

Lecture 1 - 4th Sept 2024

Statistics: the science of understanding data and making decisions in the face of variability and uncertainty.

Probability: A branch of mathematics concerned with describing and modeling uncertain events.

Preliminaries:

- Experiment: the process of obtaining an observed result of some phenomenon.
- Trial: the performance of an experiment.
- Outcome: the result of a single trial (attempt) of an experiment.
- Event: one or more outcomes of an experiment.
- Probability: the measure of how likely an event is.

Sample Space: the set of ALL possible distinct outcomes in a random experiment, denoted by S .

↳ note: one and only one of the outcomes occurs in any single trial of the experiment.

↳ example:

- Roll a six-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- Flip a coin: $S = \{\text{heads, tails}\}$
- Waiting time for a bus: $S = \{t \in \mathbb{R}, 0 \leq t \leq 10\}$
- A sample space is finite if it consists of a

finite number of outcomes, say $\{a_1, a_2, \dots, a_n\}$

- A sample space can be a set of countable infinite outcomes, say $\{a_1, a_2, \dots\}$, where the outcomes can be put into a one-on-one correspondence with positive integers.

↳ Example: suppose our experiment consists of tossing a coin until it lands on heads.

↳ the possible outcomes in the sample space are:

H, TH, TTH, TTTH, TTTTH, ...

∴ the outcomes are countable but infinite.

Discrete Sample Space: when a sample space is either finite or countably infinite. Else, it's non-discrete.

↳ Example:

- Roll a six-sided die: discrete
- Flip a coin: discrete
- Waiting time for a bus: non-discrete

Event: a subset of sample space S.

↳ we say A is an event if $A \subset S$.

↳ note: A and S are both sets!

Example: consider tossing two coins.

a) what is the event of obtaining at least one head?

$$S = \{ HH, HT, TH, TT \}$$

$$\therefore A = \{ HH, HT, TH \}$$

b) What is the event of obtaining at least one head and at least one tail?

$$A = \{ HT, TH \}.$$

Elementary / Simple Event: an event that contains only one outcome of the experiment. Eg, $A = \{ \alpha, \beta \}$.

Compound Event: an event made up of two or more simple events. Eg: $A = \{ \alpha_1, \alpha_2 \}$.

Set Notations and Terminology:

- A set is a collection of elements. Eg, $A = \{ \text{apple, orange} \}$.
- The notation $\alpha \in A$ or $\alpha \notin A$ will mean that α is or is not an element of A .
- The empty / null set is \emptyset .
- Union: $A \cup B = \{ \alpha \mid \alpha \in A \text{ or } \alpha \in B \}$
- Intersection: $A \cap B = \{ \alpha \mid \alpha \in A \text{ and } \alpha \in B \}$
- The notation $B \subset A$ means that B is a subset of A .
- Complement: $A^c = \{ \alpha \mid \alpha \in S, \alpha \notin A \} = \bar{A}$
- Two sets A and B are disjoint if $A \cap B = \emptyset$.

- The set $A \cap B^c$ is "A but not B"
- The set $A^c \cap B^c$ is "neither A nor B"

- If A_1, \dots, A_n is a finite collection of sets,

$\hookrightarrow A_1 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i$ is "in all A_i ;"

$\hookrightarrow A_1 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$ is "in at least one A_i ;"

✓ very similar to disjoint!

Mutually Exclusiveness: when two events A and B if $A \cap B = \emptyset$. If events are exclusive, they have no outcomes in common.

\hookrightarrow Example: tossing two coins. A is the event "at least one head", and B is the event "both tails".

$$\hookrightarrow S = \{HH, HT, TH, TT\}.$$

$$A = \{HH, HT, TH\}, B = \{TT\}.$$

$$A \cap B = \{HH, HT, TH\} \cap \{TT\} = \emptyset !$$

\therefore , A and B are mutually exclusive.

\hookrightarrow Events A_1, A_2, A_3, \dots are said to be mutually exclusive if they are pairwise mutually exclusive. That is,

$$A_i \cap A_j = \emptyset, \text{ wherever } i \neq j.$$

Probability Modeling : assigning a probability $P(A)$ to each event A . This probability measures how likely it is that A will happen when the experiment is conducted.

- ↳ think of $P(A)$ as a (set) function, whose domain is a collection of sets (events), and the range of which is a subset of real numbers.
- ↳ not all set functions are appropriate for assigning probabilities to events.

• Let $S = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$ be a discrete sample space. We assign to each elementary event $A_i = \{\alpha_i\}$ for $i = 1, 2, \dots$, a number $P(A_i) = P(\{\alpha_i\})$, such that

$$\bullet 0 \leq P(A_i) \text{ and } \sum_{\text{all } i} P(A_i) = 1$$

• We call $P(A)$ the probability of A , and call the set of probabilities $\{P(A_i), i=1, 2, \dots\}$ the probability distribution on S .

↳ General Properties :

- The null event (empty set) has a probability of 0. ∴ $P(\emptyset) = 0$.
- If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$
- If A_1, A_2, \dots, A_k is a finite collection of pairwise mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_R) = P(A_1) + P(A_2) + \dots + P(A_R)$$

$$\cdot P(\bar{A}) = 1 - P(A)$$

↳ Example: Suppose a six-sided die is rolled once.
If A is the event that an even number is obtained.
What is $P(A)$?

- $S: \{1, 2, 3, 4, 5, 6\}$. Since each number is equally likely, $P(\{i\}) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

$$A = \{2, 4, 6\} \rightarrow P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) \\ = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

↳ Example: Toss a coin twice. What's the probability of getting exactly one head?

- $S: \{HH, HT, TH, TT\}$. Each option is uniformly likely.
- $A = \{HT, TH\} \rightarrow P(A) = P(\{HT\}) + P(\{TH\}) \\ = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$

Random Selection: when choosing an object from a finite set where each object has an equal chance of being selected.

Uniform Probability Model: consider a discrete finite sample space $S = \{a_1, a_2, \dots, a_n\}$, where each

simple event has a probability of $\frac{1}{N}$. This is a uniform distribution over the set $\{\alpha_1, \alpha_2, \dots, \alpha_N\}$.

For a compound event A, we can calculate $P(A)$ by:

$$P(A) = \frac{n(A)}{N},$$

where $n(A)$ is the number of outcomes in A.

The formula $P(A) = \frac{n(A)}{N}$ is classical probability.

(trivial) example: draw one card at random from a well-shuffled deck without jokers. What is the probability that the card is a spade?

$$S = \{A\heartsuit, 2\heartsuit, \dots\}.$$

$$A_s = \{A\spadesuit, 2\spadesuit, \dots, K\spadesuit\}.$$

$$P(A_s) = \frac{n(A_s)}{N} = \frac{13}{52} = \frac{1}{4}.$$

Example: suppose that P is a probability, and A and B are events such that A and B are mutually exclusive, and $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$. Is each statement true or false?

A) A and B are each non-empty

Since $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$, we know that

$$P(A) = 0.2 \text{ and } P(B) = 0.3.$$

we also know that $P(\emptyset) = 0$. Since $0.2 \neq 0 \neq 0.3$, we know A and B cannot be empty.
 \therefore the statement is true.

B) $P(A \cap B) = 0$

Since we're told that A and B are mutually exclusive, we know that $A \cap B = \emptyset$.

we also know that $P(\emptyset) = 0$, so the statement is true!

C) $P(A \cup B) = 0.4$

we know that $P(A \cup B) = P(A) + P(B)$

"the probability of
A or B"

$$P(A) + P(B) = 0.2 + 0.3 = 0.5 \neq 0.4.$$

\therefore the statement is false!

Example: suppose two six-sided die are rolled where the outcomes are of the form (die 1, die 2). If A is the event that the first die is even, and B is the event that the sum of both rolls is 6, compute the event $C = \bar{A} \cap B$

$$\hookrightarrow A = \{(2,1), (2,2), \dots, (4,1), (4,2), \dots, (6,1), (6,2), \dots, (6,6)\}.$$

$$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

\bar{A} means the event that A does not happen, so in other words, \bar{A} is the event that the first die is odd.

$$\therefore \bar{A} = \{(1,1), \dots, (1,6), (3,1), \dots, (3,6), (5,1), \dots, (5,6)\}.$$

$\bar{A} \cap B$ is A AND B.

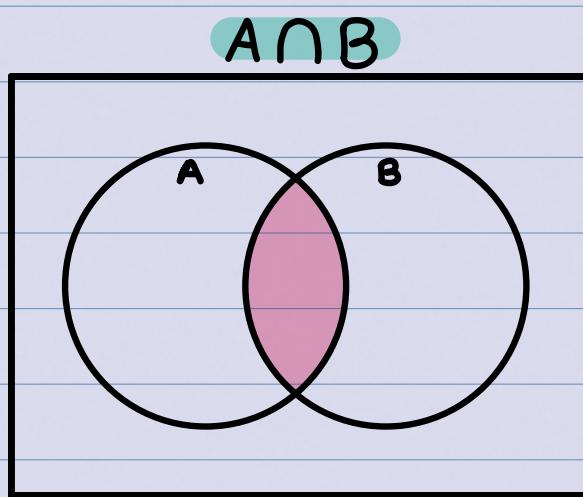
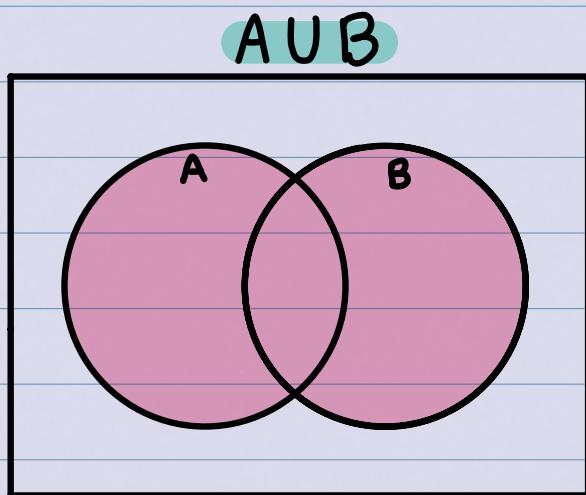
$$\hookrightarrow \therefore C = \{(1,5), (3,3), (5,1)\}.$$

Venn Diagram: a tool to illustrate the relationship among sets (or events).

↳ made up of:

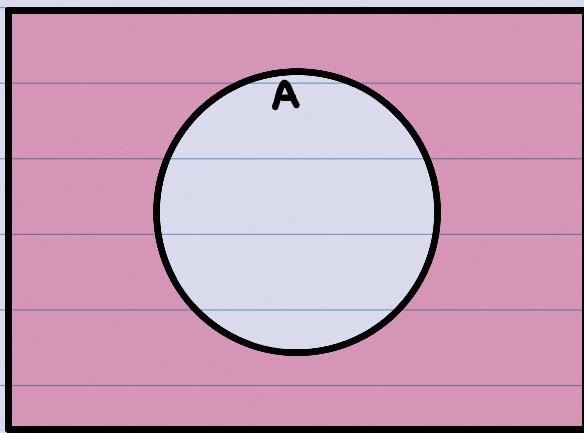
- a rectangle to represent the sample space S.
- Circles within the rectangle to illustrate the events.

The Union and Intersection of Two Events



The Compliment of an Event:

\bar{A}



"S but not A"

De Morgan's Laws:

$$\cdot \overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\cdot \overline{A \cap B} = \bar{A} \cup \bar{B}$$

Probability Rules

- Probability of the Complement of an Event
 - $P(A) = 1 - P(\bar{A})$
- Proof: $A \cup \bar{A} = S$, so $P(A \cup \bar{A}) = P(A) + P(\bar{A})$
since $P(A \cup \bar{A}) = P(S) = 1$, $1 = P(A) + P(\bar{A})$
bc mutually exclusive!

Example: an experiment involves tossing a coin four times. The event A is "at least one head". Find the probability of A, ie, find $P(A)$.

- the only case where there is NOT at least one head is $\bar{A} = \{\text{TTTT}\}$.

flipping a coin 4 times would result in 2^4 possible

outcomes, and \bar{A} is only 1 of these.

$$\therefore P(\bar{A}) = \frac{1}{16}, \text{ so } P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{16} = \frac{15}{16}.$$

probability of complement!

Example: two ordinary dice are rolled. Find the probability that at least one lands on a 6.

$$S = \{(1,1), \dots, (1,6), (2,1), \dots, (2,6), \dots, (6,1), \dots, (6,6)\}.$$

A = the event of at least one six.

$\therefore \bar{A}$ is the event that neither die is a six.

classical probability of each die to NOT be a six: $\frac{n(x)}{N}$.

where $n(x)$ is the number of outcomes that satisfy the roll not being a 6 - ie; $\{1, 2, 3, 4, 5\} \Rightarrow 5$, and N is the total number of outcomes in the set - ie; $\{1, 2, 3, 4, 5, 6\} \Rightarrow 6$.

$$\therefore \frac{n(x)}{N} = \frac{5}{6} !$$

since both dice are independent, the probability of both NOT being a six (aka \bar{A}) is $\frac{5}{6} \cdot \frac{5}{6}$.

$$\therefore P(\bar{A}) = \frac{25}{36}, \text{ so } P(A) = 1 - \frac{25}{36} = \frac{11}{36} !$$

Rules for Union:

• Union of two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: At a university, 70% of the students own a car, 60% of students live on campus, and 50% both live on campus and own a car. If a student is chosen at random, what is the probability that the student does not own a car and does not live on campus?

- Let A be the event that a student has a car.
- Let B be the event that a student lives on campus.

we want: $P(\bar{A} \cap \bar{B})$.

↳ we know that $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

$$\hookrightarrow P(\bar{A} \cap \bar{B}) = 1 - P(\overline{A \cup B}) = 1 - P(A \cup B)$$

from the union of two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\begin{aligned}\therefore P(\bar{A} \cap \bar{B}) &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 + P(A \cap B) - P(A) - P(B) \\ &= 1 + 0.5 - 0.7 - 0.6\end{aligned}$$

$$\hookrightarrow \therefore P(\bar{A} \cap \bar{B}) = 0.2 = 20\%.$$

Union of Three Events: for any 3 events A, B, and C:



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Conditional Probability: determining the probability of some event A, while knowing that some related event B has occurred.

Example: consider that a fair die has been rolled and you are asked to give the probability that it was a 5.

$$\hookrightarrow A = \{5\}, S = \{1, 2, 3, 4, 5, 6\}.$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}.$$

However, if you're told that the dice has landed on an odd number, the odds change:

$$\hookrightarrow A = \{5\}, S = \{1, 3, 5\}.$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{3}.$$

Example: a box contains 100 microchips, some produced by Factory 1 and the rest by Factory 2. Among these microchips, some are defective, while others are good. Let A be the event of "obtaining a defective microchip" and B be the event of "the microchip was produced in Factory 2."

the number of microchips in each category:

| | B | \bar{B} | totals |
|-----------|----|-----------|--------|
| A | 15 | 5 | 20 |
| \bar{A} | 45 | 35 | 80 |
| totals | 60 | 40 | 100 |

a) Find the probability of obtaining a defective microchip.

the total of A is 20, and there are 100 total microchips.

$$\hookrightarrow P(A) = \frac{n(A)}{N} = \frac{20}{100} = \frac{1}{5} \quad (20\%)$$

b) Given that a microchip is from factory 1, find the probability that it is defective.

the sample space S shrinks from all 100 microchips to only the 60 made by factory 1 (event B).

$$\hookrightarrow \therefore P(A) = \frac{n(A)}{N} = \frac{15}{60} = \frac{1}{4} \quad (25\%)$$

Conditional Probability mathematical definition:

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad (P(B) > 0)$$

↪ "the probability of A given B"

$$\hookrightarrow 0 \leq P(A|B) \leq 1$$

- $P(A|B) = 1 - P(\bar{A}|B)$

Pretty much the same as
the regular union of two events

- $P(A_1 \cup A_2 | B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2 | B)$

Multiplication Theorem of Probability: a way to compute the joint occurrence of A and B:

$$\hookrightarrow P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$$

Law of Total Probability: if B_1, B_2, \dots, B_k is a collection of mutually exclusive events and $B_1 \cup B_2 \cup \dots \cup B_k = S$, then for any event A,

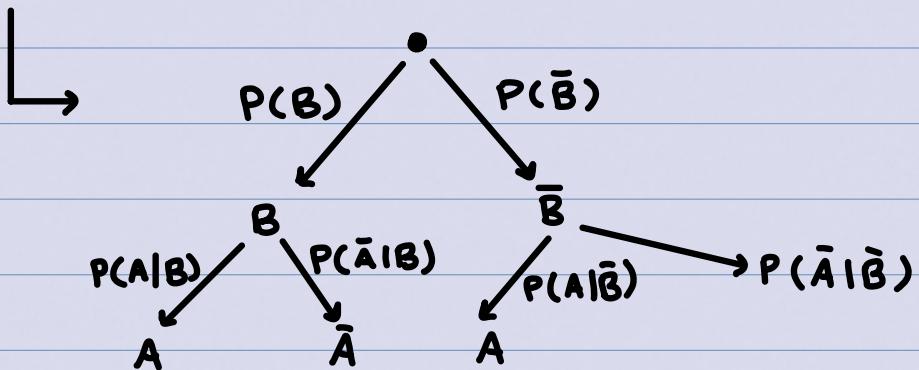
$$P(A) = \sum_{i=1}^k P(B_i) \cdot P(A|B_i)$$

Note: for a special case $k=2$:

only for
 $k=2$!!

$$P(A) = P(A|B) \cdot P(B) + P(A|\bar{B}) \cdot P(\bar{B})$$

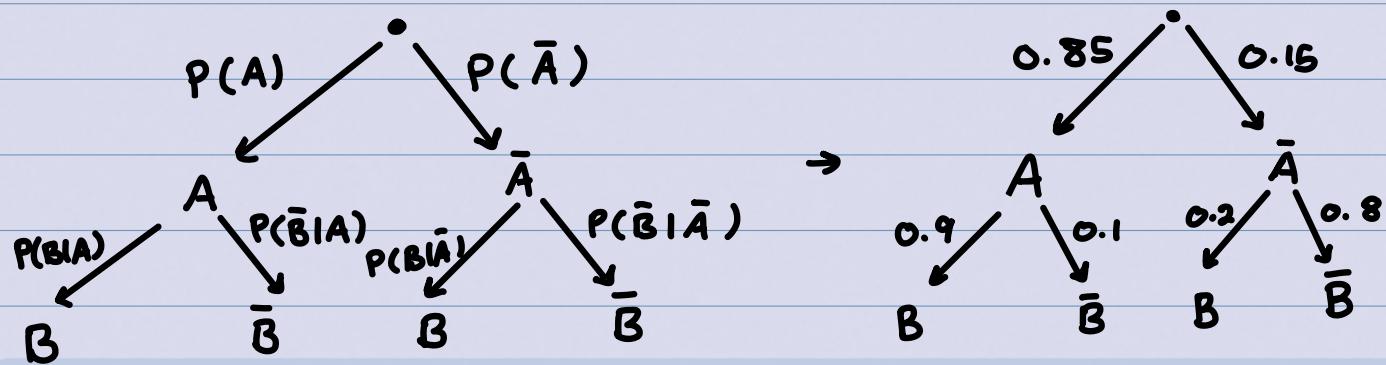
for event A to occur, it must happen with either B or \bar{B} . \therefore only $A \cap B$ or $A \cap \bar{B}$ can occur.



Example: without water, a plant will die with a

probability of 0.8, and with water, it will die with a probability of 0.1. I'll remember to water the plant with a probability of 0.85. Represent this info using a tree diagram.

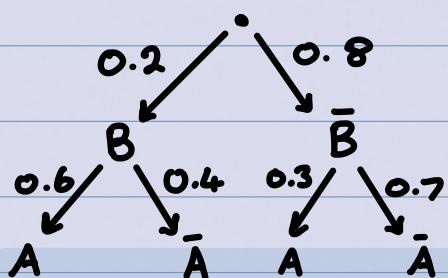
Let A be the event that I remember to water the plant, and B be the event that the plant survives.



Example: in a typical year, 20% of the days have a high temperature greater than 22°C . 40% of these days, there is no rain. During the rest of the year, when the temperature is $\leq 22^{\circ}\text{C}$, 70% of the days have no rain.

a) represent this information in a tree diagram.

let A be the event that it rains, and let B be the event that the temperature peaks above 22°C .



b) solve for the proportion of days in the year which have rain and a temperature $\leq 22^{\circ}\text{C}$.

↳ aka, find $P(A \cap \bar{B})$

$$\text{we know } P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A \cap \bar{B})}{1 - P(B)}$$

$$\begin{aligned} \therefore P(A \cap \bar{B}) &= P(A|\bar{B}) \cdot (1 - P(B)) \\ &= 0.3 \cdot (1 - 0.2) = 0.3 \cdot 0.8 = 0.24 \\ &= 24\% . \end{aligned}$$

Baye's Rule: $P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$,

and more generally, $P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^k P(B_i)P(A|B_i)}$ ↗ for each $j=1, 2, \dots, k$.

Example: revisiting the plant problem from before, if the plant is alive, what's the probability that I remembered to water it?

↗ same as previous time

↳ let A be the event that I remember to water the plant, and B be the event that the plant survives.

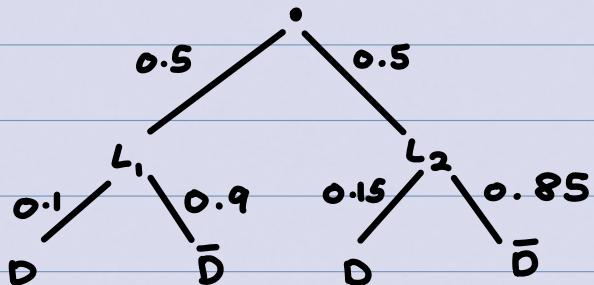
$$\begin{aligned} P(A|B) &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})} = \frac{(0.9)(0.85)}{(0.9)(0.85) + (0.2)(0.15)} \\ &= 0.9623 = 96.23\% ! \end{aligned}$$

Example: electric motors from two assembly lines are

pooled in a common stockroom, with equal numbers of motors from each line. Motors are periodically sampled for testing. It's known that 10% of the motors from line 1 are defective, while 15% of the motors from line two are defective.

a) if a motor is randomly selected, what is the probability that it is defective?

Let D be the event that a motor is defective.



$$\begin{aligned} \rightarrow P(D) &= P(D \cap L_1) + P(D \cap L_2) \\ &= 0.5 \cdot 0.1 + 0.5 \cdot 0.15 \\ &= 0.125 \end{aligned}$$

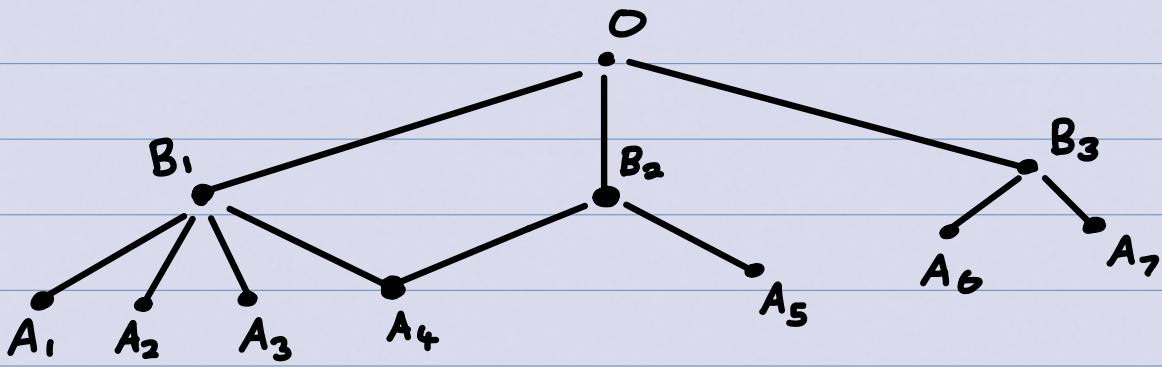
↳ 12.5% chance.

b) if the randomly selected motor is defective, what is the probability that it came from line 1?

$$P(L_1 | D) = \frac{P(D | L_1) P(L_1)}{P(D)} = \frac{0.1 \cdot 0.5}{0.125} = 0.4$$

= 40% chance.

Example: a man starts at point O on this map. He chooses a path at random and follows it to point B₁, B₂, or B₃. From that point, he chooses a new path at random and follows it to one of the points A, ... A₇.



a) what is the probability that the man arrives at point A_4 ?

$$P(A_4) = P(A_4 | B_1) + P(A_4 | B_2) = \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12} + \frac{1}{6}$$

$$= \frac{1}{4} = 0.25$$

$= 25\%$!

b) suppose that the man arrives at point A_4 . What is the probability that he passed through B_1 ?

$$P(B_1 | A_4) = \frac{P(A_4 | B_1)}{P(A_4)} = \frac{\frac{1}{4} \cdot \frac{1}{3}}{\frac{1}{4}} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{4}{12} = \frac{1}{3}$$

\therefore , there was an 33.33% chance he came from B_1 .

Independent Event: events A and B are called independent if and only if $P(A \cap B) = P(A)P(B)$. Otherwise, A and B are called dependent events.

↳ an equivalent formulation can be given in terms of conditional probability :

→ If A and B are events such that $P(A) > 0$ and $P(B) > 0$, then A and B are called independent if and only if either of the following hold:

$$P(A|B) = P(A), \text{ or } P(B|A) = P(B)$$

Note: independence \neq mutually exclusive!

unless $P(A) = 0$!

General rule: $P(A \cap B) = P(B|A) \cdot P(A)$

Independence of Compliments: Suppose events A and B are independent. Then the following events are also independent:

- \bar{A} and \bar{B}
- \bar{A} and B
- A and \bar{B}

Independence of n events: the k events A_1, A_2, \dots, A_k are said to be independent, or mutually independent, if and only if:

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \dots P(A_{i_k})$$

forall possible distinct i_1, i_2, \dots, i_k chosen from 1, 2, ..., n.

→ Note: for a sequence of events to be independent, all of their subsets must also be independent.

Example: suppose a fair die is rolled twice. Let A be the event that the first roll is a 6, and let B be the event

that the second roll is a 6. Show that A and B are independent events.

↳ we want to show that $P(A \cap B) = P(A) \cdot P(B)$

$$P(\{6,6\}) = P(A \cap B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

$$P(A) = \frac{n(A)}{N} = \frac{1}{6} = \frac{n(B)}{N} = P(B). \therefore P(A) = P(B) = \frac{1}{6}.$$

$$P(A) \cdot P(B) = \frac{1}{36} = P(A \cap B).$$

↳ ∴, the events are independent.

Counting Techniques

often, manually counting many outcomes manually can be very time-consuming and tedious.

1) Addition Rule : "OR" interpreted as addition

↳ Eg: suppose job 1 can be done in p ways and job 2 can be done in q ways. ∴, we can do either job 1 or job 2 in $p+q$ ways.

2) Multiplication Rule: "AND" interpreted as multiplication.

↳ Eg: suppose job 1 can be done in p ways and for each of these ways, we can do job 2 in q ways. ∴, we can do both Job 1 and Job 2 in $p \times q$ ways.

Example: how many are there to answer a 20

question true/false test?

$$Q_1 \cdot Q_2 \cdot \dots \cdot Q_{20} = 2^{20} \text{ ways.}$$

Sampling with Replacement: what we get on the first selection does not affect what we get on second selection.

↳ Eg: every time an object is selected, it's put back into the pool of possible objects.

Sampling without Replacement: what we get on the first selection will affect what we get on the second selection.

↳ Eg: once an object is selected, it stays out of the pool of possible objects.

Example: A bag contains 3 blue marbles and 5 red marbles. If you pick two marbles from the bag,

a) what is the probability of picking 2 blue marbles if the selection is with replacement?

$$P = \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64}$$

b) what about without replacement?

$$P = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56} = \frac{3}{28}$$