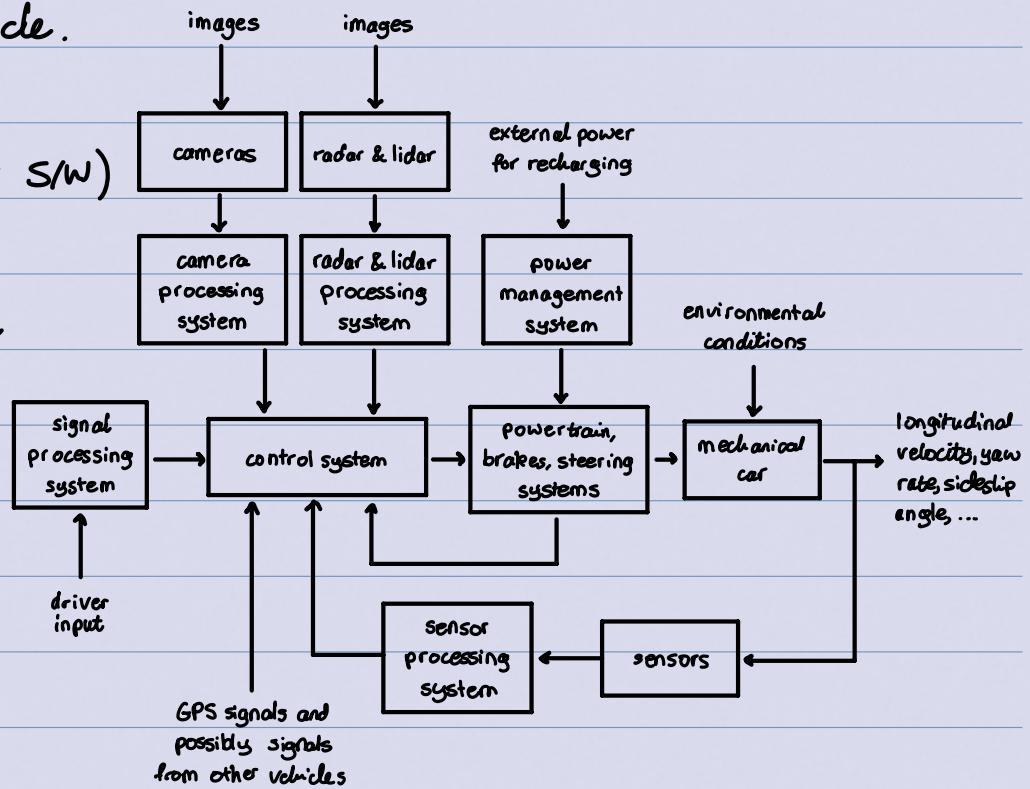


Motivating Example: design of software and hardware used to control an electric vehicle.

boxes = systems (H/W or S/W)

arrows = signals

⇒ this is an example of a block diagram



Can decompose this into 3 types of tasks:

1. Modelling

- we need some way to describe how systems process to generate outputs

2. Analysis

- we need tools to determine and study the behavior of the various systems
- e.g., is the control system stable? is it fast or slow? How does it respond in windy conditions?

3. Design

- we need to have a systematic way to create and tune the various systems (control system, image processing system, etc.)

Signal: a function of one or more independent variables, generally containing information about the behavior of some phenomenon of interest.

⇒ Example: a signal may represent a force, a torque, an angle, a

Speed, a stock price, available SSD memory, etc.

- We will deal with only the situation where there is one independent variable, namely time:

- If time is varying consistently, it's a continuous-time signal.

↳ we denote time by t and continuous-time signals as $x(t)$, $u(t)$, $y(t)$, etc

- If time jumps from one value to the next, it's a discrete-time signal.

↳ we denote time by k and discrete-time signals as $x[k]$, $u[k]$, $y[k]$, etc

System: a device, process, or algorithm that takes one or more input signals and generates one or more output signals.

⇒ Example: each of the blocks in the electric vehicle system, a rocket, a heart, a phone, a planet, etc

It's traditional to denote a generic input signal by u (ie, either $u(t)$ or $u[k]$) and a generic output signal by y (ie, either $y(t)$ or $y[k]$)

Systems that have one input signal and one output signal are called single-input single-output (SISO). Systems that have multiple inputs and multiple outputs are called multi-input multi-output (MIMO).

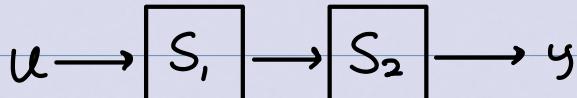
The output of the system is also called the response of the system.

If both the input signal(s) and output signal(s) are continuous-time signals, then we say the system is a continuous-time system.

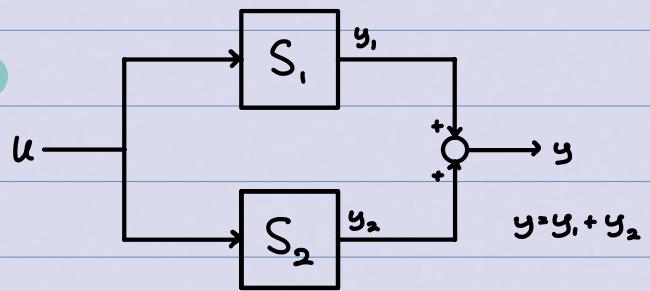
Similarly, if both are discrete-time signals, then we say the system is a discrete-time system. Any other combination results in a hybrid system.

In a block diagram, blocks can be connected in **Series** (aka a **Cascade connection**) or in **parallel** (with the help of a **Summer**):

Series:



parallel:



Differential Equation: any math equation that, in contrast to a purely algebraic equation, includes the derivatives of one or more dependent variable with respect to one or more independent variables.

Ordinary Differential Equation (ODE): a differential equation with only one independent variable.

Partial Differential Equation (PDE): a differential equation with more than one independent variable.

Order of a Differential Equation: the order of the highest derivative in the equation.

⇒ Example: Are the following algebraic, ODEs, or PDEs?

$$\cdot \frac{d^3y}{dt^3} + 4y = \frac{du}{dt} + 2u$$

ODE, 3rd order

$$\cdot F = ma$$

algebraic

$$\cdot F = m \frac{d^2y}{dt^2}$$

ODE, 2nd order

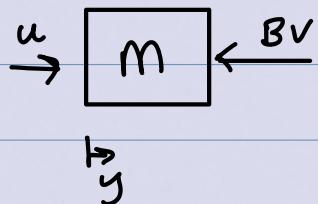
$$\cdot \ddot{y} + 2(2 - \dot{y}^3)\dot{y} + 4y = u$$

ODE, 3rd order

$$\cdot \frac{\partial y(x,t)}{\partial t} - K \frac{\partial^2 y^2(x,t)}{\partial^2 x} = u(x,t)$$

PDE, 2nd order

Example: consider the dynamics of a vehicle moving in a straight line. The system is affected mainly by the force applied by the engine and air resistance (friction). Let u = input force due to engine and v = output velocity ($= \dot{y}$)

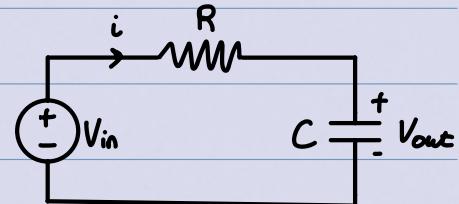


$$F = ma \rightarrow u - Bv = m\ddot{y}$$

$$\rightarrow u - Bv = m\dot{v}$$

$$\rightarrow u = m\dot{v} + Bv \quad \therefore, 1^{\text{st}} \text{ order ODE}$$

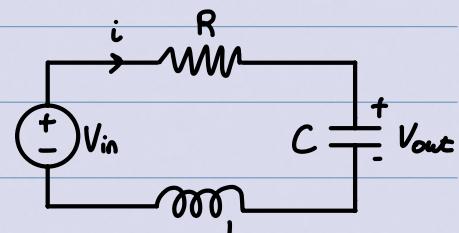
Example: consider this RC circuit. We desire to know the dynamic relationship between the output voltage V_{out} and the input voltage V_{in} . Let i be the current in the loop.



$$Ri - V_{\text{in}} + V_{\text{out}} = 0, \text{ and } i = C \frac{dV_{\text{out}}}{dt}.$$

$$\hookrightarrow V_{\text{out}} + RC \frac{dV_{\text{out}}}{dt} = V_{\text{in}} \quad \therefore, 1^{\text{st}} \text{ order ODE}$$

Example: Same as previous example, but now with an inductor included.



$$Ri - V_{\text{in}} + V_{\text{out}} + L \frac{di}{dt} = 0 \text{ and } i = C \frac{dV_{\text{out}}}{dt}$$

$$\hookrightarrow V_{\text{out}} + RC \frac{dV_{\text{out}}}{dt} + LC \frac{d^2V_{\text{out}}}{dt^2} = V_{\text{in}} \quad \therefore, 2^{\text{nd}} \text{ order ODE}$$

Static / Memoryless System: at each time instant, each possible output doesn't depend on any value of the input except perhaps for the input at the same time instant.

↳ else, the system is said to be Dynamic or to have Memory.

Example: a resistor ($V(t) = i(t)R$) is a static system

Example: a capacitor ($C \frac{dv(t)}{dt} = i(t)$ or $v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau$) is a dynamic system.

Causal / Non-Anticipative System: at each time instant, each possible output does not depend on the future values of the input.

↳ else, the system is said to be non-causal or acausal.

Example: the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k-1]$ is causal.

Example: the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k+1]$ is noncausal.

Example: the system $y(t) = K \int_{-\infty}^t u(\tau) d\tau$ is causal

⇒ A static system is causal!

Linear System: satisfies the superposition property, that is, for any input signals u_1 (w/ output y_1) and u_2 (w/ output y_2) and any constants α_1 and α_2 , a response to the input signal $u = \alpha_1 u_1 + \alpha_2 u_2$ is $y = \alpha_1 y_1 + \alpha_2 y_2$.

↳ Else, the system is said to be nonlinear.

Homogeneity Property: for any input signal u , (w/ output y_1) and any constant α_1 , a response to the input signal $u = \alpha_1 u_1$ is $y = \alpha_1 y_1$.

Additivity Property: for any input signals u_1 , (w/ output y_1) and u_2 (w/ output y_2), a response to the input signal $u = u_1 + u_2$ is $y = y_1 + y_2$.

A system satisfies the Superposition property (ie, the system is linear) if and only if it satisfies both the homogeneity property and the additivity property.

Proof:

(\Rightarrow) If superposition holds:

- set $\alpha_1 = \alpha_2 = 1$ to conclude additivity holds
- set $\alpha_2 = 0$ to conclude homogeneity holds

(\Leftarrow) If both homogeneity and additivity hold, then let $y_1 = S(u_1)$ and $y_2 = S(u_2)$. Then a response to input $\alpha_1 u_1 + \alpha_2 u_2$ is:
 $S(\alpha_1 u_1 + \alpha_2 u_2)$
= $S(\alpha_1 u_1) + S(\alpha_2 u_2)$ by additivity
= $\alpha_1 S(u_1) + \alpha_2 S(u_2)$ by homogeneity
= $\alpha_1 y_1 + \alpha_2 y_2$.

\therefore superposition property is satisfied.

\Rightarrow it's often faster to check for both homogeneity and additivity instead of superposition directly!

Example: are the following systems linear or nonlinear?

a) $y(t) = Ku(t)$

• Satisfies superposition:

Apply input u_1 to get output $y_1 = Ku_1$

Apply input u_2 to get output $y_2 = Ku_2$

Apply input $a_1u_1 + a_2u_2$ to get output $K(a_1u_1 + a_2u_2)$
 $= a_1(Ku_1) + a_2(Ku_2) = a_1y_1 + a_2y_2$

∴, the system is linear.

b) $y(t) = Ku(t) + 1$

Fails both homogeneity and additivity. Eg:

Apply input u_1 to get output $y_1 = Ku_1 + 1$

Apply input u_2 to get output $y_2 = Ku_2 + 1$

Apply input $u_1 + u_2$ to get output $K(u_1 + u_2) + 1 \neq y_1 + y_2$.

∴, the system is nonlinear.

c) $y(t) = au(t) + bu^2(t)$

- Fails additivity. Eg:

Apply input u_1 to get output $y_1 = au_1 + bu_1^2$

Apply input u_2 to get output $y_2 = au_2 + bu_2^2$

Apply input $u_1 + u_2$ to get output $a(u_1 + u_2) + b(u_1 + u_2)^2 \neq y_1 + y_2$

∴, the system is nonlinear (if $b \neq 0$)

d) $y(t) = \sin(u(t))$

- Fails homogeneity. Eg:

Apply input u_1 to get output $y_1 = \sin(u_1)$

Apply input a_1u_1 to get output $y = \sin(a_1u_1) \neq a_1u_1$

∴, the system is nonlinear

e) $y(t) = \sin(t)u(t)$

• Satisfies superposition!

Apply input u_1 to get output $y_1(t) = \sin(t) u_1(t)$

Apply input u_2 to get output $y_2(t) = \sin(t) u_2(t)$

Apply input $\alpha_1 u_1 + \alpha_2 u_2$ to get output

$$y(t) = \sin(t)(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

∴ the system is linear!

f) $y(t) = 5u(t) + u(t)y(t)$

Solving for y : $y = 5u + uy \rightarrow y = \frac{5u}{1-u}$

Fails homogeneity:

Apply input u_1 to get output $y_1 = \frac{5u_1}{1-u_1}$ ($u_1 \neq 1$)

Apply input $\alpha_1 u_1$ to get output $y = \frac{5\alpha_1 u_1}{1-\alpha_1 u_1} \neq \alpha_1 y_1$

∴ the system is nonlinear!

g) $y(t) = K \frac{du(t)}{dt}$

• Satisfies superposition:

Apply input u_1 to get output $y_1 = K \frac{du_1}{dt}$

Apply input u_2 to get output $y_2 = K \frac{du_2}{dt}$

Apply input $\alpha_1 u_1 + \alpha_2 u_2$ to get output $y = K \frac{d}{dt}(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 y_1 + \alpha_2 y_2$

∴ the system is linear!

h) $y(t) = K \int_{-\infty}^t u(\tau) d\tau$

• Satisfies superposition:

Apply input u_1 to get output $y_1 = K \int_{-\infty}^t u_1(\tau) d\tau$

Apply input u_2 to get output $y_2 = K \int_{-\infty}^t u_2(\tau) d\tau$

Apply input $\alpha_1 u_1 + \alpha_2 u_2$ to get output $y = K \int_{-\infty}^t \alpha_1 u_1(\tau) + \alpha_2 u_2(\tau) d\tau = \alpha_1 y_1 + \alpha_2 y_2$

∴ the system is linear!

$$i) M \frac{d^2y(t)}{dt^2} + B \frac{dy(t)}{dt} + K_s y(t) = u(t) \quad (\text{mass-spring damper system})$$

• Satisfies Superposition

Apply input u_1 to get output y_1 , satisfying $M \frac{d^2y_1}{dt^2} + B \frac{dy_1}{dt} + K_s y_1 = u_1$ ①

Apply input u_2 to get output y_2 satisfying $M \frac{d^2y_2}{dt^2} + B \frac{dy_2}{dt} + K_s y_2 = u_2$ ②

Compute $\alpha_1 \cdot ① + \alpha_2 \cdot ②$:

$$M \frac{d^2}{dt^2} (\alpha_1 y_1 + \alpha_2 y_2) + B \frac{d}{dt} (\alpha_1 y_1 + \alpha_2 y_2) + K_s (\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 u_1 + \alpha_2 u_2$$

$\therefore \alpha_1 y_1 + \alpha_2 y_2$ is a solution to the ODE when $u = \alpha_1 u_1 + \alpha_2 u_2$.

So, the system is linear!

$$j) \frac{dP(t)}{dt} = \alpha P(t) + b P^2(t) + u(t)$$

• Fails homogeneity.

Apply input u_1 to get output satisfying $\frac{dP_1}{dt} = \alpha P_1 + b P_1^2 + u_1$ ①

Suppose the system satisfies homogeneity. Then, $\frac{da_i P_i}{dt} = \alpha a_i P_i + b(a_i P_i)^2 + u_i$
 $\Rightarrow \frac{dP_i}{dt} = \alpha P_i + b a_i P_i^2 + u_i$ ②

If ① and ② both hold, then so does ① - ②:

$$0 = bP_1^2 - b\alpha_1 P_1^2 \Rightarrow b(\alpha_1 - 1)P_1^2 = 0 \quad ③$$

But ③ doesn't hold in general (except if $b=0$ or $\alpha_1=1$ or $P_1=0$ which are not of interest).

\therefore the system does not satisfy homogeneity by contradiction, and therefore, the system is nonlinear!

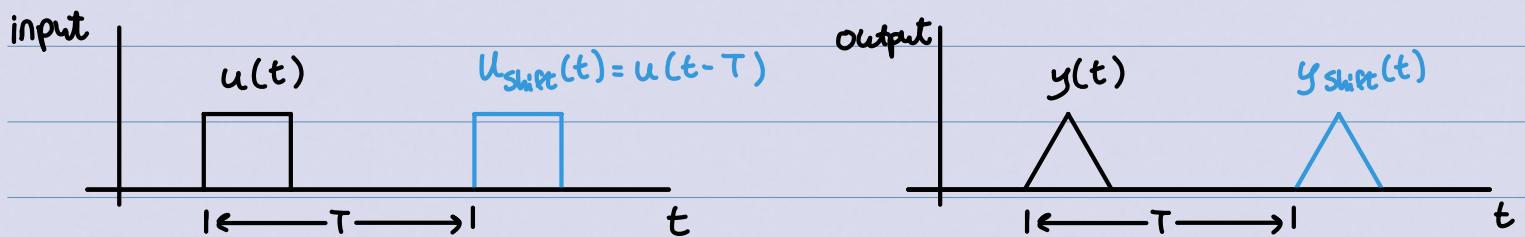
Time-Invariant Systems vs Time-Varying Systems

A system is said to be time-invariant if a time shift in the input signal always causes the same time shift (but no other distortion) in the output signal. More formally:

Assume input u is applied to the system with an associated output y . For a constant T (with $-\infty < T < \infty$), let $u_{\text{shift}}(t) = u(t-T)$ denote the shifted output. The system is said to be time-invariant if, for all u and all T , there exists an output associated with the input u_{shift} , denoted y_{shift} , such that:

$$y_{\text{shift}}(t) = y(t-T) \text{ for } -\infty < T < \infty.$$

A system that is not time-invariant is said to be time-varying.



↳ system is time-invariant if $y_{\text{shift}}(T) = y(t-T)$.

Example: indicate if the following systems are time-invariant or time-varying:

a) $y(t) = Ku(t)$

Apply input u to get output $y(t) = Ku(t)$

Apply input u_{shift} to get output $y_{\text{shift}}(t) = Ku_{\text{shift}}(t)$

\therefore the system is time-invariant! $= Ku(t-T) = y(t-T)$

b) $y(t) = Ku(t) + 1$

Apply input u to get output $y(t) = Ku(t) + 1$

Apply input u_{shift} to get output $y_{\text{shift}}(t) = Ku_{\text{shift}}(t) + 1$

\therefore the system is time-invariant! $= K(u(t-T)) + 1 = y(t-T)$

c) $y(t) = Ku(t) + t$

Apply input u to get output $y(t) = Ku(t) + t$

Apply input u_{shift} to get output $y_{\text{shift}}(t) = Ku_{\text{shift}}(t) + t$

\therefore the system is time-varying.

$$= Ku(t-T) + t \neq Ku(t-T) + t - T = y(t-T)$$

d) $y(t) = \sin(t)u(t)$

Apply input u to get output $y(t) = \sin(t)u(t)$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(t)u_{shift}(t)$

\therefore the system is time-varying.

$$\begin{aligned} &= \sin(t)u(t-T) \neq \sin(t-T)u(t-T) \\ &= y(t-T) \end{aligned}$$

e) $y(t) = \sin(u(t))$

Apply input u to get output $y(t) = \sin(u(t))$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(u_{shift}(t))$

\therefore the system is time-invariant.

$$= \sin(u(t-T)) = y(t-T).$$

f) $y(t) = K \frac{du(t)}{dt}$

Apply input u to get output $y(t) = K \frac{du(t)}{dt}$

Apply input u_{shift} to get output $y_{shift}(t) = K \frac{du_{shift}(t)}{dt}$

\therefore the system is time-invariant.

$$= K \frac{du(t-T)}{dt} = y(t-T).$$

g) $y(t) = K \int_{-\infty}^t u(\tau) d\tau$

Apply input u to get output $y(t) = K \int_{-\infty}^t u(\tau) d\tau$

Apply input u_{shift} to get output $y_{shift}(t) = K \int_{-\infty}^t u_{shift}(\tau) d\tau$

\therefore the system is time-invariant!

$$\begin{aligned} &= K \int_{-\infty}^t u(t-\tau) d\tau = K \int_{-\infty}^t u(\tau) d\tau \\ &= y(t-T) \end{aligned}$$

h) $M \frac{d^2y(t)}{dt^2} + B \frac{dy(t)}{dt} + K_s y(t) = u(t)$

Apply input u to get output y satisfying:

$$M \frac{d^2y(t)}{dt^2} \Big|_t + B \frac{dy}{dt} \Big|_t + K_s y(t) = u(t)$$

①

Apply input $u_{shift}(t)$ to get output y_{shift} satisfying:

$$M \frac{d^2y_{shift}(t)}{dt^2} \Big|_t + B \frac{dy_{shift}}{dt} \Big|_t + K_s y_{shift}(t) = u_{shift}(t) = u(t-T)$$

let $x = t-T \rightarrow dx = dt$

$$\rightarrow M \frac{d^2 y_{\text{shift}}(t)}{dx^2} \Big|_{x+T} + B \frac{dy_{\text{shift}}}{dx} \Big|_{x+T} + K_S y_{\text{shift}}(t) = u(x) \quad (2)$$

① and ② represent the same ODE with the same input.

Therefore, the outputs match: $y_{\text{shift}}(x+T) = y(x)$

$$\Leftrightarrow y_{\text{shift}}(t) = y(t-T).$$

∴, the system is time-invariant.

Definition of a solution to an ODE

A general n^{th} -order ODE with input u and output y can be written as: $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0 \quad (n \geq m)$

For a given input u , any output y which satisfies the equation is considered to be a solution to the ODE. More formally:

Assume the input $u(t)$ is defined on some time interval $t_0 < t < t_1$, (possibly with $t_0 = -\infty$ and/or $t_1 = \infty$) and that the first m derivatives of u are defined for $t_0 < t < t_1$. Then a function $y(t) = \phi(t)$ is said to be a solution to the ODE if the first n derivatives of ϕ exist for $t_0 < t < t_1$, and if $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0$ for $t_0 < t < t_1$.

Example: this is the logistic equation which models the dynamics of population growth: $\frac{dp}{dt} = \alpha p - bP^2 + u$. Consider the special case where $u=0$ and $b=0$, resulting in $\frac{dp}{dt} = \alpha p$. Show that $P(t) = 3e^{\alpha t}$ and $P(t) = 10e^{\alpha t}$ (for $-\infty < t < \infty$) are both solutions:

$$P(t) = 3e^{\alpha t} \rightarrow \frac{dp}{dt} = 3\alpha e^{\alpha t} = \alpha P(t). \quad \therefore \frac{dp}{dt} = \alpha p \quad \checkmark$$

$$P(t) = 10e^{\alpha t} \rightarrow \frac{dp}{dt} = 10\alpha e^{\alpha t} = \alpha P(t). \quad \therefore \frac{dp}{dt} = \alpha p \quad \checkmark$$

\therefore , both $P(t) = 3e^{at}$ and $P(t) = 10e^{at}$ are solutions.

↳ This shows that there need not be only one solution to an ODE. Usually, there exists a whole family of solutions, namely $P(t) = ce^{at}$ (for $-\infty < t < \infty$) for some constant c . Formally:

It's normally (but not always) true that there exists a family of solutions to $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0$, and the family is parameterised by n constants c_1, c_2, \dots, c_n . The family of solutions is said to be the general solution to the ODE. If specific values are chosen for the constants c_1, c_2, \dots, c_n , we say that the resulting $y(t)$ is a particular solution to the ODE.

Example: imagine a mass-spring damper system modeled by the ODE $M\ddot{y} + B\dot{y} + K_S y = u$. Consider the case where $u=0$, $M=1$, $B=3$, and $K_S=2$: $\ddot{y} + 3\dot{y} + 2y = 0$.

a) verify that $y(t) = -c_1 e^{-2t} + 2c_1 e^{-t} - c_2 e^{-2t} + c_2 e^{-t}$ (for $-\infty < t < \infty$), parameterised by 2 arbitrary constants c_1 & c_2 , is a family of solutions.

$$y = -c_1 e^{-2t} + 2c_1 e^{-t} - c_2 e^{-2t} + c_2 e^{-t}$$

$$\dot{y} = 2c_1 e^{-2t} - 2c_1 e^{-t} + 2c_2 e^{-2t} - c_2 e^{-t}$$

$$\ddot{y} = -4c_1 e^{-2t} + 2c_1 e^{-t} - 4c_2 e^{-2t} + c_2 e^{-t}$$

$$\therefore \ddot{y} + 3\dot{y} + 2y = 0 \Rightarrow -4c_1 e^{-2t} + 2c_1 e^{-t} - 4c_2 e^{-2t} + c_2 e^{-t} + 3(2c_1 e^{-2t} - 2c_1 e^{-t} + 2c_2 e^{-2t} - c_2 e^{-t})$$

$$+ 2(-C_1 e^{-2t} + 2C_1 e^{-t} - C_2 e^{-2t} + C_2 e^{-t}) = 0$$

$$\rightarrow \underbrace{-4C_1 e^{-2t}}_{-3C_2 e^{-t}} + \underbrace{2C_1 e^{-t}}_{-2C_1 e^{-2t}} - \underbrace{4C_2 e^{-2t}}_{4C_1 e^{-t}} + \underbrace{C_2 e^{-t}}_{-2C_2 e^{-2t}} + \underbrace{6C_1 e^{-2t}}_{2C_2 e^{-t}} - \underbrace{6C_1 e^{-t}}_{6C_2 e^{-2t}} = 0$$

→ everything cancels, get $0=0$ ✓

b) find a particular solution to $\ddot{y} + 3\dot{y} + 2y = 0$

arbitrarily, $C_1 = C_2 = 1$: $y(t) = -e^{-2t} + 2e^{-t} - e^{-2t} + e^{-t}$
 $\hookrightarrow y(t) = -2e^{-2t} + 3e^{-t}$

equally arbitrarily, $C_1 = C_2 = 0$: $y(t) = 0 + 0 + 0 + 0$
 $\hookrightarrow y(t) = 0$.

We should get used to the idea that there's a family of solutions to an ODE. However, there are exceptions:

example: consider the system described by the following

nonlinear ODE (for $-\infty < t < \infty$): $(\frac{dy}{dt})^2 + y^2 = u$

a) if $u = -1$, there are no solutions (why?)

the LHS cannot be negative!

b) if $u = 0$, there is only one solution (why?)

only solution is $\frac{dy}{dt} = 0$ and $y = 0$.

c) if $u = 2$, there exist multiple solutions (why?)

One possible solution: $y(t) = \sqrt{2}$, so $\frac{dy}{dt} = 0$.

$$\therefore (\frac{dy}{dt})^2 + y^2 = \sqrt{2}^2 + 0 = 2.$$

Another possible solution: $y(t) = \sin(t) + \cos(t)$, so $\frac{dy}{dt} = \cos(t) - \sin(t)$.

$$\begin{aligned}\therefore \left(\frac{dy}{dt}\right)^2 + y^2 &= (\cos t - \sin t)^2 + (\sin t + \cos t)^2 \\ &= \cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\cos t \sin t + \cos^2 t \\ &= 2\sin^2 t + 2\cos^2 t = 2(\sin^2 t + \cos^2 t) = 2\end{aligned}$$

The Initial-Value-Problem (IVP)

Consider the ODE $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0$. Assume the input u is given. The initial-value-problem involves finding a solution (if one exists) of the ODE for $t_0 \leq t \leq t_1$, (or $t_0 \leq t \leq t_1$), possibly with $t_1 = \infty$, subject to the following initial conditions:

$$y(t_0) = y_0, y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1}$$

(assuming that y_0, y_1, \dots, y_{n-1} are given constants)

note: initial conditions are all specified at the same time

instant, $t = t_0$. If different times are used (eg $y(0) = 2$ and $y'(3) = -4$), the problem is no longer an IVP, but a boundary-value-problem (BVP). We will not deal with BVPs.

note 2: an IVP usually has a unique solution (with exceptions)

Example: consider again the mass-spring-damper system. Find the particular solution to the following IVP:

Solve $\ddot{y} + 3\dot{y} + 2y = 0$ subject to $y(0) = 2$ and $\dot{y}(0) = 3$

We saw previously that $y(t) = -C_1 e^{-2t} + 2C_1 e^{-t} - C_2 e^{-2t} + C_2 e^{-t}$ is a family of solutions to $\ddot{y} + 3\dot{y} + 2y = 0$. at initial $t = 0$:

$$y(0) = -C_1 e^{-2(0)} + 2C_1 e^{-(0)} - C_2 e^{-2(0)} + C_2 e^{-(0)} = -C_1 + 2C_1 - C_2 + C_2 = C_1$$

$$\dot{y}(0) = 2C_1 e^{-2(0)} - 2C_1 e^{-(0)} + 2C_2 e^{-2(0)} - C_2 e^{-(0)} = 2C_1 - 2C_1 + 2C_2 - C_2 = C_2$$

$\therefore y(t) = y(0)(-e^{-2t} + 2e^{-t}) + \dot{y}(0)(-e^{-2t} + e^{-t})$ for $t \geq 0$ is

another way of writing the family of solutions.

We were given now that $y(0)=2$ and $\dot{y}(0)=3$, so:

$$y(t) = 2(-e^{-2t} + 2e^{-t}) + 3(-e^{-2t} + e^{-t}) \\ = -2e^{-2t} + 4e^{-t} - 3e^{-2t} + 3e^{-t} = -5e^{-2t} + 7e^{-t}$$

$\therefore y(t) = 7e^{-t} - 5e^{-2t}$ is the particular solution to the IVP subject to the given initial conditions.

However, it's not always true that an IVP has a unique solution. Non-uniqueness arises in the following example because of the specific form of nonlinearity.

Example: the IVP $\ddot{y}(t) - 4t\sqrt{y(t)} = 0$, subject to $y(0)=0$ has 2 solutions.

$$1. y(t) = 0 \quad (t \geq 0)$$

$$2. y(t) = t^4 \quad (t \geq 0) \rightarrow 4t^3 - 4t\sqrt{t^4} = 4t^3 - 4t^3 = 0$$

Factors that affect the solution to an IVP

There are two distinct factors that affect the solution to an IVP: the input signal and the initial conditions.

Example: Consider again the mass-spring damper system, but now with $u=2$. As before, we'll have $M=1$, $B=3$, $K_s=2$. Using tools we'll later learn, we can show that the solution to the IVP

$\ddot{y} + 3\dot{y} + 2y = 2$ subject to $y(0)=2$ and $\dot{y}(0)=3$ is, for $t \geq 0$:

$$y(t) = \underbrace{y(0)(-e^{-2t} + 2e^{-t}) + \dot{y}(0)(-e^{-2t} + e^{-t})}_{\text{response due to initial conditions}} + \underbrace{1 - 2e^{-t} + e^{-2t}}_{\text{response due to input signal}} \\ = -4e^{-2t} + 5e^{-t} + 1. \rightarrow \text{same transient but different steady-state response}$$

transient response: response for finite t

steady-state response: response as $t \rightarrow \infty$

the three approaches that we will consider to solve first-order ODEs are:

1. phase portrait sketch
2. separation of variables
3. exact differential approach

Approach 1: Phase Portrait Sketch

Simple qualitative approach for determining solutions to an ODE that has the form:

$$\frac{dy}{dt} = f(y, u)$$

for constant input u . By considering values of y and u where $f(y, u) = 0$, and the sign of $f(y, u)$ between those values, we can get a good idea (qualitatively) of how $y(t)$ behaves, at least for the situation where u is a constant.

Example: consider the logistic equation with $u = 0$, $\frac{dP}{dt} = P(a - bP)$ we can determine the sign of the derivative for different values of P :

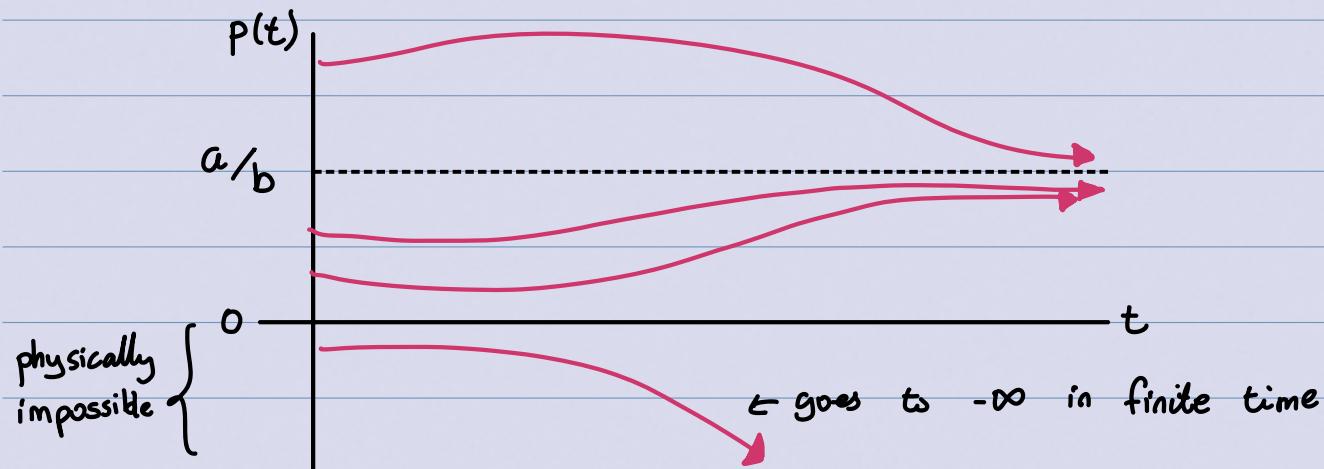
P value	Sign of dP/dt
$P < 0$	- → this case isn't possible
$P = 0$	○
$0 < P < a/b$	+
$P = a/b$	○
$P > a/b$	-

Using this information, we can sketch on an axis the

phase portrait of the ODE:



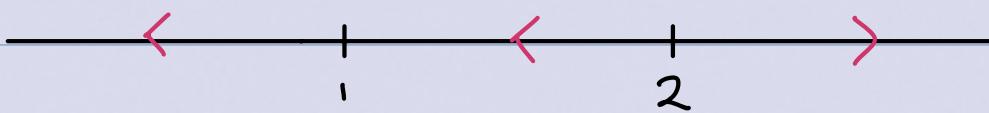
So, we can expect the solution $P(t)$ to look something like:



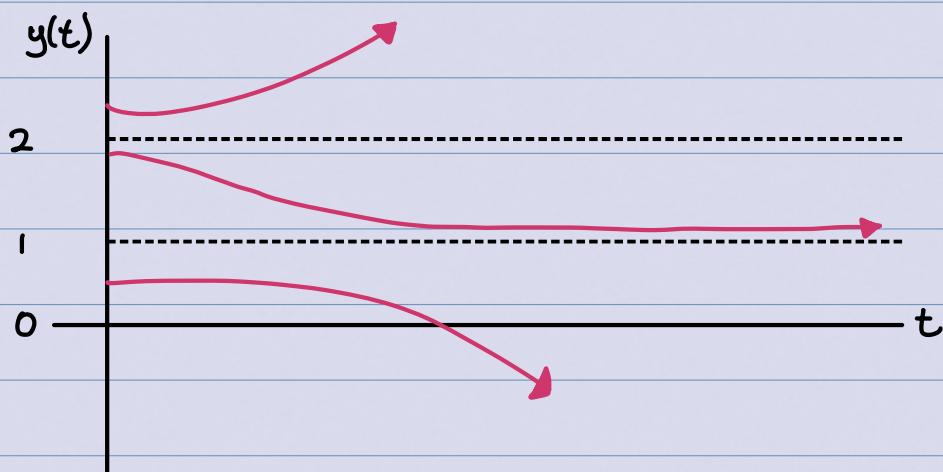
→ we expect it to always converge to a/b in steady-state

Example: consider the following ODE $\frac{dy}{dt} = (y-1)^2(y-2)$ w/ $y(0)=y_0$.
Use the phase portrait approach to characterize the solutions.

y value	sign of dy/dt
$y < 1$	-
$y = 1$	0
$1 < y < 2$	-
$y = 2$	0
$y > 2$	+



the expected solution for various values of y_0 :



Approach 2: Separation of Variables

Phase portrait method is simple to apply and usually insightful. However, if we want more detail, we need quantitative analysis.

Separation of Variables works for any first-order ODE that can be written (for nonzero u) in the form: $\frac{dy}{dt} = f(t) \cdot g(y)$.

An ODE in this form is said to be separable.

↳ for nonzero u , the structure is: $\frac{dy}{dt} = f(t, u) \cdot g(y)$ or $\frac{dy}{dt} = f(t, u) \cdot g(y, u)$ only if u is constant

This form is nice since we can rearrange into a form that can often be solved explicitly via integration.

Example: which of the following can be written in separable form?

a) $\frac{dy}{dt} = e^t y$ yes! already in separate form

b) $t \frac{dy}{dt} = ye^{y+2t} \Rightarrow \frac{dy}{dt} = \left(\frac{e^{2t}}{t}\right)(ye^y)$ yes!

c) $\frac{dy}{dt} = y + e^t$ not separable

Example: a) use separation of variables to determine the general solution to $\frac{dy}{dt} = e^t y$. b) what is the particular solution associated with the initial condition $y(0) = 2$?

a) $\frac{dy}{dt} = e^t y$

↓ !! can split up $\frac{dy}{dt}$
if $y \neq 0$

$$\Rightarrow \frac{dy}{y} = e^t dt \Rightarrow \int \frac{1}{y} dy = \int e^t dt \Rightarrow \ln|y| = e^t + C, \Rightarrow |y| = e^{e^t + C}$$

$$\therefore y(t) = C_2 e^{e^t}$$

special case $y(t) = 0$: this is a valid solution to the ODE, and the general solution addresses it when $C_2 = 0$. so C_2 can be anything!

$$\therefore y(t) = C_2 e^{e^t} \quad (-\infty < t < \infty, -\infty < C_2 < \infty)$$

b) $y(0) = 2 \Rightarrow C_2 e^{e^0} = 2 \Rightarrow C_2 e^1 = 2 \Rightarrow C_2 = 2e^{-1}$

$$\therefore y(t) = 2e^{e^{t-1}}, \quad t \geq 0.$$

Example: a) consider the logistic equation, simplified with $b, u = 0$:
 $\frac{dP(t)}{dt} = \alpha P(t)$. Use the separation of variables method to determine the general solution. b) what is the particular solution associated with the initial condition $P(0) = P_0$?

a) $\frac{dP}{dt} = \alpha P(t)$

$$\Rightarrow \int \frac{1}{P} dP = \int \alpha dt \Rightarrow \ln|P| = \alpha t + C_1 \Rightarrow |P| = e^{\alpha t + C_1} \Rightarrow P(t) = C_2 e^{\alpha t}$$

Special case: $P(t) = 0 \Rightarrow$ a solution to the ODE and the general

Solution includes this case when $C_2 = 0$.

$$\therefore P(t) = C_2 e^{at} \quad (-\infty < t < \infty, C_2 \text{ arbitrary constant})$$

b) $P(0) = P_0 : P_0 = C_2 e^{a(0)} = C_2$

$$\therefore P(t) = P_0 e^{at}, \quad t \geq 0.$$

Approach 3: Exact Differential approach

(Review) Schwartz's Theorem: consider a function $\phi(t, y)$ whose second-order (partial) derivatives $\frac{\partial^2 \phi}{\partial t^2}, \frac{\partial^2 \phi}{\partial y^2}, \frac{\partial^2 \phi}{\partial t \partial y}, \frac{\partial^2 \phi}{\partial y \partial t}$ all exist and are continuous. Then it's a fact that:

$$\frac{\partial^2 \phi}{\partial t \partial y} = \frac{\partial^2 \phi}{\partial y \partial t}.$$

In addition, the differential: $d\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial y} dy$ exists.

Example: consider $\phi(t, y) = t + 2y + e^{ty}$.

All the partial derivatives exist and are continuous:

$$\frac{\partial \phi}{\partial t} = 1 + ye^{ty}$$

$$\frac{\partial \phi}{\partial y} = 2 + te^{ty}$$

$$\hookrightarrow \frac{\partial^2 \phi}{\partial y \partial t} = e^{ty} + tye^{ty}$$

$$\hookrightarrow \frac{\partial^2 \phi}{\partial t \partial y} = e^{ty} + tye^{ty}$$

$$\therefore \frac{\partial^2 \phi}{\partial y \partial t} = \frac{\partial^2 \phi}{\partial t \partial y} !$$

& the differential of $\phi(t, y)$ is $d\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial y} dy = (1 + ye^{ty})dt + (2 + te^{ty})dy$

Consider a first-order ODE of the form (for zero u): $\frac{dy}{dt} = -\frac{M(t, y)}{N(t, y)}$

and rewrite it as $M(t, y)dt + N(t, y)dy = 0$.

If there is a function $\phi(t, y)$ with continuous second-order partial derivatives whose differentials happen to match the above equation, that is, $d\phi(t, y) = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial y} dy = M(t, y)dt + N(t, y)dy$, then we say that $M(t, y)dt + N(t, y)dy$ is an exact differential.

If $M(t, y)dt + N(t, y)dy$ is an exact differential, then $M(t, y)dt + N(t, y)dy = 0$ can be written equivalently as $d\phi(t, y) = 0$.

We conclude that the family of solutions is (for arbitrary constant c): $\phi(t, y) = c$.

Example: Use the exact differential approach to find all solutions to the ODE $\frac{dy}{dt} = -\left(\frac{1+ye^{ty}}{2+te^{ty}}\right)$

$$(1+ye^{ty})dt + (2+te^{ty})dy = 0$$

from the previous example, we saw that $(1+ye^{ty})dt + (2+te^{ty})dy$ is an exact differential with $\phi(t, y) = t + 2y + e^{ty}$.

∴ general solution of the ODE is $t + 2y + e^{ty} = c \quad (-\infty < t < \infty)$
↳ can't solve for $y \rightarrow$ an "implicit" solution

Theorem: Assume $M(t, y)$ and $N(t, y)$ are both continuous with continuous first-order partial derivatives. Then $M(t, y)dt + N(t, y)dy$ is an exact differential if and only if $\frac{\partial M(t, y)}{\partial y} = \frac{\partial N(t, y)}{\partial t}$.

↳ Proof:

(\Rightarrow) if $M(t, y)dt + N(t, y)dy$ is an exact differential, then (by definition) \exists a function $\phi(t, y)$ st. $d\phi(t, y) = M(t, y)dt + N(t, y)dy$ with $\frac{\partial^2 \phi}{\partial t \partial y} = \frac{\partial^2 \phi}{\partial y \partial t}$.

But, $\frac{\partial^2 \phi}{\partial t \partial y} = \frac{\partial^2 \phi}{\partial y \partial t}$ is equivalent to $\frac{\partial M(t, y)}{\partial y} = \frac{\partial N(t, y)}{\partial t}$ Since:

$$\frac{\partial^2 \phi}{\partial t \partial y} = \frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial y} \right) = \frac{\partial}{\partial t} (N(t, y)) \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y \partial t} = \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial t} \right) = \frac{\partial}{\partial y} (M(t, y)).$$

$\therefore \Rightarrow$ proved.

(\Leftarrow) suppose that $\frac{\partial M(t, y)}{\partial y} = \frac{\partial N(t, y)}{\partial t}$ holds. we must construct a function $\phi(t, y)$ such that $d\phi(t, y) = M(t, y)dt + N(t, y)dy$. This is hard so it'll be illustrated in the following examples:

Example) which of the following are exact differentials?

a) $ydt + tdy$ $\frac{\partial(y)}{\partial y} - \frac{\partial(t)}{\partial t} = 1 \Rightarrow$ yes!

b) $ydt - tdy$ $\frac{\partial(y)}{\partial y}(y) \neq \frac{\partial(-t)}{\partial y}(-t) ! \Rightarrow$ no

c) $-ysin(t)dt + cos(t)dy$ $\frac{\partial}{\partial y}(-ysint) = \frac{\partial}{\partial t}(cost) \Rightarrow -\sin t \Rightarrow$ yes!

d) $t^2dt + (1+y+y^2)dy$ $\frac{\partial}{\partial y}(t^2) = \frac{\partial}{\partial t}(1+y+y^2) = 0 \Rightarrow$ yes!