

Lecture 1 - 4th Sept 2024

Statistics: the science of understanding data and making decisions in the face of variability and uncertainty.

Probability: A branch of mathematics concerned with describing and modeling uncertain events.

Preliminaries:

- Experiment: the process of obtaining an observed result of some phenomenon.
- Trial: the performance of an experiment.
- Outcome: the result of a single trial (attempt) of an experiment.
- Event: one or more outcomes of an experiment.
- Probability: the measure of how likely an event is.

Sample Space: the set of ALL possible distinct outcomes in a random experiment, denoted by S .

↳ note: one and only one of the outcomes occurs in any single trial of the experiment.

↳ example:

- Roll a six-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- Flip a coin: $S = \{\text{heads, tails}\}$
- Waiting time for a bus: $S = \{t \in \mathbb{R}, 0 \leq t \leq 10\}$
- A sample space is finite if it consists of a

finite number of outcomes, say $\{a_1, a_2, \dots, a_n\}$

- A sample space can be a set of countable infinite outcomes, say $\{a_1, a_2, \dots\}$, where the outcomes can be put into a one-on-one correspondence with positive integers.

↳ Example: suppose our experiment consists of tossing a coin until it lands on heads.

↳ the possible outcomes in the sample space are:

H, TH, TTH, TTTH, TTTTH, ...

∴ the outcomes are countable but infinite.

Discrete Sample Space: when a sample space is either finite or countably infinite. Else, it's non-discrete.

↳ Example:

- Roll a six-sided die: discrete
- Flip a coin: discrete
- Waiting time for a bus: non-discrete

Event: a subset of sample space S.

↳ we say A is an event if $A \subset S$.

↳ note: A and S are both sets!

Example: consider tossing two coins.

a) what is the event of obtaining at least one head?

$$S = \{ HH, HT, TH, TT \}$$

$$\therefore A = \{ HH, HT, TH \}$$

b) What is the event of obtaining at least one head and at least one tail?

$$A = \{ HT, TH \}.$$

Elementary / Simple Event: an event that contains only one outcome of the experiment. Eg, $A = \{ \alpha, \beta \}$.

Compound Event: an event made up of two or more simple events. Eg: $A = \{ \alpha_1, \alpha_2 \}$.

Set Notations and Terminology:

- A set is a collection of elements. Eg, $A = \{ \text{apple, orange} \}$.
- The notation $\alpha \in A$ or $\alpha \notin A$ will mean that α is or is not an element of A .
- The empty / null set is \emptyset .
- Union: $A \cup B = \{ \alpha \mid \alpha \in A \text{ or } \alpha \in B \}$
- Intersection: $A \cap B = \{ \alpha \mid \alpha \in A \text{ and } \alpha \in B \}$
- The notation $B \subset A$ means that B is a subset of A .
- Complement: $A^c = \{ \alpha \mid \alpha \in S, \alpha \notin A \} = \bar{A}$
- Two sets A and B are disjoint if $A \cap B = \emptyset$.

- The set $A \cap B^c$ is "A but not B"
- The set $A^c \cap B^c$ is "neither A nor B"
- If A_1, \dots, A_n is a finite collection of sets,

$\rightarrow A_1 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i$ is "in all A_i ;"

$\rightarrow A_1 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$ is "in at least one A_i ;"

↳ very similar to disjoint!

Mutually Exclusiveness: when two events A and B if $A \cap B = \emptyset$. If events are exclusive, they have no outcomes in common.

↳ Example: tossing two coins. A is the event "at least one head", and B is the event "both tails".

$$S = \{HH, HT, TH, TT\}.$$

$$A = \{HH, HT, TH\}, B = \{TT\}.$$

$$A \cap B = \{HH, HT, TH\} \cap \{TT\} = \emptyset !$$

∴, A and B are mutually exclusive.

↳ Events A_1, A_2, A_3, \dots are said to be mutually exclusive if they are pairwise mutually exclusive. That is,

$$A_i \cap A_j = \emptyset, \text{ wherever } i \neq j.$$

Probability Modeling : assigning a probability $P(A)$ to each event A . This probability measures how likely it is that A will happen when the experiment is conducted.

- ↳ think of $P(A)$ as a (set) function, whose domain is a collection of sets (events), and the range of which is a subset of real numbers.
- ↳ not all set functions are appropriate for assigning probabilities to events.

• Let $S = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$ be a discrete sample space. We assign to each elementary event $A_i = \{\alpha_i\}$ for $i = 1, 2, \dots$, a number $P(A_i) = P(\{\alpha_i\})$, such that

$$\bullet 0 \leq P(A_i) \text{ and } \sum_{\text{all } i} P(A_i) = 1$$

• We call $P(A)$ the probability of A , and call the set of probabilities $\{P(A_i), i=1, 2, \dots\}$ the probability distribution on S .

↳ General Properties :

- The null event (empty set) has a probability of 0. ∴ $P(\emptyset) = 0$.
- If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$
- If A_1, A_2, \dots, A_k is a finite collection of pairwise mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_R) = P(A_1) + P(A_2) + \dots + P(A_R)$$

$$\cdot P(\bar{A}) = 1 - P(A)$$

↳ Example: Suppose a six-sided die is rolled once.
If A is the event that an even number is obtained.
What is $P(A)$?

- $S: \{1, 2, 3, 4, 5, 6\}$. Since each number is equally likely, $P(\{i\}) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

$$\begin{aligned} A &= \{2, 4, 6\}. \rightarrow P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}. \end{aligned}$$

↳ Example: Toss a coin twice. What's the probability of getting exactly one head?

- $S: \{HH, HT, TH, TT\}$. Each option is uniformly likely.
- $A = \{HT, TH\} \rightarrow P(A) = P(\{HT\}) + P(\{TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Random Selection: when choosing an object from a finite set where each object has an equal chance of being selected.

Uniform Probability Model: consider a discrete finite sample space $S = \{a_1, a_2, \dots, a_n\}$, where each

simple event has a probability of $\frac{1}{N}$. This is a uniform distribution over the set $\{\alpha_1, \alpha_2, \dots, \alpha_N\}$.

For a compound event A, we can calculate $P(A)$ by:

$$P(A) = \frac{n(A)}{N},$$

where $n(A)$ is the number of outcomes in A.

The formula $P(A) = \frac{n(A)}{N}$ is classical probability.

(trivial) example: draw one card at random from a well-shuffled deck without jokers. What is the probability that the card is a spade?

$$S = \{A\heartsuit, 2\heartsuit, \dots\}.$$

$$A_s = \{A\spadesuit, 2\spadesuit, \dots, K\spadesuit\}.$$

$$P(A_s) = \frac{n(A_s)}{N} = \frac{13}{52} = \frac{1}{4}.$$

Example: suppose that P is a probability, and A and B are events such that A and B are mutually exclusive, and $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$. Is each statement true or false?

A) A and B are each non-empty

Since $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$, we know that

$$P(A) = 0.2 \text{ and } P(B) = 0.3.$$

we also know that $P(\emptyset) = 0$. Since $0.2 \neq 0 \neq 0.3$, we know A and B cannot be empty.
 \therefore the statement is true.

B) $P(A \cap B) = 0$

Since we're told that A and B are mutually exclusive, we know that $A \cap B = \emptyset$.

we also know that $P(\emptyset) = 0$, so the statement is true!

C) $P(A \cup B) = 0.4$

we know that $\boxed{P(A \cup B)} = P(A) + P(B)$

"the probability of
A or B"

$$P(A) + P(B) = 0.2 + 0.3 = 0.5 \neq 0.4.$$

\therefore the statement is false!

Example: suppose two six-sided die are rolled where the outcomes are of the form (die 1, die 2). If A is the event that the first die is even, and B is the event that the sum of both rolls is 6, compute the event $C = \bar{A} \cap B$

$$\hookrightarrow A = \{(2,1), (2,2), \dots, (4,1), (4,2), \dots, (6,1), (6,2), \dots, (6,6)\}.$$

$$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

\bar{A} means the event that A does not happen, so in other words, \bar{A} is the event that the first die is odd.

$$\therefore \bar{A} = \{(1,1), \dots, (1,6), (3,1), \dots, (3,6), (5,1), \dots, (5,6)\}.$$

$\bar{A} \cap B$ is A AND B.

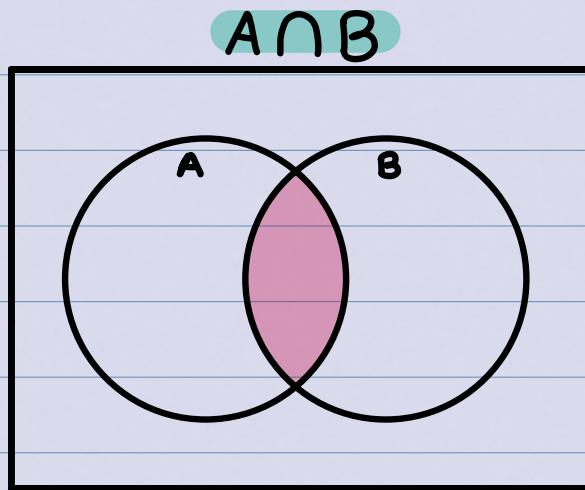
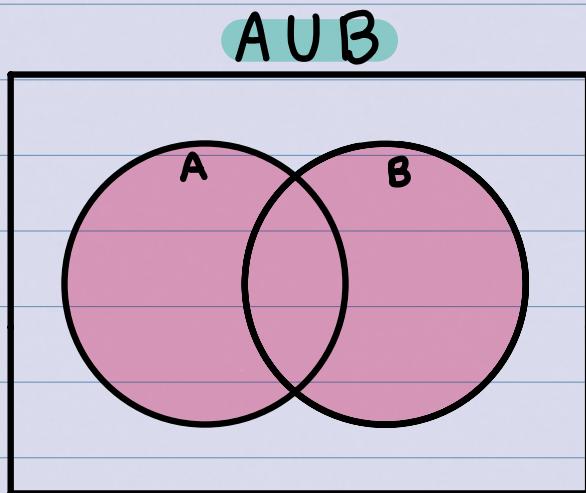
$$\hookrightarrow \therefore C = \{(1,5), (3,3), (5,1)\}.$$

Venn Diagram: a tool to illustrate the relationship among sets (or events).

↳ made up of:

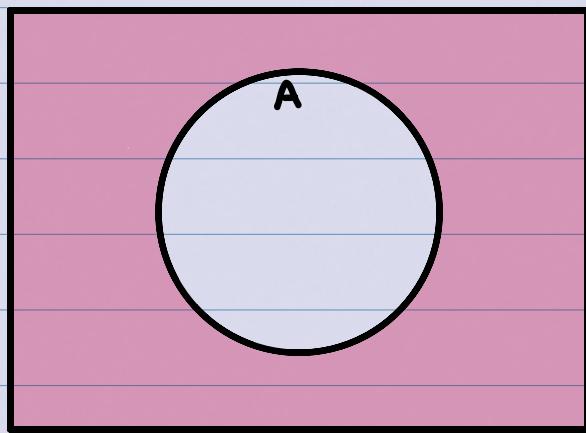
- a rectangle to represent the sample space S.
- Circles within the rectangle to illustrate the events.

The Union and Intersection of Two Events



The Compliment of an Event:

\bar{A}



"S but not A"

De Morgan's Laws:

$$\cdot \overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\cdot \overline{A \cap B} = \bar{A} \cup \bar{B}$$

Probability Rules

- Probability of the Complement of an Event
 - $P(A) = 1 - P(\bar{A})$
- Proof: $A \cup \bar{A} = S$, so $P(A \cup \bar{A}) = P(A) + P(\bar{A})$
since $P(A \cup \bar{A}) = P(S) = 1$, $1 = P(A) + P(\bar{A})$
bc mutually exclusive!

Example: an experiment involves tossing a coin four times. The event A is "at least one head". Find the probability of A, ie, find $P(A)$.

- the only case where there is NOT at least one head is $\bar{A} = \{\text{TTTT}\}$.

flipping a coin 4 times would result in 2^4 possible

outcomes, and \bar{A} is only 1 of these.

$$\therefore P(\bar{A}) = \frac{1}{16}, \text{ so } P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{16} = \frac{15}{16}.$$

probability of complement!

Example: two ordinary dice are rolled. Find the probability that at least one lands on a 6.

$$S = \{(1,1), \dots, (1,6), (2,1), \dots, (2,6), \dots, (6,1), \dots, (6,6)\}.$$

A = the event of at least one six.

$\therefore \bar{A}$ is the event that neither die is a six.

classical probability of each die to NOT be a six: $\frac{n(x)}{N}$.

where $n(x)$ is the number of outcomes that satisfy the roll not being a 6 - ie; $\{1, 2, 3, 4, 5\} \Rightarrow 5$, and N is the total number of outcomes in the set - ie; $\{1, 2, 3, 4, 5, 6\} \Rightarrow 6$.

$$\therefore \frac{n(x)}{N} = \frac{5}{6} !$$

since both dice are independent, the probability of both NOT being a six (aka \bar{A}) is $\frac{5}{6} \cdot \frac{5}{6}$.

$$\therefore P(\bar{A}) = \frac{25}{36}, \text{ so } P(A) = 1 - \frac{25}{36} = \frac{11}{36} !$$