{1999 \times 1998 \times 1997 \times 1927}{73 \times 72 \times \dots \times 1} = \text{some positive integer.}$

**Binomial Theorem:** for any  $n \geq 1$ ,  $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k$

eg:  $(x+1)^3 = (x+1)(x+1)(x+1) = x^3 + 3x^2 + 3x + 1.$

$$\begin{array}{ccccccc} & \overset{\binom{3}{3}=1}{\nearrow} & \overset{\binom{3}{2}}{\nearrow} & \overset{\binom{3}{1}}{\nearrow} & \overset{\binom{3}{0}=1}{\nearrow} \\ & & & & & & \end{array}$$

## Combinatorial Proof of the Binomial Theorem:

$$(x+1)^n = (x+1)(x+1) \dots \overset{n \text{ copies of } (x+1)}{(x+1)}$$

how do we get an  $x^k$  term? by selecting  $x$  from  $k$  binomials and  $1$  from the other  $n-k$  binomials.  $\therefore$  the total number of ways is  $\binom{n}{k}$ .

Therefore,  $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k$ .  $\square$ .

## Combinatorial Proofs

- Procedure to prove  $M=N$ :

- 1) "Cleverly" choose a set  $S$

- 2) Count the number of elements in  $S$  in two ways:

- a)  $|S|=M$  and

- b)  $|S|=N$

- 3) Conclude that  $M=N$ .

Claim:  $2^n = \sum_{k=0}^n \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

↳ note: algebraically, we can prove this by substituting  $x=1$  into the binomial theorem.

Combinatorial proof:

- let  $S$  be the set of all subsets of  $\{1, 2, \dots, n\}$ .

- a) let  $S_i$  be the subsets in  $S$  of size  $i$ , where  $0 \leq i \leq n$ .

Then  $S = S_0 \cup S_1 \cup \dots \cup S_n$  (note, a disjoint union).

$$\therefore |S| = |S_0| + |S_1| + |S_2| + \dots + |S_n|$$

$$|S| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{k=0}^n \binom{n}{k}.$$

b) to choose a subset  $X$  of  $\{1, 2, \dots, n\}$ , we do:

- for each  $i$ ,  $1 \leq i \leq n$ , we include  $i$  in  $X$ , or not.

- Therefore, the total number of choices is  $2^n$ . Hence  $|S| = 2^n$ .

$$\therefore |S| = \sum_{k=0}^n \binom{n}{k} = 2^n. \quad \square.$$

Claim: let  $1 \leq k \leq n$ . Then  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$  [Pascal's Identity].

• Algebraic Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} + \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} = \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} + \frac{(n-1)! \cdot k}{k!(n-k)!}$$

$$= \frac{(n-1)![(n-k)+k]}{k!(n-k)!} = \frac{(n-1)! \cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \quad \square.$$

• Combinatorial Proof:

let  $S$  = set of all size- $k$  subsets of  $\{1, 2, 3, \dots, n\}$ .

then,  $|S| = \binom{n}{k}$  by definition.

let  $A$  = size- $k$  subsets that don't include  $n$ .

let  $B$  = size- $k$  subsets that includes  $n$ .

Then  $S = A \cup B$  is a disjoint union. So,  $|S| = |A| + |B|$ .

Now,  $|A| = \binom{n-1}{k}$ , and  $|B| = \binom{n-1}{k-1}$ . Thus,  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ .  $\square$ .

• Claim: for  $k, n \geq 0$ ,  $\binom{n+k}{n} = \sum_{i=0}^k \binom{n+i-1}{n-1} = \binom{n-1}{n-1} + \binom{n}{n-1} + \dots + \binom{n+k-1}{n-1}$

Combinatorial Proof:

$$\therefore |S| = \binom{n+k}{n}$$

• let  $S$  = the size- $n$  subsets of  $\{1, 2, \dots, n, n+1, \dots, n+k-1, n+k\}$ .

↳ note: the largest number in a size- $n$  subset is between  $n$  and  $n+k$ .

for each  $0 \leq i \leq k$ , let  $S_i$  = subset whose largest element is  $n+i$ .

Then,  $S$  = disjoint union  $S_0 \cup S_1 \cup \dots \cup S_k$ . So,  $|S| = \sum_{i=0}^k |S_i| = \sum_{i=0}^k \binom{n+i-1}{n-1}$ .  $\square$ .

Visually:

See that the LHS is size- $n$  subsets of  $n+k$  sets. So, we choose subsets of size  $n$  ( $k$  times):

Subsets:  $\{\underbrace{\quad}_{n \text{ size}}\} \cup \{\underbrace{\quad}_n\} \cup \dots \cup \{\underbrace{\quad}_n\} \cup \{\underbrace{\quad}_n\}$

$\cdot \{\underbrace{\quad}_n\} \cup \{\underbrace{\quad}_n\} \cup \dots \cup \{\underbrace{\quad}_n\} \cup \{\underbrace{\dots, n+k}_n\}$

↳ see that the last subset can be counted as  $\binom{n+k-1}{n-1}$ .

Claim: for  $0 \leq k \leq n$ ,  $\binom{n}{k} = \binom{n}{n-k}$

↳ eg:  $\binom{100}{98} = \binom{100}{2} = \frac{100 \times 99}{2}$ .

eg:  $n=4, k=1$  : size-1 subsets of  $\{1, 2, 3, 4\} = \{1\}, \{2\}, \{3\}, \{4\}$ .

size-3 subsets of  $\{1, 2, 3, 4\} = \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}$ .

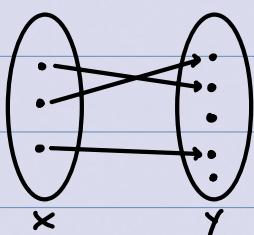
we can pair these sets by picking the size-3 subset which does not contain the element in the size-1 subset. ie:  $\{1\} \leftrightarrow \{2, 3, 4\}, \{2\} \leftrightarrow \{1, 3, 4\} \dots$

→ aka, a correspondence or a bijection!

**Bijections:** let  $f: X \rightarrow Y$  be a function.

•  $f$  is injective (1-1) if  $\forall x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ .

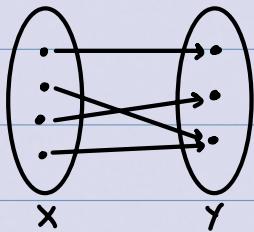
• Visually:



. see that  $|X| \leq |Y|$ .

•  $f$  is surjective (onto) if  $\forall y \in Y, \exists x \in X \text{ st } f(x) = y$ .

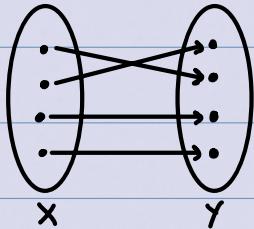
• Visually:



. see that  $|X| \geq |Y|$

•  $f$  is bijective if it is injective AND surjective.

• Visually:

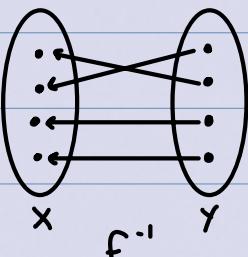
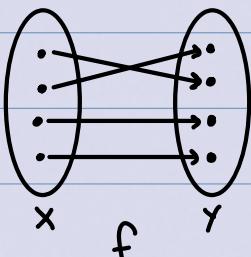


. see that  $|X| = |Y|$ .

• **Theorem:** let  $X, Y$  be finite sets. Suppose  $f: X \rightarrow Y$  is a bijection. Then,  $|X| = |Y|$ .

• **Inverse function:** the inverse of a function  $f: X \rightarrow Y$  is a function  $f^{-1}: Y \rightarrow X$  such that:

- 1) ( $f^{-1}$  reverses  $f$ )  $\Rightarrow \forall x \in X, f^{-1}(f(x)) = x$ .
- 2) ( $f$  reverses  $f^{-1}$ )  $\Rightarrow \forall y \in Y, f(f^{-1}(y)) = y$ .



• Theorem: let  $f: X \rightarrow Y$ . Then  $f$  is a bijection if and only if  $f$  has an inverse.

### Combinatorial Proofs Using Bijections that $M=N$ .

1) Select two sets  $X, Y$ , with  $|X|=M$  and  $|Y|=N$ .

2) Define  $f: X \rightarrow Y$ . ( $f$  is a bijection).

3) Define  $f^{-1}: Y \rightarrow X$ .

4) Prove that  $f^{-1}$  is the inverse function

↳ prove the two conditions of the inverse function definition are satisfied.

5) Conclude that  $M=N$ .

• A function  $f: X \rightarrow Y$  is well-defined if  $\forall x \in X, f(x) \in Y$ .

Claim: for  $0 \leq k \leq n$ ,  $\binom{n}{k} = \binom{n}{n-k}$  (continued)

let  $X$  be the set of all size- $k$  subsets of  $\{1, 2, \dots, n\}$ .

$$\therefore |X| = \binom{n}{k}.$$

let  $Y$  be the set of all size- $(n-k)$  subsets of  $\{1, 2, \dots, n\}$ .

$$\therefore |Y| = \binom{n}{n-k}.$$

Define  $f: X \rightarrow Y$  as follows:  $\forall S \in X, f(S) = S^c$  ( $S^c = \{1, 2, \dots, n\} - S$ ).

• See that  $f$  is well-defined, ie,  $f(S) = S^c \in Y$  since  $|S^c| = n-k$

Define  $f^{-1}: Y \rightarrow X$  as follows:  $\forall T \in Y, f^{-1}(T) = T^c$  ( $T^c = \{1, 2, \dots, n\} - T$ ).

• See that  $f^{-1}$  is well-defined, ie,  $f^{-1}(T) = T^c \in X$  since  $|T^c| = k$ .

now,  $\forall S \in X, f^{-1}(f(S)) = f^{-1}(S^c) = (S^c)^c = S$ .

$$\forall T \in Y, f(f^{-1}(T)) = f(T^c) = (T^c)^c = T.$$

$\therefore f$  is a bijection, and therefore  $|X|=|Y|$ , so  $\binom{n}{k} = \binom{n}{n-k}$ .

## Generating Series

↳ we'll encode solutions to counting problems as coefficients of a "generating series".

• eg: how many subsets of  $\{1, 2, 3\}$  have size  $n$ ,  $\forall n \ 0 \leq n \leq 3$ ?

let  $S = \text{all size-}n \text{ subsets of } \{1, 2, 3\} \ \forall n \ 0 \leq n \leq 3$ .

define weight function  $w: S \rightarrow \mathbb{N}$ , by  $w(\sigma) = |\sigma| \ \forall \sigma \in S$ .

$\sigma \in S$	$\{\emptyset\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$w(\sigma)$	0	1	1	1	2	2	2	3

associate each element  $\sigma \in S$  with the term  $x^{w(\sigma)}$ :

$$x^{w(\sigma)} \quad | \quad x \quad x \quad x \quad x^2 \quad x^2 \quad x^2 \quad x^3$$

the generating series of  $X$  with respect to  $w$  is the sum of the  $x^{w(\sigma)}$  terms:

$$\Phi_S^w(x) = 1 + 3x + 3x^2 + x^3 = (x+1)^3.$$

↑  
3 elements in  $S$  have weight 1  
↓  
3 elements in  $S$  have weight 2

$\therefore$  the number of size- $n$  subsets of  $\{1, 2, 3\}$  is the coefficient of  $x^n$  in  $\Phi_S^w(x)$ ,  $\forall 0 \leq n \leq 3$ .

(of any length!)

• eg: how many binary strings don't have 000 or 00111 as a substring?

• let  $S = \text{all binary strings with no 000 or 00111 as a substring}$ .

↳ we want to organize these strings by their length, so:

define  $w(\sigma)$  be the length of  $\sigma$  (where  $\sigma \in S$ ).

Definition: let  $S$  be a set.

- A function  $w: S \rightarrow \mathbb{N}$  is a weight function if for  $n \in \mathbb{N}$  there are only finitely many elements in  $S$  of weight  $n$ .
- The generating series for  $S$  with respect to  $w$  is  $\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)}$ .
- The coefficient of  $x^n$  in a  $\Phi_S^w(x)$  is denoted  $[x^n] \Phi_S^w(x)$ .  
↳ eg:  $[x^{73}] (x+1)^{97} = \binom{97}{73} \rightarrow \binom{97}{73} x^{73}$ .  
↳ result:  $[x^n] \Phi_S^w(x)$  is the number of elements in  $S$  of weight  $n$ .

Eg:  $S = \text{all subsets from } \{1, 2, \dots, n\}$ .

For  $\sigma \in S$ ,  $w(\sigma) = |\sigma|$ .

$$\text{Then, } \Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{i=0}^n \binom{n}{i} x^i = (x+1)^n$$

$$\text{So, } [x^i] \Phi_S^w(x) = \binom{n}{i}.$$

Eg:  $S = \text{set of all binary strings}$ .

for  $\sigma \in S$ , let  $w(\sigma) = \text{length of } \sigma$ .

$$\text{Then, } \Phi_S^w(x) = ? + ?x + ?x^2 + ?x^3 + \dots$$

$\nearrow \# \text{binary strings of length 1}$   
 $\nwarrow \# \text{binary strings of length 0}$        $\nearrow \# \text{binary strings of length 2}$ .

$$\therefore \Phi_S^w(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$$

## Formal Power Series (FPS)

• An FPS is an expression of the form  $A(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k \geq 0} a_k x^k$ , where  $a_i \in \mathbb{R}$ .

↳ we don't care about convergence or divergence - we only care about

the coefficients.

- So,  $\infty$  is called an "indeterminate".

## Operations on Formal Power Series

$$\left. \begin{aligned} \text{let } A(x) &= a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k \geq 0} a_k x^k \\ \text{let } B(x) &= b_0 + b_1 x + b_2 x^2 + \dots = \sum_{k \geq 0} b_k x^k. \end{aligned} \right\} \text{FPSs.}$$

$$1) (=): A(x) = B(x) \Leftrightarrow a_k = b_k \quad \forall k \geq 0.$$

$$2) (+): A(x) + B(x) = \sum_{k \geq 0} (a_k + b_k) x^k$$

$$2b) (-): A(x) - B(x) = \sum_{k \geq 0} (a_k - b_k) x^k$$

$$3) (\times): A(x) \cdot B(x) = C(x) = \sum_{n \geq 0} C_n x^n, \text{ where } C_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

4) (Inversion): the inverse of a FPS  $A(x)$ , if it exists, is a new FPS  $B(x)$  such that  $A(x)B(x) = 1$ .

↳ we'll write  $B(x) = A(x)^{-1} = \frac{1}{A(x)}$ .

Example: Show that  $(1-x)^{-1} = 1+x+x^2+\dots = \sum_{k \geq 0} x^k$ .

$$\text{let } P(x) = \text{RHS} = \sum_{k \geq 0} x^k.$$

$$\text{Then, } (1-x)P(x) = (1-x) \sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - x \sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - \sum_{k \geq 0} x^{k+1}$$

$$= \sum_{k \geq 0} x^k - \sum_{k \geq 1} x^k = 1 + \left( \sum_{k \geq 1} x^k - \sum_{k \geq 1} x^k \right) = 1 + 0 = 1.$$

$$1) (1-x)^{-1} = \frac{1}{1-x} = \sum_{k \geq 0} x^k \rightarrow [x^n](1-x)^{-1} = 1 \quad \forall n \geq 0.$$

↳ geometric series!

$$2) \text{partial geometric series: } 1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}.$$

↑ partial bc not infinite!  
bc if  $k > n, \binom{n}{k} = 0$  anyway.

$$3) \text{binomial series: } (x+1)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^n \binom{n}{k} x^k. \quad \therefore [x^k](x+1)^n = \binom{n}{k}.$$

$$4) \text{negative binomial series: } \forall n \geq 1, (1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k$$

↳ so  $[x^k](1-x)^{-n} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ .

Example: determine  $[x^n] \left( \frac{x}{1-2x} \right)^5$ .

$$[x^n] \left( \frac{x}{1-2x} \right)^5 = [x^n] x^5 (1-2x)^{-5} = [x^{n-5}] (1-2x)^{-5} \text{ for } n \geq 5.$$

$$\text{Then, see that } (1-x)^{-5} = \sum_{k \geq 0} a_k x^k. \text{ So, } (1-2x)^{-5} = \sum_{k \geq 0} a_k x^k 2^k.$$

$$= 2^{n-5} [x^{n-5}] (1-x)^{-5} = 2^{n-5} \binom{n-5+5-1}{5-1} = 2^{n-5} \binom{n-1}{4}.$$

Proving the negative binomial theorem  $(1-x)^{-n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k, n \geq 1$ .

Combinatorial Proof (sketch):

$$\begin{aligned} (1-x)^{-n} &= (1-x)^{-1} \cdot (1-x)^{-1} \cdot \dots \cdot (1-x)^{-1} \quad (\text{n times}). \\ &= (1+x+x^2+\dots) \cdot (1+x+x^2+\dots) \cdot \dots \cdot (1+x+x^2+\dots) \quad (\text{n times}) \end{aligned}$$

We get an  $x^k$  term by selecting  $x^{a_1}$  from the first series,  $x^{a_2}$  from the second series, ..., and  $x^{a_n}$  from the last series, where  $a_1, a_2, \dots, a_k$  are non-negative integers with  $a_1 + a_2 + \dots + a_n = k$ . Then, multiplying the  $x^{a_i}$  terms gives  $x^{a_1} x^{a_2} \dots x^{a_n} = x^{a_1 + a_2 + \dots + a_n} = x^k$ .

The number of ways of choosing the  $a_i$ 's is equal to the coefficient of  $x^k$  in  $(1-x)^{-n}$ .

Illustration by example:  $n=4$ ,  $k=13$ :

begin with a string of  $k=13$  0s and then insert  $n-1=3$  1s.

$\hookrightarrow \underbrace{000}_{a_1=3} \underbrace{10000}_{a_2=4} \underbrace{00}_{a_3=2} \underbrace{10}_{a_4=4} \underbrace{000}_{a_5=3}$

$$\text{note that } a_1 + a_2 + a_3 + a_4 = 3 + 4 + 2 + 4 = 13!$$

This gives a bijection between binary strings of length  $k+n-1$  with exactly  $n-1$  ones, and non-negative integers  $a_1, a_2, \dots, a_n$  st  $\sum_{i=1}^n a_i = k$ .

$\therefore$ , the number of non-negative integers  $a_1, \dots, a_k$  with  $\sum_{i=1}^n a_i = k$  is equal to the number of binary strings of length  $k+n-1$  with exactly  $n-1$  ones, which is  $\binom{k+n-1}{n-1} = \binom{n+k-1}{k}$ .

$\therefore$ , the coefficient of  $x^k$  in  $(1-x)^{-n}$  is  $\binom{n+k-1}{k}$ .  $\square$ .

### Extracting Coefficients from an FPS:

let  $A(x) = \sum_{n \geq 0} a_n x^n$  and  $B(x) = \sum_{n \geq 0} b_n x^n$ .

$$1) [x^n] (A(x) \pm B(x)) = [x^n] A(x) + [x^n] B(x).$$

$$2) [x^n] (A(x) \cdot B(x)) = \sum_{i=0}^n a_i b_{n-i} = \sum_{i=0}^n ([x^i] A(x)) ([x^{n-i}] B(x))$$

$$3) [x^n] c A(x) = c [x^n] A(x), \text{ where } c \text{ is a constant.}$$

$$4) [x^n] x^l A(x) = 0 \text{ if } n < l, \text{ and } [x^n] x^l A(x) = [x^{n-l}] A(x) \text{ if } n \geq l.$$

$$5) [x^n] A(cx) = c^n [x^n] A(x), \text{ where } c \text{ is a constant}$$

6)  $[x^n] A(x^l) = 0$  if  $l \neq n$ , and  $[x^n] A(x^l) = [x^{n-l}] A(x)$  if  $l \mid n$ .

Example: determine  $[x^n] x^3(3x+1)^7$  where  $n \geq 3$ .

$$[x^n] x^3(3x+1)^7 = [x^{n-3}] (3x+1)^7 = 3^{n-3} [x^{n-3}] (x+1)^7 \\ = 3^{n-3} \binom{7}{n-3}$$

Example: determine  $[x^n] x^3(3x+1)^7(1-4x^2)^{-m}$ ,  $m \geq 1$ ,  $n \geq 3$ .

$$[x^n] x^3(3x+1)^7(1-4x^2)^{-m}$$

$$= [x^n] x^3 \left( \sum_{i \geq 0} \binom{7}{i} (3x)^i \right) \left( \sum_{j \geq 0} \binom{m+j-1}{j} (4x^2)^j \right)$$

$$= [x^n] \sum_{i \geq 0, j \geq 0} \binom{7}{i} \binom{m+j-1}{j} 3^i 4^j x^{3+i+2j}$$

$$= \sum_{i \geq 0, j \geq 0} \binom{7}{i} \binom{m+j-1}{j} 3^i 4^j, \text{ and only care where } 3+i+2j=n.$$

$$\hookrightarrow i=n-2j-3, \quad j=\frac{n-3-i}{2}$$

Substituting:

$$= \sum_{j \geq 0} \binom{7}{n-2j-3} \binom{m+j-1}{j} 3^{n-2j-3} 4^j$$

$$\text{Since } j=\frac{n-3-i}{2}, \quad j \leq \frac{n-3}{2} \text{ (as } i \geq 0\text{). } \therefore j \leq \lfloor \frac{n-3}{2} \rfloor$$

$$\lfloor \frac{n-3}{2} \rfloor$$

$$= \sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{7}{n-2j-3} \binom{m+j-1}{j} 3^{n-2j-3} 4^j.$$

Example: how many length-64 binary strings have neither 000 nor 00111 as a substring?

• don't know how to do this for specifically length-64. So, let's generalize for length- $n$  and make a generating series.

• Roadmap:

- 1) Let  $S$  be a set we wish to count by weight
- 2) Decompose  $S$  into "simpler" sets using disjoint union and cartesian product
- 3) Determine the generating series for the simpler sets
- 4) Combine the generating series  $\Phi_{S_i}(x)$  to get  $\Phi_S(x)$ .
- 5) The number of elements of weight  $n$  in  $S$  is  $[x^n] \Phi_S(x)$ .

• Sum Lemma: let  $S = A \cup B$  be a disjoint union. Let  $w$  be a weight function  $S \rightarrow N$ .

$$\text{Then, } \Phi_S^w(x) = \Phi_A^w(x) + \Phi_B^w(x).$$

↳ Proof:

$$\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in A \cup B} x^{w(\sigma)} = \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)} = \Phi_A^w(x) + \Phi_B^w(x).$$

• Product Lemma: let  $S = A \times B = \{(a, b) : a \in A, b \in B\}$ . Let  $u: A \rightarrow N$  be a weight function on  $A$ , and let  $v: B \rightarrow N$  be a weight function on  $B$ . Define the weight function  $w: S \rightarrow N$  as follows:

$$\text{For } \sigma = (a, b) \in S, w(\sigma) = u(a) + v(b).$$

$$\text{Then, } \Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_B^v(x).$$

↳ Proof:

$$\Phi_A^u(x) \cdot \Phi_B^v(x) = \left( \sum_{a \in A} x^{u(a)} \right) \left( \sum_{b \in B} x^{v(b)} \right) = \sum_{a \in A, b \in B} x^{u(a) + v(b)} = \sum_{\sigma \in S} x^{w(\sigma)} = \Phi_S^w(x).$$

eg: find the generating series of binary strings of length  $n$  where the weight of the string is the number of 1s in it.

Method 1:

let  $S = \text{all binary strings of length } n$ .

Then,  $\Phi_S^w(x) = \sum_{k \geq 0} \alpha_k x^k$ , where  $\alpha_k = \text{number of elements of weight } k$   
 $= \text{number of binary strings of length } n \text{ with } k \text{ 1s} = \binom{n}{k}$ .

$$\therefore \Phi_S^w(x) = \sum_{k \geq 0} \binom{n}{k} x^k = (1+x)^n$$

Method 2:

let  $A = \{0, 1\}$ . Define  $u(0) = 0$  and  $u(1) = 1$ .

$$\Phi_A^u(x) = x^{u(0)} + x^{u(1)} = x^0 + x^1 = 1 + x.$$

now, we use  $A$  to describe the more complicated set  $S$ :

$S = A \times A \times A \times \dots \times A$  ( $n$  times).

define  $w: S \rightarrow \mathbb{N}$  as follows:

$\sigma = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , then  $w(\sigma) = \# \text{ 1s in } \sigma = u(\alpha_1) + u(\alpha_2) + \dots + u(\alpha_n) = \sum_{i=0}^n u(\alpha_i)$ .

→ see that the weight of the complicated set  $S$  is the sum of the weight of the less complicated set  $A$ .

$\therefore$  by the product lemma,  $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_A^u(x) \cdots \Phi_A^u(x)$  ( $n$  times).

$$= [\Phi_A^u(x)]^n = (1+x)^n.$$

☞ note: order matters!!

Compositions: A composition of  $n \in \mathbb{N}$  is a sequence of positive integers  $(\alpha_1, \alpha_2, \dots, \alpha_k)$  that add to  $n$ .  $k = \# \text{ of parts}$ .

example: the compositions of  $n=4$  are:

$(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2), (1,3), (3,1), (4)$ .

Note:  $n=0$  has one composition,  $()$ .

How many  $k$ -part compositions of  $n$  are there?

Let  $S$  = the set of all compositions with  $k$  parts.

Define  $w: S \rightarrow \mathbb{N}$  as follows:

if  $\sigma = (\alpha_1, \alpha_2, \dots, \alpha_k)$ , then  $w(\sigma) = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_k$ .

Goal: find  $\Phi_S^w(x)$ , and the answer will be  $[x^n] \Phi_S^w(x)$ .

let  $P = \{1, 2, 3, 4, \dots\}$ .

then,  $S = P \times P \times P \times \dots \times P$  ( $k$  times)  $= P^k$  ↑ cartesian product of  $k$  copies of  $P$ .

define  $u: P \rightarrow \mathbb{N}$  by  $u(a) = a \quad \forall a \in P$ .

Then,  $w(\sigma) = \alpha_1 + \alpha_2 + \dots + \alpha_k = u(\alpha_1) + u(\alpha_2) + \dots + u(\alpha_k)$ .

by the product lemma,  $\Phi_S^w(x) = \Phi_P^u(x) \times \dots \times \Phi_P^u(x)$  ( $k$  times)  
 $= [\Phi_P^u(x)]^k$ .

Now,  $\Phi_P^u(x) = \sum_{a \in P} x^{u(a)} = x + x^2 + x^3 + x^4 + \dots = x(1+x+x^2+\dots) = x(1-x)^{-1}$ .

So,  $\Phi_S^w(x) = [\Phi_P^u(x)]^k = x^k (1-x)^{-k}$

$\therefore$  the number of  $k$ -part coefficients of  $n$  is  $[x^n] x^k (1-x)^{-k}$   
 $= [x^{n-k}] (1-x)^{-k} = \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ .

String Lemma: let  $S$  be a set with weight function  $w$ .

Suppose that  $S$  has no elements of weight 0. Then,

$$\Phi_{S^*}^{\omega^*}(x) = \frac{1}{1 - \Phi_S^\omega(x)}.$$

**Proof:** let  $S$  be a set and  $\omega: S \rightarrow N$  it's weight function.

let  $S^k$  denote  $S \times S \times S \times \dots \times S$  ( $k$  times) ( $S^0 = \{\emptyset\}$ ).

Define  $S^* = S' \cup S^2 \cup S^3 \cup \dots = \bigcup_{k \geq 0} S^k$  is a disjoint union.

Define  $\omega_k: S^k \rightarrow N$ : if  $(a_1, a_2, \dots, a_k) \in S^k$ , then  $\omega_k = \omega(a_1) + \dots + \omega(a_k)$ .

Define  $\omega^*: S^* \rightarrow N$ :  $\omega^*(\sigma) = \omega_k(\sigma)$  where  $\sigma \in S^k$ .

→ we assume that  $S$  has no elements of weight 0!!

Then,  $\Phi_{S^*}^{\omega^*}(x) = \sum_{k \geq 0} \Phi_{S^k}^{\omega_k}(x)$  by the sum lemma.

Since  $\Phi_{S^k}^{\omega_k}(x) = \prod_{i=1}^k \Phi_S^\omega(x) = [\Phi_S^\omega(x)]^k$ .

We have  $\Phi_{S^*}^{\omega^*}(x) = \sum_{k \geq 0} [\Phi_S^\omega(x)]^k = \frac{1}{1 - \Phi_S^\omega(x)}$ .

How many compositions of  $n$  are there where each part is  $\geq 2$ ?

let  $S = \text{all compositions where each part is } \geq 2$ .

let  $P = \{2, 3, 4, \dots\}$

Then,  $S = P^0 \cup P \cup P^2 \cup P^3 \dots$ . So,  $S = \bigcup_{k \geq 0} P^k$ , ∴  $S = P^*$ .

By the string lemma,  $\Phi_S(x) = \frac{1}{1 - \Phi_P(x)}$ .

$$\text{Now, } \Phi_P(x) = x^2 + x^3 + x^4 + \dots$$

$$= x^2(1 + x + x^2 + \dots)$$

$$= x^2(1-x)^{-1}$$

$$\therefore \Phi_S(x) = \frac{1}{1 - \frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}. \text{ So, } [x^n] \frac{1-x}{1-x-x^2}$$

A formal power series  $A(x)$  is rational if there exist polynomials  $P(x), Q(x)$  such that  $A(x) = P(x)/Q(x)$ .

- we extract coefficients from ration power series using recurrences and partial fractions.

How many compositions have an odd number of parts, where each part  $\equiv 1 \pmod{3}$ .

Let  $S$  = the set of all compositions with an odd number of parts, with each part  $\equiv 1 \pmod{3}$ .

Let  $P = \{1, 4, 7, 10, 13, \dots\}$  (ie,  $P = \{x : x \equiv 1 \pmod{3}\}$ ).

$$\begin{aligned} \text{Then, } S &= P^1 \cup P^3 \cup P^5 \cup P^7 \cup \dots && \rightarrow \text{since } A \times (B \cup C) = (A \times B) \cup (A \times C). \\ &= P \times (P^0 \cup P^2 \cup P^4 \cup P^6 \cup \dots) \\ &= P \times (P^2)^* \end{aligned}$$

$$\begin{aligned} \therefore \Phi_S(x) &= \Phi_P(x) \cdot \Phi_{(P^2)^*}(x) \quad \text{by the product lemma} \\ &= \Phi_P(x) \cdot \frac{1}{1 - \Phi_{P^2}(x)} \quad \text{by the string lemma} \end{aligned}$$

$$\therefore \Phi_P(x) = x + x^4 + x^7 + x^{10} + \dots = \frac{x}{1 - x^3}$$

$$\begin{aligned} \Phi_{P^2}(x) &= \Phi_P(x) \cdot \Phi_P(x) \quad \text{by the product lemma} \\ &= \frac{x^2}{(1-x^3)^2} \end{aligned}$$

$$\begin{aligned} \text{So, } \Phi_S(x) &= \frac{x}{1-x^3} \cdot \frac{1}{1 - \frac{x^2}{(1-x^3)^2}} = \frac{x}{1-x^3} \cdot \frac{(1-x^3)^2}{(1-x^3)^2 - x^2} = \frac{x - x^4}{1-x^2 - 2x^3 + x^6} \\ \therefore [x^n] \frac{x - x^4}{1-x^2 - 2x^3 + x^6} & \end{aligned}$$

## Binary Strings

A binary string  $\sigma$  is a sequence of bits  $b_1, b_2, \dots, b_n$ , where  $b_i = 0$  or  $1$  for  $1 \leq i \leq n$ , and  $n$  is the length of  $\sigma$ .

- There's only one binary string of length 0: the empty string  $\epsilon$ .
- We will (almost) always use the length of a binary string as its weight!

If  $a = a_1, a_2, \dots, a_m$ ,  $b = b_1, b_2, \dots, b_n$  are binary strings, their concatenation is  $ab = a_1, a_2, \dots, a_m b_1, b_2, \dots, b_n$ .

↳ note the length of  $ab$  is  $m+n$ .

↳ also,  $\epsilon\sigma = \sigma$  ∀ strings  $\sigma$ .

$a$  is a substring of  $b$  if  $b = cad$  for some strings  $c$  and  $d$ .

→  → blocks

A block of a binary string is a non-empty maximal substring of all 0s or all 1s.

Regular Expressions: a method for generating/producing a set of binary strings.

Let  $A, B$  be binary strings.

1)  $A \cup B$  is the union of  $A$  and  $B$

↳ eg:  $A = \{00\}$ ,  $B = \{1, 11, 111\}$ .  $A \cup B = \{00, 1, 11, 111\}$  or  $00 \cup (1 \cup 11 \cup 111)$ .

2)  $AB = \{ab : a \in A, b \in B\}$  is the concatenation of  $A$  and  $B$

↳ eg:  $A = \{1, 11\}$ ,  $B = \{001\}$ .  $AB = \{1001, 11001\}$ , or  $(1 \cup 11)001$ .

3)  $A^* = \bigcup_{k \geq 0} A^k$ , where  $A^k = AAA\dots A$   <sup>$\geq k$  times</sup> and  $A^0 = \{\epsilon\}$

↳ eg:  $0^* \rightarrow$  all 0-strings of any length (including 0).

Examples:  $(00)^*$  = { $\epsilon$ , 00, 0000, 000000, ...}.

$0(00)^*$  ( $= (00)^*0$ ) = all odd-length strings of 0s

$(0 \cup 1)^*$  = all binary strings!

$(0 \cup 111)^*$  = all binary strings where each block of 1s has length a multiple of 3.

Ideally, we'd like to use the Sum, Product, and String lemmas to find the generating series for a set of strings produced by a regular expression. But you have to be careful!

eg: let  $A = \{1, 0\}$ ,  $B = \{1, 01\}$

Then  $AB = \{11, 101, 01, 001\}$  Same-ish  
 $A \times B = \{(1, 1), (1, 01), (0, 1), (0, 01)\}$ .

eg: let  $A = \{1, 10\}$ ,  $B = \{1, 01\}$

Then  $AB = \{11, 101, \cancel{101}, 1001\}$  not same!  
 $A \times B = \{(1, 1), (1, 01), (10, 1), (10, 01)\}$ .

So,  $|AB| \neq |A \times B|$ !! Issue when we're trying to count. Removing commas from  $A \times B$  might cause ambiguity.

A regular expression is unambiguous if every string generated by the expression can be generated in exactly one way.

↳ eg:  $(1 \cup 0)(1 \cup 01)$  is unambiguous.

$(1 \cup 10)(1 \cup 01)$  is ambiguous, since 101 can be generated in two ways: 101 and 10, 1.

Example: Is  $0^* \cup 1^*$  ambiguous or not?

↳ ambiguous, because the empty string  $\epsilon$  can be generated in two ways. (So,  $\Phi_{0^* \cup 1^*} \neq \Phi_{0^*} + \Phi_{1^*}$ ).

Example:  $S = (0^*11 \cup 001^*)^*$ . Is this an ambiguous expression?

↳ ambiguous, since 0011 can be generated in two ways.

Let A, B be sets of strings.

- 1) Then  $A \cup B$  is unambiguous if the union is disjoint (ie  $A \cap B = \emptyset$ ).
- 2) Then  $AB$  is unambiguous if  $\forall \sigma \in AB$ , there's exactly one pair of strings  $a, b \in A \times B$  such that  $\sigma = ab$ .
- 3) Then  $A^*$  is unambiguous if  $A^R$  is unambiguous  $\forall R$ , and  $A^0, A^1, A^2, \dots$  are disjoint.

- If  $A \cup B$  is unambiguous, then  $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$ . SUM LEMMA
- If  $AB$  is unambiguous, then  $\Phi_{AB}(x) = \Phi_A(x) \cdot \Phi_B(x)$ . PRODUCT LEMMA
- If  $A^*$  is unambiguous, then  $\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$  STRING LEMMA

How many binary strings don't have 000 or 00111 as a string?