eg: $\{a, b, c, b\}$ is not a set!

Sets: an unordered collection of distinct objects.

↳ $\{a, b, c\} = \{c, b, a\}$ is a set of size 3

↳ $\{0, 1, 2, \dots\} = \mathbb{N}$ is an infinite set

Set operations: let A, B be finite sets:

- **Cartesian Product:** $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$
- **Size of Cartesian Product:** $|A \times B| = |A| \cdot |B|$

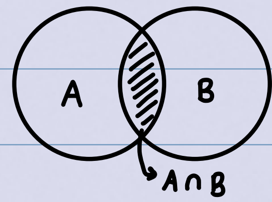
↳ eg: how many binary strings have length 4?

$\underline{0/1}$ $\underline{0/1}$ $\underline{0/1}$ $\underline{0/1}$ → $2 \times 2 \times 2 \times 2 = 2^4 = \underline{16}$.

$\#\text{strings} = |\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}| = |\{0,1\}| \times |\{0,1\}| \times |\{0,1\}| \times |\{0,1\}| = 2 \times 2 \times 2 \times 2 = 16.$

In general, the number of binary strings of length n is 2^n.

- **Union:** $A \cup B = \{x : x \in A \text{ or } x \in b\}$.
- **Size of Union:** $|A \cup B| = |A| + |B| - |A \cap B|$



↳ if $|A \cap B| = 0$, then $|A \cup B| = |A| + |B|$.

↳ $A \cup B$ with $|A \cap B| = 0$ is a disjoint union.

↳ eg: how many binary strings of length 8 are there which begin with 001 or 1011?

let S be the set of all strings.

let A_1 = set of length-8 binary strings that begin with 001.

let A_2 = set of length-8 binary strings that begin with 1011.

then $S = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. So, $|S| = |A_1| + |A_2| = 2^5 + 2^4 = 48$.

Counting Subsets

Permutations: a permutation of a set S is an ordered listing of the elements in S.

↳ eg: the permutations of $\{a, b, c\}$ are abc, acb, bac, bca, cab, cba.

- the number of permutations of a set S of size n is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \cdots \times \underline{2 \text{ choices}} \times \underline{1 \text{ choice}} = n!$$

Partial Permutations: a permutation of a subset of S.

- number of partial permutations of S (with $|S| = n$) of size k is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \cdots \times \underline{n-k+2 \text{ choices}} \times \underline{n-k+1 \text{ choices}}$$

$\therefore\ n(n-1)(n-2)\ldots(n-k+2)(n-k+1).$

↳ note: this works even if $k > n$.

How many subsets of S (with $|S| = n$) are there of size k?

- number of partial permutations of S of size k is $n(n-1)(n-2)\ldots(n-k+1)$.
- each size-k subset of S has $k!$ permutations.

↳ $\therefore$ $$\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

- **Combinations:** let $0 \leq k \leq n$. Then "n choose k" is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$ (aka, binomial coefficient)

↳ note, if $k > 1$, then $\binom{n}{k} = 0$.

eg: $\binom{n}{0} = \dfrac{n!}{0!\,(n-0)!} = \dfrac{n!}{0!\,n!} = \dfrac{1}{0!} = 1.$

eg: $\binom{n}{n} = \dfrac{n!}{n!\,(n-n)!} = \dfrac{n!}{n!\,0!} = \dfrac{1}{0!} = 1.$

eg: $\binom{10}{3} = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$

eg: $\binom{1999}{73} = \dfrac{1999 \times 1998 \times 1997 \times 1927}{73 \times 72 \times \ldots \times 1}$ = Some positive integer.

Binomial Theorem: for any $n \geq 1$, $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

eg: $(x+1)^3 = (x+1)(x+1)(x+1) = x^3 + 3x^2 + 3x + 1.$

$\binom{3}{3}=1 \quad \binom{3}{2} \quad \binom{3}{1} \quad \binom{3}{0}=1.$

Combinatorial Proof of the Binomial Theorem:

$(x+1)^n = (x+1)(x+1)\ldots(x+1)$ ← n copies of $(x+1)$

how do we get an x^k term? by selecting x from k binomials and 1 from the other $n-k$ binomials. ∴, the total number of ways is $\binom{n}{k}$.

Therefore, $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. □.

Combinatorial Proofs

- **Procedure to prove $M=N$:**
  1) "Cleverly" choose a set S
  2) Count the number of elements in S in two ways:
     a) $|S| = M$ and
     b) $|S| = N$
  3) Conclude that $M=N$.

Claim: $2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

↳ note: algebraically, we can prove this by substituting $x=1$ into the binomial theorem.

Combinatorial proof:

- let S be the set of all subsets of $\{1, 2, \ldots, n\}$.
  a) let S_i be the subsets in S of size i, where $0 \leq i \leq n$.

Then $S = S_0 \cup S_1 \cup \ldots \cup S_n$ (note, a disjoint union).

$\therefore$, $|S| = |S_0| + |S_1| + |S_2| + \ldots + |S_n|$

$|S| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$.

b) to choose a subset X of $\{1, 2, \ldots, n\}$, we do:

- for each i, $1 \le i \le n$, we include i in X, or not.
- Therefore, the total number of choices is 2^n. Hence $|S| = 2^n$.

$\therefore$, $|S| = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. $\square$.

Claim: let $1 \le k \le n$. Then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ [Pascal's Identity].

- Algebraic Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!\,(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} + \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} = \frac{(n-1)! \cdot (n-k)}{k!\,(n-k)!} + \frac{(n-1)! \cdot k}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,[(n-k)+k]}{k!\,(n-k)!} = \frac{(n-1)! \cdot n}{k!\,(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \quad \square.$$

- Combinatorial Proof:

let S = set of all size-k subsets of $\{1, 2, 3, \ldots, n\}$.

then, $|S| = \binom{n}{k}$ by definition.

let A = size-k subsets that don't include n.

let B = size-k subsets that includes n.

Then $S = A \cup B$ is a disjoint union. So, $|S| = |A| + |B|$.

Now, $|A| = \binom{n-1}{k}$, and $|B| = \binom{n-1}{k-1}$. Thus, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. $\square$.

- Claim: for $k, n \ge 0$, $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n-1}{n-1} + \binom{n}{n-1} + \ldots + \binom{n+k-1}{n-1}$

Combinatorial Proof:

$\therefore, |S| = \binom{n+k}{n}$

- let S = the size-n subsets of $\{1, 2, \dots, n, n+1, \dots, n+k-1, n+k\}$.
  - ↳ note: the largest number in a size-n subset is between n and $n+k$.

for each $0 \leq i \leq k$, let S_i = subset whose largest element is $n+i$.
Then, S = disjoint union $S_0 \cup S_1 \cup \dots \cup S_k$. So, $|S| = \sum_{i=0}^{k} |S_i| = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$. $\square$.

Visually:

See that the LHS is size-n subsets of $n+k$ sets. So, we choose subsets of size n (k times):

Subsets: $\{\overbrace{\qquad\qquad}^{n \text{ size}}\} \cup \{\overbrace{\qquad\qquad}^{n}\} \cup \dots \cup \{\overbrace{\qquad\qquad}^{n}\} \cup \{\overbrace{\qquad\qquad}^{n}\}$

$\cdot \{\qquad\qquad\} \cup \{\qquad\qquad\} \cup \dots \cup \{\qquad\qquad\} \cup \{\overbrace{\dots, n+k}^{n}\}$

↳ see that the last subset can be counted as $\binom{n+k-1}{n-1}$.

Claim: for $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$

↳ eg: $\binom{100}{98} = \binom{100}{2} = \frac{100 \times 99}{2}$.

eg: $n=4$, $k=1$: Size-1 subsets of $\{1,2,3,4\} = \{1\}, \{2\}, \{3\}, \{4\}$.

Size-3 subsets of $\{1,2,3,4\} = \{1,2,3\}, \{2,3,4\}, \{1,2,4\}, \{1,3,4\}$.

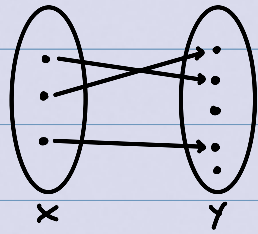
we can pair these sets by picking the size-3 subset which does not contain the element in the size-1 subset. ie: $\{1\} \Leftrightarrow \{2,3,4\}$, $\{2\} \Leftrightarrow \{1,3,4\}$...

→ aka, a correspondence or a bijection!

Bijections: let $f: X \to Y$ be a function.

- f is **injective (1-1)** if $\forall x_1, x_2 \in X,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

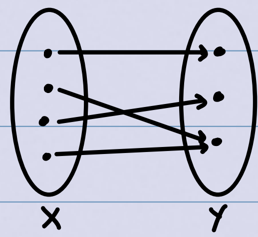
- Visually:



. see that $|X| \leq |Y|$.

- f is **surjective (onto)** if $\forall y \in Y,\ \exists x \in X$ st $f(x) = y$.

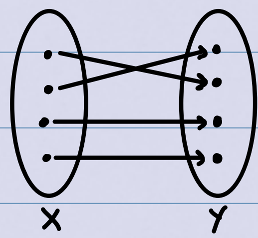
- Visually:



. see that $|X| \geq |Y|$

- f is **bijective** if it is injective AND surjective.

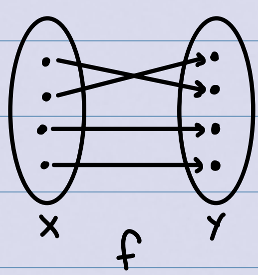
- Visually:

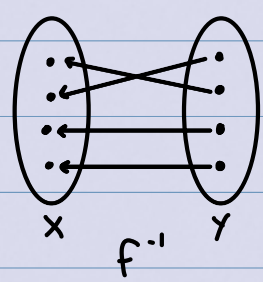


. see that $|X| = |Y|$.

- Theorem: let X, Y be finite sets. Suppose $f: X \to Y$ is a bijection. then, $|X| = |Y|$.

- **Inverse function:** the inverse of a function $f: X \to Y$ is a function $f^{-1}: Y \to X$ such that:
  1) (f^{-1} reverses f) $\Rightarrow \forall x \in X,\ f^{-1}(f(x)) = x$.
  2) (f reverses f^{-1}) $\Rightarrow \forall y \in Y,\ f(f^{-1}(y)) = y$.





• Theorem: let $f: X \to Y$. Then f is a bijection if and only if f has an inverse.

Combinatorial Proofs Using Bijections that $M = N$.

1) Select two sets X, Y, with $|X| = M$ and $|Y| = N$.
2) Define $f: X \to Y$. (f is a bijection).
3) Define $f^{-1}: Y \to X$.
4) Prove that f^{-1} is the inverse function
   ↳ prove the two conditions of the inverse function definition are satisfied.
5) Conclude that $M = N$.

• A function $f: X \to Y$ is well-defined if $\forall x \in X, f(x) \in Y$.

Claim: for $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$ (continued)

let X be the set of all size-k subsets of $\{1, 2, \dots, n\}$.
$\therefore, |X| = \binom{n}{k}$.
let Y be the set of all size-$(n-k)$ subsets of $\{1, 2, \dots, n\}$.
$\therefore, |Y| = \binom{n}{n-k}$.

size-k subset

Define $f: X \to Y$ as follows: $\forall S \in X$, $f(S) = S^c$ ($S^c = \{1, 2, \dots n\} - S$).
• See that f is well-defined, ie, $f(S) = S^c \in Y$ since $|S^c| = n-k$

size-$(n-k)$ subset

Define $f^{-1}: Y \to X$ as follows: $\forall T \in Y$, $f^{-1}(T) = T^c$ ($T^c = \{1, 2, \dots, n\} - T$).
• see that f^{-1} is well-defined, ie, $f^{-1}(T) = T^c \in X$ since $|T^c| = k$.

now, $\forall S \in X$, $f^{-1}(f(S)) = f^{-1}(S^c) = (S^c)^c = S$.
$\forall T \in Y$, $f(f^{-1}(T)) = f(T^c) = (T^c)^c = T$.

$\therefore$, f is a bijection, and therefore $|X| = |Y|$, so $\binom{n}{k} = \binom{n}{n-k}$.

Generating Series

↳ we'll encode solutions to counting problems as coefficients of a "generating series".

• eg: how many subsets of $\{1,2,3\}$ have size n, $\forall n$ $0 \le n \le 3$?

let S = all size-n subsets of $\{1,2,3\}$ $\forall n$ $0 \le n \le 3$.

define weight function $\omega: S \to \mathbb{N}$, by $\omega(\sigma) = |\sigma|$ $\forall \sigma \in S$.

$\sigma \in S$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2,3\}$
$\omega(\sigma)$	0	1	1	1	2	2	2	3

associate each element $\sigma \in S$ with the term $x^{\omega(\sigma)}$:

$x^{\omega(\sigma)}$	1	x	x	x	x^2	x^2	x^2	x^3

the generating series of X with respect to ω is the sum of the $x^{\omega(\sigma)}$ terms:

3 elements in S have weight 1

$$\Phi_S^{\omega}(x) = 1 + 3x + 3x^2 + x^3 = (x+1)^3.$$

3 elements in S have weight 2

$\therefore$, the number of size-n subsets of $\{1,2,3\}$ is the coefficient of x^n in $\Phi_S^{\omega}(x)$, $\forall$ $0 \le n \le 3$.

(of any length!)

•eg: how many binary strings don't have 000 or 00111 as a substring?

· let S = all binary strings with no 000 or 00111 as a substring.

↳ we want to organize these strings by their length, so:

define $w(\sigma)$ be the length of σ (where $\sigma \in S$).

Definition: let S be a set.

- A function $w: S \to \mathbb{N}$ is a weight function if $\forall n \in \mathbb{N}$ there are only finitely many elements in S of weight n.
- The generating series for S with respect to w is $\Phi_w^S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$.
- The coefficient of x^n in a $\Phi_w^S(x)$ is denoted $[x^n]\Phi_w^S(x)$.
  - ↳ eg: $[x^{73}](x+1)^{97} = \binom{97}{73} \to \binom{97}{73}x^{73}$.

↳ result: $[x^n]\Phi_w^S(x)$ is the number of elements in S of weight n.

Eg: S = all subsets from $\{1, 2, \dots, n\}$.

For $\sigma \in S$, $w(\sigma) = |\sigma|$.

Then, $\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n$

So, $[x^i]\Phi_S^w(x) = \binom{n}{i}$.

Eg: S = set of all binary strings.

for $\sigma \in S$, let $w(\sigma)$ = length of σ.

\# binary strings of length 1

Then, $\Phi_S^w(x) = ? + ?x + ?x^2 + ?x^3 + \dots$

\# binary strings of length 0   \# binary strings of length 2.

$\therefore$, $\Phi_S^w(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$

Formal Power Series (FPS)

- An FPS is an expression of the form $A(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{k \geq 0} a_k x^k$, where $a_i \in \mathbb{R}$.

↳ we don't care about convergence or divergence - we only care about the coefficients.

- So, x is called an "indeterminate".

Operations on Formal Power Series

let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k \geq 0} a_k x^k$
let $B(x) = b_0 + b_1 x + b_2 x^2 + \dots = \sum_{k \geq 0} b_k x^k$. } FPSs.

1) $(=)$: $A(x) = B(x) \iff a_k = b_k \;\; \forall k \geq 0$.

2) $(+)$: $A(x) + B(x) = \sum_{k \geq 0} (a_k + b_k) x^k$

2b) $(-)$: $A(x) - B(x) = \sum_{k \geq 0} (a_k - b_k) x^k$

3) $(\times)$: $A(x) \cdot B(x) = C(x) = \sum_{n \geq 0} c_n x^n$, where $c_n = a_0 b_n + a_1 b_{n-1} + \dots a_n b_0 = \sum_{i=0} a_i b_{n-i}$

4) (Inversion): the inverse of a FPS $A(x)$, if it exists, is a new FPS $B(x)$ such that $A(x)B(x) = 1$.

↳ we'll write $B(x) = A(x)^{-1} = 1/A(x)$.

Example: show that $(1-x)^{-1} = 1 + x + x^2 + \dots = \sum_{k \geq 0} x^k$.

let $P(x) = \text{RHS} = \sum_{k \geq 0} x^k$.

Then, $(1-x)P(x) = (1-x)\sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - x\sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - \sum_{k \geq 0} x^{k+1}$

re-index the Σ

$= \sum_{k \geq 0} x^k - \sum_{k \geq 1} x^k = 1 + \left(\sum_{k \geq 1} x^k - \sum_{k \geq 1} x^k\right) = 1 + 0 = 1.$

Common Formal Power Series

1) $(1-x)^{-1} = \frac{1}{1-x} = \sum_{k\geq 0} x^k \rightarrow [x^n](1-x)^{-1} = 1 \ \forall n \geq 0.$

↳ geometric series!

2) partial geometric series: $1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$.

partial bc not infinite!

3) binomial series: $(x+1)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$. $\therefore [x^k](x+1)^n = \binom{n}{k}$.

bc if $k>n$, $\binom{n}{k}=0$ anyway.

4) negative binomial series: $\forall n \geq 1$, $(1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k\geq 0} \binom{n+k-1}{k} x^k$

↳ so $[x^k](1-x)^{-n} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

Example: determine $[x^n]\left(\frac{x}{1-2x}\right)^5$.

$[x^n]\left(\frac{x}{1-2x}\right)^5 = [x^n]\, x^5(1-2x)^{-5} = [x^{n-5}](1-2x)^{-5}$ for $n \geq 5$.

Then, see that $(1-x)^{-5} = \sum_{k\geq 0} a_k x^k$. So, $(1-2x)^{-5} = \sum_{k\geq 0} a_k x^k 2^k$.

$= 2^{n-5}[x^{n-5}](1-x)^{-5} = 2^{n-5}\binom{n-5+5-1}{5-1} = 2^{n-5}\binom{n-1}{4}$.

Proving the negative binomial theorem $(1-x)^{-n} = \sum_{k\geq 0}\binom{n+k-1}{k}x^k$, $n \geq 1$.

Combinatorial Proof (sketch):

$(1-x)^{-n} = (1-x)^{-1} \cdot (1-x)^{-1} \cdot \ldots \cdot (1-x)^{-1}$ (n times).

$= (1+x+x^2+\dots)\cdot(1+x+x^2+\dots)\cdot\ldots\cdot(1+x+x^2+\dots)$ (n times)

- we get an x^k term by selecting x^{a_1} from the first series, x^{a_2} from the second series, ..., and x^{a_n} from the last series, where $a_1, a_2, \dots, a_k$ are non-negative integers with $a_1+a_2+\dots+a_n = k$.

Then, multiplying the x^{a_i} terms gives $x^{a_1}x^{a_2}\dots x^{a_n} = x^{a_1+a_2+\dots+a_n} = x^k$.

- The number of ways of choosing the a_i's is equal to the coefficient of x^k in $(1-x)^{-n}$.

- Illustration by example: $n=4$, $k=13$:
  - begin with a string of $k=13$ 0s and then insert $n-1=3$ 1s.

    ↳ $\underbrace{000}_{a_1=3}1\underbrace{0000}_{a_2=4}1\underbrace{00}_{a_3=2}1\underbrace{0000}_{a_4=4}.$

    note that $a_1+a_2+a_3+a_4 = 3+4+2+4 = 13$!
  - this gives a bijection between binary strings of length $k+n-1$ with exactly $n-1$ ones, and non-negative integers $a_1, a_2, \dots, a_n$ st $\sum_{i=1}^{n} a_i = k$.
  - ∴, the number of non-negative integers $a_1, \dots, a_k$ with $\sum_{i=1}^{n} a_i = k$ is equal to the number of binary strings of length $k+n-1$ with exactly $n-1$ ones, which is $\binom{k+n-1}{n-1} = \binom{n+k-1}{k}$.
  - ∴, the coefficient of x^k in $(1-x)^{-n}$ is $\binom{n+k-1}{k}$. □.

Extracting Coefficients from an FPS:

- let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$.

1) $[x^n](A(x) \pm B(x)) = [x^n]A(x) + [x^n]B(x)$.

2) $[x^n](A(x)\cdot B(x)) = \sum_{i=0}^{n} a_i b_{n-i} = \sum_{i=0}^{n} ([x^i]A(x))([x^{n-i}]B(x))$

3) $[x^n]cA(x) = c[x^n]A(x)$, where c is a constant.

4) $[x^n]x^l A(x) = 0$ if $n < l$, and $[x^n]x^l A(x) = [x^{n-l}]A(x)$ if $n \geq l$.

5) $[x^n]A(cx) = c^n[x^n]A(x)$, where c is a constant

6) $[x^n]A(x^\ell)=0$ if $\ell \nmid n$, and $[x^n]A(x^\ell) = [x^{n/\ell}]A(x)$ if $\ell \mid n$.

Example: determine $[x^n](x^3(3x+1)^7)$ where $n \geq 3$.

$$[x^n]\,x^3(3x+1)^7 = [x^{n-3}](3x+1)^7 = 3^{n-3}[x^{n-3}](x+1)^7$$

$$= 3^{n-3}\binom{7}{n-3}.$$

Example: determine $[x^n]\,x^3(3x+1)^7(1-4x^2)^{-m}$, $m \geq 1$, $n \geq 3$.

$$[x^n]\,x^3(3x+1)^7(1-4x^2)^{-m}$$

$$= [x^n]\,x^3\Big(\sum_{i\geq 0}\binom{7}{i}(3x)^i\Big)\Big(\sum_{j\geq 0}\binom{m+j-1}{j}(4x^2)^j\Big)$$

$$= [x^n]\sum_{i\geq 0,\, j\geq 0}\binom{7}{i}\binom{m+j-1}{j}3^i\,4^j\,x^{3+i+2j}$$

$$= \sum_{i\geq 0,\, j\geq 0}\binom{7}{i}\binom{m+j-1}{j}3^i\,4^j,$$ and only care where $3+i+2j=n$.

$\hookrightarrow \therefore$, $i = n-2j-3$, $j = \frac{n-3-i}{2}$

Substituting:

$$= \sum_{j=0}\binom{7}{n-2j-3}\binom{m+j-1}{j}3^{n-2j-3}\,4^j$$

Since $j = \frac{n-3-i}{2}$, $j \leq \frac{n-3}{2}$ (as $i \geq 0$). $\therefore$, $j \leq \left\lfloor \frac{n-3}{2} \right\rfloor$

$$= \sum_{j=0}^{\left\lfloor \frac{n-2}{3} \right\rfloor}\binom{7}{n-2j-3}\binom{m+j-1}{j}3^{n-2j-3}\,4^j.$$

Generating Series: Counting

example: how many length-64 binary strings have neither 000 nor 00111 as a substring?

- don't know how to do this for specifically length-64. So, let's generalize for length-n and make a generating series.

- Roadmap:
  1) Let S be a set we wish to count by weight
  2) Decompose S into "simpler" sets using disjoint union and cartesian product
  3) Determine the generating series for the simpler sets
  4) Combine the generating series $\Phi_{S_i}(x)$ to get $\Phi_S(x)$.
  5) The number of elements of weight n in S is $[x^n]\Phi_S(x)$.

- Sum Lemma: let $S = A \cup B$ be a disjoint union. Let ω be a weight function $S \to \mathbb{N}$.

  Then, $\Phi_S^\omega(x) = \Phi_A^\omega(x) + \Phi_B^\omega(x)$.

  ↳ Proof:

$$\Phi_S^\omega(x) = \sum_{\sigma \in S} x^{\omega(\sigma)} = \sum_{\sigma \in A \cup B} x^{\omega(\sigma)} = \sum_{\sigma \in A} x^{\omega(\sigma)} + \sum_{\sigma \in B} x^{\omega(\sigma)} = \Phi_A^\omega(x) + \Phi_B^\omega(x).$$

- Product Lemma: let $S = A \times B = \{(a,b) : a \in A, b \in B\}$. Let $u : A \to \mathbb{N}$ be a weight function on A, and let $v : B \to \mathbb{N}$ be a weight function on B. Define the weight function $\omega : S \to \mathbb{N}$ as follows:
  - $\forall \sigma = (a,b) \in S$, $\omega(\sigma) = u(a) + v(b)$.

  Then, $\Phi_S^\omega(x) = \Phi_A^\omega(x) \cdot \Phi_B^\omega(x)$.

  ↳ Proof:

$$\Phi_A^\omega(x) \cdot \Phi_B^\omega(x) = \left(\sum_{a \in A} x^{u(a)}\right)\left(\sum_{b \in B} x^{v(b)}\right) = \sum_{a \in A, b \in B} x^{u(a)+v(b)} = \sum_{\sigma \in S} x^{\omega(\sigma)} = \Phi_S^\omega(x).$$

eg: find the generating series of binary strings of length n where the weight of the string is the number of 1s in it.

- Method 1:

let S= all binary strings of length n.

Then, $\Phi_S^w(x) = \sum_{k\geq 0} a_k x^k$, where a_k = number of elements of weight k = number of binary strings of length n with k 1s $= \binom{n}{k}$.

$$\therefore,\ \Phi_S^w(x) = \sum_{k\geq 0} \binom{n}{k} x^k = (1+x)^n$$

- Method 2:

let $A = \{0, 1\}$. Define $u(0) = 0$ and $u(1) = 1$.

$$\Phi_A^u(x) = x^{u(0)} + x^{u(1)} = x^0 + x^1 = 1 + x.$$

now, we use A to describe the more complicated set S:

$S = A \times A \times A \times \dots \times A$ (n times).

define $w: S \rightarrow \mathbb{N}$ as follows:

$\sigma = (a_1, a_2, \dots, a_n)$, then $w(\sigma) = \#$ 1s in σ $= u(a_1) + u(a_2) + \dots + u(a_n) = \sum_{i=0}^{n} u(a_i)$.

→ see that the weight of the complicated set S is the sum of the weight of the less complicated set A.

$\therefore$, by the product lemma, $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_A^u(x) \cdot \dots \cdot \Phi_A^u(x)$ (n times).

$$= \left[\Phi_A^u(x)\right]^n = (1+x)^n.$$

note: order matters!!

Compositions: A composition of $n \in \mathbb{N}$ is a sequence of positive integers $(a_1, a_2, \dots, a_k)$ that add to n. k = # of parts.

- example: the compositions of n = 4 are:

(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2), (1,3), (3,1), (4).

• note: $n=0$ has one composition, ().

How many k-part compositions of n are there?

let S = the set of all compositions with k parts.
Define $w: S \to \mathbb{N}$ as follows:
if $\sigma = (a_1, a_2, \dots, a_k)$, then $w(\sigma) = a_1 + a_2 + a_3 + \dots + a_k$.
Goal: find $\Phi_S^w(x)$, and the answer will be $[x^n]\,\Phi_S^w(x)$.

let $P = \{1, 2, 3, 4, \dots\}$.
then, $S = P \times P \times P \times \dots \times P$ (k times) $= P^k$ cartesian product of k copies of P.

define $u: P \to \mathbb{N}$ by $u(a) = a$ $\forall a \in P$.
Then, $w(\sigma) = a_1 + a_2 + \dots + a_k = u(a_1) + u(a_2) + \dots + u(a_k)$.

by the product lemma, $\Phi_S^w(x) = \Phi_P^u(x) \times \dots \times \Phi_P^u(x)$ (k times)
$= [\Phi_P^u(x)]^k$.

Now, $\Phi_P^u(x) = \sum_{a \in P} x^{u(a)} = x + x^2 + x^3 + x^4 + \dots = x(1 + x + x^2 + \dots) = x(1-x)^{-1}$.

So, $\Phi_S^w(x) = [\Phi_P^u(x)]^k = x^k (1-x)^{-k}$

∴, the number of k-part coefficients of n is $[x^n]\, x^k (1-x)^{-k}$
$= [x^{n-k}](1-x)^{-k} = \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$.

String Lemma: let S be a set with weight function w.
Suppose that S has no elements of weight 0. Then,

$$\Phi_{S^*}^{w^*}(x) = \frac{1}{1-\Phi_S^w(x)}.$$

Proof: let S be a set and $w: S \to \mathbb{N}$ it's weight function.
let S^k denote $S \times S \times S \times \ldots \times S$ (k times) $(S^0 = \{\})$.
Define $S^* = S^1 \cup S^2 \cup S^3 \cup \ldots = \bigcup_{k\geq 0} S^k$ is a disjoint union.
Define $w_k: S^k \to \mathbb{N}$: if $(a_1, a_2, \ldots, a_k) \in S^k$, then $w_k = w(a_1) + \ldots + w(a_k)$.
Define $w^*: S^* \to \mathbb{N}$: $w^*(\sigma) = w_k(\sigma)$ where $\sigma \in S^k$.
→ we assume that S has no elements of weight 0!!

Then, $\Phi_{S^*}^{w^*}(x) = \sum_{k\geq 0} \Phi_{S^k}^{w_k}(x)$ by the sum lemma.

Since $\Phi_{S^k}^{w_k}(x) = \prod_{i=1}^{k} \Phi_S^w(x) = [\Phi_S^w(x)]^k$.

We have $\Phi_{S^*}^{w^*}(x) = \sum_{k\geq 0} [\Phi_S^w(x)]^k = \frac{1}{1-\Phi_S^w(x)}$.

How many compositions of n are there where each part is ≥ 2?

let S = all compositions where each part is ≥ 2.
let $P = \{2, 3, 4, \ldots\}$
Then, $S = P^0 \cup P \cup P^2 \cup P^3$. So, $S = \bigcup_{k\geq 0} P^k$, $\therefore$ $S = P^*$.

By the string lemma, $\Phi_S(x) = \frac{1}{1-\Phi_P(x)}$.

Now, $\Phi_P(x) = x^2 + x^3 + x^4 + \ldots$
$= x^2(1 + x + x^2 + \ldots)$
$= x^2(1-x)^{-1}$

$\therefore$, $\Phi_S(x) = \frac{1}{1-\frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}$. So, $[x^n]\frac{1-x}{1-x-x^2}$

A formal power series $A(x)$ is rational if there exist polynomials $P(x), Q(x)$ such that $A(x) = P(x)/Q(x)$.

- we extract coefficients from ration power series using recurrences and partial fractions.

How many compositions have an odd number of parts, where each part $\equiv 1 \pmod 3$.

Let S = the set of all compositions with an odd number of parts, with each part $\equiv 1 \pmod 3$.

Let $P = \{1, 4, 7, 10, 13, \ldots\}$ (ie, $P = \{x : x \equiv 1 \pmod 3\}$).

Then, $S = P^1 \cup P^3 \cup P^5 \cup P^7 \cup \ldots$

$= P \times (P^0 \cup P^2 \cup P^4 \cup P^6 \cup \ldots)$ since $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$= P \times (P^2)^*$

$\therefore$, $\Phi_S(x) = \Phi_P(x) \cdot \Phi_{(P^2)^*}(x)$ by the product lemma

$= \Phi_P(x) \cdot \dfrac{1}{1 - \Phi_{P^2}(x)}$ by the string lemma

$\therefore$, $\Phi_P(x) = x + x^4 + x^7 + x^{10} + \ldots = \dfrac{x}{1 - x^3}$

$\Phi_{P^2}(x) = \Phi_P(x) \cdot \Phi_P(x)$ by the product lemma

$= \dfrac{x^2}{(1-x^3)^2}$

So, $\Phi_S(x) = \dfrac{x}{1-x^3} \cdot \dfrac{1}{1 - \frac{x^2}{(1-x^3)^2}} = \dfrac{x}{1-x^3} \cdot \dfrac{(1-x^3)^2}{(1-x^3)^2 - x^2} = \dfrac{x - x^4}{1 - x^2 - 2x^3 + x^6}$.

$\therefore$, $[x^n] \dfrac{x - x^4}{1 - x^2 - 2x^3 + x^6}$

Binary Strings

A binary string σ is a sequence of bits $b_1 b_2 \dots b_n$ where $b_i = 0$ or 1 for $1 \le i \le n$, and n is the length of σ.

- There's only one binary string of length 0: the empty string ε.
- We will (almost) always use the length of a binary string as it's weight!

If $a = a_1 a_2 \dots a_m$, $b = b_1 b_2 \dots b_n$ are binary strings, their concatenation is $ab = a_1 a_2 \dots a_m b_1 b_2 \dots b_n$.

↳ note the length of ab is $m+n$.

↳ also, $\varepsilon\sigma = \sigma$ $\forall$ strings σ.

a is a substring of b if $b = cad$ for some strings c and d.

00111010000001 → blocks

A block of a binary string is a non-empty maximal substring of all 0s or all 1s.

Regular Expressions: a method for generating/producing a set of binary strings.

Let A, B be binary strings.

1) $A \cup B$ is the union of A and B

↳ eg: $A = \{00\}$, $B = \{1, 11, 111\}$. $A \cup B = \{00, 1, 11, 111\}$ or $00 \cup (1 \cup 11 \cup 111)$.

2) $AB = \{ab : a \in A,\ b \in B\}$ is the concatenation of A and B

↳ eg: $A = \{1, 11\}$, $B = \{001\}$. $AB = \{1001, 11001\}$, or $(1 \cup 11)001$.

3) $A^* = \bigcup_{k \ge 0} A^k$, where $A^k = \overbrace{AAA\dots A}^{k \text{ times}}$ and $A^0 = \{\varepsilon\}$

↳ eg: 0^* → all 0-strings of any length (including 0).

Examples: $(00)^* = \{\varepsilon, 00, 0000, 000000, \ldots\}$.

$0(00)^*$ $(= (00)^*0)$ = all odd-length strings of 0s

$(0 \cup 1)^*$ = all binary strings!

$(0 \cup 111)^*$ = all binary strings where each block of 1s has length a multiple of 3.

- Ideally, we'd like to use the Sum, Product, and String lemmas to find the generating series for a set of strings produced by a regular expression. But you have to be careful!

eg: let $A = \{1, 0\}$, $B = \{1, 01\}$

Then $AB = \{11, 101, 01, 001\}$ same(ish)

$A \times B = \{(1,1), (1,01), (0,1), (0,01)\}$.

eg: let $A = \{1, 10\}$, $B = \{1, 01\}$

→ sets can't have duplicates!

Then $AB = \{11, 101, \cancel{101}, 1001\}$ not same!

$A \times B = \{(1,1), (1,01), (10,1), (10,01)\}$.

So, $|AB| \neq |A \times B|$!! Issue when we're trying to count. Removing commas from $A \times B$ might cause ambiguity.

A regular expression is unambiguous if every string generated by the expression can be generated in exactly one way.

↳ eg: $(1 \cup 0)(1 \cup 01)$ is unambiguous.

$(1 \cup 10)(1 \cup 01)$ is ambiguous, since 101 can be generated in two ways: $\underline{1}\,\underline{01}$ and $\underline{10}\,\underline{1}$.

Example: Is $0^* \cup 1^*$ ambiguous or not?

↳ ambiguous, because the empty string ε can be generated in two ways. (So, $\Phi_{0^* \cup 1^*} \neq \Phi_{0^*} + \Phi_{1^*}$).

Example: $S = (0^*11 \cup 001^*)^*$. Is this an ambiguous expression?
↳ ambiguous, since 0011 can be generated in two ways.

Let A, B be sets of strings.
1) Then $A \cup B$ is unambiguous if the intersection is disjoint (ie $A \cap B = \emptyset$).
2) Then AB is unambiguous if $\forall \sigma \in AB$, there's exactly one pair of strings $a, b \in A \times B$ such that $\sigma = ab$.
3) Then A^* is unambiguous if A^k is unambiguous $\forall k$, and $A^0, A^1, A^2, \ldots$ are disjoint.

- If $A \cup B$ is unambiguous, then $\Phi_{A\cup B}(x) = \Phi_A(x) + \Phi_B(x)$. SUM LEMMA
- If AB is unambiguous, then $\Phi_{AB}(x) = \Phi_A(x) \cdot \Phi_B(x)$. PRODUCT LEMMA
- If A^* is unambiguous, then $\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)}$ STRING LEMMA

Counting Binary Strings

eg: find the generating series for all binary strings where weight = length.

1) $\Phi_S(x) = \sum_{n\geq 0} a_n x^n$, where $a_n = \#$ strings of length $n = 2^n$.

$$= \sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}.$$

2) $(0 \cup 1)^*$ is an unambiguous expression for S.
So, $\Phi_S(x) = \Phi_{(0\cup 1)^*}(x) = \frac{1}{1-\Phi_{(0\cup 1)}(x)}$ by string lemma

$$= \frac{1}{1-(x^1+x^1)} = \frac{1}{1-2x}$$

3) Claim: $1^*(00^*11^*)^*0^*$ is an unambiguous expression for S.

eg: $\underbrace{}_{1^*}\underbrace{000011}_{00^*11^*}\underbrace{011}_{00^*11^*}\underbrace{001}_{00^*11^*}\underbrace{011}_{00^*11^*}\underbrace{00}_{0^*}$

justification: the decomposition of a string into it's blocks is unique.

So, $\Phi_S(x) = \Phi_{1^*(00^*11^*)0^*}(x)$

- $\Phi_{1^*}(x) = \dfrac{1}{1-\Phi_1(x)} = \dfrac{1}{1-x} = \Phi_{0^*}(x)$.

- $\Phi_{00^*11^*}(x) = \Phi_0(x) \cdot \Phi_{0^*}(x) \cdot \Phi_1(x) \cdot \Phi_{1^*}(x)$

$$= x \cdot \frac{1}{1-x} \cdot x \cdot \frac{1}{1-x} = \frac{x^2}{(1-x)^2}$$

So, $\Phi_{(00^*11^*)^*}(x) = \dfrac{1}{1-\Phi_{00^*11^*}(x)} = \dfrac{1}{1-\frac{x^2}{(1-x)^2}} = \dfrac{(1-x)^2}{(1-x)^2-x^2}$

$\therefore$, $\Phi_{1^*(00^*11^*)^*0^*}(x) = \Phi_{1^*}(x) \cdot \Phi_{(00^*11^*)^*}(x) \cdot \Phi_{0^*}(x)$

$$= \frac{1}{1-x} \cdot \frac{(1-x)^2}{(1-x)^2-x^2} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2-x^2} = \frac{1}{1-2x}.$$

eg: find the generating series for all strings where every block of 0s has even length and must be followed by a block of at least 3 1s.

let S = set of all binary strings where every block of 0s has even length and is followed by a block of at least 3 1s.

idea: begin with $1^*(00^*11^*)^*0^*$. If we replace any portition of this block decomposition with an unambiguous expression for a non-empty subset of that portion, then the new expression is also unambiguous.

$$\underset{1^*}{\underbrace{1^*}}(\underset{00(00)^*}{\underbrace{00^*}}\,\underset{1111^*}{\underbrace{11^*}})^*\,\underset{\varepsilon}{\underbrace{0^*}}$$

$\therefore$, $1^*(00(00)^*1111^*)^*$ is an unambiguous expression for S.

$$\Phi_S(x) = \frac{1}{1-x} \cdot \frac{1}{1-\left(x^2 \cdot \frac{1}{1-x^2} \cdot x^3 \cdot \frac{1}{1-x}\right)} = \frac{1}{(1-x)\left(1 - \frac{x^5}{(1-x^2)(1-x)}\right)}$$

$$= \frac{(1-x)(1-x^2)}{(1-x)((1-x^2)(1-x)-x^5)} = \frac{1-x^2}{1-x-x^2+x^3-x^5}$$

Let A, B be sets such that $B \subseteq A$. Then, $\Phi_{A \setminus B}(x) = \Phi_A(x) - \Phi_B(x)$, since $|A \setminus B| = |A| - |B|$.

Eg: how many length-n binary strings don't have 111 as a substring?

$(\varepsilon \cup 1 \cup 11)(00^*(1 \cup 11 \cup \varepsilon))^* 0^*$

Eg: how many length-n binary strings don't have 110100 as a substring?
↳ we must use recursive decomposition here!!

Recursive Decompositions

- express a set in terms of itself

Eg: find a recursive decomposition for A = all binary strings.
every string σ either begins with a 0 or a 1 followed by a binary string, except if $\sigma = \varepsilon$.
So, $A = \underbrace{\varepsilon}_{\text{base case}} \underbrace{(0 \cup 1) A}_{\text{recursive step}}$ is an unambiguous expression for A.

$\therefore$, $\Phi_A(x) = 1 + (2x \cdot \Phi_A(x))$. Solving, we get $\Phi_A(x) = \frac{1}{1-2x}$.

Eg: find a recursive decomposition which don't have 111 as a substring.

let A be the set of all strings which don't have 111 as a substring.
let $\sigma \in A$. Either σ has a 0 or it doesn't.

- If it doesn't, then $\sigma \in \{\varepsilon, 1, 11\}$.
- If it does, then either σ begins with a 0, 10, or 110, followed by a string in A.

So, $A = \underbrace{(\varepsilon \cup 1 \cup 11)}_{\text{base case}} \cup \underbrace{(0 \cup 10 \cup 110)\, A}_{\text{recursive step}}$ is unambiguous.

$\Phi_A(x) = (1+x+x^2) + (x+x^2+x^3)\,\Phi_A(x)$, so $\Phi_A(x) = \dfrac{1+x+x^2}{1-x-x^2-x^3}$.

Eg: how many length-4 binary strings have neither 000 nor 00111 as a substring?

i) find a block decomposition for the set of all such strings.
ii) find $\Phi_S(x) = \dfrac{1+x+x^2}{1-x-x^2-x^3+x^5}$
iii) answer is $[x^4]\,\Phi_S(x)$.

we'll see that the number of such strings of length n is given by the recurrence relation $a_n - a_{n-1} - a_{n-2} - a_{n-3} + a_{n-5} = 0 \ \forall n \geq 5$, which comes from the denominator of $\Phi_S(x)$.
So, $a_n = a_{n-1} + a_{n-2} + a_{n-3} - a_{n-5}$, but we need initial conditions s.t. $n \leq 4$.
By inspection, $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 7$, and $a_4 = 13$.
Then, $a_5 = 13+7+4-1 = 23$, $a_6 = 41$, $a_7 = 73$, $a_{64} = 13{,}076{,}512{,}262{,}747{,}676$.

Rational Power Series

(I) Rational series $\rightarrow$ recurrence relation:

eg: consider $A(x) = \dfrac{P(x)}{Q(x)} = \dfrac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \sum_{n\geq 0} a_n x^n$. Find a recurrence relation for a_n.

↳ Strategy: multiply both sides by $Q(x)$ and equate coefficients of x^n on both sides:

$$(1-8x+21x^2-18x^3)\sum_{n\geq 0} a_n x^n = 1-2x+3x^2$$

↳ $$\sum_{n\geq 0} a_n x^n - 8\sum_{n\geq 0} a_n x^{n+1} + 21\sum_{n\geq 0} a_n x^{n+2} - 18\sum_{n\geq 0} a_n x^{n+3} = 1-2x+3x^2$$

↳ $$\sum_{n\geq 0} a_n x^n - 8\sum_{n\geq 1} a_{n-1} x^n + 21\sum_{n\geq 2} a_{n-2} x^n - 18\sum_{n\geq 3} a_{n-3} x^n = 1-2x+3x^2$$

extracting $[x^n]$ of both sides:

$n=0$: $a_0 = 1$ $\Rightarrow$ $a_0 = 1$

$n=1$: $a_1 - 8a_0 = -2$ $\Rightarrow$ $a_1 = 6$

$n=2$: $a_2 - 8a_1 + 21a_0 = 3$ $\Rightarrow$ $a_2 = 30$

$n\geq 3$: $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$.

$\therefore$, for $n\geq 3$, we have that $a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n+3}$.

↳ NOTE!! : $Q(x)$ was $1-8x+21x^2-18x^3$, so these coefficients will always match!

Let $a_0, a_1, a_2, \ldots$ be a sequence of complex numbers.
Let $c_1, c_2, \ldots, c_d$ be constants.
Then, a_n satisfies a (linear homogeneous) recurrence relation if

$a_n + c_1 a_{n+1} + c_2 a_{n+2} + \ldots + c_d a_{n+d} = 0 \quad \forall n \geq d$.

↳ note: $a_0, a_1, \ldots, a_{d-1}$ are the initial conditions

Theorem: let $A(x) = P(x)/Q(x)$ be a rational power series with $Q(x) = 1 + c_1 x + c_2 x^2 + \dots + c_d x^d$ AND degree $(P) <$ degree (Q)

Then the a_n satisfy a recurrence relation:

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d} \quad \forall n \geq d$$

with initial conditions $a_0, a_1, a_2, \dots, a_{d-1}$.

(II) reccurrence relation → rational series:

eg: consider $a_n - 7a_{n-1} - 16a_{n-2} + 12a_{n-3} = 0$, with $a_0 = 1$, $a_1 = 0$, and $a_2 = 2$.
find $P(x)$, $Q(x)$ such that $A(x) = \sum_{n\geq 0} a_n x^n = \frac{P(x)}{Q(x)}$.

Strategy: multiply both sides of the reccurrence relation by $A(x)$ and sum for $n \geq 3$:

$$\hookrightarrow \sum_{n\geq 3} a_n x^n - 7 \sum_{n\geq 3} a_{n-1} x^n - 16 \sum_{n\geq 3} a_{n-2} x^n + 12 \sum_{n\geq 3} a_{n-3} x^n = 0$$

$$\sum_{n\geq 3} a_n x^n - 7x \sum_{n\geq 3} a_{n-1} x^{n-1} - 16x^2 \sum_{n\geq 3} a_{n-2} x^{n-2} + 12x^3 \sum_{n\geq 3} a_{n-3} x^{n-3} = 0$$

$$\sum_{n\geq 3} a_n x^n - 7x \sum_{n\geq 2} a_n x^n - 16x^2 \sum_{n\geq 1} a_n x^n + 12x^3 \sum_{n\geq 0} a_n x^n = 0$$

$$\therefore (A(x) - a_0 - a_1 x - a_2 x^2) - 7x(A(x) - a_0 - a_1 x) - 16x^2(A(x) - a_0) + 12x^3 A(x) = 0$$

$$\hookrightarrow A(x)[1 - 7x - 16x^2 + 12x^3] = a_0 + a_1 x + a_2 x^2 - 7x(a_0 + a_1 x) - 16x^2 a_0$$

since $a_0 = 1$, $a_1 = 0$, and $a_2 = 2$, we get:

$$A(x) = [1 - 7x - 16x^2 + 12x^3] = 1 - 7x - 14x^2,$$ so:

$$A(x) = \frac{1 - 7x - 14x^2}{1 - 7x - 16x^2 + 12x^3}.$$

Theorem: let $a_0, a_1, a_2, \ldots$ be a sequence that satisfies a reccurrence relation $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d} = 0 \quad \forall n \geq d$ with initial conditions $a_0, a_1, \ldots, a_{d-1}$. Let $A(x) = \sum_{n \geq 0} a_n x^n$. Then, $A(x) = \frac{P(x)}{Q(x)}$ where:

$$Q(x) = 1 + c_1 x + c_2 x^2 + \ldots + c_d x^d \text{ and } \deg(P) < d.$$

(III) Partial Fractions (rational series → "explicit formula")

eg: let $A(x) = \frac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \frac{P(x)}{Q(x)} = \sum_{n \geq 0} a_n x^n$. Find an "explicit formula" for a_n.

1) Factor $Q(x)$ into a product of linear factors:

$$Q(x) = (1-\lambda_1 x)^{d_1} (1-\lambda_2 x)^{d_2} \ldots (1-\lambda_m x)^{d_m}$$

we have $Q(x) = (1-2x)(1-3x)^2$.

2) Partial Fractions : $A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-3x} + \frac{C_3}{(1-3x)^2}$

where C_1, C_2, C_3 are unique constants.

3) Find the constants (C_1, C_2, C_3):

multiply of sides of the partial series expression by $Q(x) = (1-2x)(1-3x)^2$

$\hookrightarrow C_1(1-3x)^2 + C_2(1-2x)(1-3x) + C_3(1-2x) = 1-2x+3x^2$

$C_1(1-6x+9x^2) + C_2(1-5x+6x^2) + C_3(1-2x) = 1-2x+3x^2$

equating coefficients of x^0, x^1, x^2 on both sides:

$x^0 \to C_1 + C_2 + C_3 = 1$

$x^1 \to -6C_1 - 5C_2 - 2C_3 = -2$

$x^2 \to 9C_1 + 6C_2 = 3$

solving the systems of linear equations, we get that
$c_1 = 3,\ c_2 = -4,\ c_3 = 2$.

So, $A(x) = \frac{3}{1-2x} - \frac{4}{1-3x} + \frac{2}{(1-3x)^2}$

4) Extract coefficients

geometric series!! negative binomial

$$a_n = [x^n]A(x) = [x^n]\frac{3}{1-2x} - [x^n]\frac{4}{1-3x} + [x^n]\frac{2}{(1-3x)^2}$$

$$= 3\cdot 2^n - 4\cdot 3^n + 2[x^n]\frac{1}{(1-3x)^2}$$

$$= 3\cdot 2^n - 4\cdot 3^n + 2\cdot 3^n [x^n]\frac{1}{(1-x)^2} \qquad = 3\cdot 2^n - 4\cdot 3^n + 2\cdot 3^n\binom{n+2-1}{2-1}$$

$$= 3\cdot 2^n - 4\cdot 3^n + 2\cdot 3^n\binom{n+1}{1} = 3\cdot 2^n - 4\cdot 3^n + 2\cdot 3^n\cdot(n+1)$$

$$= 3\cdot 2^n + (2n-2)3^n$$

$\therefore$, $a_n = 3\cdot 2^n + (2n-2)3^n \ \forall\, n \geq 0$.

Note: such a factorisation always exists if you allow λ_i to be complex, by the fundamental theorem of algebra!
↳ if you cannot factor $Q(x)$, then the method fails.

Notation: $Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2}\cdots(1-\lambda_m x)^{d_m}$, where λ_i is an "inverse root" of $Q(x)$, and d_i is its multiplicity.

eg: suppose $A(x) = \frac{1-3x^2+7x^3}{(1+4x)^2(1-3x)^4}$
↳ $Q(x)$ has two inverse roots, -4 with multiplicity 2, and 3 with

multiplicity 4.

$$A(x) = \frac{C_1}{1+4x} + \frac{C_2}{(1+4x)^2} + \frac{C_3}{1-3x} + \frac{C_4}{(1-3x)^2} + \frac{C_5}{(1-3x)^3} + \frac{C_6}{(1-3x)^4}$$

$$\text{Then } a_n = [x^n]A(x) = \underbrace{(B + Cn)}_{\text{polynomial of } n \text{ in degree} < 2,\ \lambda_1}(-4)^n + \underbrace{(D + En + Fn^2 + Gn^3)}_{\text{polynomial in } n \text{ of degree} < 4,\ \lambda_2} \cdot 3^n$$

Theorem: let $A(x) = \frac{P(x)}{Q(x)}$, where $\deg(P) < \deg(Q)$ and the constant term of $Q(x)$ is 1. Factor $Q(x)$ into a product of linear polynomials $Q(x) = (1-\lambda_1 x)^{d_1} \dots (1-\lambda_m x)^{d_m}$ where λ_i are the inverse roots of $Q(x)$, with multiplicities d_i. Then, there exist unique constants C_{ij}, where $1 \le i \le m$, $1 \le j \le d_i$ such that:

$$A(x) = \left(\frac{C_{1,1}}{1-\lambda_1 x} + \frac{C_{1,2}}{(1-\lambda_1 x)^2} + \frac{C_{1,d_1}}{(1-\lambda_1 x)^{d_1}}\right) + \dots + \left(\frac{C_{m,1}}{1-\lambda_m x} + \frac{C_{m,2}}{(1-\lambda_m x)^2} + \dots + \frac{C_{m,d_m}}{(1-\lambda_m x)^{d_m}}\right).$$

Moreover, $[x^n]A(x) = p_1(n)\lambda_1^n + \dots + p_m(n)\lambda_m^n$ $\forall n \ge 0$, where $p_i(n)$ is a polynomial of degree $< d_i$.

(IV) recurrence → explicit formula

eg: given $a_n - 6a_{n-1} + 12a_{n-2} - 10a_{n-3} + 3a_{n-4} = 0$ $\forall n \ge 4$, with $a_0 = 5$, $a_1 = 11$, $a_2 = 23$, $a_3 = 49$. Find an explicit formula for a_n.

let $A(x) = \sum_{k \ge 0} a_k x^k = P(x)/Q(x)$,

where $Q(x) = 1 - 6x + 12x^2 - 10x^3 + 3x^4$ ← directly from recurrence!

$= (1-3x)(1-x)^3$ ← given!

the inverse roots of $Q(x)$ are $\lambda_1 = 3$ and $\lambda_2 = 1$ with multiplicities $d_1 = 1$ and $d_2 = 3$, respectively.

$\therefore$, an explicit formula for a_n is: $a_n = \left(\substack{\text{polynomial in } n \\ \text{of degree} < d_1 = 1}\right) 3^n + \left(\substack{\text{polynomial in } n \\ \text{of degree} < d_2 = 3}\right) 1^n$

So, $a_n = C \cdot 3^n + (D + En + Fn^2) \quad \forall n \geq 0.$

Now, we use the initial conditions to find the constants C, D, E, and F:

- $n=0$: $a_0 = 5 = C + D$
- $n=1$: $a_1 = 11 = 3C + D + E + F$
- $n=2$: $a_2 = 23 = 9C + D + 2E + 4F$
- $n=3$: $a_3 = 49 = 27C + D + 3E + 9F$

Solving, we get:
$C=1, D=4, E=3, F=1$

$\therefore$, $a_n = 3^n + (4 + 3n + n^2) \quad \forall n \geq 0.$

General Method: given a recurrence relation $a_n + C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_d n_{n-d} \quad \forall n \geq d$ with initial conditions $a_0, a_1, \ldots, a_{d-1}$, we find the explicit formula by:

1) Defining the denominator $Q(x) = 1 + C_1 x + C_2 x^2 + \ldots + C_d x^d$
2) Find the inverse roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ and their multiplicities $d_1, d_2, \ldots, d_m$ of $Q(x)$ by factoring $Q(x)$ into a product of linear factors.
3) Set up the general form $a_n = p_1(n)\lambda_1^n + \ldots + p_m(n)\lambda_m^n$ for all $n \geq 0$, where $p_i(n)$ has degree $< d_i$ with unknown coefficients.
4) Use the initial conditions to determine these coefficients.

(V) explicit formula $\rightarrow$ recurrence

eg: given $a_n = (2n-3)(-1)^n + (2n^2+n-5)4^n \quad \forall n \geq 0$. Find the recurrence for a_n.

let $A(x) = \sum_{k \geq 0} a_k x^k = P(x)/Q(x)$.

Then, $Q(x)$ must be a polynomial with inverse roots $\lambda_1 = -1$ and $\lambda_2 = 4$, with multiplicities $d_1 = 2$ and $d_2 = 3$. $2n-3$ and $2n^2+n-5$

↳ note, we get multiplicities by seeing degree of respective polynomials, plus one!

So, $Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2)^{d_2} = (1+x)^2(1-4x)^3$.

expanding, we get: $Q(x) = 1 - 10x + 25x^2 + 20x^3 - 80x^4 - 64x^5$, so:

$a_n - 10a_{n-1} + 25a_{n-2} + 20a_{n-3} - 80a_{n-4} - 64a_{n-5} = 0 \quad \forall n \geq 5$.

let's find initial conditions by plugging $n \in \{0,1,2,3,4\}$ into the given explicit formula: $a_0 = -8$, $a_1 = -7$, $a_2 = 81$, $a_3 = 1021$, $a_4 = 7941$.

General Method: given an explicit formula of the form $a_n = p_1(n)\lambda_1^n + \dots + p_m(n)\lambda_m^n$ for all $n \geq 0$, where $p_i(n)$ has degree $< d_i$, we get the recurrence relation by:

1) Define the denominator polynomial $Q(x) = (1-\lambda_1 x)^{d_1} \dots (1-\lambda_m x)^{d_m}$
2) Compute $Q(x) = 1 + c_1 x + c_2 x^2 + \dots + c_d x^d$
3) A recurrence relation for the a_n is then:

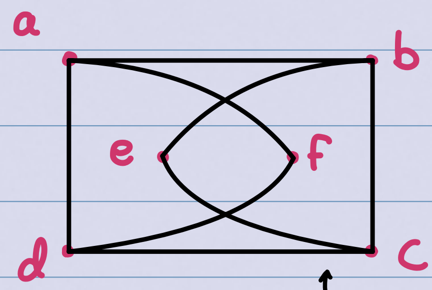
   $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d} = 0 \quad \forall n \geq d$.
4) The initial conditions $a_0, a_1, \dots, a_{d-1}$ can be determined from the explicit formula.

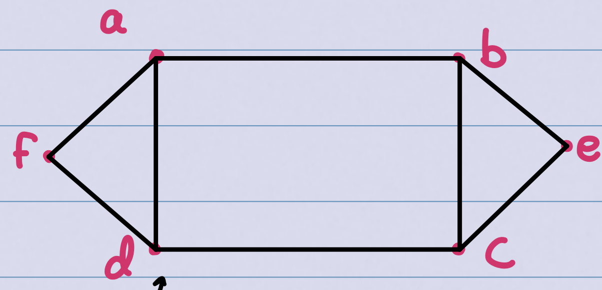
Graph Theory

A graph has a finite set of vertices $V(G)$.

Vertices that are connected are paired in the Edge Set $E(G)$.

Example: $V(G) = \{a, b, c, d, e, f\}$, $E(G) = \{(a,b), (b,c), (c,d), (d,a), (a,f), (d,f), (b,e), (c,e)\}$



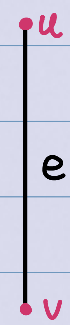


Same graph!

A graph G consists of a pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of objects called vertices, and $E(G)$ is a set of unordered pairs of distinct vertices called edges.

Terminology: for an edge $(u,v) \in E(G)$, we say:

- u is adjacent to v
- u is a neighbor of v
- e is incident with u and v
- u and v are the ends of e,
- e joins u and v



→ we use the shorthand $e = uv$ to represent the edge $\{u, v\}$.

→ note, if $e = uv$, then $e = vu$.

The degree of a vertex v in a graph G is the number of edges incident with v in G, denoted $\deg(v)$.

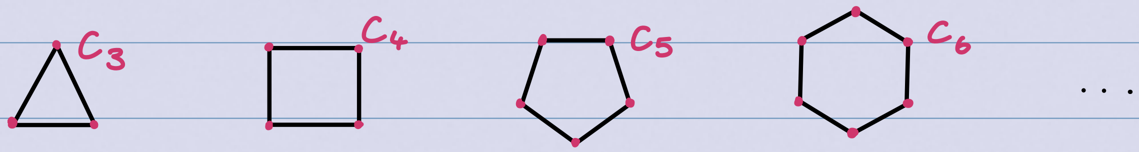
Handshaking Lemma: for any graph G, $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

Paths:

A path of length n:

- $P_n = (\{v_0, v_1, \ldots, v_n\}, \{v_0v_1, v_1v_2, v_2v_3, \ldots, v_{n-1}, v_n\})$
- number of vertices is $n+1$
- number of edges is n

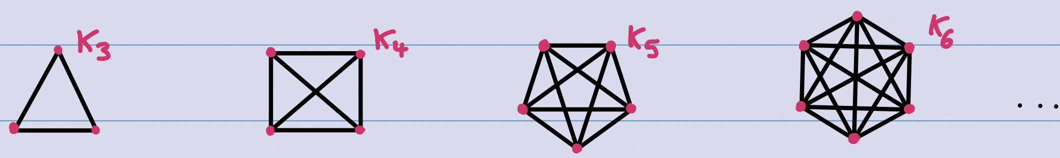
Cycles:



A cycle of length n:

- $C_n = (\{v_1, v_2, \ldots, v_n\}, \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\})$
- number of vertices is n
- number of edges is n

Complete Graphs:



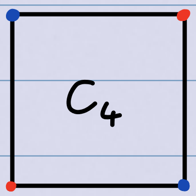
A complete graph on n vertices:

- Every pair of vertices is adjacent
- $K_n = (\{v_1, v_2, \ldots, v_n\}, \{v_iv_j : 1 \leq i \leq j \leq n\})$
- number of vertices is n
- number of edges is $\binom{n}{2} = \frac{n(n-1)}{2}$

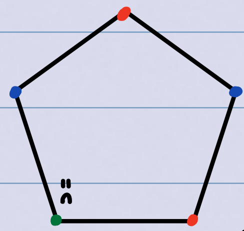
A graph is bipartite if there is a partition (A, B) of it's vertices such that every edge joins a vertex in A with a vertex in B.

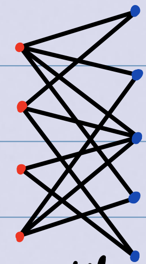
- we can (informally) check if a graph is bipartite by coloring it's vertices. If two adjacent vertices have the same color, then the graph is not bipartite!

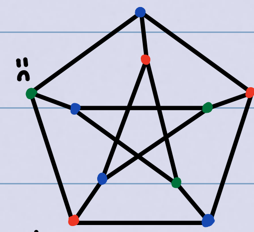
C_n is bipartite if and only if n is even.
Moreover, a graph is bipartite if and only if it does not contain an odd cycle.



bipartite!


not bipartite!


bipartite!

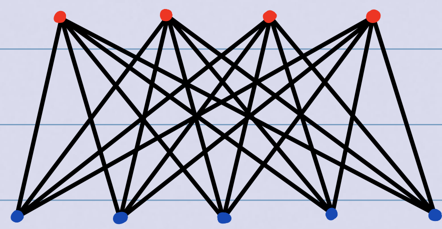

not bipartite!

Complete Bipartite Graph: a bipartite graph with bipartition (A, B), where $|A| = m$, $|B| = n$, and every vertex in A is adjacent to every vertex in B

↳ $K_{m,n} = \left(\{u_1, u_2, \dots, u_m\} \cup \{v_1, v_2, \dots, v_n\},\ \{u_i v_j : 1 \le i \le m \le j \le n\} \right)$.

- number of vertices is $m+n$
- number of edges is mn

$K_{4,5}$:



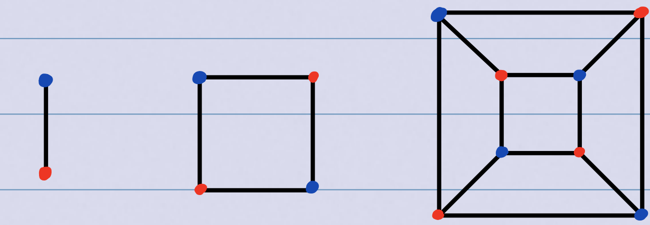
A graph is regular if every vertex has the same degree.
↳ more precisely, a graph is k-regular if every vertex has degree k.

- Paths → P_0 is 0-regular, P_1 is 1-regular, P_n is not regular for $n \ge 2$
- Cycles → cycles are 2-regular
- The complete graph K_n → $(n-1)$-regular
- The complete bipartite graph $K_{m,n}$ → m-regular and n-regular
- Number of edges in a k-regular graph with n vertices → $kn/2$.

n-Cubes (Hypercubes): graphs whose vertices are the length-n binary strings, and two vertices are adjacent if and only if their binary strings differ in exactly one bit.

Properties of the n-cube:

- number of vertices is 2^n
- n-cubes are n-regular
- number of edges is $n2^{n-1}$
- n-cubes are bipartite

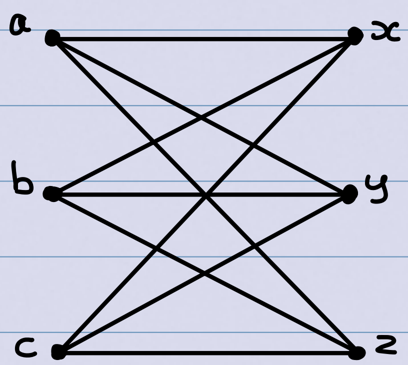


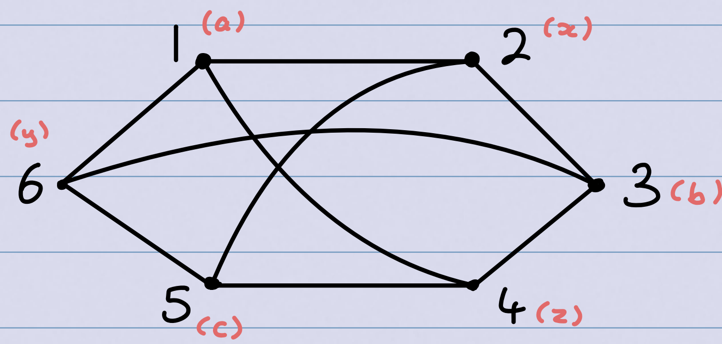
1-cube 2-cube 3-cube

Isomorphic Graphs

Informally: two graphs are isomorphic (ie, essentially the same) if one can be obtained from the other by "relabelling" vertices.

Eg: Are the following graphs isomorphic?



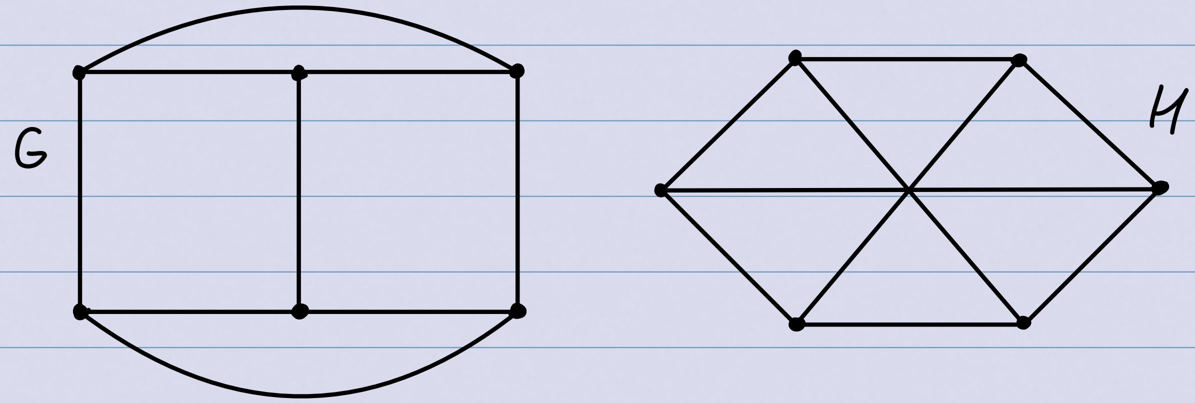


Yes! we rename vertices as follows:

1)	1	2	3	4	5	6
2)	a	x	b	z	c	y

Formally, two graphs G and H are isomorphic if there is a bijection $f: V(G) \to V(H)$ which preserves adjacency, ie, $uv \in E(G)$ if and only if $f(u)f(v) \in H$.

Eg: are the following graphs isomorphic?



No! G has 3 mutually adjacent vertices (a triangle), but H doesn't. So, there doesn't exist a bijection from V(G) to V(H) that preserves adjacency.

Graph Invariants

- A graph invariant is a property of a graph that does not change over time.
  - ↳ examples of possible graph invariants:
    1) number of vertices (isomorphic graphs have the same number of vertices)
    2) number of edges (isomorphic graphs have the same number of edges)
    3) the degree sequence: a list of all vertex degrees in non-decreasing sequence (isomorphic graphs have the same degree sequence)
    4) bipartiteness (isomorphic graphs are either both bipartite or both non-bipartite.
    5) containment of a cycle of a certain length
    6) etc

Determining if two graphs are isomorphic

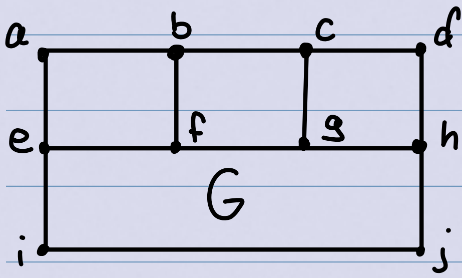
- to show that two graphs G and H are isomorphic, give a bijection from $V(G)$ to $V(H)$ that preserves adjacency.

- to show that two graphs G and H are not isomorphic, find a graph invariant that is different for G and H (eg, # vertices, # edges, degree sequence, etc), or find an adjacency structure that is in G but not in H.

- note: no efficient algorithm is known for determining whether two graphs are isomorphic or not!

Fundemental Concepts in Graph Theory

Walks and Paths



- A walk in G of length 5 b, f, g, f, e, i is called a b, i walk
- a, b, c, b, a is a closed a, a walk
- b is a path (and a closed b, b walk)
  ↳ length 0 walk

Definition: let $u, v \in V(G)$. A u,v-walk in G is a sequence of vertices $v_0, v_1, \dots, v_k$, where $v_0 = u$, $v_k = v$, and $v_i v_{i+1} \in E(G)$
↳ $\forall\ 0 \le i \le k+1$

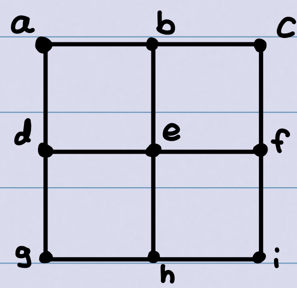
- The length of the walk is k
- The walk is closed if $v_0 = v_k$ (end where we start)
- A u,v-path in G is a u,v-walk with no repeated vertices (and no repeated edges).

Theorem: let $u, v \in V(G)$. If there exists a u,v-walk in G, then there exists a u,v-path in G.
↳ Proof: let $w = v_0, v_1, v_2, \dots, v_k$ be a u-v walk in G (so

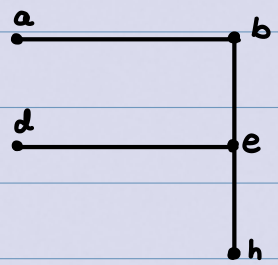
$v_0 = u$ and $v_k = v$) of shortest length. Suppose there exists indices i and j with $i<j$ and $v_i = v_j$. Then, $w' = v_0, v_1, \dots, v_{j+1}, \dots, v_k$ is a u,v-walk in G that is shorter than w. So, this contradicts the choice of w. $\therefore$, the vertices in w are distinct, and so w is a u,v-path in G.

↳ **Corolary:** let $u, v, w \in V(G)$. If there exists a u,v-path and a v,w-path in G, then there exists a u,w-path in G.

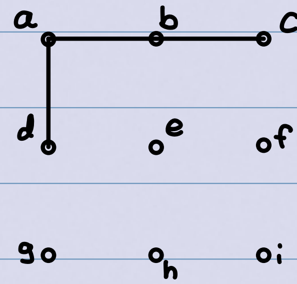
Subgraphs



h o subgraph



subgraph of G



spanning subgraph

A graph H is a subgraph of G if:

1) $V(H) \subseteq V(G)$
2) $E(H) \subseteq E(G)$
3) each edge in $E(H)$ has both ends in $V(H)$

→ if $V(H) = V(G)$ then H is a spanning subgraph of G

note: a u,v-path in G is a subgraph of G.

Cycles: A cycle in a graph G is a closed walk $v_0, v_1, v_2, \dots, v_k$ and where $v_1, v_2, \dots, v_k$ are distinct and $k \geq 2$ and $v_0 = v_k$ (the length of the cycle is k).

Theorem: let G be a graph where each vertex has degree ≥ 2. Then G contains a cycle!

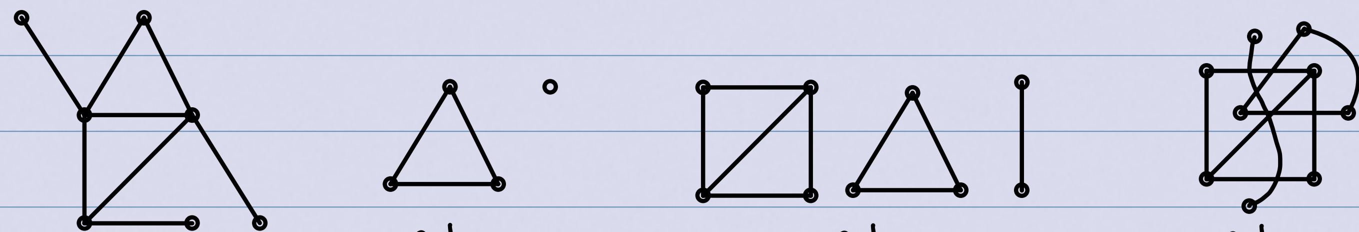
proof: let $P = v_0, v_1, v_2, \dots, v_k$ be the longest path in G. note that $k \geq 2$, since every vertex has degree at least 2, so any vertex v, together with it's two neighbors, gives a path of at least 2.

now, v_0 is adjacent to v_1 since $\deg(v_0) \geq 2$, v_0 is adjacent to at least one other vertex, say $x \neq v_1$. If x is not on path P, then $x, v_0, \dots, v_k$ is a path that is longer than P, contradicting the choice of P. $\therefore$, x is on P, so $x = v_i$ for some $2 \leq i \leq k$. Then $v_0, v_1, \dots, v_i = x, v_0$ is a cycle in G (of length ≥ 3).

Connectedness + Components

A graph G is connected if $\forall x, y \in V(G)$, there exists an x,y-path in G.

eg:



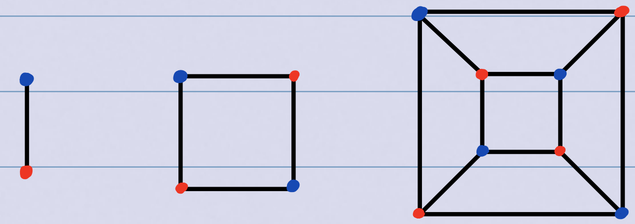
connected — not connected — not connected — not connected

Theorem: let $v \in V(G)$. If there exists a v,w-path in G $\forall w \in V(G)$, then G is connected.

proof: let $x, y \in V(G)$. By assumption, there is a v,x-path in G, so there's also a x,v-path in G. Also by assumption, there's a v,y-path in G. By the previous corollary, there exists an x,y-path in G.

$\therefore$, G is connected.

eg: Show that the n-cube is connected.

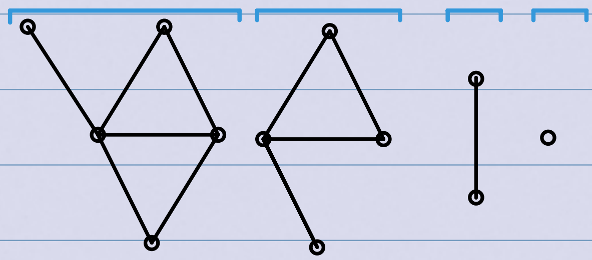


1-cube 2-cube 3-cube

consider $v = 000000$, $w = 010110$. Then, 000000, 010000, 010100, 010110 is a v,w-walk in the 6-cube.

Formally, let $v = \overbrace{000\ldots00}^{n}$ and $w = \{0,1\}^n$. Suppose w has exactly k 1s $(0 \le k \le n)$ with 1s in positions $1 \le i_1 < i_2 < \ldots < i_3 \le n$. For each $1 \le j \le k$, define v_j to be a length-n binary string with 1s in positions $i_1, i_2, \ldots, i_j$ and 0s elsewhere. Then $v, v_1, v_2, \ldots, \overset{= v_k}{w}$ is a v,w-path in the n-cube. $\therefore$ by the previous theorem, the n-cube is connected.

Components



4 components

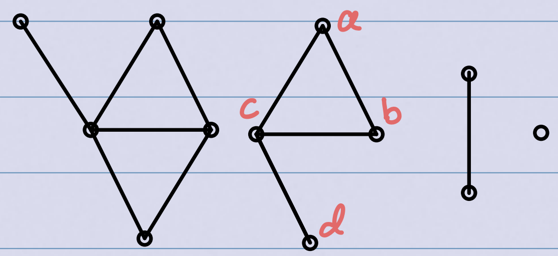
A component of a graph G is a maximal connected subgraph of G.

Observation: a graph is connected if and only if it has only one component!

Cuts

Let $X \subseteq V(G)$. The cut induced by X, is the set of all edges in $E(G)$ with exactly one end in X.

Eg:



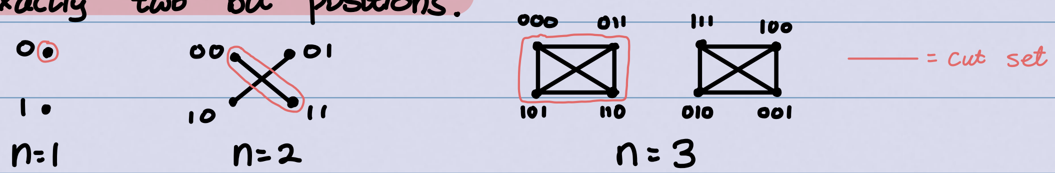
if $X = \{a, b, c, d\}$, then the cut induced by X is empty.

Theorem: a graph is connected if and only if <u>for all</u> non-empty proper subset $X \subseteq V(G)$, the cut induced by X is non-empty.

↳ subset that cannot be equal

Eg: let $n \geq 1$. Define G_n to be a graph whose vertex set is the of length-n binary strings, with two vertices being adjacent iff they differ in exactly two bit positions.

examples:



Claim: G_n is not connected!

let X = vertices in G_n labelled with an even number of 1s. Then, X is <u>non-empty</u> (since the string $000...000 \in X$) and is a <u>proper subset</u> of $V(G_n)$ (since $1000...000 \notin X$). We'll argue that the cut induced by X is empty:

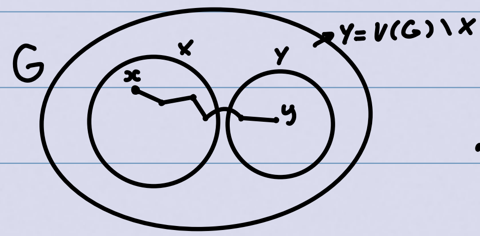
ie, $y \in V(G) \setminus X$

let $x \in X$ and $y \notin X$ (but $y \in V(G)$). Now x has an even number of 1s (by defn. of X). Flipping one bit in X gives a vertex in $V(G) \setminus X$ (not in X). Flipping a second bit of this vertex gives a vertex with an even number of 1s, so it's in X. ∴, there's no edge from a vertex in X to a vertex outside of X.

Therefore, G_n is not connected!

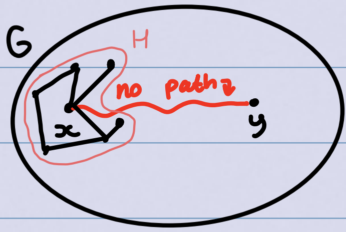
Proof w/ a sketch:

($\Rightarrow$)



- x exists bc X non-empty
- y exists bc X is proper subset

($\Leftarrow$)
contrapositive



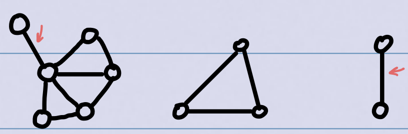
- if the cut induced by X is empty, the graph is not connected.

(Formal) Proof:

($\Rightarrow$) Suppose that G is connected. Let X be any non-empty proper subset of $V(G)$, and let $Y = V(G) \setminus X$. Let $x \in X$ (which exists since X is non-empty) and $y \in Y$ (which exists since X is proper). Since G is connected, there exists an x,y-path in G: $x = x_0, x_1, x_2, \dots, x_k = y$ $(k \geq 1)$.
Let i be the largest index such that $x_i \in X$ and $x_{i+1} \notin X$. Such an index exists since $x_0 \in X$ (so $i \geq 0$) and $x_k \notin X$ (so $i \leq k-1$). Then, $x_i x_{i+1}$ is an edge with one end in X and the other end not in X. $\therefore$, the cut induced by X is non-empty.

contrapositive
($\Leftarrow$) Suppose G is not connected. So, there exists $x, y \in V(G)$ such that there's no x,y-path. Let H be the component of the graph that contains x, and let $X = V(H)$. Then, X is non-empty (since $x \in X$), and is proper (since $y \notin X$). Since H is a maximal connected subgraph of G, there's no edge in the graph with one end in X and the other end not in X. $\therefore$, the cut induced by X is empty!

Bridges



$G - e$ is the graph obtained by deleting the edge e from G.

An edge $e = uv$ is a bridge if $G - e$ has more components than G.

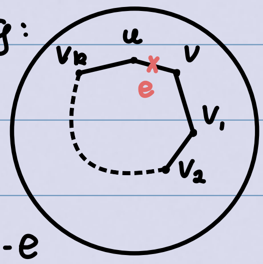
Theorem: let e be a bridge of a connected graph G. Then, $G-e$ must have 2 components.

↳ Equivalently, e is contained in a cycle G if and only if G is not a bridge.

• **Proof:**

($\Rightarrow$) Assume that $e=uv$ is contained in a cycle C of G. Say $C = u, v, v_1, v_2, \ldots, v_k, u$.

eg:



G-e

There is a path ~u & v in $G-e$, namely $v, v_1, \ldots, v_k, v$. Let H be the component of G that contains e. Now, u & v are still components of $G-e$.

So, $G-e$ has the same number of components, and therefore, e is not a bridge of G.

($\Leftarrow$) Assume $e=uv$ is not a bridge of G.

Let H be the component of G that contains e.

Since e is not a bridge of G, $G-e$ has the same number of components as G.

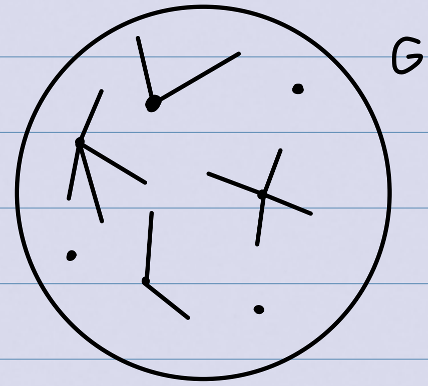
In particular, u and v are in the same component of $G-e$. So, there is a path p ~u & v in $G-e$.

Then, p together with e gives a cycle in G. So, e is contained in a cycle of G.

Recall the theorem that states that if every vertex of G has degree ≥ 2, then G has a cycle.

↳ **Corollary:** suppose G has no vertex of degree one, then G has a cycle, except if G has no edges (ie, $E(G) = \emptyset$ or $V(G) = 0$).

Example: let G be a graph where each vertex has an even degree. Prove that G has no bridges.



If G has no edges, then G has no bridges.
Suppose that G has at least one edge. By the corollary, G has a cycle C.
Remove the edges of C from G to get a graph G_1. Every vertex of G still has vertices of even degree. So, if G_1 is not empty, it must contain a cycle C_1.
Then, we remove the edges of C_1 from G_1 to yield G_2. Again, G_2 must have vertices of even degree, and therefore a cycle C_2.
... continue the process ...
The process will terminate, bc the sequence $|E(G)|$, $|E(G_2)|$, $|E(G_3)|$, ... is a strictly decreasing sequence of non-negative integers, and therefore must eventually reach 0.
This shows that every edge of G is on at least 1 cycle, and therefore, G has no bridges.

Trees

- A tree is a connected graph with no cycles.
- root and leaves ⇒ a leaf is a vertex of degree 1.

Properties of Trees

1) Every edge of a tree is a bridge

↳ Proof: trees have no cycles

2) A tree is a minimally connected graph. ie, if any edge is deleted, the resulting graph is disconnected (since every edge is a bridge).

3) if a tree T has n vertices, it has $m = n-1$ edges.

- Proof: by induction on # vertices n:
  - Base Case: if T has $n=1$ vertex, it has 0 edges. $\therefore$, $m = n-1 = 0$, so BC holds.
  - Induction Hypothesis: let $n \geq 2$. Assume the statement is true $\forall$ trees with fewer than n vertices.
  - Inductive Step: let T be a tree with n vertices. We will show that it has $m = n-1$ edges.

  Let e be any edge in T.

  Since e is a bridge, $T-e$ has exactly 2 components T_1 and T_2. However, T_1 and T_2 have no cycles, so T_1 and T_2 are trees.

  Let $n_1 = |T_1|$ and $n_2 = |T_2|$. Note, $n = n_1 + n_2$, $n_1 \geq 1$, $n_2 \geq 1$.

  So, $n_1 < n$ and $n_2 < n$.

  By the inductive hypothesis, T_1 has $n_1 - 1$ edges and T_2 has $n_2 - 1$ edges.

  $\therefore$, the total number of edges in T is $\overbrace{n_1 - 1}^{|T_1|} + \overbrace{n_2 - 1}^{|T_2|} + \overset{\text{the edge we deleted } e}{1}$

  $= n_1 + n_2 - 1 = n - 1$.

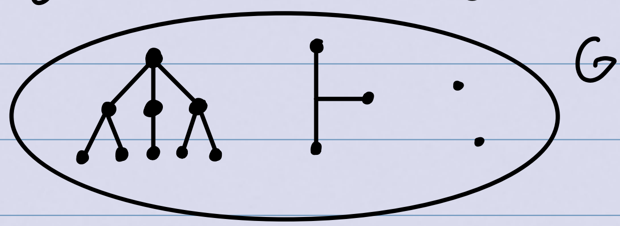
  So, the statement is true for trees with n vertices.

  Therefore, by induction, the statement is true $\forall$ trees.

4) if a tree T be a tree with at least 2 vertices, then T has at least 2 leaves.

- Proof: let P be a longest-path in T, say $v_1, v_2, \ldots, v_k$ where $k \geq 2$. Now, v_1 cannot be adjacent to any vertex in P other than v_2 as otherwise, we'd have a cycle in T. So, v_1 and similarly v_k have degree 1, so they are leaves.

Forests



A forest is a graph with no cycles.

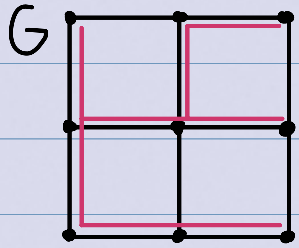
Properties:

1) Each component of a forest is a tree
2) Every edge of a forest is a bridge
3) If a forest has n vertices and c components, then it has $m = n - c$ edges.
   ↳ Proof: each of the c components of the forest has one less edge than vertices, since each component is a tree. So, in total, the number of edges in the forest is c less than the # of vertices.

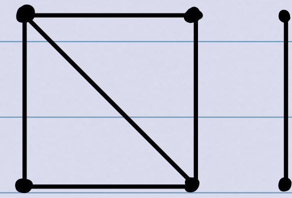
Spanning Trees

A spanning tree of a graph G is a spanning subgraph of G that is also a tree.

↳ Recall: a subgraph H of G is a spanning subgraph if $V(H) = V(G)$.



• = spanning subgraph of G



doesn't have a spanning subgraph!

Theorem: a graph is connected if and only if it has a spanning tree.

Proof:

($\Rightarrow$) let G be a connected graph.

- if G has no cycles, then G is a tree. So, G itself is a spanning tree of G.
- Suppose now that G has at least one cycle C. Let e be any edge in C, then G-e is connected (since e is not a bridge). Then, G-e is a spanning subgraph of G. note that G-e has at least one less cycle than G. Repeat the process, which must eventually terminate with a graph T. Then T is a spanning subgraph of G with no cycles, and is connected. So, T is a spanning subgraph of G.

($\Leftarrow$) Suppose a graph G has a spanning subtree T. Let $x, y \in V(G)$. Since T is a spanning subgraph of G, we have that $x, y \in V(T)$. Since T is connected, there is an x,y-path in T. Since T is a subgraph of G, this path is also a path in G. So, G is connected.

Corollary: let G be a graph with n vertices and $n-1$ edges. Then G is a tree.

Proof: Since G is connected, it has a spanning tree T. Since T is a spanning subgraph of G, T also has n vertices. Since T is a tree, it has $n-1$ edges. These are all the vertices and edges in G, so $G = T$. Therefore, G is a tree.

Summary: Let G be a graph with n vertices. Then, G is a tree if and only if any two of the following conditions are true:

1) G is connected
2) G has no cycles
3) G has $n-1$ edges.

Justification:

- (1) and (2): definition of a tree
- (1) and (3): by previous corollary
- (2) and (3): G is a forest, say with c components and hence $n-c$ edges. Since G has $n-c$ components, c must be 1. So, G is connected and so it's a tree.

A few results on Trees

- Formula for the number of leaves on a tree:

  Let T be a tree with $n \geq 2$ vertices.

  Let n_i be the number of vertices in the tree of degree i, where $1 \leq i \leq n-1$

  Count vertices: $n_1 + n_2 + n_3 + \ldots + n_{n-1} = n$ (1)

  $m = \#\text{edges}!$

  Count edges: $n_1 + 2n_2 + 3n_3 + \ldots + (n-1)n_{n-1} = 2m = 2(n-1)$ (2)

  So, $2 \times (1) - (2)$ gives: $n_1 - n_3 - 2n_4 - \ldots - (n-3)n_{n-1} = 2$.

$\therefore,\ n_1 = 2 + n_3 + 2n_4 + \dots + (n-3)n_{n-1}.$

So, $n_1 = 2 + \sum_{i=3}^{n-1} (i-2)n_i.$

↳ **Observations:**

i) $n_1 \geq 2$ (as $n_i \geq 0$)

ii) n_1 doesn't depend on n_2! $\nearrow (n_d \geq 1)$

iii) if T has a vertex of degree d, then T has at least d leaves

iv) Trees with exactly 2 leaves are paths (of length ≥ 1)

Theorem: let T be a tree, and $x, y \in V(T)$. Then there is a unique x,y-path in T.

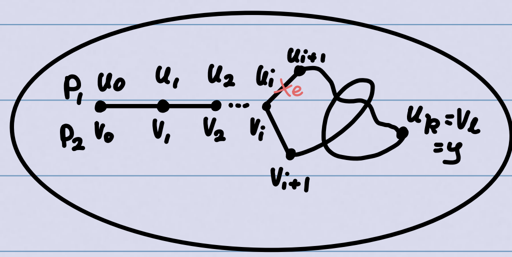
Proof:

- Since T is connected, there exists at least one x,y-path in T!
- But, T cannot have two distinct x,y-paths. If it did, T would have a cycle, which contradicts the definition of a tree.

Lemma: let G be a graph, and let $x, y \in V(G)$. If G has two distinct x,y-paths, then G has a cycle.

Proof:

let the two distinct x,y-paths be:



$P_1 = x = u_0, u_1, u_2, \dots, u_k = y$

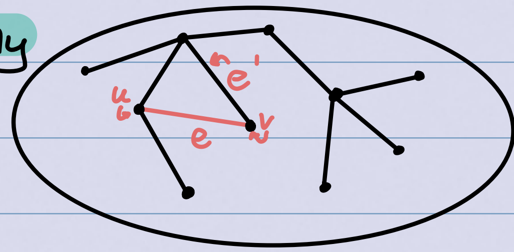
$P_2 = x = v_0, v_1, v_2, \dots, v_l = y.$

let $i \geq 0$ be the smallest index such that $u_i = v_i$ but $u_{i+1} \neq v_{i+1}$.

Consider the graph $G-e$ (where e is the edge connecting u_i with u_{i+1}). Then there is a walk from u_i to u_{i+1} in $G-e$, namely $u_i = v_i, v_{i+1}, \ldots, v_\ell = u_k, u_{k-1}, \ldots, u_{i+2}, u_{i+1}$.

So, $G-e$ also has a u_i, u_{i+1}-path (as there is a u_i, u_{i+1}-walk). So, u_i and u_{i+1} are in the same component of $G-e$. Therefore, e was not a bridge of G, and therefore, e is on a cycle! Therefore, G has a cycle.

Theorem: Let T be a spanning tree of a connected graph G. Let $e \in E(G)$ which is not in T. Then $T+e$ has a unique cycle C. If e' is any edge in C, then $T+e-e'$ is also a spanning tree.



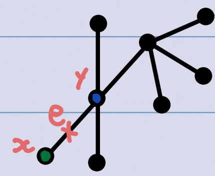
Proof:

Let $e = uv$. Since T has no cycles, any cycle in $T+e$ must use the edge e. If it didn't, we'd already have a cycle in T (contradiction!).
Such a cycle must be composed of a u,v-path in T along with e. Since there's a unique u,v-path in T, there is a unique cycle in $T+e$.

Since e' is on the cycle in $T+e$, e' is not a bridge in $T+e$. So, $T+e-e'$ also has one component. Since $T+e-e'$ has the same number of vertices and edges as T, it is also a spanning tree of G.

Observations:

1) A graph is bipartite if and only if all of it's components are bipartite
2) If H is a subgraph of G, then if G is bipartite, then so is H.
3) Odd cycles are not bipartite



aside proof: imagine lifting an arbitrary vertex out of the plane. it's neighbors can be one color, next-neighbors can be another.

Lemma: All trees are bipartite

Proof (by induction on the # vertices):

- The unique tree with one vertex ($n=1$) is bipartite
- Let $n \geq 2$, and suppose that all trees on fewer than n vertices are bipartite.
- Let T be a tree with n vertices. Let x be a leaf in T, and let $e = xy$ be the edge incident with x. Consider the graph $T' = T - x$. Then T' has no cycles (since T doesn't), and T' has one less edge than vertices (since T does). Therefore, T' is a tree with $n-1$ vertices. By the induction hypothesis, T' has a bipartition (G, B). Without loss of generality, suppose that $y \in B$ (blue). Then, $(G \cup \{x\}, B)$ is a bipartition of T.

$\therefore$ By induction, all trees are bipartite!

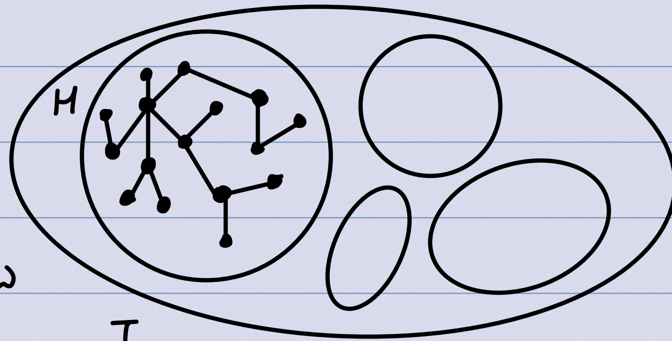
Bipartite Characterisation Theorem: A graph G is bipartite if and only if G has no odd cycles.

Proof:

($\Rightarrow$) contrapositive: Suppose G has an odd cycle. Since odd cycles are not bipartite, then neither is G.

($\Leftarrow$) contrapositive: Suppose G is not bipartite. We'll show that G has an odd cycle.

Let H be a component of G that is not bipartite.



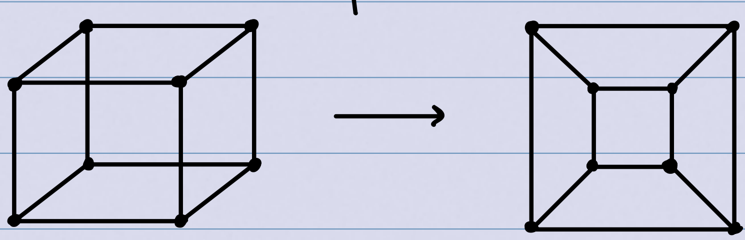
Since H is connected, we know H must have a spanning tree, T.
Since T is a tree, it is bipartite, say with bipartition (A, B). But, H is not bipartite! So, there must exist an edge $e = uv$ such that $u \in B$ and $v \in B$ (or $u \in A$ and $v \in A$). Note that e is not in T.
Since T is a tree, there is a (unique) path in T between u and v: $P = v_0 v_1 v_2 \dots v_k$ where $u_0 = u$, $v_k = v$.
Now, $v_0 \in B$, $v_1 \in A$, $v_2 \in B$, $v_3 \in A$, $\dots$, $v_k \in B$. So, v_i is in B if and only if i is even! So, $v_0 \dots v_k$ is an even length path in T. Adding e (ie, $P' = v_0 v_1 v_2 \dots v_k v_0$) creates a cycle of odd length!
So, H (and therefore G) has an odd cycle $v_0 v_1 v_2 \dots v_k v_0$.

Planar Graphs

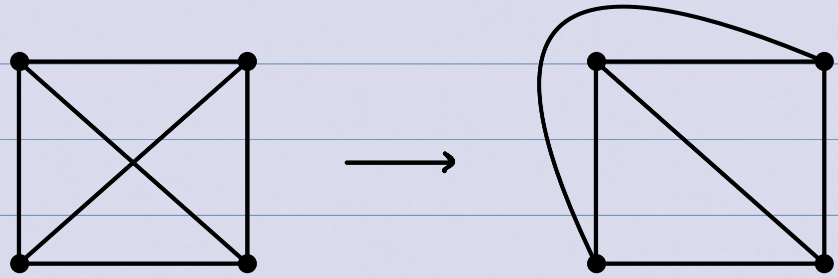
A planar embedding of a graph G is a drawing of G on the plane so that no two edges intersect (except at their ends).

A graph is planar if it has a planar embedding
A graph is nonplanar if it doesn't have a planar embedding

Example: the 3-cube is planar

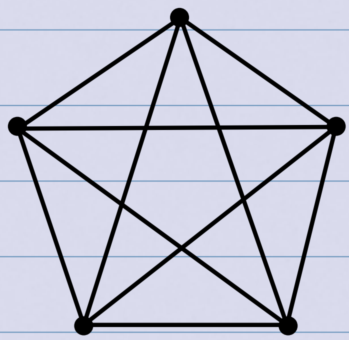


Example: is K_4 planar?



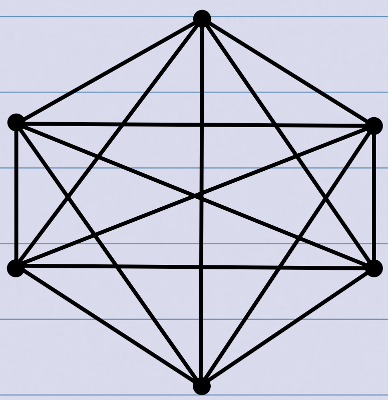
yes! We can draw it without overlapping any edges.

Example: is K_5 planar?



No! There's no way to draw K_5 without overlapping edges (proof later)

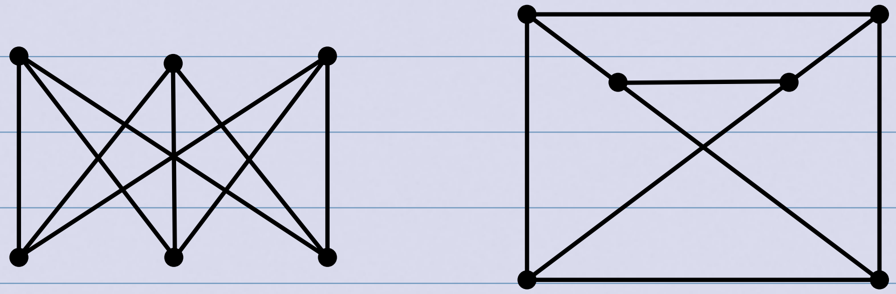
Example: is K_6 planar?



No! K_5 is a subgraph of K_6 and K_5 is nonplanar

Theorem: If H is a subgraph of G and G is planar, then so is H.

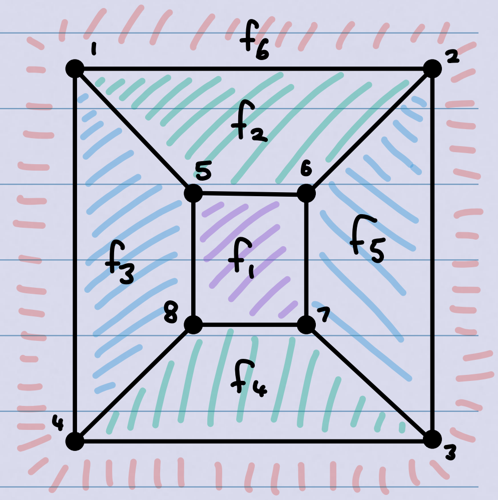
Example: is $K_{3,3}$ planar?



No! Proof later.

A graph is planar if and only if its components are planar.

A planar embedding divides the plane into regions called faces.



Informal definition: a face is the region bounded by a cycle.

One face, the outer face, is unbounded; the other faces are bounded.

Two faces are adjacent if they are incident with a common edge.

The boundary of a face is the set of vertices and edges incident with the face.

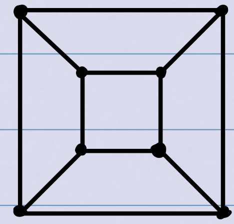
Example: boundary of f_1 is $\{5, 6, 7, 8\}$, $\{56, 67, 78, 85\}$

Planar Embeddings of Forests

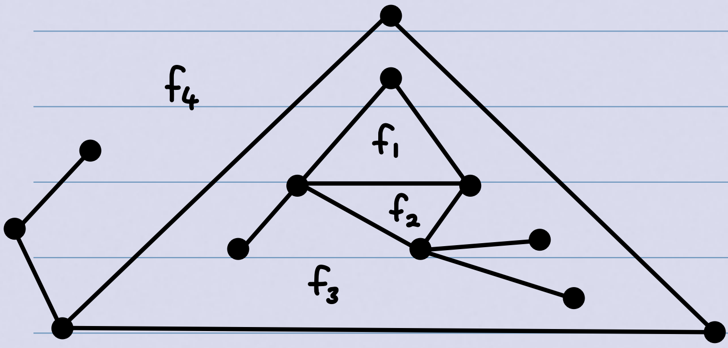
1. All forests (and trees) are planar
2. A planar embedding of a forest has only one face (an outer face), and no cycle in its boundary.

The degree of a face is the number of edges incident with the face, with bridges counted twice!

↳ Example: each face of the 3-cube has a degree of 4.



Example: find the degrees of the faces in this graph:



this planar embedding has $n=12$ vertices, $m=13$ edges, and $s=4$ faces

$\deg(f_1)=3$, $\deg(f_2)=3$, $\deg(f_3)=13$, and $\deg(f_4)=7$.

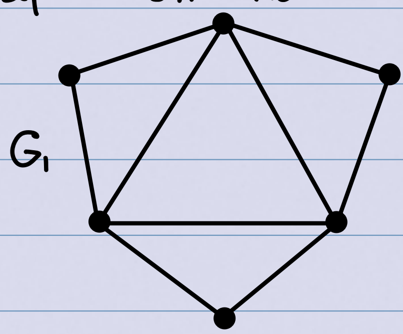
Faceshaking Lemma: for a planar embedding with faces $f_1, f_2, \dots, f_s$ and m edges, $\sum_{i=1}^{s} \deg(f_i) = 2m$.

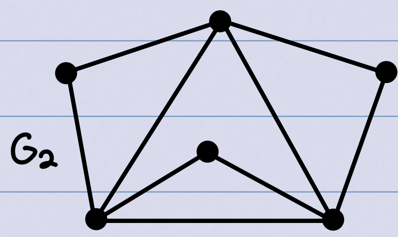
Proof:

each edge that is not a bridge is incident with exactly two faces. Each edge that is a bridge is incident with exactly one face, but contributes 2 to the degree of that face. Thus, each edge contributes 2 to the total degree of the faces.

Planar Embeddings of a Graph

- A planar graph can have different planar embeddings:



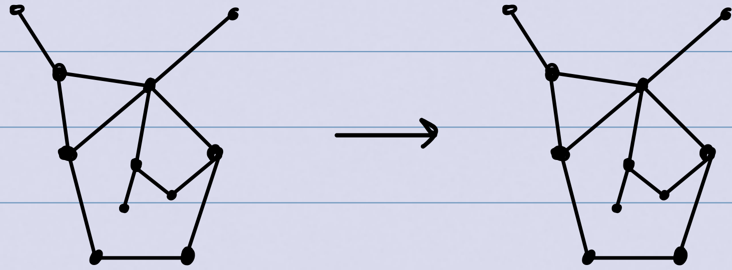


G_1 and G_2 are isomorphic graphs!
However, G_1 has a face of degree 6 (the outer face), whereas G_2 has no face of degree 6.
Therefore, these planar embeddings are different.
Note, however, that the two planar embeddings have the same number of faces.

Theorem (Euler's Formula for Planar Connected Graphs):
Let G be a connected planar graph with n vertices and m edges. Consider a planar embedding of G with s faces. Then, $n-m+s=2$ ($s=2-n+m$).
↳ See that Euler's formula generalises the result that a tree with n vertices has $m=n-1$ edges (since a tree has a planar embedding with $s=1$ face).

Proof:
we fix $n \geq 1$. We'll use induction on $m \geq n-1$.
(Since G is connected, it has a spanning tree, which has $n-1$ edges. So, G has $\geq n-1$ edges).
Base Case: Suppose G has $m=n-1$ edges. Then G is a tree, so any planar embedding of G has $s=1$.
Thus, $n-m+s=n-(n-1)+1=2$, as required. So, Euler's formula holds for the base case of $m=n-1$.
Inductive Hypothesis: let $m \geq n$. Suppose that Euler's formula is true for all connected planar graphs with n vertices and fewer than m edges.
Inductive Step: Let G be a connected planar graph with n vertices and exactly m edges. Consider any planar embedding of G with s faces. (We'll show $n-m+s=2$).

- connected + planar
- n vertices
- m edges
- s faces



- connected + planar
- n vertices
- $m-1$ edges
- $s-1$ faces

Since G is connected, it has a spanning tree T with $n-1$ edges. Since $m > n-1$, G has an edge which is not in T. Then, $T+e$ has a cycle containing e, so e is not a bridge of G.

Consider $G-e$. Then $G-e$ is connected (since e is not a bridge of G), is planar (since G has a planar embedding), n vertices, and $m-1$ edges. Moreover, $G-e$ has $s-1$ faces because the two sides of e are two different faces of G which merge into one when e is removed.

By the induction hypothesis for $G-e$: $n-(m-1)+(s-1)=2$.

So, $n-m+s=2$ as required. Therefore, Euler's formula is true!

Theorem (Generalisation of Euler's Formula):

Let G be a planar graph with n vertices, m edges, and c components. Consider a planar embedding of G with s faces. Then, $n-m+s=c+1$ (so $s=c+1-n+m$).

↳ Corollary: let G be a planar graph. Then every planar embedding of G has the same number of faces, namely, $s=c+1-n+m$.

Proof:

Let the components of G be $H_1, H_2, \ldots, H_c$. Suppose H_i has n_i vertices and m_i edges. Suppose that any planar embedding of H_i has s_i faces. Then Euler's formula for planar and connected graphs gives $n_i - m_i + s_i = 2$.

Summing gives:

$$\sum_{i=1}^{c} n_i - \sum_{i=1}^{c} m_i + \sum_{i=1}^{c} s_i = \sum_{i=1}^{c} 2.$$

$$\therefore\ n - m + \sum_{i=1}^{c} s_i = 2c$$

But, $\sum_{i=1}^{c} s_i = s + (c-1)$. So, $n - m + s + c - 1 = 2c$.

$$\therefore\ n - m + s = c + 1.$$

We know that a graph on n vertices can have at most $\binom{n}{2} = \frac{n(n+1)}{2}$ edges. But, how many edges can a planar graph on n vertices have?

Lemma: let G be a planar embedding with n vertices and m edges. Suppose that each face of the embedding has degree at least d (where $d \geq 3$). Then, $m \leq \frac{d(n-2)}{d-2}$.

Proof:

Let $f_1, f_2, \ldots, f_s$ be the faces of G.

Then, $2m = \sum_{i=1}^{s} \deg(f_i) \geq \sum_{i=1}^{s} d = sd$, so $2m \geq sd$.

By Euler's Formula, $n - m + s = c + 1$, where $c = \#$ components in G.

Since $c \geq 1$, we have $n - m + s \geq 2$, so $s \geq m - n + 2$.

Thus, $2m \geq d(m-n+2)$, so $dm - 2m \leq dn - 2d$.

We conclude that $m \leq \frac{d(n-2)}{d-2}$.

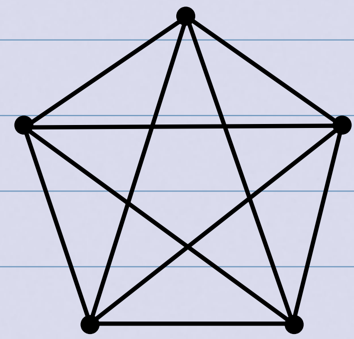
Theorem: let G be a planar graph with $n \geq 3$ vertices and m edges. Then $m \leq 3n-6$.

Proof: Suppose first that G has no cycles, so G has $m \leq n-1$ edges. Then $3n-6 = n-1 + (2n-5) \geq n-1 \geq m$ (since $2n-5 \geq 0$)

Suppose next that G has at least one cycle. The boundary of every face of a planar embedding of G has at least one cycle. Since a cycle has at least 3 edges, the degree of each face is at least 3. Thus, $m \leq \frac{3(n-2)}{3-2} = 3n-6$.

Corollary: K_5 is not planar.

Proof: K_5 has $n=5$ vertices and $m=10$ edges, so $m \nleq 3n-6$. Thus, K_5 is not planar.



Theorem: Let G be a planar bipartite graph with $n \geq 3$ vertices and m edges. Then, $m \leq 2n-4$.

Proof:

Suppose first that G has no cycles, so G has $m \leq n-1$ edges. Then $2n-4 = n+1+(n-3) \geq n-1 \geq m$ (since $n-3 \geq 0$)

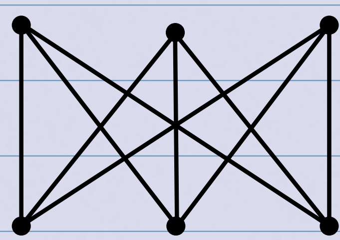
Suppose next that G has at least one cycle. The boundary of every face of a planar embedding of G has at least one cycle. Since G is bipartite, a cycle has at least 4 edges, and hence the degree of each face is at least 4. Thus,

$m \leq \frac{4(n-2)}{4-2} = 2n-4$.

Corollary: $K_{3,3}$ is not planar.

Proof: The bipartite graph $K_{3,3}$ has $n=6$ vertices and $m=9$ edges, so $m \nleq 2n-4$.

Thus, $K_{3,3}$ is not planar.



Summary: Let G be a planar graph with n vertices and m edges.

(I) Suppose a planar embedding of G has all its faces of degree $\geq d$ (where $d \geq 3$). Then, $m \leq \frac{d(n-2)}{d-2}$

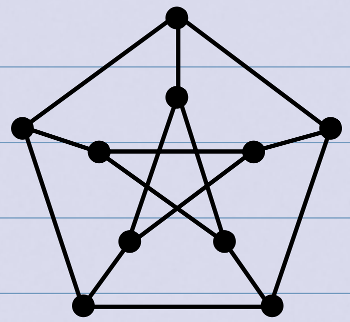
(II) If $n \geq 3$, then $m \leq 3n-6$.

(III) K_5 is not planar (since $m=10$, $n=5 \Rightarrow 10 \leq 9 \rightarrow$ false)

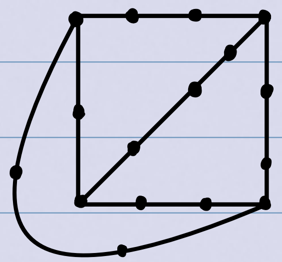
(IV) If $n \geq 3$ and G is bipartite, then $m \leq 2n-4$.

(V) $K_{3,3}$ is not planar (since $m=9$, $n=6$, $9 \leq 8 \rightarrow$ false)

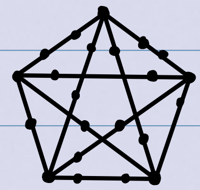
Example: is the Peterson Graph planar?



- $n=10$, $m=15$
- since $m \leq 3n-6$, we can't conclude anything about it's planarity
- Since G is not bipartite, we can't use the $m \leq 2n-4$ result.



→ an edge subdivision of K_4, which is planar



→ an edge subdivision of K_5, which is nonplanar

An **edge subdivision** of a graph G is a graph obtained by repeating the following process (maybe 0 times):

- replace an edge x — y with x — z — y, where z is a new vertex
  ↳ note: the new vertices must have degree 2.

Observations:

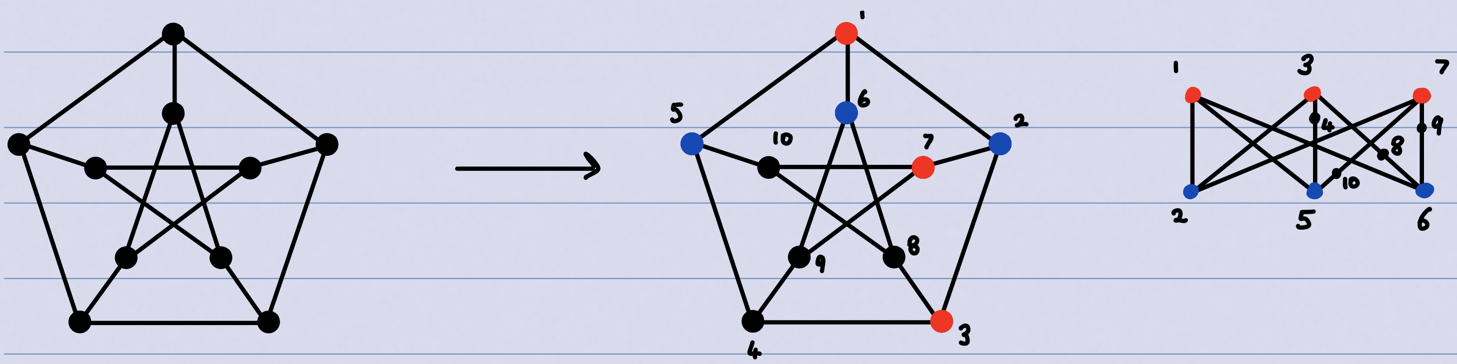
1. A graph is planar if and only if every edge subdivision of G is planar
2. Any edge subdivision of K_5 or $K_{3,3}$ is nonplanar
3. If a graph G has an edge subdivision as a subgraph, then G is nonplanar.
4. The converse is also true:

Kuratowki's Theorem (a characterisation of planar graphs):

A graph is planar if and only if G does not contain any edge subdivisions of K_5 or $K_{3,3}$ as a subgraph.

Equivalently, a graph is nonplanar if and only if it has an edge subdivision of K_5 or $K_{3,3}$ as a subgraph.

Example: show that the peterson graph is nonplanar.



So, the Peterson graph has an edge subdivision of $K_{3,3}$ as a subgraph, so it is nonplanar.

Summary: how to decide if a graph is planar?

1. Determine n and m
2. If $m \not\leq 3n-6$, then conclude that G is nonplanar

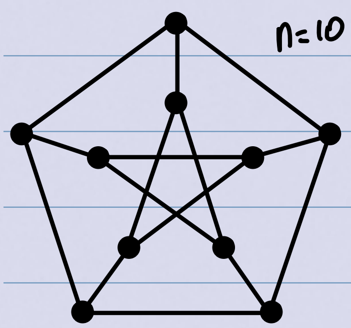
3. If G is bipartite and $m \nleq 2n-4$, then conclude that G is nonplanar
4. If G has an edge subdivision of K_5 or $K_{3,3}$ as a subgraph, then conclude that G is nonplanar
5. G is probably planar, so try finding a planar embedding.

Graph Coloring

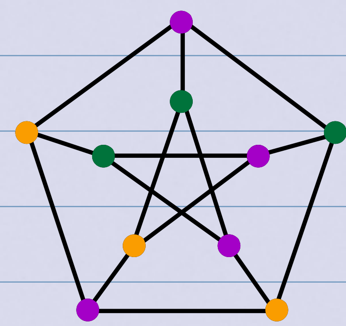
A k-coloring of a graph G is a function $f: V(G) \to \underbrace{\{1,2,\dots,k\}}_{\text{"colors"}}$ such that adjacent vertices get different colors.
↳ ie, $\forall uv \in E(G)$, $f(u) \neq f(v)$.

A graph is k-colorable if it has a k-coloring.

Example: coloring the Peterson graph:



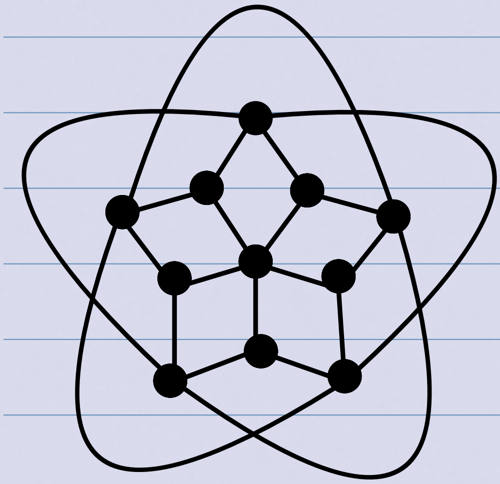
- trivially, G is 10-colorable
- is G 1-colorable? No, since it has edges!
- not 2-colorable, since there's odd-length cycle!
- G is 3-colorable!



Observations:

1. A graph with n vertices is n-colorable
2. G is 1-colorable if and only if G is empty ↗ any #vertices, but 0 edges
3. G is 2-colorable if and only if G is bipartite
4. Deciding whether a graph is 3-colorable is "NP-complete"
5. For any fixed $k \geq 3$, determining whether a graph is k-colorable is also "NP-Complete"

The Grotzsch Graph



$n = 11$ vertices

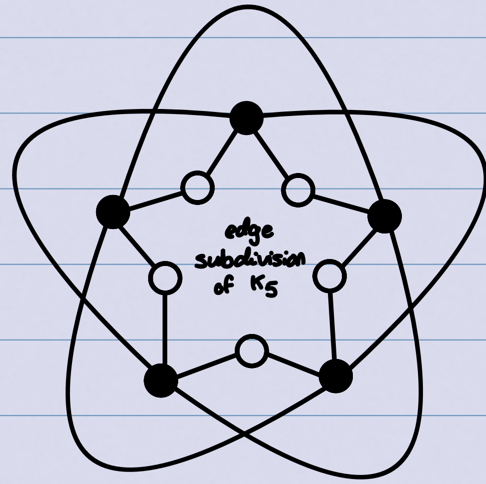
$m = 20$ edges

not regular

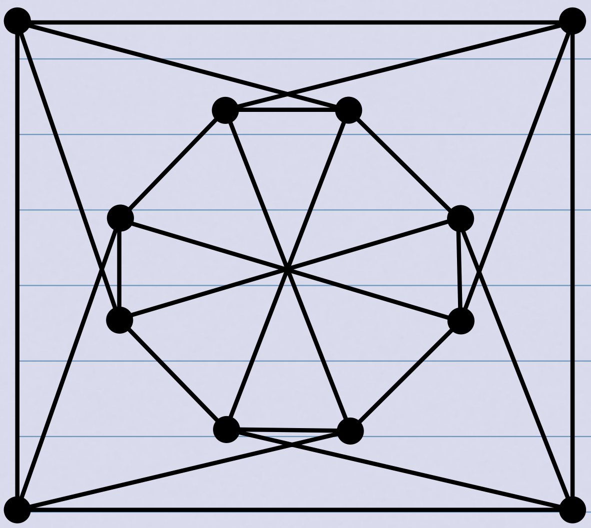
triangle-free

not bipartite

not planar



The Chvátal Graph



$n = 12$ vertices

$m = 24$ edges

4-regular

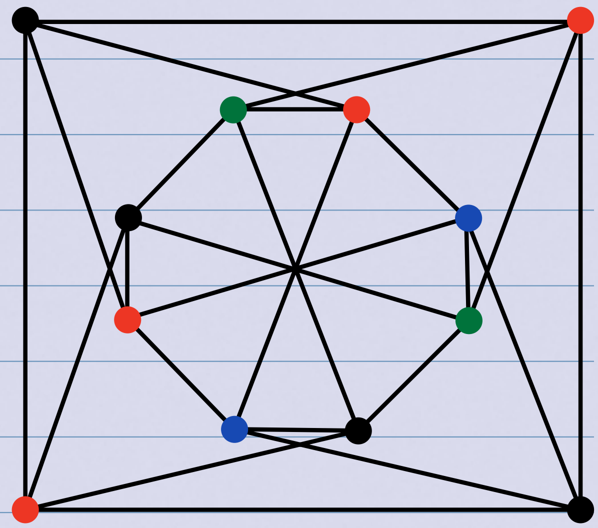
triangle-free

not bipartite

not 2/3-colorable

4 colorable

not planar



Four Color Theorem: every planar graph is 4-colorable

↳ Very hard to prove, so we'll prove that every planar graph is 6-colorable

Lemma: every planar graph has a vertex of degree at most 5.

Proof: let G be a planar graph with n vertices so, suppose $n \geq 6$. Suppose that all of the vertices in G have degree ≥ 6. By the handshaking lemma, G has at least $6n/2 = 3n$ edges. But, $m \leq 3n-6$ for a planar graph! Since $3n \nleq 3n-6$, we have a contradiction, so there must be at least one vertex of degree less than 6.

Six Color Theorem: every planar graph is 6-colorable

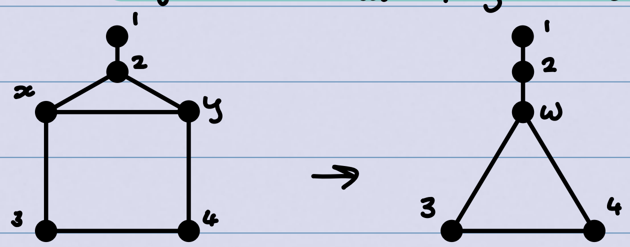
Proof (by induction on the number of vertices, n, in a planar graph G):

Base Case: if $n=1$, then G is trivially 6-colorable.

Induction Hypothesis: let $n \geq 2$. Suppose that all planar graphs with fewer than n vertices is 6-colorable.

Inductive Step: let G be a planar graph with n vertices. By the previous lemma, we know G has a vertex v of degree ≤ 5. Let G' be the graph obtained by deleting v and all of it's incident edges. Then, G' has $n-1$ vertices and is therefore 6-colorable by the induction hypothesis. The neighbors of v in G were colored with ≤ 5 colors (since there's ≤ 5 neighbors). So, at least one color remains to color v. This coloring of v, together with the 6-coloring of G', is a 6-coloring of G.

Let G be a graph, and $e = xy \in E(G)$. Then the contraction of e gives a graph G/e obtained by combining x and y into one new vertex w, which is adjacent to all neighbors of both x and y.



Fact: if G is planar, then so is G/e!

Five-Color Theorem: Every planar graph is 5-colorable

Proof (by induction on the number of vertices, n):

Base Case: Planar graphs with $n=1$ and $n=2$ are trivially 5-colorable.

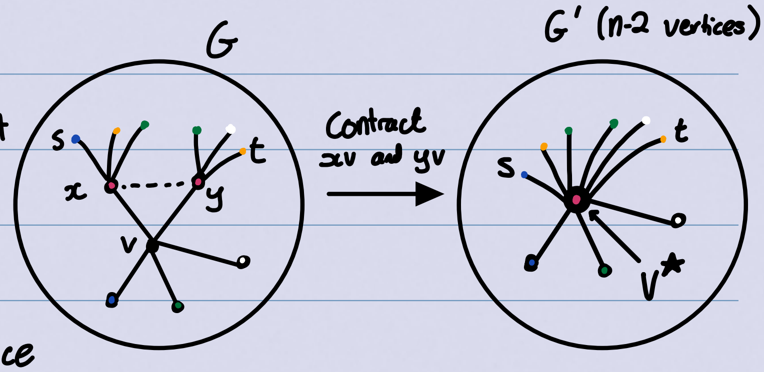
Inductive Hypothesis: let $n \geq 3$, and assume that all planar graphs with fewer than n vertices are 5-colorable.

Inductive Step: let G be a planar graph on n vertices. Let v be a vertex of degree at most 5.

Case 1: $\deg(v) \leq 4$:

Consider $G' = G - v$. Then G' is planar (since G is planar), with n-1 vertices (as we deleted v). So, G is 5-colorable by the induction hypothesis. Now, color the vertices of G with the same colors as used in the 5-coloring of G', and coloring the vertex v with one of the colors that wasn't used to color the neighbors of v.

Case 2: $\deg(v) = 5$:



the 5 neighbors of v in G cannot all be adjacent to each other (because then we'd have K_5 as a subgraph, which is impossible since G is planar). Suppose x and y are two neighbors of v that are not adjacent to each other. Now consider the graph G' obtained by contracting xv and yv; let v^* be the new vertex. Then G' is planar, and has n-2 vertices, So G' has a 5-coloring by the induction hypothesis. Now, we color the vertices of

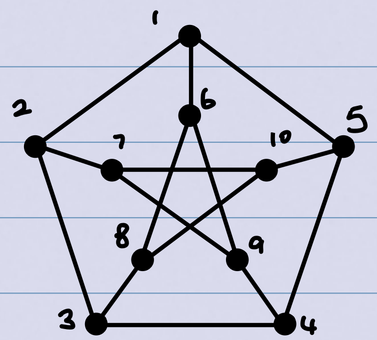
G with the same colors used for G'; color x and y with the color assigned to v^*; and color v with a color not used for the neighbors of v.
This is a 5-coloring of G since x and y are not adjacent, and all neighbors of x and y are adjacent to v^* in G' (so they must have different colors than x and y).

$\therefore$, G is 5-colorable!

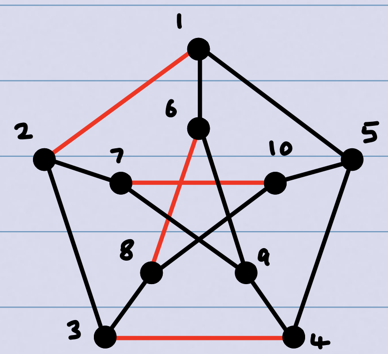
Matchings

A **matching M** of a graph G is a **subset of the edges** of G, **no two of which share a common vertex**

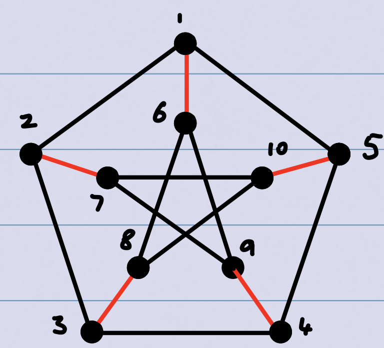
Example:



$M_1 = \{\}$ is a matching



$M_2 = \{12, 68, 710, 34\}$ is a maximal matching



$M_3 = \{16, 510, 27, 38, 49\}$ is a maximum matching

Question: given G, how to (efficiently) find a maximum matching? How to (efficiently) identify that a matching has maximum size?

Let M be a matching of G. A vertex **v is saturated by M** if **v is incident to some edge in M**; **otherwise** it's unsaturated.

M is a perfect matching if every vertex is saturated

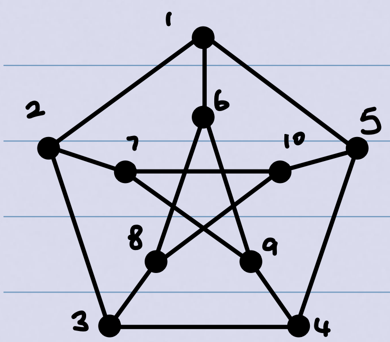
Observations:

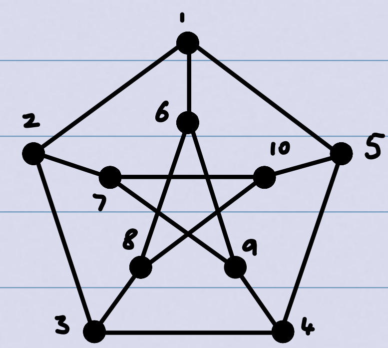
- If G has n vertices, then $|M| \le n/2$
- If G has a perfect matching, then n is even

Covers

A cover C of a graph G is a subset of the vertices of G, such that each edge of G has at least one end in the set C.

Example:





Fact: a cover of size 5 does not exist for the Peterson graph!

$C_1 = \{1, 2, 3, \dots, 8, 9, 10\}$ is a trivial cover

$C_2 = \{1, 4, 10, 7, 6, 3\}$ is a minimum cover

Question: how to (efficiently) find a minimum cover?

minimum size

Matching-Cover Duality

Lemma: let M be a matching of G, and C a cover of G. Then $|M| \le |C|$.

Proof: let $M = \{e_1, e_2, \ldots, e_k\}$, with $e_i = u_i v_i$. For each $1 \le i \le k$, at least one of u_i and v_i must be in C (since C is a cover). Without loss of generality, suppose that each $u_i \in C$. So, $u_1, u_2, \ldots, u_k \in C$ and the u_i are distinct (since M is a matching). So, $|C| \ge k$, and $|M| = k$.

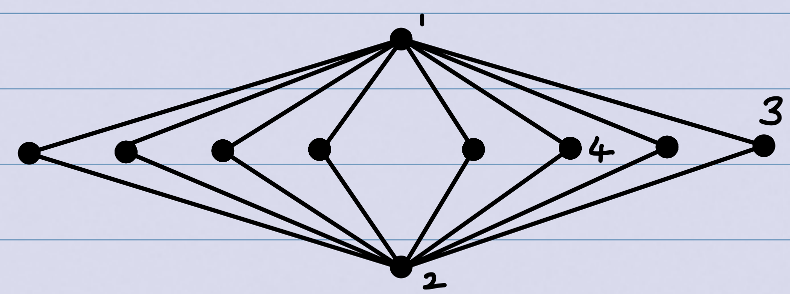
$\therefore$, $k = |M| \le |C|$.

Corollary: If M is a matching of G, and C is a cover of G, and $|M| = |C|$, then M is a maximum matching and C is a minimum covering.

Proof: Let M' be any matching of G. Then, $|M|' \le |C|$ by the previous lemma. So, $|M|' \le |M|$. Since M' is arbitrary, we conclude that M is a maximum matching. Now, let C' be any cover of G. Then, $|C|' \ge |M|$ by the previous lemma. So, $|C|' \ge |C|$. Since C' is arbitrary, we conclude that C is a minimum cover.

Observation: to prove that a matching M has maximum size, just find a cover of the same size!

Example:

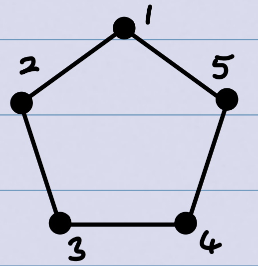


$M = \{13, 24\}$ is a matching

$C = \{1, 2\}$ is a cover

since $|M| = |C| = 2$, M is a max matching and C is a min cover.

Example:

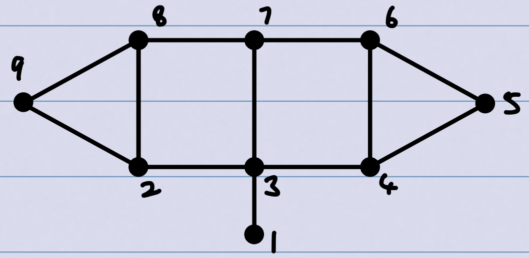


$M = \{12, 34\}$ is a max matching
$C = \{1, 4, 2\}$ is a min cover
But $|M| < |C|$.

So, we can't use the corollary to prove that M is maximum!

König's Theorem: let G be a bipartite graph. Then a maximum matching in G has the same size as a minimum covering of G.

Example:



$M = \{28, 37, 46\}$
$P = \{1, 3, 7, 6, 4, 5\}$ augmenting path
$M' = \{28, 13, 76, 45\}$

Definition: let M be a matching in a graph G. An M-alternating path is a path in G whose edges are alternately in M and not in M.

Definition: an M-augmenting path is an M-alternating path whose end vertices are not saturated by M.

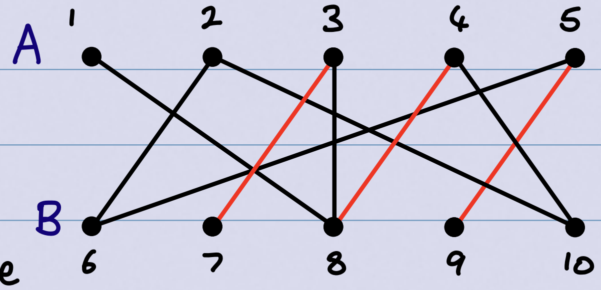
Lemma: let M be a matching in a graph G. If G has an M-augmenting path, then M is not a maximum matching.

Proof: replace the matching edges on the path with the non-matching edges on the path. The resulting matching M' satisfies $|M'| = |M| + 1$.

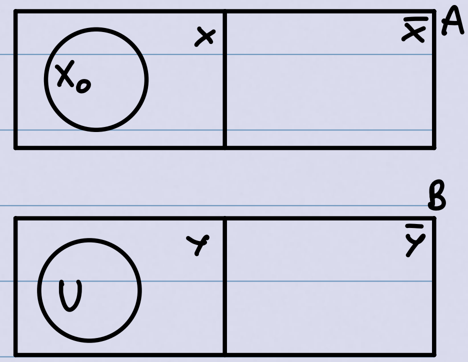
The XY-Construction

- Let G be a bipartite graph, and let's call it's bipartition (A, B).

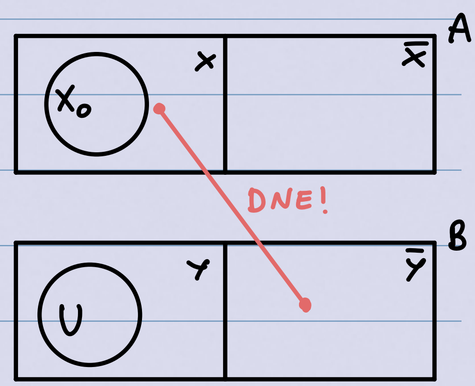
- Let M be a matching of G
- Let X_0 = vertices in A that are unsaturated by M
- Let X = vertices in A which are reachable by an M-alternating path that originates in $x_0 \in X$
- Let $\bar{X}$ be the vertices in A that aren't in X
- Let Y be the vertices in B which are reachable by an M-alternating path that begins in X_0
- Let $\bar{Y}$ be the vertices in B which are not in Y
- Let U be the unsaturated vertices in Y.



$x_0 = \{1,2\}$
$x = \{1,2,4\}$
$\bar{x} = \{3,5\}$
$y = \{6,8,10\}$
$\bar{y} = \{7,9\}$
$u = \{6,10\}$



Claim 1: there is no edge with one end in X and the other end in $\bar{Y}$.



Proof: Suppose there's an edge uv with $u \in X$ and $v \in \bar{Y}$.

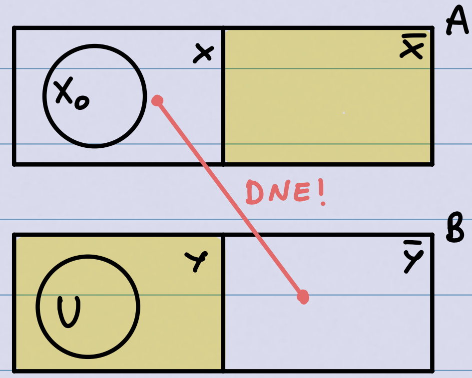
If $u \in X_0$, then uv means that $v \in Y$, which is a contradiction since $v \in \bar{Y}$.

If $u \in X \setminus X_0$, then there is an M-alternating x_0, u-path P, for some $x_0 \in X_0$. This path can't use any vertices in $\bar{X}$ or $\bar{Y}$. Also, the last edge in P is in M. So, uv is not in M. Then the path P, together with the edge uv, is an M-alternating x_0, v-path

So, $v \in Y$, but $v \in \bar{Y}$, so we have a contradiction!

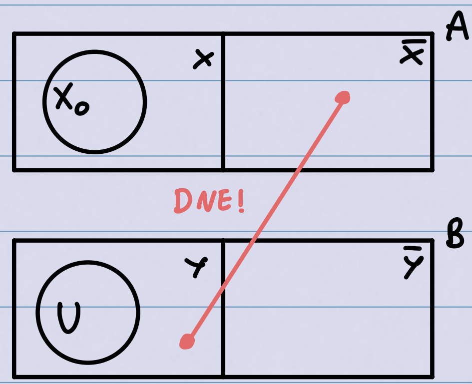
$\therefore$ the claim holds!

Claim 2: $C = \bar{X} \cup Y$ is a cover of the graph



Proof: follows from claim 1, since there are no edges between X and $\bar{Y}$.

Claim 3: There is no matching edge with one end in $\bar{X}$ and the other end in Y.

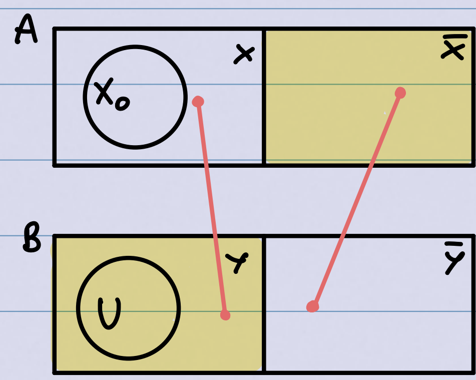


Proof: Suppose that $st \in M$, where $s \in Y \setminus U$ and $t \in \bar{X}$.

Since $s \in Y$, there exists an M-alternating x_0, s-path P for some $x_0 \in X_0$. This path cannot use any vertices in $\bar{X}$ or $\bar{Y}$. The first edge of P is non-matching, so the last edge is non-matching.

Adding st to P gives an M-alternating x_0, t-path, so $t \in X$, which is a contradiction!

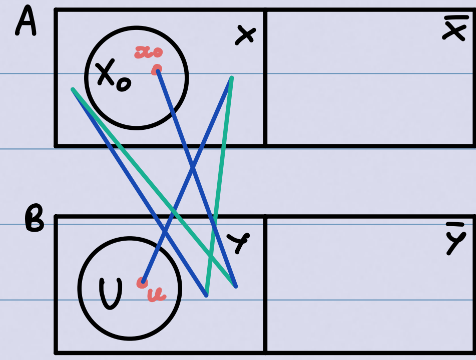
Claim 4: $|M| = |C| - |U|$



Proof: matching edges must have one end in $X \setminus X_0$ and the other end in $Y \setminus U$, or, one end in $\bar{X}$ and the other end in $\bar{Y}$, from claim 3.

- The # of matching edges of the first kind is $|Y| - |U|$
- The # of matching edges of the second kind is $|\bar{X}|$
- Thus, $|M| = (|Y| - |U|) + |\bar{X}| = (|\bar{X}| + |Y|) - |U| = |C| - |U|$.

Claim 5: There is an M-augmenting path from X_0 to each vertex in U.



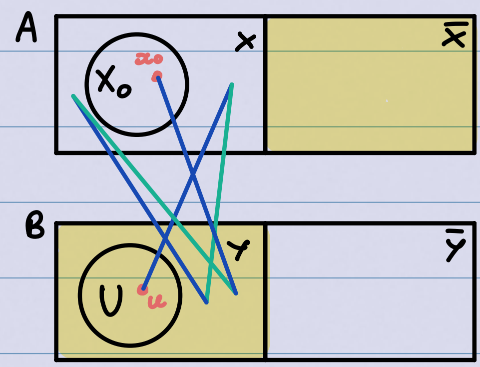
Proof: let $u \in U$. Since $u \in Y$, there exists an M-alternating x_0, u-path for some $x_0 \in X_0$.

Since $x_0 \in X_0$, the first edge in this path is not in M. $\therefore$, the path is M-augmenting.

Proof of Konig's Theorem:

Theorem: let G be a bipartite graph. Then the size of a maximum matching is equal to the size of a minimum cover of G.

Proof: let M be a maximum matching of G. Let $A, B, X_0, X, \bar{X}, Y, \bar{Y}, U, C$ be defined as earlier.



Now, suppose that $U \neq \emptyset$, and let $u \in U$.

By claim 5, there exists an M-augmenting x_0, u-path for some $x_0 \in X_0$.

Thus, we can get a larger matching by replacing the matching edges on the path with the non-matching edges, which contradicts M being a maximum matching.

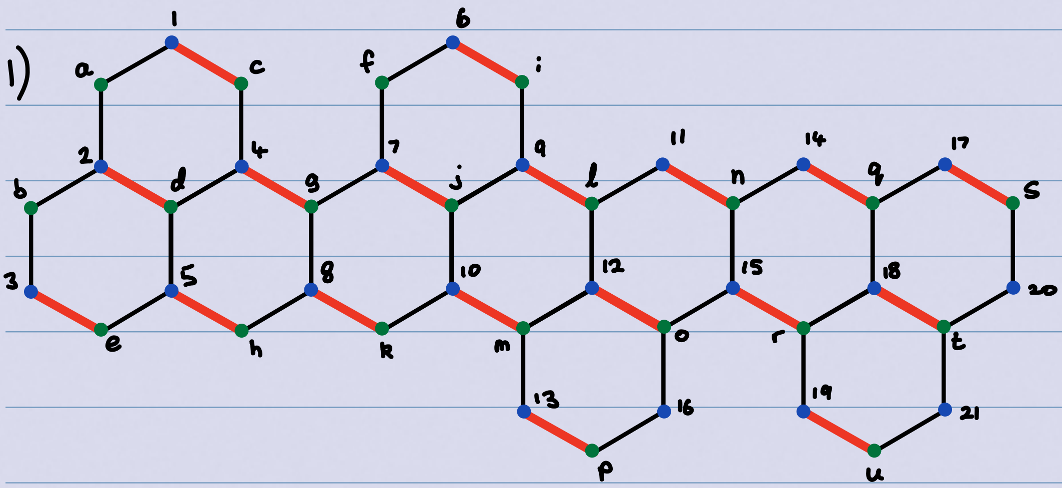
$\therefore$, $U = \emptyset$, so $|M| = |C|$.

Algorithm (basic version) to find a maximum matching M and a minimum cover C in a bipartite graph with bipartition (A, B) such that $|M| = |C|$.

1. begin with any matching M (eg, a single edge)

2. while there's an M-augmenting path in G, do:
   - obtain a larger matching M by replacing edges in P with the non-matching edges in P
3. let X_0 = unsaturated vertices in A,
   X = vertices in A reachable by an M-alternating path beginning in X_0,
   $\bar{X} = A \setminus X$,
   Y = vertices in B reachable by an M-alternating path beginning in X_0,
   $C = \bar{X} \cup \bar{Y}$
4. Output (M, C).

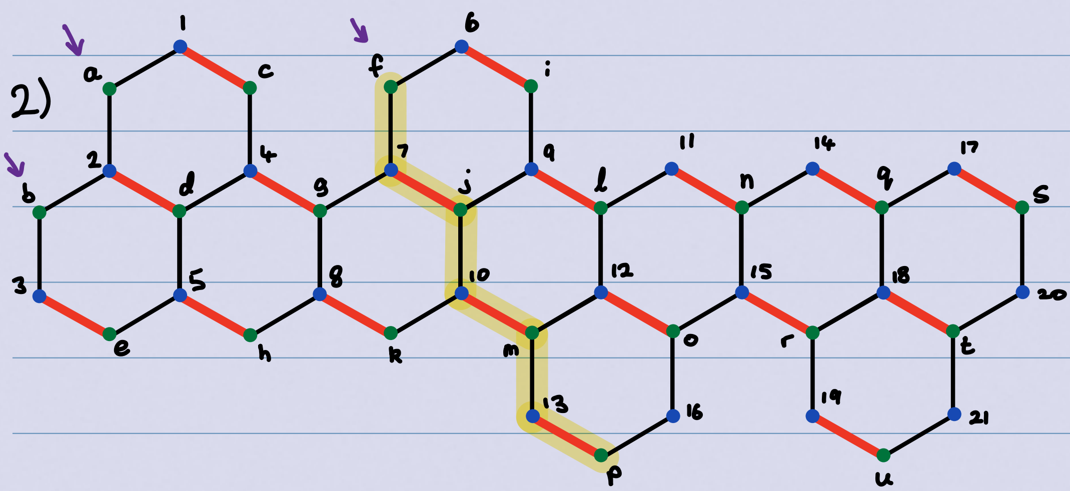
Example: Bipartite max-matching min-cover

1) 

$A^{\bullet} = \{a, b, \ldots, u\}$

$B^{\bullet} = \{1, 2, \ldots, 21\}$

$|M| = 18$

2) 

M-augmenting path:

f, 7, j, 10, m, 13, p, 16

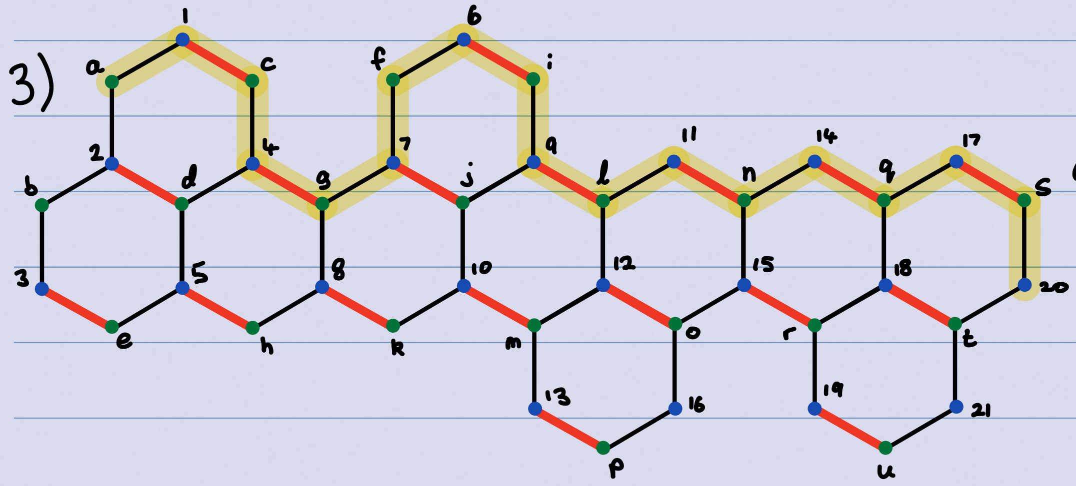
$A^{\bullet} = \{a, b, \ldots, u\}$

$B^{\bullet} = \{1, 2, \ldots, 21\}$

$|M| = 18$

$X_0 = \{a, b, f\}$ unsaturated

3)



M-augmenting path

$a, 1, c, 4, g, 7, f, 6, i, 9, l, 11, n, 14, q, 17, s, 20$

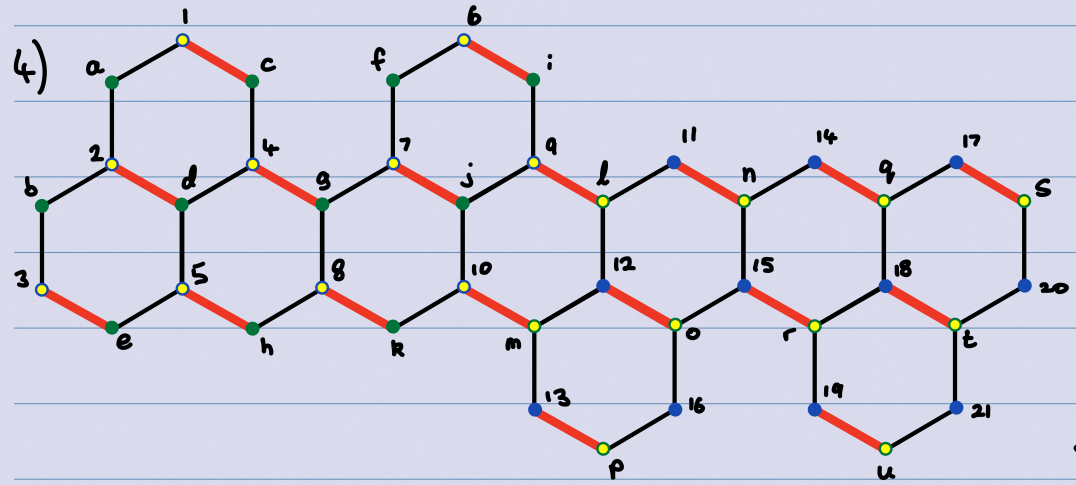
$A = \{a, b, \dots, u\}$

$B = \{1, 2, \dots, 21\}$

$|M| = 19$

- $X_0 = \{a, b\}$

4)



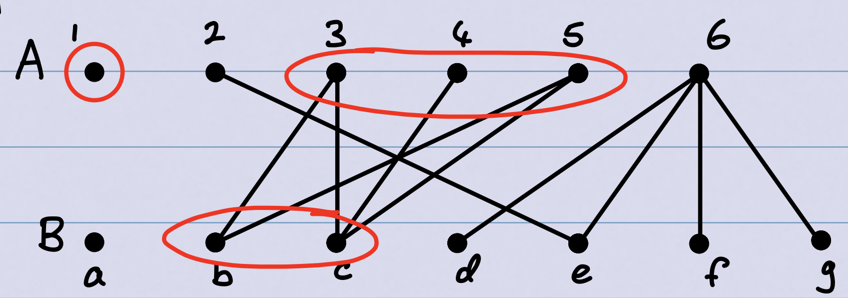
- $C = \bar{X} \cup Y = \{l, \dots, u, 1, \dots, 10\}$
- $|C| = 20 = |M|$
- M is a max matching, C is a min cover
- $A = \{a, b, \dots, u\}$, $B = \{1, 2, \dots, 21\}$
- $|M| = 20$, $x_0 = \{b\}$
- $X = \{a, b, \dots, k\}$, $Y = \{1, 2, \dots, 10\}$

Question: given a bipartite graph G with bipartition (A, B), does G have a perfect matching?

Answer: No, if $|A| \neq |B|$!

Question: Suppose $|A| \leq |B|$. Does there exist a matching that saturates all of the vertices in A?

Example:



is there a matching that saturates every vertex in A?

Answer: No, the 3 vertices $\{3, 4, 5\}$ have only two neighbors, $\{b, c\}$

let G be a graph and $D \subseteq V(G)$. The neighborhood of D, denoted $N(D)$, is the set of all vertices adjacent to at least one vertex in D.

Hall's Theorem: A bipartite graph with bipartition (A, B) has a matching which saturates every vertex in A if and only if for every $D \subseteq A$, $|N(D)| \geq |D|$.

Corollary: Let G be a bipartite graph with bipartition (A, B). Then G has a perfect matching if and only if:

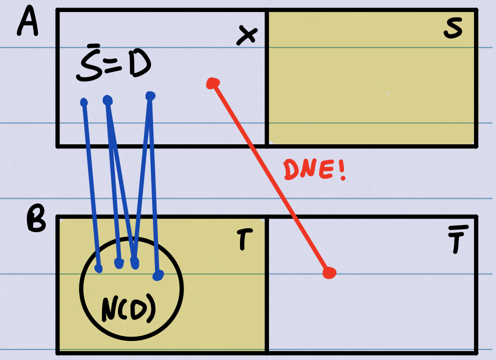
1) $|A| = |B|$
2) $\forall D \subseteq A$, $|N(D)| \geq |D|$

Proof of Hall's Theorem:

($\Rightarrow$) Suppose G has a matching that saturates every vertex in A. Let $D \subseteq A$. Every edge of the matching that has one end in D has it's other end in $N(D)$, and those neighbors are distinct. So, $|N(D)| \geq |D|$

($\Leftarrow$) Suppose G doesn't have a matching that saturates every vertex in A (we'll produce $D \subseteq A$, st $|N(D)| < |D|$).
Let M be a maximum size matching in G. Then $|M| < |A|$ since M doesn't saturate every vertex in A.
Let C be a minimum cover of G. Then, by König's Theorem, $|C| = |M| < |A|$.
Let $S = A \cap C$, $\bar{S} = A - S$, $T = B \cap C$, and $\bar{T} = B - T$.

Note: $C = S \cup T$!



Let $D = \bar{S}$. Consider $N(D)$. Then, $N(D) \subseteq T$, since G can't have any edges whose ends are in D and $\bar{T}$, since otherwise C is not a cover of G.

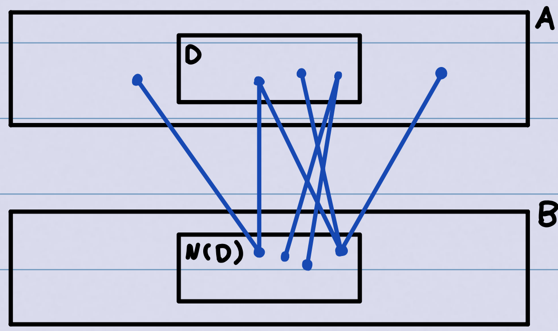
So, $|N(D)| \leq |T| = (|T| + |S|) - |S| = |C| - |S| < |A| - |S|$, since $|C| < |A|$.

So, $|N(D)| \leq |T| < |A| - |S| = |\bar{S}| = |D|$.

Thus, $|N(D)| < |D|$

Theorem: every k-regular bipartite graph $(k \geq 1)$ has a perfect matching.

↳ eg: the n-cube is bipartite and n-regular. So, the n-cube has perfect matching



Proof: let G be a k-regular bipartite graph with bipartition (A, B).

Claim: $|A| = |B|$

Then, $\sum_{v \in A} \deg(v) = k|A|$ and $\sum_{v \in B} \deg(v) = k|B|$. But, $k|A| = k|B|$, so $|A| = |B|$.

Let $D \subseteq A$. Then every edge which has one end in D has it's other end in $N(D)$, but not necessarily the other way around. Thus, $\sum_{v \in D} \deg(v) \leq \sum_{v \in N(D)} \deg(v)$. $\therefore$, $k|D| \leq k|N(D)|$, so $|D| \leq |N(D)|$. By the corollary to Hall's theorem, G has a perfect matching!