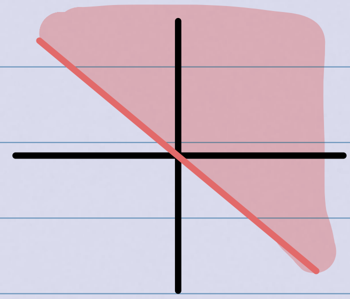
Lecture 1 - 8th Jan

Multivariable functions

→ functions w/ more than one independent variable!

- **Domain:** the set of numbers $(x, y, z, \ldots)$ such that the outcome number $f(x, y, z)$ exists.

Example: what is the domain of the function

$f(x, y) = \sqrt{x+y}$ ⟶ 

since $x + y \geq 0$, $y \geq -x$

Lecture 2: 10th Jan

Example: consider $z = y\cos x$. Find the equation of the level curve that goes through $(\pi, 2)$.

- when $x = \pi$ and $y = 2$, $z = 2\cos\pi = -2$.

$\therefore -2 = y\cos x$, so $y = -\dfrac{2}{\cos x}$

- if we want to take a cross section, we can do so by setting one variable to a constant.

↳ **Example:** trace of $x=1$ for $z = f(x, y) = 10 - 4x^2 - y^2$.

$\hookrightarrow z = f(1,y) = 10 - 4 - y^2 = 6 - y^2.$

Example: find the contour lines of $|x + y - 10| = z$ for $z = 0, 4$.

$z = 0$: $|x + y - 10| = 0$

$\hookrightarrow x + y = 10 \rightarrow y = 10 - x.$

$z = 4$: $|x + y - 10| = 4$

$\hookrightarrow x + y - 10 = 4 \rightarrow y = 14 - x$

$\hookrightarrow x + y - 10 = -4 \rightarrow y = 6 - x$

Limits

- In multiple variables, there are often infinite ways to approach a point with a limit.

to prove a limit exists, we must prove that for every x and y, $\lim_{(x,y)\to(a,b)} = L$, a constant.

to prove a limit does not exist, we must prove that there are at least two sets of x and y ST $\lim_{(x,y)\to(a,b)}$ outputs different values.

Example: does $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + 4y^4}$ exist?

some number

$\hookrightarrow$ let $x = my$.

$$\lim_{y\to 0} \frac{m^2y^2y^2}{m^4y^4+4y^4} = \lim_{y\to 0} \frac{m^2y^4}{m^4y^4+4y^4} = \lim_{y\to 0} \frac{m^2}{m^4+4} = \frac{m^2}{m^4+4}.$$

Since $\frac{m^2}{m^4+4}$ relies on m, an arbitrary number that can change, the limit will result in different outputs.

$\therefore$ the limit does not exist!

- proving that the limit exists is more tricky, so we will focus on proving that they don't exist!

Partial Derivatives

- The partial derivative of $f(x,y)$ with respect to x at the point (a,b) is:

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \partial_x f(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

- The partial derivative of $f(x,y)$ with respect to y at the point (a,b) is:

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \partial_y f(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

Notation: $f(x) \to \frac{df}{dx}$, but $f(x,y) \to \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Example: determine the partial derivatives $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ for $f(x,y) = yx^2+y$.

$$\lim_{h\to 0} \frac{y(x+h)^2+y-(yx^2+y)}{h} \qquad \left(\frac{\partial f}{\partial x}\right)$$

$$\lim_{h \to 0} \frac{y(x+h)^2 - yx^2}{h}$$

$$\lim_{h \to 0} \frac{y(x^2+2xh+h^2) - yx^2}{h}$$

$$\lim_{h \to 0} \frac{yx^2 + 2xhy + h^2 - yx^2}{h}$$

$$\lim_{h \to 0} \frac{2xyh + h^2}{h}$$

$$\lim_{h \to 0} 2xy + h \rightarrow \frac{\partial f}{\partial x} = 2xy.$$

$$\lim_{h \to 0} \frac{(y+h)x^2 + y + h - (yx^2+y)}{h}$$

$$\lim_{h \to 0} \frac{x^2y + x^2h + y + h - x^2y - y}{h}$$

$$\lim_{h \to 0} \frac{x^2h + h}{h}$$

$$\lim_{h \to 0} x^2 + 1 \rightarrow \frac{\partial f}{\partial y} = x^2 + 1$$

$$\therefore \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2xy, x^2+1)$$

- To perform a partial derivative, we just pretend that everything except the variable we are differentiating with respect to is constant.
  - eg, if doing $\frac{\partial f}{\partial x}$, pretend y is some constant.

- Product and Quotient laws are the same!

Example: Calculate the partial derivatives $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ for $f(x,y) = \ln|x+y^2|$ at $(1,2)$.

$\cdot\ \partial_x f(x,y) = \frac{1}{x+y^2}$ and $\partial_y f(x,y) = \frac{2y}{x+y^2}$

$\therefore\ \partial_x f(1,2) = \frac{1}{1+4} = \frac{1}{5}$, $\partial_y f(1,2) = \frac{4}{5}$

$\therefore\ (\partial_x f(1,2), \partial_y f(1,2)) = \left(\frac{1}{5}, 4/5\right)$.

Higher Order Partial Derivatives:

↳ $\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$

Example: determine $\left(\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}\right)$ for $f(x,y) = x^2y^3 + x + y$ at the point $(1,1)$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2xy^3 + 1) = 2y^3.$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 2(1^3) = 2.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}(2xy^3 + 1) = 6xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x}(1,1) = 6(1)(1) = 6.$$

notice that $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$!!!

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(3x^2y^2 + 1) = 6xy^2$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,1) = 6(1,1) = 6.$$

$$\therefore \left(\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}\right) = (2y^3, 6xy^2, 6xy^2)$$

$$\left(\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}\right)(1,1) = (2, 6, 6)$$

Note: $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$ $\forall$ functions!

Example: $\partial^5_{xyxxy}\left(x^2 \cos\sqrt{y^2+4}\right)$

↳ order does not matter!

$$\partial^4_{yxxy} \frac{\partial}{\partial x}\left(x^2 \cos\sqrt{y^2+4}\right) = \partial^4_{yxxy}\left(2x \cos\sqrt{y^2+4}\right)$$

$$\partial^3_{yxy} \frac{\partial}{\partial x}\left(2x \cos\sqrt{y^2+4}\right) = \partial^3_{yxy}\left(2 \cos\sqrt{y^2+4}\right)$$

$$\partial^2_{yy} \frac{\partial}{\partial x}\left(2 \cos\sqrt{y^2+4}\right) = \underline{0}.$$

Interpretation of Partial Derivatives

- For a single variable, derivative = slope
- For many variables, the partial derivative is the slope of the tangent lines of the cross section.
  ↳ this will be a tangent plane!

- tangent lines: $y = f(x_0) + f'(x_0)(x - x_0)$
- tangent planes: $z = z_0 + \underbrace{f_x(x_0, y_0)}_{\text{rate of change in the } x\text{-direction}}(x - x_0) + \underbrace{f_y(x_0, y_0)}_{\text{rate of change in the } y\text{-direction}}(y - y_0)$

Example: Find the tangent plane equation at the point $(\frac{\pi}{6}, 1)$ for $z = \cos 2x + y$.

$\cdot f_x(x_0, y_0) = f_x(\frac{\pi}{6}, 1) = -2\sin 2x = -2\sin\frac{\pi}{3} = -\sqrt{3}$

$\cdot f_y(x_0, y_0) = f_y(\frac{\pi}{6}, 1) = 1$

$f(x_0, y_0) = f(\frac{\pi}{6}, 1) = \cos\frac{\pi}{3} + 1 = 3/2.$

$\therefore z = \frac{3}{2} - \sqrt{3}(x - \frac{\pi}{6}) + 1(y-1)$

$\hookrightarrow z = \frac{3}{2} - \sqrt{3}(x - \pi/6) + y - 1$

· tangent planes can be used to approximate z.

↓

Example: find the linear approximation of $z = \ln(2x+y)$ at $(-1, 3)$.

$\cdot f_x(-1, 3) = \frac{2}{2x+y} = \frac{2}{-2+3} = 2.$

$\cdot f_y(-1, 3) = \frac{1}{2x+y} = \frac{1}{-2+3} = 1.$

$\cdot f(-1, 3) = \ln(-2+3) = \ln 1 = 0.$

$\therefore z \approx 0 + 2(x-(-1)) + 1(y-3)$

$\approx 2x + 2 + y - 3$

$\approx 2x + y - 1$

$\Delta z = \partial_x f(x_0, y_0)\Delta x + \partial_y f(x_0, y_0)\Delta y$

↓ ↓ ↓

$dz = \partial_x f(x_0, y_0)\, dx + \partial_y f(x_0, y_0)\, dy$ ↗ total differential!

Example: find the differential of $f(x,y) = e^{2x+y^2}$

$\partial_x f(x,y) = 2e^{2x+y^2}$

$\partial_y f(x,y) = 2ye^{2x+y^2}$

$\therefore \partial f = 2e^{2x+y^2}\,dx + 2ye^{2x+y^2}\,dy.$

Now, approximate $f(x,y) = e^{2x+y^2}$ at $(-0.1, 0.1)$.

· take $(x_0, y_0) = (0, 0)$.
and $(x, y) = (-0.1, 0.1)$.

$f(x,y) \approx f(x_0,y_0) + \partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0)$

$\downarrow$

$f(-0.1,0.1) \approx f(0,0) + \partial_x f(0,0)(-0.1-0) + \partial_y f(0,0)(0.1-0)$

$\downarrow$

$f(-0.1,0.1) \approx f(0,0) + 2e^0(-0.1) + 0(0.1)$

$\approx e^0 - e^0/5 \approx 1 - 0.2 \approx 0.8.$

Example: $P = \frac{RnT}{V}$, and we know that $\underbrace{\left|\frac{\Delta P}{P}\right| < \frac{1}{10} \text{ and } \left|\frac{\Delta T}{T}\right| < \frac{1}{10}}_{\text{error is less than } 10\%}$.

Find the upper error in % for V.
↳ find $\left|\frac{\Delta V}{V}\right|$!

$V = \frac{RnT}{P}$, $\Delta V = ?$. $V = V(P, T)$.

$\Delta V = \partial_P V\,\Delta P + \partial_T V\,\Delta T$ where $\partial_P V = -\frac{RnT}{P^2}$
and $\partial_T V = Rn/P$.

$$\therefore \Delta V = -\frac{RnT}{p^2}\Delta p + \frac{Rn}{p}\Delta T$$

$$\frac{\Delta V}{V} = -\frac{RnT}{p^2 V}\Delta p + \frac{Rn}{pV}\Delta T$$

$$\frac{\Delta V}{V} = -\frac{RnT}{p^2\frac{RnT}{p}}\Delta p + \frac{Rn}{p\frac{RnT}{p}}\Delta T$$

$$\frac{\Delta V}{V} = -\frac{\Delta p}{p} + \frac{\Delta T}{T}$$

triangle inequality!

$$\left|\frac{\Delta V}{V}\right| \approx \left|-\frac{\Delta p}{p} + \frac{\Delta T}{T}\right| \leq \left|\frac{\Delta p}{p}\right| + \left|\frac{\Delta T}{T}\right|$$

$$\left|\frac{\Delta V}{V}\right| \leq \frac{1}{10} + \frac{1}{10} \leq \frac{1}{5} \leq 20\%.$$

$$\therefore \left|\frac{\Delta V}{V}\right| \leq 20\% !$$

Vector Valued Functions

- Each component is a function (ie each component satisfies the vertical line test), but the full graph might not satisfy the vertical line test.

- Domain is ST each component is defined as a function.

Example: find the parametric equations for the straight line going through $A(2,6)$ and $B(4,8)$.

$\vec{r}(t) = \begin{vmatrix} x(t) = \\ y(t) = \end{vmatrix}$ $\quad P = \begin{vmatrix} x \\ y \end{vmatrix}$ on the line

$$\vec{OA} = \begin{vmatrix} x_A - x_0 \\ y_A - y_0 \end{vmatrix} = \begin{vmatrix} 2 - 0 \\ 6 - 0 \end{vmatrix} = \begin{vmatrix} 2 \\ 6 \end{vmatrix}.$$

$$\vec{AB} = \begin{vmatrix} x_B - x_A \\ y_B - y_A \end{vmatrix} = \begin{vmatrix} 4 - 2 \\ 8 - 6 \end{vmatrix} = \begin{vmatrix} 2 \\ 2 \end{vmatrix}$$

$$\vec{AP} = t\,\vec{AB} = t \begin{vmatrix} 2 \\ 2 \end{vmatrix}$$

$$\vec{r}(t) = \vec{OP} = \vec{OA} + \vec{AP}. \rightarrow \vec{r}(t) = \begin{vmatrix} 2 \\ 6 \end{vmatrix} + t \begin{vmatrix} 2 \\ 2 \end{vmatrix}$$

$$\therefore \vec{r}(t) = \begin{vmatrix} x(t) = 2 + 2t \\ y(t) = 6 + 2t \end{vmatrix}.$$

· the derivative of a vector-valued function is simply the derivative of each component.

↳ Example:

$$\vec{r}(t) = \begin{vmatrix} x(t) = \cos(2t) \\ y(t) = -\sin(2t) \\ z(t) = t \end{vmatrix}$$

→ position

$\vec{r}''(t) = \vec{a}(t)$

→ acceleration

$$\vec{a}(t) = \begin{vmatrix} x''(t) = -4\cos(2t) \\ y''(t) = 4\sin(2t) \\ z''(t) = 0. \end{vmatrix}$$

Example: a particle moves in space with the following acceleration:

$$\vec{a}(t) = \begin{vmatrix} x''(t) = \cos(2t) \\ y''(t) = 2 \\ z''(t) = t \end{vmatrix}, \text{ and } \vec{v}(0) = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}, \quad \vec{r}(0) = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

Find a vector-valued function for its position.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \left| \begin{array}{l} x'(t) = \frac{\sin(2t)}{2} + C \\ y'(t) = 2t + C \\ z'(t) = t^2/2 + C \end{array} \right.$$

$$\vec{V}(0): \left| \begin{array}{l} 0 = 0 + C \to C = 0 \\ 1 = 0 + C \to C = 1 \\ 0 = 0 + C \to C = 0 \end{array} \right. \quad \to \quad \vec{V}(t) = \left| \begin{array}{l} x'(t) = \sin(2t)/2 \\ y'(t) = 2t + 1 \\ z'(t) = t^2/2 \end{array} \right.$$

$$\vec{r}(t) = \int \vec{V}(t)\,dt = \left| \begin{array}{l} x(t) = -\frac{\cos(2t)}{4} + C \\ y(t) = t^2 + t + C \\ z(t) = t^3/6 + C \end{array} \right.$$

$$\vec{r}(0): \left| \begin{array}{l} 1 = -\frac{1}{4} + C \to C = 5/4 \\ 0 = 0 + 0 + C \to C = 0 \\ 0 = 0 + C \to C = 0 \end{array} \right. \quad \to \quad \vec{r}(t) = \left| \begin{array}{l} x(t) = \frac{5}{4} - \cos(2t)/4 \\ y(t) = t^2 + t \\ z(t) = t^3/6 \end{array} \right.$$

- for $f(x(t), y(t))$, $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

use normal d, as f is a function with only one variable — t!

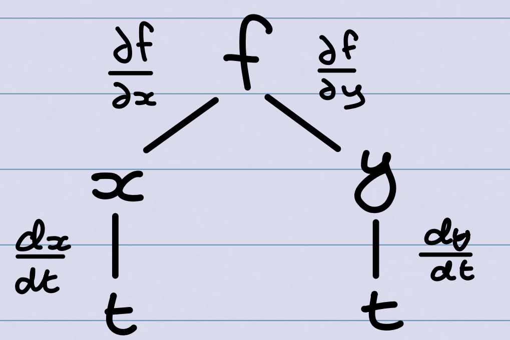
Example: find $\frac{df}{dt}$ for $f(x,y) = ye^{2x}$, $x(t) = \cos t$, $y(t) = t^2 + 1$.

using $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$:

$$\frac{df}{dt} = (2ye^{2x})(-\sin t) + (e^{2x})(2t)$$

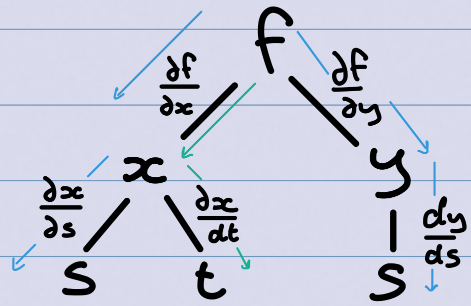
$$\hookrightarrow \frac{df}{dt} = -2(t^2+1)e^{2\cos t}\sin t + 2te^{2\cos t}$$

a way to remember this formula:



product along the branch, sum all the branch contributions.

This tree can extend to many variables; eg $f(x(s,t),\ y(s))$:



to find $\frac{\partial f}{\partial s}$, consider all branches ending in s: $\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$

to find $\frac{\partial f}{\partial t}$, consider all branches ending in t: $\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}$.

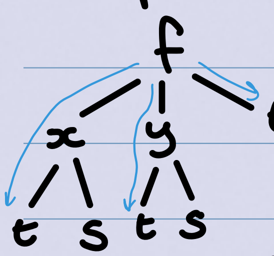
Example: find $\frac{\partial f}{\partial s}$ for $f(x,y) = y\cos x$, $x(s,t) = t^2 + \ln s$, $y(s) = e^s$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$= (-y\sin x)\left(\frac{1}{s}\right) + (\cos x)(e^s)$$

$$\therefore \frac{\partial f}{\partial s} = -\frac{y\sin x}{s} + e^s\cos x$$

Example: find $\frac{\partial f}{\partial t}$ for $f(x,y,t) = \sin y + \cos x + t$, $x(s,t) = 2t + s$, $y(s,t) = st$.



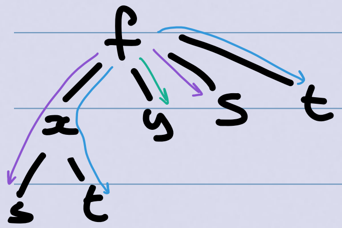
$$\therefore \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial t} f$$

x, y are constant as function of t.

$$\rightarrow \frac{\partial f}{\partial t} = (-\sin x)(2) + (\cos y)(s) + 1$$

$$\frac{\partial f}{\partial t} = s\cos y - 2\sin x + 1.$$

Example: find $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial s}$ for $f(x,y,s,t) = e^y + \ln x + t^2 + s^3$ and $x(s,t) = s + \ln t^2$.



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial t} f\Big|_x$$

↳ $\frac{\partial f}{dt} = \frac{1}{x} \cdot \frac{1}{t^2} \cdot 2t + 2t = 2t\left(\frac{1}{t^2 x} + t\right)$

$\frac{\partial f}{\partial y} = e^y$, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial s} f\Big|_x = \frac{1}{x} \cdot 1 + 3s^2 = \frac{1}{x} + 3s^2$

Gradient: the vector collecting all partial derivatives.

↳ denoted by: $\vec{\nabla} f = \left| \begin{matrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{matrix} \right.$ → $\vec{\nabla}$ is pronounced "nabla"

we have already encountered the gradient! :

for $f(x(t), y(t), z(t))$, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$,

which can also be written as: $\left| \begin{matrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{matrix} \right. \cdot \left| \begin{matrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{matrix} \right.$,

or equivalently $\vec{\nabla} f \cdot \frac{d\vec{r}}{dt}$, where $r = \left| \begin{matrix} x(t) \\ y(t) \\ z(t) \end{matrix} \right.$.

Also, in $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$,
we see that $f(\vec{r}) \approx f(\vec{a}) + \vec{\nabla} f(\vec{a})(\vec{r} - \vec{a})$,
where $\vec{r} = \left| \begin{matrix} x \\ y \end{matrix} \right.$ and $\vec{a} = \left| \begin{matrix} a \\ b \end{matrix} \right.$.

dot product!!!

Directional Derivative: $D_u f = \vec{\nabla} f \cdot \hat{u}$ ↪ aka, rate of change in direction u

↳ note: $\hat{u}$ must be a <u>unit vector</u>!

Example: find rate of change of $f(x,y,z)$ in only the x direction.

$$\hat{u} = \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} . \rightarrow \vec{\nabla} f \cdot \hat{u} = \vec{\nabla} f \cdot \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \partial_f x .$$

Limit Definition of $D_u f$:

$$\hookrightarrow D_{\hat{u}} f(a,b) = \lim_{h \to 0} \frac{f(\vec{a} + h\hat{u}) - f(\vec{a})}{h} = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

Example: the rate of change in the direction $\hat{u} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ at the point $(2,3)$ of $f(x,y) = x^2 e^y + \ln x$ is ?

$$\vec{\nabla} f = \begin{vmatrix} \partial_x f = 2xe^y + 1/x \\ \partial_y f = x^2 e^y \end{vmatrix} \rightarrow \vec{\nabla} f(2,3) = \begin{vmatrix} 4e^3 + 1/2 \\ 4e^3 \end{vmatrix}$$

$$\therefore \vec{\nabla} f \cdot \hat{u} = \vec{\nabla} f \cdot \begin{vmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{vmatrix} = \frac{1}{\sqrt{2}} \left(4e^3 + \frac{1}{2}\right) + \frac{1}{\sqrt{2}} (4e^3)$$

$$\therefore = \frac{1}{\sqrt{2}} \left(8e^3 + \frac{1}{2}\right)$$

Example: the rate of change in the direction from $A:(3,4)$ to $B(3,6)$ at the point $(1,2)$ of the function $f(x,y) = x^2 e^y + \ln x$ is?

$$\vec{AB} = \begin{vmatrix} x_B - x_A \\ y_B - y_A \end{vmatrix} = \begin{vmatrix} 3-3 \\ 6-4 \end{vmatrix} = \begin{vmatrix} 0 \\ 2 \end{vmatrix} , \quad |\vec{AB}| = \sqrt{0 + 2^2}$$

$$\hat{u} = \frac{\vec{AB}}{|AB|} = \frac{1}{2} \begin{vmatrix} 0 \\ 2 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix} .$$

$$\vec{\nabla} f = \begin{vmatrix} 2xe^y + 1/x \\ x^2 e^y \end{vmatrix} \rightarrow \vec{\nabla} f(1,2) = \begin{vmatrix} 2e^2 + 1 \\ e^2 \end{vmatrix}$$

$\therefore \vec{\nabla} f \cdot \hat{u} = \begin{vmatrix} 2e^2+1 \\ e^2 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix} = 0 + e^2 = e^2.$

- Gradient indicates the direction where the rate of change is maximized!

Example: in which direction is the rate of change of $f(x,y) = \ln(x^2+y)$ maximized at $(1,1)$?

$$\vec{\nabla} f = \begin{vmatrix} \partial_x f \\ \partial_y f \end{vmatrix} = \begin{vmatrix} \frac{1}{x^2+y} \cdot 2x \\ \frac{1}{x^2+y} \cdot 1 \end{vmatrix} = \begin{vmatrix} \frac{2x}{x^2+y} \\ \frac{1}{x^2+y} \end{vmatrix}$$

$$\vec{\nabla} f(1,1) = \begin{vmatrix} 2/2 \\ 1/2 \end{vmatrix} = \begin{vmatrix} 1 \\ 1/2 \end{vmatrix}.$$

any scalar multiple of this vector is also in the same direction, and is $\therefore$ a correct answer

- Gradient ($\vec{\nabla} f$) indicates the direction of steepest ascent
- Opposite gradient vector ($-\vec{\nabla} f$) indicates the direction of steepest descent.

Example: is the rate of change of the elevation biggest towards or away from the top of the hill modelled by $f = 1000 - 0.01x^2 - 0.02y^2$?

$$\vec{\nabla} f = \begin{vmatrix} \partial_x f \\ \partial_y f \end{vmatrix} = \begin{vmatrix} -2(0.01)x \\ -2(0.02)y \end{vmatrix} \rightarrow \vec{\nabla} f(60,100) = \begin{vmatrix} -1.2 \\ -4 \end{vmatrix}$$

vector $\langle -1.2, -4 \rangle$ points towards the origin (the top of the hill) from $(60, 100)$.

$\therefore$ towards the top of the hill!

Example: is there a direction where the rate of change is bigger than 1 at (2,1) for the function $f(x,y) = \ln(x^2+y^2)$?

$$\vec{\nabla} f = \begin{vmatrix} \partial_x f \\ \partial_y f \end{vmatrix} = \begin{vmatrix} 2x/x^2+y^2 \\ 2y/x^2+y^2 \end{vmatrix} \rightarrow \vec{\nabla} f(2,1) = \begin{vmatrix} 4/5 \\ 2/5 \end{vmatrix}$$

magnitude of gradient is max rate of change!

$$|\vec{\nabla} f(2,1)| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{4}{25}} = \frac{2\sqrt{5}}{5}.$$

since $|\vec{\nabla} f(2,1)| = \frac{2\sqrt{5}}{5} < 1$, no!

Optimization: Finding Max and Min

A function $f(x,y)$ has a local max at (x_0, y_0) if $f(x_0, y_0) \geq f(x,y)$ for all (x,y) in some disk centered at (x_0, y_0).

A function $f(x,y)$ has a local min at (x_0, y_0) if $f(x_0, y_0) \leq f(x,y)$ for all (x,y) in some disk centered at (x_0, y_0).

- A critical point (a,b) is ST $\vec{\nabla} f(a,b) = \vec{0}$, aka, all partial derivatives are 0 at (a,b).
  - tangent plane at critical point is horizontal.
- A saddle point is a critical point which is neither a local max or min.

Example: how many critical points does the function $f(x,y) = 4 + x^3 + y^3 - 3xy$ have?

$\partial_x f = 3x^2 - 3y = 0 \rightarrow x^2 - y = 0 \Rightarrow x^2 = y$
$\partial_y f = 3y^2 - 3x = 0 \rightarrow y^2 - x = 0 \Rightarrow y^2 = x.$

since $x^2 = y$ and $y^2 = x$, $x^4 = x$, true when $x = 0$ or 1. holds when $y = 0$ and 1 as well.

$\therefore$ two critical points: $(0,0)$ and $(1,1)$.

The Second Derivative Test for Local Extrema:

- Suppose P_0 is a critical point of $f(x,y)$ and suppose that the second-order partial derivatives of f are continuous in some neighborhood of P_0.

- Let $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

- If $D(P_0) > 0$, the f has an extremum at P_0.
- If $f_{xx}(P_0) < 0$, then this extremum is a max
- If $f_{xx}(P_0) > 0$, then this extremum is a min

- If $D(P_0) < 0$, then f does not have an extremum at point P_0 (that is, it has a saddle point instead).

- If $D(P_0) = 0$, the test is inconclusive!

Example: determine the nature of the critical points $(0,0)$ and $(1,1)$ in $f(x,y) = 4 + x^3 + y^3 - 3xy$

$\partial_x f = 3x^2 - 3y \rightarrow \partial_{xx} f = 6x$
$\partial_y f = 3y^2 - 3x \rightarrow \partial_{yy} f = 6y$

$\partial_{xy} f = -3.$

at $P_0(0,0)$: $D(P_0) = 0 \cdot 0 - (-3)^2 = -9 < 0.$
$\therefore (0,0)$ is a saddle point!

at $P_0(1,1)$: $D(P_0) = 6 \cdot 6 - (-3)^2 = 36 - 9 = 27 > 0.$
$\therefore$ there is an extremum at $(1,1)$.

since $f_{xx}(1,1) = 6 > 0$, the point $(1,1)$ is a local minimum!

Example: determine the nature of the critical points of $f(x,y) = 8x^2 - 2y$:

$\partial_x f = 16x, \quad \partial_y f = -2.$

since $\partial_y f = -2$ can never be 0, there are no critical points!

Example: determine the nature of the critical points of $f(x,y) = xy$.

$\partial_x f = y, \quad \partial_y f = x.$

$\partial_x f = 0$ when $y = 0$, and $\partial_y f = 0$ when $x = 0$.

$\therefore (0,0)$ is a critical point.

$\partial_{xx} f = 0$, $\partial_{yy} f = 0$, $\partial_{xy} f = 1$

$D(0,0) = 0 \cdot 0 - (1)^2 = -1 < 0$.

Since $D(0,0) = -1 < 0$, $(0,0)$ is a saddle point!

Optimization Under Constraint:

- To find the critical points of $f(x,y)$ subject to a constraint $g(x,y) = K$ (where K is a constant), find the values of x and y for which:

$\vec{\nabla} f = \lambda \nabla g$ and $g(x,y) = K$ for some constant λ,
or, $\nabla g = \vec{0}$ and $g(x,y) = K$

- If the curve has endpoints, calculate $f(x,y)$ at the endpoints and include them in the comparison.
- If it's an infinite curve, then $f(x,y)$ might not even be bounded. $\therefore$, consider the limits of $f(x,y)$ to see if there are any absolute extrema at all.

Example: Determine if $f(x,y) = 8x^2 - 2y$ has extrema when $x^2 + y^2 = 1$.

$$\overbrace{x^2 + y^2 - 1}^{g(x,y)} = 0 \longrightarrow \vec{\nabla} g = \begin{vmatrix} 2x \\ 2y \end{vmatrix}$$

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y) = 0 \end{cases} \rightarrow \begin{cases} \begin{vmatrix} 16x \\ -2 \end{vmatrix} = \lambda \begin{vmatrix} 2x \\ 2y \end{vmatrix} \\ x^2 + y^2 = 1 \end{cases} \longrightarrow \begin{cases} 16x = 2\lambda x \\ -2 = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

$16(0) = 2\lambda(0) \rightarrow 0 = 0!$

1) $x = 0 \longrightarrow 0^2 + y^2 = 1 \rightarrow y = \pm 1$

2) $x \neq 0 \longrightarrow 16 = 2\lambda \rightarrow \lambda = 8. \rightarrow -2 = 2(8)y \rightarrow y = -1/8.$

$16\cancel{x} = 2\lambda\cancel{x} \rightarrow 16 = 2\lambda!$

$y = -\frac{1}{8}$: $x^2 + (\frac{1}{8})^2 = 1 \rightarrow x^2 = \frac{64}{64} - \frac{1}{64} \rightarrow x = \pm \frac{\sqrt{63}}{8}.$

$\therefore (0,1), (0,-1), (\frac{\sqrt{63}}{8}, -\frac{1}{8}), (\frac{\sqrt{63}}{8}, -\frac{1}{8})$

$f(0,1) : -2, \; f(0,-1) = 2, \; f(\frac{\sqrt{63}}{8}, -\frac{1}{8}) = f(-\frac{\sqrt{63}}{8}, -\frac{1}{8}) = \frac{63}{8} + \frac{1}{4}$

min!

max!

$\therefore$ 2 maxes at $(\frac{\sqrt{63}}{8}, -1/8)$ and $(-\frac{\sqrt{63}}{8}, -1/8)$, and 1 min at $(0,1)$.

But, from the constraint, y must ≥ 0.

$\therefore$, $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g$, $g(x,y) = 0$ @ only $x = 0, y = 1$.

the endpoints will be $y = 0$ and $x = \pm 1$.

$f(0,1) = -2 \rightarrow$ minimum, $f(1,0) = f(-1,0) = 8 \rightarrow$ maximum

$\therefore$ there is a minimum at $(0,1)$ and maximums at $(1,0)$ and $(-1,0)$.

Example: Find the dimensions of the box with the largest volume if the total surface area is 66 cm^2.

$V(x,y,z) = xyz, \quad A(x,y,z) = 2xy + 2yz + 2xz = 66.$

$$\vec{\nabla} V = \lambda \vec{\nabla} A \rightarrow \left| \begin{array}{l} yz = \lambda(2y+2z) \\ xz = \lambda(2x+2z) \\ xy = \lambda(2x+2y) \end{array} \right. = \left| \begin{array}{l} yz = 2\lambda(y+z) \\ xz = 2\lambda(x+z) \\ xy = 2\lambda(x+y) \end{array} \right.$$

works when $x = y$ and $x = z$, $\therefore y = z = x$.

$A = 2x^2 + 2x^2 + 2x^2 = 66 \rightarrow 6x^2 = 66 \rightarrow x^2 = 11 \rightarrow x = \sqrt{11}.$

$\therefore x = y = z = \sqrt{11}.$

$A = xy \rightarrow dA = dx\,dy$

Double Integrals: $\int f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A.$

Example: Consider $f(x,y) = x + e^{x/2} - y$. Find the antiderivative of $f(x,y)$ as a function of x and then as a function of y.

C is a function of y!

$$\int f(x,y)\,dx = \int x + e^{x/2} - y\,dx = \frac{x^2}{2} + 2e^{x/2} - xy + C(y)$$

C is a function of x!

$$\int f(x,y)\,dy = \int x + e^{x/2} - y\,dy = yx + ye^{x/2} - \frac{y^2}{2} + C(x)$$

- When integrating in x, everything in y is taken as a constant!
- When integrating in y, everything in x is taken as a constant!

Example: evaluate $\int_{x=2}^{x=3} \frac{x^2}{y} - \frac{y}{x+1}\,dx$

call this $f(x,y)$!

Since we are integrating w/ respect to x, take y as a constant!

$$\frac{1}{y}\int_2^3 x^2\,dx + y\int_2^3 \frac{1}{x+1}\,dx$$

$$= \frac{1}{y}\left[\frac{x^3}{3}\right]_2^3 + y\left[\ln|x+1|\right]_2^3$$

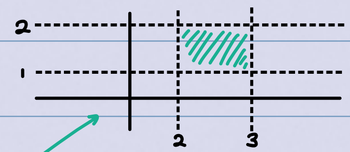
$$= \underbrace{\frac{1}{y}\left(9-\frac{8}{3}\right) + y\ln 4/3}$$

no more x-dependence!!

Now, evaluate $\int_{y=1}^{y=2} f(y)\,dy$: $\int_1^2 \frac{19}{3y} - y\ln\frac{4}{3}\,dy$.

$$= \frac{19}{3}\left[\ln|y|\right]_1^2 - \ln\frac{4}{3}\left[\frac{y^2}{2}\right]_1^2$$

$$= \frac{19}{3}\ln 2 - \frac{3}{2}\ln\frac{4}{3}$$



- This is the same as doing: $\int_{y=1}^{y=2}\int_{x=2}^{x=3} f(x,y)\,dx\,dy$!

integrate over x with y constant

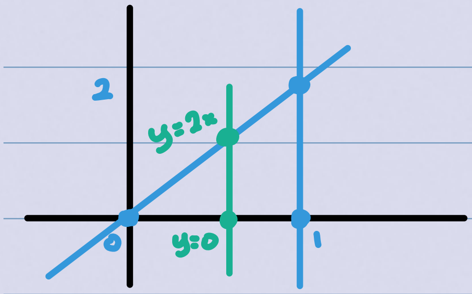
integrate over y (should be no x left)

- Order of integration does NOT matter!!!

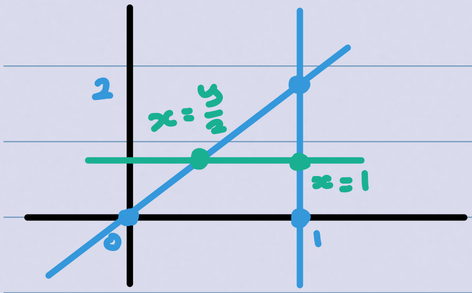
$$\int_{y=0}^{y=1}\int_{x=0}^{x=1} x+y\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1} x+y\,dy\,dx$$

- But, if due to the shape of the domain, the bounds depend on the variables, we cannot typically seperate the integrals this way!

Such as, integrating over the following triangle:



- take x constant
- y varies from 0 to $2x$
- x varies from 0 to 1



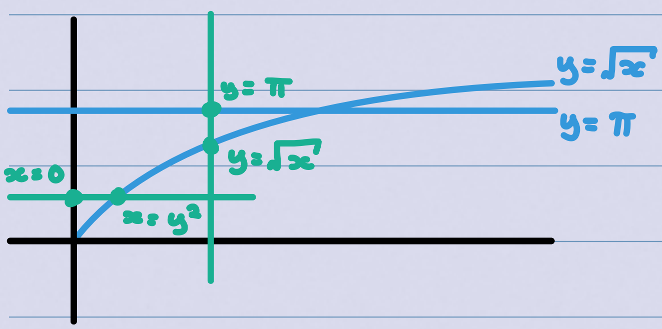
- take y constant
- x varies from $y/2$ to 1
- varies from 0 to 2

this can be solved either as: $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=2x}(x+y)\,dy\,dx$,

or as $\displaystyle\int_{y=0}^{y=2}\int_{x=y/2}^{x=1}(x+y)\,dx\,dy$.

- the outermost integral should NOT have a function or variable in the bounds!

Example: slice the domain given by $x=0$, $y=\pi$, and $y=\sqrt{x}$.



when x is constant;

$y: \sqrt{x} \to \pi$, $x: 0 \to \pi^2$

when y is constant;

$x: 0 \to y^2$, $y: 0 \to \pi$.

On such domain, set up both integrals for $f(x,y) = e^{x/y}$.

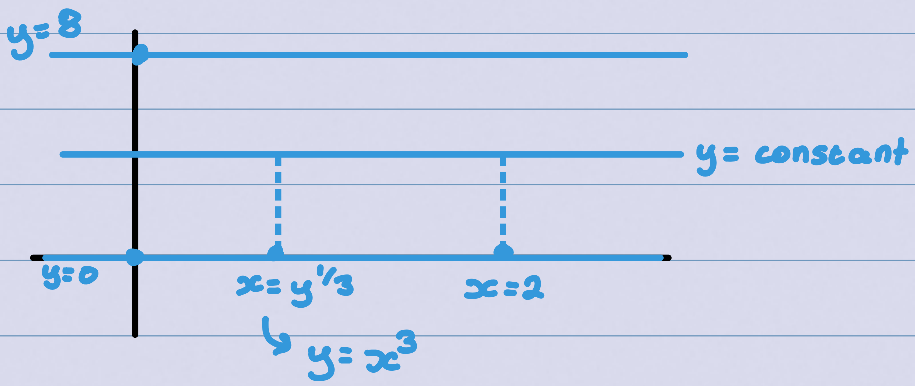
- either : $\displaystyle\int_{x=0}^{x=\pi^2}\int_{y=\sqrt{x}}^{y=\pi} e^{x/y}\,dy\,dx$,

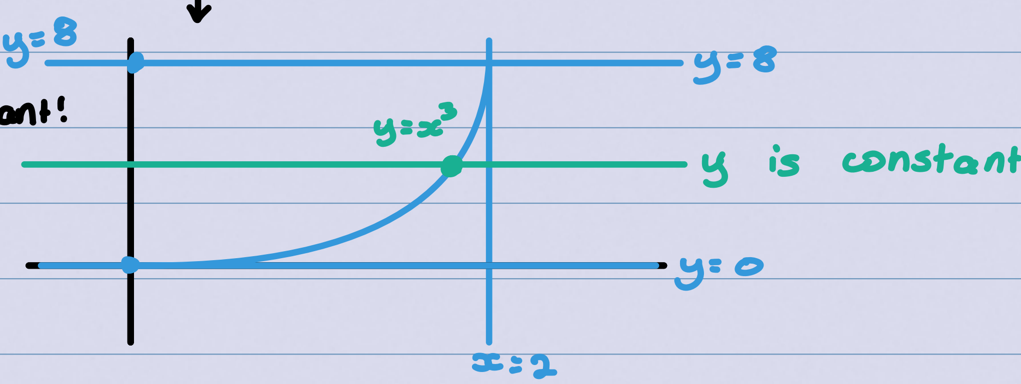
or : $\displaystyle\int_{y=0}^{y=\pi}\int_{x=0}^{x=y^2} e^{x/y}\,dx\,dy$.

Example: Reverse the order of integration of $\int_0^8 \int_{y^{1/3}}^2 (x^4+1)^{1/2}\,dx\,dy$

y is constant!

- sketch!





now, x is constant!

$y: 0 \to x^3$ and

$x: 0 \to 2$.

$$\therefore \int_0^8 \int_{y^{1/3}}^2 (x^4+1)^{1/2}\,dx\,dy = \int_0^2 \int_0^{x^3} (x^4+1)^{1/2}\,dy\,dx.$$

• Converting from Cartesian to Polar Coordinates

- $(x, y) \to (r, \theta)$!

↳ $r = \sqrt{x^2+y^2}$, and $\theta = \tan^{-1} \frac{y}{x}$!

↳ $x = r\cos\theta$, $y = r\sin\theta$.

Example: what point (x, y) is $(r=2, \theta = \frac{\pi}{4})$?

↳ $x = 2\cos\frac{\pi}{4}$, $y = 2\sin\frac{\pi}{4}$

↳ $\therefore, (x, y) = (\sqrt{2}, \sqrt{2})$

In polar, $\overbrace{dA}^{\text{general area element}} = \underbrace{dx\,dy}_{\text{cartesian area element}} = \overbrace{r\,dr\,d\theta}^{\text{polar area element}}$!!!

Example: give the domain by $(x-1)^2 + y^2 \leq 4$ in polar coordinates.

$(x-1)^2 + y^2 \leq 4$

↳ $x^2 - 2x + 1 + y^2 \leq 4$

$r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta \leq 3$

$r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta \leq 3$

↳ $r^2 - 2r\cos\theta \leq 3, \quad \theta \in [0, 2\pi)$

Example: Determine the area of a disk with Radius R using polar coordinates:

$\{(r,\theta) : [0,R] \times [0,2\pi)\}$.

↳ $\displaystyle\int_0^R \int_0^{2\pi} \underbrace{r\,d\theta\,dr}_{dA}$

↳ $\displaystyle\int_0^R 2\pi r\,dr = 2\pi\left[\frac{r^2}{2}\right]_0^R$

$\therefore$ radius $= \pi R^2$!

The Change of Variable Formula:

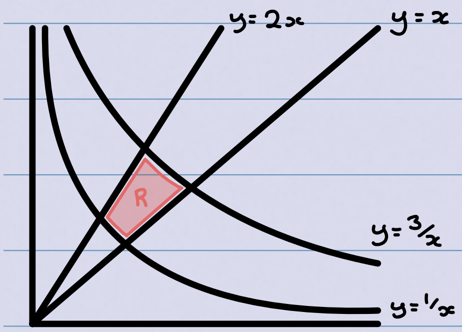
↳ Suppose that the variables x and y are related to the variables u and v by the equations $x = x(u,v),\ y = y(u,v)$. Then:

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{uv}} f[x(u,v),\ y(u,v)] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

where $\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

- This function is called the Jacobian of the transformation, and is also denoted by J.

Example: Find the area of the region R bounded by the curves $y = 2x$, $y = x$, $y = 1/x$, $y = 3/x$.



let $u = xy$, and let $v = \frac{y}{x}$.

↳ $uv = xy \cdot \frac{y}{x} = y^2$. $\therefore$, $y = \sqrt{uv}$

$\frac{u}{v} = xy \cdot \frac{x}{y} = x^2$. $\therefore$, $x = \sqrt{\frac{u}{v}}$

$y = \frac{1}{x} \to u = 1$, $y = \frac{3}{x} \to u = 3$

$y = x \to v = 1$, $y = 2x \to v = 2$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \det \begin{bmatrix} \frac{\partial}{\partial u} u^{1/2} v^{-1/2} & \frac{\partial}{\partial v} u^{1/2} v^{-1/2} \\ \frac{\partial}{\partial u} u^{1/2} v^{1/2} & \frac{\partial}{\partial v} u^{1/2} v^{1/2} \end{bmatrix}$$

$$= \det \begin{bmatrix} \frac{1}{2} u^{-1/2} v^{-1/2} & -\frac{1}{2} u^{1/2} v^{-3/2} \\ \frac{1}{2} u^{-1/2} v^{1/2} & \frac{1}{2} u^{1/2} v^{-1/2} \end{bmatrix}$$

$$= \left(\tfrac{1}{2} u^{-1/2} v^{-1/2} \cdot \tfrac{1}{2} u^{1/2} v^{-1/2}\right) - \left(-\tfrac{1}{2} u^{1/2} v^{-3/2} \cdot \tfrac{1}{2} u^{-1/2} v^{1/2}\right)$$

$$= \frac{1}{4}\frac{1}{\sqrt{u}}\frac{1}{\sqrt{v}} \cdot \sqrt{u}\frac{1}{\sqrt{v}} + \frac{1}{4}\sqrt{u}\frac{1}{\sqrt{v^3}} \cdot \frac{1}{\sqrt{u}}\sqrt{v}$$

$$= \frac{1}{4v} + \frac{\sqrt{u}\, v^{1/2}}{\sqrt{u}\, 4 v^{3/2}} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}.$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}.$$

$$\iint_R x^2+y^2\,dx\,dy = \int_{u=1}^{u=3}\int_{v=1}^{v=2}\left(\frac{u}{v}+uv\right)\left|\frac{1}{2v}\right| dv\,du$$

$$= \int_{u=1}^{u=3}\frac{1}{2}\int_{v=1}^{v=2}\left(\frac{u}{v^2}+u\right)dv\,du$$

$$= \int_{u=1}^{u=3}\frac{1}{2}\left[-\frac{u}{v}+uv\right]_1^2 du$$

$$= \frac{1}{2}\int_{u=1}^{u=3}\left(-\frac{u}{2}+2u+u-u\right)du$$

$$= \frac{1}{2}\int_{u=1}^{u=3}\left(2u-\frac{u}{2}\right)du = \frac{1}{2}\left[2\frac{u^2}{2}-\frac{u^2}{4}\right]_1^3$$

$$= \frac{1}{2}\left[9-\frac{9}{4}-1+\frac{1}{4}\right] = \frac{1}{2}\left[8-\frac{8}{4}\right] = \frac{1}{2}(6) = 3\,!$$

Example: prove $dx\,dy \to r\,dr\,d\theta$:

↳ In polar, $x = r\cos\theta$ and $y = r\sin\theta$.

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

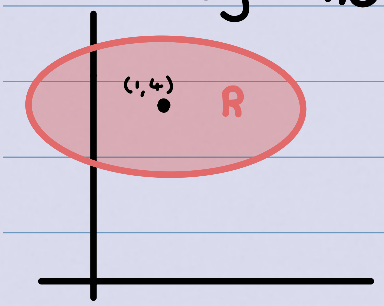
$$= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta+\sin^2\theta) = r\,!$$

$\therefore$ special case of change of variables - polar! :

$$\hookrightarrow \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

• Note: the Jacobian must not be 0 on the interior of the bounds!

Example: Evaluate $\iint_R x\,dA$ in the region R defined by the inequality $2(x-1)^2 + 3(y-4)^2 \leq 4$.



let $x - 1 = \dfrac{r\cos\theta}{\sqrt{2}}$, $y - 4 = \dfrac{r\sin\theta}{\sqrt{3}}$

↳ transforming into a sphere centered at the origin!

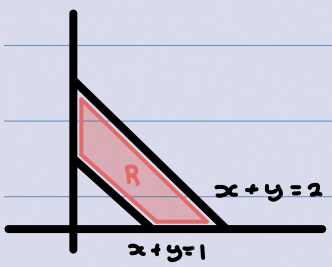
$\therefore 2(x-1)^2 + 3(y-4)^2 \leq 4 \rightarrow r^2\cos^2\theta + r^2\sin^2\theta \leq 4$,

so $0 \leq r \leq 2$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \dots = \frac{r}{\sqrt{6}}.$$

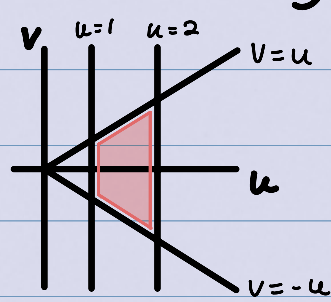
$$\therefore \iint_{R_{xy}} x\,dA = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} \left(1 + \frac{r\cos\theta}{\sqrt{2}}\right)\frac{r}{\sqrt{6}}\,dr\,d\theta = \dots = \frac{4\pi}{\sqrt{6}}.$$

Example: evaluate $\iint_R \cos\left(\dfrac{y-x}{y+x}\right) dA$ over R, the region enclosed by the x-axis, the y-axis, and the curves $x + y = 1$ and $x + y = 2$.



let $u = x + y \rightarrow u = 1$, $u = 2$

$y = 0 \rightarrow u = v$, $x = 0 \rightarrow v = -u$

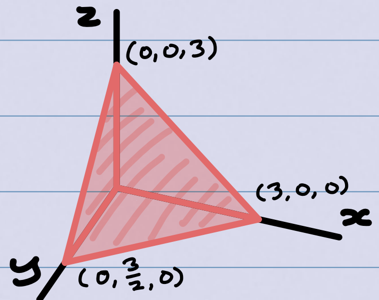


$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \int_{u=1}^{u=2}\int_{v=-u}^{v=u} \cos\left(-\frac{v}{u}\right) dv\,du$$

Average Value of $f(x, y)$ $= \frac{1}{\text{Area}} \iint_R f(x,y)\,dx\,dy$

Triple Integrals

Example: Evaluate $\iiint_D z\,dV$ on the region $D \in \mathbb{R}^3$ defined by $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + 2y + z \leq 3$.



$$0 \leq x \leq 3$$
$$0 \leq y \leq \frac{3-x}{2}$$
$$0 \leq z \leq 3 - x - 2y$$

$$\int_{x=0}^{x=3} \int_{y=0}^{y=\frac{3-x}{2}} \int_{z=0}^{z=3-x-2y} (z)\,dz\,dy\,dx$$ → easily solvable!

Newton's Method: an iterative method for tangent line approximation.

↓

Equation: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Example: use Newton's method to estimate the solution of $\sin(x) = e^x - 2$ accurate to 3 decimals.

Start with $x_1 = 1$, and use $f(x) = \sin x - e^x + 2$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{\sin 1 - e + 2}{\cos 1 - e} \approx 1.057$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.057 - \frac{\sin(1.057) - e^{1.057} + 2}{\cos(1.057) - e^{1.057}} \approx 1.054$$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 1.054.$

Since x_3 and x_4 are equal to 3 decimal places, $x \approx 1.054.$

Example: Use Newton's Method to approximate $\sqrt[8]{500}$

↳ let $f(x) = x^8 - 500$, as $f(\sqrt[8]{500}) = 0.$

$x=2: 256 - 500 = -244.$ → closeset to 0 from $x \in \mathbb{N}$, so take $x_1 = 2.$
$x = 3: 6561 - 500 = 6061.$

$$x_2 = 2 - \frac{2^8 - 500}{8(2)^7} = \frac{573}{256} \approx 2.23828125$$

$$x_3 = \frac{573}{256} - \frac{\left(\frac{573}{256}\right)^8 - 500}{8\left(\frac{573}{256}\right)^7} \approx 2.180559717$$

$$x_4 = x_3 - \frac{x_3^8 - 500}{8(x_3)^7} \approx 2.174616$$

$$x_5 = x_4 - \frac{x_4^8 - 500}{8(x_4)^7} \approx 2.174559281$$

$$x_6 = x_5 - \frac{x_5^8 - 500}{8(x_5)^7} \approx 2.174559276$$

x_5 and x_6 are equal to 6 decimal places, so x_6 is a very close approximation: $\therefore$, $\sqrt[8]{500} \approx 2.174559.$

↗ relies on IVT!

Bisection Method:

Interpolation: estimating $f(x)$ from known values.

↳ Through 2 points, the simplest curve is a line.
Through 3 points, the simplest curve is a parabola.

Through $n+1$ points, the simplest curve is a polynomial of the n^{th} degree.

- We will assume equidistant points for MATH 119!

Example: find $y = a + bx + cx^2$ for $\begin{vmatrix} x_1 = 0 \\ y_1 \end{vmatrix}, \begin{vmatrix} x_2 = 0 \\ y_2 \end{vmatrix}, \begin{vmatrix} x_3 = 0 \\ y_3 \end{vmatrix}$

↳ $y_1 = a + bx_1 + cx_1^2 \rightarrow x_1 = 0$

$y_2 = a + bx_2 + cx_2^2 \rightarrow x_2 = 1$

$y_3 = a + bx_3 + cx_3^2 \rightarrow x_3 = 2$

↓

$y_1 = a + 0b + 0c$

$y_2 = a + b + c$

$y_3 = a + 2b + 4c$

$$\Delta y_{12} = y_2 - y_1 = a + b + c - a = b + c$$

$$\Delta y_{23} = y_3 - y_2 = a + 2b + 4c - a - b + c = b + 3c$$

$$\Delta y_{123} = \Delta y_{23} - \Delta y_{12} = b + 3c - b - c = 2c$$

$$\therefore\ c = \frac{\Delta y_{123}}{2}, \quad b = \Delta y_{12} - \frac{\Delta y_{123}}{2}, \quad a = y_1$$

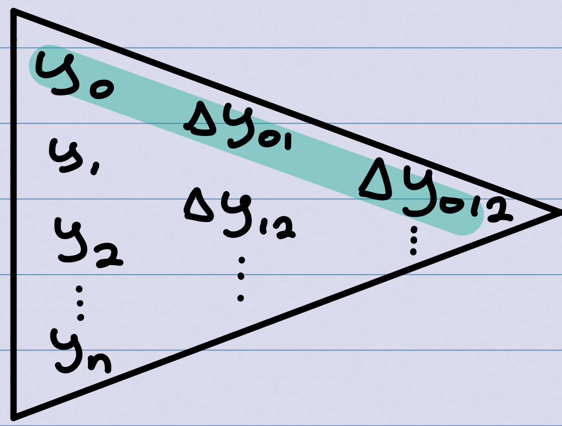
$$\therefore\ y = a + bx + cx^2$$
$$= y_1 + \left(\Delta y_{12} - \frac{\Delta y_{123}}{2}\right)x + \frac{\Delta y_{123}}{2}x^2$$

General Formula:

$$\hookrightarrow y = y_0 + x\Delta y_{12} + x(x-1)\frac{\Delta y_{123}}{2!} + x(x-1)(x-2)\frac{\Delta y_{1234}}{3!}$$
$$+ \dots + x(n+1)(n+2)\cdots(x-n+1)\frac{\Delta y_{123\dots n+1}}{n!}$$

To get the coefficients:

$\hookrightarrow$ only need the top row of values!



Example: Consider the interpolating polynomial $P(x)$ going through $(0,2)$, $(1,2)$, $(2,12)$, $(3,62)$, $(4,206)$. Determine $P(5)$.

	y	Δ	Δ^2	Δ^3	Δ^4
1 →	2	0 (2-2)	10 (10-0)	30 (40-10)	24 (54-30)
2 →	2	10	40	54	
3 →	12	50	94		
4 →	62	144			
5 →	206				

$$P(x) = y_0 + x\Delta y_{12} + x(x-1)\frac{\Delta y_{123}}{2!} + x(x-1)(x-2)\frac{\Delta y_{1234}}{3!} + x(x-1)(x-2)(x-3)\frac{\Delta y_{12345}}{4!}$$

$$P(x) = 2 + 0 + x(x-1)\frac{10}{2!} + x(x-1)(x-2)\frac{30}{3!} + x(x-1)(x-2)(x-3)\frac{24}{4!}$$

$$\therefore\ P(x) = 2 + 5x(x-1) + 5x(x-1)(x-2) + x(x-1)(x-2)(x-3)$$

$$\hookrightarrow P(5) = 2 + 25(4) + 25(4)(3) + 5(4)(3)(2) = 522.$$

This can also be generalized to non-integer values!

↳ $x_n = x_0 + nh \rightarrow 0 < h < 1$

$$y = y_0 + \frac{(x - x_0)}{h}\Delta y_{12} + \frac{(x - x_0)(x - x_1)}{h^2}\frac{\Delta y_{123}}{2!}$$

$$+ \dots + \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{h^n}\frac{\Delta y_{1\dots n+1}}{n!}$$

Taylor Polynomials:

↳ where we take $h \to 0$ and $\Delta y_{\dots} \to 0$!

aka $\frac{\Delta y}{h} = \frac{\Delta y}{\Delta x} = f'(x)$

$$P_{n,x_0}(x) = y_0 + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

order n, centered at x_0

$$+ \dots + \frac{f^n(x_0)}{n!}(x - x_0)^n.$$

- A taylor series centered at 0 is called a maclaurin series!

Example: find $T_{6,0}(x)$ for $f(x) = \cos(x)$.

$f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$,
$f'''(x) = \sin x$, $f^4(x) = f(x)$, $f^5(x) = f'(x)$, $f^6(x) = f''(x)$.

$$P_{6,0}(x) = \cos 0 + (-\sin 0)(x - 0) + (-\cos 0)\frac{(x-0)^2}{2!} + \sin 0\frac{(x-0)^3}{3!}$$
$$+ \cos 0\frac{(x-0)^4}{4!} + (-\sin 0)\frac{(x-0)^5}{5!} + (-\cos 0)\frac{(x-0)^6}{6!}$$

↳ $P_{6,0}(x) = 1 - 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} - 0 - \frac{x^6}{6!}$

↳ $P_{6,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$.

Maclaurins to memorize!!

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

When trying to find a taylor series for a product of two functions, you can do them seperately and multiply their TSs!

↓

Example: find $T_{2,0}(x)$ for $e^x \cos x$.

↳ let $f(x) = e^x$ and $g(x) = \cos x$.

$$f(x) = e^x = 1 + x + \frac{x^2}{2!}, \qquad g(x) = \cos x = 1 - \frac{x^2}{2!}$$

$$\therefore f(x)g(x) = e^x \cos x = \left(1 + x + \frac{x^2}{2!}\right)\left(1 - \frac{x^2}{2!}\right)$$

$$= 1 - \frac{x^2}{2!} + x - \frac{x^3}{2!} + \frac{x^2}{2!} - \frac{x^4}{2!} = 1 + x + \frac{x^3}{2!} - \frac{x^4}{2!}.$$

However, since we only need terms of degree ≤ 2, we can disregard $\frac{x^3}{2!}$ and $-\frac{x^4}{2!}$.

$\therefore\ T_{2,0}(x) = 1 + x$!

We can take a similar approach to composition!

Example: find $T_{10,0}(x)$ for $\cos(x^2)$:

we know $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$, so

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!}$$

$$= 1 - \frac{x^4}{2} + \frac{x^8}{24} + \frac{x^{12}}{720}.$$

But since $x^{12} > x^{10}$, $T_{10,0}(x) = 1 - \frac{x^4}{2} + \frac{x^8}{24}$.

Taylor Series can also be used as a (major) improvement to linear approximations!

↓

Example: Approximate $\sqrt{4.5}$ with $P_{2,x_0}(x)$.

take $f(x) = \sqrt{x}$ and $x_0 = 4$

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$.

↓

$f(4) = 2$, $f'(4) = 1/4$, $f''(4) = -\frac{1}{32}$.

$$\therefore P_{2,4}(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\,\frac{(x-4)^2}{2!}$$

$$\sqrt{4.5} \approx P_{2,4}(4.5) = 2 + \frac{1}{4}\left(\frac{1}{2}\right) - \frac{1}{32}\cdot\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2$$

$$= 2 + \frac{1}{8} - \frac{1}{64}\left(\frac{1}{4}\right) = 2.121.$$

- There are certain functions that can be very hard or even impossible to integrate, so we can just integrate their taylor polynomials to get a good approximation.

Example: $\int_0^{1/3} \cos(x^2)\,dx$

↳ Since $\cos(x^2)$ cannot be integrated normally, we can just integrate it's taylor expansion!

$$\cos(x^2) \approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}$$

$$\int_0^{1/3} \cos(x^2)\,dx \approx \int_0^{1/3} 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}\,dx$$

$$= \left[x - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{9360}\right]_0^{1/3} \approx 0.3329.$$

The Taylor Theorem with Integral Remainder

↳ Suppose that f has $n+1$ derivatives at x_0. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!}(x-x_0)^k + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

this expression is labelled $R_n(x)$!

Taylor's Inequality:

↳ The error in using an n^{th}-order Taylor Polynomial $P_{n,x_0}(x)$ as an approximation to $f(x)$ satisfies the inequality:

$$|R_n(x)| \leq K \frac{|x - x_0|^{n+1}}{(n+1)!},$$

where $|f^{n+1}(z)| \leq K$ $\forall$ values of z between x_0 and x.

Example: for which value of x can we replace $\sin x$ by $P_{3,0}(x) = x - \frac{x^3}{3!}$ with an error magnitude of no greater than $4 \cdot 10^{-3}$?

$|R_n(x)| \leq K \frac{|x - x_0|^{n+1}}{(n+1)!}$, where $n = 3$, $x_0 = 0$.

↳ $|R_{3,0}(x)| \leq K \frac{|x|^4}{4!}$, $|f^4(x)| \leq K$.

$f(x) = \sin x$, so $f^4(x) = -\cos x$, so $|f^4(x)| = \cos x$.

$\cos x$ maximum at $1 \rightarrow K = 1$ because $|f^4(x)| \leq 1$.

↳ $|R_{3,0}(x)| \leq \frac{|x|^4}{4!}$ for which $\frac{|x|^4}{4!} \leq 4 \cdot 10^{-3}$

$\therefore |x| \leq (4! \cdot 4 \cdot 10^{-3})^{1/4} \leq 0.55$

• Upper bound on error in the integral approx. is:

$$\hookrightarrow \int_a^b K \frac{|t-x_0|^{n+1}}{(n+1)!} \, dt$$

Example: Approximate $\int_0^1 e^x \, dx$ using $P_{3,0}(x)$ e^x and evaluate the associated error.

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

stop here, since $P_{3,0}$

$$\int_0^1 e^x \, dx = \int_0^1 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \, dx$$

$$= \left[x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} \right]_0^1 = 1.708.$$

$|R_n(x)| \leq K \frac{|x-x_0|^{n+1}}{(n+1)!}$, where $n=3$, $x_0 = 0$.

$$\hookrightarrow |R_{3,0}(x)| \leq K \frac{|x|^4}{4!}$$

K is upper bound for $f^4(x)$ on $[0,1]$.

since $f(x) = e^x$, $f^4(x) = e^x$, so we can simply take $K = 3$ for simplicity.

take 3, as it's simpler

$\therefore |f^4(x)| \leq e < 3$ on $[0,1]$

or use e

$$\hookrightarrow \int_0^1 3 \frac{x^4}{4!} \, dx = \frac{3}{5!} \approx 0.02.$$

Example: the length of the curve $y = x^3$ from $(0,0)$ to $(\frac{1}{2}, \frac{1}{8})$ is given by $L = \int_0^{1/2} \sqrt{1 + 9x^4} \, dx$. Using two nonzero terms of a taylor polynomial, estimate the value of L, and use Taylor's Inequality to find an upper bound on the magnitude of the error.

take $f(u) = \sqrt{1+u}$, where $u = 9x^2$ and $x_0 = 0$.

$f(u) = (1+u)^{1/2} \to f'(u) = \frac{1}{2}(1+u)^{-1/2} \to f'(0) = 1/2$

$\hookrightarrow f''(u) = -\frac{1}{4}(1+u)^{-3/2} \to f''(0) = -1/4$

2 terms centered at 0 for P(u).

$P_{1,0}(u) = f(0) + f'(0)u = 1 + \frac{u}{2}$ $\quad (\approx \sqrt{1+u})$

composition

$\hookrightarrow P_{1,0}(9x^4) = 1 + \frac{9x^4}{2}$ $\quad (\approx \sqrt{1+9x^2})$

integration

$\hookrightarrow \int_0^{1/2} P_{1,0}(9x^4)\,dx = \int_0^{1/2} 1 + \frac{9}{2}x^4\,dx = \frac{1}{2} + \frac{9}{10 \cdot 2^5}$ $\quad (\approx \int_0^{1/2} \sqrt{1+9x^2}\,dx)$ $= L$

we know $f(u) = \sqrt{1+u} = 1 + \frac{u}{2} + R_{1,0}(u)$, where

x_0

$2 \to 1+1$, since $R_{1,0}$

$|R_{1,0}(u)| \le K \frac{|u-0|^2}{2!}$, from Taylor's Inequality.

Since we're integrating $x \in [0, \frac{1}{2}]$, we're integrating $u \in [0, 9/16]$, as $u = 9x^4$.

$\hookrightarrow$ Therefore, K is the upper bound of $|f''(x)|$ on $[0, \frac{9}{16}]$.

let $g(u) = |f''(u)| = \frac{1}{4}(1+u)^{-3/2} = \frac{1}{4(1+u)^{3/2}}$.

Since u is always ≥ 0 $(u = 9x^4)$, g(u) will be maximized at $u = 0 \to K = \frac{1}{4}$.

$\hookrightarrow$ Error $= \int_0^{1/2} K \frac{|u|^2}{2!} dx = \int_0^{1/2} \frac{1}{4} \frac{81x^8}{2} dx$

$= \frac{81}{8} \left[\frac{x^9}{9} \right]_0^{1/2} = \frac{81}{72} \cdot \frac{1}{2^9}$

$\therefore L = \int_0^{1/2} \sqrt{1+9x^4}\, dx \approx \frac{1}{2} + \frac{9}{10 \cdot 2^5} \pm \frac{81}{72 \cdot 2^9}$

Infinite Series: the sum of the elements of a sequence a_i while keeping the order.

$\hookrightarrow \sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

Partial Sum: $S_n = \sum_{i=1}^{n} a_i$

Geometric Series: $\sum_{n=0}^{\infty} a(r)^n = a \sum_{n=0}^{\infty} (r)^n$

$\hookrightarrow$ if $|r| < 1$, a geometric series will always converge to: $\frac{1}{1-r}$.

Example: is $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots$ converging? if yes, what is the sum?

$\hookrightarrow$ Looks like a geometric series: $\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n$

we know that $\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n - \left(\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1\right)$

Since $|r| = \frac{1}{3} < 1$, we know this series will converge by the geometric series test.

$\hookrightarrow \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} - \left(1+\frac{1}{3}\right) = \frac{1}{\frac{2}{3}} - \frac{4}{3}$

$= \frac{3}{2} - \frac{4}{3} = \frac{9}{6} - \frac{8}{6} = \frac{1}{6}.$

Example: $\sum_{n=0}^{\infty} 2^n = ?$

$\hookrightarrow$ since $|r| = 2 > 1$, this series will diverge by the geometric series test.

Example: what is the fraction associated to 1.2313131... ?

$\hookrightarrow$ $1.2313131... = 1.2 + 0.031 + 0.00031 + \dots$

$= 1.2 + 3.1 \times 10^{-2} + 3.1 \times 10^{-4} + \dots$

$1.2313131... = \frac{12}{10} + \sum_{n=0}^{\infty} 3.1 \times 10^{-2n}$

$= \frac{6}{5} + 3.1 \sum_{n=1}^{\infty} (10^{-2})^n$

$= \frac{6}{5} + 3.1\left(\sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n - 1\right)$

aka $(10^{-2})^0$

$\therefore, = \frac{6}{5} + \frac{31}{10}\left(\frac{1}{1-\frac{1}{100}} - 1\right) = \frac{6}{5} + \frac{31}{10}\left(\frac{100}{99} - 1\right)$

$= \frac{6}{5} + \frac{31}{10}\left(\frac{1}{99}\right) = \frac{1219}{990}.$

$\therefore, 1.2313131... = \frac{1219}{990}$!

Divergence Test: Given a series $\sum_{n=0}^{\infty} c_n$, if $\lim_{n\to\infty} c_n \neq 0$, then the series diverges!

↳ this test cannot prove convergence!

Example: is $\sum_{n=1}^{\infty} \frac{\ln(2+e^n)}{2n}$ convergent?

↳ $\lim_{n\to\infty} \frac{\ln(2+e^n)}{2n}$ is of the form $\frac{\infty}{\infty}$

↳ $\lim_{n\to\infty} \frac{\frac{1}{2+e^n}\cdot e^n}{2} = \lim_{n\to\infty} \frac{1}{2}\,\frac{e^n}{2+e^n} = \frac{1}{2}.$

Since $\lim_{n\to\infty} \frac{\ln(2+e^n)}{2n} = \frac{1}{2} \neq 0$, this series diverges by the divergence test.

↱ always do divergence test first!

Integral Test: let $\sum_{n=1}^{\infty} a_n$ be such that $a_n \geq 0$ and a_n monotonically decreases as a function of n. Then, suppose $f(x)$ is a continuous positive decreasing function such that $f(n) = a_n$.

↳ $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)\,dx$ converges!

↳ $\sum_{n=1}^{\infty} a_n$ diverges if $\int_1^{\infty} f(x)\,dx$ diverges!

Note: if $\int_1^{\infty} f(x)\,dx = L$, it does NOT mean that the sum $\sum_{n=1}^{\infty} a_n = L$!!

Example: is $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent or divergent?

↳ divergence test fails as $\lim_{n\to\infty} \frac{1}{n} = 0$!

we can apply the integral test as $\frac{1}{n}$ is always greater than or equal to 0 and monotonically decreases as a function of n.

↳ let $f(x) = \frac{1}{x}$ → decreasing for $x \geq 1$, as $f'(x) = -1/x^2 < 0$ for $x \geq 1$.

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln b - \ln x\right]_1^b$$

$$= \lim_{b\to\infty} \ln\left|\frac{b}{1}\right| = \lim_{b\to\infty} \ln|b| = \infty.$$

∴, since $\int_1^\infty \frac{1}{x}\,dx$ is divergent, $\sum_{n=1}^\infty \frac{1}{n}$ also diverges by the integral test!

Example: is $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ convergent?

↳ divergence test fails as $\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$!

we can apply the integral test as $\frac{1}{n(\ln n)^2}$ is always greater than or equal to 0 and monotonically decreasing as a function of n.

↳ let $f(x) = \frac{1}{x(\ln x)^2}$

↳ $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ $\quad u = \ln x \to du = \frac{1}{x}dx \to dx = x\,du$

$\hookrightarrow \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{x}{x\,u^2}\,du = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-2}\,du$

$= \lim_{b\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \left[-\frac{1}{\ln b} + \frac{1}{\ln 2}\right]$

$= \frac{1}{\ln 2}.$

$\therefore$ Since $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ converges, $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ also converges by the integral test!

p-series Test: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \le 1$.

Example: is $\sum_{n=4}^\infty \frac{1}{n^2}$ convergent or divergent?

$\hookrightarrow$ since $p = 2 > 1$, $\sum_{n=4}^\infty$ is convergent by the p-series test!

- Note, we do not care what value n starts at.

Comparison Test: consider 2 sequences a_n and b_n such that $0 \le a_n \le b_n$. If:

- $\hookrightarrow \sum_{n=1}^\infty b_n$ converges $\rightarrow \sum_{n=1}^\infty a_n$ also converges.
- $\hookrightarrow \sum_{n=1}^\infty a_n$ diverges $\rightarrow \sum_{n=1}^\infty b_n$ also diverges.

Example: is $\sum_{n=1}^\infty \frac{n}{n^3+1}$ convergent?

$\hookrightarrow$ divergence test fails as $\lim_{n\to\infty} \frac{n}{n^3+1} = 0.$

we know that $n^3+1 \geq n^3$, so $\frac{1}{n^3+1} \leq \frac{1}{n^3}$.
Therefore, $\frac{n}{n^3+1} \leq \frac{n}{n^3} \to \frac{n}{n^3+1} \leq \frac{1}{n^2}$.

$\therefore \quad 0 \leq \frac{n}{n^3+1} \leq \frac{1}{n^2}$.

we know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series test with $p=2$.

Since $0 \leq \frac{n}{n^3+1} \leq \frac{1}{n^2}$ and $\frac{1}{n^2}$ is convergent, we see that the series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ must also be convergent by the comparison test.

Example: is $\sum_{n=1}^{\infty} \frac{1}{3^n+n}$ convergent?

divergence test fails, as $\lim_{n\to\infty} \frac{1}{3^n+n} = 0.$

we know $3^n+n \geq n$, so $\frac{1}{3^n+n} \leq \frac{1}{n}$.
But, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the p-series test, so the comparison test fails for $b_n = n$.

we also know $3^n+n \geq 3^n$, so $\frac{1}{3^n+n} \leq \frac{1}{3^n}$

since $0 \leq \frac{1}{3^n+n} \leq \frac{1}{3^n}$, and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent as it is a geometric series with $|r| = \frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{1}{3^n+n}$ is also convergent by the comparison test!

Limit Comparison Test: if $a_n \geq 0$ and $b_n > 0$:

↳ if $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$, then either both a_n and b_n converge or both diverge!

Example: is $\sum_{n=0}^{\infty} \frac{n}{n^3+3}$ convergent?

↳ divergence test fails, as $\lim_{n\to\infty} \frac{n}{n^3+3} = 0$.

let $a_n = \frac{n}{n^3+3}$ (≥ 0) and $b_n = \frac{n}{n^3} = \frac{1}{n^2}$ (>0).

↳ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3+3} \cdot n^2 = \lim_{n\to\infty} \frac{n^3}{n^3+3} = 1 \neq 0$.

we know $b_n = \frac{1}{n^2}$ is convergent by the p-series test with $p=2$, and therefore, as $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n}{n^3+3}$ is also convergent by the LCT.

Example: is $\sum_{n=2}^{\infty} \frac{4n^2+n}{(n^7+n^3)^3}$ convergent?

↳ divergence test fails, as $\lim_{n\to\infty} \frac{4n^2+n}{(n^7+n^3)^3} = 0$.

let $a_n = \frac{4n^2+n}{(n^7+n^3)^3}$ (≥ 0) and $b_n = \frac{n^2}{n^{21}} = \frac{1}{n^{19}}$ (>0).

↳ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^2+n}{(n^7+n^3)^3} \cdot n^{19} = \lim_{n\to\infty} \frac{4n^{21}+n^{20}}{(n^7+n^3)^3} = 4 \neq 0$.

we know $b_n = \frac{1}{n^{19}}$ is convergent by the p-series test with $p=19$, and therefore, as $\lim_{n\to\infty} \frac{a_n}{b_n} = 4$,

$\sum_{n=2}^{\infty} \frac{4n^2+n}{(n^7+n^3)^3} = \sum_{n=2}^{\infty} a_n$ is also convergent by LCT.

Alternating Series Test: An alternating series is of the form $\sum_{n=0}^{\infty} (-1)^n c_n$ where $c_n \geq 0$.

if $c_n \geq 0$, $\lim_{n\to\infty} c_n = 0$, and c_n monotonically decreases as a function of n, then the alternating series $\sum_{n=0}^{\infty} (-1)^n c_n$ converges!

Example: does $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converge?

↳ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = -\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ → alternating series!

as $\lim_{n\to\infty} \frac{1}{n} = 0$, $\frac{1}{n} \geq 0$ for all $n \geq 1$, and $\frac{1}{n}$ monotonically decreases as a function of n, $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges by the AST.

Absolute and Conditional Convergence:

↳ $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=0}^{\infty} |a_n|$ does not.

Example: is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ absolutely or conditionally convergent?

|

↳ since $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the p-series test $(p = 2 > 1)$, we see that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

Example: is $\sum_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n^3+1}}$ absolutely convergent?

↓

aka, is $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{\sqrt{n^3+1}}\right| = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{\sqrt{n^3+1}}$ convergent?

we know that $n^3+1 \geq n^3$, so $\sqrt{n^3+1} \geq \sqrt{n^3}$, and therefore, $\frac{1}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3}} \rightarrow = \frac{1}{n^{3/2}}$

we also know that $|\sin(n)|$ is always ≤ 1.

$$\therefore \frac{|\sin(n)|}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3+1}} \leq \frac{1}{n^{3/2}}.$$

Since $0 \leq \frac{|\sin(n)|}{\sqrt{n^3+1}} \leq \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-series test, we know that $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{\sqrt{n^3+1}}$ also converges by the comparison test.

Since $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{\sqrt{n^3+1}} = \sum_{n=1}^{\infty} \left|\frac{\sin(n)}{\sqrt{n^3+1}}\right|$ converges, we see that the series is absolutely convergent!

Ratio Test: Consider the series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = R$. If:

↳ $R = 1 \rightarrow$ the ratio test fails!

$R < 1 \to$ the series converges absolutely

$R > 1 \to$ the series diverges

Example: is $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ convergent ?

let $a_n = (-1)^n \frac{2^n}{n!}$.

since we have abs value, $(-1)^n$ and $(-1)^{n+1} = 1$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n (-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n\to\infty} \left| \frac{2 \cdot 2^n}{(n+1)\, n!} \cdot \frac{n!}{2^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0 < 1.$$

since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$ where $a_n = (-1)^n \frac{2^n}{n!}$, $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ converges absolutely by the ratio test!

Telescoping Series: terms in the partial sum cancel.

Example: $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = ?$

$$\hookrightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \sum_{n=0}^{\infty} \frac{1}{n+1} - \sum_{n=0}^{\infty} \frac{1}{n+2}$$

$$= \underbrace{\left(1 - \tfrac{1}{2}\right)}_{n=0} + \underbrace{\left(\tfrac{1}{2} - \tfrac{1}{3}\right)}_{n=1} + \underbrace{\left(\tfrac{1}{3} - \tfrac{1}{4}\right)}_{n=2} + \underbrace{\ldots}_{\ldots}$$

$= 1 + \ldots$ where things will keep infinitely cancelling out!

$$\therefore \sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = 1.$$

Power Series: of the form $\sum_{n=0}^{\infty} c_n (x - x_0)^n$, with variable x and centered at x_0.

Example: interval of convergence of $\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$?

let $a_n = \frac{1}{2^n} x^n$.

$\hookrightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x \cdot x^n}{2 \cdot 2^n} \cdot \frac{2^n}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x}{2} \right| = \frac{|x|}{2}$.

for $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ to converge by the ratio test, we need $\frac{|x|}{2}$ to be less than one.

$\hookrightarrow$ Radius of convergence is 2!

$\hookrightarrow \frac{|x|}{2} < 1 \rightarrow |x| < 2 \rightarrow -2 < x < 2$.

$x = -2$: $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n \cdot (-1)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$.

$\hookrightarrow$ divergent, so $x = -2$ not included in IoC!

$x = 2$: $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$

$\hookrightarrow$ divergent, so $x = 2$ not included in IoC!

$\therefore$, interval of convergence is $(-2, 2)$.

Example: $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\sqrt{n}\, 3^n}$ interval of convergence ?

$\hookrightarrow$ let $a_n = \dfrac{(x-2)^n}{n\sqrt{n}\,3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}}\cdot\frac{n\sqrt{n}\,3^n}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)(x-2)^n}{(n+1)\sqrt{n+1}\cdot 3\cdot 3^n}\cdot\frac{n\sqrt{n}\cdot 3^n}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{n\sqrt{n}(x-2)}{3(n+1)\sqrt{n+1}}\right|$$

$$= \frac{|x-2|}{3}\lim_{n\to\infty}\left|\frac{n\sqrt{n}}{(n+1)\sqrt{n+1}}\right| = \frac{|x-2|}{3}.$$

for $\sum_{n=1}^{\infty} a_n$ to converge, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ must be <1.

$$\therefore \frac{|x-2|}{3} < 1 \rightarrow |x-2| < 3 \rightarrow -1 < x < 5$$

$$x=-1: \sum_{n=1}^{\infty}\frac{(x-2)^n}{n\sqrt{n}\,3^n} = \sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}\,3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n\sqrt{n}\,3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$$

$\hookrightarrow$ convergent by AST, so included in IoC!

$$x=5: \sum_{n=1}^{\infty}\frac{(x-2)^n}{n\sqrt{n}\,3^n} = \sum_{n=1}^{\infty}\frac{3^n}{n\sqrt{n}\,3^n} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

$\hookrightarrow$ converges by p-series test ($p = 3/2 > 1$), so included in IoC:

$\therefore$ interval of convergence : $[-1, 5]$.

Using Power Series to get Taylor Series:

Example: e^{-x^3}

↳ let $u = -x^3$ → we know $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$.

$$\therefore e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!}$$

Example: find the Taylor Series of $\frac{1}{x}$ centered at 2.

$\frac{1}{x}$ is quite close to $\frac{1}{1-x}$, for which we know the Taylor Expansion!

$$\frac{1}{x} = \frac{1}{x-2+2} = \frac{1}{2+(x-2)} = \frac{1}{2} \frac{1}{1+\frac{x-2}{2}}$$

let $-u = \frac{x-2}{2}$ → $u = \frac{2-x}{2}$. → $\frac{1}{2} \frac{1}{1-u}$.

$$\frac{1}{2} \frac{1}{1-u} = \frac{1}{2} \sum_{n=0}^{\infty} u^n = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2-x}{2}\right)^n = -\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{2^n}$$

Example: find the Taylor Series for $\frac{x}{(1+x)^2}$ centered at 2.

Note that $-\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{1}{(1+x)^2}$. → we need to write $\frac{1}{1+x}$ as a taylor series centered at 2 and then differentiate, giving us the Taylor series for $\frac{1}{(1+x)^2}$ centered at 2.

$$↳ \frac{1}{1+x} = \frac{1}{1+x-2+2} = \frac{1}{3+(x-2)} = \frac{1}{3} \frac{1}{1+\frac{x-2}{3}}$$

let $-u = \frac{x-2}{3}$ → $u = -\frac{x-2}{3} = \frac{2-x}{3}$

$$\frac{1}{3}\frac{1}{1-u} = \frac{1}{3}\sum_{n=0}^{\infty} u^n = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n \frac{(x-2)^n}{3^n}$$

→ Taylor Series for $\frac{1}{1+x}$ centered at 2.

$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{3}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n \frac{(x-2)^n}{3^n}$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}\frac{d}{dx}(x-2)^n = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n} n(x-2)^{n-1}$$

$$\therefore \frac{1}{(1+x)^2} = -\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n \frac{n(x-2)^{n-1}}{3^n}.$$

$$\frac{x}{(1+x)^2} = x \cdot \frac{1}{(1+x)^2} = (x-2+2)\cdot\frac{1}{(1+x)^2}$$

$$= (x-2)\frac{1}{(1+x)^2} + 2\frac{1}{(1+x)^2}$$

$$= (x-2)\left[-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n \frac{n(x-2)^{n-1}}{3^n}\right] + 2\left[-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n \frac{n(x-2)^{n-1}}{3^n}\right]$$

$$= \frac{2-x}{3}\sum_{n=0}^{\infty}(-1)^n \frac{n(x-2)^{n-1}}{3^n} - \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n \frac{n(x-2)^{n-1}}{3^n}$$

Operations on Power Series:

- If the series $\sum c_n (x-x_0)^n$ has radius of convergence R, then we can:
  - differentiate it (term-by-term)
  - integrate it (term-by-term)
  - multiply through by a constant (term-by-term)
  - add it (term-by-term) to another series with radius of convergence $\geq$ R.

and the result will also have R_oC R.

- Note: end points of I_oC might change!

Addition and Subtraction

note both centered at a!

Consider 2 series $\sum_{n=0}^{\infty} b_n (x-a)^n$ and $\sum_{n=0}^{\infty} c_n (x-a)^n$ which converge to some functions $f(x)$ and $g(x)$ on intervals I_1 and I_2. Then the series

$$\sum_{n=0}^{\infty} (b_n \pm c_n)(x-a)^n \text{ converges to } f(x) \pm g(x) \text{ on } I_1 \cap I_2$$

Example: what is the IoC of $\sum_{n=1}^{\infty} \left(1+\frac{1}{n3^n}\right) x^n$?

$$\hookrightarrow \sum_{n=1}^{\infty} \left(1+\frac{1}{n3^n}\right) x^n = \sum_{n=1}^{\infty} x^n + \frac{x^n}{n3^n} = \sum_{n=1}^{\infty} x^n + \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$$

IoC of x^n : $(-1, 1)$ (from geometric series)

IoC of $\frac{x^n}{n3^n}$: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x \cdot x^n}{(n+1)\cdot 3 \cdot 3^n} \cdot \frac{n3^n}{x^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{xn}{(n+1)\cdot 3} \right| = \frac{|x|}{3} \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \frac{|x|}{3}.$$

$$\frac{|x|}{3} < 1 \rightarrow |x| < 3 \rightarrow -3 < x < 3.$$

$$x=-3: \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

↳ converges by AST, so include in IoC!

$$x=3: \sum_{n=1}^{\infty} \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

↳ diverges by p-series test ($p=1 \leq 1$), so do not include in IoC!

$\therefore$ IoC of $\frac{x^n}{n3^n}$: $[-3, 3)$, IoC of x^n: $(-1, 1)$

$[-3, 3) \cap (-1, 1) = (-1, 1)$.

Product

note both centered at a!

Consider 2 series $\sum_{n=0}^{\infty} b_n (x-a)^n$ and $\sum_{n=0}^{\infty} c_n (x-a)^n$ which converge to some functions $f(x)$ and $g(x)$ with radius of convergence R_1 and R_2. Then the radius of convergence of the product:

$$\left(\sum_{n=0}^{\infty} b_n (x-a)^n\right)\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) \text{ is } R = \min(R_1, R_2)$$

Example: RoC of $\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right)\left(\sum_{n=1}^{\infty} 2^n x^n\right) = ?$

RoC of $\frac{x^n}{n}$: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x \cdot x^n}{n+1} \cdot \frac{n}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{xn}{n+1} \right| = |x| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = |x|$.

$\hookrightarrow |x| < 1 \to -1 < x < 1 \to$ RoC $= 1$.

RoC of $2^n x^n$: $\lim_{n\to\infty} \left| 2^{n+1} x^{n+1} \cdot \frac{1}{2^n x^n} \right| = \lim_{n\to\infty} \left| \frac{2x\, 2^n x^n}{2^n x^n} \right|$

$= 2|x| \lim_{n\to\infty} \left| \frac{2^n x^n}{2^n x^n} \right| = 2|x|$

$\hookrightarrow 2|x| < 1 \to -1 < 2x < 1 \to -\frac{1}{2} < x < \frac{1}{2} \to$ RoC $= \frac{1}{2}$.

RoC is $\min(\frac{1}{2}, 1) = \frac{1}{2}$!

Integration and Differentiation:

consider $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ with RoC R, then f is differentiable on $|x-a| < R$.

↳ $f'(x) = \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=0}^{\infty} c_n\, n\, (x-a)^{n-1}$

↳ check this starting value!!

integration of f on interval of convergence:

↳ $\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n\, dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

↳ the radius of convergence of $f'(x)$ and $\int f(x)\,dx$ is also R! But, the bounds might stop being included in the IoC after differentiating or integrating!

Example: $\sum_{n=0}^{\infty} (n+1)x^n = ?$

note that $(n+1)x^n = \frac{d}{dx} x^{n+1} = \frac{d}{dx}(x \cdot x^n)$!

$$\sum_{n=0}^{\infty} x\, x^n = x \sum_{n=0}^{\infty} x^n = x\left(\frac{1}{1-x}\right) = \frac{x}{1-x}$$

$$\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{d}{dx}\left(\frac{1}{1-x} - 1\right) = \frac{1}{(1-x)^2}$$

↳ $\therefore,\ \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$ with RoC $= 1$.

Example: find a Maclaurin series for $\tan^{-1} x$.

|

↳ we know that $\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt$.

$$\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} \rightarrow \text{let } u = -t^2 \rightarrow \frac{1}{1-u}.$$

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}.$$

$$\therefore \tan^{-1}(x) = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x = \sum_{n=0}^{\infty} (-1) \frac{x^{2n+1}}{2n+1}$$

Example: Maclaurin series for $g(x) = \int_0^x t^2 e^{-t^2}\, dt$?

we know that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$. let $u = -t^2$:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}.$$

↳ $t^2 e^{-t^2} = t^2 \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+2}}{n!}$

$$g(x) = \int_0^x t^2 e^{-t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+2}}{n!}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n+2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{t^{2n+3}}{2n+3}\right]_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+3)n!}$$

Big-O Notation

↳ given two functions f and g, we say that "f is of order g as $x \to x_0$" and write:

$$f(x) = O(g(x)) \text{ as } x \to x_0$$

if there exists a constant $A > 0$ such that

$$|f(x)| \leq A|g(x)|$$

on some interval around x_0.

↳ Big-O makes Taylor's Inequality much easier!

Taylor's Inequality: $|R_n(x)| \leq K \frac{|x-x_0|^{n+1}}{(n+1)!}$

↳ AKA, $R_n(x) = O((x-x_0)^{n+1})$ as $x \to x_0$!!

↳ ∴, $f(x) = P_{n,x_0}(x) + O((x-x_0)^{n+1})$!

Example: write the second-degree taylor polynomial for e^x using big-O notation.

↳ we know $e^x = 1 + x + \frac{x^2}{2!} + R_n(x)$

$n=2$, so $n+1=3$

aka, $e^x = 1 + x + \frac{x^2}{2} + O((x-x_0)^{n+1})$

↳ $x_0 = 0$

$= 1 + x + \frac{x^2}{2} + O(x^3)$ as $x \to 0$

Example: write the second degree Taylor

Polynomial for $\cos x$ using big-O notation.

↳ we know $\cos x = 1 - \frac{x^2}{2!} + R_n(x)$

aka, $\cos x = 1 - \frac{x^2}{2!} + O((x - x_0)^{n+1})$ ↳ $x_0 = 0$

$n=2$, but the $n=3$ term is 0, so use $n=4$!

$= 1 - \frac{x^2}{2!} + O(x^4)$ as $x \to 0$.

The algebra of Big-O:

the following hold as $x \to 0$:

- $C\,O(x^n) = O(x^n) \quad \forall\, C \in \mathbb{R}$
- $O(x^m) + O(x^n) = O(x^{\min(n,m)})$
- $O(x^m) \cdot O(x^n) = O(x^{m+n})$
- $[O(x^n)]^m = O(x^{nm})$
- $\frac{O(x^m)}{x^n} = O(x^{m-n})$

Example: write the second-degree Taylor Polynomial for $e^x + \cos x$ using Big-O Notation.

↳ from above examples, we see that:

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3) \text{ as } x \to 0$$

$$\cos x = 1 - \frac{x^2}{2!} + O(x^4) \text{ as } x \to 0$$

$$\therefore e^x + \cos x = \left(1 + x + \frac{x^2}{2}\right) + \left(1 - \frac{x^2}{2}\right) + \overbrace{O(x^3) + O(x^4)}^{\text{as } x \to 0}$$

$\hookrightarrow \therefore e^x + \cos x = 2 + x + O(x^{\min(3,4)})$

$= 2 + x + O(x^3)$ as $x \to 0$.

Evaluating limits with Big-O

Example: $\lim_{x \to 0} \frac{\sin x}{x} = ?$ (using Big-O!)

we know that $\sin x = x - \frac{x^3}{3!} + R_n(x)$

$\hookrightarrow \sin x = x + O(x^2) = x + O(x^3)$

$$\frac{\sin x}{x} = \frac{x + O(x^2)}{x} = 1 + O(x)$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} 1 + O(x) = 1 + \lim_{x \to 0} O(x) = 1.$$

Example: $\lim_{x \to 0} \left(\frac{1 - \cos(x^2)}{x^8} - \frac{1}{2x^4} \right) = ?$

we know that $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{4!} + O(u^5)$

$\hookrightarrow \cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} + O((x^2)^5)$

$$= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} + O(x^{10})$$

$$\frac{1 - \cos(x^2)}{x^8} = \frac{1 - 1 + \frac{x^4}{2} - \frac{x^8}{4!} + O(x^{10})}{x^8}$$

$$= \frac{\frac{x^4}{2} - \frac{x^8}{4!} + O(x^{10})}{x^8} = \frac{1}{2x^4} - \frac{1}{4!} + O(x^2)$$

$$\frac{1-\cos(x^2)}{x^8} - \frac{1}{2x^4} = \frac{1}{2x^4} - \frac{1}{4!} + O(x^2) - \frac{1}{2x^4}$$

$$= -\frac{1}{4!} + O(x^2)$$

$$\lim_{x \to 0} \left(\frac{1-\cos(x^2)}{x^8} - \frac{1}{2x^4} \right) = \lim_{x \to 0} \left(-\frac{1}{4!} + O(x^2) \right) = -\frac{1}{4!}$$

Example: $\lim_{x \to 0} \left(\frac{1-\sqrt{1+x^2}}{1+x-e^x} \right) = ?$

using Taylor's Theorem: $\sqrt{1+x^2}$:

$f(x) = \sqrt{1+x^2} \rightarrow f'(x) = x(1+x^2)^{-1/2}$

$\hookrightarrow f'(0) = 0$

$f''(x) = (1+x^2)^{-1/2} + -\frac{1}{2}(1+x^2)^{-3/2}(2x)(x)$

$\hookrightarrow f''(0) = 1$

$$\sqrt{1+x^2} = \frac{f(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + O(x^3)$$

$$\hookrightarrow \sqrt{1+x^2} = \frac{1}{1}(x)^0 + 0 + \frac{1}{2}(x)^2 + O(x^3)$$

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} + O(x^3).$$

$$\hookrightarrow \therefore, \ 1 - \sqrt{1+x^2} = -\frac{x^2}{2} + O(x^3)$$

we know that $1+x-e^x = 1+x-\left(1+x+\frac{x^2}{2}+O(x^3)\right)$

$\hookrightarrow 1+x-e^x = -\frac{x^2}{2} + O(x^3)$

$$\lim_{x \to 0} \frac{1-\sqrt{1+x^2}}{1+x-e^x} = \lim_{x \to 0} \frac{-\frac{x^2}{2} + O(x^3)}{-\frac{x^2}{2} + O(x^3)} = 1.$$

Note: never simplify $\frac{O(x^m)}{O(x^n)}$!!

The Alternating Series Estimation Theorem (ASET):

Consider a convergent alternating series $\sum (-1)^k a_k$.
If we use the n^{th} partial sum S_n as an estimate of the sum s, then the error satisfies the inequality:
$|s - s_n| \leq a_{n+1}$.

$\hookrightarrow$ aka, the truncation error is less than the first term omitted.

Example: Use the ASET to estimate the sum of $\sum_{k=3}^{\infty} (-1)^{k-1} \frac{8}{k^4 \ln k}$.

by the AST, we know that this series is convergent, as $\frac{8}{k^4 \ln k} \geq 0$, $\lim_{k \to \infty} \frac{8}{k^4 \ln k} = 0$, and it monotonically decreases as a function of k.

Using S_8 as an estimate for S:

$\hookrightarrow \frac{8}{3^4 \ln 3} - \frac{8}{4^4 \ln 4} + \frac{8}{5^4 \ln 5} - \frac{8}{6^4 \ln 6} + \frac{8}{7^4 \ln 7} - \frac{8}{8^4 \ln 8} \approx 0.0735$

the first term unused is $a_9 = \frac{8}{9^4 \ln 9} \approx 0.0009 < 0.001$.

$\therefore, \sum_{k=3}^{\infty} (-1)^k \frac{8}{k^4 \ln k} = 0.074 \pm 0.001$.