Lecture 1 - Complex numbers

Fundamental Number Sets

excluding 0!

Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational Numbers: $\mathbb{Q} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \}$

Real numbers: $\mathbb{R}$: the set of all rational or irrational numbers.

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ subset (may be equal)

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ proper subset

Examples:

$x + 3 = 5 \rightarrow x = 2$ $\mathbb{N}$

$x + 4 = 3 \rightarrow x = -1$ $\mathbb{Z}$

$2x = 1 \rightarrow x = \frac{1}{2}$ $\mathbb{Q}$

$x^2 = 2 \rightarrow x = \sqrt{2}$ $\mathbb{R}$

$x^2 + 1 = 0 \rightarrow x = ?$ no real solution!

A complex number in standard form is an expression of the form $x + yj$ where $x, y \in \mathbb{R}$ and $j^2 = -1$.

The set of all complex numbers is denoted by:

$$\mathbb{C} = \{ x + yj \mid x, y \in \mathbb{R}, j^2 = -1 \}$$

Real and Imaginary Parts

Let $z = x + yj \in \mathbb{C}$ with $x, y \in \mathbb{R}$, we call x the real part and y the imaginary part of z.

↳ $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$

If $x = 0$, $z = yj$ is purely imaginary.

If $y = 0$, $z = x$ is purely real.

Lecture 2

Algebraic operations on complex numbers

↳ Def: **Equality of complex numbers**

↳ let $z = x + yj$, $w = u + vj$ with $x, y, u, v \in \mathbb{R}$ if and only if $x = u$ and $y = v$, aka iff $\mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$

↳ Def: **Operations on complex numbers**

↳ let $z = w + yj$ and $w = u + vj$.

↳

$$z + w = (x + yj) + (u + vj) = (x + u) + (y + v)j$$

$$z - w = (x + yj) - (u + vj) = (x - u) + (y - v)j$$

$$zw = (x + yj)(u + vj) = (xu - yv) + (xv + yu)j$$

Example: $z = 3 - 2j, \quad w = -2 + j$

$\hookrightarrow \; z + w = (3 - 2) + (-2 + 1)j = 1 - j$

$z - w = (3 + 2) + (-2 - 1)j = 5 - 3j$

$zw = (-6 + 2) + (3 + 4)j = -4 + 7j$

Conjugates: the conjugate of $z = x + yj$ is $\bar{z} = x - yj$.

the conjugate of $z = x$ is $\bar{z} = x$.

the conjugate of $z = yj$ is $\bar{z} = -yj$.

Division: to divide $z = x + yj$, $z \neq 0$, multiply numerator and denominator by $\bar{z}$.

$\hookrightarrow$ **Example:** $\frac{1}{z}$. $\quad = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

$\hookrightarrow \; \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{x - yj}{(x + yj)(x - yj)} = \frac{x - yj}{x^2 - (yj)^2} = \frac{x - yj}{x^2 + y^2}$

Example: $\frac{7 + j}{3 - j}$

$\hookrightarrow \; \frac{7 + j}{3 - j} \cdot \frac{3 + j}{3 + j} = \frac{(7 + j)(3 + j)}{9 + 1} = \frac{20 + 10j}{10} = 2 + j.$

Properties of conjugates:

- Let $z, w \in \mathbb{C}$ with $z = x + yj$ where $x, y \in \mathbb{R}$.

$\hookrightarrow \; \bar{\bar{z}} = z$

$\hookrightarrow \; z + \bar{z} = 2x = 2\mathrm{Re}(z)$

$\hookrightarrow \; z - \bar{z} = 2j = 2\mathrm{Im}(z)$

$\hookrightarrow \; z \in \mathbb{R} \iff \bar{z} = z$

$\hookrightarrow$ z is purely imaginary $\iff \bar{z} = -z$.

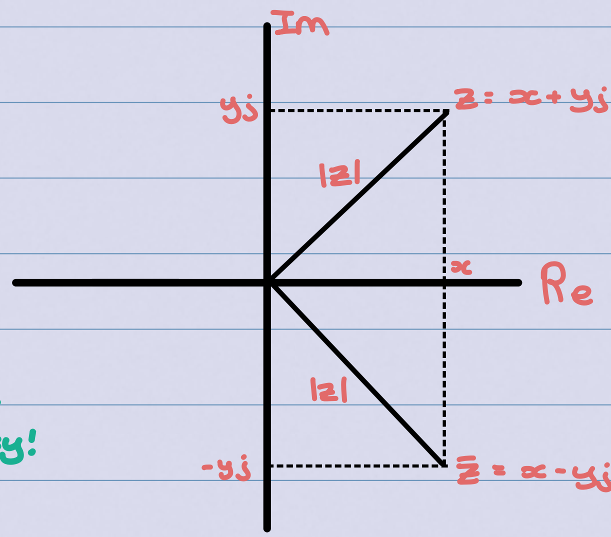
↳ $\overline{z+w} = \bar{z} + \bar{w}$
↳ $\overline{zw} = \bar{z}\,\bar{w}$
↳ $\overline{z^k} = \bar{z}^k$ for $k \in \mathbb{R}$ ($k > 0$ if $z = 0$).
↳ $z\bar{z} = x^2 + y^2$.

Modulus: The modulus of $z = x + yj$ with $x, y \in \mathbb{R}$ is the nonnegative real number $|z| = \sqrt{x^2 + y^2}$.

↳ **Example:** $z = 3 + 4j \rightarrow |z| = \sqrt{3^2 + 4^2} \rightarrow |z| = 5$

↳ **Properties:**

- $|z| = 0$
- $|\bar{z}| = |z|$
- $|z\bar{z}| = |z|^2$
- $|zw| = |z| \cdot |w|$
- $|z + w| \leq |z| + |w|$ → triangle inequality!



Polar form

↳ Let $|z| = r > 0$, and θ denotes the angle between the line segment and θ (from 0 to z).
↳ r is the radius and θ is the argument of z.

↳ **Polar form of a complex number is:**

$$z = r(\cos\theta + j\sin\theta)$$

where $r = |z|$ and θ is an argument of z.

aka, $z = r(\cos(\theta + 2k\pi) + j\sin(\theta + 2k\pi))$.

$\hookrightarrow x = r\cos\theta$ $y = r\sin\theta$

$\hookrightarrow \cos\theta = x/r$ $\hookrightarrow \sin\theta = y/r$

$\hookrightarrow$ Example: write $z = 1+\sqrt{3}j$ in polar form

$\hookrightarrow r = |z| = \sqrt{1+\sqrt{3}^2} = 2.$

$\hookrightarrow z = 2\left(\frac{x}{2} + \frac{y}{2}j\right)$

$\hookrightarrow z = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}j\right)$

$\hookrightarrow \cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}$

$\hookrightarrow \theta = \pi/3.$

$\therefore z = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right).$

Multiplication in Polar Form:

$z_1 = r_1(\cos\theta_1 + j\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + j\sin\theta_2).$

$\hookrightarrow z_1 z_2 = r_1(\cos\theta_1 + j\sin\theta_1)\,r_2(\cos\theta_2 + j\sin\theta_2)$

$r_1 r_2(\cos\theta_1\cos\theta_2 + j\cos\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2 + j^2\sin\theta_1\sin\theta_2)$

$\hookrightarrow r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + j(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$\hookrightarrow z_1 \times z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2))$

Example: $r_1 = 7$, $\theta_1 = \pi/2$, $r_2 = 3$, $\theta_2 = 8\pi/3$.

↳ $z_1 = 7\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)$, $z_2 = 3\left(\cos\frac{8\pi}{3} + j\sin\frac{8\pi}{3}\right)$.

↳ $z_1 \times z_2 = 21\left(\cos\left(\frac{17\pi}{6}\right) + j\sin\left(\frac{17\pi}{6}\right)\right)$.

Division in Polar Form:

↳ $z_1/z_2 = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)\right)$

↳ where $z_2 \neq 0$!

Powers of complex numbers:

↳ let $z = r(\cos\theta + j\sin\theta)$

↳ $z^2 = r^2(\cos\theta + j\sin\theta)(\cos\theta + j\sin\theta)$

↳ $z^2 = r^2(\cos(2\theta) + j\sin(2\theta))$

↳ $z^3 = z^2 \cdot z$

↳ $z^3 = r^3(\cos(3\theta) + j\sin(3\theta))$

↳ $z^{-1} = 1/z$

↳ $z^{-1} = \frac{1}{r}(\cos(-\theta) + j\sin(-\theta))$.

↳ $z^n = r^n(\cos(n\theta) + j\sin(n\theta))$ → De Moivre's Theorem!

Example : $(1-j)^8$ in standard form

↳ $z = 1 - j$ (standard form)

$r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

$\cos\theta = x/r = 1/\sqrt{2} = \sqrt{2}/2$.
$\sin\theta = y/r = -1/\sqrt{2} = -\sqrt{2}/2$.
$\Big\} \theta = -\pi/4$.

↳ $z = \sqrt{2}\left(1/\sqrt{2} - 1/\sqrt{2}\, j\right)$

↳ $z = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + j\sin(-\pi/4)\right)$.

↳ $z^8 = \sqrt{2}^8\left(\cos(-2\pi) + j\sin(-2\pi)\right)$

↳ $z^8 = 16(1 + j0)$

↳ $z^8 = 16$

n^{th} Roots of complex numbers : $w^n = z$

↳ let $z = r(\cos\theta + j\theta)$, $w = R(\cos\phi + j\sin\phi)$

↳ $w^n = R^n(\cos(n\phi) + j\sin(n\phi)) = r(\cos\theta + j\sin\theta)$.

↳ $R^n = r$ and $n\phi = \theta + 2\pi k$

↳ $\phi = \frac{\theta + 2\pi k}{n}$

$\hookrightarrow \omega_k = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + j\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$ → for $k = 0, 1, \ldots, n-1$)

$\hookrightarrow$ where $n = 5$ for 5^{th} root.

Example: 3^{rd} root of -1 $\forall\, \omega \in \mathbb{C}$ st. $\omega^3 = -1$.

$\hookrightarrow k = 0, 1, 2.$

$\hookrightarrow \omega_0 = r^{1/3}\left(\cos\left(\frac{\pi + 2\pi 0}{3}\right) + j\sin\left(\frac{\pi + 2\pi 0}{3}\right)\right).$

$\hookrightarrow \omega_1 = -1$

$\hookrightarrow \omega_2 = 1/2 - \frac{\sqrt{3}}{2} j.$

$k = 0, 1, \ldots, n-1$ because it starts looping again!

Complex Exponentials

$\hookrightarrow$ Let $\theta \in \mathbb{R}$. The expression $e^{j\theta}$ is $e^{j\theta} = \cos\theta + j\sin\theta.$

$\hookrightarrow \therefore re^{j\theta} = r(\cos\theta + j\sin\theta) = r(\cos(\theta + 2k\pi) + j\sin(\theta + 2k\pi))$

$\hookrightarrow z_1 \cdot z_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)} = r_1 r_2(\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$

Powers

$\hookrightarrow z^n = (re^{j\theta})^n = (r(\cos\theta + j\sin\theta))^n \to z^n = r^n e^{jn\theta}.$

Euler's identity

↳ let $z=-1$. → standard form

$\cos\theta = \frac{x}{r}$, $\sin\theta = 0$

↳ $\theta = \pi$

so, $z = \cos(\pi)$ → polar form

↳ $\cos(\pi) = e^{j\pi}$ → exponential form

↳ $e^{j\pi} + 1 = 0$.

Example: find the 6^{th} root of -64 in standard form.

↳ $-64 = 64(\cos(\pi + 2\pi k) + j\sin(\pi + 2\pi k)) = 64e^{jk}$.

↳ $w_k = 64^{1/6} e^{j\frac{\pi + 2\pi k}{6}}$, $k = 0, 1, 2, 3, 4, 5$.

↳ $w_0 = 2e^{j\pi/6} = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) = \sqrt{3} + j$.

$w_1 = 2e^{j\pi/2} = 2(0 + j) = 2j$

$w_2 = 2e^{j5\pi/6} = -\sqrt{3} + j$

$w_3 = -\sqrt{3} - j$

$w_4 = -2j$

$w_5 = \sqrt{3} - j$

Complex Polynomials

↳ A polynomial $P(x)$ of degree n is given by

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ $(a_n \neq 0)$.

↳ assuming coefficients $\in \mathbb{C}$.

A value C is a root of $P(x)$ if $P(C) = 0$, aka, $(x-c)$ is a factor of $p(x)$.

If $\forall(a_0, a_1, \ldots, a_{n-1}, a_n) \in \mathbb{R}$, real polynomial! $\mathbb{R} \in \mathbb{C}$.

If $\exists(a_0, a_1, \ldots, a_{n-1}, a_n) \in \mathbb{C}$, complex polynomial!

complex polynomial, $n=3$.

Example: $P(z) = \underbrace{j}_{a_3}z^3 - \underbrace{(1-j)}_{a_2}z^2 + \underbrace{3}_{a_1}z + \underbrace{(3-j)}_{a_0}$

$a_n \ldots a_0,\ b_n \ldots b_0 \in \mathbb{C}$

Two polynomials
$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z_1 + a_0$$
$$Q(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z_1 + b_0$$
are equal if $a_n = b_n,\ a_{n-1} = b_{n-1},\ \ldots,\ a_1 = b_1,\ a_0 = b_0$

$P(z) + Q(z) = (P+Q)(z)$

↳ $(a_n + b_n)z^n + \ldots + (a_1 + b_1)z^1 + (a_0 + b_0)$.

Scalar Multiplication: let $c \in \mathbb{C}$.

↳ $(cP)(z) = c\,P(z) = c a_n z^n + \ldots + c a_1 z^1 + c a_0$.

Theorem 5.1 (Fundemental theorem of algebra)

↳ let $p(z)$ be a complex polynomial of degree at least one. Then $p(z)$ has at least one complex root.

Corolary 5.1

↳ let $p(z)$ be a complex polynomial of degree $n \geq 1$. Then $p(z)$ has exactly n complex roots, including multiplicities.

Theorem 5.2 (Conjugate Root Theorem)

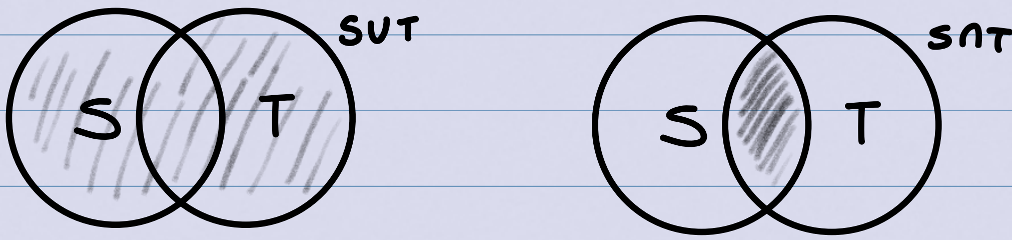
↳ Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a real polynomial. If $\omega \in \mathbb{C}$ is a root of $p(x)$, then $\bar{\omega}$ is as well.

Set Theory

Unions and Intersections:

↳ Let S, T be sets. The union of S and T is the set $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

↳ The intersection of sets S and T is:
$S \cap T = \{x \mid x \in S \text{ AND } x \in T\}$

↳ 

Lecture 6:

↳ Vector Algebra

$$3\hat{i} + 5\hat{j} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \ x_1, x_2, x_3, \ldots x_n \in \mathbb{R} \right\}$$

Example: $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$

Zero Vector

↳ the zero vector in $\mathbb{R}^n$ is defined by:

↳ $\vec{0}_{\mathbb{R}^n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0_n \end{bmatrix}$ eg. $\vec{0}_{\mathbb{R}^2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Addition of Vectors:

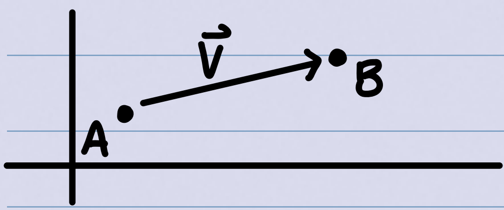
↳ $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ + \\ \vdots \\ + \\ x_n + y_n \end{bmatrix}$

↳ Example: $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$

Two nonzero vectors are parallel if they are scalar multiples.

↳ eg. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is parrellel to $\begin{bmatrix} 10 \\ 30 \end{bmatrix}$

Vectors between Points:



p is in standard position:

$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \overrightarrow{OP}$

↳ from the origin

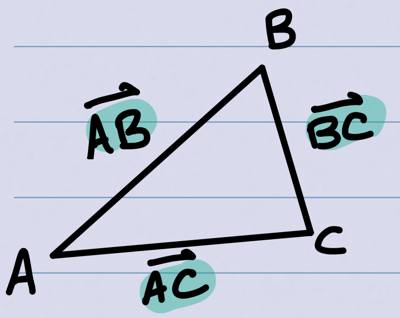
For points (a_1, a_2) and $B\ (b_1, b_2)$

↳ $\overrightarrow{AB} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\overrightarrow{OB}} - \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\overrightarrow{OA}} = \underbrace{\begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix}}_{\overrightarrow{AB}}$ → holds in general in $\mathbb{R}^n$.

Find the vector from: $A(1,4,3)$ and $B(5,0,2)$

$\hookrightarrow \vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix}$

Given 3 points A, B, C



$\vec{AC} = \vec{OC} - \vec{OA} = (\vec{OB} - \vec{OA}) + (\vec{OC} - \vec{OB})$

$\hookrightarrow \vec{AB} + \vec{BC} = \vec{AC}$

Also, $\vec{AB} = \vec{AC} - \vec{BC} = -\vec{CA} - (-\vec{CB}) = -\vec{CA} + \vec{CB} = \vec{CB} - \vec{CA}$

Theorem 6.1:

$\hookrightarrow$ let $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and let $c, d \in \mathbb{R}$.

We have:

1) $\vec{x} + \vec{y} \in \mathbb{R}^n$
2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
3) $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + \vec{y} + \vec{w}$
4) $\exists$ a vector $\vec{0} \in \mathbb{R}^n$ st $\vec{v} + \vec{0} = \vec{v}$ $\forall \vec{v} \in \mathbb{R}^n$.
5) $\forall \vec{x} \in \mathbb{R}^n$, $\exists (-x) \in \mathbb{R}^n$ st $\vec{x} + (-\vec{x}) = 0$
6) $c(\vec{x}) \in \mathbb{R}^n$
7) $c(d\vec{x}) = cd(\vec{x}) \in \mathbb{R}^n$.
8) $(c+d)\vec{x} = c\vec{x} + d\vec{x}$

Linear Combination

↳ let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{R}^n$ and $c_1, c_2, \dots, c_k \in \mathbb{R}$ for some positive integer k. We call the vector $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k$.

Let $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ and $\vec{w}$ be $\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ be two vectors in $\mathbb{C}^n$.

↓

Norm: $(\|\vec{z}\|) = \sqrt{z_1^2 + \dots + z_n^2}$

Complex inner product of $\vec{z}$ and $\vec{w}$:

$\langle \bar{z}, \bar{w} \rangle = \bar{z}_1 w_1 + \dots + \bar{z}_n w_n$

Dot product of $\vec{z}$ and $\vec{w}$:

$\vec{z} \cdot \vec{w} = z_1 w_1 + \dots + z_n w_n$.

Properties of Norms:

- $\|\vec{x}\| \geq 0$ iff $\vec{x} = 0$.
- $\|c\vec{x}\| = |c| \times \|\vec{x}\|$.
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle property)

Unit Vector: $\vec{x} \in \mathbb{R}^n$ is a unit vector if $\|\vec{x}\| = 1$

Example: Find the unit vector of $\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

↳ $\|\vec{x}\| = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{77}$.

↳ $\vec{y} = \frac{1}{\sqrt{77}} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

Dot Product Example: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -4 \\ 5 \end{bmatrix}$

$\hookrightarrow (1)(-3) + (1)(-4) + (2)(5)$

$\quad \hookrightarrow -3 - 4 + 10$

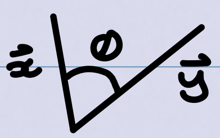
$\quad\quad \hookrightarrow 3.$

Properties of Dot Products:

(let $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$).

- $\vec{x} \cdot \vec{y} \in \mathbb{R}$
- $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
- $\vec{x} \cdot 0 = 0$
- $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2$!!!
- $(c\vec{x}) \cdot \vec{y} = c(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (c\vec{y})$.
- $\vec{w} \cdot (\vec{x} \pm \vec{y}) = \vec{w} \cdot \vec{x} \pm \vec{w} \cdot \vec{y}$.

Theorem 7.3: For two nonzero vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ determining an angle θ,



$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos \theta.$$

Theorem 7.4: Cauchy - Schwarz Inequality:

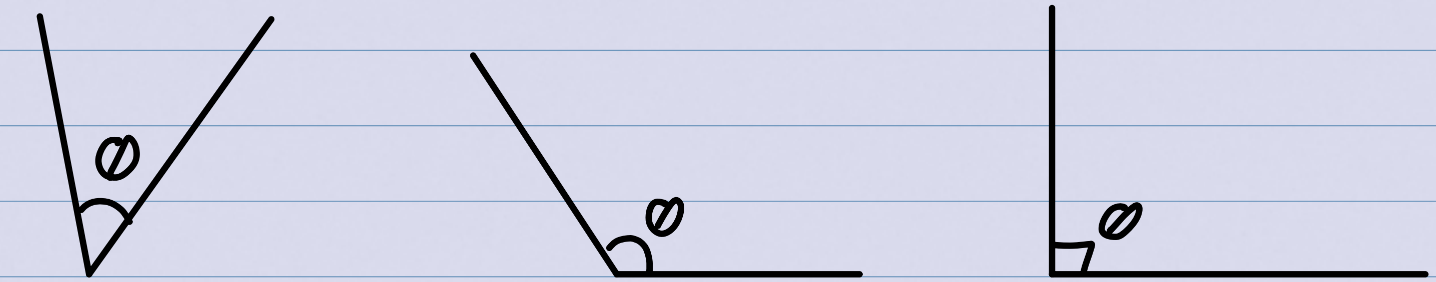
For any two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$

Example 7.6: Find angle between $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$$\hookrightarrow \cos \theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \frac{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}{\left\| \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \right\|} = \frac{(2)(1) + (-1)(1) + (-1)(-2)}{\sqrt{2^2 + 1^2 + (-1)^2} \cdot \sqrt{1^2 + (-1)^2 + (-2)^2}}.$$

↳ $\dfrac{2-1+2}{\sqrt{6}\sqrt{6}} = \dfrac{3}{6} = \dfrac{1}{2}$.

↳ $\cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}$.



Acute	Obtuse	Orthogonal
$0 < \theta < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta < \pi$	$\theta = \frac{\pi}{2}$.
$\vec{x}\cdot\vec{y} > 0$	$\vec{x}\cdot\vec{y} < 0$	$\vec{x}\cdot\vec{y} = 0$

Example: Find if $\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}6\\-2\end{bmatrix}$ is acute, obtuse, etc:

↳ $(1)(6)+(2)(-2) = 6-4 = 2 > 0$, so acute!

Definition 8.1: $\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \;\middle|\; z_1, \dots, z_n \in \mathbb{C} \right\}$

Definition 8.2: the zero vector in $\mathbb{C}^n$ is denoted by $\vec{0}_{\mathbb{C}^n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, or just $\vec{0}$.

8.4: the norm of a vector in $\mathbb{C}^n$ is
$\|\vec{x}\| = \sqrt{\bar{x}_1 x + \bar{x}_2 x_2 + \dots + \bar{x}_n x_n}$

the complex inner product of vectors in $\mathbb{C}^n$:
$\langle \vec{z}, \vec{w} \rangle = \bar{z}_1 w_1 + \dots + \bar{z}_n w_n$

the dot product of vectors in $\mathbb{C}^n$:
$\vec{z} \cdot \vec{w} = z_1 w_1 + \ldots + z_n w_n$

Example: compute $\vec{z} \cdot \vec{z}$, $\langle \vec{z}, \vec{z} \rangle$, and $\|\vec{z}\|$ for $\vec{z} = \begin{bmatrix} 1+j \\ 2-j \end{bmatrix}$

↳ $\vec{z} \cdot \vec{z} = (1+j)(1+j) + (2-j)(2-j) = 3 - 2j$

↳ $\langle \vec{z}, \vec{z} \rangle = (\overline{1+j})(1+j) + (\overline{2-j})(2-j)$
$= (1-j)(1+j) + (2+j)(2-j) = 7$

↳ $\|\vec{z}\| = \sqrt{(\overline{1+j})(1+j) + (\overline{2-j})(2+j)} = \sqrt{7}$

Properties of norms and complex inner products:
↳ let $\vec{v}, \vec{w}, \vec{z} \in \mathbb{C}^n$ and $a \in \mathbb{C}$. Then,

1) $\langle \vec{z}, \vec{z} \rangle \geq 0$
2) $\|\vec{z}\|^2 = \langle \vec{z}, \vec{z} \rangle$
3) $\langle \vec{z}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{z} \rangle}$
4) $\langle \vec{v} + \vec{w}, \vec{z} \rangle = \langle \vec{v}, \vec{z} \rangle + \langle \vec{w}, \vec{z} \rangle$
5) $\langle a\vec{z}, \vec{w} \rangle$ and $\langle \vec{z}, a\vec{w} \rangle$ both $= a \langle \vec{z}, \vec{w} \rangle$.
6) $|\langle \vec{z}, \vec{w} \rangle| \leq \|\vec{z}\| \|\vec{w}\|$ → cauchy-schwarz inequality
7) $\|\vec{z} + \vec{w}\| \leq \|\vec{z}\| + \|\vec{w}\|$ → triangle inequality

→ aka product vector!
Cross Product in $\mathbb{R}^3$:

↳ let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be two vectors in $\mathbb{R}^3$.

The cross product of $\vec{x}$ and $\vec{y}$ is:

$$\hookrightarrow \vec{x} \times \vec{y} = \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ -(x_1 y_3 - y_1 x_3) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

Example: $\vec{x} = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

$$\hookrightarrow \vec{x} \times \vec{y} = \begin{bmatrix} 6(2) - 3(3) \\ -(1(2) - (-1)(3)) \\ 1(3) - (-1)(6) \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 9 \end{bmatrix}$$

Properties of Cross Products:

$\hookrightarrow$ let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^3$, $c \in \mathbb{R}$.

1) $\vec{x} \times \vec{y} \in \mathbb{R}^3$
2) $\vec{x} \times \vec{y}$ is orthogonal to both $\vec{x}$ and $\vec{y}$.
3) $\vec{x} \times \vec{0} = \vec{0} = \vec{0} \times \vec{x}$.
4) $\vec{x} \times \vec{x} = \vec{0}$
5) $\vec{x} \times \vec{y} = -(\vec{y} \times \vec{x})$
6) $(c\vec{x}) \times \vec{y} = c(\vec{x} \times \vec{y}) = \vec{x} \times (c\vec{y})$
7) $\vec{w} \times (\vec{x} \pm \vec{y}) = (\vec{w} \times \vec{x}) \pm (\vec{w} \times \vec{y})$
8) $(\vec{x} \times \vec{y}) \times \vec{w} = (\vec{x} \times \vec{w}) \pm (\vec{y} \times \vec{w})$

not the same!

Theorem 9.2 → Lagrange Identity

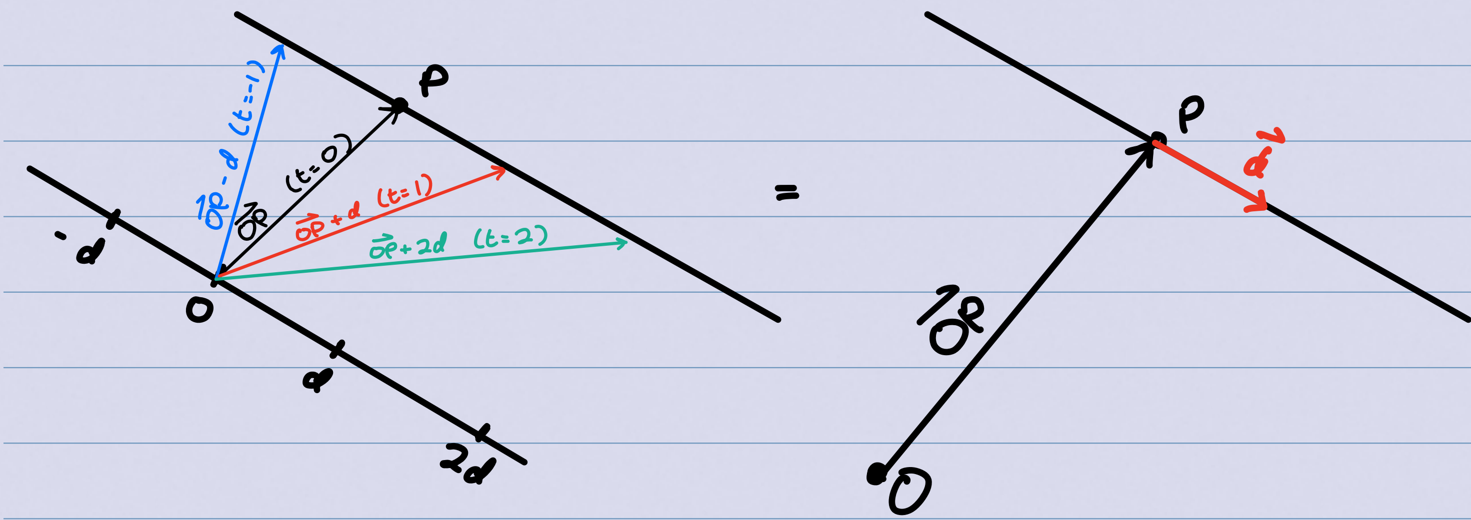
$\hookrightarrow$ Let $\vec{x}, \vec{y} \in \mathbb{R}^3$. Then $\|\vec{x} \times \vec{y}\|^2 = \|\vec{x}\|^2 \|\vec{y}\|^2 - (\vec{x} \cdot \vec{y})^2$.

$\hookrightarrow \|\vec{x} \times \vec{y}\| = \|\vec{x}\| \|\vec{y}\| \sin\theta$ → area of a parellelogram!

Definition 9.1: Vector Equation of a Line

↳ a line in $\mathbb{R}^n$ through a point P with direction $\vec{d}$, where $\vec{d} \in \mathbb{R}^n \neq 0$, is given by the vector equation:

↳ $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \overrightarrow{OP} + t\vec{d}$, $t \in \mathbb{R}$.



Example: Find the vector equation of the line through points $A(1, 1, -1)$ and $B(4, 0, -3)$

↳ $\vec{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

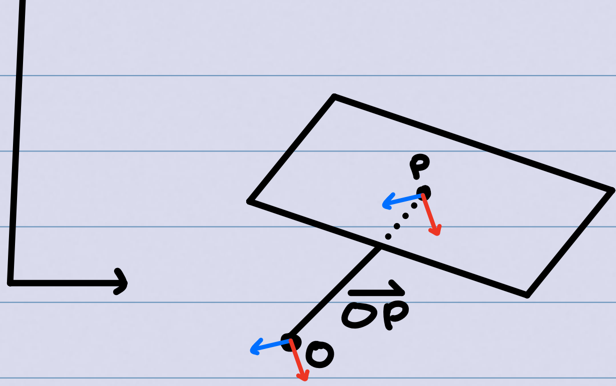
then, using point A:

↳ $\vec{x} = \overrightarrow{OA} + t\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, $t \in \mathbb{R}$.

Vector Equation of a Plane

↳ The vector equation for a plane in $\mathbb{R}^n$ through a point P is given by $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \overrightarrow{OP} + s\vec{u} + t\vec{v}$, $s, t \in \mathbb{R}$.

↳ $\vec{u}, \vec{v}$ are nonzero nonparallel vectors.



Example: find a vector equation for the plane containing points $A(1,1,1)$, $B(1,2,3)$, and $C(-1,1,2)$

↳ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

nonzero and nonparallel!

$s, t \in \mathbb{R}$.

↳ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overrightarrow{OA} + s\overrightarrow{AB} + t\overrightarrow{AC} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Note: if we set either s or t to 0 and let the other be arbitrary, we get vector equations for two lines, each of which lie in the plane:

or replace w/ t

$\vec{x} = \overrightarrow{OA} + s\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad s \in \mathbb{R}$

Example: Find a vector equation of the plane containing the point $P(1,-1,2)$ and the line with vector equation: $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $r \in \mathbb{R}$.

↳ first, we need to find two vectors in the plane.

1) take the direction vector of the given line
2) take a vector from a known point on the given line to point P.

↳ $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -3 \end{bmatrix}$

↳ $\vec{x} = \vec{OP} + s\vec{u} + t\vec{v} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + t\begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$, $s, t \in \mathbb{R}$.

The Scalar Equation of a Plane in $\mathbb{R}^3$

Definition 10.2: Normal vector on a plane:

↳ a nonzero vector $\vec{n} \in \mathbb{R}^3$ is a normal vector for a plane if for any two points P and Q on the plane, $\vec{n}$ is orthogonal to $\vec{PQ}$.

Note: consider a plane in $\mathbb{R}^3$ with $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$. and suppose $P(a, b, c)$ is a given point on this plane. For any point $Q(x_1, x_2, x_3)$, Q lies on the plane if and only if:

↳ $0 = \vec{n} \cdot \vec{PQ} = \vec{n}(\vec{OQ} - \vec{OP}) = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 - a \\ x_2 - b \\ x_3 - c \end{bmatrix}$

↳ $= n_1(x_1 - a) + n_2(x_2 - b) + n_3(x_3 - c)$

Definition 10.3: Scalar Equation of a Plane:

↳ The scalar equation of a plane in $\mathbb{R}^3$ with $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$

containing the point $P(a, b, c)$ is given by:

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = n_1 a + n_2 b + n_3 c.$$

Example 10.3: Find a scalar equation of the plane containing the points $A(3,1,2)$, $B(1,2,3)$, and $C(-2,1,3)$.

↳ $\vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}-2\\1\\1\end{bmatrix}$

→ nonzero and nonparallel

$\vec{AC} = \vec{OC} - \vec{OA} = \begin{bmatrix}-2\\1\\3\end{bmatrix} - \begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}-5\\0\\1\end{bmatrix}$

cross product!!

↳ $\vec{n} = \vec{AB} \times \vec{AC} = \begin{bmatrix}-2\\1\\1\end{bmatrix} \times \begin{bmatrix}-5\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-3\\5\end{bmatrix}$

↳ $n_1 x_1 + n_2 x_2 + n_3 x_3 = n_1 a + n_2 b + n_3 c.$

↳ $1(x_1 - 3) - 3(x_2 - 1) + 5(x_3 - 2) = 0$

↳ x_1, x_2, x_3 from $A(3,1,2)$.

↳ $x_1 - 3x_2 + 5x_3 = 10.$

Note: given a vector equation $\vec{x} = \vec{OP} + s\vec{u} + t\vec{v}$ for a plane in $\mathbb{R}^3$ containing a point P, we can find a normal vector $\vec{n} = \vec{u} \times \vec{v}$

↳ cross product!

Definition 10.4: Parrellel Lines in Parrellel Planes.

↳ Two lines in $\mathbb{R}^n$ are parrellel if their direction vectors are parrellel. Two planes in $\mathbb{R}^3$ are parellel if their normal vectors are parrellel.

Hyperplanes:

Definition 10.5: Scalar Equation of a Hyperplane

↳ a hyperplane in $\mathbb{R}^n$ is defined by the scalar equation $a_1x_1 + \dots + a_nx_n = d$ where $a_1, \dots a_n, d \in \mathbb{R}$. The vector $\vec{n} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is called a normal vector for the hyperplane. Given the point $P(p_1, \dots, p_n)$ lying on the hyperplane, $d = a_1p_1 + \dots + a_np_n$.

↳ In $\mathbb{R}^2$, a hyperplane is a line. In $\mathbb{R}^3$, it's a plane.

Exercise: Find the vector equation and scalar equation for the line in $\mathbb{R}^2$ passing through $P(1,-3)$ and $Q(2,4)$.

↳ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

↳ $\vec{x} = \overrightarrow{OP} + t\overrightarrow{PQ} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 7 \end{bmatrix}, \quad t \in \mathbb{R}.$

Since $\vec{n} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ is orthogonal to $\overrightarrow{PQ}$, our scalar equation is in the form $7x_1 - x_2 = d$.

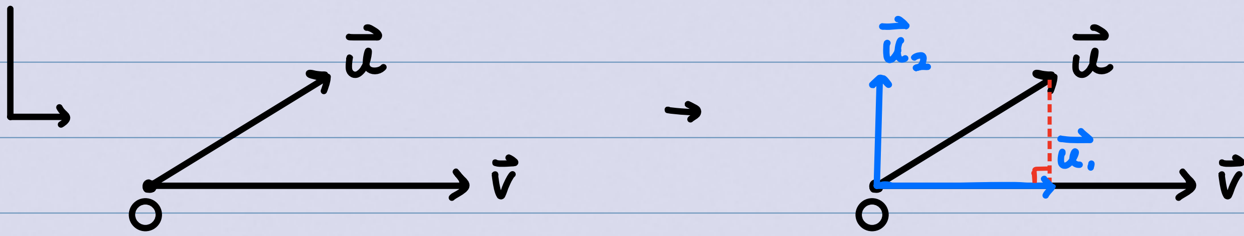
↳ using point P, $d = (7)(1) - 1(-3) = 10$, so our scalar equation is $7x_1 - x_2 = 10$

Note: given a vector equation of a line in $\mathbb{R}^2$, $\vec{x} = \overrightarrow{OP} + t\vec{d}, \quad t \in \mathbb{R}$ with $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \neq \vec{0}$, we can take $\vec{n} = \begin{bmatrix} d_2 \\ -d_1 \end{bmatrix}$ when finding the scalar equation.

Projections

↳ given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq 0$, we can write $\vec{u} = \vec{u}_1 + \vec{u}_2$ where $\vec{u}_1$ is a scalar multiple of $\vec{v}$ and $\vec{u}_2$ is orthogonal to $\vec{v}$.

↳ **note:** in physics, this is often done when you want to split a force into it's horizontal and vertical components.



↳ for $\vec{u}, \vec{v} \in \mathbb{R}$, $\vec{v} \neq \vec{0}$:

$\vec{u} = \vec{u}_1 + \vec{u}_2$

for some $t \in \mathbb{R}$.

$\vec{u}_2$ is orthogonal to $\vec{v}$, so: $\vec{u}_2 \cdot \vec{v} = 0$.

$\vec{u}_1$ is a scalar multiple of $\vec{v}$, so: $\vec{u}_1 = t\vec{v}$

↳ $0 = \vec{u}_2 \cdot \vec{v} = (\vec{u} - \vec{u}_1) \cdot \vec{v} = \vec{u} \cdot \vec{v} - \vec{u}_1 \cdot \vec{v} = \vec{u} \cdot \vec{v} - (t\vec{v}) \cdot \vec{v}$.

↳ $0 = \vec{u} \cdot \vec{v} - t(\vec{v} \cdot \vec{v}) = \vec{u} \cdot \vec{v} - t\,\|\vec{v}\|^2$

↳ since $\vec{v} \neq \vec{0}$, $t = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$.

Definition 11.1: Projection and Perpendicular

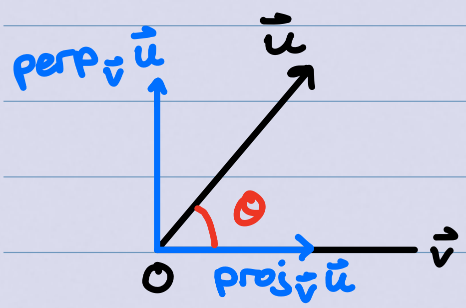
↳ let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. The projection of $\vec{u}$ onto $\vec{v}$ is:

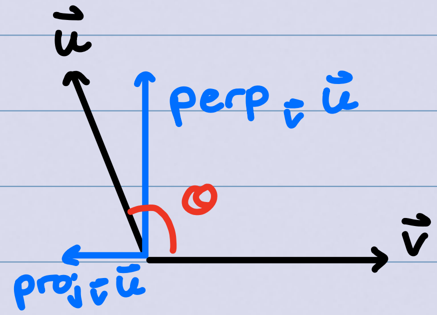
$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\,\vec{v}.$$

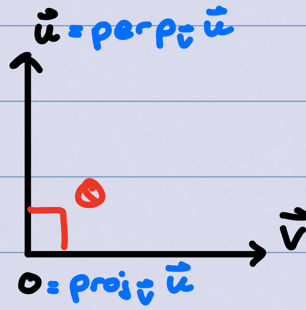
And the projection of $\vec{u}$ perpendicular to $\vec{v}$ (or the perpendicular of $\vec{u}$ onto $\vec{v}$) is:

$$\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u}.$$

↳ $\vec{u}_1 = \operatorname{proj}_{\vec{v}} \vec{u}$, and $\vec{u}_2 = \operatorname{perp}_{\vec{v}} \vec{u}$.







$0 \le \theta \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \theta \le \pi$	$\theta = \pi/2$
$\vec{u} \cdot \vec{v} > 0$	$\vec{u} \cdot \vec{v} < 0$	$\vec{u} \cdot \vec{v} = 0$
$\operatorname{proj}_{\vec{v}} \vec{u}$ in same direction as $\vec{v}$	$\operatorname{proj}_{\vec{v}} \vec{u}$ opposite direction as $\vec{v}$	$\operatorname{proj}_{\vec{v}} \vec{u} = \vec{0}$, $\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u}$.

Example 11.1: let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Find $\operatorname{perp}_{\vec{v}} \vec{u}$ and $\operatorname{proj}_{\vec{v}} \vec{u}$:

↳ $\operatorname{proj}_{\vec{v}} \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \dfrac{(-1)(1) + (2)(1) + (3)(2)}{\left(\sqrt{(-1)^2 + (1)^2 + (2)^2}\right)^2} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

↳ $\dfrac{-1+2+6}{1+1+4} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \dfrac{7}{6} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

↳ $\operatorname{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} -7/6 \\ 7/6 \\ 7/3 \end{bmatrix}$ → $\vec{u} = \operatorname{proj}_{\vec{v}} \vec{u} + \operatorname{perp}_{\vec{v}} \vec{u}$

$$\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -7/6 \\ 7/6 \\ 7/3 \end{bmatrix} = \begin{bmatrix} 13/6 \\ 5/6 \\ 2/3 \end{bmatrix}$$

The shortest distance from a point to a line:

↳ **Example 11.2:** Find the shortest distance from the point $P(1,2,3)$ to the line L which passes through the point $P_0(2,-1,2)$ with direction vector $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Also, find the point Q on L that is closest to P.

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

Projecting $\overrightarrow{P_0P}$ onto $\vec{d}$:

↳ $$\text{proj}_{\vec{d}} \overrightarrow{P_0P} = \frac{\overrightarrow{P_0P} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} = \frac{(-1)(1) + (3)(1) + (1)(-1)}{\sqrt{(1)^2 + (1)^2 + (-1)^2}^{\,2}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

↳ $$\text{proj}_{\vec{d}} \overrightarrow{P_0P} = \frac{-1+3-1}{1+1+1} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix}$$

it follows that: $\text{perp}_{\vec{d}} \overrightarrow{P_0P} = \overrightarrow{P_0P} - \text{proj}_{\vec{d}} \overrightarrow{P_0P} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} -4/3 \\ 8/3 \\ 4/3 \end{bmatrix}$

The shortest distance from P to L is given by:

↳ $$\|\text{perp}_{\vec{d}} \overrightarrow{P_0P}\| = \frac{1}{3}\sqrt{4^2 + 8^2 + 4^2} = \frac{1}{3}\sqrt{16(1+4+1)} = \frac{4}{3}\sqrt{6}.$$

To find Q: either use

↳ $$\overrightarrow{OQ} = \overrightarrow{OP_0} + \overrightarrow{\text{proj}_{\vec{d}} P_0P} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -2/3 \\ 5/3 \end{bmatrix}$$

or

↳ $$\overrightarrow{OQ} = \overrightarrow{OP} - \text{perp}_{\vec{d}} \overrightarrow{P_0P} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4/3 \\ 8/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -2/3 \\ 5/3 \end{bmatrix}$$

$\therefore Q\left(\frac{7}{3}, -\frac{2}{3}, \frac{5}{3}\right)$ is the closest point on L to P.

The shortest distance from a point to a plane:

Example 11.3: Find the shortest distance from the point $P(1,2,3)$ to the plane T with equation $x_1 + x_2 - 3x_3 = -2$. Also, find the point Q on T that is closest to P.

- Find a $P_0(x, y, z)$ that lies on T:
  ↳ $P_0(-2, 0, 0)$ works because $-2+0-3(0) = -2$

- Find the normal vector of the plane: $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

$$\text{proj}_{\vec{n}} \overrightarrow{P_0P} = \frac{\overrightarrow{P_0P} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} = \frac{(3)(1) + (2)(1) + (3)(-3)}{\left(\sqrt{(1)^2 + (1)^2 + (-3)^2}\right)^2} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

↳ $\frac{3+2-9}{1+1+9} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = -\frac{4}{11} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

- The shortest distance from P to T is:
  ↳ $\|\text{proj}_{\vec{n}} \overrightarrow{P_0P}\| = \left| -\frac{4}{11} \right| \sqrt{1+1+9} = \frac{4\sqrt{11}}{11}$.

To find Q:
↳ $\overrightarrow{OQ} = \overrightarrow{OP} - \text{proj}_{n} \overrightarrow{P_0P} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{4}{11} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 15/11 \\ 26/11 \\ 21/11 \end{bmatrix}$

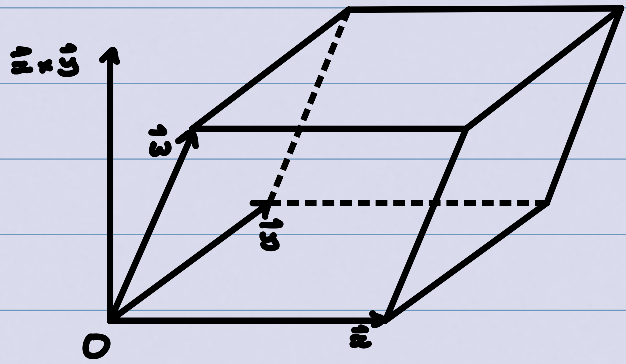
$\therefore Q\left(\frac{15}{11}, \frac{26}{11}, \frac{21}{11}\right)$ is the point on T closest to P.

Volumes of Parallelpipeds in $\mathbb{R}^3$.

↳ made from three non-zero vectors $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^3$ s.t they are nonzero, nonparellel, and none of them lies on the plane defined by the other two.

↳ basically, a 3-D parellelogram!

↳



Volume of a parellelipiped:

↳ $V = \underbrace{\|\vec{x} \times \vec{y}\|}_{\text{base}} \underbrace{\|\text{proj}_{\vec{x} \times \vec{y}} \vec{w}\|}_{\text{height}}$

↳ $V = \|\vec{x} \times \vec{y}\| \left\| \dfrac{\vec{w} \cdot (\vec{x} \times \vec{y})}{\|\vec{x} \times \vec{y}\|^2} (\vec{x} \times \vec{y}) \right\|$

↳ $V = \|\vec{x} \times \vec{y}\| \dfrac{|\vec{w} \cdot (\vec{x} \times \vec{y})|}{\|\vec{x} \times \vec{y}\|^2} \|\vec{x} \times \vec{y}\|$

↳ $V = |\vec{w} \cdot (\vec{x} \times \vec{y})|$

Example 11.4: let $\vec{w} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{x} = \begin{bmatrix}1\\1\\2\end{bmatrix}$, and $\vec{y} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$. Find the volume of the parellelipid they determine.

↳ $|\vec{w} \cdot (\vec{x} \times \vec{y})| = \left| \begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \left(\begin{bmatrix}1\\1\\2\end{bmatrix} \times \begin{bmatrix}1\\2\\-3\end{bmatrix} \right) \right| = \left| \begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-7\\5\\1\end{bmatrix} \right|$

↳ $|\vec{w} (\vec{x} \times \vec{y})| = V = |(1)(-7) + (1)(5) + (1)(1)| = 1$.

Note: the volume of a parellelipipide can also be written as:

$$V = |\vec{w} \cdot (\vec{y} \times \vec{x})| = |\vec{x} \cdot (\vec{w} \times \vec{y})| = |\vec{x} \cdot (\vec{y} \times \vec{w})| = |\vec{y} \cdot (\vec{x} \times \vec{w})| = |\vec{y} \cdot (\vec{w} \times \vec{x})|$$

Lecture 12:

Exercise 12.1: Find all points that lie on all three planes with scalar equations:

$$2x_1 + x_2 + 9x_3 = 31,$$
$$x_2 + 2x_3 = 8,$$
$$x_1 + 3x_3 = 10.$$

$\hookrightarrow x_1 = 10 - 3x_3$

$x_2 = 8 - 2x_3$

$\hookrightarrow 2(10 - 3x_3) + (8 - 2x_3) + 9x_3 = 31.$

$20 - 6x_3 + 8 - 2x_3 + 9x_3 = 31$

$28 + x_3 = 31$

$\hookrightarrow x_3 = 3.$

$x_1 = 10 - 3x_3 \rightarrow x_1 = 10 - 3(3) \rightarrow x_1 = 1.$

$x_2 = 8 - 2x_3 \rightarrow x_2 = 8 - 2(3) \rightarrow x_2 = 2.$

$\hookrightarrow x_1 = 1, \; x_2 = 2, \; x_3 = 3 \rightarrow (1, 2, 3).$

Definition 12.1: System of Linear Equations:

↳ A linear equation in n variables is an equation of the form: $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$. where $x_1, \dots, x_n \in \mathbb{R}$ are variables, $a_1, \dots, a_n \in \mathbb{R}$ are coefficients, and $b \in \mathbb{R}$ is some constant.

More generally:

↳ a system of m linear equations in n variables is written as:

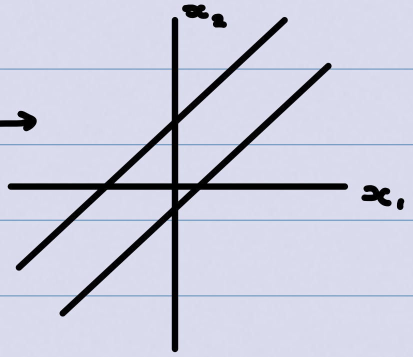
↳
$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2. \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_n. \end{aligned}$$

↳ the number a_{ij} is the coefficient of x_j in the ith equation and b_i is the constant term in the ith equation. Each of the m equations is a scalar equation of a hyperplane in $\mathbb{R}^n$.

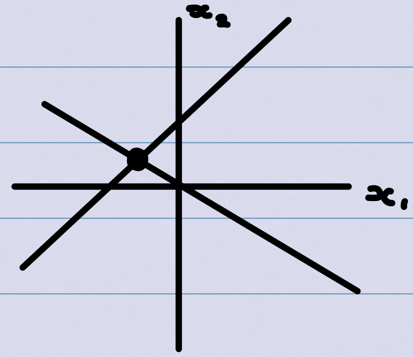
a system of two linear equations in two variables:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 &= b_1, \\ a_{21} x_1 + a_{22} x_2 &= b_2. \end{aligned}$$

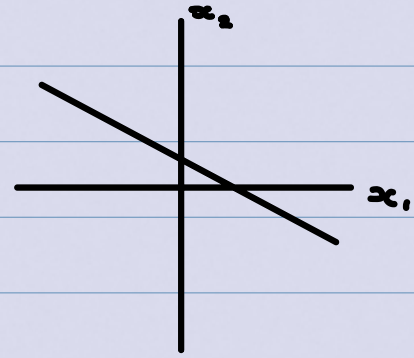
can be solved by finding the points of intersection:



lines are parallel and distinct, so no solutions!



lines are not parallel, so one unique solution!



lines are parallel but not distinct, so infinite solutions!

Definition 12.3: Consistent and Inconsistent:

↳ We call a linear system of equations consistent if it has at least one solution. Otherwise, we call the linear system inconsistent.

Example 12.3: solve the linear system:

$$x_1 + 3x_2 = -1$$
$$x_1 + x_2 = 3.$$

↳

$$x_1 + x_2 = 3$$
$$-(x_1 + 3x_2) = -1$$

subtracting first equation from second.

↳ $-2x_2 = 4$

multiply equation by $-\frac{1}{2}$.

↳ $x_2 = -2$

$x_1 + 3x_2 = -1$

↳ $x_1 + 3(-2) = -1$

↳ $x_1 = -1 + 6$

↳ $x_1 = 5$

subtract 3 times the second equation from the first.

$\therefore$, the system is consistent and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

↳ this solution can be written much more compactly as such:

↳ $\left[\underbrace{\begin{matrix} 1 & 3 \\ 1 & 1 \end{matrix}}_{\text{the coefficient matrix, } A} \middle| \overbrace{\begin{matrix} -1 \\ 3 \end{matrix}}^{\text{constant matrix, } \vec{b}} \right] \xrightarrow[R_2 - R_1]{} \left[\begin{matrix} 1 & 3 \\ 0 & -2 \end{matrix} \middle| \begin{matrix} -1 \\ -4 \end{matrix}\right] \xrightarrow[-\frac{1}{2}R_2]{} \left[\begin{matrix} 1 & 3 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} -1 \\ -2 \end{matrix}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 5 \\ -2 \end{matrix}\right]$

aka, $[A \mid \vec{b}]$.

Definition 12.4: Coefficient and Augmented Matricies, with Constant Vectors.

↳ For the system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m, \end{aligned}$$

the coefficient matrix is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The constant vector is $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

The augmented matrix is:

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Definition 12.5: Elementary Row Operations (EROs):

We are allowed to:

- Swap two rows
- Add a scalar multiple of one row to another
- Multiply any row by a nonzero scalar.

Example 12.4: Solve the linear system of equations:

$$\begin{aligned} 2x_1 + x_2 + 9x_3 &= 31 \\ x_2 + 2x_3 &= 8 \\ x_1 + 3x_3 &= 10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 9 & 31 \\ 0 & 1 & 2 & 8 \\ 1 & 0 & 3 & 10 \end{array}\right] \xrightarrow[R_3 \leftrightarrow R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 2 & 1 & 9 & 31 \end{array}\right]$$

swap row 1 and row 3!

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 2 & 1 & 9 & 31 \end{array}\right] \xrightarrow[R_3 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & 3 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & 3 & 11 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\substack{R_1 - 3R_3 \\ R_2 - 2R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

↳ $\therefore$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Lecture 13:

Definition 13.1: Row Echelon Form and Reduced Row Echelon Form:

- The first nonzero entry in each row of a matrix is called a leading entry or a pivot.
- A matrix is in Row Echelon Form (REF) if:
  - ↳ all rows whose entries are all 0 appear below all the rows that contain nonzero entries.
  - ↳ each leading entry is to the right of the leading entries above it.
- A matrix is in Reduced Row Echelon Form (RREF) if it is in REF and:
  - ↳ each leading entry is a 1 (called a leading one)
  - ↳ each leading one is the only nonzero entry in the column.

$\therefore$, if a matrix is in RREF, it is also in REF.

Example 13.1: Solve the system of linear equations:

$$\begin{aligned} 3x_1 + x_2 &= 10 \\ 2x_1 + x_2 + x_3 &= 6 \\ -3x_1 + 4x_2 + 15x_3 &= -20. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 10 \\ 2 & 1 & 1 & 6 \\ -3 & 4 & 15 & -20 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 2 & 1 & 1 & 6 \\ -3 & 4 & 15 & -20 \end{array}\right] \xrightarrow[R_3+3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 4 & 12 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - x_3 &= 4 \\ x_2 + 3x_3 &= -2 \\ 0x_3 &= 0 \end{aligned}$$

$\rightarrow$ no restriction on x_3, so let $x_3 = t \in \mathbb{R}$.

$$\left.\begin{aligned} x_1 &= 4 + t \\ x_2 &= -3t - 2 \\ x_3 &= t \end{aligned}\right\} t \in \mathbb{R}.$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Definition 13.2: Leading Variable and Free Variable:

↳ Let $[R \mid \vec{c}]$ be any REF of the augmented matrix $[A \mid \vec{b}]$ of some consistent system of linear equations. If the j^{th} column of R has a leading entry in it, then the variable x_j is called a leading variable. If the j^{th} column of R does not have a leading entry, then x_j is a free variable.

Example 13.2: Solve the linear system of equations:

$$\begin{aligned} x_1 + 6x_2 \qquad\quad - x_4 &= -1 \\ x_3 + 2x_4 &= 7 \end{aligned}$$

↳ $\left[\begin{array}{cccc|c} 1 & 6 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 7 \end{array}\right]$

↳ Since the leading entries are in the first and third columns, so x_1 and x_3 are leading variables while x_2 and x_4 are free variables.

↳ so, we have: $\left.\begin{aligned} x_1 &= -1 - 6s + t \\ x_2 &= s \\ x_3 &= 7 - 2t \\ x_4 &= t \end{aligned}\right\} s, t \in \mathbb{R}.$

or, as a vector equation: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Example 13.3: Solve the linear system of equations:

$$2x_1 + 12x_2 - 8x_3 = -4$$
$$2x_1 + 13x_2 - 6x_3 = -5$$
$$-2x_1 - 14x_2 + 4x_3 = 7$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 2 & 12 & -8 & -4 \\ 2 & 13 & -6 & -5 \\ -2 & -14 & 4 & 7 \end{array}\right] \begin{array}{c} \longrightarrow \\ R_2 - R_1 \\ R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 12 & -8 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 3 \end{array}\right]$$

$$\begin{array}{c} \longrightarrow \\ \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|c} 2 & 12 & -8 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\hookrightarrow \therefore \; 2x_1 + 12x_2 - 8x_3 = -4$$
$$x_2 + 2x_3 = -1$$
$$0 = 1$$

$\downarrow$

impossible!

$\therefore$ no solutions!

Example 13.4: Solve the linear system of equations:

$$jz_1 - z_2 - z_3 + (-1+j)z_4 = -1$$
$$-(1+j)z_3 - 2jz_4 = -1-3j$$
$$2jz_1 - 2z_2 - z_3 - (1-3j)z_4 = j$$

$$\left[\begin{array}{cccc|c} j & -1 & -1 & -1+j & -1 \\ 0 & 0 & -1-j & -2j & -1-3j \\ 2j & -2 & -1 & -1+3j & j \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{cccc|c} j & -1 & -1 & -1+j & -1 \\ 0 & 0 & -1-j & -2j & -1-3j \\ 0 & 0 & 1 & 1+j & 2+j \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{-jR_1} \left[\begin{array}{cccc|c} 1 & j & j & 1+j & j \\ 0 & 0 & 1 & 1+j & 2+j \\ 0 & 0 & -1-j & -2j & -1-3j \end{array}\right] \xrightarrow{R_3+(1+j)R_2} \left[\begin{array}{cccc|c} 1 & j & j & 1+j & j \\ 0 & 0 & 1 & 1+j & 2+j \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-jR_2} \left[\begin{array}{cccc|c} 1 & j & 0 & 2 & 1-j \\ 0 & 0 & 1 & 1+j & 2+j \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

→ z_1 and z_3 are leading variables.
z_2 and z_4 are free variables.

$$\therefore \left.\begin{aligned} z_1 &= (1-j) - js - 2t \\ z_2 &= s \\ z_3 &= (2+j) - (1+j)t \\ z_4 &= t \end{aligned}\right\} s,t \in \mathbb{C}$$

Lecture 14:

Definition 14.1: Rank

↳ The rank of matrix A, denoted by $\operatorname{rank}(A)$, is the number of leading entries in any REF of A.

Note: given a matrix and two of it's REFs, the number of leading entries in both of these REFs will be the same.

Example 14.1: find the ranks of the matrices:

$A = \left[\begin{array}{ccc|c} 2 & 1 & 9 & 31 \\ 0 & 1 & 2 & 8 \\ 1 & 0 & 3 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} \underline{1} & 0 & 3 & 10 \\ 0 & \underline{1} & 2 & 8 \\ 0 & 0 & \underline{1} & 3 \end{array}\right]$

augmented matrices

$B = \left[\begin{array}{cccc|c} 2 & 0 & 1 & 3 & 4 \\ 5 & 1 & 6 & -7 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \underline{1} & 1 & 4 & -13 & -5 \\ 0 & \underline{-2} & -7 & 29 & 14 \end{array}\right]$

coefficient matrix

$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{1} & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ rank$(A) = 3$, rank$(B) = 2$, rank$(C) = 1$.

Note: if a matrix A has m rows and n columns, then rank$(A) =$ min$\{m, n\}$.

Theorem 14.1: System-Rank Theorem

- Let $[A \mid \vec{b}]$ be the augmented matrix of a system of m linear equations in n variables.

1) The system is only consistent if rank$(A) =$ rank$([A \mid \vec{b}])$.
2) If the system is consistent, then the number of parameters in the general solution is the number of variables minus the rank of A.
   - # parameters $= n -$ rank(A).
3) The system is consistent $\forall \vec{b} \in \mathbb{R}^m$ iff rank$(A) = m$.

Example 14.2: Demonstrating the System-Rank Theorem:

↳ From example 12.4:

→ m=3 linear equations in n=3 variables

↳ $[A|\vec{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 9 & 31 \\ 0 & 1 & 2 & 8 \\ 1 & 0 & 3 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

↳ The System-Rank Theorem says that:

1) $\text{rank}(A) = 3 = \text{rank}([A|\vec{b}])$, so it's consistent.
2) # parameters $= n - \text{rank}(A) = 3 - 3 = 0$, so there are no parameters in the solution (unique solution).
3) $\text{rank}(A) = 3 = m$, so the system will be consistent for any $b \in \mathbb{R}^3$.
   ↳ aka, any $b_1, b_2, b_3 \in \mathbb{R}^3$.

Example 14.6: Find an equation that $b_1, b_2, b_3 \in \mathbb{R}$ must satisfy so that the following system is consistent.

$$2x_1 + 12x_2 - 8x_3 = b_1$$
$$2x_1 + 13x_2 - 6x_3 = b_2$$
$$-2x_1 - 14x_2 + 4x_3 = b_3$$

→ step 1 is to bring the augmented matrix to REF.

$$\left[\begin{array}{ccc|c} 2 & 12 & -8 & b_1 \\ 2 & 13 & -6 & b_2 \\ -2 & -14 & 4 & b_3 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 2 & 12 & -8 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & -2 & -4 & b_3 + b_1 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \\ \\ R_3+2R_2 \end{array} \left[\begin{array}{ccc|c} 2 & 12 & -8 & b_1 \\ 0 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & b_3+b_1+2(b_2-b_1) \end{array}\right]$$

↳ Since rank$(A)=2$, rank$([A|\vec{b}])$ must be 2 for the system to be consistent.

So, $b_3+b_1+2(b_2-b_1)$ must be 0.

↳ $b_3+b_1+2b_2-2b_1=0$

↳ $\therefore -b_1+2b_2+b_3=0.$

Example 14.7: For which values of the parameters $k, l \in \mathbb{R}$ does the following system have no solutions, a unique solution, and infinitely many solutions?

$$2x_1+6x_2=5$$
$$4x_1+(k+15)x_2=l+8.$$

$A=\begin{bmatrix} 2 & 6 \\ 4 & k+15 \end{bmatrix}$ and $\vec{b}=\begin{bmatrix} 5 \\ l+8 \end{bmatrix}$

Carry $[A|\vec{b}]$ to REF: $\left[\begin{array}{cc|c} 2 & 6 & 5 \\ 4 & k+15 & l+8 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} 2 & 6 & 5 \\ 0 & k+3 & l-2 \end{array}\right].$

↱ $2-\text{rank}(A)=0$ and $\text{rank}(A)=\text{rank}([A|\vec{b}])$.

$\therefore$ unique solution: $k\neq -3$

↱ $\text{rank}(A)\neq \text{rank}([A|\vec{b}])$.

no solutions: $k=-3$ and $l\neq 2$

↱ $\text{rank}(A)=\text{rank}([A|\vec{b}])$,

infinitely many solutions: $k=-3$ and $l=2$. $2-\text{rank}(A)=1$.

Definition 14.2: Undetermined Linear Systems of Equations.

↳ A linear system of m equations in n variables is undetermined if $n > m$

↳ aka, if there are more variables than equations.

Example 14.8:

$$x_1 + x_2 - x_3 + x_4 - x_5 = 1$$
$$x_1 - x_2 - 3x_3 + 2x_4 + 2x_5 = 7$$

this system is undetermined as $n = 5 > m = 2$.

Theorem 14.2:

↳ A consistent undetermined linear system of equations has infinitely many solutions.

Definition 14.3: Overdetermined Linear System of Equations

↳ A linear system of m equations in n variables is overdetermined if $n < m$.

↳ aka, if there are less variables than equations.

Example 14.9:

$$-2x_1 + x_2 = 2$$

$$x_1 - 3x_2 = 4$$
$$3x_1 + 2x_2 = 7.$$

This system is overdetermined as $n = 2 < m = 3$.

Note: overdetermined systems tend to be inconsistent.

Lecture 15:

Definition 15.1: Homogeneous Linear System of Equations:

↳ A homogeneous linear equation is a linear equation where the constant term is zero. A system of homogeneous linear equations is a collection of finitely many homogeneous equations.

A homogeneous system of m linear equations written in n variables is written as:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n &= 0.
\end{aligned}$$

Note: Every homogeneous system is consistent, as $x_1 = x_2 = \dots = x_n = 0$ always holds.

↳ this is called the trivial solution.

Example 15.1: Solve the homogeneous linear system:

$$x_1 + x_2 + x_3 = 0$$
$$3x_2 - x_3 = 0$$

↳ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/3 & 0 \end{array}\right]$

$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4/3 & 0 \\ 0 & 1 & -1/3 & 0 \end{array}\right].$

$\therefore \left.\begin{aligned} x_1 &= -\tfrac{4}{3}t \\ x_2 &= 1/3\, t \\ x_3 &= t \end{aligned}\right\} t \in \mathbb{R}$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$

Definition 15.2: Associated Homogeneous System of Linear Equations

↳ Given a non-homogeneous linear system of equations with augmented matrix $[A \mid \vec{b}]$ (so $\vec{b} \neq 0$), the homogeneous system with augmented matrix $[A \mid \vec{0}]$ is called the associated homogeneous system.

$$x_1 + x_2 + x_3 = 1$$
$$3x_2 - x_3 = 3$$

↓

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & -1 & 3 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4/3 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$

$\hookrightarrow$ $\left.\begin{aligned} x_1 &= -\tfrac{4}{3}t \\ x_2 &= 1 + \tfrac{1}{3}t \\ x_3 &= t \end{aligned}\right\} t \in \mathbb{R}.$

or

$$\overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{t\begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}}_{\text{associated homogeneous solution}}, \quad t \in \mathbb{R}.}^{\text{Solution to the system of equations}}$$

Example 15.2: $\begin{aligned} x_1 + 6x_2 - x_4 &= -1 \\ x_3 + 2x_4 &= 7 \end{aligned}$

the solution is: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix} + s\begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, s,t \in \mathbb{R}.$

The solution is as a plane through $(-1,0,7,0)$ in $\mathbb{R}^4$ since the vectors $\begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ are nonzero and not parallel.

$\hookrightarrow$ $\therefore$ the solution to the associated homogeneous system is:

$\hookrightarrow$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}.$ $\rightarrow$ a plane through the origin in $\mathbb{R}^4$!

Note: given two solutions of a homogeneous system of linear equations, say $\vec{x}_1$ and $\vec{x}_2$, any linear combination of them is also a solution to the system.

Example 15.3:

↳ consider a homogeneous system of m equations in n unknowns. Suppose $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ are solutions to this system.

↳ $c_1\vec{y} + c_2\vec{z}$ is also a solution $\forall\, c_1, c_2 \in \mathbb{R}$!

Lecture 16:

Application of homogeneous linear equations: Balancing Chemical Reactions!

photosynthesis!

Example: balance $CO_2 + H_2O \longrightarrow C_6H_{12}O_6 + O_2$.

↳ $x_1\, CO_2 + x_2\, H_2O \longrightarrow x_3\, C_6H_{12}O_6 + x_4\, O_2$.

↳ $C: \; x_1 = 6x_3$

$O: \; 2x_1 + x_2 = 6x_3 + 2x_4$

$H: \; 2x_2 = 12x_3$

$\hookrightarrow x_1 - 6x_3 = 0$

$2x_1 + x_2 - 6x_3 - 2x_4 = 0$

$2x_2 - 12x_3 = 0$

now, we reduced the augmented matrix to RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 1 & -6 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & -12 & 2 & 0 \end{array}\right] \xrightarrow{\cdot -\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_4 = t \in \mathbb{R}. \\ 6x_3 = t \rightarrow x_3 = t/6. \\ x_2 = -t - (-2t) = t \\ x_1 = t \end{array}\right\} x_1, x_2, x_4 = t \text{ and } x_3 = \frac{t}{6}.$$

Since all xs must be an integer, t must be an integer multiple of 6. For simplicity, let $t = 6$.

$\hookrightarrow \therefore x_1 = x_2 = x_4 = 6$, and $x_3 = 1$.

$\hookrightarrow 6CO_2 + 6H_2O \longrightarrow C_6H_{12}O_6 + 6O_2$.

Application 2: Linear Models:

Example 16.1:

An industrial city has four heavy industries (A_1, A_2, A_3, and A_4), each of which burns coal to manufacture products. By law, no industry can burn more than 45 units of coal per day. Each industry produces Pb, SO_2, and NO_2 at daily rates per unit of coal burned, and release them into the atmosphere.

Industry	A_1	A_2	A_3	A_4
Pb	1	0	1	7
SO_2	2	1	2	9
NO_2	0	2	2	0

It's leaked that on one day last year, 250 units of Pb, 550 units of SO_2, and 400 units of NO_2 were measured in the atmosphere. An inspector reported that A_3 did not break the law on that day. Which industry(ies) broke the law on that day?

Let a_i denote the number of units of coal burned by Industry A_i.

↳ Pb: $a_1 + 0a_2 + a_3 + 7a_4 = 250$

SO_2: $2a_1 + a_2 + 2a_3 + 9a_4 = 550$

NO_2: $0a_1 + 2a_2 + 2a_3 + 0a_4 = 400$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 7 & 250 \\ 2 & 1 & 2 & 9 & 550 \\ 0 & 2 & 2 & 0 & 400 \end{array}\right] \cdots \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 100 \\ 0 & 1 & 0 & -5 & 50 \\ 0 & 0 & 1 & 5 & 150 \end{array}\right]$$

$a_4 = t, \quad a_3 = 150 - 5t, \quad a_2 = 50 + 5t, \quad a_1 = 100 - 2t.$

Since A_3 did not break the law, $0 \le 150 - 5t \le 45$

$\to -150 \le -5t \le -105 \quad \to \quad 30 \ge t \ge 21$

A_4 could not break the law as $a_4 = t$.

$21 \le t \le 30$

$105 \le 5t \le 150$

$155 \le 50 + 5t \le 200$

$155 \le a_2 \le 200 \quad \to$ A_2 is breaking the law!

$21 \le t \le 30$

$-42 \ge -2t \ge -60$

$58 \ge 100 - 2t \ge 40$

$58 \ge a_1 \ge 40 \quad \to$ A_1 might be breaking the law!

A_2 is breaking the law, and A_1 might be!

Lecture 17

Application 3: Network Flow

Definition 17.1 : Networks

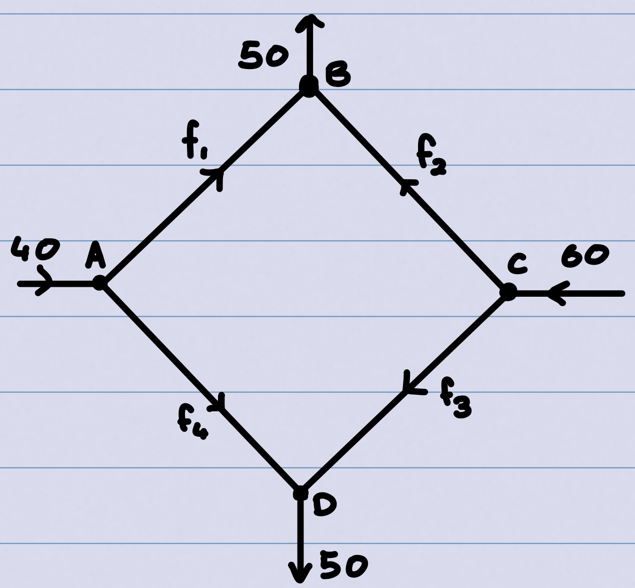
↳ A network consists of a system of junctions or nodes that are connected by directed line segments.

Junction Rule : the flow into a junction must equal the flow out of the junction.

Definition 17.2 : Equilibrium

↳ A network where every node obeys the Junction Rule is said to be in equilibrium.

Example 17.1 :



Compute all f_1, f_2, f_3, f_4 so the system is in equilibrium.

Using the Junction Rule,

	Flow In		Flow Out
A:	40	$=$	$f_1 + f_4$
B:	$f_1 + f_2$	$=$	50
C:	60	$=$	$f_2 + f_3$
D:	$f_3 + f_4$	$=$	50

$$
\begin{aligned}
f_1 + 0f_2 + 0f_3 + f_4 &= 40 \\
f_1 + f_2 + 0f_3 + 0f_4 &= 50 \\
0f_1 + f_2 + f_3 + 0f_4 &= 60 \\
0f_1 + 0f_2 + f_3 + f_4 &= 50
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 40 \\ 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 60 \\ 0 & 0 & 1 & 1 & 40 \end{array}\right] \cdots \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 40 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$f_4 = t$, $f_3 = 50 - t$, $f_2 = 10 + t$, $f_1 = 40 - t$.

$$
\begin{aligned}
f_1 \geq 0 &\Rightarrow 40 - t \geq 0 \Rightarrow t \leq 40 \\
f_2 \geq 0 &\Rightarrow 10 + t \geq 0 \Rightarrow t \geq -10 \\
f_3 \geq 0 &\Rightarrow 50 - t \geq 0 \Rightarrow t \leq 50 \\
f_4 \geq 0 &\Rightarrow t \geq 0.
\end{aligned}
$$

Lecture 18

Matrix Algebra

Definition 18.1 : Matrix

a $m \times n$ matrix A is a rectangular array with m rows and n columns. The entry in the i^{th} row and j^{th} column is denoted by either a_{ij} or $(A)_{ij}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & & a_{mn} \end{bmatrix}$$

The set of all $m \times n$ matrices with real entries is denoted by $M_{m \times n}(\mathbb{R})$. For a matrix $A \in M_{m \times n}(\mathbb{R})$, we say that A has size $m \times n$ and call a_{ij} the (i,j)-entry of A.

If $m=n$, we call A a square matrix.

Example 18.1

$$A = \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \sin\pi \end{bmatrix}$$

A is a 3×2 matrix and B is a 2×2 square matrix.

Definition 18.2: Zero Matrix

↳ the $m \times n$ matrix with all 0 entries is called a zero matrix, denoted by $O_{m \times n}$.

Definition 18.3: Matrix Addition and Scalar Multiplication

↳ For $A, B \in M_{m \times n}(\mathbb{R})$ we define matrix addition as:

↳ $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$

and for $c \in \mathbb{R}$, scalar multiplication is defined by:

↳ $(cA)_{ij} = c(A)_{ij}$

Also, we define $A - B = A + (-1)B$.

Example 18.2: Find $a, b, c \in \mathbb{R}$ ST:

$$\begin{bmatrix} a & b & c \end{bmatrix} - 2\begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} -3 & 3 & 6 \end{bmatrix}$$

↳ $\begin{bmatrix} -3 & 3 & 6 \end{bmatrix} = \begin{bmatrix} a-2c & b-2a & c-2b \end{bmatrix}$

↳ $a - 2c = -3$
$b - 2a = 3$
$c - 2b = 6$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ -2 & 1 & 0 & 3 \\ 0 & -2 & 1 & 6 \end{array}\right]$$

↳ $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$a = b = -3, \quad c = 0$ ⟵

Example 18.3

↳ Let $c \in \mathbb{R}$ and $A \in M_{m\times n}(\mathbb{R})$ be such that $cA = O_{m\times n}$.
Prove that either $c = 0$ or $A = O_{m\times n}$.

Since $cA = O_{m\times n}$, $ca_{ij} = 0$ $\forall i = 1, \ldots, m$ and $j = 1, \ldots, n$.

If $c = 0$, then the result holds, so assume $c \neq 0$. But since $a_{ij} = 0$ $\forall i = 1, \ldots m$, and $j = 1, \ldots, n$, so $A = O_{m\times n}$.

Theorem 18.1:

↳ Let $A, B, C \in M_{m\times n}(\mathbb{R})$ and let $c, d \in \mathbb{R}$.

1) $A + B \in M_{m\times n}$
2) $A + B = B + A$
3) $A + (B + C) = (A + B) + C$
4) $\exists\ O_{m\times n} \in M_{m\times n}(\mathbb{R})$ ST $\forall A_{m\times n} \in M_{m\times n}(\mathbb{R})$, $A + O_{m\times n} = A$.
5) $\forall\ A \in M_{m\times n}(\mathbb{R})$, $\exists\ (-A) \in M_{m\times n}(\mathbb{R})$ ST $A + (-A) = O$.
6) $cA \in M_{m\times n}(\mathbb{R})$
7) $c(dA) = cd(A)$
8) $(c + d)A = cA + dA$
9) $c(A + B) = cA + cB$
10) $1 \cdot A = A$.

Definition 18.4: Transpose of a Matrix

↳ Let $A \in M_{m\times n}(\mathbb{R})$. The transpose of A, denoted by A^T, is the $n\times m$ matrix satisfying $(A^T)_{ij} = (A)_{ji}$.

Example 18.4: Find the transpositions of A, B, and C.

↳ $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$

↳ $A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$, $C^T = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$

Theorem 18.2: Properties of Transpose

↳ Let $A, B \in M_{m \times n}(\mathbb{R})$ and $c \in \mathbb{R}$.

1) $A^T \in M_{n \times m}(\mathbb{R})$
2) $(A^T)^T = A$
3) $(A+B)^T = A^T + B^T$
4) $(cA)^T = cA^T$.

Example 18.5: Solve for A if:

↳ $\left(2A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

↳ $(2A^T)^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

↳ $2A - 3\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

↳ $2A - \begin{bmatrix} 3 & -3 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

↳ $2A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 6 & 3 \end{bmatrix}$

$\hookrightarrow 2A = \begin{bmatrix} 5 & 0 \\ 5 & 5 \end{bmatrix}$

$\hookrightarrow A = \begin{bmatrix} 2.5 & 0 \\ 2.5 & 2.5 \end{bmatrix}$

Definition 18.5: Symmetric Matrix

↳ A matrix A is symmetric if $A^T = A$.

↓

Note: if $A \in M_{m \times n}(\mathbb{R})$, then $A^T \in M_{n \times m}(\mathbb{R})$.

↳ $\therefore$ $A^T = A$ implies $n = m$, so a symmetric matrix must be a square matrix.

Example 18.6: $A = \begin{bmatrix} 1 & 6 \\ 6 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix}$.

↳ $A^T = \begin{bmatrix} 1 & 6 \\ 6 & 9 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$

Since $A = A^T$ and $B \neq B^T$, A is symmetric and B is not.

Lecture 19

↳ **The Matrix-Vector Product**

Consider this linear system of equations:

↳ $x_1 + 3x_2 - 2x_3 = -7$

$-x_1 - 4x_2 + 3x_3 = 8.$

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & -4 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$.

↳ let $\vec{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, and $\vec{a}_3 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

↳ these are the columns of A, so $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$.

The system is consistent if:

↳ $\vec{b} = \begin{bmatrix} -7 \\ 8 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 - 2x_3 \\ -x_1 - 4x_2 + 3x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$\vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3.$$

Definition 19.1: Matrix-Vector Product

$\therefore \vec{a}_1, \dots \vec{a}_n \in \mathbb{R}^m$.

↳ Let $A = [\vec{a}_1 \ \dots \ \vec{a}_n] \in M_{m \times n}(\mathbb{R})$ and $\vec{x} = [x_1 \ \dots \ x_n] \in \mathbb{R}^n$. Then the vector $A\vec{x}$ is defined by:

$$A\vec{x} = x_1 \vec{a}_1 + \dots + x_n \vec{a}_n \in \mathbb{R}^m.$$

Example 19.1:

$$\begin{bmatrix} 1 & 5 \\ -1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + (2) \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 4 \end{bmatrix}$$

$\therefore \vec{x} = [-1 \ \ 2]^T$ is a solution to the linear system:

$$\begin{aligned} x_1 + 5x_2 &= 9 \\ -x_1 + 2x_2 &= 5 \\ -2x_1 + x_2 &= 4 \end{aligned}$$

Theorem 19.1:

1) Every linear system of equations can be expressed as $A\vec{x} = \vec{b}$ for some matrix A and some vector $\vec{b}$.
2) The system is consistent iff $\vec{b}$ can be expressed as a linear combination of the columns of A
3) If $\vec{a}_1, \dots, \vec{a}_n$ are the columns of $A \in M_{m \times n}(\mathbb{R})$ and $\vec{x} = [x_1 \dots x_n]^T$, then $\vec{x}$ satisfies $A\vec{x} = \vec{b}$ iff $x_1\vec{a}_1 + \dots x_n\vec{a}_n = \vec{b}$.

Keep sizes of matrices and vectors in mind!!:

$$\underbrace{A}_{m \times n}\ \underbrace{\vec{x}}_{\mathbb{R}^n} = \underbrace{\vec{b}}_{\mathbb{R}^m}.$$

Example: $A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

$A\vec{x}$ is not defined since A has two columns but $\vec{x} \notin \mathbb{R}^2$.

Example 19.2: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\hookrightarrow 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\therefore \vec{x} = [1 \ \ -1]^T$ is a solution to the homogeneous system:

$$x_1 + x_2 = 0$$
$$2x_1 + 2x_2 = 0.$$

Theorem 19.2: Properties of the Matrix-Vector Product:

↳ Let $A, B \in M_{m \times n}(\mathbb{R})$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

↳ 1) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.

2) $A(c\vec{x}) = c(A\vec{x}) = cA(\vec{x})$.

3) $(A + B)\vec{x} = A\vec{x} + B\vec{x}$.

Dot Product to Compute Matrix-Vector:

↳ Consider $A = \begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\therefore A\vec{x} = (1)\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + (2)\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} (1)(1) + (1)(-1) + (2)(6) \\ (1)(0) + (1)(2) + (2)(1) \\ (1)(4) + (1)(-3) + (2)(2) \end{bmatrix}}_{\text{pretty much a dot product!}}$$

$$= \begin{bmatrix} 12 \\ 4 \\ 5 \end{bmatrix}.$$

In general, given $A \in M_{m \times n}(\mathbb{R})$, there are vectors $\vec{r}_1, \ldots, \vec{r}_m \in \mathbb{R}^n$ st

$$A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix},$$

and for any $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{r}_1^{\,T}\vec{x} \\ \vdots \\ \vec{r}_m^{\,T}\vec{x} \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$

↳ the i^{th} entry of $A\vec{x}$ is the dot product $\vec{r}_i \cdot \vec{x}$ where $\vec{r}_i^{\,T}$ is the i^{th} row of A.

Exercise 19.1: Let $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & -3 & 2 \end{bmatrix}$ and let $\vec{x} = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}^T$.

↳ compute $A\vec{x}$ as: a linear combination of A's columns

a dot product.

- **Linear Combinations:** $A\vec{x} = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

$$= (1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ -3 \end{bmatrix} + (0)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

- **Dot Product:** $A\vec{x} = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} (1)(1) + (1)(2) + (2)(1) + (-1)(0) \\ (2)(1) + (1)(2) + (-3)(1) + (2)(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Definition 19.2

↳ the $n \times n$ identity matrix, denoted by I_n or $I_{n\times n}$, is the square matrix of size $n \times n$ with $(I_n)_{ii} = 1$ for $i = 1, 2, \ldots, n$. Every other entry is 0.

↳ For example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\vec{e}_1, \vec{e}_2, \vec{e}_3]$.

Example 19.5: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

↳ $A\vec{x} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

$B\vec{x} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

↳ $\therefore A\vec{x} = B\vec{x}$, but $\vec{x} \neq 0$ and $A \neq B$!

Theorem 19.3: Matrices Equal Theorem

↳ Let $A, B \in M_{m\times n}(\mathbb{R})$. If $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$, then $A = B$.

Lecture 20:

Matrix Multiplication

Definition 20.1

↳ If $A \in M_{m\times n}(\mathbb{R})$ and $B = [\vec{b}_1 \ \dots \ \vec{b}_k] \in M_{n\times k}(\mathbb{R})$, then the matrix product AB is the $m\times k$ matrix:

$$AB = [A\vec{b}_1 \ \dots \ A\vec{b}_k].$$

Example 20.1: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$

$\therefore \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

↳ $A\vec{b}_1 = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$.

↳ $A\vec{b}_2 = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$.

$$\therefore AB = [A\vec{b}_1, A\vec{b}_2] = \begin{bmatrix} 9 & 6 \\ 0 & 1 \end{bmatrix}.$$

Note: we had $(A_{3\times 2})(B_{3\times 2}) = (AB)_{2\times 2}$.

↳ In general, for AB to be defined, the number of columns in A must equal the number of rows of B.

Example 20.2: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$.

↳ $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} (1)(1)+(4)(2) & (1)(1)+(2)(-2) & (1)(3)+(2)(1) \\ (3)(1)+(4)(4) & (3)(1)+(4)(-2) & (3)(3)+(4)(1) \end{bmatrix}$

$= \begin{bmatrix} 9 & -3 & 5 \\ 19 & -5 & 13 \end{bmatrix}$.

↳ Note: AB is defined, but BA is not defined!

Example 20.3: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

↳ $AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$.

↳ $AB \neq BA$! → matrix multiplication is not commutative.

Example 20.4: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

↳ $(AB)^T = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \right)^T = \begin{bmatrix} -1 & 5 \\ -1 & 11 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 \\ 5 & 11 \end{bmatrix}$.

$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 6 \end{bmatrix}$.

$B^T A^T = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 5 & 11 \end{bmatrix}$.

↳ $\therefore (AB)^T = B^T A^T \neq A^T B^T$!

Theorem 20.1: Properties of Matrix Multiplication.

↳ Let $c \in \mathbb{R}$ and A, B, C be matrices of appropriate sizes.

↳ 1) $IA = A$ → identity matrix

2) $AI = A$

3) $A(BC) = (AB)C$. → Matrix multiplication is associative.

4) $A(B+C) = AB + AC$. → left-distributive law

5) $(B+C)A = BA + CA$. → right-distributive law.

6) $(cA)B = c(AB) = A(cB)$.

7) $(AB)^T = B^T A^T \neq A^T B^T$. !!!

↳ generalizable to: $(A_1 A_2 \dots A_k)^T = A_k^T \dots A_2^T A_1^T$

Exercise 20.1: simplify $A(3B-C) + (A-2B)C + 2B(C+2A)$.

↳ $3AB - AC + AC - 2BC + 2BC + 4BA$

↳ $3AB + 4BA$ → $\neq 7AB$, we cannot assume $AB = BA$.

Lecture 22:

Complex Matricies

Example 22.1: Let $A = \begin{bmatrix} j & 2-j \\ 4+j & 1-2j \end{bmatrix}$, $B = \begin{bmatrix} 1 & j \\ 2j & 1-j \end{bmatrix}$

↳ $AB = \begin{bmatrix} j & 2-j \\ 4+j & 1-2j \end{bmatrix} \begin{bmatrix} 1 & j \\ 2j & 1-j \end{bmatrix}$ → $\vec{b}_1 = \begin{bmatrix} 1 \\ 2j \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} j \\ 1-j \end{bmatrix}$

$\hookrightarrow A\vec{b}_1 = \begin{bmatrix} j & 2-j \\ j+4 & 1-2j \end{bmatrix}\begin{bmatrix} 1 \\ 2j \end{bmatrix} = \begin{bmatrix} j \\ j+4 \end{bmatrix} + \begin{bmatrix} 4j+2 \\ 2j+4 \end{bmatrix} = \begin{bmatrix} 5j+2 \\ 3j+8 \end{bmatrix}$

$A\vec{b}_2 = \begin{bmatrix} j & 2-j \\ j+4 & 1-2j \end{bmatrix}\begin{bmatrix} j \\ 1-j \end{bmatrix} = \begin{bmatrix} -1 \\ 4j+3 \end{bmatrix} + \begin{bmatrix} 1-3j \\ -1-3j \end{bmatrix} = \begin{bmatrix} -3j \\ j+2 \end{bmatrix}$

$\hookrightarrow AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix} = \begin{bmatrix} 5j+2 & -3j \\ 3j+8 & j+2 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & j \\ 2j & 1-j \end{bmatrix}\begin{bmatrix} j & 2-j \\ 4+j & 1-2j \end{bmatrix} = \begin{bmatrix} (1)(j)+(j)(4+j) & (1)(2-j)+(j)(1-2j) \\ (2j)(j)+(1-j)(4+j) & (2j)(2-j)+(1-j)(1-2j) \end{bmatrix}$

$\hookrightarrow BA = \begin{bmatrix} 5j-1 & 4 \\ 3-3j & j+1 \end{bmatrix}$

So, $AB \neq BA \rightarrow$ multiplication of complex matrices is also not commutative.

Definition 22.1: Conjugate of a Matrix

$\hookrightarrow$ Let $A = [a_{ij}] \in M_{m\times n}(\mathbb{C})$.

$\hookrightarrow \bar{A} = [\bar{a}_{ij}]$

Theorem 22.1

$\hookrightarrow$ If $A \in M_{m\times n}(\mathbb{C})$ and $\vec{z} \in \mathbb{C}^n$, then $\overline{A\vec{z}} = \bar{A}\,\bar{\vec{z}}$

Example 22.2: $A = \begin{bmatrix} j & 1+j \\ 1+j & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2-j \\ 2+j & 6 \end{bmatrix}$

$\hookrightarrow A^T = \begin{bmatrix} j & 1+j \\ 1+j & 3 \end{bmatrix} = A$, $B^T = \begin{bmatrix} 3 & 2+j \\ 2-j & 6 \end{bmatrix} \neq B$

$\hookrightarrow$ A is symmetric and B is not!

Definition 22.2: Inverse Matrix

$n \times n$ identity matrix

↳ Let $A \in M_{n\times n}(\mathbb{R})$. If $\exists\, B \in M_{n\times n}(\mathbb{R})$ ST $AB = I = BA$, then A is invertible and B is an inverse of A. Also, B is invertible with A an inverse of B.

Example 22.3: $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

↳ $AB = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2-1 & 0 \\ 0 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2-1 & 0 \\ 2-2 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore AB = I = BA$.

↳ A is invertible and B is an inverse of A.

Example 22.4: $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

↳ then $\forall\, b_1, b_2, b_3, b_4 \in \mathbb{R}$,

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} b_1 + 2b_3 & b_2 + 2b_4 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore A$ is not invertible.

Theorem 22.2:

↳ Let $A, B \in M_{m\times n}(\mathbb{R})$ be such that $AB = I$. Then,

$BA = I$. Moreover, $\text{rank}(A) = \text{rank}(B) = n$.

Theorem 22.3

↳ Let $A \in M_{m \times n}(\mathbb{R})$ be invertible. If $B, C \in M_{m \times n}(\mathbb{R})$ are both inverses of A, then $B = C$.

Theorem 22.4: Properties of Matrix Inverses

↳ Let $A, B \in M_{m \times n}(\mathbb{R})$ be invertible and let $c \in \mathbb{R} \neq 0$.

↳ 1) $(cA)^{-1} = \frac{1}{c} A^{-1}$

2) $(AB)^{-1} = B^{-1} A^{-1}$

3) $(A^k)^{-1} = (A^{-1})^k \quad \forall\, k \in \mathbb{N}$.

4) $(A^T)^{-1} = (A^{-1})^T$

5) $(A^{-1})^{-1} = A$

Note: 2) generalizes to: $(A_1 A_2 \ldots A_k)^{-1} = (A_k^{-1} \ldots A_2^{-1} A_1^{-1})$.

Lecture 23: Matrix Inversion Algorithm

Consider $A \in M_{3 \times 3}(\mathbb{R})$. If A is invertible, then $\exists\, X = [\vec{x}_1 \;\; \vec{x}_2 \;\; \vec{x}_3] \in M_{3 \times 3}(\mathbb{R})$ such that:

↳ $AX = I$

$A[\vec{x}_1 \;\; \vec{x}_2 \;\; \vec{x}_3] = [\vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3]$

$[A\vec{x}_1 \;\; A\vec{x}_2 \;\; A\vec{x}_3] = [\vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3]$.

↳ $A\vec{x}_1 = \vec{e}_1 \;,\quad A\vec{x}_2 = \vec{e}_2 ,\quad A\vec{x}_3 = \vec{e}_3$.

We have three equations with the same coefficient matrix, so we have the following augmented matrix:

$\hookrightarrow [A \mid I] = [I \mid \vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3]$

where $B = [\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3] \in M_{3\times 3}(\mathbb{R})$ is the matrix that I reduces to under the same elementary row operations that carry A to I. $\therefore \vec{b}_1$ is the solution to $A\vec{x}_1 = \vec{e}_1$, etc.

$\therefore \vec{x}_1 = \vec{b}_1$, $\vec{x}_2 = \vec{b}_2$, and $\vec{x}_3 = \vec{b}_3$.

$\therefore AX = AB = A[\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3] = [A\vec{b}_1 \quad A\vec{b}_2 \quad A\vec{b}_3] = I.$

$\hookrightarrow$ So, $A^{-1} = B$.

Example 23.1: Find A^{-1} if it exists.

$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

$$\hookrightarrow \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow{R_1 + 3R_2} \left[\begin{array}{cc|cc} 2 & 0 & -5 & 3 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{cc|cc} 2 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{RREF(A)=I} & \underbrace{\begin{matrix} -5/2 & 3/2 \\ 2 & -1 \end{matrix}}_{A^{-1}=B} \end{array}\right]$$

$\therefore A$ is invertible (since $RREF(A) = I$) and $A^{-1} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$.

Example 23.2: Find A^{-1} if it exists.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$\hookrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$.

RREF(A) $\neq I$

Since $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, A is not invertible.

$\hookrightarrow$ note $\text{rank}(A) = 1 < 2$.

Properties of Matrix Inverses:

Theorem 23.1: Cancellation Laws:

$\hookrightarrow$ Let $A \in M_{n \times n}(\mathbb{R})$ be invertible.

$\hookrightarrow$

left cancellation

1) $\forall B, C \in M_{n \times k}(\mathbb{R})$, if $AB = AC$, then $B = C$.
2) $\forall B, C \in M_{n \times k}(\mathbb{R})$, if $BA = CA$, then $B = C$.

right cancellation

Example 23.3: If $AB = CA$, does $B = C$?

invertible

$\hookrightarrow A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$.

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

$CA = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

$\therefore AB = CA$ but $B \neq C$. → there is no mixed cancellation!

Example 23.4:

↳ $\forall A, B \in M_{n\times n}(\mathbb{R})$ with A, B and $A+B$ invertible, does $(A+B)^{-1} = A^{-1} + B^{-1}$.

Let $A = B = I$. $\therefore A + B = 2I$.

↳ $(A+B)^{-1} = (2I)^{-1} = \frac{1}{2}I^{-1} = \frac{I}{2}$.

But, $A^{-1} + B^{-1} = I^{-1} + I^{-1} = I + I = 2I$.

Since $\frac{I}{2} \neq 2I$, $(A+B)^{-1} \neq A^{-1} + B^{-1}$.

Theorem 23.2: Invertible Matrix Theorem

↳ Let $A \in M_{n\times n}(\mathbb{R})$. The following are equivalent:

* updated version → theorem 29.2!

↳ 1) A is invertible
2) $\text{rank}(A) = n$
3) $\text{RREF}(A) = I$
4) $\forall \vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ and has a unique solution.
5) A^T is invertible.

↳ $A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

Example 23.5: $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

↳ $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -23/2 \\ 9 \end{bmatrix}$

Lecture 24: Spanning Sets

Definition 24.1: Span

↳ Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. The span of B is:

↳ $$\text{Span } B = \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

Note: we say that the set span B is spanned by B and that B is a spanning set for span B.

Example 24.1: Determine if $\begin{bmatrix}2\\3\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}4\\5\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}\right\}$.

↳ $$\begin{bmatrix}2\\3\end{bmatrix} = c_1\begin{bmatrix}4\\5\end{bmatrix} + c_2\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}4c_1 + 3c_2\\5c_1 + 3c_2\end{bmatrix}$$

↳ $$4c_1 + 3c_2 = 2$$
$$5c_1 + 3c_2 = 3$$

↳ $$\left[\begin{array}{cc|c}4 & 3 & 2\\5 & 3 & 3\end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c}5 & 3 & 3\\4 & 3 & 2\end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c}1 & 0 & 1\\4 & 3 & 2\end{array}\right]$$

$$\xrightarrow[R_2 - 4R_1]{} \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 3 & -2\end{array}\right] \xrightarrow[\frac{1}{3}R_2]{} \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & -2/3\end{array}\right] \begin{array}{l}\to c_1\\ \to c_2\end{array}$$

Since the system is consistent, $\begin{bmatrix}2\\3\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}4\\5\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}\right\}$.

$$\therefore \begin{bmatrix}2\\3\end{bmatrix} = 1\begin{bmatrix}4\\5\end{bmatrix} - \frac{2}{3}\begin{bmatrix}3\\3\end{bmatrix}$$

Example 24.2: determine if $\begin{bmatrix}1\\2\\3\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

$\hookrightarrow \begin{bmatrix}1\\2\\3\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}c_1+c_2\\c_2\\c_1\end{bmatrix}$

$\hookrightarrow c_1 + c_2 = 1$

$c_2 = 2$

$c_1 = 3$

$\hookrightarrow \left[\begin{array}{cc|c}1&1&1\\0&1&2\\1&0&3\end{array}\right] \xrightarrow[R_3-R_1]{} \left[\begin{array}{cc|c}1&1&1\\0&1&2\\0&-1&2\end{array}\right] \xrightarrow[R_3+R_2]{} \left[\begin{array}{cc|c}1&1&1\\0&1&2\\0&0&4\end{array}\right]$

Since the system is inconsistent, $\begin{bmatrix}1\\2\\3\end{bmatrix} \notin \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

Theorem 24.1

$\hookrightarrow$ Let $B = \{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, $\vec{c} \in \mathbb{R}^k$, and let $A = [\vec{v}_1 \; \cdots \; \vec{v}_k] \in M_{n\times k}(\mathbb{R})$.

$\hookrightarrow$ $\vec{x} \in \text{Span } B$ iff $A\vec{c} = \vec{x}$ is consistent.

Example 24.3: describe $S = \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ geometrically.

$\hookrightarrow S = \left\{ s\begin{bmatrix}1\\2\\3\end{bmatrix} \;\middle|\; s \in \mathbb{R}\right\}$.

$\therefore \vec{x} \in S$ iff $\vec{x} = s\begin{bmatrix}1\\2\\3\end{bmatrix}$ for some $s \in \mathbb{R}$.

$\vec{x} = s\begin{bmatrix}1\\2\\3\end{bmatrix}$, $s \in \mathbb{R}$ is a vector equation for S. It is also a vector equation for a line in $\mathbb{R}^3$ through the origin.

↳ S is a line in $\mathbb{R}^3$ through the origin with direction vector $\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Example 24.5: $S_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $S_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Show that $S_1 = S_2$.

↳ First, show that $S_1 \subseteq S_2$. let $\vec{x} \in S_1$.

$$\vec{x}_1 = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

since $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$

$$\therefore \vec{x} = (c_1 + c_3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (c_2 + c_3) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \underbrace{\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}_{S_2\,!}$$

Now, show that $S_2 \subseteq S_1$. $\vec{y} \in S_2$.

↳ $\vec{y} = d_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

↳ $\vec{y} = d_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in \overbrace{\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}}^{S_1!}$.

$\therefore$ Since $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$, $S_1 = S_2$!

Theorem 24.2

↳ Let $\vec{v}_1, \ldots \vec{v}_k \in \mathbb{R}^n$.

One of these vectors, say $\vec{v}_i$, can be expressed as a linear combination of $\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k$ iff:

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{span}\{\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$$

Example 24.6: $S = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

Since $\begin{bmatrix}5\\0\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}2\\4\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}5\\0\end{bmatrix}$ can be removed.

$\therefore$, $S = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

Since $\begin{bmatrix}2\\4\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}2\\4\end{bmatrix}$ can be removed.

$\therefore$ $S = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$. → not scalar multiples, so we have to keep both.

$$\vec{x} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}.$$

$$\vec{x} = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \mathbb{R}^2 = S.$$

Lecture 25: Linear Dependence and Linear Independence

Example 25.1:

Determine if $A = \left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$ is linearly dependent or linearly independent.

$$c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

↳ $2c_1 - c_2 = 0$
$3c_1 + 2c_2 = 0$

↳ $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 3 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -7 & 0 \end{array}\right]$.

no free variables, so it's a unique solution.
↳ linearly independent.

Example 25.2:
↳ determine if $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly dependent or linearly independent.

↳ $c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

↳ $c_1 + 2c_2 + c_3 = 0$
$c_2 + c_3 = 0$
$-c_1 + c_3 = 0$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

↳ c_3 is free, so B is linearly dependent.

Theorem 25.1

↳ Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$ and let $A = [\vec{v}_1 \dots \vec{v}_k]$. Then B is linearly independent iff $k = \text{rank}(A)$.

Exercise 25.2: is $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ linearly independent?

↳ $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$.

↳ $\text{rank}(A) = 2 < 3 = k$, so B is linearly dependent.

Theorem 25.2

↳ A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent iff

$$\vec{v}_i \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$$

Lecture 26

Definition 26.1: Subspaces of $\mathbb{R}^n$:

↳ A subset $\mathbb{S}$ of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if $\forall \vec{w}, \vec{x}, \vec{y} \in \mathbb{S}$ and $\forall c, d \in \mathbb{R}$,

1) $\vec{x} + \vec{y} \in \mathbb{S}$
2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
3) $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
4) $\exists\ \vec{0} \in \mathbb{S}$ ST $\vec{v} + \vec{0} = \vec{v}\ \ \forall \vec{v} \in \mathbb{S}$.
5) $\forall\ \vec{x} \in \mathbb{S}$, $\exists\ -\vec{x} \in \mathbb{S}$ ST $\vec{x} + (-\vec{x}) = \vec{0}$.
6) $c\vec{x} \in \mathbb{S}$
7) $c(d\vec{x}) = (cd)\vec{x}$
8) $(c+d)\vec{x} = c\vec{x} + d\vec{x}$

9) $c(\vec{x}+\vec{y}) = c\vec{x} + c\vec{y}$

10) $1\vec{x} = \vec{x}$.

However, we do not need to test <u>all</u> of these!

Theorem 26.1: Subspace Test

↳ A subset S is a subspace of $\mathbb{R}^n$ if:

↳ 1) $\vec{0}_{\mathbb{R}^n} \in S$ → S contains the 0 vector of $\mathbb{R}^n$

2) if $\vec{x}, \vec{y} \in S$, then $\vec{x}+\vec{y} \in S$ → S is closed under vector addition

3) if $\vec{x} \in S$ and $c \in \mathbb{R}$, then $c\vec{x} \in S$

↳ S is closed under scalar multiplication

Example 26.1: $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

S is not a subset of $\mathbb{R}^2$ since $\vec{0} \notin S$.

Example 26.2: $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 - x_2 + 2x_3 = 0 \right\}$

↳ Show that S is a subspace of $\mathbb{R}^3$.

- Since $0 - 0 + 2(0) = 0$, $\vec{0} \in S$.
- Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in S$.

↳ $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} y_1+z_1 \\ y_2+z_2 \\ y_3+z_3 \end{bmatrix} \in S$.

↳ Show that $(y_1+z_1) - (y_2+z_2) + 2(y_3+z_3) = 0$

↳ $(y_1 - y_2 + 2y_3) + (z_1 - z_2 + 2z_3) = 0 + 0 = 0.$

↳ so, $\vec{y} + \vec{z} \in S$ and S is closed under vector addition.

- Finally, for $c \in \mathbb{R}$, show that $c\vec{y} = \begin{bmatrix} cy_1 \\ cy_2 \\ cy_3 \end{bmatrix} \in S$ by showing that $(cy_1 - cy_2 + 2cy_3) = 0$.

↳ $cy_1 - cy_2 + 2y_3 = c(y_1 - y_2 + 2y_3) = 0 \in S.$

↳ S is closed under scalar multiplication.

$\therefore S$ is a subspace of $\mathbb{R}^3$ by the subspace test.

Example 26.3: $S = \left\{ c \begin{bmatrix} 1 \\ 3 \end{bmatrix} \middle| c \in \mathbb{R} \right\}$.

↳ show that S is a subspace of $\mathbb{R}^2$.

- $c = 0$ gives $0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$. $\therefore \vec{0} \in S$.

- $\exists\, c_1, c_2 \in \mathbb{R}$ ST: $\vec{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{y} = c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

↳ $\vec{x} + \vec{y} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (c_1 + c_2) \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

↳ if $c = c_1 + c_2$, then $\vec{x} + \vec{y} \in S$.

- $\forall\, c \in \mathbb{R}$, $c\vec{x} = c\left(c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = (cc_1) \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

↳ so $c\vec{x} \in S$.

$\therefore$, S is a subspace of $\mathbb{R}^2$ by the Subspace Test.

Theorem 26.2

↳ Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. Then $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$.

Lecture 27

Definition 27.1: Basis

↳ Let S be a subspace of $\mathbb{R}^n$ and let $B = \{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq S$. We say that B is a basis for S if:

1) B is linearly independent
2) $S = \text{span } B$.

If $S = \{\vec{0}\}$, then we define $B = \emptyset$ to be the basis for S.

Example 27.1: Show that $B = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

First, show that B is linearly independent.

↳ Consider the matrix: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

↳ $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Since $\text{rank}(A) = 2$, B is linearly independent.

Since A has two rows and $\text{rank}(A) = 2$, the system

$A\vec{c} = \vec{x}$ is consistent for every $\vec{x} \in \mathbb{R}^2$, so $\vec{x} \in \text{span } B \ \forall \ x \in \mathbb{R}^2$.

↳ $\therefore$, $\mathbb{R}^2 \subseteq \text{span } B$.

Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \in \mathbb{R}^2$, and $\mathbb{R}^2$ is closed under linear combinations, span $B \subseteq \mathbb{R}^2$.

$\therefore$ span $B = \mathbb{R}^2$, so B is a basis for $\mathbb{R}^2$.

Definition 27.2: Standard Basis for $\mathbb{R}^n$

↳ Let $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n$ be the columns of the $n \times n$ identity matrix I. The set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$, called the standard basis for $\mathbb{R}^n$.

↳ In $\mathbb{R}^2$, the standard basis is $\{\vec{e}_1, \vec{e}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

In $\mathbb{R}^3$, the standard basis is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Note: it is easy to write any vector in $\mathbb{R}^n$ as a linear combination of the standard basis vectors in $\mathbb{R}^n$.

Theorem 27.1

↳ Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. If $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for $\mathbb{R}^n$, then $k = n$.

Theorem 27.2:

↳ let $B = \{\vec{v}_1, \dots, \vec{v}_n\} \subseteq \mathbb{R}^n$. Then B is a basis for $\mathbb{R}^n$ if and only if $[\vec{v}_1 \ \dots \ \vec{v}_k]$ has rank n.

Example 27.4: Find a basis for the subspace

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x_1 - x_2 + 2x_3 = 0 \right\} \text{ of } \mathbb{R}^3.$$

↳ $x_1 = x_2 - 2x_3$.

↳ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

let $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

↳ $S \subseteq \operatorname{span} B \rightarrow \therefore,$ $S = \operatorname{span} B$.

Since neither vector in B is a scalar multiple of the other, B is linearly independent.

$\therefore$ B is a basis for S.

Example 27.5: Find a basis for $S = \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$.

$\vec{x} = \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

↳ $S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Since $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, S can be shortened:

↳

↳ $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

So, $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a spanning set for S.

Since B is linearly independent and $S = \text{span } B$, B is a basis for S.

Theorem 27.3:

↳ If $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subspace $S \subseteq \mathbb{R}^n$, then every $\vec{x} \in S$ can be expressed as a linear combination $\vec{v}_1, \dots, \vec{v}_k$ in a unique way.

Lecture 28

Dimension of a Subspace:

Theorem 28.1

↳ Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for a subspace S of $\mathbb{R}^n$. If $C = \{\vec{w}_1, \dots, \vec{w}_l\}$ is a set in S with $l > k$, then C is linearly dependent.

Theorem 28.2:

↳ If $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ and $C = \{\vec{w}_1, \dots, \vec{w}_l\}$ are both bases for a subspace S of $\mathbb{R}^n$, then $k = l$.

Definition 28.1: Dimension

↳ If $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subspace S of $\mathbb{R}^n$, then we say the dimension of S is k, and we write $\dim(S) = k$.

If $S = \{0\}$, then $\dim(S) = 0$ since ϕ is a basis for S.

Example 28.2:

↳ We saw in example 27.5 that the subspace $S: \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} \right\}$ ($a, b, c \in \mathbb{R}$) of $\mathbb{R}^3$ had basis: $B: \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

↳ $\therefore \dim(S) = 2$.

Theorem 28.3:

↳ If S is a k-dimensional subspace of $\mathbb{R}^n$ with $k > 0$, then:

1) A set of more than k vectors in S is linearly dependent.
2) A set of fewer than k vectors cannot span S.
3) A set of k vectors in S spans S iff it is linearly independent.

Example 28.3:

↓

Let S be a subspace of $\mathbb{R}^3$ with $\dim(S) = 2$ and suppose that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ belong to S. Find a basis for S.

Since $\vec{v}_1$ and $\vec{v}_2$ are nonzero and nonparallel, we have that $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set of two vectors in S. Since $\dim(S) = 2$, $S = \text{span}\{\vec{v}_1, \vec{v}_2\}$.

↳ $\therefore \{\vec{v}_1, \vec{v}_2\}$ is a basis for S.

Three Fundamental Subspaces Associated with a Matrix

Definition 28.2: Nullspace

↳ Let $A \in M_{m\times n}(\mathbb{R})$. The nullspace of A (also reffered to as the kernel of A) is the subset of $\mathbb{R}^n$ defined by: $\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}$.

Definition 28.3: Column Space

↳ Let $A = [\vec{a}_1 \ \dots \ \vec{a}_n] \in M_{m\times n}(\mathbb{R})$. The column space of A is the subset of $\mathbb{R}^m$ defined by:

$\text{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$.

Definition 28.4: Row Space

↳ Let $A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} \in M_{m\times n}(\mathbb{R})$. The row space of A is the subset of $\mathbb{R}^n$ defined by:

$\text{Row}(A) = \{A^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\} = \text{Span}\{\vec{r}_1^{\,T}, \dots, \vec{r}_m^{\,T}\}$.

Theorem 28.4:

↳ Let $A \in M_{m\times n}(\mathbb{R})$. Null(A) and Row(A) are subspaces of $\mathbb{R}^n$ and Col(A) is a subspace of $\mathbb{R}^m$.

Theorem 28.5:

↳ Let $A \in M_{m\times n}(\mathbb{R})$. If $B \in M_{m\times n}(\mathbb{R})$ is obtained by A from a series of $\underbrace{\text{EROs}}_{\text{elementary row operations}}$, then Row(B) = Row(A).

Theorem 28.6:

↳ Let $A = [\vec{a}_1 \cdots \vec{a}_n] \in M_{m\times n}(\mathbb{R})$ and suppose $B = [\vec{b}_1 \cdots \vec{b}_n] \in M_{m\times n}(\mathbb{R})$ is obtained from A by a series of EROs. Then for any $c_1, \ldots, c_n \in \mathbb{R}$, $c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{0}$ iff $c_1\vec{b}_1 + \cdots + c_n\vec{b}_n = \vec{0}$.

Lecture 29

Example 29.1: $A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 3 & 12 & 3 \end{bmatrix}$

Find a basis for Null(A), Col(A), and Row(A), and state the dimensions of each of these subspaces.

$$\begin{bmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 3 & 12 & 3 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3-R_1]{R_1-R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution to the homogeneous system $A\vec{x} = \vec{0}$ is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R},$$

so $B_1 = \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a spanning set for Null(A). B_1 is linearly independent and is therefore a basis for Null(A), so $\dim(\text{Null}(A)) = 2$.

Since $\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,

Col(A) = Span B_2 where $B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

↳ $\dim(\text{Col}(A)) = 2$. ↪ linearly independent

Row(A) = span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

↳ $B_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for Row(A)

$\dim(\text{Row}(A)) = 2$.

Theorem 29.1:

↳ Let $A \in M_{m \times n}(\mathbb{R})$. Then:

1) $\dim(\text{Null}(A)) = n - \text{rank}(A)$
2) $\dim(\text{Col}(A)) = \text{rank}(A)$
3) $\dim(\text{Row}(A)) = \text{rank}(A)$

Theorem 29.2: Invertible Matrix Theorem Revisited:

↳ Let $A \in M_{n \times n}(\mathbb{R})$. The following are equivalent:

↳ 1) A is invertible
2) $\text{rank}(A) = n$
3) $\text{RREF}(A) = I$
4) $\forall \vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ and has a unique solution. ↳ $\vec{x} = A^{-1}\vec{b}$
5) A^T is invertible.
6) $\text{Null}(A) = \{0\}$
7) The columns of A form a linearly independent set
8) The columns of A span $\mathbb{R}^n$
9) $\text{Col}(A) = \mathbb{R}^n$
10) $\text{Null}(A^T) = \{\vec{0}\}$
11) The rows of A form a linearly independent set
12) The rows of A span $\mathbb{R}^n$
13) $\text{Row}(A) = \mathbb{R}^n$

Lecture 30

Vector Spaces

Definition 30.1 : Vector Space

↳ A set $\mathbb{V}$ is called a vector space over $\mathbb{R}$ if for every $\vec{v}, \vec{x}, \vec{y} \in \mathbb{V}$ and for every $c, d \in \mathbb{R}$:

1) $\vec{x} + \vec{y} \in \mathbb{V}$ — $\mathbb{V}$ is closed under addition
2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ — addition is commutative
3) $(\vec{x} + \vec{y}) + \vec{v} = \vec{x} + (\vec{y} + \vec{z})$ — addition is associative

4) $\exists \vec{0} \in \mathbb{V}$ ST $\vec{x} + \vec{0} = \vec{x}$ $\forall \vec{x} \in \mathbb{V}$. zero vector

5) $\forall \vec{x} \in \mathbb{V}, \exists (-\vec{x}) \in \mathbb{V}$ ST $\vec{x} + (-\vec{x}) = \vec{0}$ additive inverse

6) $c\vec{x} \in \mathbb{V}$ $\mathbb{V}$ is closed under scalar multiplication

7) $c(d\vec{x}) = (cd)\vec{x}$ Scalar multiplication is associative

8) $(c+d)\vec{x} = c\vec{x} + d\vec{x}$ distributive law

9) $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$ distributive law

10) $1\vec{x} = \vec{x}$ Scalar multiplicative identity

Theorem 30.1:

↳ If $\mathbb{V}$ is a vector space, then $\forall \vec{x} \in \mathbb{V}$,

- $\vec{0}\vec{x} = \vec{0}$
- $-\vec{x} = (-1)(\vec{x})$

Definition 30.2:

↳ Let $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. The span of B is:

$$\text{Span } B = \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}.$$

The set Span B is spanned by B, and B is a spanning set for Span B.

Definition 30.3: Linear Dependence and Independence

↳ Let $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. We say that B is linearly dependent if there exist $c_1, \dots, c_k \in \mathbb{R}$, not all zero so that

$$\vec{0} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$$

we say that B is linearly independent if the only solution to $\vec{0} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ is $c_1 = 0 = \dots = c_k$.

Definition 30.4 : Subspace :

↳ A subset S is called a subspace of a vector space $\mathbb{V}$ if S is itself a vector space under the same operations of vector addition and scalar multiplication as $\mathbb{V}$.

Theorem 30.2 : Subspace Test

↳ A subset S is called a subspace of vector space $\mathbb{V}$ if:

→ S contains the $\vec{0}$ vector of $\mathbb{V}$

1) $\vec{0}_{\mathbb{V}} \in S$

→ S is closed under vector addition

2) if $\vec{x}, \vec{y} \in S$, then $\vec{x} + \vec{y} \in S$
3) if $\vec{x} \in S$ and $c \in \mathbb{R}$, then $c\vec{x} \in S$.

↳ S is closed under scalar multiplication

Definition 30.5 : Basis

↳ Let S be a subspace of $\mathbb{V}$, and let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in S. Then B is a basis for

S if B is linearly independent and $S = \text{Span } B$. If $S = \{\vec{0}\}$, then we define $B = \emptyset$ to be a basis for S.

Definition 30.6: Dimension

↳ If $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for a subspace S of $\mathbb{V}$, then we say the dimension of S is k, and we write $\dim(S) = k$. If $S = \{\vec{0}\}$, then $\dim(S) = 0$.

Theorem 30.3:

↳ If S is a k-dimensional subspace of $\mathbb{V}$ with $k > 0$, then:

1) A set of more than k vectors is linearly dependent,
2) A set of fewer than k vectors in S cannot span S,
3) A set of exactly k vectors in S spans S iff it is linearly independent.

Example 30.4:

↳ $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Show that B is a basis for the vector space $M_{2\times2}(\mathbb{R})$.

- First, show that B is linearly independent:

↳ $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

this gives $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore c_1 = c_2 = c_3 = c_4 = 0$, so B is linearly independent.

· Next, show that Span $B = M_{2\times 2}(\mathbb{R})$.

↳ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

So span $B = M_{2\times 2}(\mathbb{R})$.

$\therefore B$ is a basis for $M_{2\times 2}(\mathbb{R})$.

Definition 30.7: Standard Basis for $M_{2\times 2}(\mathbb{R})$

↳ The set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is called the standard basis for $M_{2\times 2}$.

Example 30.5:

↳ Let $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$. Show B is a basis for $M_{2\times 2}(\mathbb{R})$ and express $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a linear combination of the vectors in B.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = c_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

↳ $$\begin{aligned} c_1 + c_2 + c_4 &= 1 \\ c_1 + c_2 + c_3 &= 2 \\ c_2 + c_3 + c_4 &= 3 \\ c_1 + c_3 + c_4 &= 4 \end{aligned} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 1 & 0 & 1 & 1 & 4 \end{array}\right]$$

this reduces to: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & -2/3 \\ 0 & 0 & 1 & 0 & 7/3 \\ 0 & 0 & 0 & 1 & 4/3 \end{array}\right]$

so $c_1 = \frac{1}{3}$, $c_2 = -\frac{2}{3}$, $c_3 = \frac{7}{3}$, $c_4 = \frac{4}{3}$.

$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \frac{7}{3}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Also, since the coefficient matrix reduced to I,

$c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

has only the trivial solution, so B is linearly independent.

Since B has 4 vectors and $\dim(M_{2\times2}(\mathbb{R})) = 4$, Span $B = M_{2\times2}(\mathbb{R})$, so B is a basis for $M_{2\times2}(\mathbb{R})$.

Example 30.6:

↳ Let $B \in M_{n\times k}(\mathbb{R})$ be fixed and let

$$S = \{A \in M_{m\times n}(\mathbb{R}) \mid AB = O_{m\times k}\}.$$

Show S is a subspace of $M_{m\times n}(\mathbb{R})$.

- Since $O_{m\times n} B = O_{m\times k}$, we have that $O_{m\times n} \in S$.
- Let $A_1, A_2 \in S$.

↳ $A_1B = O_{m\times k} = A_2B$. Then:

$$(A_1 + A_2)B = A_1B + A_2B = O_{m\times k} + O_{m\times k} = O_{m\times k}.$$

$\therefore A_1 + A_2 \in S.$

$\cdot \ \forall c \in \mathbb{R},\ (cA_1)B = c(A_1B) = c(O_{m\times k}) = O_{m\times k}.$

$\therefore cA_1 \in S.$

$\therefore S$ is a subspace of $M_{m\times n}(\mathbb{R})$ by the subspace test.

Example 30.7:

↳ Consider the subspace $S = \{A \in M_{2\times 2}(\mathbb{R}) \mid A^T = A\}$ of $M_{2\times 2}(\mathbb{R})$. Find a basis for S and state the dimension of S.

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in S$. Then $A^T = A$, so:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}, \text{ so } a_2 = a_3.$$

$$\therefore A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} = a_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a spanning set for S. Since each vector in B contains a nonzero entry where others contain a zero entry, B is linearly independent and thus a basis for S. It follows that $\dim(S) = 3$.

Lecture 31

Definition 31.1 : $P_n(\mathbb{R})$

↳ For each nonnegative integer n, the set

$$P_n(\mathbb{R}) = \{a_0 + a_1x + \dots + a_nx^n \mid a_0, a_1, \dots, a_n \in \mathbb{R}\}$$

denotes the set of all real polynomials of degree at most n.

We denote the zero polynomial by $0 = 0 + 0x + \dots + 0x^n \in P_n(\mathbb{R})$.

Note that:

$P_0(\mathbb{R}) = \{a_0 \mid a_0 \in \mathbb{R}\} = \mathbb{R}$ → constant polynomials

$P_1(\mathbb{R}) = \{a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\}$ → constant and first degree polynomials

$P_2(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$

↳ constant, first, and second degree polynomials.

$\therefore P_0(\mathbb{R}) \subseteq P_1(\mathbb{R}) \subseteq P_2(\mathbb{R}) \subseteq \dots$

Definition 31.2 : Equality, Addition, and Scalar Multiplication

↳ Let $p(x), q(x) \in P_n(\mathbb{R})$ with

$$p(x) = a_0 + a_1x + \dots + a_nx^n$$
$$q(x) = b_0 + b_1x + \dots + b_nx^n$$

We say that p and q are equal iff $a_i = b_i$ and write $p = q$. Otherwise, we write $p \neq q$.

We define addition and scalar multiplication by:

$(p+q)(x) = p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$

$(kp)(x) = kp(x) = ka_0 + ka_1 x + \ldots + ka_n x^n \quad \forall k \in \mathbb{R}$.

Theorem 31.1:

With the operations of addition and scalar multiplication, $P_n(\mathbb{R})$ is a vector space for each nonnegative integer n.

Example 31.1:

Consider the set $B = \{1, x, \ldots, x^n\} \subseteq P_n(\mathbb{R})$. Show that B is a basis for the vector space $P_n(\mathbb{R})$.

For $c_0(1) + c_1 x + \ldots + c_n x^n = 0 + 0x + \ldots + 0x^n$.

We have the $c_0 = c_1 = \ldots = c_n = 0$ so B is linearly independent. Also, for any polynomial $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in P_n(\mathbb{R})$, $p(x)$ is trivially a linear combination of the elements in B:

$p(x) = a_0(1) + a_1(x) + \ldots + a_n(x^n)$, so $\operatorname{Span} B = P_n(\mathbb{R})$.

$\therefore$, B is a basis for $P_n(\mathbb{R})$.

Definition 31.3: Standard Basis for $P_n(\mathbb{R})$.

↳ The set $B = \{1, x, \dots, x^n\}$ is called the standard basis for $P_n(\mathbb{R})$.

Example 31.2:

↳ Let $B = \{1, 1+x, 1+x+x^2\} \subseteq P_2(\mathbb{R})$. Show B is a basis for $P_2(\mathbb{R})$.

For $c_1, c_2, c_3 \in \mathbb{R}$, consider

$$c_1(1) + c_2(1+x) + c_3(1+x+x^2) = 0$$

Rearranging gives:

$$(c_1 + c_2 + c_3) + (c_2 + c_3)x + c_3 x^2 = 0 + 0x + 0x^2$$

Thus:
$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

$\therefore c_1, c_2, c_3 = 0$, so B is linearly independent. Since B has 3 elements and $\dim(P_2(\mathbb{R})) = 3$, we see that $\operatorname{Span} B = P_2(\mathbb{R})$ and so B is a basis for $P_2(\mathbb{R})$.

Vector Spaces of $\mathbb{C}$

- $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. For $\vec{z} \in \mathbb{C}^n$,

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = z_1\vec{e}_1 + \dots + z_n\vec{e}_n.$$

we call $\{\vec{e}_1, \dots, \vec{e}_n\}$ (where $\vec{e}_i$ is the i^{th} column of $n \times n$ identity matrix) the standard basis for $\mathbb{C}^n$, so $\dim(\mathbb{C}^n) = n$.

- $M_{m\times n}(\mathbb{C})$ is a vector space over $\mathbb{C}$. The standard basis for $M_{m\times n}(\mathbb{C})$ is the same as for $M_{m\times n}(\mathbb{R})$, so $\dim(M_{m\times n}(\mathbb{C})) = mn$.

- $P_n(\mathbb{C})$ is a vector space over $\mathbb{C}$. The standard basis is $\{1, z, \dots, z^n\}$ (where $z \in \mathbb{C}$), so $\dim(P_n(\mathbb{C})) = n+1$.

Lecture 32

Linear Transformations

hello

Definition 32.1: Matrix Transformation

For $A \in M_{m\times n}(\mathbb{R})$, the function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f_A(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$ is called the matrix transformation corresponding to A. We call $\mathbb{R}^n$ the domain of f_A and $\mathbb{R}^m$ the codomain of f_A. We

Say that f_A maps $\vec{x}$ to $A\vec{x}$ and say that $A\vec{x}$ is the image of $\vec{x}$ under f_A.

Example 32.1: let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$.

Then $A \in M_{2\times 3}(\mathbb{R})$ and so $f_A : \mathbb{R}^3 \to \mathbb{R}^2$. We can compute

$$f_A(1,1,4) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = ((1)(1)+(2)(1)+(3)(4), (1)(1)+(-1)(1)+4)$$
$$= (15, 4).$$

more generally, $f_A(x_1, x_2, x_3) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= (x_1 + 2x_2 + 3x_3,\ x_1 - x_2 + x_3)$$

Theorem 32.1: Properties of Matrix Transformations

↳ Let $A \in M_{m\times n}(\mathbb{R})$ and let f_A be the matrix transformation corresponding to A. For every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and for every $c \in \mathbb{R}$,

1) $f_A(\vec{x} + \vec{y}) = f_A(\vec{x}) + f_A(\vec{y})$

2) $f_A(c\vec{x}) = c f_A(\vec{x})$

Definition 32.2:

↳ A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$ and $\forall s, t \in \mathbb{R}$,

$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$.

For $m = n$, a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ is often called a linear operator on $\mathbb{R}^n$.

Example 32.2: show that $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$L(x_1, x_2) = (x_1 - x_2, 2x_1 + x_2)$$

is a linear transformation.

· let $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$. With $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we have

$$\begin{aligned} L(s\vec{x} + t\vec{y}) &= L(sx_1 + ty_1, sx_2 + ty_2) \\ &= ((sx_1 + ty_1) - (sx_2 + ty_2), 2(sx_1 + ty_1) + (sx_2 + ty_2)) \\ &= (sx_1 - sx_2, 2sx_1 + sx_2) + (ty_1 - ty_2, 2ty_1 + ty_2) \\ &= s(x_1 - x_2, 2x_1 + x_2) + t(y_1 - y_2, 2y_1 + y_2) \\ &= sL(\vec{x}) + tL(\vec{y}). \end{aligned}$$

As $L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$, we see that L is a linear transformation

Example 32.3:

↳ Show that $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_3^2 + 3)$ is not linear.

Consider $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then

$L(\vec{x} + \vec{y}) = L(1, 1, 0) = (2, 3)$

but $L(\vec{x}) + L(\vec{y}) = L(1,0,0) + L(0,1,0) = (1,3) + (1,3) = (2,6)$
which shows that L is not linear (taken $s = t = 1$).

Theorem 32.2:

↳ If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then L is a matrix transformation with corresponding matrix

$$[L] = [L(\vec{e}_1) \ldots L(\vec{e}_n)] \in M_{m \times n}(\mathbb{R}),$$

that is, $L(\vec{x}) = [L]\vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$.

Example 32.5:

↳ let $\vec{d} \in \mathbb{R}^2$ be nonzero and define $L: \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = \text{proj}_{\vec{d}} \vec{x} \quad \forall \vec{x} \in \mathbb{R}^2$. Show L is linear, and then find the standard matrix of L with $\vec{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

We first show L is linear. Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$.

$$L(s\vec{x} + t\vec{y}) = \text{proj}_{\vec{d}}(s\vec{x} + t\vec{y})$$

$$= \frac{(s\vec{x} + t\vec{y}) \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d}$$

$$= s\frac{\vec{x}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d} + t\frac{\vec{y}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d}$$ → by properties of the dot product

$$= s\,\text{proj}_{\vec{d}}\,\vec{x} + t\,\text{proj}_{\vec{d}}\,\vec{y}$$

$$= sL(\vec{x}) + tL(\vec{y})$$ → L is linear

Now with $\vec{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$,

$$L(\vec{e}_1) = \text{proj}_{\vec{d}}\,\vec{e}_1 = \frac{\vec{e}_1\cdot\vec{d}}{\|\vec{d}\|^2}\cdot\vec{d} = -\frac{1}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/10 \\ -3/10 \end{bmatrix}$$

$$L(\vec{e}_2) = \text{proj}_{\vec{d}}\,\vec{e}_2 = \frac{\vec{e}_2\cdot\vec{d}}{\|\vec{d}\|^2}\cdot\vec{d} = \frac{3}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/10 \\ 9/10 \end{bmatrix}$$

so, $[L] = [L(\vec{e}_1) \;\; L(\vec{e}_2)] = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$

Lecture 33

Example 33.2: Find the vector that results from from rotating $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ counterclockwise about the origin by an angle of $\pi/6$.

$$R_{\frac{\pi}{6}}(\vec{x}) = [R_{\frac{\pi}{6}}]\vec{x} = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3}-2 \\ 1+2\sqrt{3} \end{bmatrix}.$$

Note: a clockwise rotation by θ is a counterclockwise rotation by $-\theta$.

↳ $\therefore$ a clockwise rotation by θ is given by the linear transformation:

$$[R_{-\theta}] = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

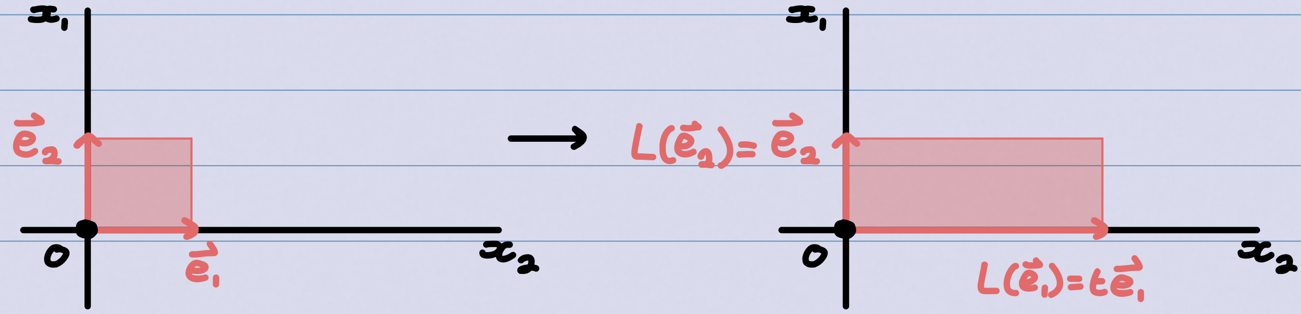
Stretches and Compressions:

For $t \in \mathbb{R} > 0$, let $A = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$, and define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = A\vec{x}$ $\forall$ $\vec{x} \in \mathbb{R}^2$. Then L is a matrix transformation and hence a linear transformation. For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$L(\vec{x}) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 t \\ x_2 \end{bmatrix}.$$

- If $t > 1$, then L is a stretch in the x_1 direction by a factor of t
- If $0 < t < 1$, then L is a compression in the x_1 direction by a factor of t.
- If $t = 0$, then L is a projection onto the x_2-axis.
- If $t < 0$, then L is a reflection in the x_2-axis followed by a stretch or compression by a factor of $-t > 0$.

Stretch in the x_1-direction by a factor of $t>1$:



Dilations and Contractions:

For $t \in \mathbb{R}$ with $t>0$, let $B = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$ and define $L(\vec{x}) = B\vec{x}$ $\forall\, x \in \mathbb{R}^2$. Then L is a matrix transformation and thus a linear transformation.

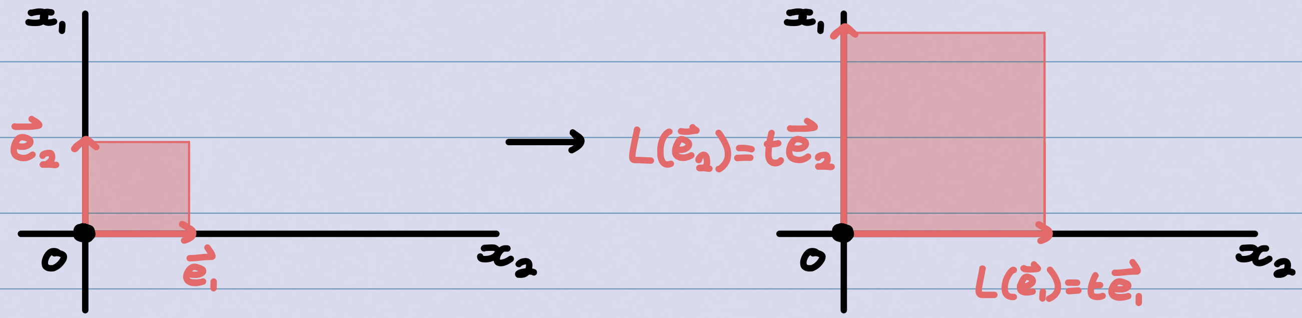
For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$L(\vec{x})$ is simply a scalar multiple of $\vec{x}$.

$$L(\vec{x}) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} = t\vec{x}$$

- If $t>1$, L is a dilation
- If $0<t<1$, L is a contraction.
- If $t=1$, B is the identity matrix and $L(\vec{x}) = \vec{x}$.

A dilation by a factor of $t>1$:



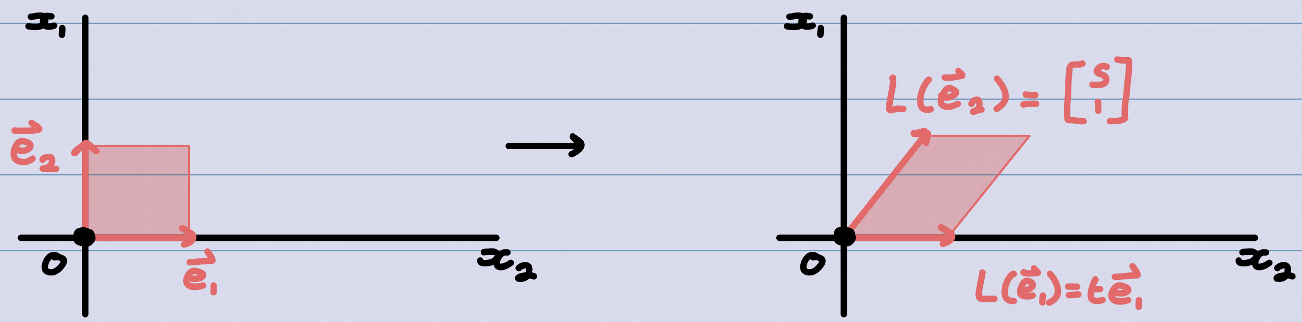
Shears:

For $s \in \mathbb{R}$, let $C = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ and define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = C\vec{x}$ $\forall\, \vec{x} \in \mathbb{R}^2$. Then L is a matrix transformation and hence a linear transformation. For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$L(\vec{x}) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + s x_2 \\ x_2 \end{bmatrix}$$

- L is a shear in the x_1-direction by a factor of s (aka, a horizontal shear).

A shear in the x_1-direction by a factor of $s > 0$:



A shear in the x_2-direction (aka, a vertical shear) by a factor of $s \in \mathbb{R}$ has standard matrix:

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

Lecture 34:

Operations on Linear Transformations:

Definition 34.1: Equality of Linear Transformations:

↳ Let $L, M : \mathbb{R}^n \to \mathbb{R}^m$ be (linear) transformations.
If $L(\vec{x}) = M(\vec{x})$ $\forall x \in \mathbb{R}^n$, then L and M are equal and $L = M$.

If $\exists \vec{x} \in \mathbb{R}^n$ ST $L(\vec{x}) \neq M(\vec{x})$, then $L(\vec{x}) \neq M(\vec{x})$.

Note: $L = M \iff [L] = [M]$.

Definition 34.2: Operations on Linear Transformations:

↳ Let $L, M : \mathbb{R}^n \to \mathbb{R}^m$ be (linear transformations and let $c \in \mathbb{R}$.

aka $[L] + [M]$

$L + M : \mathbb{R}^n \to \mathbb{R}^m$ is defined by: $(L+M)(\vec{x}) = L(\vec{x}) + M(\vec{x})$

aka $c[L]$.

$cL : \mathbb{R}^n \to \mathbb{R}^m$ is defined by: $(cL)(\vec{x}) = cL(\vec{x})$.

Example 34.1: Let $L, M : \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations such that:

$$L(x_1, x_2, x_3) = (2x_1 + x_2,\ x_1 - x_2 + x_3)$$
$$M(x_1, x_2, x_3) = (x_3,\ x_1 + 2x_2 + 3x_3)$$

Calculate $L + M$ and $-2L$

$\hookrightarrow (L+M)(\vec{x}) = L(\vec{x}) + M(\vec{x})$

$= (2x_1 + x_2,\ x_1 - x_2 + x_3) + (x_3,\ x_1 + 2x_2 + 3x_3)$

$$= \begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 + x_3 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 + x_3 \\ 2x_1 + x_2 + 4x_3 \end{bmatrix}$$

$$= (2x_1 + x_2 + x_3,\ 2x_1 + x_2 + 4x_3).$$

$$(-2)L(\vec{x}) = -2(2x_1 + x_2,\ x_1 - x_2 + x_3)$$
$$= (-4x_1 - 2x_2,\ -2x_1 + 2x_2 - 2x_3)$$

Theorem 34.1:

$\hookrightarrow$ Let $L, M : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations and let $c \in \mathbb{R}$. Then $L+M : \mathbb{R}^n \to \mathbb{R}^m$ and $cL : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations.

Also, $[L+M] = [L] + [M]$ and $[cL] = c[L]$.

Theorem 34.2:

$\hookrightarrow$ Let $L, M, N : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations and let $c, d \in \mathbb{R}$. We have:

1) $L+M : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation $\rightarrow$ closed under addition
2) $L+M = M+L$ $\rightarrow$ addition is commutative
3) $(L+M)+N = L+(M+N)$ $\rightarrow$ addition is associative

4) $\forall\ 0: \mathbb{R}^n \to \mathbb{R}^m$ st $L + 0 = L$ $\forall\ L: \mathbb{R}^n \to \mathbb{R}^m$ → zero transformation

5) $\forall L: \mathbb{R}^n \to \mathbb{R}^m$, $\exists\ (-L): \mathbb{R}^n \to \mathbb{R}^m$ st $L + (-L) = 0$ → additive inverse

6) $cL: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation → closure under scalar multiplication

7) $c(dL) = (cd)L$ → scalar multiplication is associative

8) $(c+d)L = cL + dL$ → distributive law

9) $c(L+M) = cL + cM$ → distributive law

10) $1L = L$ → scalar multiplicative identity.

Definition 34.3: Composition of Linear Transformations

↳ Let $L: \mathbb{R}^n \to \mathbb{R}^m$ and $M: \mathbb{R}^m \to \mathbb{R}^p$ be (linear) transformations.

The composition $M \circ L: \mathbb{R}^n \to \mathbb{R}^p$ is defined by:

$$(M \circ L)(\vec{x}) = M(L(\vec{x})).$$

Note: the domain of m must contain the codomain of L.

Example 34.2: Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ and $M: \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations defined by:

$$L(x_1, x_2, x_3) = (x_1 + x_2,\ x_2 + x_3)$$
$$M(x_1, x_2) = (x_1 - 3x_2,\ 2x_1).$$

Calculate $M \circ L$.

$$\begin{aligned} M \circ L &= M(L(x_1, x_2, x_3)) \\ &= M(x_1 + x_2,\ x_2 + x_3) \\ &= (x_1 + x_2 - 3(x_2 + x_3),\ 2(x_1 + x_2)) \end{aligned}$$

$= (x_1 + x_2 - 3x_2 - 3x_3,\ 2x_1 + 2x_2)$
$= (x_1 - 2x_2 - 3x_3,\ 2x_1 + 2x_2).$

Theorem 34.3:

↳ Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations. Then $M \circ L : \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation, and:

$$[M \circ L] = [M][L].$$

Example 34.3: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a counterclockwise rotation about the origin by an angle of $\frac{\pi}{4}$, and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a projection onto the x_1-axis. Find the standard matricies $M \circ L$ and $L \circ M$.

$$[L] = \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$[M] = [\operatorname{proj}_{\vec{e}_1} \vec{e}_1 \quad \operatorname{proj}_{\vec{e}_1} \vec{e}_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore [M \circ L] = [M][L] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 \end{bmatrix}$$

$$\therefore [L \circ M] = [L][M] = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 \end{bmatrix}$$

Example 34.4: Let $L, M : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations defined by $L(x_1, x_2) = (2x_1 + x_2, x_1 + x_2)$ and $M(x_1, x_2) = (x_1 - x_2, -x_1 + 2x_2)$. Find $[M \circ L]$ and $[L \circ M]$.

$$[L] = [L(\vec{e}_1) \quad L(\vec{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$[M] = [M(\vec{e}_1) \quad M(\vec{e}_2)] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore [M \circ L] = [M][L] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore [L \circ M] = [L][M] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we see that $[M \circ L] = I = [L \circ M] \therefore M \circ L = L \circ M.$

↳ this implies that M and L are inverses of each other

Lecture 35

Inverse Linear Transformations

Definition 35.1: Identity Operator

↳ The linear operator Id on $\mathbb{R}^n$ defined by $Id(\vec{x}) = \vec{x}$ $\forall x \in \mathbb{R}^n$ is called the identity operator or identity transformation.

$\therefore [Id] = [Id(\vec{e}_1) \; \cdots \; Id(\vec{e}_n)] = [\vec{e}_1 \cdots \vec{e}_n] = I.$

Definition 35.2: Invertible Linear Operator

↳ If L is a linear operator on $\mathbb{R}^n$ and there exists another linear operator M on $\mathbb{R}^n$ such that $M \circ L = Id = L \circ M$, then we say L is invertible and call M the inverse of L, and write $L^{-1} = M$ (and that $M^{-1} = L$).

Theorem 35.1:

↳ If $L, M : \mathbb{R}^n \to \mathbb{R}^n$ are linear operators, then M is the inverse of L iff $[M]$ is the inverse of $[L]$.

Note: if L is an invertible operator on $\mathbb{R}^n$, then

$$[L^{-1}] = [L]^{-1}$$

Example 35.1: Recall that $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a counterclockwise rotation about the origin by an angle of θ. Describe the inverse transformation of R_θ and find it's standard matrix.

· The inverse transformation of R_θ is $R_\theta^{-1} = R_{-\theta}$:

↳ $[R_{-\theta}] = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

· We can also use the inversion algorithm to find $[R_\theta]^{-1}$.

↳ $\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right]$

Exercise 35.1: Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $L(x_1, x_2) = (2x_1 + 5x_2,\ x_1 + 3x_2)$. Find L^{-1}, that is, find an expression for $L^{-1}(x_1, x_2)$.

↳ $[L] = [L(\vec{e}_1)\ \ L(\vec{e}_2)] = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right] \xrightarrow[R_2 - 2R_1]{} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[-R_2]{} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\therefore \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 5x_2 \\ -x_1 + 2x_2 \end{bmatrix}$$

$$\therefore L^{-1}(x_1, x_2) = (3x_1 - 5x_2,\ -x_1 + 2x_2)$$

The Kernel and Range of a Linear Transformation

Definition 35.3: The Kernel of a Linear Transformation

↳ Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a (linear) transformation.
The kernel of L is $\text{Ker}(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{0}\}$.

↳ $\text{Ker}(L) = \text{Null}(L)$ (the nullspace of L).

Example 35.2:

↳ Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $L(x_1, x_2) = (x_1 - x_2, -3x_1 + 3x_2)$

Determine which of $\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ belong to $\text{Ker}(L)$.

$L(\vec{x}_1) = L(0, 0) = (0 - 0, -3(0) + 3(0)) = (0, 0)$
$L(\vec{x}_2) = L(1, 1) = (1 - 1, -3(1) + 3(1)) = (0, 0)$
$L(\vec{x}_3) = L(3, 2) = (3 - 2, -3(3) + 3(2)) = (1, -3)$.

$\therefore \vec{x}_1, \vec{x}_2 \in \text{Ker}(L)$, but $\vec{x}_3 \notin \text{Ker}(L)$.

Definition 35.4: The Range of a Linear Transformation

↳ Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a (linear) transformation. The range of L is $\text{Range}(L) = \{L(\vec{x}) \mid \vec{x} \in \mathbb{R}\}$.

Note: $\text{Range}(L) \subseteq \mathbb{R}^m$.

Example 35.3: Let $L: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $L(x_1, x_2) = (x_1 + x_2, 2x_1 + x_2, 3x_2)$.

Determine which of $\vec{y}_1 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ and $\vec{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ belong to $\text{Range}(L)$.

↳ $\vec{y}_1$: $L(x_1, x_2) = (x_1 + x_2, 2x_1 + x_2, 3x_2) = (2,3,3)$.

↳ $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 0 & 3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore x_1 = x_2 = 1$, so $L(1,1) = (2,3,3)$.

$\therefore \vec{y}_1 \in$ Range (L).

$\vec{y}_2$: $L(x_1, x_2) = (x_1 + x_2, 2x_1 + x_2, 3x_2) = (1,1,2)$.

↳ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{array}\right]$

$\therefore$ the system is inconsistent, so there is no $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ST $L(x_1, x_2) = (1,1,2)$.

$\therefore \vec{y}_2 \notin$ Range (L).

Theorem 35.2:

↳ Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix $[L]$. Then:

1) $\text{Ker}(L) = \text{Null}([L])$.
2) $\text{Range}(L) = \text{Col}([L])$.

In particular, Ker (L) is a subspace of $\mathbb{R}^n$ and Range (L) is a subspace of $\mathbb{R}^m$.

Example 35.4: Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a projection onto the line through the origin with direction vector $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find a basis for Ker(L) and Range (L).

$$[L] = [L(\vec{e}_1) \;\; L(\vec{e}_2) \;\; L(\vec{e}_3)]$$
$$= [\operatorname{proj}_{\vec{d}}\, \vec{e}_1 \;\; \operatorname{proj}_{\vec{d}}\, \vec{e}_2 \;\; \operatorname{proj}_{\vec{d}}\, \vec{e}_3].$$
$$= \left[\frac{1}{\sqrt{1+1+1}^{\,2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\; \frac{1}{\sqrt{1+1+1}^{\,2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\; \frac{1}{\sqrt{1+1+1}^{\,2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$
$$= \left[\frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\; \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\; \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$
$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

To find a basis for Ker(L), we solve the homogeneous system of equations given by $[L]\vec{x} = \vec{0}$. Carrying [L] to RREF gives:

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{3R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \therefore \vec{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad s, t \in \mathbb{R}.$$

So, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\ker(L)$.

The RREF of $[L]$ has a leading one in the first column only, and so a basis for $\operatorname{Range}(L)$ is $\left\{ \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}$.

Lecture 36

Determinants, Adjugates, and Matrix Inverses

the 2×2 case!

Definition 36.1: The Determinant and the Adjugate

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$.

The determinant of A is:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The adjugate of A is:

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 36.1: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then:

$$\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2.$$

Also,

$$A\,(\text{adj } A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (\det A)\, I.$$

$$(\text{adj } A)A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (\det A)\, I$$

$$\therefore A\,(\text{adj } A) = (\text{adj } A)\, A = (\det A)\, I.$$

Theorem 36.1

Let $A \in M_{2\times 2}(\mathbb{R})$. Then:

$$A(\text{adj } A) = (\det A)\, I = (\text{adj } A).$$

A is only invertible if $\det A \neq 0$, and:

$$A^{-1} = \frac{1}{\det A}\, \text{adj } A.$$

Definition 36.2: Cofactors of an $n\times n$ matrix

Let $A \in M_{n\times n}(\mathbb{R})$ and let $A(i,j)$ be the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column of A. The (i,j)-cofactor of A, denoted by

C_{ij}, is:

$$C_{ij} = (-1)^{i+j} \det A(i,j)$$

Example 36.2: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 4 \\ 4 & 1 & 1 \end{bmatrix}$. Then the (3, 2)-cofactor of A is:

$$C_{32} = (-1)^{3+2} \det A(3,2) = (-1)^5 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= (-1)(4-3) = -1.$$

and the (2, 2)-cofactor of A is:

$$C_{22} = (-1)^{2+2} \det A(2,2) = (-1)^4 \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= (1)(1-12) = -11.$$

Definition 36.3: The Determinant: the $n \times n$ case.

↳ Let $A = [a_{ij}] \in M_{n \times n}(\mathbb{R})$. $\forall i = 1, \dots, n$, we define the determinant of A as

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

which we refer to as a cofactor expansion of A along the i^{th} row of A. Equivalently, $\forall j = 1, \dots, n$,

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$$

which we refer to as a cofactor expansion of A along the j^{th} column of A.

Example 36.3: Compute $\det A$; $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{bmatrix}$

- Doing a cofactor expansion along the first row:

$$\det A = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} -5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - 3\begin{vmatrix} 4 & -5 \\ -7 & 8 \end{vmatrix}$$

$$= 1(-45-48) - 2(36+42) - 3(32+35)$$

$$= -93 - 2(78) - 3(-3)$$

$$= -93 - 156 + 9$$

$$= -240$$

- Doing a cofactor expansion along the second column:

$$\det A = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{vmatrix} = -2\begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - 5\begin{vmatrix} 1 & -3 \\ -7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & -3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(36+42) - 5(9-21) - 8(6+12)$$

$$= -2(78) - 5(12) - 8(18)$$

$$= -240 \text{ (as before!)}$$

- Note: doing a cofactor expansion along a row or column with many 0s makes it much easier to compute the determinant.

Lecture 37

↱ the $n \times n$ case!

Definition 37.1: The Cofactor Matrix and The Adjugate

↳ Let $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$

↓

1) $C_{ij} = (-1)^{i+j} \det A(i,j)$ is the (i,j)-cofactor of A

2) The cofactor matrix of A is:

$$\operatorname{cof} A = [C_{ij}] \in M_{n\times n}(\mathbb{R})$$

3) The adjugate of A is:

$$\operatorname{adj} A = [C_{ij}]^T \in M_{n\times n}(\mathbb{R})$$

Example 37.1: Find adj A if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 1 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

Example 37.2: Find $\det A$ and A^{-1} if $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix}$

$\hookrightarrow \det A = 1\begin{vmatrix}1&4\\2&4\end{vmatrix} - 1\begin{vmatrix}1&4\\1&4\end{vmatrix} + 2\begin{vmatrix}1&1\\1&2\end{vmatrix}$

$$= 1(4-8) - 1(4-4) + 2(2-1)$$
$$= 1(-4) - 0 + 2$$
$$= -4+2 = -2.$$

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix}1&4\\2&4\end{vmatrix} & -\begin{vmatrix}1&4\\1&4\end{vmatrix} & \begin{vmatrix}1&1\\1&2\end{vmatrix} \\ -\begin{vmatrix}1&2\\2&4\end{vmatrix} & \begin{vmatrix}1&2\\1&4\end{vmatrix} & -\begin{vmatrix}1&2\\1&2\end{vmatrix} \\ \begin{vmatrix}1&2\\1&4\end{vmatrix} & -\begin{vmatrix}1&2\\1&4\end{vmatrix} & \begin{vmatrix}1&1\\1&1\end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix}-4&0&1\\0&2&-1\\2&-2&0\end{bmatrix}^T = \begin{bmatrix}-4&0&2\\0&2&-2\\1&-1&0\end{bmatrix}.$$

since $A^{-1} = \frac{1}{\det A}\operatorname{adj} A = -\frac{1}{2}\begin{bmatrix}-4&0&2\\0&2&-2\\1&-1&0\end{bmatrix}$

$$A^{-1} = \begin{bmatrix}2&0&1\\0&1&-1\\1/2&1/2&0\end{bmatrix}$$

Example 37.3: $A = \begin{bmatrix}1&2\\1&4\end{bmatrix}$, $B = \begin{bmatrix}2&1\\4&1\end{bmatrix}$, $C = \begin{bmatrix}1&3\\1&5\end{bmatrix}$, $D = \begin{bmatrix}2&2\\2&4\end{bmatrix}$

$\therefore \det = 2$, $\det B = -2$, $\det C = 2$, $\det D = 4$.

Notice that B, C, and D can be derived from A by one elementary column operation.

$\hookrightarrow A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow[C_1 \leftrightarrow C_2]{} \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} = B$ and $\det B = -\det A$.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow[C_1 + C_2 \to C_2]{} \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} = C$ and $\det C = \det A$.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{2C_1 \to C_2} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} = D$ and $\det D = 2 \det A$.

Theorem 37.2:

$\hookrightarrow$ Let $A \in M_{n \times n}(\mathbb{R})$:

1) If A has a row/column of 0s, then $\det A = 0$.
2) If B is obtained from A by swapping two rows or two columns, then $\det B = -\det A$.
3) If B is obtained from A by adding a multiple of one row/column to another row, then $\det B = \det A$.
4) If two rows/columns of A are equal, then $\det A = 0$.
5) If B is obtained from A by multiplying a row/column by $c \in \mathbb{R}$, then $\det B = c \det A$.

Example 37.4: Find $\det A$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$

$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} \xrightarrow[\substack{R_2 - 4R_1 \\ R_3 - 7R_1}]{} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} - 0 + 0.$

$\det A = \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} \xrightarrow{-\frac{1}{3} C_1 \to C_1} (-3) \begin{vmatrix} 1 & -6 \\ 2 & -11 \end{vmatrix} = (-3)(-11+12) = -3.$

Example 37.5: Let $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$. Show $\det A = (b-a)(c-a)(c-b)$.

$$\det A = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \underset{\substack{R_2 - R_1 \\ R_3 - R_1}}{=} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = 1 \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a) \begin{vmatrix} 1 & (b+a) \\ c-a & (c+a)(c-a) \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c+a-b-a)$$
$$= (b-a)(c-a)(c-b)$$

Exercise 37.1: When does $A = \begin{bmatrix} x & x & 1 \\ x & 1 & x \\ 1 & x & x \end{bmatrix}$ fail to be invertible?

↳ A fails to be invertible when $\det A = 0$.

$$\therefore 0 = \begin{vmatrix} x & x & 1 \\ x & 1 & x \\ 1 & x & x \end{vmatrix} \underset{\substack{R_1 - xR_3 \\ R_2 - xR_3}}{=} \begin{vmatrix} 0 & x-x^2 & 1-x^2 \\ 0 & 1-x^2 & x-x^2 \\ 1 & x & x \end{vmatrix} = 1 \begin{vmatrix} x(1-x) & (1+x)(1-x) \\ (1+x)(1-x) & x(1-x) \end{vmatrix}$$

$$= (1-x)^2 \begin{vmatrix} x & 1+x \\ 1+x & x \end{vmatrix}$$

$$= (1-x)^2 \left(x^2 - (1+x)^2 \right) = (1-x)^2 \left(x^2 - 1 - 2x - x^2 \right)$$

$$= -(1-x)^2 (2x+1)$$

$\therefore (x-1)^2(2x+1) = 0$ when $x = 1, -\frac{1}{2}$.

Example 37.6: compute $\det A$ if $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$.

↳ $\det A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{vmatrix} = 1 \begin{vmatrix} 3 & 0 & 0 \\ 5 & 6 & 0 \\ 8 & 9 & 10 \end{vmatrix} = (1)(3) \begin{vmatrix} 6 & 0 \\ 9 & 10 \end{vmatrix}$

$= 3(60 - 0) = 180$

Definition 37.2: Upper and Lower Triangular Matricies

↳ Let $A \in M_{m \times n}(\mathbb{R})$. A is upper triangular if every entry below the main diagonal is zero. A is lower triangular if every entry above the main diagonal is zero.

Example 37.7:

↳ The matricies $\begin{bmatrix} 4 & -7 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & -2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

are upper triangular.

The matricies $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

are lower triangular.

Theorem 37.3

↳ If $A = [a_{ij}] \in M_{n \times n}(\mathbb{R})$ is a triangular matrix (upper or lower triangular), then

$$\det A = a_{11} a_{22} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii} .$$

Lecture 38

Properties of Determinants

Theorem 38.1:

↳ If $A \in M_{n \times n}(\mathbb{R})$ and $k \in \mathbb{R}$, then $\det(kA) = k^n \det A$.

Example 38.1:

↳ Find $(\det A)(\det B)$ and $\det(AB)$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$.

$$(\det A)(\det B) = (4-6)(2+1) = (-2)(3) = -6.$$

$$\det(AB) = \begin{vmatrix} -1 & 5 \\ -1 & 11 \end{vmatrix} = (-11+5) = -6.$$

Theorem 38.2:

↳ If $A, B \in M_{n \times n}(\mathbb{R})$, then $\det(AB) = (\det A)(\det B)$.

Theorem 38.3:

↳ Let $A \in M_{n \times n}(\mathbb{R})$ be invertible. Then $\det(A^{-1}) = \dfrac{1}{\det A}$

Example 38.2:

↳ Let $A_1, A_2, \dots, A_k \in M_{n \times n}(\mathbb{R})$ be such that $A_1 A_2 \dots A_k$ is invertible.

Then, $0 \neq \det(A_1 A_2 \cdots A_k) = (\det A_1)(\det A_2) \cdots (\det A_k)$.

↳ $\therefore$, for $i = 1, 2, \ldots, k$, we have $\det A_i \neq 0$, so A_i is invertible.

Theorem 38.4:

↳ Let $A \in M_{n \times n}(\mathbb{R})$. Then $\det(A^T) = \det(A)$.

Example 38.3: If $\det(A) = 3$, $\det(B) = -2$, and $\det(C) = 4$ for $A, B, C \in M_{n \times n}(\mathbb{R})$, find $\det(A^2 B^T C^{-1} B^2 (A^{-1})^2)$.

↳ $$\det(A^2 B^T C^{-1} B^2 (A^{-1})^2) = \det A^2 \det B^T \det C^{-1} \det B^2 \det (A^{-1})^2$$
$$= (\det A)^2 (\det B) \frac{1}{\det C} (\det B)^2 \frac{1}{(\det A)^2}$$
$$= \frac{(\det B)^3}{\det C} = \frac{(-2)^3}{4} = -2.$$

Example 38.4: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

↳ $\det A + \det B = 0 + 0 = 0$.

$\det(A + B) = \det(I) = 1$.

$\therefore \det(A + B) \neq \det A + \det B$.

Lecture 39

Eigenvalues and Eigenvectors

Definition 39.1: Eigenvalues and Eigenvectors

↳ For $A \in M_{n\times n}(\mathbb{R})$, a scalar λ is an eigenvalue of A if $A\vec{x} = \lambda\vec{x}$ for some nonzero vector $\vec{x}$. The vector $\vec{x}$ is then an eigenvector of A corresponding to λ.

Example 39.1: $A = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$$A\vec{x} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\vec{x}.$$

$\therefore$ $\lambda = 1$ is an eigenvalue of A and $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the corresponding eigenvector.

Theorem 39.1

↳ Let $A \in M_{n\times n}(\mathbb{R})$. A number λ is an eigenvalue of A if and only if λ satisfies the equation:

$$\det(A - \lambda I) = 0.$$

If λ is an eigenvalue of A, then all nonzero solutions of the homogeneous system of equations:

$$(A - \lambda I)\vec{x} = 0$$

are the eigenvectors of A corresponding to λ.

Definition 39.2: Characteristic Polynomial

↳ Let $A \in M_{n\times n}(\mathbb{R})$. The characteristic polynomial of A is:

$$C_A(\lambda) = \det(A - \lambda I)$$

Example 39.3: Find the eigenvalues and all corresponding eigenvectors for $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

↳ $C_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - (-2)$

$= 4 - 4\lambda - \lambda + \lambda^2 + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2).$

λ is only an eigenvalue of A iff $C_A(\lambda) = 0$.

↳ $(\lambda-3)(\lambda-2) = 0$ @ $\lambda = 3, 2$.

$\therefore$, $\lambda_1 = 2$ and $\lambda_2 = 3$ are the eigenvalues of A.

To find the eigenvectors of A corresponding to $\lambda_1 = 2$, we solve the homogeneous system $(A - 2I)\vec{x} = 0$

↳ $A - 2I = \begin{bmatrix} 1-2 & 2 \\ -1 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \xrightarrow[R_2 - R_1]{} \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$

so, $\vec{x} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

$\therefore$ the eigenvectors of A corresponding to $\lambda_1 = 2$ are:

↳ $t\begin{bmatrix}2\\1\end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

To find the eigenvectors of A corresponding to $\lambda_2 = 3$, we solve the homogeneous system $(A-3I)\vec{x} = 0$.

↳ $A - 3I = \begin{bmatrix}1-3 & 2\\-1 & 4-3\end{bmatrix} = \begin{bmatrix}-2 & 2\\-1 & 1\end{bmatrix} \xrightarrow[R_2 - \frac{R_1}{2}]{} \begin{bmatrix}-2 & 2\\0 & 0\end{bmatrix} \xrightarrow{-\frac{R_1}{2}} \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$.

So, $\vec{x} = \begin{bmatrix}s\\s\end{bmatrix} = s\begin{bmatrix}1\\1\end{bmatrix}$, $s \in \mathbb{R}$.

$\therefore$ the eigenvectors of A corresponding to $\lambda_2 = 3$ are:

↳ $s\begin{bmatrix}1\\1\end{bmatrix}$, $s \in \mathbb{R}$, $s \neq 0$.

Lecture 40:

Definition 40.1: Eigenspace

↳ Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{R})$. The set containing all of the eigenvectors of A corresponding to λ together with the zero vector is called the eigenspace of A corresponding to λ, and is denoted by $E_\lambda(A)$.

Note: $E_\lambda(A) = \text{Null}(A - \lambda I)$.

Theorem 40.1:

↳

↳ Let λ be an eigenvalue of $A \in M_{n \times n}(\mathbb{R})$.

- If $\lambda \in \mathbb{R}$, then $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.
- If $\lambda \notin \mathbb{R}$, then $E_\lambda(A)$ is a subspace of $\mathbb{C}^n$.

Example 40.1: Find the eigenvalues and a basis for each eigenspace of A where $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

↳ $$C_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \underset{R_3 - R_2}{\overset{R_1 + \lambda R_2}{=}} \begin{vmatrix} 0 & 1-\lambda^2 & 1+\lambda \\ 1 & -\lambda & 1 \\ 0 & 1+\lambda & -\lambda-1 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 1-\lambda^2 & 1+\lambda \\ 1+\lambda & -1-\lambda \end{vmatrix} = (-1)(1+\lambda) \begin{vmatrix} 1-\lambda^2 & 1 \\ 1+\lambda & -1 \end{vmatrix}$$

$$= (-1)(1+\lambda)^2 \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1 \end{vmatrix} = (-1)(1+\lambda)^2(\lambda - 1 - 1)$$

$$= (-1)(\lambda+1)^2(\lambda-2).$$

$\therefore$ the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$.

For $\lambda_1 = 2$: we solve $(A - 2I)\vec{x} = 0$:

↳ $$A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 + 2R_2} \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow[R_3 + R_1]{} \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_1/3} \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\vec{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

$\therefore$ a basis for $E_{\lambda_1}(A)$ is $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = -1$: we solve $(A + I) = 0$:

$\hookrightarrow A+I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_1 - R_3]{R_1 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so, $\vec{x} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $t, s \in \mathbb{R}$.

$\therefore$ a basis for $\lambda_2 = -1$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Theorem 40.2:

$\hookrightarrow$ Let $A \in M_{n \times n}(\mathbb{R})$. Then $C_A(\lambda)$ is a real polynomial of degree n.

Definition 40.2:

$\hookrightarrow$ Let $A \in M_{n \times n}(\mathbb{R})$ with eigenvalue λ. The algebraic multiplicity, denoted by a_λ, is the number of times λ appears as a root of $C_A(\lambda)$.

Example 40.2:

$\hookrightarrow$ In example 40.1, we found that $\lambda_1 = 2$ and $\lambda_2 = -1$ were the only two eigenvalues of A since $C_A(\lambda) = (-1)(\lambda+1)^2(\lambda-2)$.

The exponent of 2 on the $(\lambda+1)$ term means that $\lambda_2 = -1$ has an algebraic multiplicity of 2, while the exponent of 1 on the $(\lambda - 2)$ term means that $\lambda_1 = 2$ has an algebraic multiplicity of 1.

$\therefore a_{\lambda_1} = 1, \ a_{\lambda_2} = 2.$

Definition 40.3: Geometric Multiplicity

↳ Let $A \in M_{n \times n}(\mathbb{R})$ with eigenvalue λ. The geometric multiplicity of λ, denoted by g_λ, is the dimension of the eigenspace $E_\lambda(A)$.

Example 40.3:

↳ From example 40.1, we found that $\lambda_1 = 2$ and $\lambda_2 = -1$ were the only two eigenvalues of A.

\# of vector in the basis

We saw that $\dim(E_{\lambda_1}(A)) = 1$ and $\dim(E_{\lambda_2}(A)) = 2$.

$\therefore g_{\lambda_1} = 1, \ g_{\lambda_2} = 2.$

Theorem 40.3:

↳ $\forall \ A \in M_{n \times n}(\mathbb{R})$ and any eigenvalue of A,

$$1 \leq g_\lambda \leq a_\lambda \leq n.$$

Example 40.5: $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find the eigenvalues for A, and for each eigenvalue, find one corresponding eigenvector.

$$C_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1$$

$$= 1 - 2\lambda + \lambda^2 + 1 = \lambda^2 - 2\lambda + 2$$

$$\frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2j}{2} = 1 \pm j.$$

$\therefore \lambda_1 = 1 + j, \quad \lambda_2 = 1 - j.$

λ_1: $(A - (1+j)I)\vec{x} = 0$

$$A - (1+j)I : \begin{bmatrix} j & -1 \\ 1 & j \end{bmatrix} \xrightarrow{-jR_1} \begin{bmatrix} 1 & j \\ 1 & j \end{bmatrix} \xrightarrow[R_2 - R_1]{} \begin{bmatrix} 1 & j \\ 0 & 0 \end{bmatrix}$$

so $\vec{x} = \begin{bmatrix} -jt \\ t \end{bmatrix} = t \begin{bmatrix} -j \\ 1 \end{bmatrix}, \quad t \in \mathbb{C}.$

λ_2: $(A - (1-j)I)\vec{x} = 0$

$$A - (1-j)I : \begin{bmatrix} -j & -1 \\ 1 & -j \end{bmatrix} \xrightarrow{jR_1} \begin{bmatrix} 1 & -j \\ 1 & -j \end{bmatrix} \xrightarrow[R_2 - R_1]{} \begin{bmatrix} 1 & -j \\ 0 & 0 \end{bmatrix}$$

so $\vec{x} = \begin{bmatrix} jt \\ t \end{bmatrix} = t \begin{bmatrix} j \\ 1 \end{bmatrix}, \quad t \in \mathbb{C}.$

$\therefore \begin{bmatrix} j \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1 + j$.

$\begin{bmatrix} j \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 1 - j$.

Lecture 41

Diagonalization

Definition 41.1: Diagonal Matrix

↳ an $n \times n$ matrix D such that $d_{ij} = 0 \ \forall i \neq j$ is a diagonal matrix and is denoted by $D = \text{diag}(d_{11}, \dots, d_{nn})$.

Diagonal Matricies are both upper and lower triangular.

Example 41.1:

↳ The matricies $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

are diagonal matrices.

Lemma 41.1:

↳ If $D = \text{diag}(d_{11}, \dots, d_{nn})$ and $E = \text{diag}(e_{11}, \dots, e_{nn})$,

1) $D + E = \text{diag}(d_{11} + e_{11}, \dots, d_{nn} + e_{nn})$.
2) $DE = \text{diag}(d_{11}e_{11}, \dots, d_{nn}e_{nn}) = ED$
3) $\forall k \in \mathbb{N}, \ D^k = \text{diag}(d_{11}^k, \dots, d_{nn}^k)$.

↳ holds $\forall k \in \mathbb{Z}$ if none of $d_{11}, \dots, d_{nn} = 0$

↳ aka, if D is invertible.

Definition 41.2: Diagonalizable Matrix

↳ An $n \times n$ matrix A is diagonalizable if there exists an $n \times n$ invertible matrix P and an $n \times n$ diagonal matrix D so that $P^{-1}AP = D$

↳ We say P diagonalizes A to D.

Lemma 41.2:

↳ Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of A. If B_i is a basis for the eigenspace $E_{\lambda_i}(A)$, then $B = B_1 \cup B_2 \cup \ldots \cup B_k$ is linearly independent.

Theorem 41.1: Diagonalization Theorem

↳ A matrix $A \in M_{n \times n}(\mathbb{R})$ with every eigenvalue being real is diagonalizable iff there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Corollary 41.1:

↳ An $n \times n$ matrix is diagonalizable iff $a_\lambda = g_\lambda$ for every eigenvalue λ of A.

Corollary 41.2:

↳ If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Example 41.2: Diagonalize $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

↳ From example 39, $\lambda_1 = 2$ and $\lambda_2 = 3$.
$a_{\lambda_1} = 1$ and $a_{\lambda_2} = 1$

Since $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for λ_1, $g_{\lambda_1} = 1$.
Since $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for λ_2, $g_{\lambda_2} = 1$.

Since $a_{\lambda_1} = 1 = g_{\lambda_1}$ and $a_{\lambda_2} = 1 = g_{\lambda_2}$, A is diagonalizable.

We take $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and have that P diagonalizes A, that is, $P^{-1}AP = \text{diag}(2,3) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Lecture 42

Powers of Matricies

↳ $A^k = PD^kP^{-1}$

Example 42.1: Find a formula for A^k $\forall k \in N$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

|

↳ From example 41.2, A is diagonalizable with

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

So,
$$A^k = P D^k P^{-1}$$
$$A^k = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$A^k = \begin{bmatrix} 2 \cdot 2^k + 0 & 0 + 3^k \\ 2^k + 0 & 0 + 3^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$A^k = \begin{bmatrix} 2^{k+1} & 3^k \\ 2^k & 3^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$A^k = \begin{bmatrix} 2^{k+1} - 3^k & 2 \cdot 3^k - 2^{k+1} \\ 2^k - 3^k & 2 \cdot 3^k - 2^k \end{bmatrix}$$

Definition 42.1 : Trace

↳ Let $A \in M_{n \times n}(\mathbb{R})$. The trace of A, denoted by $\text{tr } A$, is the sum of the entries on the main diagonal.

↳ $$\text{tr } A = \sum_{i=1}^{n} (A)_{ii} = (A)_{11} + \dots + (A)_{nn}$$

$$\text{tr } A = \lambda_1 a_{\lambda_1} + \dots + \lambda_k a_{\lambda_k} = \sum_{i=1}^{k} (\lambda_i a_{\lambda_i}).$$

Example 42.2: Find the det and trace of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

↳ $\lambda_1 = -1$ and $\lambda_2 = 2$ with $a_{\lambda_1} = 2$ and $a_{\lambda_2} = 1$.

↳ $\det A = (-1)^2 (2)^1 = 2.$

$\text{tr } A = (-1)(2) + (2)(1) = 0.$

Lecture 43

Definition 43.1 : Orthogonal Set

↳ A set $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is an orthogonal set if $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$.

Example 43.1:

↳ the standard basis $\{\vec{e}_1, \ldots, \vec{e}_k\}$ for $\mathbb{R}^n$ is an orthogonal set

the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^4$.

Theorem 43.1:

↳ If all vectors in an orthogonal set are nonzero, then the set is linearly independent.

Definition 43.2: Orthogonal Basis

↳ If an orthogonal set B is a basis for a subspace S of $\mathbb{R}^n$, then B is an orthogonal basis for S.

↳ $\vec{x} = \dfrac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 + \cdots + \dfrac{\vec{x} \cdot \vec{v}_k}{\|\vec{v}_k\|^2} \vec{v}_k$

Example 43.2: let $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right\}$ be an orthogonal basis for $\mathbb{R}^2$. Write $\vec{x}$ as a linear combination of the vectors in B.

↳ For $c_1, c_2 \in \mathbb{R}$, consider $\vec{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

Then
$$c_1 = \frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} = \frac{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}}{\left(\sqrt{1^2 + 3^2}\right)^2} = \frac{-1+3}{1+9} = \frac{2}{10} = \frac{1}{5}$$

$$c_2 = \frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2} = \frac{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \end{bmatrix}}{\left(\sqrt{6^2 + (-2)^2}\right)^2} = \frac{-6-2}{36+4} = \frac{-8}{40} = -\frac{1}{5}$$

$$\therefore \vec{x} = \frac{1}{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

Definition 43.3: Orthonormal Set, Orthoromal Basis

↳ An orthogonal set $\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is called an orthormal set if $\|\vec{v}_i\| = 1$ for $i = 1, \dots, k$. If an orthonormal set B is a basis for subspace S of $\mathbb{R}^n$, then B is an orthonormal basis for S.

Example 43.3:

↳ The standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$ is an orthogonal set (and an orthonormal basis for $\mathbb{R}^n$).
The set:

$\hookrightarrow \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$

is also an orthonormal set (and also an orthonormal basis for $\mathbb{R}^3$).

Exercise 43.1: From $B = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^2$. Obtain an orthonormal basis C for $\mathbb{R}^2$ from B and express $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in C.

Since $\left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\| = \sqrt{1+9} = \sqrt{10}$ and $\left\| \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right\| = \sqrt{40} = 2\sqrt{10}$,

we have $C = \left\{ \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

Since $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} = -\frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}} = \frac{2}{\sqrt{10}}$ and

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix} = -\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}} = -\frac{4}{\sqrt{10}}$,

we obtain $\vec{x} = \frac{2}{\sqrt{10}} \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} - \frac{4}{\sqrt{10}} \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$

Definition 43.4: Orthogonal Matrix

$\hookrightarrow$ Let $P \in M_{n \times n}(\mathbb{R})$. P is an orthogonal matrix if

$$P^T P = I \rightarrow P^{-1} = P^T$$

Theorem 43.2:

↳ Let $P \in M_{n\times n}(\mathbb{R})$. The following are equivalent:

1) P is an orthogonal matrix
2) the columns of P form an orthonormal basis for $\mathbb{R}^n$
3) the rows of P form an orthonormal basis for $\mathbb{R}^n$
4) P^{-1} is an orthogonal matrix

Example 43.4:

↳ • The $n\times n$ identity matrix is an orthogonal matrix since $I^{-1} = I = I^T$

• $[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix since:

$$[R_\theta]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = [R_\theta]^T$$

• The matrix $P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ is an orthogonal matrix since its columns form an orthonormal basis for $\mathbb{R}^3$.

Lecture 44

Lemma 44.1:

↳ Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$. $\forall\, t_1, \dots, t_{k-1} \in \mathbb{R}$,

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k + t_1\vec{v}_1 + \dots + t_{k-1}\vec{v}_{k-1}\}$$

Example 44.1: Let $B = \{\vec{v}_1, \vec{v}_2\}$ with $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ be a basis for $\mathbb{R}^2$. Since $\vec{v}_1 \cdot \vec{v}_2 = 7 \neq 0$, B is not an orthogonal basis for $\mathbb{R}^2$.

↳ Let $\vec{\omega}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$\vec{\omega}_2 = \vec{v}_2 - \text{proj}_{\vec{\omega}_1} \vec{v}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{\omega}_1}{\|\vec{\omega}_1\|^2}\vec{\omega}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \frac{7}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

↳ $\vec{\omega}_2 = \begin{bmatrix} -2/5 \\ 1/5 \end{bmatrix}$.

Then $\{\vec{\omega}_1, \vec{\omega}_2\}$ is an orthogonal basis for $\mathbb{R}^2$ since it contains two nonzero nonparallel vectors and $\omega_1 \cdot \omega_2 = 0$. We may then normalize $\vec{\omega}_1$ and $\vec{\omega}_2$:

$$\vec{u}_1 = \frac{1}{\|\vec{\omega}_1\|}\vec{\omega}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{\omega}_2\|}\vec{\omega}_2 = \frac{1}{1/\sqrt{5}}\begin{bmatrix} -2/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$$

so $\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

Theorem 44.1 : Gram-Schmidt Procedure

↳ Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for a subspace S of $\mathbb{R}^n$. Define:

$\vec{w}_1 = \vec{v}_1$

$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}\vec{v}_2$

$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}\vec{v}_3 - \text{proj}_{\vec{w}_2}\vec{v}_3$

$\vdots$

$\vec{w}_k = \vec{v}_k - \text{proj}_{\vec{w}_1}\vec{v}_k - \text{proj}_{\vec{w}_2}\vec{v}_k - \cdots - \text{proj}_{\vec{w}_{k-1}}\vec{v}_k$

Then $\{\vec{w}_1, \dots, \vec{w}_k\}$ is an orthogonal basis for S and $\forall j = 1, \dots, k$, $\text{Span}\{\vec{v}_1, \dots, \vec{v}_j\} = \text{Span}\{\vec{w}_1, \dots, \vec{w}_k\}$.

Example 44.2: let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for a subspace S of $\mathbb{R}^5$ with:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Find an orthonormal basis for S.

$\vec{w}_1 = \vec{v}_1$.

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}\vec{v}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1+2+1}{4}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\hookrightarrow \vec{w}_2 = \begin{bmatrix} -3/2 \\ 3/2 \\ 1 \\ -1/2 \\ 1/2 \end{bmatrix} \xrightarrow{\times 2} \vec{w}_2 = \begin{bmatrix} -3 \\ 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

$$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}\vec{v}_3 - \text{proj}_{\vec{w}_2}\vec{v}_3 = \vec{v}_3 - \frac{\vec{w}_1 \cdot \vec{v}_3}{\|\vec{w}_1\|^2}\vec{w}_1 - \frac{\vec{w}_2 \cdot \vec{v}_3}{\|\vec{w}_2\|^2}\vec{w}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{24}\begin{bmatrix} -3 \\ 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} \rightarrow \vec{w}_3 = \begin{bmatrix} -1/4 \\ -3/4 \\ 1/2 \\ 1/4 \\ 3/4 \end{bmatrix} \xrightarrow{\times 4} \begin{bmatrix} -1 \\ -3 \\ 2 \\ 1 \\ 3 \end{bmatrix}.$$

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthogonal basis for S.

We now normalize the vectors:

$$\vec{u}_1 = \frac{1}{\|w_1\|} w_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{2\sqrt{6}}\begin{bmatrix} -3 \\ 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\|w_3\|} w_3 = \frac{1}{2\sqrt{6}}\begin{bmatrix} -1 \\ -3 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis for S.

Lecture 45

Definition 45.1: Orthogonal Subspaces

↳ Let S_1 and S_2 be two subspaces of $\mathbb{R}^n$. We say S_1 and S_2 are orthogonal if $\vec{x} \cdot \vec{y} = 0$ for every $\vec{x} \in S_1$ and every $\vec{y} \in S_2$.

Exercise 45.1:

↳ let S_1 and S_2 be two subspaces of $\mathbb{R}^n$ with $\{\vec{v}_1, \dots, \vec{v}_k\}$ a basis for S_1 and $\{\vec{w}_1, \dots, \vec{w}_k\}$ a basis for S_2. Then S_1 and S_2 are orthogonal iff $\vec{v}_i \cdot \vec{w}_j = 0$.

Definition 45.2: Orthogonally Diagonalizable Matrix

↳ An $A \in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalizable if there exists an $n\times n$ orthogonal matrix P and an $n\times n$ diagonal matrix D so that $P^TAP = D$. In this case, we say that P orthogonally diagonalizes A to D.

Example 45.1: From Example 41.2, the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ has eigenvalue $\lambda_1 = 2$ and $\lambda_2 = 3$ with

$\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ a basis for $E_{\lambda_1}(A)$ and

$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ a basis for $E_{\lambda_2}(A)$.

Since the basis vectors from the two eigenspaces are not orthogonal, the eigenspaces are not orthogonal. As a result, we cannot find an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of A, and $\therefore$ A is not orthogonally diagonalizable.

Theorem 45.1:

↳ Let $A \in M_{n\times n}(\mathbb{R})$ be a $\overbrace{\text{symmetric matrix}}^{A = A^T}$. Then every eigenvalue of A is real.

Lemma 45.1:

↳ Let $A \in M_{n\times n}(\mathbb{R})$. Then A is symmetric iff

$\vec{x} \cdot (A\vec{y}) = (A\vec{x}) \cdot \vec{y} \quad \forall\, \vec{x}, \vec{y} \in \mathbb{R}^n.$

Theorem 45.2:

↳ Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix. If λ_1, λ_2 are distinct eigenvalues of A, then $E_{\lambda_1}(A)$ and $E_{\lambda_2}(A)$ are orthogonal subspaces of $\mathbb{R}^n$.

Theorem 45.3:

↳ Let $A \in M_{n\times n}(\mathbb{R})$. Then A is symmetric iff A is orthogonally diagonalizable.

Lecture 46

Example 46.1: orthogonally diagonalize $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$.

$C_A(\lambda) = \det(A - \lambda I)$.

↳ $C_A(\lambda) = \begin{vmatrix} 5-\lambda & -4 & -2 \\ -4 & 5-\lambda & -2 \\ -2 & -2 & 8-\lambda \end{vmatrix} \overset{R_1 - R_2}{=} \begin{vmatrix} 9-\lambda & -9+\lambda & 0 \\ -4 & 5-\lambda & -2 \\ -2 & -2 & 8-\lambda \end{vmatrix}$

$\overset{C_1 + C_2 \to C_2}{=} \begin{vmatrix} 9-\lambda & 0 & 0 \\ -4 & 1-\lambda & -2 \\ -2 & -4 & 8-\lambda \end{vmatrix} = (9-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ -4 & 8-\lambda \end{vmatrix}$

$= (9-\lambda)((1-\lambda)(8-\lambda) - 8) = (9-\lambda)(8 - 9\lambda + \lambda^2 - 8)$

$= (9-\lambda)(\lambda^2 - 9\lambda) = (9-\lambda)(\lambda)(\lambda-9) = -\lambda(\lambda-9)^2$

$\therefore \lambda_1 = 0, \ \lambda_2 = 9.$

$a_{\lambda_1} = 1, \ a_{\lambda_2} = 2.$

λ_1: solve $(A - \lambda_1 I)\vec{x} = \vec{0}$

$\hookrightarrow A - 0I = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 1 & -4 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \xrightarrow[R_3 + 2R_1]{R_2 + 4R_1} \begin{bmatrix} 1 & 1 & -4 \\ 0 & 9 & -18 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{1/9 R_2} \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $\rightarrow x_3 = t$

$x_2 = 2t$

$x_1 = 2t$

$\therefore \vec{x} = \begin{bmatrix} 2t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$ $\rightarrow g_{\lambda_1} = 1 = a_{\lambda_1}.$

$\therefore$ a basis for $E_{\lambda_1}(A)$ is $B_1 = \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$

λ_2: solve $(A - \lambda_2 I)\vec{x} = 0$

$\hookrightarrow A - 9I = \begin{bmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ -2 & -2 & -1 \end{bmatrix} \begin{matrix} -1/4 R_1 \\ -1/4 R_2 \\ -1/2 R_3 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/2 \\ 1 & 1 & 1/2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so, $\vec{x} = \begin{bmatrix} -s - t/2 \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}, \ s, t \in \mathbb{R}.$

$\therefore$ a basis for $E_{\lambda_2}(A)$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ $\rightarrow g_{\lambda_2} = 2 = a_{\lambda_2}$

let $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$

Find an orthogonal basis for $E_{\lambda_2}(A)$

$\hookrightarrow$ let $\vec{w}_2 = \vec{v}_2.$

$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_2} \vec{v}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\|\vec{w}_2\|^2} \vec{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \vec{w}_3 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}.$

$$\vec{u}_1 = \frac{1}{\|w_1\|} w_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\|w_3\|} w_3 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/(3\sqrt{2}) \\ 1/(3\sqrt{2}) \\ -4/(3\sqrt{2}) \end{bmatrix}$$

$P^{-1}AP = D$ where $P^{-1} = P^T$ since P is orthogonal.

$\hookrightarrow \therefore P^T A P = D.$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad P = \begin{bmatrix} 2/3 & -1/\sqrt{2} & 1/(3\sqrt{2}) \\ 2/3 & 1/\sqrt{2} & 1/(3\sqrt{2}) \\ 1/3 & 0 & -4/(3\sqrt{2}) \end{bmatrix}$$

2)

b) $\det(A) = 2.$ $\quad \det((A^{-1})^3). = \dfrac{1}{\det(A)^3} = \dfrac{1}{2^3} = 1/8.$

$$z = 7e^{j\frac{8\pi}{9}}$$

$$r = 7, \quad \theta = \frac{8\pi}{9}.$$

$$z = 7\left(\cos\frac{8\pi}{9} + j\sin\frac{8\pi}{9}\right).$$

$x_5 = t$

$x_4 = -3t$

$x_3 = s$

$x_2 = -s - t$

$x_1 = -t - 3s$

$\ker(L) = \operatorname{span}\{[$

$$t\begin{bmatrix} -1 \\ -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} + s\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$M(\vec{x}) = perp_{\vec{d}}(\vec{x})$.

$M(s\vec{x} + t\vec{y}) = perp_{\vec{d}}(s\vec{x} + t\vec{y})$

$= s\vec{x} + t\vec{y} - proj_{\vec{d}}(s\vec{x} + t\vec{y})$

$= s\vec{x} + t\vec{y} - s\,proj_{\vec{d}}\vec{x} - t\,proj_{\vec{d}}\vec{y}$

$= s(\vec{x} - proj_{\vec{d}}\vec{x}) + t(\vec{y} - proj_{\vec{d}}\vec{y})$

$= s(perp_{\vec{d}}\, x) + t(perp_{\vec{d}}\, x)$

$= s\,M(\vec{x}) + t\,M(\vec{y})$

$M(\vec{x}) = perp_{\begin{bmatrix}1\\2\end{bmatrix}} \vec{x}$.

$$= \frac{x_1 \cdot 1 + x_2 \cdot 2}{1+2^2} \begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{x_1 + 2x_2}{5} \begin{bmatrix}1\\2\end{bmatrix}$$

$\frac{4}{5}x_1 - \frac{2}{5}x_2$

$$[R_\theta] = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

$$\begin{bmatrix}x_2\\-x_1\end{bmatrix}$$

$$[R_{\frac{3\pi}{2}}] = \begin{bmatrix}\cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2}\\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2}\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$perp_{\begin{bmatrix}1\\2\end{bmatrix}} \begin{bmatrix}x_2\\-x_1\end{bmatrix} = \begin{bmatrix}x_2\\-x_1\end{bmatrix} - proj_{\begin{bmatrix}1\\2\end{bmatrix}} \begin{bmatrix}x_2\\-x_1\end{bmatrix}$$

$$= \begin{bmatrix}x_2\\-x_1\end{bmatrix} - \frac{x_2 - 2x_1}{5}\begin{bmatrix}x_2\\-x_1\end{bmatrix}$$

$$= \begin{bmatrix}x_2\\x_1\end{bmatrix} - \begin{bmatrix}(x_2^2 - 2x_1x_2)/5\\(x_1x_2 + 2x_1^2)/5\end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = 3\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

$a = 3a \rightarrow$ impossible

basis $= \phi$.

dim (0).

$$\text{refl}_{\vec{d}}\, \vec{x} = 2\,\text{proj}_{\vec{d}}\, \vec{x} - \vec{x}$$

$$2\,\text{proj}_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \vec{e}_1 - \vec{e}_1$$

$$2\ \frac{1\cdot 0 + 1\cdot 1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$2\,\text{proj}_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \vec{e}_2 - \vec{e}_2$$

$$2\ \frac{1\cdot 0 + 1\cdot 1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \cdot \frac{\sqrt{2}}{2} & + \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2} \begin{bmatrix} -1 & +1 \\ 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \quad u \times v = \begin{bmatrix} u_2 v_3 - v_2 u_3 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - v_1 u_2 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 5 & 6 & 7 & 8 \end{vmatrix}$$

14 - 18

24 - 28

20 - 8

6 - 10

14 - 18

$L(0,0,1) = (1,0,0)$ $\quad L(0,1,1) = (1,1,0)$ $\quad L(1,1,1) = (1,1,1)$

$x_3 = x_1$ $\quad x_2 = x_2$ $\quad x_1 = x_3$

$L(x_1, x_2, x_3) = (x_3, x_2, x_1).$

$[L(\vec{e}_1)\ L(\vec{e}_2)\ L$ $\quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\det L = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(0-1) = -1.$$

$-1 \neq 0$ so it's invertible

$$\operatorname{adj}(L) = \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$L(\vec{v}) = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$$

$$[L] = \begin{bmatrix} 3 & 6 & 1 & -1 \\ 1 & 2 & 0 & -1 \\ -2 & -4 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 1 & -1 \\ 1 & 2 & 0 & -1 \\ 2 & 4 & -2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 1 & -1 \\ 1 & 2 & 0 & -1 \\ 3 & 6 & -2 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 1 & -1 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 1 & -1 \\ 3 & 6 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1/3 & -1/3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_4 = t$

$x_3 = -2t$

$x_2 = s$

$x_1 = t\frac{1}{3} + t\frac{2}{3} - 2s = t - 2s$

$$= t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker(L) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

$$\text{range}(L) = \left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 1 & 0 & a & 1 & d \\ -1 & -1 & b & -2 & e \\ 3 & 1 & c & 0 & f \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

$2a_1 - 5a_2 \quad -2a_1 - 3a_2 + 6a_3$

$$\begin{bmatrix} 1 & 0 & a & -1 & d \\ 0 & -1 & a+b & -1 & d+e \\ 0 & 1 & c-3a & 3 & f-3d \end{bmatrix}$$

$a_3 = 2a_1 - 5a_2$

$a_5 = -2a_1 - 3a_2 + 6a_3$

$$\begin{bmatrix} 1 & 0 & a & -1 & d \\ 0 & -1 & a+b & -1 & d+e \\ 0 & 0 & b+c-2a & & f-6d \end{bmatrix}$$

$f(x) \mid h(x)$
$g(x) \mid h(x)$

$f(x)u(x) + g(x)V(x) = 1$

$\hookrightarrow \gcd(f(x), g(x)) = 1.$

$h(x)f(x)u(x) + h(x)g(x)V(x) = h(x)$

$n \equiv 237 \pmod{1000}$
$n \equiv 100 \pmod{343}$

$n \equiv n_0 \pmod{343000}.$

x	y	q	r
1	0	1000	0
0	1	343	0
1	-2	314	2
-1	3	29	1
11	-32	24	10
-12	35	5	1
59	-172	4	4
-71	207	1	1
		0	4

$\therefore \gcd(1000, 343) = 1.$

$\therefore\ n \equiv n_0 \pmod{343000}$.

$n \equiv 237 \pmod{1000}$

$\hookrightarrow n = 1000x + 237$

$n \equiv 100 \pmod{343}$

$1000x + 237 \equiv 100 \pmod{343}$

$1000x \equiv -137 \pmod{343}$

$1000x + 343y = -137$ (LDE).

$1000(-71) + 343(207) = 1$

$1000(71 \cdot 137) - 343(207 \cdot 137) = -137$

$1000(9727) - 343(28359) = -137$

$\therefore\ x = 9727$.

$n_0 = 9727(1000) + 237$

$(e, n) = (7, 407)$

$(d, n) = \ ?$

$407 = 11 \cdot 37$

$\quad\ \ \downarrow \quad \downarrow$

$\quad\ \ p \quad q$

$ed \equiv 1 \pmod{360}$.

$7d \equiv 1 \pmod{360}$

$7d + 360y = 1$

x	y	q	r
1	0	360	0
0	1	7	0
1	-51	3	51
-2	103	1	2
		0	3

$$\therefore d = 103$$

$$\left.\begin{aligned}&[4][x] + [3][y] = [2]\\&[2][x] + [4][y] = [2].\end{aligned}\right\} \mathbb{Z}_6$$

$$4x + 3y = 2$$
$$4x = 4 - 8y$$
$$4 - 8y + 3y = 2$$
$$-5y = -2$$
$$[1][y] = [4].$$

$\gcd(1, 6) = 1$, so

$$[1][1]^{-1}[y] = [4][1]^{-1}$$
$$[y] = [4].$$

$$[2][x] + [16] = [2]$$
$$[2][x] + [4] = [2]$$
$$[x] = 5.$$

$$\begin{array}{r|l} & x^3 - 2x^2 + 2x \\ \hline x - 1 & x^4 - 3x^3 + 4x^2 - 2x \\ & x^4 - x^3 \\ \hline & -2x^3 + 4x^2 \end{array}$$

$$-2x^3+2x$$

$$2x^2-2x$$

$$2x^2-2x$$

$$0$$

$(x-1)(x^3-2x^2+2x) = x^4-3x^3+4x^2-2x$

$x(x-1)(x^2-2x+2) = x^4-3x^3+4x^2-2x$

$x(x-1)(x-1+i)(x-1-i).$

$$\frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2} = 1\pm i$$

$\gcd(a,14) \mid 2$

$a = 2, 4, 6, 8, 10, 12, 16, 18, 20, 22,$

1 2 3 4 5 6 7 8 9

1, 3, 5,

$5 \mid x(y+z)$

$$
\begin{array}{r|l}
 & z^{12} \\
\hline
1+i & z^{12} - 2 + 65 + i \\
 & z^{12} - (1+i)z^{10} \\
\hline
 & 2 + (1+i)z^{1}
\end{array}
$$

$a \mid b$ $ac \mid d$ $\rightarrow$ $a \mid (3b-5d).$

$a \mid ac$ by def

since $a \mid ac \mid d$, $a \mid d$ (TD).

since $a \mid b$ and $a \mid d$, $a \mid (bs+dt)$ (DIC)

take $s=3$ and $t=-5$

$\therefore a \mid (3b-5d).$

$\gcd(a,b)=1$ and $c|a$, then $\gcd(a,bc)=c$.

$c|a$ and $c|bc$

by BL, $as+bt=1$ for some s,t since $\gcd(a,b)=1$

$cas+cbt=c$

$acs+bct=c$

$\gcd(a,bc)=c$.

$w=z^3-3z^2\bar{z}+3z(\bar{z})^2-(\bar{z})^3$.

let $z=x+yi$

$(x+yi)^3-3z\bar{z}z+3\bar{z}z\bar{z}-(x-yi)^3$

$(x+yi)^3-3|z|^2z+3|z|^2\bar{z}-(x-yi)^3$

$(x+yi)^3-3|z|^2(z-\bar{z})-(x-yi)^3$

$(x+yi)^3-3|z|^2(2\mathrm{Im}(z)i)-(x-yi)^3$

$(x+yi)(x+yi)(x+yi)-$ imaginary $-(x-yi)^3$

$(x^2+2xyi-y^2)(x+yi)-(x-yi)^3$

$(x^3+\underset{\mathrm{Im}}{\underset{\downarrow}{x^2yi}}+\underset{\mathrm{im}}{\underset{\downarrow}{2x^2yi}}-2xy^2-xy^2-\underset{\mathrm{im}}{\underset{\downarrow}{y^3i}})-(x-yi)^3$

$x^3-2xy^2-xy^2-(x-yi)(x-yi)(x-yi)$

$x^3-2xy^2-xy^2-(x^2-2xyi-y^2)(x-yi)$

$x^3-2xy^2-xy^2-(x^3-\underset{\mathrm{im}}{\underset{\downarrow}{xyi}}-\underset{\mathrm{im}}{\underset{\downarrow}{2x^2yi}}-2xy^2-xy^2+\underset{\mathrm{im}}{\underset{\downarrow}{y^3i}})$

$x^3-2xy^2-xy^2-x^3+2xy^2+xy^2=0$.

no real part, so it's purely imaginary!

$a_1 = 14,\ a_2 = 21,\ a_m = 3a_{m-1} + a_{m-2}$

Posi :

BC :

$a_0 = 1,\ a_1 = 3, \quad n \geq 2:\ a_n = 3a_{n-1} - 2a_{n-2} - 1$

show $a_n = 2^n + n$

BC: $n=0$: $a_0 = 2^0 + 0 = 1$ ✓
$a_1 = 2^1 + 1 = 3$ ✓ } BCs pass

IH: $\forall k \in \mathbb{Z},\ k = 0, \ldots, n,\ a_k = 2^k + k$

IS: show $a_{k+1} = 2^{k+1} + (k+1)$

$a_{k+1} = 3a_k - 2_{k-1} - 1$

$2^{k+1} + k + 1 = 3(2^k + k) -$

$a_{k+1} = 3(2^k + k) - 2(2^{k-1} + k - 1) - 1$

$3 \cdot 2^k + 3k - 2^k - 2k + 2 - 1$

$3 \cdot 2^k - 2^k + k + 1$

$2^k(3-1) + k + 1 = 2 \cdot 2^k + k + 1$

$$\underbrace{4\left(x+\frac{n}{\gcd}\right)}_{\leq 50}$$

$$[5][x]^2 = [6]$$

$$[5][x^2] = [32]$$

$$5x^2 \equiv 6 \pmod{13}$$

$x \pmod{13}$	0	1	2	3	4	5	6	7	8	9	10	11	12
$x^2 \pmod{13}$	0	1	4	9	3	12	10	10	12	3	9	1	1
$5x^2 \pmod{13}$	0	5	7	6							6		

$$\therefore [x] = [3], [10]$$

$x \equiv 5 \pmod 9$

$10x \equiv 6 \pmod{28}$.

prove $\gcd(5^{98}+3, 5^{99}+1) = 14$

$$(5^{98}+3)s + (5^{99}+1)t = 14$$

$$2^5 + 16\overline{z} = 0$$

$z = x + yi$

$z = r(\cos\theta + i\sin\theta)$

$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$

$\gcd(a, 77) = 1 \implies a^{30} \equiv 1 \pmod{77}$

$as + 77t = 1$ $\quad a^{30} \equiv 1 \pmod{7}$

$as + 7 \cdot 11t = 1$ $\quad a^{30} \equiv 1 \pmod{11}$

$\gcd(a, 7) = 1$

$\gcd(a, 11) = 1$ $\quad a^{10} \equiv 1 \pmod{7}$

FLT: $a^6 \equiv 1 \pmod{7}$ $\quad (a^{10})^3 \equiv 1^3 \pmod{7}$

$a^{10} \equiv 1 \pmod{11}$ $\quad 1^3 \equiv 1^3 \pmod{7}$

$(a^4)^5 \equiv 1 \pmod{7}$

$a^{20} \equiv 1$