Lecture 1 - 4th Sept 2024

Statistics: the science of understanding data and making decisions in the face of variability and uncertainty.

Probability: A branch of mathematics concerned with describing and modeling uncertain events.

Preliminaries:

- **Experiment**: the process of obtaining an observed result of some phenomenon.
- **Trial**: the performance of an experiment.
- **Outcome**: the result of a single trial (attempt) of an experiment.
- **Event**: one or more outcomes of an experiment.
- **Probability**: the measure of how likely an event is.

Sample Space: the set of ALL possible distinct outcomes in a random experiment, denoted by S.

↳ note: one and only one of the outcomes occurs in any single trial of the experiment.

↳ example:

- Roll a six-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- Flip a coin: $S = \{\text{heads}, \text{tails}\}$
- Waiting time for a bus: $S = \{t \in \mathbb{R}, 0 \leq t \leq 10\}$

- a sample space is finite if it consists of a finite number of outcomes, say $\{a_1, a_2, \ldots, a_N\}$

- a sample space can be a set of countable infinite outcomes, say $\{a_1, a_2, \ldots\}$, where the outcomes can be put into a one-on-one correspondence with positive integers.
  - Example: Suppose our experiment consists of tossing a coin until it lands on heads.
    - the possible outcomes in the sample space are:

      H, TH, TTH, TTTH, TTTTH, ...

      ∴, the outcomes are countable but infinite.

Discrete Sample Space: when a sample space is either finite or countably infinite. Else, it's non-discreet.

- Example:
  - Roll a six-sided die: discreet
  - Flip a coin: discreet
  - Waiting time for a bus: non-discreet

Event: a subset of sample space S.

- we say A is an event if $A \subset S$.
  - note: A and S are both sets!

↳ Example: consider tossing two coins.

a) what is the event of obtaining at least one head?

$S = \{HH, HT, TH, TT\}$

$\therefore A = \{HH, HT, TH\}$

b) What is the event of obtaining at least one head and at least one tail?

$A = \{HT, TH\}$.

Elementary / Simple Event: an event that contains only one outcome of the experiment. Eg, $A = \{a_1\}$.

Compound Event: an event made up of two or more simple events. Eg: $A = \{a_1, a_2\}$.

Set Notations and Terminology:

- A set is a collection of elements. Eg, $A = \{apple, orange\}$.
- The notation $x \in A$ or $x \notin A$ will mean that x is or is not an element of A.
- The empty / null set is $\emptyset$.
- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The notation $B \subset A$ means that B is a subset of A.
- Complement: $A^c = \{x \mid x \in S, x \notin A\} = \bar{A}$
- Two sets A and B are disjoint if $A \cap B = \emptyset$.

- The set $A \cap B^c$ is "A but not B"
- The set $A^c \cap B^c$ is "neither A nor B"

- If $A_1, \ldots, A_n$ is a finite collection of sets,
  - → $A_1 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$ is "in all A_i"
  - → $A_1 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$ is "in at least one A_i"

very similar to disjoint!

Mutually Exclusiveness: when two events A and B if $A \cap B = \emptyset$. If events are exclusive, they have no outcomes in common.

↳ Example: tossing two coins. A is the event "at least one head", and B is the event "both tails".

↳ $S = \{HH, HT, TH, TT\}$.

$A = \{HH, HT, TH\}$, $B = \{TT\}$.

$$A \cap B = \{HH, HT, TH\} \cap \{TT\} = \emptyset \ !$$

$\therefore$, A and B are mutually exclusive.

↳ Events $A_1, A_2, A_3, \ldots$ are said to be mutually exclusive if they are pairwise mutually exclusive. That is,

$$A_i \cap A_j = \emptyset, \text{ wherever } i \neq j.$$

Probability Modeling: assigning a probability $P(A)$ to each event A. This probability measures how likely it is that A will happen when the experiment is conducted.

↳ think of $P(A)$ as a (set) function, whose domain is a collection of sets (events), and the range of which is a subset of real numbers.

↳ not all set functions are appropiate for assigning probabilities to event.

- Let $S = \{a_1, a_2, a_3, \dots\}$ be a discrete sample space. We assign to each elementary event $A_i = \{a_i\}$ for $i = 1, 2, \dots$, a number $P(A_i) = P(\{a_i\})$, such that

  - $0 \le P(A_i)$ and $\sum_{\text{all } i} P(A_i) = 1$

- We call $P(A)$ the probability of A, and call the set of probabilities $\{P(A_i), i = 1, 2, \dots\}$ the probability distribution on S.

↳ **General Properties:**

- The null event (empty set) has a probability of 0. $\therefore$ $P(\emptyset) = 0$.
- If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$
- If $A_1, A_2, \dots, A_k$ is a finite collection of pairwise mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_k) = P(A_1) + P(A_2) + \dots + P(A_k)$$

- $P(\bar{A}) = 1 - P(A)$

↳ Example: suppose a six-sided die is rolled once. If A is the event that an even number is obtained. What is $P(A)$?

- $S = \{1, 2, 3, 4, 5, 6\}$. Since each number is equally likely, $P(\{i\}) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

$A = \{2, 4, 6\}$. → $P(A) = P(\{2\}) + P(\{4\}) + P(\{6\})$
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.

↳ Example: Toss a coin twice. What's the probability of getting exactly one head?

- $S = \{HH, HT, TH, TT\}$. Each option is uniformly likely.
- $A = \{HT, TH\}$ → $P(A) = P(\{HT\}) + P(\{TH\})$
$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Random Selection: when choosing an object from a finite set where each object has an equal chance of being selected.

Uniform Probability Model: consider a discrete finite sample space $S = \{a_1, a_2, \dots, a_N\}$, where each simple event has a probability of $1/N$. This is a uniform distribution over the set $\{a_1, a_2, \ldots, a_N\}$.

↳ For a compound event A, we can calculate P(A) by:

$$P(A) = \frac{n(A)}{N},$$

where $n(A)$ is the number of outcomes in A.

- The formula $P(A) = \frac{n(A)}{N}$ is classical probability.

(trivial) example: draw one card at random from a well-shuffled deck without jokers. What is the probability that the card is a spade?

↳ $S = \{A\heartsuit, 2\heartsuit, \ldots\}$.

$A_s = \{A\spadesuit, 2\spadesuit, \ldots, K\spadesuit\}$.

↳ $P(A_s) = \frac{n(A_s)}{N} = \frac{13}{52} = \frac{1}{4}$.

Example: suppose that P is a probability, and A and B are events such that A and B are mutually exclusive, and $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$. Is each statement true or false?

A) A and B are each non-empty

Since $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$, we know that

$P(A) = 0.2$ and $P(B) = 0.3$.

we also know that $P(\emptyset) = 0$. Since $0.2 \neq 0 \neq 0.3$, we know A and B cannot be empty.
$\therefore$, the statement is true.

B) $P(A \cap B) = 0$

Since we're told that A and B are mutually exclusive, we know that $A \cap B = \emptyset$.

we also know that $P(\emptyset) = 0$, so the statement is true!

C) $P(A \cup B) = 0.4$

we know that $\underbrace{P(A \cup B)}^{\text{"the probability of A or B"}} = P(A) + P(B)$

$P(A) + P(B) = 0.2 + 0.3 = 0.5 \neq 0.4$.
$\therefore$, the statement is false!

Example: suppose two six-sided die are rolled where the outcomes are of the form (die 1, die 2). If A is the event that the first die is even, and B is the event that the sum of both rolls is 6, compute the event $C = \bar{A} \cap B$

↳ $A = \{(2,1), (2,2), \ldots, (4,1), (4,2), \ldots, (6,1), (6,2), \ldots, (6,6)\}$.

$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$\bar{A}$ means the event that A does not happen, so in other words, $\bar{A}$ is the event that the first die is odd.

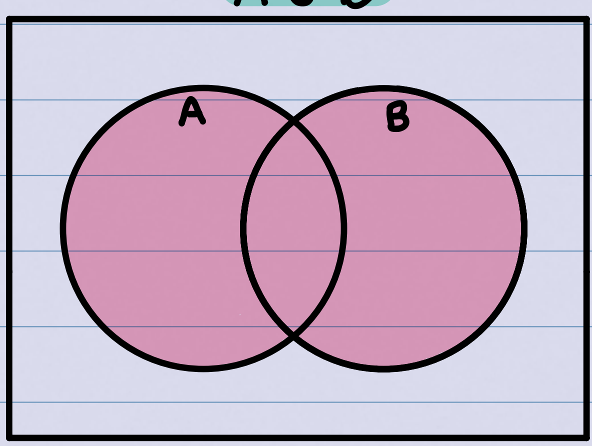
$\therefore, \bar{A} = \{(1,1), \ldots, (1,6), (3,1), \ldots, (3,6), (5,1), \ldots, (5,6)\}$.

$\bar{A} \cap B$ is A AND B.
↳ $\therefore, C = \{(1,5), (3,3), (5,1)\}$.

Venn Diagram: a tool to illustrate the relationship among sets (or events).
↳ made up of:
- a rectangle to represent the sample space S.
- Circles within the rectangle to illustrate the events.

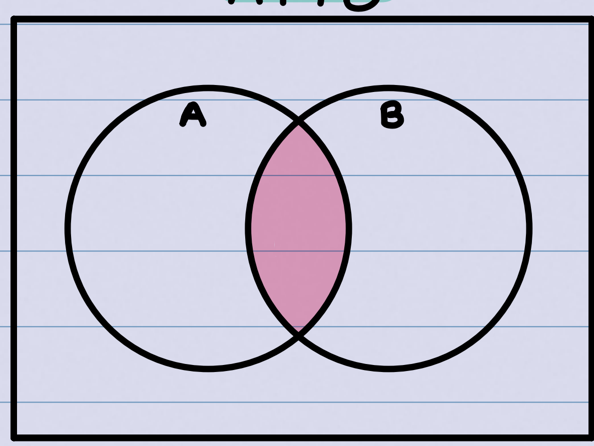
The Union and Intersection of Two Events

$A \cup B$



"A or B"

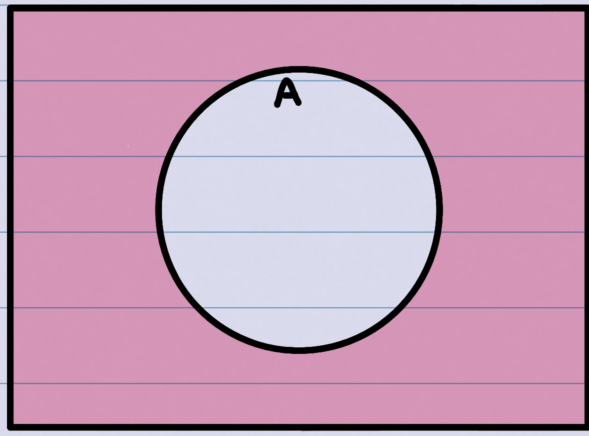
$A \cap B$



"A and B"

The Compliment of an Event:

$\bar{A}$



"S but not A"

De Morgan's Laws:

- $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Probability Rules

- **Probability of the Complement of an Event**

  ↳ $P(A) = 1 - P(\bar{A})$

bc mutually exclusive!

Proof: $A \cup \bar{A} = S$, so $P(A \cup \bar{A}) = P(A) + P(\bar{A})$

since $P(A \cup \bar{A}) = P(S) = 1$, $1 = P(A) + P(\bar{A})$!

Example: an experiment involves tossing a coin four times. The event A is "at least one head". Find the probability of A, ie, find $P(A)$.

- the only case where there is NOT at least one head is $\bar{A} = \{TTTT\}$.

Flipping a coin 4 times would result in 2^4 possible outcomes, and $\bar{A}$ is only 1 of these.

$\therefore$, $P(\bar{A}) = \frac{1}{16}$, so $\overbrace{P(A) = 1 - P(\bar{A})}^{\text{probability of complement!}} = 1 - \frac{1}{16} = \frac{15}{16}$.

Example: two ordinary dice are rolled. Find the probability that at least one lands on a 6.

$S = \{(1,1), \ldots, (1,6), (2,1), \ldots, (2,6), \ldots, (6,1), \ldots, (6,6)\}$.

A = the event of at least one six.
$\therefore$, $\bar{A}$ is the event that neither die is a six.

classical probability of each die to NOT be a six: $\frac{n(x)}{N}$.
where $n(x)$ is the number of outcomes that satisfy the roll not being a 6 – ie; $\{1,2,3,4,5\} \Rightarrow 5$, and N is the total number of outcomes in the set – ie; $\{1,2,3,4,5,6\} \Rightarrow 6$.

$\therefore \frac{n(x)}{N} = \frac{5}{6}$!

Since both dice are independent, the probability of both NOT being a six (aka $\bar{A}$) is $\frac{5}{6} \cdot \frac{5}{6}$.

$\therefore$, $P(\bar{A}) = \frac{25}{36}$, so $P(A) = 1 - \frac{25}{36} = \frac{11}{36}$!

Rules for Union:

· Union of two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: At a university, 70% of the students own a car, 60% of students live on campus, and 50% both live on campus and own a car. If a student is chosen at random, what is the probability that the student does not own a car and does not live on campus?

- Let A be the event that a student has a car.
- Let B be the event that a student lives on campus.

we want: $P(\bar{A} \cap \bar{B})$.

↳ we know that $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

↳ $P(\bar{A} \cap \bar{B}) = 1 - P(\overline{\bar{A} \cap \bar{B}}) = 1 - P(A \cup B)$

from the union of two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\begin{aligned} \therefore,\ P(\bar{A} \cap \bar{B}) &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 + P(A \cap B) - P(A) - P(B) \\ &= 1 + 0.5 - 0.7 - 0.6 \end{aligned}$$

↳ $\therefore,\ P(\bar{A} \cap \bar{B}) = 0.2 = 20\%$.

Union of Three Events: for any 3 events A, B, and C:

↓

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Conditional Probability: determining the probability of some event A, while knowing that some related event B has occured.

Example: consider that a fair die has been rolled and you are asked to give the probability that it was a 5.

↳ $A = \{5\}$, $S = \{1, 2, 3, 4, 5, 6\}$.

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}$.

However, if you're told that the dice has landed on an odd number, the odds change:

↳ $A = \{5\}$, $S = \{1, 3, 5\}$.

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{3}$.

Example: a box contains 100 microchips, some produced by Factory 1 and the rest by Factory 2. Among these microchips, some are defective, while others are good. Let A be the event of "obtaining a defective microchip" and B be the event of "the microchip was produced in Factory 2."

the number of microchips in each category:

	B	$\bar{B}$	totals
A	15	5	20
$\bar{A}$	45	35	80
totals	60	40	100

a) Find the probability of obtaining a defective microchip.

the total of A is 20, and there are 100 total microchips.

$$P(A) = \frac{n(A)}{N} = \frac{20}{100} = \frac{1}{5} \quad (20\%)$$

b) Given that a microchip is from factory 1, find the probability that it is defective.

the sample space S shrinks from all 100 microchips to only the 60 made by factory 1 (event B).

$$\therefore,\ P(A) = \frac{n(A)}{N} = \frac{15}{60} = \frac{1}{4} \quad (25\%)$$

Conditional Probability mathematical definition:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) > 0)$$

"the probability of A given B"

↳ • $0 \leq P(A \mid B) \leq 1$

• $P(A \mid B) = 1 - P(\bar{A} \mid B)$

• $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

pretty much the same as the regular union of two events

Multiplication Theorem of Probability: a way to compute the joint occurence of A and B:

$$P(A \cap B) = P(B) \cdot P(A \mid B) = P(A) \cdot P(B \mid A)$$

• **Law of Total Probability:** if $B_1, B_2, \ldots, B_k$ is a collection of mutually exclusive events and $B_1 \cup B_2 \cup \ldots \cup B_k = S$, then for any event A,

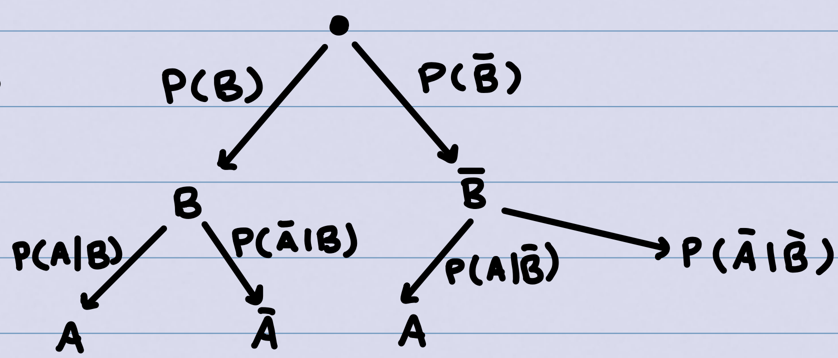
$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A \mid B_i)$$

Note: for a special case $k = 2$:

only for $k = 2$!!

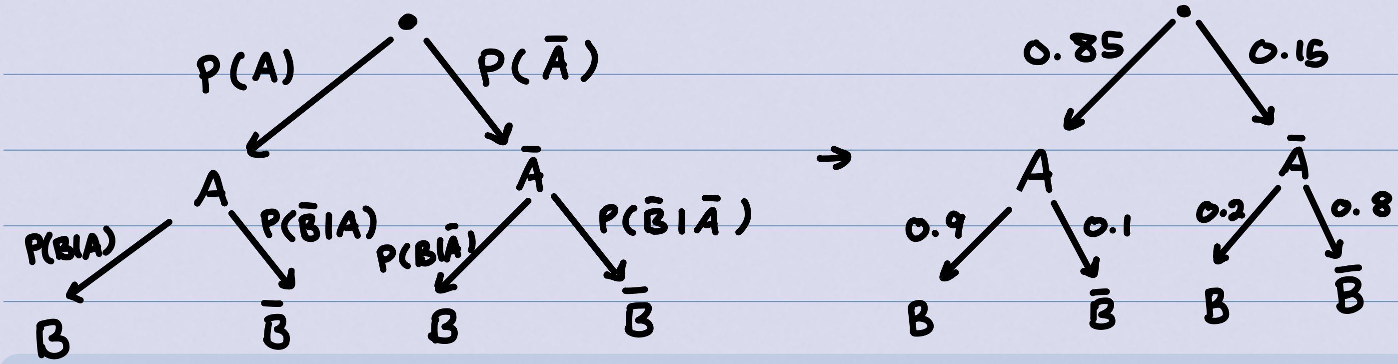
$$P(A) = P(A \mid B) \cdot P(B) + P(A \mid \bar{B}) \cdot P(\bar{B})$$

• for event A to occur, it must happen with either B or $\bar{B}$. ∴, only $A \cap B$ or $A \cap \bar{B}$ can occur.



Example: without water, a plant will die with a probability of 0.8, and with water, it will die with a probability of 0.1. I'll remember to water the plant with a probability of 0.85. Represent this info using a tree diagram.

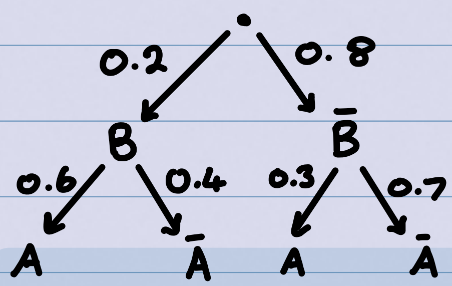
↳ let A be the event that I remember to water the plant, and B be the event that the plant survives.



Example: in a typical year, 20% of the days have a high temperature greater than 22°C. 40% of these days, there is no rain. During the rest of the year, when the temperature is ≤ 22°C, 70% of the days have no rain.

a) represent this information in a tree diagram.

let A be the event that it rains, and let B be the event that the temperature peaks above 22°C.



b) solve for the proportion of days in the year which have rain and a temperature $\leq 22°C$.

↳ aka, find $P(A \cap \bar{B})$

we know $P(A|\bar{B}) = \dfrac{P(A \cap \bar{B})}{P(\bar{B})} = \dfrac{P(A \cap \bar{B})}{1 - P(B)}$

$\therefore$, $P(A \cap \bar{B}) = P(A|\bar{B}) \cdot (1 - P(B))$

$= 0.3 \cdot (1 - 0.2) = 0.3 \cdot 0.8 = 0.24$

$= 24\%$.

Baye's Rule: $P(B|A) = \dfrac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$,

and more generally, $P(B_j|A) = \dfrac{P(B_j)P(A|B_j)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$ → for each $j = 1, 2, \ldots, k$.

Example: revisiting the plant problem from before, if the plant is alive, what's the probability that I remembered to water it?

↳ let A be the event that I remember to water the plant, and B be the event that the plant survives.

→ same as previous time

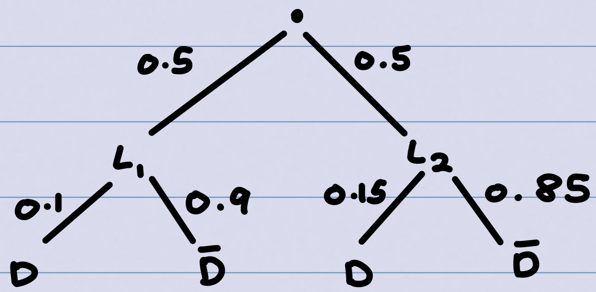
$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})} = \frac{(0.9)(0.85)}{(0.9)(0.85) + (0.2)(0.15)}$$

$= 0.9623 = 96.23\%$!

Example: electric motors from two assembly lines are pooled in a common stockroom, with equal numbers of motors from each line. Motors are periodically sampled for testing. It's known that 10% of the motors from line 1 are defective, while 15% of the motors from line two are defective.

a) if a motor is randomly selected, what is the probability that it is defective?

- Let D be the event that a motor is defective.



$\rightarrow P(D) = P(D \cap L_1) + P(D \cap L_2)$

$= 0.5 \cdot 0.1 + 0.5 \cdot 0.15$

$= 0.125$

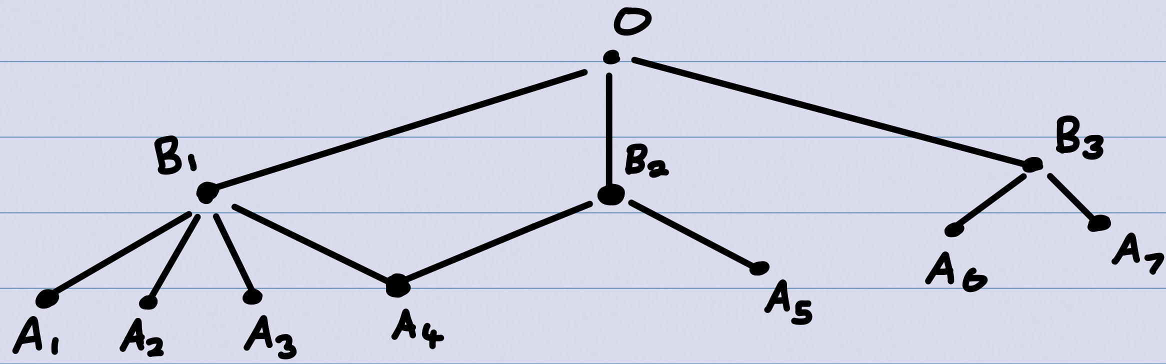
↳ 12.5% chance.

b) if the randomly selected motor is defective, what is the probability that it came from line 1 ?

$$P(L_1 \mid D) = \frac{P(D \mid L_1)P(L_1)}{P(D)} = \frac{0.1 \cdot 0.5}{0.125} = 0.4$$

= 40% chance.

Example: a man starts at point 0 on this map. He chooses a path at random and follows it to point B_1, B_2, or B_3. From that point, he chooses a new path at random and follows it to one of the points $A_1 \ldots A_7$.



a) what is the probability that the man arrives at point A_4?

$$P(A_4) = P(A_4 \mid B_1) + P(A_4 \mid B_2) = \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12} + \frac{1}{6}$$
$$= 1/4 = 0.25$$

= 25% !

b) suppose that the man arrives at point A_4. What is the probability that he passed through B_1?

$$P(B_1 \mid A_4) = \frac{P(A_4 \mid B_1)}{P(A_4)} = \frac{\frac{1}{4} \cdot \frac{1}{3}}{1/4} = \frac{1/12}{1/4} = \frac{4}{12} = \frac{1}{3}$$

∴, there was an $33.3\dot{3}\%$ chance he came from B_1.

Independent Event: events A and B are called independent if and only if $P(A \cap B) = P(A)P(B)$. Otherwise, A and B are called dependent events.

↳ an equivalent formulation can be given in terms of conditional probability:

↳ If A and B are events such that $P(A) > 0$ and $P(B) > 0$, then A and B are called independent if and only if either of the following hold:

$$P(A|B) = A \quad , \quad \text{or} \quad P(B|A) = P(B)$$

Note: independence ≠ mutually exclusive!

↱ unless $P(A) = 0$!

General rule: $P(A \cap B) = P(B|A) \cdot P(A)$

Independence of Compliments: suppose events A and B are independent. Then the following events are also independent:

↳
- $\bar{A}$ and $\bar{B}$
- $\bar{A}$ and B
- A and $\bar{B}$

Independence of n events: the k events $A_1, A_2, \ldots, A_k$ are said to be independent, or mutually independent, if and only if:

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \ldots P(A_{i_k})$$

$\forall$ possible distinct $i_1, i_2, \ldots, i_k$ chosen from $1, 2, \ldots, n$.

↳ Note: for a sequence of events to be independent, all of their subsets must also be independent.

Example: suppose a fair die is rolled twice. Let A be the event that the first roll is a 6, and let B be the event that the second roll is a 6. Show that A and B are independent events.

↳ we want to show that $P(A \cap B) = P(A) \cdot P(B)$

$P(\{6,6\}) = P(A \cap B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

$P(A) = \frac{n(A)}{N} = \frac{1}{6} = \frac{n(B)}{N} = P(B)$. $\therefore$, $P(A) = P(B) = \frac{1}{6}$.

$P(A) \cdot P(B) = \frac{1}{36} = P(A \cap B)$.

↳ $\therefore$, the events are independent.

Counting Techniques

often, manually counting many outcomes manually can be very time-consuming and tedious.

1) Addition Rule : "OR" interpreted as addition

↳ Eg: suppose job 1 can be done in p ways and job 2 can be done in q ways. $\therefore$, we can do either job 1 or job 2 in $p+q$ ways.

2) Multiplication Rule: "AND" interpreted as multiplication.

↳ Eg: suppose job 1 can be done in p ways and for each of these ways, we can do job 2 in q ways. $\therefore$, we can do both Job 1 and Job 2 in $p \times q$ ways.

Example: how many are there to answer a 20 question true/false test?

$$\overset{Q_1}{2} \times \overset{Q_2}{2} \times \dots \times \overset{Q_{20}}{2} = 2^{20} \text{ ways.}$$

Sampling with Replacement: what we get on the first selection does not affect what we get on second selection.

↳ Eg: every time an object is selected, it's put back into the pool of possible objects.

Sampling without Replacement: what we get on the first selection will affect what we get on the second selection.

↳ Eg: once an object is selected, it stays out of the pool of possible objects.

Example: A bag contains 3 blue marbles and 5 red marbles. If you pick two marbles from the bag,

a) what is the probability of picking 2 blue marbles if the selection is with replacement?

let A be the event of picking two marbles.

using independence property!

↳ P(A) = P(blue on first pick) + P(blue on second pick)

↳ $P(A) = \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64} = 14.06\%$

b) what about without replacement?

$P(A) = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56} = \frac{3}{28} = 10.71\%!$

Example: a personal identification number (PIN) of length 4 is formed by randomly selecting 4 digits from the set of digits $\{0, 1, 2, \ldots, 9\}$. If selection is done with replacement, find the probability that:

a) The PIN is even:

↳ for the PIN to be even, the last digit must be a multiple of 2. ∴, $A = \{0, 2, 4, 6, 8\}$.

the first 3 digits don't matter!

$$\therefore, P(A) = \frac{n(A)}{N} = \frac{5}{10} = \frac{1}{2} \rightarrow 50\%$$

b) the PIN contains at least one 0?

↳ P(at least one 0) = 1 - P(no 0s in PIN)

∴, there are 9 ways to fill each of the digits, as we want the probability of no zeros.

multiplication counting rule

↳ not 0 AND not 0 AND...

$$\therefore, P(\text{no } 0\text{s}) = 9 \cdot 9 \cdot 9 \cdot 9 = 9^4$$

total sample space is 10^4 (including 0s).

$$\therefore P(\text{at least one zero}) = 1 - \frac{n(\text{no } 0\text{s})}{n(S)} = 1 - \frac{9^4}{10^4}$$

c) Redo a and b but without replacement:

i) even PIN:

A: event of getting an even number. $\{0, 2, 4, 6, 8\}$.

S: the sample space $10 \times 9 \times 8 \times 7$ → we have less options for each digit, bc no replacement!

A: $\underbrace{9 \times 8 \times 7}$ 5 → bc len({0, 2, 4, 6, 8}) = 5. → $n(A) = 9 \times 8 \times 7 \times 5$

↳ $P(A) = \dfrac{9 \times 8 \times 7 \times 5}{10 \times 9 \times 8 \times 7} = \dfrac{5}{10} = \dfrac{1}{2} = 50\%$!

ii) at least one zero:

$P(B) = 1 - P(\bar{B}) = 1 - \dfrac{9 \times 8 \times 7 \times 6}{10 \times 9 \times 8 \times 7} = 1 - \dfrac{6}{10} = \dfrac{2}{5} = 40\%$

no 0 in PIN

Example: a fair die is tossed 3 times. What is the probability that exactly one of the tosses produces a number greater than 4?

let A be the event that the first toss is greater than 4, and the second and third tosses are ≤ 4.

↳ $\{5, 6\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}$.

"AND"

∴, $n(A) = 2 \times 4 \times 4 = 32$.

let B be the event where the 2nd toss is > 4, and let C be the event where it's the 3rd toss. → $n(A) = n(B) = n(C)$

∴, total number of ways to get exactly one number greater than 4 is $3(2 \times 4 \times 4) = 96$.

$\therefore$, P(exactly one toss > 4) = $\frac{96}{6^3} \approx 44.44\%$.

Counting Permutations:

- **Permutation:** an ordered arrangement of a set of objects.

Example: how many different ordered arrangements of the letters a, b, and c are possible?

↳ abc, acb, bac, bca, cab, cba.

$\therefore$, 6 arrangements!

This will very quickly become very tedious and slow.

- The number of permutations of n distinguishable objects is:

$$n \times (n-1) \times \dots \times 1 = n!$$

- Given n distinct objects, a permutation of length r is an ordered subset of r objects.

- The number of permutations of length r taken from n objects is denoted as $n^{(r)}$, where:

"n to r factors"

$$n^{(r)} = n \times (n-1) \times \dots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Example: five seperate awards are to be presented to some students from a class of size 30. How many different outcomes are possible if

a) a student can receive any number of awards?

$$\underset{1^{st}}{\underline{30}} \times \underset{2^{nd}}{\underline{30}} \times \underset{3^{rd}}{\underline{30}} \times \underset{4^{th}}{\underline{30}} \times \underset{5^{th}}{\underline{30}} = 30^5.$$

$\therefore,\ n(A) = 30^5$

b) each student can receive, at most, one award?

$$\underset{1^{st}}{\underline{30}} \times \underset{2^{nd}}{\underline{29}} \times \underset{3^{rd}}{\underline{28}} \times \underset{4^{th}}{\underline{27}} \times \underset{5^{th}}{\underline{26}} = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$$

$\hookrightarrow n(A) = 30^{(5)} = \dfrac{30!}{25!}$

· Sometimes the outcomes in the sample space are subsets of a fixed size:

Example: suppose we have three books: b_1, b_2, and b_3. We choose two of the books to read. In how many ways can the two books be read if:

a) the order that the books are read in matters:

(b_1, b_2), (b_2, b_1)
(b_1, b_3), (b_3, b_1) → 6 ways!
(b_2, b_3), (b_3, b_2)

b) the order that the books are read in does not matter?

$\{b_1, b_2\}$
$\{b_1, b_3\}$ → 3 ways!
$\{b_2, b_3\}$

Given n distinct objects, a combination of size r is an unordered subset of r objects chosen from n distinct objects.

↳ The number of combinations (or subsets) of size r chosen from n distinct objects is given by:

$$\binom{n}{r} = \frac{n^{(r)}}{r!} = \frac{n!}{(n-r)r!}$$

↳ "n choose r"

Example: we randomly select a subset of 3 digits from $\{0, 1, \ldots, 9\}$. All the digits in each outcome are unique, but the order of the elements in a subset is not relevant.

aka, without replacement

Find the probability that:

a) All the digits in the selected subset are even:

$$S = \{\{0,1,2\}, \{0,1,3\}, \ldots, \{7,8,9\}\}$$

$$n(S) = \binom{10}{3}.$$

length of 5

the set of even digits is $\{0, 2, 4, 6, 8\}$.

let A be the event that all 3 digits are even.

$\hookrightarrow n(A) = \binom{5}{3}$

$\hookrightarrow P(A) = \dfrac{n(A)}{n(S)} = \dfrac{\binom{5}{3}}{\binom{10}{3}} = \dfrac{1}{12} = 8.3\dot{3}\ \%.$

b) At least one of the digits in the selected subset is less than or equal to 5.

P(at least one digit ≤ 5)
$= 1 - P(\text{no digit} \leq 5)$
$= 1 - P(\text{all digits} > 5)$

the set of digits > 5: $\{6, 7, 8, 9\}$.

$= 1 - \dfrac{\binom{4}{3}}{\binom{10}{3}} = \dfrac{29}{30} = 96.6\dot{6}\ \%.$

Example: there are 30 geese of which 6 were tagged. Later, 5 of the geese are randomly captured.

a) How many samples of 5 are possible? $\binom{30}{5}$

b) How many samples of 5, which include 2 of the tagged geese, are possible?

$\hookrightarrow \binom{6}{2} \cdot \binom{24}{3}$

c) if the five captured geese represent a simple random sample drawn from the 30 geese, find the probability that:

i) two of the 5 captured geese are tagged.

$$P(A) = \frac{\binom{6}{2}\binom{24}{3}}{\binom{30}{5}} = 0.2130 = 21.3\%$$

ii) none of the 5 captured geese are tagged.

$$P(B) = \frac{\binom{24}{5}}{\binom{30}{5}} = 0.2983 = 29.83\%$$

→ Key Properties of Permutations and Combinations:

1) $n^{(r)} = n \times (n-1)^{(r-1)}$ for $r \geq 1$.

2) $\binom{n}{r} = \frac{n^{(r)}}{r!}$

3) Symmetry Property: $\binom{n}{r} = \binom{n}{n-r}$ for $r \geq 0$.

4) since $0! = 1$, $\binom{n}{0} = \binom{n}{n} = 1$.

5) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

6) Binomial Theorem:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \text{ for } x \in \mathbb{R}.$$

Summary of Counting Techniques:

- Addition rule: OR (+)
- Multiplication rule: AND (×)

- **Factorial:** $n!$ is the number of arrangements of n distinct objects where order matters.
- **Permutation:** $n^{(k)}$ is the number of ways to choose k objects from n distinct objects where order matters.
- **Combination:** $\binom{n}{k}$ is the number of ways to choose k objects from n objects where order does <u>NOT</u> matter.

Example: a person has 10 friends, of whom 6 will be invited to a party. How many choices are there if:

a) 2 of the friends will only attend together?

↳ $\binom{8}{4} \cdot \binom{2}{2}$

b) 2 of the friends will not attend if the other is also attending?

OR (+)! → only one or the other can occur.

↳ $\binom{8}{6} \cdot \binom{2}{0} + \binom{8}{5} \cdot \binom{2}{1}$

Example: 13 cards are picked at random from a standard deck of 52 cards without replacement. Find the probability that we pick:

a) at least one ace:

↳ A = the event we pick at least one Ace, so

$\bar{A}$ = the event we pick no aces

↳ $P(A) = 1 - P(\bar{A}) = 1 - \dfrac{\binom{48}{13}}{\binom{52}{13}}$

b) 6 spades, 4 hearts, 2 diamonds, and one club:

$$\hookrightarrow \frac{\binom{13}{6}\cdot\binom{13}{4}\cdot\binom{13}{2}\cdot\binom{13}{1}}{\binom{52}{13}}$$

Example: consider drawing 3 numbers at random with replacement from the digits $\{0, 1, \ldots, 9\}$. What is the probability that there is a repeated number among the 3?

$S = 10 \cdot 10 \cdot 10 = 10^3$

A = event of repeated number

$\bar{A}$ = event of no repeated number

$\hookrightarrow 10 \cdot 9 \cdot 8$ options

$$\therefore P(A) = 1 - P(\bar{A}) = 1 - \frac{n(\bar{A})}{n(S)} = 1 - \frac{10^{(3)}}{10^3} = \frac{7}{25}.$$

Example: Melissa selects 7 numbers between 1 and 50, and then a computer does the same. What is the probability that at least 5 of Melissa's numbers match the computer's?

$\hookrightarrow S = \binom{50}{7}$, and Melissa must match 5, 6, or 7 of the numbers: $\binom{7}{5}\binom{43}{2} + \binom{7}{6}\binom{43}{1} + \binom{7}{7}\binom{43}{0}$

$$\therefore P = \frac{\binom{7}{5}\binom{43}{2} + \binom{7}{6}\binom{43}{1} + \binom{7}{7}\binom{43}{0}}{\binom{50}{7}}$$

Example: a box contains 4 coins - 3 fair coins and 1 biased coin which lands on heads 80% of the time. A coin is randomly picked and tossed 6 times. It lands on heads 5 times. What is the probability the coin is fair?

let H5 be the event of getting 5 heads in 6 tosses. let F be the event the coin is fair, and let B be the event the coin is biased.

$$P(F \mid H5) = \frac{P(F \cap H5)}{P(H5)} = \frac{P(H5 \mid F)P(F)}{P(H5 \mid F)P(F) + P(H5 \mid B)P(B)}$$

$$= \frac{6 \times 0.5^5 \times 0.5 \times 3/4}{6 \times 0.5^5 \times 0.5 \times 3/4 + 6 \times 0.8^5 \times 0.2 \times 1/4} = 0.417$$

Example: there are 4 passengers on a 5-floor elevator. What is the probability that:

a) the passenges all get off on different floors?

↳ 5 people getting off on 4 different floors is $5^{(4)}$, out of total possibilities $S = 5^4$.

$$\therefore, \frac{5^{(4)}}{5^4} = \frac{24}{125} = 19.2\%$$

b) 2 passengers get off on floor two and 2 get off on floor 3?

↳ $$\frac{\binom{4}{2}\binom{2}{2}}{5^4}$$

c) 2 passengengers get off one a floor and 2 passengers get off on a different floor?

↳ $$\frac{\binom{4}{2}\binom{2}{2} \times \binom{5}{2}}{5^4}$$