Lecture 1 - 4th Sept 2024

Statistics: the science of understanding data and making decisions in the face of variability and uncertainty.

Probability: A branch of mathematics concerned with describing and modeling uncertain events.

Preliminaries:

- **Experiment**: the process of obtaining an observed result of some phenomenon.
- **Trial**: the performance of an experiment.
- **Outcome**: the result of a single trial (attempt) of an experiment.
- **Event**: one or more outcomes of an experiment.
- **Probability**: the measure of how likely an event is.

Sample Space: the set of ALL possible distinct outcomes in a random experiment, denoted by S.

↳ note: one and only one of the outcomes occurs in any single trial of the experiment.

↳ example:

- Roll a six-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- Flip a coin: $S = \{\text{heads}, \text{tails}\}$
- Waiting time for a bus: $S = \{t \in \mathbb{R}, 0 \leq t \leq 10\}$

- a sample space is finite if it consists of a finite number of outcomes, say $\{a_1, a_2, \ldots, a_N\}$

- a sample space can be a set of countable infinite outcomes, say $\{a_1, a_2, \ldots\}$, where the outcomes can be put into a one-on-one correspondence with positive integers.

↳ Example: suppose our experiment consists of tossing a coin until it lands on heads.

↳ the possible outcomes in the sample space are:

H, TH, TTH, TTTH, TTTTH, ...

∴, the outcomes are countable but infinite.

Discrete Sample Space: when a sample space is either finite or countably infinite. Else, it's non-discreet.

↳ Example:
- Roll a six-sided die: discreet
- Flip a coin: discreet
- Waiting time for a bus: non-discreet

Event: a subset of sample space S.
↳ we say A is an event if $A \subset S$.

↳ note: A and S are both sets!

↳ Example: consider tossing two coins.

a) what is the event of obtaining at least one head?

$S = \{HH, HT, TH, TT\}$

$\therefore A = \{HH, HT, TH\}$

b) What is the event of obtaining at least one head and at least one tail?

$A = \{HT, TH\}$.

Elementary / Simple Event: an event that contains only one outcome of the experiment. Eg, $A = \{a_1\}$.

Compound Event: an event made up of two or more simple events. Eg: $A = \{a_1, a_2\}$.

Set Notations and Terminology:

- A set is a collection of elements. Eg, $A = \{\text{apple, orange}\}$.
- The notation $x \in A$ or $x \notin A$ will mean that x is or is not an element of A.
- The empty / null set is $\emptyset$.
- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The notation $B \subset A$ means that B is a subset of A.
- Complement: $A^c = \{x \mid x \in S, x \notin A\} = \bar{A}$
- Two sets A and B are disjoint if $A \cap B = \emptyset$.

- The set $A \cap B^c$ is "A but not B"
- The set $A^c \cap B^c$ is "neither A nor B"

- If $A_1, \ldots, A_n$ is a finite collection of sets,
  - $A_1 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$ is "in all A_i"
  - $A_1 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$ is "in at least one A_i"

very similar to disjoint!

Mutually Exclusiveness: when two events A and B if $A \cap B = \emptyset$. If events are exclusive, they have no outcomes in common.

Example: tossing two coins. A is the event "at least one head", and B is the event "both tails".

$S = \{HH, HT, TH, TT\}$.
$A = \{HH, HT, TH\}$, $B = \{TT\}$.

$$A \cap B = \{HH, HT, TH\} \cap \{TT\} = \emptyset \; !$$

$\therefore$, A and B are mutually exclusive.

Events $A_1, A_2, A_3, \ldots$ are said to be mutually exclusive if they are pairwise mutually exclusive. That is,

$$A_i \cap A_j = \emptyset, \text{ wherever } i \neq j.$$

Probability Modeling: assigning a probability $P(A)$ to each event A. This probability measures how likely it is that A will happen when the experiment is conducted.

↳ think of $P(A)$ as a (set) function, whose domain is a collection of sets (events), and the range of which is a subset of real numbers.

↳ not all set functions are appropiate for assigning probabilities to event.

- Let $S = \{a_1, a_2, a_3, \dots\}$ be a discrete sample space. We assign to each elementary event $A_i = \{a_i\}$ for $i = 1, 2, \dots,$ a number $P(A_i) = P(\{a_i\})$, such that

$$0 \leq P(A_i) \text{ and } \sum_{\text{all } i} P(A_i) = 1$$

- We call $P(A)$ the probability of A, and call the set of probabilities $\{P(A_i), i = 1, 2, \dots\}$ the probability distribution on S.

↳ **General Properties:**

- The null event (empty set) has a probability of 0. $\therefore$ $P(\emptyset) = 0$.
- If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$
- If $A_1, A_2, \dots, A_k$ is a finite collection of pairwise mutually exclusive events, then

$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k)$

- $P(\bar{A}) = 1 - P(A)$

↳ Example: suppose a six-sided die is rolled once. If A is the event that an even number is obtained. What is $P(A)$?

- $S = \{1, 2, 3, 4, 5, 6\}$. Since each number is equally likely, $P(\{i\}) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

$A = \{2, 4, 6\}$. $\rightarrow P(A) = P(\{2\}) + P(\{4\}) + P(\{6\})$
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.

↳ Example: Toss a coin twice. What's the probability of getting exactly one head?

- $S = \{HH, HT, TH, TT\}$. Each option is uniformly likely.
- $A = \{HT, TH\} \rightarrow P(A) = P(\{HT\}) + P(\{TH\})$
$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Random Selection: when choosing an object from a finite set where each object has an equal chance of being selected.

Uniform Probability Model: consider a discrete finite sample space $S = \{a_1, a_2, \ldots, a_N\}$, where each simple event has a probability of $1/N$. This is a uniform distribution over the set $\{a_1, a_2, \ldots, a_N\}$.

↳ For a compound event A, we can calculate P(A) by:

$$P(A) = \frac{n(A)}{N},$$

where n(A) is the number of outcomes in A.

- The formula $P(A) = \frac{n(A)}{N}$ is classical probability.

(trivial) example: draw one card at random from a well-shuffled deck without jokers. What is the probability that the card is a spade?

↳ $S = \{A\heartsuit, 2\heartsuit, \ldots\}$.
$A_s = \{A\spadesuit, 2\spadesuit, \ldots, K\spadesuit\}$.

↳ $P(A_s) = \frac{n(A_s)}{N} = \frac{13}{52} = \frac{1}{4}$.

Example: suppose that P is a probability, and A and B are events such that A and B are mutually exclusive, and $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$. Is each statement true or false?

A) A and B are each non-empty

since $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$, we know that

$P(A) = 0.2$ and $P(B) = 0.3$.

we also know that $P(\emptyset) = 0$. Since $0.2 \neq 0 \neq 0.3$, we know A and B cannot be empty.
$\therefore$, the statement is true.

B) $P(A \cap B) = 0$

Since we're told that A and B are mutually exclusive, we know that $A \cap B = \emptyset$.

we also know that $P(\emptyset) = 0$, so the statement is true!

C) $P(A \cup B) = 0.4$

we know that $\underbrace{P(A \cup B)}_{\text{"the probability of A or B"}} = P(A) + P(B)$

$P(A) + P(B) = 0.2 + 0.3 = 0.5 \neq 0.4$.
$\therefore$, the statement is false!

Example: suppose two six-sided die are rolled where the outcomes are of the form (die 1, die 2). If A is the event that the first die is even, and B is the event that the sum of both rolls is 6, compute the event $C = \bar{A} \cap B$

↳ $A = \{(2,1), (2,2), \ldots, (4,1), (4,2), \ldots, (6,1), (6,2), \ldots, (6,6)\}$.

$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$\bar{A}$ means the event that A does not happen, so in other words, $\bar{A}$ is the event that the first die is odd.

$\therefore, \bar{A} = \{(1,1), \ldots, (1,6), (3,1), \ldots, (3,6), (5,1), \ldots, (5,6)\}$.

$\bar{A} \cap B$ is A AND B.

↳ $\therefore$, $C = \{(1,5), (3,3), (5,1)\}$.

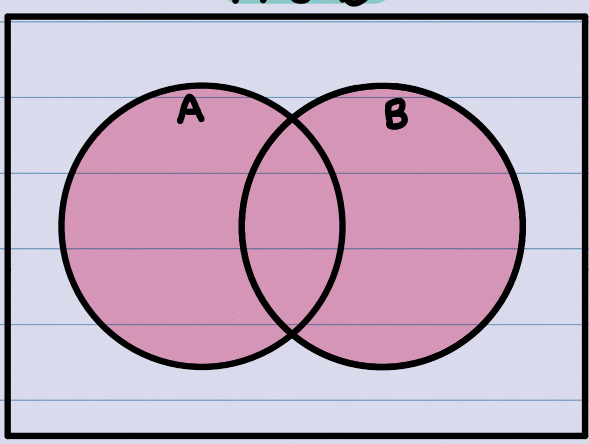
Venn Diagram: a tool to illustrate the relationship among sets (or events).

↳ made up of:

- a rectangle to represent the sample space S.
- Circles within the rectangle to illustrate the events.

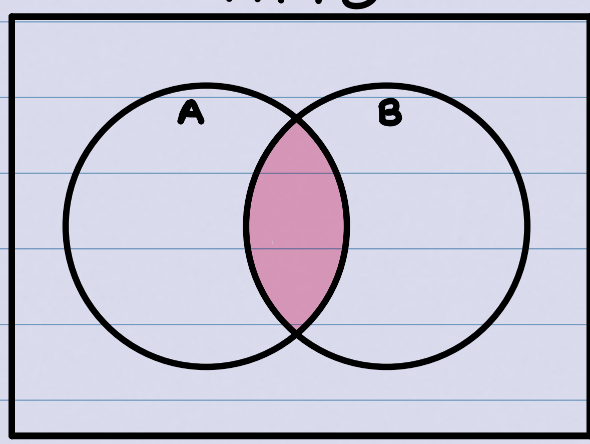
The Union and Intersection of Two Events

$A \cup B$



"A or B"

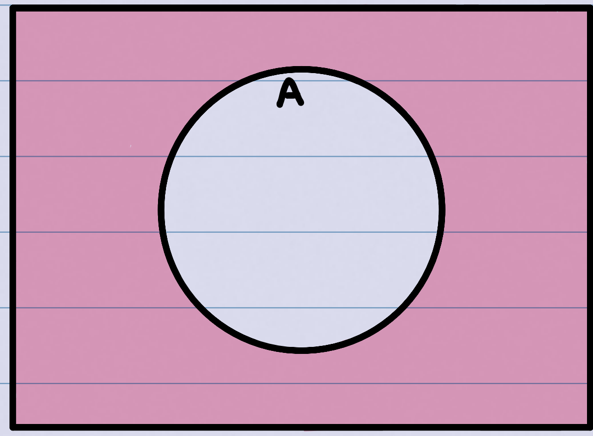
$A \cap B$



"A and B"

The Compliment of an Event:

$\bar{A}$



"S but not A"

De Morgan's Laws:

- $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Probability Rules

- Probability of the Complement of an Event

↳ $P(A) = 1 - P(\bar{A})$

Proof: $A \cup \bar{A} = S$, so $P(A \cup \bar{A}) = P(A) + P(\bar{A})$ bc mutually exclusive!
since $P(A \cup \bar{A}) = P(S) = 1$, $1 = P(A) + P(\bar{A})$!

Example: an experiment involves tossing a coin four times. The event A is "at least one head". Find the probability of A, ie, find P(A).

- the only case where there is NOT at least one head is $\bar{A} = \{TTTT\}$.

Flipping a coin 4 times would result in 2^4 possible outcomes, and $\bar{A}$ is only 1 of these.

$\therefore$, $P(\bar{A}) = \frac{1}{16}$, so $\overbrace{P(A) = 1 - P(\bar{A})}^{\text{probability of complement!}} = 1 - \frac{1}{16} = \frac{15}{16}$.

Example: two ordinary dice are rolled. Find the probability that at least one lands on a 6.

$S = \{(1,1), \dots, (1,6), (2,1), \dots, (2,6), \dots, (6,1), \dots, (6,6)\}$.

A = the event of at least one six.
$\therefore$, $\bar{A}$ is the event that neither die is a six.

classical probability of each die to NOT be a six: $\frac{n(x)}{N}$.
where $n(x)$ is the number of outcomes that satisfy the roll not being a 6 - ie; $\{1,2,3,4,5\} \Rightarrow 5$, and N is the total number of outcomes in the set - ie; $\{1,2,3,4,5,6\} \Rightarrow 6$.

$\therefore \frac{n(x)}{N} = \frac{5}{6}$!

Since both dice are independent, the probability of both NOT being a six (aka $\bar{A}$) is $\frac{5}{6} \cdot \frac{5}{6}$.

$\therefore$, $P(\bar{A}) = \frac{25}{36}$, so $P(A) = 1 - \frac{25}{36} = \frac{11}{36}$!

Rules for Union:

· Union of two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: At a university, 70% of the students own a car, 60% of students live on campus, and 50% both live on campus and own a car. If a student is chosen at random, what is the probability that the student does not own a car and does not live on campus?

· Let A be the event that a student has a car.
· Let B be the event that a student lives on campus.

we want: $P(\bar{A} \cap \bar{B})$.
↳ we know that $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

↳ $P(\bar{A} \cap \bar{B}) = 1 - P(\overline{\bar{A} \cap \bar{B}}) = 1 - P(A \cup B)$

from the union of two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\begin{aligned} \therefore, P(\bar{A} \cap \bar{B}) &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 + P(A \cap B) - P(A) - P(B) \\ &= 1 + 0.5 - 0.7 - 0.6 \end{aligned}$$

↳ $\therefore, P(\bar{A} \cap \bar{B}) = 0.2 = 20\%$.

Union of Three Events: for any 3 events A, B, and C:
↓
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Conditional Probability: determining the probability of some event A, while knowing that some related event B has occured.

Example: consider that a fair die has been rolled and you are asked to give the probability that it was a 5.

↳ $A = \{5\}$, $S = \{1, 2, 3, 4, 5, 6\}$.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}.$$

However, if you're told that the dice has landed on an odd number, the odds change:

↳ $A = \{5\}$, $S = \{1, 3, 5\}$.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{3}.$$

Example: a box contains 100 microchips, some produced by Factory 1 and the rest by Factory 2. Among these microchips, some are defective, while others are good. Let A be the event of "obtaining a defective microchip" and B be the event of "the microchip was produced in Factory 2."

the number of microchips in each category:

	B	$\bar{B}$	totals
A	15	5	20
$\bar{A}$	45	35	80
totals	60	40	100

a) Find the probability of obtaining a defective microchip.

the total of A is 20, and there are 100 total microchips.

$$P(A) = \frac{n(A)}{N} = \frac{20}{100} = \frac{1}{5} \quad (20\%)$$

b) Given that a microchip is from factory 1, find the probability that it is defective.

the sample space S shrinks from all 100 microchips to only the 60 made by factory 1 (event B).

$$\therefore, \; P(A) = \frac{n(A)}{N} = \frac{15}{60} = \frac{1}{4} \quad (25\%)$$

Conditional Probability mathematical definition:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) > 0)$$

"the probability of A given B"

$\hookrightarrow$ • $0 \le P(A \mid B) \le 1$

• $P(A \mid B) = 1 - P(\bar{A} \mid B)$

• $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$ ↗ pretty much the same as the regular union of two events

Multiplication Theorem of Probability: a way to compute the joint occurence of A and B:

$\hookrightarrow$ $P(A \cap B) = P(B) \cdot P(A \mid B) = P(A) \cdot P(B \mid A)$

• Law of Total Probability: if $B_1, B_2, \dots, B_k$ is a collection of mutually exclusive events and $B_1 \cup B_2 \cup \dots \cup B_k = S$, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A \mid B_i)$$

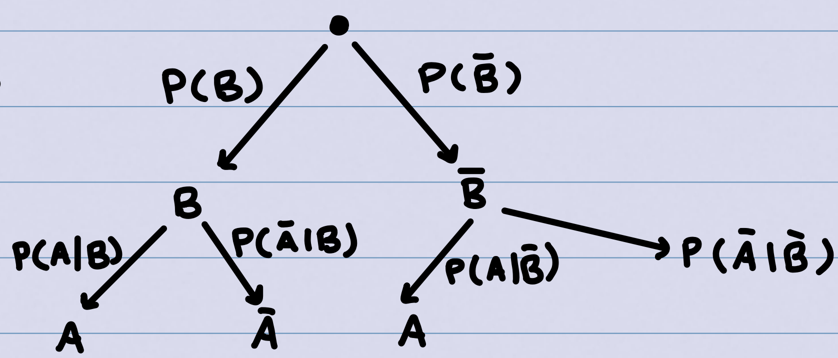
Note: for a special case $k = 2$:

↗ only for $k = 2$!!

$$P(A) = P(A \mid B) \cdot P(B) + P(A \mid \bar{B}) \cdot P(\bar{B})$$

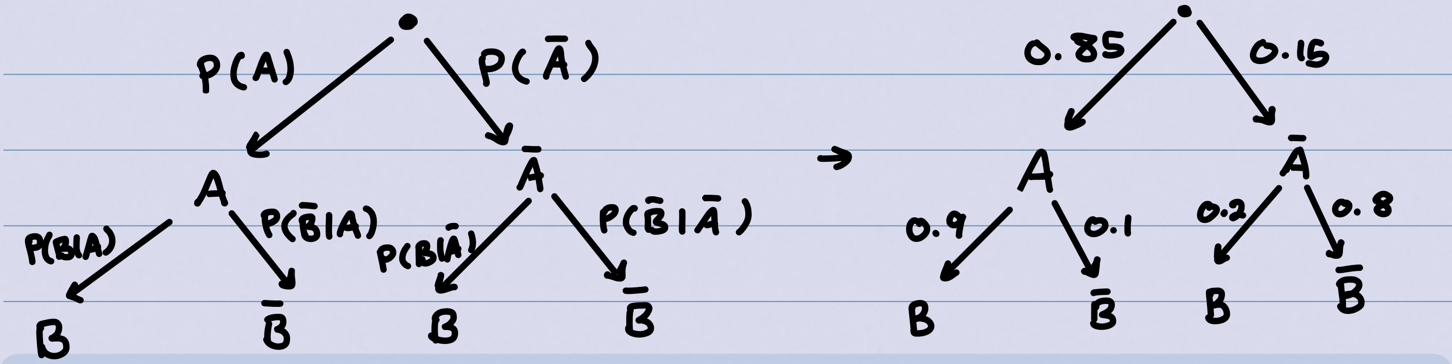
• for event A to occur, it must happen with either B or $\bar{B}$. ∴, only $A \cap B$ or $A \cap \bar{B}$ can occur.

$\hookrightarrow$



Example: without water, a plant will die with a probability of 0.8, and with water, it will die with a probability of 0.1. I'll remember to water the plant with a probability of 0.85. Represent this info using a tree diagram.

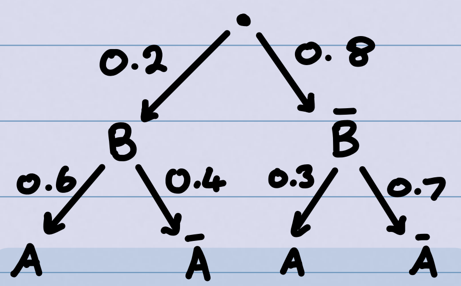
↳ let A be the event that I remember to water the plant, and B be the event that the plant survives.



Example: in a typical year, 20% of the days have a high temperature greater than 22°C. 40% of these days, there is no rain. During the rest of the year, when the temperature is ≤ 22°C, 70% of the days have no rain.

a) represent this information in a tree diagram.

let A be the event that it rains, and let B be the event that the temperature peaks above 22°C.



b) solve for the proportion of days in the year which have rain and a temperature $\leq 22°C$.

↳ aka, find $P(A \cap \bar{B})$

we know $P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A \cap \bar{B})}{1 - P(B)}$

$\therefore$, $P(A \cap \bar{B}) = P(A|\bar{B}) \cdot (1 - P(B))$

$= 0.3 \cdot (1 - 0.2) = 0.3 \cdot 0.8 = 0.24$

$= 24\%$.

Baye's Rule: $P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$,

and more generally, $P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$ ↝ for each $j = 1, 2, \dots, k$.

Example: revisiting the plant problem from before, if the plant is alive, what's the probability that I remembered to water it?

↳ let A be the event that I remember to water the plant, and B be the event that the plant survives.

↱ same as previous time

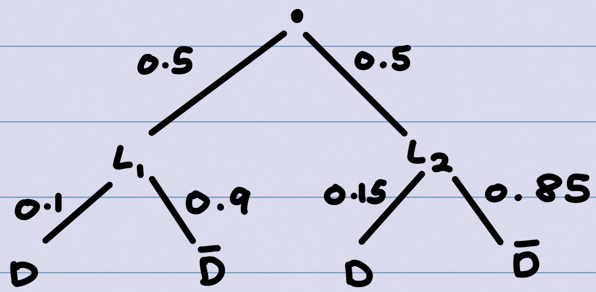
$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})} = \frac{(0.9)(0.85)}{(0.9)(0.85) + (0.2)(0.15)}$$

$= 0.9623 = 96.23\%$!

Example: electric motors from two assembly lines are pooled in a common stockroom, with equal numbers of motors from each line. Motors are periodically sampled for testing. It's known that 10% of the motors from line 1 are defective, while 15% of the motors from line two are defective.

a) if a motor is randomly selected, what is the probability that it is defective?

• Let D be the event that a motor is defective.



$\rightarrow P(D) = P(D \cap L_1) + P(D \cap L_2)$

$= 0.5 \cdot 0.1 + 0.5 \cdot 0.15$

$= 0.125$

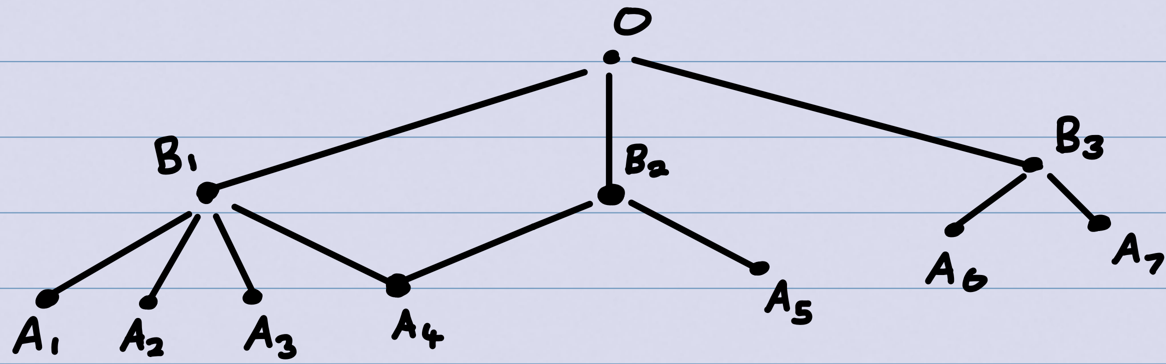
↳ 12.5% chance.

b) if the randomly selected motor is defective, what is the probability that it came from line 1?

$$P(L_1 \mid D) = \frac{P(D \mid L_1)P(L_1)}{P(D)} = \frac{0.1 \cdot 0.5}{0.125} = 0.4$$

= 40% chance.

Example: a man starts at point O on this map. He chooses a path at random and follows it to point B_1, B_2, or B_3. From that point, he chooses a new path at random and follows it to one of the points $A_1 \dots A_7$.



a) what is the probability that the man arrives at point A_4?

$$P(A_4) = P(A_4 \mid B_1) + P(A_4 \mid B_2) = \frac{1}{4}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{12} + \frac{1}{6}$$
$$= 1/4 = 0.25$$

$= 25\%$!

b) suppose that the man arrives at point A_4. What is the probability that he passed through B_1?

$$P(B_1 \mid A_4) = \frac{P(A_4 \mid B_1)}{P(A_4)} = \frac{\frac{1}{4}\cdot\frac{1}{3}}{1/4} = \frac{1/12}{1/4} = \frac{4}{12} = \frac{1}{3}$$

$\therefore$, there was an $33.3\dot{3}\%$ chance he came from B_1.

Independent Event: events A and B are called independent if and only if $P(A \cap B) = P(A)P(B)$. Otherwise, A and B are called dependent events.

↳ an equivalent formulation can be given in terms of conditional probability:

↳ If A and B are events such that $P(A) > 0$ and $P(B) > 0$, then A and B are called independent if and only if either of the following hold:

$$P(A|B) = A, \quad \text{or} \quad P(B|A) = P(B)$$

Note: independence ≠ mutually exclusive!

unless $P(A) = 0$!

General rule: $P(A \cap B) = P(B|A) \cdot P(A)$

Independence of Compliments: suppose events A and B are independent. Then the following events are also independent:

- $\bar{A}$ and $\bar{B}$
- $\bar{A}$ and B
- A and $\bar{B}$

Independence of n events: the k events $A_1, A_2, \ldots, A_k$ are said to be independent, or mutually independent, if and only if:

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \ldots P(A_{i_k})$$

$\forall$ possible distinct $i_1, i_2, \ldots, i_k$ chosen from $1, 2, \ldots, n$.

↳ Note: for a sequence of events to be independent, all of their subsets must also be independent.

Example: suppose a fair die is rolled twice. Let A be the event that the first roll is a 6, and let B be the event that the second roll is a 6. Show that A and B are independent events.

↳ we want to show that $P(A \cap B) = P(A) \cdot P(B)$

$P(\{6,6\}) = P(A \cap B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

$P(A) = \frac{n(A)}{N} = \frac{1}{6} = \frac{n(B)}{N} = P(B)$. $\therefore$, $P(A) = P(B) = \frac{1}{6}$.

$P(A) \cdot P(B) = \frac{1}{36} = P(A \cap B)$.

↳ $\therefore$, the events are independent.

Counting Techniques

often, manually counting many outcomes manually can be very time-consuming and tedious.

1) Addition Rule : "OR" interpreted as addition

↳ Eg: suppose job 1 can be done in p ways and job 2 can be done in q ways. $\therefore$, we can do either job 1 or job 2 in $p+q$ ways.

2) Multiplication Rule: "AND" interpreted as multiplication.

↳ Eg: suppose job 1 can be done in p ways and for each of these ways, we can do job 2 in q ways. $\therefore$, we can do both Job 1 and Job 2 in $p \times q$ ways.

Example: how many are there to answer a 20 question true/false test?

$$\overset{Q_1}{2} \times \overset{Q_2}{2} \times \dots \times \overset{Q_{20}}{2} = 2^{20} \text{ ways.}$$

Sampling with Replacement: what we get on the first selection does not affect what we get on second selection.

↳ Eg: every time an object is selected, it's put back into the pool of possible objects.

Sampling without Replacement: what we get on the first selection will affect what we get on the second selection.

↳ Eg: once an object is selected, it stays out of the pool of possible objects.

Example: A bag contains 3 blue marbles and 5 red marbles. If you pick two marbles from the bag,

a) what is the probability of picking 2 blue marbles if the selection is with replacement?

let A be the event of picking two marbles.

using independence property!

↳ P(A) = P(blue on first pick) + P(blue on second pick)

↳ $P(A) = \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64} = 14.06\%$

b) what about without replacement?

$P(A) = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56} = \frac{3}{28} = 10.71\%$!

Example: a personal identification number (PIN) of length 4 is formed by randomly selecting 4 digits from the set of digits $\{0, 1, 2, \ldots, 9\}$. If selection is done with replacement, find the probability that:

a) The PIN is even:

the first 3 digits don't matter!

↳ for the PIN to be even, the last digit must be a multiple of 2. ∴, $A = \{0, 2, 4, 6, 8\}$.

$$\therefore, P(A) = \frac{n(A)}{N} = \frac{5}{10} = \frac{1}{2} \rightarrow 50\%$$

b) the PIN contains at least one 0?

↳ P(at least one 0) = 1 − P(no 0s in PIN)

∴, there are 9 ways to fill each of the digits, as we want the probability of no zeros.

multiplication counting rule

↳ not 0 AND not 0 AND...

$$\therefore, P(\text{no } 0\text{s}) = 9 \cdot 9 \cdot 9 \cdot 9 = 9^4$$

total sample space is 10^4 (including 0s).

$$\therefore P(\text{at least one zero}) = 1 - \frac{n(\text{no } 0\text{s})}{n(S)} = 1 - \frac{9^4}{10^4}$$

c) Redo a and b but without replacement:

i) even PIN:

A: event of getting an even number. $\{0, 2, 4, 6, 8\}$.

S: the sample space $\square\square\square\square$ $10 \times 9 \times 8 \times 7$ → we have less options for each digit, bc no replacement!

A: $\underbrace{\square\square\square}_{9 \times 8 \times 7}\square$ → $n(A) = 9 \times 8 \times 7 \times 5$

5 → bc len({0, 2, 4, 6, 8}) = 5.

↳ $P(A) = \frac{9 \times 8 \times 7 \times 5}{10 \times 9 \times 8 \times 7} = \frac{5}{10} = \frac{1}{2} = 50\%$!

ii) at least one zero:

no 0 in PIN

$P(B) = 1 - P(\bar{B}) = 1 - \frac{9 \times 8 \times 7 \times 6}{10 \times 9 \times 8 \times 7} = 1 - \frac{6}{10} = \frac{2}{5} = 40\%$

Example: a fair die is tossed 3 times. What is the probability that exactly one of the tosses produces a number greater than 4?

let A be the event that the first toss is greater than 4, and the second and third tosses are ≤ 4.
↳ $\{5, 6\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4\}$.

"AND"

∴, $n(A) = 2 \times 4 \times 4 = 32$.

let B be the event where the 2^{nd} toss is > 4, and let C be the event where it's the 3^{rd} toss. → $n(A) = n(B) = n(C)$

∴, total number of ways to get exactly one number greater than 4 is $3(2 \times 4 \times 4) = 96$.

$\therefore$, P(exactly one toss > 4) = $\frac{96}{6^3} \approx 44.44\%$.

Counting Permutations:

· Permutation: an ordered arrangement of a set of objects.

Example: how many different ordered arrangements of the letters a, b, and c are possible?

↳ abc, acb, bac, bca, cab, cba.

$\therefore$, 6 arrangements!

This will very quickly become very tedious and slow.

· The number of permutations of n distinguishable objects is:

$$n \times (n-1) \times \dots \times 1 = n!$$

· Given n distinct objects, a permutation of length r is an ordered subset of r objects.

· The number of permutations of length r taken from n objects is denoted as $n^{(r)}$, where:

"n to r factors"

$$n^{(r)} = n \times (n-1) \times \dots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Example: five seperate awards are to be presented to some students from a class of size 30. How many different outcomes are possible if

a) a student can receive any number of awards?

$$\underset{1^{st}}{30} \times \underset{2^{nd}}{30} \times \underset{3^{rd}}{30} \times \underset{4^{th}}{30} \times \underset{5^{th}}{30} = 30^5.$$

$\therefore, n(A) = 30^5$

b) each student can receive, at most, one award?

$$\underset{1^{st}}{30} \times \underset{2^{nd}}{29} \times \underset{3^{rd}}{28} \times \underset{4^{th}}{27} \times \underset{5^{th}}{26} = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$$

↳ $n(A) = 30^{(5)} = \dfrac{30!}{25!}$

· Sometimes the outcomes in the sample space are subsets of a fixed size.

Example: suppose we have three books: b_1, b_2, and b_3. We choose two of the books to read. In how many ways can the two books be read if:

a) the order that the books are read in matters:

(b_1, b_2), (b_2, b_1)
(b_1, b_3), (b_3, b_1) → 6 ways!
(b_2, b_3), (b_3, b_2)

b) the order that the books are read in does not matter?

$\{b_1, b_2\}$
$\{b_1, b_3\}$ → 3 ways!
$\{b_2, b_3\}$

Given n distinct objects, a combination of size r is an unordered subset of r objects chosen from n distinct objects.

↳ The number of combinations (or subsets) of size r chosen from n distinct objects is given by:

$$\binom{n}{r} = \frac{n^{(r)}}{r!} = \frac{n!}{(n-r)r!}$$

↳ "n choose r"

Example: we randomly select a subset of 3 digits from $\{0, 1, \ldots, 9\}$. All the digits in each outcome are unique, but the order of the elements in a subset is not relevant.

aka, without replacement

Find the probability that:

a) All the digits in the selected subset are even:

$S = \{\{0,1,2\}, \{0,1,3\}, \ldots, \{7,8,9\}\}$

$n(S) = \binom{10}{3}$.

length of 5

the set of even digits is $\{0, 2, 4, 6, 8\}$.

let A be the event that all 3 digits are even.

$\hookrightarrow n(A) = \binom{5}{3}$

$\hookrightarrow P(A) = \frac{n(A)}{n(S)} = \frac{\binom{5}{3}}{\binom{10}{3}} = \frac{1}{12} = 8.3\dot{3}\,\%.$

b) At least one of the digits in the selected subset is less than or equal to 5.

P(at least one digit ≤ 5)
$= 1 - P(\text{no digit} \leq 5)$
$= 1 - P(\text{all digits} > 5)$

the set of digits > 5: $\{6, 7, 8, 9\}$.

$= 1 - \frac{\binom{4}{3}}{\binom{10}{3}} = \frac{29}{30} = 96.6\dot{6}\,\%.$

Example: there are 30 geese of which 6 were tagged. Later, 5 of the geese are randomly captured.

a) How many samples of 5 are possible? $\binom{30}{5}$

b) How many samples of 5, which include 2 of the tagged geese, are possible?

$\hookrightarrow \binom{6}{2} \cdot \binom{24}{3}$

c) if the five captured geese represent a simple random sample drawn from the 30 geese, find the probability that:

i) two of the 5 captured geese are tagged.

$$P(A) = \frac{\binom{6}{2}\binom{24}{3}}{\binom{30}{5}} = 0.2130 = 21.3\%$$

ii) none of the 5 captured geese are tagged.

$$P(B) = \frac{\binom{24}{5}}{\binom{30}{5}} = 0.2983 = 29.83\%$$

→ Key Properties of Permutations and Combinations:

1) $n^{(r)} = n \times (n-1)^{(r-1)}$ for $r \geq 1$.

2) $\binom{n}{r} = \frac{n^{(r)}}{r!}$

3) Symmetry Property: $\binom{n}{r} = \binom{n}{n-r}$ for $r \geq 0$.

4) since $0! = 1$, $\binom{n}{0} = \binom{n}{n} = 1$.

5) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

6) Binomial Theorem:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \text{ for } x \in \mathbb{R}.$$

Summary of Counting Techniques:

- Addition rule: OR (+)
- Multiplication rule: AND (×)

- **Factorial:** $n!$ is the number of arrangements of n distinct objects where order matters.
- **Permutation:** $n^{(k)}$ is the number of ways to choose k objects from n distinct objects where order matters.
- **Combination:** $\binom{n}{k}$ is the number of ways to choose k objects from n objects where order does <u>NOT</u> matter.

Example: a person has 10 friends, of whom 6 will be invited to a party. How many choices are there if:

a) 2 of the friends will only attend together?

$\hookrightarrow \binom{8}{4} \cdot \binom{2}{2}$

b) 2 of the friends will not attend if the other is also attending?

→ only one or the other can occur.

$\hookrightarrow \binom{8}{6} \cdot \binom{2}{0} \overset{\text{OR } (+)!}{+} \binom{8}{5} \cdot \binom{2}{1}$

Example: 13 cards are picked at random from a standard deck of 52 cards without replacement. Find the probability that we pick:

a) at least one ace:

$\hookrightarrow$ A = the event we pick at least one Ace, so

$\bar{A}$ = the event we pick no aces

$\hookrightarrow P(A) = 1 - P(\bar{A}) = 1 - \dfrac{\binom{48}{13}}{\binom{52}{13}}$

b) 6 spades, 4 hearts, 2 diamonds, and one club:

$$\hookrightarrow \frac{\binom{13}{6}\cdot\binom{13}{4}\cdot\binom{13}{2}\cdot\binom{13}{1}}{\binom{52}{13}}$$

Example: consider drawing 3 numbers at random with replacement from the digits $\{0, 1, \ldots, 9\}$. What is the probability that there is a repeated number among the 3?

$S = 10 \cdot 10 \cdot 10 = 10^3$

A = event of repeated number

$\bar{A}$ = event of no repeated number

$\hookrightarrow 10 \cdot 9 \cdot 8$ options

$$\therefore P(A) = 1 - P(\bar{A}) = 1 - \frac{n(\bar{A})}{n(S)} = 1 - \frac{10^{(3)}}{10^3} = \frac{7}{25}.$$

Example: Melissa selects 7 numbers between 1 and 50, and then a computer does the same. What is the probability that at least 5 of Melissa's numbers match the computer's?

$\hookrightarrow S = \binom{50}{7}$, and Melissa must match 5, 6, or 7 of the numbers: $\binom{7}{5}\binom{43}{2} + \binom{7}{6}\binom{43}{1} + \binom{7}{7}\binom{43}{0}$

$$\therefore P = \frac{\binom{7}{5}\binom{43}{2} + \binom{7}{6}\binom{43}{1} + \binom{7}{7}\binom{43}{0}}{\binom{50}{7}}$$

Example: a box contains 4 coins - 3 fair coins and 1 biased coin which lands on heads 80% of the time. A coin is randomly picked and tossed 6 times. It lands on heads 5 times. What is the probability the coin is fair?

let H5 be the event of getting 5 heads in 6 tosses. let F be the event the coin is fair, and let B be the event the coin is biased.

$$P(F \mid H5) = \frac{P(F \cap H5)}{P(H5)} = \frac{P(H5 \mid F)P(F)}{P(H5 \mid F)P(F) + P(H5 \mid B)P(B)}$$

$$= \frac{6 \times 0.5^5 \times 0.5 \times 3/4}{6 \times 0.5^5 \times 0.5 \times 3/4 + 6 \times 0.8^5 \times 0.2 \times 1/4} = 0.417$$

Example: there are 4 passengers on a 5-floor elevator. What is the probability that:

a) the passenges all get off on different floors?

↳ 5 people getting off on 4 different floors is $5^{(4)}$, out of total possibilities $S = 5^4$.

$$\therefore, \frac{5^{(4)}}{5^4} = \frac{24}{125} = 19.2\%$$

b) 2 passengers get off on floor two and 2 get off on floor 3?

↳ $$\frac{\binom{4}{2}\binom{2}{2}}{5^4}$$

c) 2 passengengers get off one a floor and 2 passengers get off on a different floor?

↳ $$\frac{\binom{4}{2}\binom{2}{2} \times \binom{5}{2}}{5^4}$$

Random Variable: a function, denoted by X, that assigns a real number $x = X(a)$ to each possible outcome a in a sample space S.
aka, we say X is a random variable if $X: S \to \mathbb{R}$.

- Random variables are denoted by capital letters such as X, Y, and Z.
- Their possible values/realizations are denoted by lowercase letters such as x, y, and z.

Example: consider an experiment where a fair coin is tossed 3 times.

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

For each outcome a in S, the value of the function $X(a)$ is the number of heads corresponding to a.
↳ Eg: if $a = THH$, then $X(a) = 2$.

∴

Value of X	Definition of the event
$X = 0$	$\{TTT\}$
$X = 1$	$\{TTH, THT, HTT\}$
$X = 2$	$\{HHT, HTH, THH\}$
$X = 3$	$\{HHH\}$

↱ basically, if in the naturals

Discrete Variable: if a random variable's range is a discrete subset of $\mathbb{R}$ (a finite set $\{x_1, \ldots, x_n\}$ or a countably infinite set $\{x_1, x_2, \ldots\}$).

↳ eg: number of coin flips, number of people, ...

eg: monthly rainfall, time, ...

Continuous Variable: if a random variable's range is an interval that is a subset of $\mathbb{R}$.

also referred to as discrete probability density function (discrete pdf)

Probability Mass Function (pmf / pf): the probability function of a discrete random variable X is:

$P(\{\omega \in S : X(\omega) = x\})$

$$f_X(x) = P(X = x)$$

Probability distribution of X: the set of pairs $\{(x, f(x)) : x \in A\}$.

Example: a fair coin is tossed 3 times. $X(a)$ is the number of heads corresponding to a. Find the probability distribution of a.

↳ $f(0) = P(X=0) = P(\{TTT\}) = 1/8$
$f(1) = P(X=1) = P(\{TTH, THT, HTT\}) = 3/8$
$f(2) = P(X=2) = P(\{HHT, HTH, THH\}) = 3/8$
$f(3) = P(X=3) = P(\{HHH\}) = 1/8$

↳ $f(0) + f(1) + f(2) + f(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{8}{8} = 1$!

- A function $f(x)$ is a discrete pf if and only if it satisfies both of the following properties:

1) $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$
2) $\sum_{\text{all } x} f(x) = 1$

Probability Histogram: the graph of a probability mass function of a discrete random variable.

Example: the random variable X has a pmf:

x	0	1	2	3	4
$f(x)$	$0.1c$	$0.2c$	$0.5c$	c	$0.2c$

a) Find the value of c:

↳ $\sum f(x) = 1 \rightarrow 0.1c + 0.2c + 0.5c + c + 0.2c = 2c.$

$\therefore$, $2c = 1$, so $c = 1/2 = 0.5$.

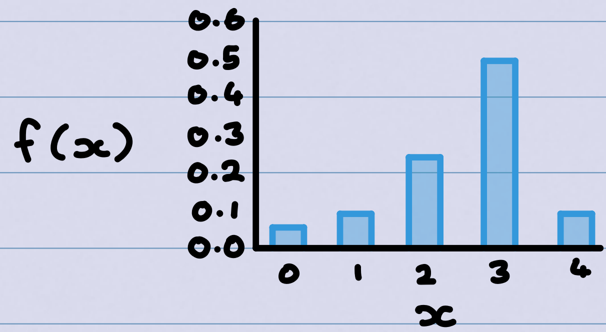
b) Find $P(X > 2)$

↳ $P(X > 2) = P(X = 3) + P(x = 4)$

$= c + 0.2c = 1.2c$

$\rightarrow 1.2c = 0.6$

c) Plot the probability histogram of $f(x)$



→ after substituting c in!

Cumulative Distribution Function (cdf): the cdf of a random variable x is defined for any real x by:

$$F(x) = P(X \leq x)$$

If X is a discrete random variable with probability function $f(x)$, then:

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u)$$

Example: find the cdf of random variable X from the previous example:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.05 & 0 \leq x < 1 \\ 0.15 & 1 \leq x < 2 \\ 0.4 & 2 \leq x < 3 \\ 0.9 & 3 \leq x < 4 \\ 1 & 4 \leq x \end{cases}$$

- For discrete random variables, the CDF $F(x)$ is represented as a step function.

(General) Example: Consider rolling two fair six-sided dice, and let the random variable X be the minimum of the rolls. What is $f_x(2)$?

↳ $f_x(2) = P(X = 2)$

$= P(\{(2,2), (2,3), (2,4), (2,5), (2,6), (3,2),$
$(4,2), (5,2), (6,2)\}$

↳ $P(X = 2) = 9/36.$

·Properties of Cumulative Distribution Functions:

↳ let $F(\cdot)$ be a cdf. Then:

1) $0 \leq F(x) \leq 1 \quad \forall x \in \mathbb{R}$
2) $F(x) \leq F(y)$ for $x < y$ → $F(x)$ is a non-decreasing function of x $\forall x \in \mathbb{R}$.
3) $\lim_{h \to 0^+} F(x+h) = F(x)$ → $F(x)$ is continuous on the right
4) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1.$

Example: Students A, B, and C each independently answer a question. The probability of getting the correct answer is 0.9 for A, 0.7 for B, and 0.4 for C. Let X be the number of people who get the answer correct.

a) Compute the probability function of X:

Since the events are independent, $P(ABC) = P(A)P(B)P(C)$
↓

$f(0) = P(X=0) = P(\bar{A}\bar{B}\bar{C}) = 0.1 \times 0.3 \times 0.6 = 0.018$

$f(1) = P(X=1) = P(A\bar{B}\bar{C}) + P(\bar{A}B\bar{C}) + P(\bar{A}\bar{B}C)$

$= 0.9 \times 0.3 \times 0.6 + 0.1 \times 0.7 \times 0.6 + 0.1 \times 0.3 \times 0.4$

$= 0.216$

$f(2) = P(X=2) = P(AB\bar{C}) + P(A\bar{B}C) + P(\bar{A}BC)$

$= 0.9 \times 0.7 \times 0.6 + 0.9 \times 0.3 \times 0.4 + 0.1 \times 0.7 \times 0.4$

$= 0.514$

$f(3) = P(x=3) = P(ABC) = 0.9 \times 0.7 \times 0.4 = 0.252$

$$\hookrightarrow f_X(x) = \begin{cases} 0.018 & x = 0 \\ 0.216 & x = 1 \\ 0.514 & x = 2 \\ 0.252 & x = 3 \end{cases}$$

b) compute the cdf of X

$$\hookrightarrow F_X(x) = \begin{cases} 0.018 & x < 0 \\ 0.234 & 0 \leq x < 1 \\ 0.748 & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

c) compute the probability that at least one person gets the answer correct.

$\hookrightarrow P(X \geq 1) = P(X=1) + P(X=2) + P(X=3)$

$= 0.216 + 0.514 + 0.252 = 0.982.$

OR, $P(X \geq 1) = 1 - P(X=0) = 1 - 0.018 = 0.982.$

$\cdot F(x)$ can be obtained from $f(x)$, and vice versa!

$\downarrow$

Example: the random variable X has a cdf given by:

x	1	2	3	4
$F(x)$	0.2	0.5	0.8	1

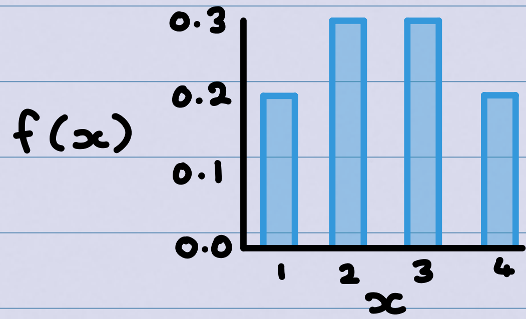
a) Find $f(3)$.

$$\hookrightarrow f_X(3) = F_X(3) - F_X(2) = 0.8 - 0.5 = 0.3$$

b) Calculate $P(1 < X \leq 3)$.

$$\hookrightarrow P(1 < x \leq 3) = P(X \leq 3) - P(X \leq 1)$$
$$= F(3) - F(1) = 0.8 - 0.2 = 0.6.$$

c) Plot $f(x)$.



aka, "mean of X" or "first moment of X"

Expected Value: suppose X is a discrete random variable with probability function $f_X(x)$. Then, $E(X)$ is called the expected value of X and is defined by:

$$E(X) = \sum_{x \in A} x \cdot f_X(x)$$

Example: X denotes the outcome of one fair six-sided die roll. Compute $E(X)$.

aka, the odds of $x = 1, \dots, 6$ $= 1/6$.

we know $f(1) = f(2) = \dots = f(6) = 1/6$

$$\therefore, E(X) = \sum_{x \in \{1,2,\dots,6\}} x \cdot f(x) = 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + 4 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 6 \cdot \tfrac{1}{6} = 3.5.$$

• Note(s):

1) The expected value of X may be a value that X can never actually take.
2) both μ and $E(x)$ refer to the same quantity which is the expected value of random variable X.

Example: a random variable X only takes two values, 0 or 1. If $P(X=0) = 0.4$, find $E(x)$.

$$P(X=1) = \overline{P(X=0)} = 1 - P(X=0) = 1 - 0.4 = 0.6$$

$$E(x) = \sum_{x \in \{0,1\}} x \cdot f(x) = 0 \times 0.4 + 1 \times 0.6 = 0.6.$$

Example: consider rolling 2 fair dice. Let x be the sum of the rolls, and let $Y = X \pmod 4$. Find Y's pmf.

↳ we know $S = \{(1,1), (1,2), \ldots, (2,1), (2,2), \ldots, (6,1), (6,2), \ldots (6,6)\}$

$\therefore$, $A = \{2, 3, 4, \ldots, 12\}$

x	$f_X(x)$	$y = x \pmod 4$
2	1/36	2
3	2/36	3
4	3/36	0
5	4/36	1
6	5/36	2
7	6/36	3
8	5/36	0
9	4/36	1
10	3/36	2
11	2/36	3
12	1/36	0

$$f_Y(0) = P(Y=0) = P(X=4) + P(X=8) + P(X=12)$$
$$= \frac{3}{36} + \frac{5}{36} + \frac{1}{36} = 9/36 = 1/4.$$

$$f_Y(1) = P(Y=1) = P(X=5) + P(X=9)$$
$$= \frac{4}{36} + \frac{4}{36} = \frac{8}{36} = 2/9$$

$$f_Y(2) = P(Y=2) = P(X=2) + P(X=6) + P(X=10)$$
$$= \frac{1}{36} + \frac{5}{36} + \frac{3}{36} = 9/36 = 1/4$$

$$f_Y(3) = P(Y=3) = P(X=3) + P(X=7) + P(X=11)$$
$$= \frac{2}{36} + \frac{6}{36} + \frac{2}{36} = 10/36 = 5/18$$

$\forall y \notin \{0, 1, 2, 3\}$, $f_Y(y) = 0$.

- General Formula when $Y = aX + b$

↳ in the case of a linear function of the form $Y = g(X) = aX + b$, where $a \neq 0$:

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)$$

- General Formula when $Y = g(X)$

↳ Let $Y = g(X)$ where $g(\cdot)$ is an injective function.

aka one-to-one

Since $g(\cdot)$ is injective, it's inverse $g^{-1}(\cdot)$ exists.

Then, we see that:

$$
\begin{aligned}
f_Y(y) &= P(Y=y) \\
&= P(g(X)=y) \\
&= P(X = g^{-1}(y)) \\
&= f_X(g^{-1}(y))
\end{aligned}
$$

$$\therefore,\ f_Y(y) = f_X(g^{-1}(y))$$

- Let A be a discrete random variable with the range A and pmf $f(x)$. The expected value of some function $g(\cdot)$ of X is given by:

$$E[g(X)] = \sum_{x \in A} g(x) \cdot f(x)$$

Example: a TV station sells 15, 30, and 60 second advertising spots. X is the length of a random commercial and suppose that it's probability distribution is given by:

x	15	30	60
$f(x)$	0.1	0.3	0.6

a) Find $E(x)$.

$\hookrightarrow E(x) = \sum_{x \in \{15,30,60\}} x \cdot f(x) = 15 \times 0.1 + 30 \times 0.3 + 60 \times 0.6 = 46.5$ seconds.

b) if a 15s spot is \$500, a 30s spot is \$800, and a 60s spot is \$1000, find the average amount paid for a commercial on this station.

↳ E (amount paid) = $500 \times 0.1 + 800 \times 0.3 + 1000 \times 0.6 = \$890.$

Example: consider the discrete random variable X with pmf of the form $f_X(x) = 1/3$, for $x = -1, 0, 1$. If we define the random variable $Y = X^2$, calculate $E(Y)$.

$$E(Y) = \sum_{x \in A} g(x) f(x) = \frac{1}{3}(-1)^2 + \frac{1}{3}(0)^2 + \frac{1}{3}(1)^2 = 2/3 \, !$$

Linearity Property of Expectation

The expectation $E(\cdot)$ is a linear operator, satisfying the properties:

1) For constants a and b,

$$E[a\,g(X) + b] = a\,E[g(x)] + b$$

2) For constants a and b and for functions g_1 and g_2,

$$E[a\,g_1(X) + b\,g_2(X)] = a\,E[g_1(X)] + b\,E[g_2(X)]$$

- If $g(X)$ is a linear function, then $E[g(x)] = g[E(x)]$.
  This is not necessarily true if $g(X)$ is non-linear!

Example: let random variable X have the following probability function:

x	1	2	3	4
$f(x)$	0.4	0.3	0.1	0.2

a) Find $E(x)$.

↳ $\sum_{x \in \{1,2,3,4\}} x \cdot f(x) = 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.1 + 4 \times 0.2 = 2.1$

b) Find $E(1/x)$.

$\hookrightarrow \sum_{x \in \{1,2,3,4\}} \frac{1}{x} \cdot f(x) = \frac{1}{1} \times 0.4 + \frac{1}{2} \times 0.3 + \frac{1}{3} \times 0.1 + \frac{1}{4} \times 0.2 = 0.633.$

• **Variance:** the variance of a random variable X is:

$$\sigma^2 = Var(X) = E[(X-\mu)^2]$$

aka μ!

$\hookrightarrow$ can also be calculated as: $\sigma^2 = Var(X) = E(X^2) - E(X)^2$

Standard Deviation: the standard deviation of a random variable X is:

$$\sigma = sd(X) = \sqrt{Var(X)} = \sqrt{E[(X-\mu)^2]}$$

• **Important Properties:**

• for real constants a and b, $Var(aX+b) = a^2 Var(X)$

• if $Y = aX + b$, then

1) $E(Y) = a\,E(X) + b$

2) $Var(Y) = a^2 Var(X)$

3) $sd(Y) = |a|\,sd(X)$

Example: X has $Var(X) = 2$. Compute the variance of random variable Y, where $Y = -2X + 3$.

$\hookrightarrow$ we know that $Var(Y) = a^2 Var(X) = (-2)^2(2) = 8$.

• **Properties of Variance:**

1) $\forall$ rvs X, $Var(X) \geq 0$.

2) big variances indicate that the probability distribution of X is more "spread out" around the mean.

3) $Var(X) = 0$ if and only if $P(X = E(x)) = 1$.

↳ no variance if the expected value is guaranteed!

Example: X is a random variable with probability function given by:

x	1	2	3	4	5
$f(x)$	0.1	0.1	0.3	0.35	0.15

If $Y = 4-2X$, find $E(Y)$, $Var(Y)$, and $sd(Y)$:

- $E(X) = \sum_{x\in\{1,\dots,5\}} 1\times 0.1 + 2\times 0.1 + 3\times 0.3 + 4\times 0.35 + 5\times 0.15 = 3.35$

$E(Y) = 4 - 2E(X) = 4 - 2(3.35) = -2.7$

$E(X^2) = \sum_{x\in\{1,\dots,5\}} x^2 \cdot f_X(x) = 1^2\times 0.1 + 2^2\times 0.1 + 3^2\times 0.3 + 4^2\times 0.35 + 5^2\times 0.15$
$= 12.55$

$Var(X) = E(X^2) - \mu^2 = E(X^2) - E(X)^2 = 12.55 - 3.35^2$
$= 1.3275$

$Var(Y) = (-2)^2\, Var(X) = 4\,(1.3275) = 5.31$

$Sd(Y) = \sqrt{Var(Y)} = \sqrt{5.31} = 2.3043.$

$\therefore$, $E(Y) = -2.7$, $Var(Y) = 5.31$, and $sd(Y) = 2.3043$!

Discrete Uniform Distribution: suppose the range of random variable X is $\{a, a+1, \dots, b\}$, where a and b are integers and all values are equally likely. Then, X is said to be a discrete uniform distribution on the set $\{a, a+1, \dots, b\}$. We may write $X \sim U[a, b]$, where a and b are the parameters of the distribution.

↳ Example: if X is the number obtained from rolling a fair die, then X has a discrete uniform distribution over the set $\{1, 2, \dots, 6\}$. Here, $a=1$, $b=6$, and $X \sim U[1, 6]$.

- Suppose $X \sim U[a, b]$. This means that we have $b-a+1$ possible values for X, which are all equally likely. Therefore, the probability function of X is:

otherwise, $f(x) = 0$

$$f(x) = P(X = x) = \frac{1}{b-a+1} \quad \text{for } X = a, a+1, \dots, b!$$

$$E(X) = \frac{a+b}{2}$$

$$Var(X) = \frac{(b-a+1)^2 - 1}{12}$$

$$F(X) = \frac{\lfloor x \rfloor - a + 1}{b-a+1}$$

$\{1, 4, 9, 16, 25, 36\}$
$\downarrow \ \downarrow \ \dots \ \downarrow$
$1 \ \ 2 \ \dots \ 6$

$\frac{1+6}{2} = \underline{3.5}$

Example: X is the outcome of a 6-sided die roll.

a) Compute $E(X)$

↳ $X \sim U[1, 6] \rightarrow E(X) = \frac{a+b}{2}$, where $a=1$, $b=6$

↳ $\frac{1+6}{2} = 7/2 = 3.5$

b) Compute $Var(X)$

↳ $X \sim U[1, 6] \rightarrow Var(X) = \dfrac{(b-a+1)^2 - 1}{12}$

↳ $Var(X) = \dfrac{(6-1+1)^2 - 1}{12} = \dfrac{35}{12}$

c) If $g(x) = x^2$, then compute $E(g(X))$.

↳ $E(g(x)) = E(x^2)$.

↳ $E(x^2) = 1 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6}$

$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = 91/6 \approx 15.17.$

Bernoulli Trial: a random experiment that the result is either a success or failure and the probability of success in every trial is p for $0 < p < 1$. Then the probability of failure is $1-p$ in every trial.

↓

closely related to a bernoulli trial is a **bionomial experiment**

↳ same as bernoulli, but repeated independently for a number of n times, and record X as the number of successes.

↳ then the RV X has binomial distribution as denoted by $X \sim Binomial(n, p)$

↳ p is probability of success!

- **Key assumptions:**

1) There are multiple trials, and on each trial, there are only two outcomes (S or F)
2) the probability of success (p) is constant over all n trials
3) the outcome (S/F) is independent of all other trials.

- **Probability Function**

for a RV $X \sim \text{Binomial}(n, p)$, we want to determine $f(x) = P(X = x)$.

- if we have x successes, then there must be $n-x$ failures. The total number of different arrangements of S's and F's is:

$$\frac{n!}{x!(n-x)!} = \binom{n}{x}$$

since the trials are independent, each arrangement (of x successes and $n-x$ failures) is achieved with probability:

$$p^{x}(1-p)^{n-x}.$$

$$\therefore,\ f(x) = \binom{n}{x} p^{x}(1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

Expected Value / Variance of Binomial Distribution:

↓

if $X \sim \text{Binomial}(n, p)$, then:

↳ $E(X) = n \cdot p$

$Var(X) = np(1-p)$

Example: 75% of students use instagram. A sample of 5 students are chosen. What is the probability that at least 3 students use instagram?

↳ $X \sim Binomial(n, p)$, where $n = 5$ and $p = 0.75$

$\therefore$, $X \sim Binomial(5, 0.75)$

$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5)$

$= \binom{5}{3} 0.75^3 \cdot 0.25^2 + \binom{5}{4} 0.75^4 \cdot 0.25^1 + \binom{5}{5} 0.75^5 \cdot 0.25^0$

$= 0.8965.$

Example: in a weekly lottery, you have a 0.02 chance of winning a prize with a single ticket. If I buy one ticket a week for 52 weeks, what is the probability that I win no prizes?

↳ $X \sim Binomial(n, p)$ where $n = 52$ and $p = 0.02$

↳ $X \sim Binomial(52, 0.02)$

$P(X=0) = \binom{52}{0} 0.02^0 (1-0.02)^{52} = 0.3498$

What's the probability I win 2 or more prizes?

$$1-P(X<2) = 1-(P(X=0)+P(X=1))$$

$$\hookrightarrow 1-\left(0.3498+\binom{52}{1}0.02^1(1-0.02)^{51}\right)$$

$$= 1-0.3498-0.3712 = 0.2790\,!$$

- Geometric Series Formula: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ if $|r|<1$
- Partial Geometric Series: $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$
- Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$
- Taylor Series for e^x: $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

- Geometric Distribution:

↳ same as bernoulli's, but repeat the experiment independently until the first success occurs. record X as the number of failures before the first success.

Then the RV X has a geometric distribution denoted as $X \sim Geo(p)$.

Geometric Distribution Probability Function

for a RV $X \sim Geo(p)$, we want to compute $f(x) = P(X=x)$.

- there is only one way to arrange x failures!

the geometric distribution has the following pmf:

$$f(x) = (1-p)^x p \quad , \quad x = 0, 1, 2, \dots$$

- If $X \sim Geo(p)$, then
  - $E(X) = \dfrac{1-p}{p}$
  - $Var(X) = \dfrac{1-p}{p^2}$

Example: at campus pizza, 60 % of orders are for takeout.

a) What is the probability that the 5^{th} order is the first takeout order?

$X \sim Geo(p)$ where $p = 0.6$

$X \sim Geo(0.6)$

4 non-takeouts first

$$P(X=4) = (1-0.6)^4 \cdot 0.6 = 0.01536$$

b) What is the probability that more than 3 orders are ordered before the first takeout?

$$P(X > 3) = 1 - P(X \leq 3)$$

$$= 1 - (P(X=3) + P(X=2) + P(X=1) + P(X=0))$$

$$= 1 - ((1-0.6)^3 \cdot 0.6 + (1-0.6)^2 \cdot 0.6 + (1-0.6)^1 \cdot 0.6 + (1-0.6)^0 \cdot 0.6) = 0.0256$$

Binomial / Geometric Models Summary

1) Two outcomes in each Bernoulli trial,
2) Each trial has same probability of success,
3) Trials are independent of each other.

Poisson Distribution

- The RV X represents the number of events of some type.

- The event occurs with a known constant mean rate, denoted by μ, where $\mu > 0$.

- One occurence is independent of the next

↳ then the RV X has a Poisson Distribution, denoted by $X \sim \text{Poisson}(\mu)$

↳ Poisson Distribution pmf: $f(x) = \dfrac{e^{-\mu}\mu^{x}}{x!}$ for $x = 0, 1, 2, \ldots$

- If $X \sim \text{Poisson}(\mu)$, then:
  - $E(X) = \mu$
  - $Var(X) = \mu$

Poisson Distribution from Binomial:

↳ If $X \sim \text{Binomial}(n, p)$, then for each value pf $x = 0, 1, 2, \ldots$, and $p \to 0$ with $np = \mu$ constant,

$$\lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} = \frac{e^{-\mu}\mu^x}{x!}$$

- This allows us to express the Poisson Distribution with $\mu = np$ as a close approximation to the binomial distribution when n is very large and p is small.

- As a general rule, the approximation gives reasonable results provided $n \geq 100$ and $p \leq 0.01$, and when x is close to np.

- When $n \geq 100$ and $np \leq 10$, the approximation will generally be excellent.

- When $n \geq 20$ and $p \leq 0.05$, the Poisson Distribution will be a good approximation.

Example: 1 out of 20 cups are winners. Suppose you buy 100 cups.

a) What is the probability that there are exactly two winning cups?

$X \sim \text{Binomial}(100, 0.05)$

$$P(X=2) = \binom{100}{2} 0.05^2 (1-0.05)^{98} = 0.08118$$

b) Approximate the probability using a Poisson approximation.

$$\mu = np = 100(0.05) = 5$$

$$P(X=2) = \frac{e^{-\mu}\mu^2}{2!} = \frac{e^{-5}5^2}{2} = 0.0842$$

Example: you roll a fair dice until you get a 6. Let RV x denote the number of rolls to get the first 6 (including the trial where the 6 occurs). What is the expected value of x?

Y = # trials before first 6: $Y \sim Geo(1/6)$

$$E(Y) = \frac{1-p}{p} = \frac{1-1/6}{1/6} = \frac{5/6}{1/6} = 5$$

including the trial where the 6 occurs:

$$E(X) = E(Y) + 1 = 6.$$

- Hypergeometric Distribution:
  - suppose a collection consists of a finite number of N items which can be classified into two distinct types:
    - There are r items of type 1 (or success),
    - The remaining N-r items are of type 2 (or failure).

Suppose $n \le N$ items are picked at random without replacement.

↓

If X represents the number of type 1 items that are drawn, then X has a hypergeometric distribution with parameters N, r, and n.

↳ $X \sim HG(N, r, n)$

• Hypergeometric Probability Mass Function:

For a RV $X \sim HG(N, r, n)$, we want to compute $f(x) = P(X = x)$

• There are $\binom{N}{n}$ points in the sample space
• There are $\binom{r}{x}$ ways to choose x type-1 objects from the r type-1 objects available
• There are $\binom{N-r}{n-x}$ ways to choose the remaining $n-x$ type-2 objects from the $N-r$ type-2 objects available.

$$\therefore, \; f(x) = P(X = x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

where $x \ge \max\{0, n-(N-r)\}$ and $x \le \min\{r, n\}$.

• If $X \sim HG(N, r, n)$,

↳ • $E(X) = \frac{nr}{N}$

• $Var(X) = n \frac{r}{N} \left(1 - \frac{r}{n}\right) \frac{N-n}{N-1}$

Example: a box has 100 microchips, 80 are good and 20 are defective. I select 10 microchips at random without replacement.

a) What's the probability that the 10 chips selected include no more than 2 defectives?

X = # defective chips in sample of 10.

↳ $\therefore$, $X \sim HG(N, r, n)$, where $N=100$, $r=20$, and $n=10$.

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \sum_{x=0}^{2} \frac{\binom{20}{x}\binom{80}{10-x}}{\binom{100}{10}} = 0.681$$

b) now we look at 30 chips instead of 10. Find the probability of getting exactly 6 defectives.

↳ $X \sim HG(N, r, n)$, where $N = 100$, $r = 20$, $n = 30$

$$P(X=6) = \frac{\binom{20}{6}\binom{80}{24}}{\binom{100}{30}} = 0.214.$$

- Hypergeometric vs Binomial Distributions
  - Both have 2 outcomes
  - Experiment repeated n times in both
  - The RV X records the number of successes

- But, the binomial distribution requires n independent trials where p is constant

- In hypergeometric, n draws are made w/o replacement, so trials are NOT independent!

- Negative Binomial Distribution
  - ↳ same as regular binomial, but repeated independently until a specified number of r successes have been obtained.
  - record X as the number of trials to obtain r successes.

Then the RV X has a negative binomial distribution, denoted as $X \sim NB(r, p)$

Negative Binomial Probability Mass Function

For a RV $X \sim NB(r, p)$, we want to compute $f(x) = P(X = x)$.

- One must obtain the r^{th} success on the x^{th} trial by obtaining $(r-1)$ successes in the first $(x-1)$ trials, in any order.
  - ↳ There are $\binom{x-1}{r-1}$ different orders.
- Then obtain a success on the x^{th} trial

↳ $f(x) = \binom{x-1}{r-1} p^r (p-1)^{x-r}, \quad x = r, r+1, \ldots$

If $X \sim NB(r, p)$,

↳ • $E(X) = r/p$

- $Var(X) = \frac{r(1-p)}{p^2}$

Example: a start-up wants 5 investors. Each investor will independently agree to invest with a 20% probability. The founder asks one investor at a time until he gets 5 "yes"s. Let X represent the total number of investors asked.

a) what is the pmf of X?

↳ $P(X = x) = \binom{x-1}{4}(0.2)^5(0.8)^{x-5}$, $x = 5, 6, \ldots$

b) Find $P(X=6)$

$P(X=6) = \binom{5}{4}(0.2)^5(0.8)^1 = 0.00128$

Binomial vs Negative Binomial Distribution

- Can be distinguished by looking at what is specified / known in advance
- Binomial: n independent trials but we do not know the number of successes we will obtain
- NB: we know the number of successes to be obtained, but don't know the number of trials needed to obtain the r successes.

→ Continuous Probability Distributions (Chapter 3)

- Continuous random variables

↳ take on values from $\mathbb{R}$ (no longer countable)

↳ they are treated differently than discrete random variables because now $P(x=a)=0$, since the chance of obtaining the specific value of a among an infinite number of possibilities is extremely remote!

↳ ∴, we specify the probability over intervals rather than individual points.

Cumulative Distribution Function (CDF)

↳ we can use the cdf to describe the distribution of a continuous RV: $F(x) = P(X \leq x)$

- must satisfy the following properties:

1) $F(x)$ is defined $\forall\ x$
2) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
3) $F(x)$ is a non-decreasing function of x.
4) $P(a < X \leq b) = F(b) - F(a)$

For a continuous RV X, the probability density function (pdf) $f(x)$ is the derivative of the cdf and is given by:

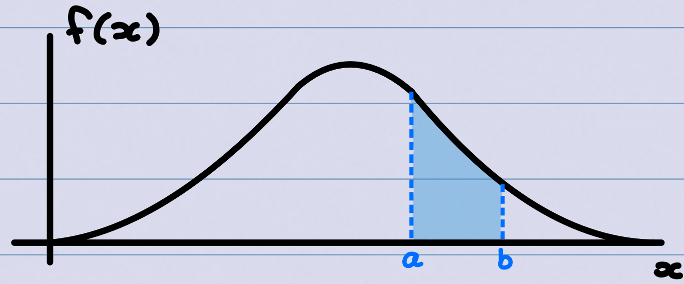
$$f(x) = \frac{dF(x)}{dx}$$

Properties of a PDF:

1) $P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)\, dx$
2) $f(x) \geq 0$ (since $F(x)$ non-decreasing)
3) $\int_{-\infty}^{\infty} f(x) = \int_{\text{all } x} f(x) = 1$
4) $F(x) = \int_{-\infty}^{x} f(u)\, du$

Interval Probability

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$



Percentile: suppose X is a continuous RV, with the cdf $F(x)$. The $100p$ th percentile of X (or the distribution) is the value q such that:

$$P(X \le q) = p$$

Example: if $F(1) = 0.8$, then the 80% percentile of the distribution is 1.

- If $p = 0.5$, then q is the median of x!

Example: Let X be a RV with cdf given by:

$$F(x) = \begin{cases} 0, & x \le 0 \\ x/4, & 0 < x \le 4 \\ 1, & x > 4 \end{cases}$$

a) find the pdf $f(x)$.

↳ $f(x) = \dfrac{dF(x)}{dx} \rightarrow f(x) = \begin{cases} 1/4, & 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$

b) find the 90^{th} percentile, which is the value of x such that the area under the curve to the left of x is 90%

$\hookrightarrow F(x) = \frac{x}{4} = 0.9 \rightarrow \quad x = 3.6.$

Example: let X be a RV with the following pdf:

$$f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

a) find c.

$\hookrightarrow$ Since $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 f(x)\,dx = 1$, we know

$$\int_0^2 c(4x - 2x^2)\,dx = 1.$$

$$\hookrightarrow c\int_0^2 4x - 2x^2\,dx = c\left[2x^2 - \frac{2x^3}{3}\right]_0^2 = 1$$

$$\hookrightarrow c\left(8 - \frac{8}{3}\right) = 1 \rightarrow \quad c = \frac{3}{8}$$

b) find $F(x)$

$\hookrightarrow F(x)$ must be 0 for $x \leq 0$.

If $0 < x < 2$:

$$\hookrightarrow F(x) = \int_{-\infty}^{u} f(u)\,du = \int_0^x \frac{3}{8}(4u - u^2)\,du$$

$$= \frac{3}{8}\left[2u^2 - \frac{2}{3}u^3\right]_0^x = \frac{3}{4}x^2 - \frac{x^3}{4}$$

If $x > 2$, $F(x) = 1$.

$$\therefore,\ F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{3}{4}x^2 - \frac{x^3}{4}, & 0 < x < 2 \\ 1, & x \geq 2 \end{cases}$$

c) Find $P(X > 1)$

↳ $P(X > 1) = 1 - P(X \leq 1) = 1 - F(1)$

$= 1 - \left(\frac{3}{4} - \frac{1}{4}\right) = 1 - \frac{1}{2} = 1/2.$

- When X is a continuous RV with pdf $f(x)$, the expected value of X is given by:

$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

In general, we define:

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

The variance of X is defined as:

$$\sigma^2 = Var(X) = E[(X-\mu)^2] = E(X^2) - E(X)^2,$$

where:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

The standard deviation of X is:

$$\sigma = sd(X) = \sqrt{Var(X)}$$

Example: suppose a continuous RV X has pdf $f(x) = cx^2$ for $0 < x < 2$, and 0 otherwise.
Find $P(X > 1)$:

↳ we know $\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{2} f(x)\,dx = 1$

↳ $c\int_0^2 x^2\,dx = c\left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}c = 1$

$\therefore,\ c = \frac{3}{8}.$

$P(X > 1) = \int_1^{\infty} f(x)\,dx = \int_1^2 \frac{3}{8}x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_1^2$

$= \frac{1}{8}(8-1) = 7/8.$

• All of the earlier properties still hold in the continuous case!

↓

• $E(a\,g(X)+b) = a\,E(g(X)) + b$

• $E(a\,g_1(X) + b\,g_2(x)) = a\,E(g_1(X)) + b\,E(g_2(X))$

• $Var(a\,g(X)+b) = a^2\,Var(g(X))$

Example: let X be a RV with the following pdf:

$$f(x) = \begin{cases} kx^2, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Find k: $\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow k\int_{-1}^{1} x^2\,dx = 1$

$\therefore,\ k\left[\frac{x^3}{3}\right]_{-1}^{1} = k\left[\frac{1}{3}+\frac{1}{3}\right] = \frac{2k}{3} = 1 \rightarrow k = \frac{3}{2}.$

b) Find the cdf of x for all values of x:

↳ $F(x) = 0$ if $x \leq -1$.

If $-1 < x < 1$:

↳ $F(x) = \int_{-\infty}^{x} f(u)\,du = \frac{3}{2}\int_{-1}^{x} u^2\,du = \frac{3}{2}\left[\frac{u^3}{3}\right]_{-1}^{x}$

$= \frac{1}{2}(x^3 + 1)$.

$F(x) = 1$ if $x \geq 1$.

$\therefore, F(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(x^3+1), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}$

c) Find $P(-\frac{1}{3} < X < 0.5)$:

↳ $P(-\frac{1}{3} < x < \frac{1}{2}) = F(\frac{1}{2}) - F(\frac{1}{3})$

$= \frac{1}{2}\left((\frac{1}{2})^3 - (\frac{1}{3})^3\right) = \frac{35}{432}$

d) Find $E(X)$

↳ $E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \frac{3}{2}\int_{-1}^{1} x \cdot x^2\,dx$

$= \frac{3}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{3}{2}\left[\frac{1}{4} - \frac{1}{4}\right] = 0.$

e) Find $Var(X)$

↳ $Var(X) = E(X^2) - E(X)^2.$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \frac{3}{2}\int_{-1}^{1} x^2 \cdot x^2\, dx$$

$$= \frac{3}{2}\left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{3}{2}\left(\frac{1}{5} + \frac{1}{5}\right) = 3/5.$$

↳ $Var(X) = \frac{3}{5} - 0^2 = 3/5 = 0.6.$

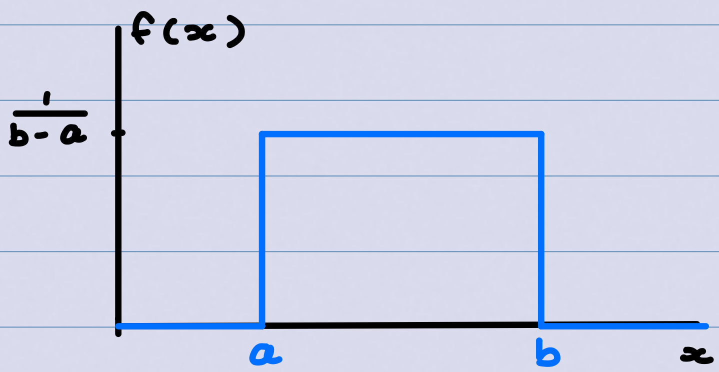
Continuous Uniform Distribution: a RV X has a continuous uniform distribution on $[a, b]$ if X has a pdf that is constant on $[a, b]$ and 0 elsewhere.

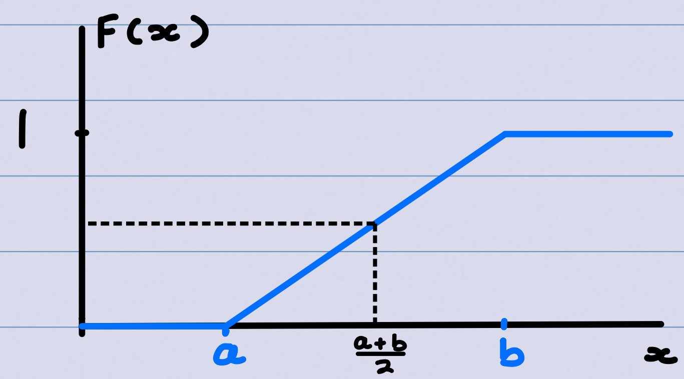
↳ The pdf of X is defined as:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

• Note: it does not matter whether the interval is open or closed.

↓





→ For a continuous RV X where $X \sim U[a, b]$,

• $E(X) = \frac{a+b}{2}$, • $Var(X) = \frac{(b-a)^2}{12}$

Exponential Distribution: a continuous RV X is said to have an exponential distribution with the rate $\lambda > 0$ if X has the probability density:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

the cdf of X is given by:

$$F(x) = 1 - e^{-\lambda x} \quad \text{for } x > 0$$

It's also common to use parameter $\theta = \frac{1}{\lambda}$ in the exponential distribution.

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta > 0$, and $F(x) = 1 - e^{-x/\theta}$ for $x > 0$

We write $X \sim Exp(\theta)$ or $X \sim Exponential(\theta)$

For a continuous RV X where $X \sim Exp(\theta)$,

- $E(X) = \theta$, $Var(X) = \theta^2$

Example: the amount of time in hours that a computer survives breaking down is exponentially distributed with a mean of 100 hours.

a) Find the probability that a computer will function between 50 and 150 hours before breaking down.

↳ let X be the time a computer functions before breaking down.

$X \sim Exp(\theta)$, where $\theta = 100$.

$P(50 < X < 150) = F(150) - F(50)$

$= (1 - e^{-150/100}) - (1 - e^{-50/100})$

$= e^{-1/2} - e^{-3/2} = 0.383$

b) Find the probability that will function for fewer than 100 hours.

↳ $P(X < 100) = F(100) = 1 - e^{-100/100} = 1 - e^{-1}$

c) If a computer survives more than 100 hours, what is the probability that it'll survive 50 more?

$$P(X > 100 + 50 \mid X > 100) = \frac{1 - F(150)}{1 - F(100)} = \frac{e^{-3/2}}{e^{-1}} = e^{-1/2}$$

• Memoryless Property of the Exponential Distribution

↳ shown in part c of previous example

↳ $P(X > b + c \mid X > b) = P(X > c)$

↳ basically, given that you've waited b units of time for the next event, the probability that you wait an additional c units of time does NOT depend on b and only depends on c.

Example: the length of a phone call is an exponential RV with a mean of 10 minutes. If someone arrives immediately ahead of you at a telephone booth, find the probability you'll wait:

a) more than 10 minutes

↳ let X be the minutes you wait → $X \sim Exp(\theta)$, where $\theta = 10$.

↳ $P(X > 10) = 1 - P(X \leq 10) = 1 - F(10)$

$$= 1 - (1 - e^{-x/\theta}) = e^{-10/10} = e^{-1} = 0.3679$$

b) between 10 and 20 minutes:

↳ $P(10 < X < 20) = F(20) - F(10)$

$$= 1 - e^{-20/10} - 1 - e^{-10/10} = e^{-1} - e^{-2} = 0.2325$$

c) more than an additional 7 minutes, given that you've already waited for over 10.

$\hookrightarrow P(X > 10+7 \mid X > 10) = P(X > 7)$

$= 1 - P(X \leq 7) = 1 - F(7) = 1 - (1 - e^{-7/10})$

$= e^{-7/10} = 0.4966$

Normal Distribution: a continuous RV X is said to have a normal distribution if X has the pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are parameters.
We write $X \sim N(\mu, \sigma^2)$

For a continuous RV X where $X \sim N(\mu, \sigma^2)$, we write:

$$E(X) = \mu, \qquad Var(X) = \sigma^2$$

- the mean, μ, shifts the distribution along the x-axis.
  $\hookrightarrow$ μ is a location parameter.

- the variance, σ^2, stretches out or pulls in the distribution
  $\hookrightarrow$ σ^2 is a scale parameter.

- The standard normal distribution is $Z \sim N(0, 1)$.

↳ every normal distribution is a version of the standard normal distribution that has been transformed / scaled!

· The cdf of the normal distribution $N(\mu, \sigma^2)$ is:

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du, \quad x \in \mathbb{R}$$

↳ horrible to solve, so let's use a different method!

Finding Normal Probabilities via a $N(0,1)$ table

↳ let $X \sim N(\mu, \sigma^2)$, and define:

$$Z = \frac{X-\mu}{\sigma}.$$

Then, $Z \sim N(0,1)$, and

$$P(X \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right)$$

Example: let $Z \sim N(0,1)$. Find the following probabilities:

a) $P(Z < 2.11)$

↳ from the table of $N(0,1)$ cdf values, we get $P(Z < 2.11) = 0.98257$.

b) $P(Z > 1.06)$

↳ from the table, we get:

$P(z > 1.06) = 1 - P(z < 1.06) = 1 - 0.85543$
$= 0.14457$

c) $P(z < -1.06)$

since a $N(0,1)$ is symmetrical about the y-axis,
$P(z < -1.06) = P(z > 1.06) = 0.14457$!

d) $P(-1.06 < x < 2.11)$

↳ $P(-1.06 < x < 2.11) = P(z < 2.11) - P(z < -1.06)$
$= 0.982 - 0.144 = 0.838$

Example: let $z \sim N(0,1)$.
a) Find 85% percentile of z

↳ the 85% percentile of z is the level u such that $F(u) = 0.85$. $\therefore$, $u = F^{-1}(0.85)$.

From the table, we see $u = 1.0364$.

b) Find a number b such that $P(z > b) = 0.9$
↳ since 90% of the distribution lies to the right of b, and since $N(0,1)$ is symmetrical, we know b must be negative!

$P(z < b) = 1 - P(z > b) = 1 - 0.9 = 0.1$

$\therefore P(Z > |b|) = 0.1 \rightarrow P(Z < |b|) = 0.9$

$\rightarrow |b| = F^{-1}(0.9) = 1.282 \rightarrow b = -1.282!$

c) Find a number c such that $P(|Z| < c) = 0.95$

$\hookrightarrow P(|Z| < c) = P(-c < Z < c) = 0.95$

$\rightarrow P(Z < c) = 0.95 + \frac{0.05}{2} = 0.975$

$\rightarrow c = F^{-1}(0.975) = 1.96$

Example: $X \sim N(10, 2)$. Calculate $P(|X-10| \leq 3)$

For a RV $X \sim N(10, 2)$, it follows that $\frac{X-10}{\sqrt{2}} \sim N(0, 1)$

$P(|X-10| \leq 3) = P(-3 \leq X-10 \leq 3)$

$= P\left(-\frac{3}{\sqrt{2}} \leq \frac{X-10}{\sqrt{2}} \leq \frac{3}{\sqrt{2}}\right)$

$= P(-2.12 \leq Z \leq 2.12)$

$= P(Z \leq 2.12) - (1 - P(Z \leq 2.12))$

$= 2 \cdot P(Z \leq 2.12) - 1$

$= 2 \cdot 0.983 - 1$

$= 0.966!$

Example: $X \sim N(-7, 14)$. Calculate $P(|X+7| \geq 8)$

For an RV $X \sim N(-7, 14)$, it follows that $\frac{X+7}{\sqrt{14}} \sim N(0,1)$

$$P(|X+7| \geq 8) = P\left(\frac{|X+7|}{\sqrt{14}} \geq \frac{8}{\sqrt{14}}\right)$$

$$= P(|Z| \geq 2.138)$$ bc symmetrical!

$$= 2 \cdot P(Z \geq 2.138)$$

$$= 2 \cdot (1 - P(Z < 2.138))$$

$$= 2 \cdot (1 - 0.9837) = 0.032.$$

Example: IQ is normally distributed. The mean is 100 and the standard deviation is 15.

a) What percentage of the population has IQ scores of 140 or more?

↳ let the RV X represent the IQ score → $X \sim N(100, 15^2)$

$$P(X \geq 140) = P(X - 100 \geq 140 - 100)$$

$$= P\left(\frac{X-100}{15} \geq \frac{140-100}{15}\right)$$

$$= P(Z \geq 8/3)$$

$$= 1 - P(Z < 8/3)$$

$$= 1 - P(Z < 2.66)$$

$$= 1 - 0.996 = 0.004.$$

b) What percentage of the population have IQ scores between 80 and 120?

↳ $P(80 < x < 120)$

$$P(80 < x < 120) = P(80-100 < X-100 < 120-100)$$

$$= P\left(\frac{-20}{15} < \frac{X-100}{15} < \frac{20}{15}\right)$$

$$= P(-4/3 < Z < 4/3)$$

$$= P(Z < 4/3) - P(Z < -4/3)$$

$$= P(Z < 4/3) - P(Z > 4/3)$$

$$= P(Z < 4/3) - (1 - P(Z < 4/3))$$

$$= 2P(Z < 4/3) - 1$$

$$= 2 \cdot 0.908 - 1 = 0.816$$

c) Solve for the 95th percentile, which is the IQ level such that 95% of the population has an IQ below it.

$$P(X \leq x) = P\left(\frac{X-100}{15} \leq \frac{x-100}{15}\right) = 0.95$$

$$\rightarrow \frac{x-100}{15} = F^{-1}(0.95) = 1.6449$$

$$\hookrightarrow x = (15)(1.6449) + 100 = 124.67$$

Multivariate Probability Distributions (Chapter 4)

multiple RVs can be denoted by X, Y, Z / X_1, X_2, X_3

The joint pmf of 2 discrete RVs: suppose X and Y are two discrete random variables. The joint probability mass function of (X, Y) is:

$$f(x,y) = P(\{X=x\} \cap \{Y=y\})$$

as shorthand, $f(x,y) = P(X=x, Y=y)$

Properties:

1) $f(x,y) \geq 0 \quad \forall \ (x,y)$
2) $\sum f(x,y) = \sum_{\text{all } x} \sum_{\text{all } y} f(x,y) = 1$

more generally, $f(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$

Example: I either get a small or a medium coffee along with two or three timbits. Let X be the size of the coffee and let Y be the number of timbits. The joint pmf of what I buy is shown below:

		coffee	
		S	M
timbits	2	0.1	0.2
	3	0.3	c

Show that the probability I buy a medium coffee and 3 timbits is 0.4.

$$\sum P(X=x, Y=y) = 1 \rightarrow \sum_{\text{all } x} \sum_{\text{all } y} P(X=x, Y=y) = 1$$

$\rightarrow P(X=S, Y=2) + P(X=S, Y=3) + P(X=M, Y=2) + P(X=M, Y=3)$

$= 0.1 + 0.3 + 0.2 + c = 1$

$\hookrightarrow c = 0.4.$

Marginal Distribution: given the joint pmf of X and Y, the marginal probability function of X is:

$$f_X(x) = \sum_{\text{all } y} f(x, y)$$

Similarly, the marginal probability function of Y is:

$$f_Y(y) = \sum_{\text{all } x} f(x, y)$$

Note: always check that these add up to 1!

Example: X and Y have a joint probability function given by:

$y \backslash x$	0	1	2
0	0.2	0.3	0.1
2	0.25	0.13	0.02

Compute the marginal probability function of X.

$P(X=0) = P(X=0, Y=1) + P(X=0, Y=2) = 0.2 + 0.25 = 0.45$
$P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) = 0.3 + 0.13 = 0.43$
$P(X=2) = P(X=2, Y=1) + P(X=2, Y=2) = 0.1 + 0.02 = 0.12$

also we have that $\sum_{x=0,1,2} f_X(x) = 1$!

Independent Random Variables: suppose that X and Y are discrete random variables with joint probability mass function $f(x, y)$ and marginal probability functions $f_X(x)$ and $f_Y(y)$.

RVs X and Y are independent if and only if:

$$f(x,y) = f_X(x)\, f_Y(y) \quad \forall \text{ pairs of values } (x,y)$$

or equivalently,

$$P(X=x, Y=y) = P(X=x)\,P(Y=y) \quad \forall\ (x,y).$$

Example: X and Y have the following joint pmf:

$y \backslash x$	0	1	2	$f(y)$
0	0.2	0.3	0.1	0.6
2	0.25	0.13	0.02	0.4
$f(x)$	0.45	0.43	0.12	1

Are X and Y independent?

↳ for X and Y to be independent, we must have $f(x, y) = f_X(x)\, f_Y(y)\ \forall\ (x,y)$ pairs.

let's just check $f(0,0) = f_X(0)\, f_Y(0)$

↳ $0.2 = 0.45 \times 0.6 = 0.27 \rightarrow 0.2 \neq 0.27!$

$\therefore$, $f(x,y) \neq f_X(x)\, f_Y(y)$, so they're dependent!

Conditional Probability Functions

↳ the conditional probability of X given $Y=y$ is:

$$f(x \mid y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{f(x,y)}{f_Y(y)},$$

provided that $P(Y=y) > 0$.

· Note that this conditional pf is defined over all x in the range of the RV X with y fixed.

· Also, $f(x \mid y) \geq 0$ and $\sum_{\text{all } x} f(x \mid y) = 1$!

Example: continuing the coffee problem:

		coffee S	coffee M
timbits	2	0.1	0.2
	3	0.3	0.4 (=c)
$f(x)$		0.4	0.6

Find the conditional probability of Y, given $X=M$:

$$P(Y=2 \mid X=M) = \frac{P(Y=2, X=M)}{P(X=M)} = \frac{0.2}{0.6} = 1/3$$

$$P(Y=3 \mid X=M) = \frac{P(Y=3, X=M)}{P(X=M)} = \frac{0.3}{0.6} = 1/2.$$

Conditional Probability Function for 2 Independent RVs

↓

since $f(x,y) = f_X(x) f_Y(y)$,

$$f(x \mid y) = P(X=x, Y=y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$$

|

↳ the conditional pmf becomes the marginal pmf in the case of independence!

Functions of Random Variables: suppose that $g: \mathbb{R}^2 \to \mathbb{R}$, and X and Y are RVs with a joint pmf $f(x, y)$. The probability function of RV $U = g(X, Y)$ is given by:

$$f_U(u) = P(U = u) = \sum_{\substack{\text{all } (x,y) \\ g(x,y)=u}} f(x, y)$$

1) If $X \sim \text{Poisson}(\mu_1)$ and $Y \sim \text{Poisson}(\mu_2)$, and X and Y are independent, then $X + Y \sim \text{Poisson}(\mu_1 + \mu_2)$

2) If $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$, and X and Y are independent, then $X + Y \sim \text{Binomial}(n+m, p)$

Example: suppose X and Y have joint pmf:

$y \backslash x$	0	1	2	$f(y)$
0	0.2	0.3	0.1	0.6
2	0.25	0.13	0.02	0.4
$f(x)$	0.45	0.43	0.12	1

Let $U = X + Y$.

a) Calculate the probability function of U.

the possible values of RV U are 0, 1, 2, 3, and 4. Therefore, we have:

$P(U=0) = P(X=0, Y=0) = f(0,0) = 0.2$

$P(U=1) = P(X=1, Y=0) = f(0,1) = 0.3$

$P(U=2) = P(X=2, Y=0) + P(X=0, Y=2) = f(0,2) + f(2,0)$
$= 0.35$

$P(U=3) = P(X=1, Y=2) = f(1,2) = 0.13$

$P(U=4) = P(X=2, Y=2) = f(2,2) = 0.02$

we have $\sum_{u=1,2,3,4,0} f(u) = 1$!

b) Find $P(X+Y>3)$

$P(U>3) = P(U=4) = 0.02.$

c) Find $P(XY) = 0$

we are looking for the event $X=0$ or $Y=0$, which would be the probability of the union of these two events.

$\hookrightarrow P(XY=0) = P(X=0) + P(Y=0) - P(X=0, Y=0)$
$= 0.6 + 0.45 - 0.2 = 0.85.$

Expectation for Multivariate Distributions:

suppose X and Y are discrete RVs with joint pmf $f(x,y)$. Then for a function $g: \mathbb{R}^2 \to \mathbb{R}$,

$$E[g(X,Y)] = \sum_{\text{all } (x,y)} g(x,y) f(x,y)$$

more generally, for n discrete random variables

$X_1, X_2, \ldots, X_n$ with joint pmf $f(x_1, x_2, \ldots, x_n)$, if $g: \mathbb{R}^n \to \mathbb{R}$, then

$$E[g(X_1, X_2, \ldots, X_n)] = \sum_{\text{all } (x_1, x_2, \ldots, x_n)} g(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n)$$

Properties of Expected Value:

$$E(X+Y) = \sum_x x f_X(x) + \sum_y y f_Y(y) = E(X) + E(Y)$$

and in general,

$$E[a g_1(X,Y) + b g_2(X,Y)] = a E[g_1(X,Y)] + b E[g_2(X,Y)]$$

Example: X and Y have the joint pmf:

$f(x,y)$	$x = 0$	1	2
$y = 0$	0.2	0.3	0.1
2	0.25	0.13	0.02

Compute $E(XY)$.

$g(x,y) = xy$	$x = 0$	1	2
$y = 0$	0	0	0
2	0	2	4

$$E(XY) = 0 + 0 + 0 + 0 + 2(0.13) + 4(0.02)$$
$$= 0.34$$

Multinomial Distribution Definition: Physical Setup

- Consider an experiment which is repeated independently n times.
  - ↳ Each trial has k possible outcomes
  - ↳ let $p_1, p_2, \dots, p_k$ denote the probability of each outcome $(p_1 + \dots + p_n = 1)$
  - ↳ let RV X_i denote the number of outcomes of type i for $i = 1, \dots, k$.

We say $X_1, X_2, \dots, X_k$ have a multinomial distribution with parameters n and $p_1, p_2, \dots, p_k$.

Shorthand : $(X_1, X_2, \dots, X_n) \sim \text{Multinomial}(n; p_1, p_2, \dots, p_n)$

- Multinomial distribution is a generalization of binomial distribution.

Multinomial Distribution Definition: PMF

If $X_1, X_2, \dots, X_k$ have a joint multinomial distribution with parameters n and $p_1, \dots p_n$, then their joint probability function is:

$$f(x_1, x_2, \dots, x_k) = \frac{n!}{x_1! \, x_2! \dots x_k!} p_1^{x_1} \, p_2^{x_2} \dots p_k^{x_k}$$

where $x_i = 0, 1, \dots, n$ and $x_1 + x_2 + \dots x_k = n$.

Note: the term $\frac{n!}{x_1! x_2! \dots x_k!}$ is called a multinomial coefficient. It counts the number of ways we can arrange x_1 items of type-1, x_2 items of type-2, ...,

x_k items of type-k among the k type of total items.

Example: the probability that a component will last less than 50 hours, between 50 and 90 hours, or more than 90 hours, is 0.2, 0.5, and 0.3, respectively. The time to failure of 8 such components is recorded.

a) What is the probability that 1 will last less than 50 hours, 5 will last between 50 and 90 hours, and 2 will last more than 90 hours?

$$(X_1, X_2, X_3) \sim \text{Multinomial}(8; 0.2, 0.5, 0.3)$$

$$P(X_1=1, X_2=5, X_3=2) = \frac{8!}{1! \cdot 5! \cdot 2!} \cdot 0.2^1 \cdot 0.5^5 \cdot 0.3^2$$

$$= 0.0945.$$

b) What is the probability that exactly 3 components will last between 50 and 90 hours?

$$X_2 \sim \text{Binomial}(8, 0.5)$$

$$P(X_2 = 3) = \binom{8}{3} 0.5^3 (1-0.5)^5$$

c) Determine the joint pmf of the number of components that last less than 50 hours and the number of components that last between 50 and 90 hours.

$f(x_1, x_2) = P(X_1 = 1, X_2 = 2)$

$= P(X_1 = x_1, X_2 = x_2, X_3 = 8 - x_1 - x_2)$

$$= \frac{8!}{x_1! x_2! (8 - x_1 - x_2)!} 0.2^{x_1} \cdot 0.5^{x_2} \cdot 0.3^{(8 - x_1 - x_2)}$$

↳ the same as the pmf of (X_1, X_2, X_3)!

Marginal Probability Distributions

Suppose that we want to find the marginal distribution of one RV, X_i, in the multinomial distribution.

↓

We can divide the outcomes into success (type i) and 'failure' (not type i).

Therefore, $X_i \sim \text{Binomial}(n, p_i)$

Example: a bag contains 6 red balls, 5 blue balls, and 5 green balls. 6 balls are drawn from the bag independently and with replacement. Let X denote the number of red balls drawn, and let Y represent the number of blue balls drawn. What is $E(X + Y)$?

$X \sim \text{Binomial}(6, 6/16)$, $Y \sim \text{Binomial}(6, 5/16)$

$E(X+Y) = E(X) + E(Y) = 6 \cdot \frac{6}{16} + 6 \cdot \frac{5}{16} = 4.125.$

Covariance: if X and Y are jointly distributed, the covariance of X and Y is

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))]$$

A shortcut to compute $Cov(X,Y)$ is:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Example: X and Y have the joint pmf:

$f(x,y)$	$x = 0$	$x = 1$	$x = 2$
$y = 0$	0.2	0.3	0.1
$y = 2$	0.25	0.13	0.02

Calculate $Cov(X,Y)$.

$Cov(X,Y) = E(XY) - E(X)E(Y)$

- $E(XY) = 2 \cdot 0.13 + 4 \cdot 0.02 = 0.34$
- $E(X) = 0 \cdot 0.45 + 1 \cdot 0.43 + 2 \cdot 0.12 = 0.67$
- $E(Y) = 0 \cdot 0.6 + 2 \cdot 0.4 = 0.8$

↳ $Cov(X,Y) = 0.34 - 0.67 \times 0.8 = -0.196$!

If X and Y are independent RVs, then

$$E(XY) = E(X)E(Y)$$

$Cov(X, Y) = 0$

Note: the converse is not necessarily true

↳ ie: $Cov(X, Y) = 0 \not\Rightarrow X$ and Y are independent!!

The Correlation Coefficient: the correlation coefficient between X and Y is denoted by ρ, and is defined by:

$$\rho = \frac{Cov(X, Y)}{sd(X)\, sd(Y)}$$

- measures the strength of the linear relationship between X and Y!
- essentially a rescaled version of the covariance, it now lies in the interval $[-1, 1]$.

- Note: $Cov(X, Y) \rightarrow$ interpret the sign

  $\rho \rightarrow$ interpret the magnitude and the sign.

- Properties:

→ ρ will have the same sign as $Cov(X, Y)$.

→ $-1 \leq \rho \leq 1$, and as $\rho \rightarrow \pm 1$, the relationship between X and Y becomes closer to linear.

→ if $\rho \rightarrow +1$, then X and Y will have an approximately positive linear relationship

→ if $\rho \rightarrow -1$, then X and Y will have an approximately negative linear relationship.

→ if $\rho \approx 0$, then X and Y are said to be uncorrelated!

• Note: just because $\rho \approx 0$, we <u>cannot</u> say that there is no relationship between X and Y, only that there doesn't seem to be any evidence of a linear relationship.

Example: RVs X and Y have the following joint pmf:

$f(x,y)$	$x=0$	$x=1$	$x=2$
$y=0$	0.2	0.3	0.1
$y=2$	0.25	0.13	0.02

Calculate the correlation coefficient. What does this indicate about X and Y?

• from a previous example, we know $E(XY) = 0.34$, $E(X) = 0.67$, $E(Y) = 0.8$, and $Cov(X, Y) = -0.196$

$$E(X^2) = 0 \cdot 0.45 + 1^2 \cdot 0.43 + 2^2 \cdot 0.12 = 0.91$$
$$Var(X) = 0.91 - (0.67)^2 = 0.4611$$
$$E(Y^2) = 0 \cdot 0.6 + 2^2 \cdot 0.4 = 1.6$$
$$Var(Y) = 1.6 - (0.8)^2 = 0.96$$

$\rho = \dfrac{-0.196}{\sqrt{0.4611} \cdot \sqrt{0.96}} = -0.295$. There's a weak negative relationship between X and Y.

Linear Combination of RVs: suppose that $X_1, \ldots, X_n$ are RVs with joint probability function $f(x_1, \ldots, x_n)$. A linear combination of the $X_1, \ldots, X_n$ is any RV of the form:

$$\sum_{i=1}^{n} a_i X_i \quad \text{where } a_1, \ldots, a_n \in \mathbb{R}.$$

The expected value of a linear combination of RVs $X_1, \dots, X_n$ is:

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

↳ some special cases:

1) $E(aX + bY) = aE(X) + bE(Y)$
2) $E(X+Y) = E(X) + E(Y)$
3) $E(X-Y) = E(X) - E(Y)$

Example: if $X \sim N(2,1)$ and $Y \sim \text{Uniform}(0,2)$, $E(2X-4Y) = ?$

→ for a normal distribution, $E(X) = \mu$, so $E(X) = 2$.
→ for a uniform distribution, $E(Y) = (a+b)/2$, so $E(Y) = 1$

$E(2X-4Y) = 2E(X) - 4E(Y) = 2(2) - 4(1) = 0.$

Sample Mean: for n RVs $X_1, \dots, X_n$, the sample mean is:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

If $X_1, \dots, X_n$ are RVs with identical mean μ (that is, $E(X_i) = \mu \; \forall i$), then $E(\overline{X}_n) = \mu$.

· Results of Covariance: two useful results:

↓

1) $Cov(X, X) = Var(X)$

2) $Cov(aX+bY, cU+dV) = ac\,Cov(X,Y) + ad\,Cov(X,V) + bc\,Cov(Y,U) + bd\,Cov(Y,V)$

- **Results for the Variance:**

1) for any RVs X and Y, we have:

$$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y)$$

2) if X and Y are independent RVs, $Cov(X,Y)=0$, and:
- $Var(X+Y) = Var(X) + Var(Y)$
- $Var(X-Y) = Var(X) - Var(Y)$

3) if $X_1, \dots, X_n$ are independent and $Var(X_i) = \sigma_i^2$, then:

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2\, Var(X_i) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

Remark: using the previous results, if $X_1, \dots, X_n$ are <u>independent</u> RVs with the same mean μ and the same variance σ^2, the sample mean $\bar{X}$ has:

$$E(\bar{X}) = \mu, \quad Var(\bar{X}) = \frac{\sigma^2}{n}$$

↳ this shows that the average of n RVs is less variable than a single observation. In other words, $\bar{X}$ becomes less variable as n gets larger.

↳ $Var(\bar{X}) \to 0$ as $n \to \infty$