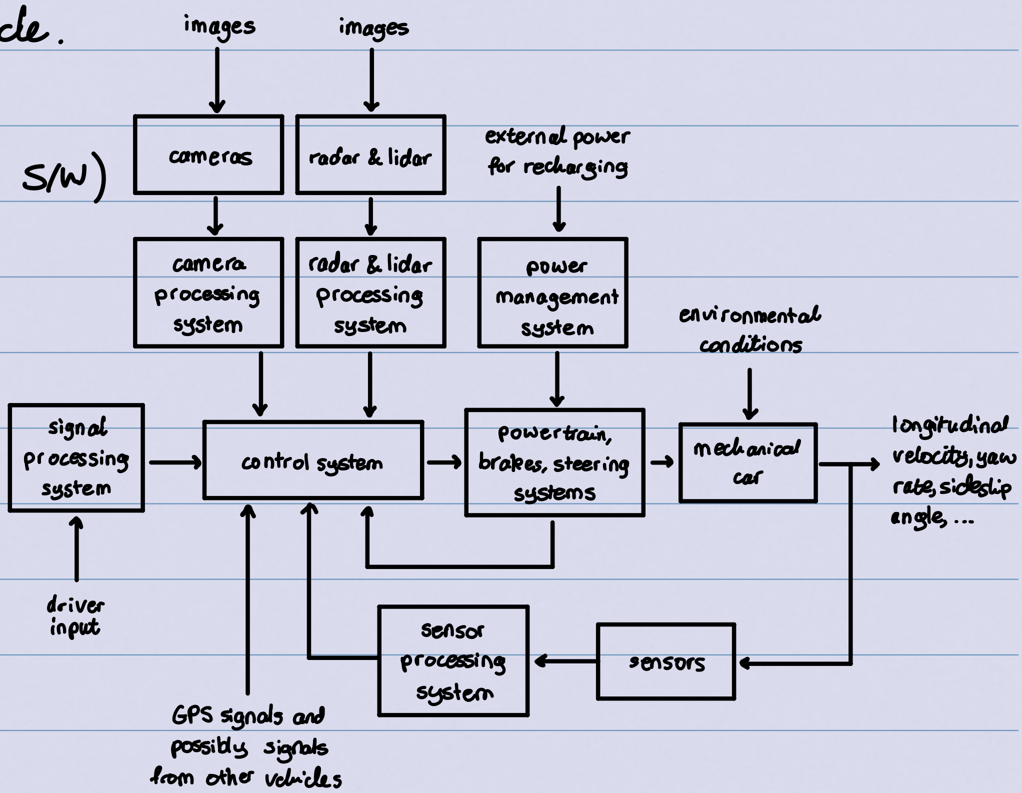
Motivating Example: design of software and hardware used to control an electric vehicle.

boxes = systems (H/W or S/W)

arrows = signals

⇒ this is an example of a block diagram



Can decompose this into **3 types of tasks**:

1. **Modelling**
   - we need some way to describe **how systems process to generate outputs**
2. **Analysis**
   - we need tools to **determine and study the behavior** of the various systems
   - eg, is the control system stable? is it fast or slow? How does it respond in windy conditions?
3. **Design**
   - we need to have a systematic way to **create and tune the various systems** (control system, image processing system, etc)

Signal: a **function of one or more independent variables**, generally containing information about the behavior of some phenomenon of interest.

⇒ **Example:** a signal may represent **a force, a torque, an angle, a** speed, a stock price, available SSD memory, etc.

- We will deal with only the situation where there is one independent variable, namely time:
  - If time is varying consistently, it's a continuous-time signal.
    ↳ we denote time by t and continuous-time signals as $x(t), u(t), y(t)$, etc
  - If time jumps from one value to the next, it's a discrete-time signal.
    ↳ we denote time by k and discrete-time signals as $x[k], u[k], y[k]$, etc

System: a device, process, or algorithm that takes one or more input signals and generates one or more output signals.

⇒ Example: each of the blocks in the electric vehicle system, a rocket, a heart, a phone, a planet, etc

It's traditional to denote a generic input signal by u (ie, either $u(t)$ or $u[k]$) and a generic output signal by y (ie, either $y(t)$ or $y[k]$)

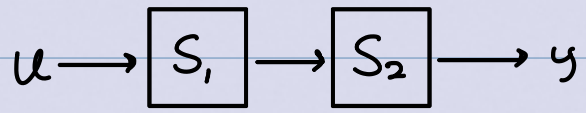
Systems that have one input signal and one output signal are called single-input single-output (SISO). Systems that have multiple inputs and multiple outputs are called multi-input multi-output (MIMO).

The output of the system is also called the response of the system.

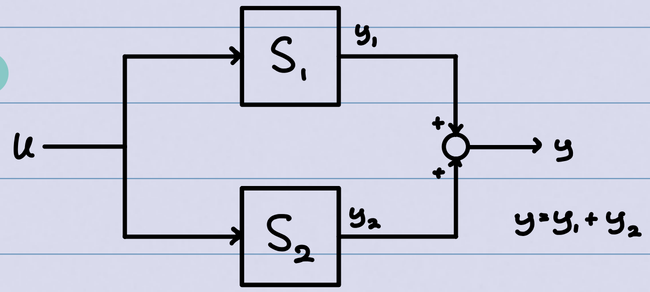
If both the input signal(s) and output signal(s) are continuous-time signals, then we say the system is a continuous-time system. Similarly, if both are discrete-time signals, then we say the system is a discrete-time system. Any other combination results in a hybrid system.

In a block diagram, blocks can be connected in series (aka a cascade connection) or in parallel (with the help of a summer):

series:



parallel:



Differential Equation: any math equation that, in contrast to a purely algebraic equation, includes the derivatives of one or more dependent variable with respect to one or more independent variables.

Ordinary Differential Equation (ODE): a differential equation with only one independent variable.

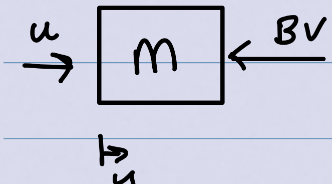
Partial Differential Equation (PDE): a differential equation with more than one independent variable.

Order of a Differential Equation: the order of the highest derivative in the equation.

⇒ Example: are the following algebraic, ODEs, or PDEs?

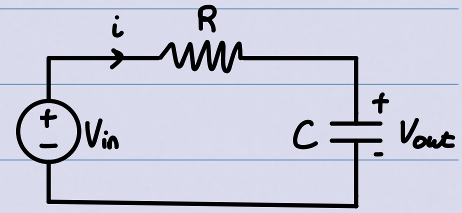
- $\frac{d^3y}{dt^3} + 4y = \frac{du}{dt} + 2u$ — ODE, 3rd order
- $F = ma$ — algebraic
- $F = m\frac{d^2y}{dt^2}$ — ODE, 2nd order
- $\ddot{y} + 2(2-\ddot{y}^3)\dot{y} + 4y = u$ — ODE, 3rd order
- $\frac{\partial y(x,t)}{\partial t} - K\frac{\partial y^2(x,t)}{\partial^2 x} = u(x,t)$ — PDE, 2nd order

Example: consider the dynamics of a vehicle moving in a straight line. The system is affected mainly by the force applied by the engine and air resistance (friction). Let
u = input force due to engine and v = output velocity $(= \dot{y})$



$$F = ma \;\rightarrow\; u - Bv = m\ddot{y}$$
$$\rightarrow u - Bv = m\dot{v}$$
$$\rightarrow u = m\dot{v} + Bv$$
$\therefore$, 1st order ODE

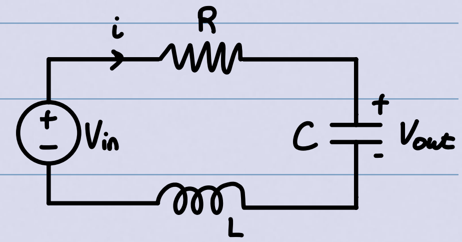
Example: consider this RC circuit. We desire to know the dynamic relationship between the output voltage V_{out} and the input voltage V_{in}. Let i be the current in the loop.



$$Ri - V_{in} + V_{out} = 0, \quad \text{and} \quad i = C\frac{dV_{out}}{dt}.$$

$$\hookrightarrow V_{out} + RC\frac{dV_{out}}{dt} = V_{in}$$
$\therefore$, 1st order ODE

Example: same as previous example, but now with an inductor included.



$$Ri - V_{in} + V_{out} + L\frac{di}{dt} = 0 \quad \text{and} \quad i = C\frac{dV_{out}}{dt}$$

$$\hookrightarrow V_{out} + RC\frac{dV_{out}}{dt} + LC\frac{d^2V_{out}}{dt^2} = V_{in}$$
$\therefore$, 2nd order ODE

Static / Memoryless System: at each time instant, each possible output doesn't depend on any value of the input except perhaps for the input at the same time instant.

↳ else, the system is said to be Dynamic or to have Memory.

Example: a resistor ($V(t) = i(t)R$) is a static system

Example: a capacitor ($C\frac{dv(t)}{dt} = i(t)$ or $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$) is a dynamic system.

Causal / Non-Anticipative System: at each time instant, each possible output does not depend on the future values of the input.
↳ else, the system is said to be non-causal or acausal.

Example: the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k-1]$ is causal.

Example: the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k+1]$ is noncausal.

Example: the system $y(t) = K\int_{-\infty}^{t} u(\tau)d\tau$ is causal

⇒ A static system is causal!

Linear System: satisfies the superposition property, that is, for any input signals u_1 (w/ output y_1) and u_2 (w/ output y_2) and any constants a_1 and a_2, a response to the input signal $u = a_1u_1 + a_2u_2$ is $y = a_1y_1 + a_2y_2$.
↳ Else, the system is said to be nonlinear.

Homogeneity Property: for any input signal u_1 (w/ output y_1) and any constant a_1, a response to the input signal $u = a_1 u_1$ is $y = a_1 y_1$.

Additivity Property: for any input signals u_1 (w/ output y_1) and u_2 (w/ output y_2), a response to the input signal $u = u_1 + u_2$ is $y = y_1 + y_2$.

A system satisfies the superposition property (ie, the system is linear) if and only if it satisfies both the homogeneity property and the additivity property.

Proof:

($\Rightarrow$) If superposition holds:

- set $a_1 = a_2 = 1$ to conclude additivity holds
- set $a_2 = 0$ to conclude homogeneity holds

($\Leftarrow$) If both homogeneity and additivity hold, then let $y_1 = S(u_1)$ and $y_2 = S(u_2)$. Then a response to input $a_1 u_1 + a_2 u_2$ is:

$S(a_1 u_1 + a_2 u_2)$

$= S(a_1 u_1) + S(a_2 u_2)$ by additivity

$= a_1 S(u_1) + a_2 S(u_2)$ by homogeneity

$= a_1 y_1 + a_2 y_2$.

$\therefore$ superposition property is satisfied.

$\Rightarrow$ it's often faster to check for both homogeneity and additivity instead of superposition directly!

Example: are the following systems linear or nonlinear?

a) $y(t) = K u(t)$

- Satisfies superposition:

Apply input u_1 to get output $y_1 = Ku_1$

Apply input u_2 to get output $y_2 = Ku_2$

Apply input $a_1u_1 + a_2u_2$ to get output $K(a_1u_1 + a_2u_2)$

$= a_1(Ku_1) + a_2(Ku_2) = a_1y_1 + a_2y_2$

$\therefore$, the system is linear.

b) $y(t) = Ku(t) + 1$

- Fails both homogeneity and additivity. Eg:

Apply input u_1 to get output $y_1 = Ku_1 + 1$

Apply input u_2 to get output $y_2 = Ku_2 + 1$

Apply input $u_1 + u_2$ to get output $K(u_1 + u_2) + 1 \neq y_1 + y_2$.

$\therefore$, the system is nonlinear.

c) $y(t) = au(t) + bu^2(t)$

- Fails additivity. Eg:

Apply input u_1 to get output $y_1 = au_1 + bu_1^2$

Apply input u_2 to get output $y_2 = au_2 + bu_2^2$

Apply input $u_1 + u_2$ to get output $a(u_1 + u_2) + b(u_1 + u_2)^2 \neq y_1 + y_2$

$\therefore$, the system is nonlinear (if $b \neq 0$)

d) $y(t) = \sin(u(t))$

- Fails homogeneity. Eg:

Apply input u_1 to get output $y_1 = \sin(u_1)$

Apply input a_1u_1 to get output $y = \sin(a_1u_1) \neq a_1u_1$

$\therefore$, the system is nonlinear

e) $y(t) = \sin(t)u(t)$

· Satisfies superposition!

Apply input u_1 to get output $y_1(t) = \sin(t)\,u_1(t)$

Apply input u_2 to get output $y_2(t) = \sin(t)\,u_2(t)$

Apply input $a_1u_1 + a_2u_2$ to get output

$y(t) = \sin(t)(a_1u_1(t) + a_2u_2(t)) = a_1y_1(t) + a_2y_2(t)$.

$\therefore$, the system is linear!

f) $y(t) = 5u(t) + u(t)y(t)$

Solving for y: $y = 5u + uy \rightarrow y = \dfrac{5u}{1-u}$

Fails homogeneity:

Apply input u_1 to get output $y_1 = \dfrac{5u_1}{1-u_1}$ $(u_1 \neq 1)$

Apply input a_1u_1 to get output $y = \dfrac{5a_1u_1}{1-a_1u_1} \neq a_1y_1$

$\therefore$ the system is nonlinear!

g) $y(t) = K\dfrac{du(t)}{dt}$

· Satisfies superposition:

Apply input u_1 to get output $y_1 = K\dfrac{du_1}{dt}$

Apply input u_2 to get output $y_2 = K\dfrac{du_2}{dt}$

Apply input $a_1u_1 + a_2u_2$ to get output $y = K\dfrac{d}{dt}(a_1u_1 + a_2u_2)$

$\therefore$ the system is linear! $= a_1y_1 + a_2y_2$

h) $y(t) = K\displaystyle\int_{-\infty}^{t} u(\tau)\,d\tau$

· Satisfies superposition!

Apply input u_1 to get output $y_1 = K\int_{-\infty}^{t} u_1(\tau)\,d\tau$

Apply input u_2 to get output $y_2 = K\int_{-\infty}^{t} u_2(\tau)\,d\tau$

Apply input $a_1u_1 + a_2u_2$ to get output $y = K\int_{-\infty}^{t} a_1u_1(\tau) + a_2u_2(\tau)\,d\tau$

$\therefore$, the system is linear! $= a_1y_1 + a_2y_2$

i) $M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + K_s y(t) = u(t)$ (mass-spring damper system)

· Satisfies superposition

Apply input u_1 to get output y_1 satisfying $M\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_s y_1 = u_1$ ①

Apply input u_2 to get output y_2 satisfying $M\frac{d^2y_2}{dt^2} + B\frac{dy_2}{dt} + K_s y_2 = u_2$ ②

Compute $a_1 \cdot ① + a_2 \cdot ②$:

$$M\frac{d^2}{dt^2}(a_1y_1 + a_2y_2) + B\frac{d}{dt}(a_1y_1 + a_2y_2) + K_s(a_1y_1 + a_2y_2) = a_1u_1 + a_2u_2$$

$\therefore$, $a_1y_1 + a_2y_2$ is a solution to the ODE when $u = a_1u_1 + a_2u_2$.

So, the system is linear!

j) $\frac{dP(t)}{dt} = aP(t) + bP^2(t) + u(t)$

· Fails homogeneity.

Apply input u_1 to get output satisfying $\frac{dP_1}{dt} = aP_1 + bP_1^2 + u_1$ ①

Suppose the system satisfies homogeneity. Then, $\frac{da_1P_1}{dt} = aa_1P_1 + b(a_1P_1)^2 + u_1$

$$\Rightarrow \frac{dP_1}{dt} = aP_1 + ba_1P_1^2 + u_1 \quad ②$$

If ① and ② both hold, then so does ① − ②:

$$0 = bP_1^2 - ba_1P_1^2 \Rightarrow b(a_1 - 1)P_1^2 = 0 \quad ③$$

But ③ doesn't hold in general (except if $b = 0$ or $a_1 = 1$ or $P_1 = 0$ which are not of interest).

$\therefore$, the system does not satisfy homogeneity by contradiction, and therefore, the system is nonlinear!

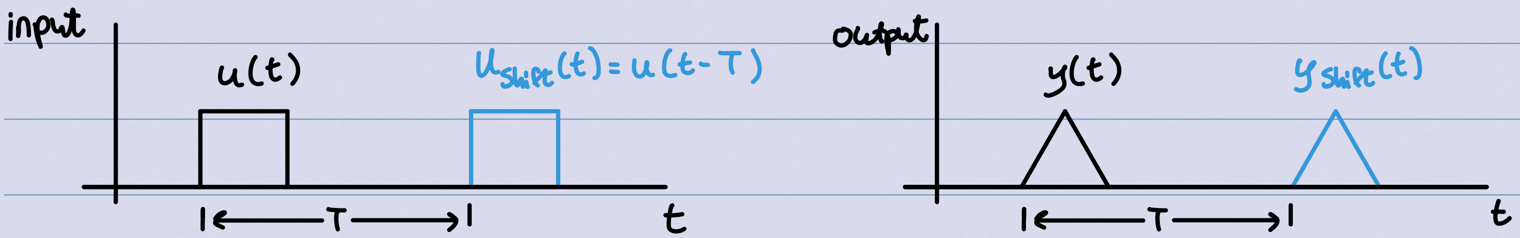
Time-Invariant Systems vs Time-Varying Systems

A system is said to be time-invariant if a time shift in the input signal always causes the same time shift (but no other distortion) in the output signal. More formally:

Assume input u is applied to the system with an associated output y. For a constant T (with $-\infty < T < \infty$), let $u_{shift}(t) = u(t-T)$ denote the shifted output. The system is said to be time-invariant if, for all u and all T, there exists an output associated with the input u_{shift}, denoted y_{shift}, such that:

$$y_{shift}(t) = y(t-T) \text{ for } -\infty < T < \infty.$$

A system that is not time-invariant is said to be time-varying.



↳ system is time-invariant if $y_{shift}(T) = y(t-T)$.

Example: indicate if the following systems are time-invariant or time-varying:

a) $y(t) = Ku(t)$

Apply input u to get output $y(t) = Ku(t)$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t)$

$\therefore$, the system is time-invariant! $\qquad = Ku(t-T) = y(t-T)$

b) $y(t) = Ku(t) + 1$

Apply input u to get output $y(t) = Ku(t) + 1$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t) + 1$

$\therefore$, the system is time-invariant! $\qquad = K(u(t-T)) + 1 = y(t-T)$

c) $y(t) = Ku(t) + t$

Apply input u to get output $y(t) = Ku(t) + t$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t) + t$

$\therefore$, the system is time-varying. $= Ku(t-T)+t \neq Ku(t-T)+t-T = y(t-T)$

d) $y(t) = \sin(t)u(t)$

Apply input u to get output $y(t) = \sin(t)u(t)$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(t)u_{shift}(t)$

$\therefore$, the system is time-varying.

$$= \sin(t)u(t-T) \neq \sin(t-T)u(t-T)$$
$$= y(t-T)$$

e) $y(t) = \sin(u(t))$

Apply input u to get output $y(t) = \sin(u(t))$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(u_{shift}(t))$

$\therefore$, the system is time-invariant.

$$= \sin(u(t-T)) = y(t-T).$$

f) $y(t) = K\frac{du(t)}{dt}$

Apply input u to get output $y(t) = K\frac{du(t)}{dt}$

Apply input u_{shift} to get output $y_{shift}(t) = K\frac{du_{shift}(t)}{dt}$

$\therefore$, the system is time-invariant.

$$= K\frac{du(t-T)}{dt} = y(t-T).$$

g) $y(t) = K\int_{-\infty}^{t} u(\tau)\,d\tau$

Apply input u to get output $y(t) = K\int_{-\infty}^{t} u(\tau)\,d\tau$

Apply input u_{shift} to get output $y_{shift}(t) = K\int_{-\infty}^{t} u_{shift}(\tau)\,d\tau$

$\therefore$, the system is time-invariant!

$$= K\int_{-\infty}^{t} u(t-T)\,d\tau = K\int_{-\infty}^{t-T} u(\tau)\,d\tau$$
$$= y(t-T)$$

h) $M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + K_s y(t) = u(t)$

Apply input u to get output y satisfying:

$$M\frac{d^2y(t)}{dt}\bigg|_t + B\frac{dy}{dt}\bigg|_t + K_s y(t) = u(t) \qquad (1)$$

Apply input $u_{shift}(t)$ to get output y_{shift} satisfying:

$$M\frac{d^2y_{shift}(t)}{dt}\bigg|_t + B\frac{dy_{shift}}{dt}\bigg|_t + K_s y_{shift}(t) = u_{shift}(t) = u(t-T)$$

let $x = t-T \rightarrow dx = dt$

$$\Rightarrow M\frac{d^2 y_{shift}(t)}{dx^2}\Big|_{x+T} + B\frac{dy_{shift}}{dx}\Big|_{x+T} + K_s\, y_{shift}(t) = u(x) \qquad (2)$$

(1) and (2) represent the same ODE with the same input.
Therefore, the outputs match: $y_{shift}(x+T) = y(x)$

$$\Leftrightarrow y_{shift}(t) = y(t-T).$$

$\therefore$, the system is time-invariant.

Definition of a solution to an ODE

A general n^{th}-order ODE with input u and output y can be written as: $F(t, y, y', y'', \ldots, y^{(n)}, u, u', u'', \ldots, u^{(m)}) = 0 \quad (n \geq m)$

For a given input u, any output y which satisfies the equation is considered to be a solution to the ODE. More formally:

Assume the input $u(t)$ is defined on some time interval $t_0 < t < t_1$ (possibly with $t_0 = -\infty$ and/or $t_1 = \infty$) and that the first m derivatives of u are defined for $t_0 < t < t_1$. Then a function $y(t) = \phi(t)$ is said to be a solution to the ODE if the first n derivatives of ϕ exist for $t_0 < t < t_1$ and if $F(t, y, y', y'', \ldots, y^{(n)}, u, u', u'', \ldots, u^{(m)}) = 0$ for $t_0 < t < t_1$.

Example: this is the logistic equation which models the dynamics of population growth: $\frac{dP}{dt} = aP - bP^2 + u$. Consider the special case where $u = 0$ and $b = 0$, resulting in $\frac{dP}{dt} = aP$. Show that $P(t) = 3e^{at}$ and $P(t) = 10e^{at}$ (for $-\infty < t < \infty$) are both solutions:

$P(t) = 3e^{at} \rightarrow \frac{dP}{dt} = 3ae^{at} = aP(t)$. $\therefore$, $\frac{dP}{dt} = aP$ ✓

$P(t) = 10e^{at} \rightarrow dP/dt = 10ae^{at} = aP(t)$. $\therefore$, $dP/dt = aP$ ✓

$\therefore$, both $P(t) = 3e^{at}$ and $P(t) = 10e^{at}$ are solutions.

↳ this shows that there need not be only one solution to an ODE. Usually, there exists a whole family of solutions, namely $P(t) = ce^{at}$ (for $-\infty < t < \infty$) for some constant c. Formally:

It's normally (but not always) true that there exists a family of solutions to $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0$, and the family is parameterised by n constants $c_1, c_2, \dots, c_n$. The family of solutions is said to be the general solution to the ODE. If specific values are chosen for the constants $c_1, c_2, \dots, c_n$, we say that the resulting $y(t)$ is a particular solution to the ODE.

Example: imagine a mass-spring damper system modeled by the ODE $M\ddot{y} + B\dot{y} + K_s y = u$. Consider the case where $u=0$, $M=1$, $B=3$, and $K_s = 2$: $\ddot{y} + 3\dot{y} + 2y = 0$.

a) verify that $y(t) = -c_1e^{-2t} + 2c_1e^{-t} - c_2e^{-2t} + c_2e^{-t}$ (for $-\infty < t < \infty$), parameterised by 2 arbitrary constants c_1 & c_2, is a family of solutions.

$$y = -c_1e^{-2t} + 2c_1e^{-t} - c_2e^{-2t} + c_2e^{-t}$$

$$\dot{y} = 2c_1e^{-2t} - 2c_1e^{-t} + 2c_2e^{-2t} - c_2e^{-t}$$

$$\ddot{y} = -4c_1e^{-2t} + 2c_1e^{-t} - 4c_2e^{-2t} + c_2e^{-t}$$

$$\therefore, \ddot{y} + 3\dot{y} + 2y = 0 \Rightarrow -4c_1e^{-2t} + 2c_1e^{-t} - 4c_2e^{-2t} + c_2e^{-t}$$
$$+ 3(2c_1e^{-2t} - 2c_1e^{-t} + 2c_2e^{-2t} - c_2e^{-t})$$

$$+2(-c_1e^{-2t}+2c_1e^{-t}-c_2e^{-2t}+c_2e^{-t})=0$$

$$\rightarrow -4c_1e^{-2t}+2c_1e^{-t}-4c_2e^{-2t}+c_2e^{-t}+6c_1e^{-2t}-6c_1e^{-t}+6c_2e^{-2t}$$
$$-3c_2e^{-t}-2c_1e^{-2t}+4c_1e^{-t}-2c_2e^{-2t}+2c_2e^{-t}=0$$

→ everything cancels, get $0=0$ ✓

b) find a particular solution to $\ddot{y}+3\dot{y}+2y=0$

arbitrarily, $c_1=c_2=1$: $y(t)=-e^{-2t}+2e^{-t}-e^{-2t}+e^{-t}$
↳ $y(t)=-2e^{-2t}+3e^{-t}$

equally arbitrarily, $c_1=c_2=0$: $y(t)=0+0+0+0$
↳ $y(t)=0$.

We should get used to the idea that there's a family of solutions to an ODE. However, there are exceptions:

example: consider the system described by the followers nonlinear ODE (for $-\infty<t<\infty$): $\left(\frac{dy}{dt}\right)^2+y^2=u$

a) if $u=-1$, there are no solutions (why?)
the LHS cannot be negative!

b) if $u=0$, there is only one solution (why?)
only solution is $dy/dt=0$ and $y=0$.

c) if $u=2$, there exist multiple solutions (why?)

One possible solution: $y(t)=\sqrt{2}$, so $dy/dt=0$.
$\therefore \left(\frac{dy}{dt}\right)^2+y^2=\sqrt{2}^2+0=2$.

another possible solution: $y(t)=\sin(t)+\cos(t)$, so $\frac{dy}{dt}=\cos(t)-\sin(t)$.

$\therefore, \left(\frac{dy}{dt}\right)^2 + y^2 = (\cos t - \sin t)^2 + (\sin t + \cos t)^2$

$= \cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\cos t \sin t + \cos^2 t$

$= 2\sin^2 t + 2\cos^2 t = 2(\sin^2 t + \cos^2 t) = 2$

The Initial-Value-Problem (IVP)

Consider the ODE $F(t, y, y', y'', \dots, y^{(n)}, u, u', u'', \dots, u^{(m)}) = 0$

Assume the input u is given. The initial-value-problem involves finding a solution (if one exists) of the ODE for $t_0 \leq t \leq t_1$ (or $t_0 \leq t \leq t_1$), possibly with $t_1 = \infty$, subject to the following initial conditions:

$$y(t) = y_0,\ y'(t_0) = y_1,\ \dots,\ y^{(n-1)}(t_0) = y_{n-1}$$

(assuming that $y_0, y_1, \dots, y_{n-1}$ are given constants)

note: initial conditions are all specified at the same time instant, $t = t_0$. If different times are used (eg $y(0) = 2$ and $y'(3) = -4$), the problem is no longer an IVP, but a boundary-value-problem (BVP). We will not deal with BVPs.

note 2: an IVP usually has a unique solution (with exceptions)

Example: consider again the mass-spring-damper system. Find the particular solution to the following IVP:

Solve $\ddot{y} + 3\dot{y} + 2y = 0$ subject to $y(0) = 2$ and $\dot{y}(0) = 3$

we saw previously that $y(t) = -c_1 e^{-2t} + 2c_1 e^{-t} - c_2 e^{-2t} + c_2 e^{-t}$ is a family of solutions to $\ddot{y} + 3\dot{y} + 2y = 0$. at initial $t = 0$:

$y(0) = -c_1 e^{-2(0)} + 2c_1 e^{-(0)} - c_2 e^{-2(0)} + c_2 e^{-(0)} = -c_1 + 2c_1 - c_2 + c_2 = c_1$

$\dot{y}(0) = 2c_1 e^{-2(0)} - 2c_1 e^{-(0)} + 2c_2 e^{-2(0)} - c_2 e^{-(0)} = 2c_1 - 2c_1 + 2c_2 - c_2 = c_2$

$\therefore, y(t) = y(0)(-e^{-2t} + 2e^{-t}) + \dot{y}(0)(-e^{-2t} + e^{-t})$ for $t \geq 0$ is another way of writing the family of solutions.

we wee given now that $y(0)=2$ and $\dot{y}(0)=3$, so:

$$y(t)=2(-e^{-2t}+2e^{-t})+3(-e^{-2t}+e^{-t})$$
$$=-2e^{-2t}+4e^{-t}-3e^{-2t}+3e^{-t}=-5e^{-2t}+7e^{-t}$$

$\therefore$, $y(t)=7e^{-t}-5e^{-2t}$ is the particular solution to the IVP subject to the given initial conditions.

However, it's not always true that an IVP has a unique solution. Non-uniqueness arises in the following example because of the specific form of nonlinearity.

Example: the IVP $\dot{y}(t)-4t\sqrt{y(t)}=0$, subject to $y(0)=0$ has 2 solutions.

1. $y(t)=0$ $(t\geq 0)$
2. $y(t)=t^4$ $(t\geq 0)$ $\rightarrow 4t^3-4t\sqrt{t^4}=4t^3-4t^3=0$

Factors that affect the solution to an IVP

There are two distinct factors that affect the solution to an IVP: the input signal and the initial conditions.

Example: consider again the mass-spring damper system, but now with $u=2$. As before, we'll have $M=1$, $B=3$, $K_s=2$. Using tools we'll later learn, we can show that the solution to the IVP $\ddot{y}+3\dot{y}+2y=2$ subject to $y(0)=2$ and $\dot{y}(0)=3$ is, for $t\geq 0$:

$$y(t)=\overbrace{y(0)(-e^{-2t}+2e^{-t})+\dot{y}(0)(-e^{-2t}+e^{-t})}^{\text{response due to initial conditions}}+\overbrace{1-2e^{-t}+e^{-2t}}^{\text{response due to input signal}}$$

$$=-4e^{-2t}+5e^{-t}+1. \rightarrow \text{same transient but different steady-state response}$$

transient response: response for finite t

steady-state response: response as $t \to \infty$

the three approaches that we will consider to solve first-order ODEs are:

1. phase portrait sketch
2. seperation of variables
3. exact differential approach

Approach 1: Phase Portrait Sketch

Simple qualitative approach for determining solutions to an ODE that has the form:

$$\frac{dy}{dt} = f(y,u)$$

for constant input u. By considering values of y and u where $f(y,u)=0$, and the sign of $f(y,u)$ between those values, we can get a good idea (qualitatively) of how $y(t)$ behaves, at least for the situation where u is a constant.

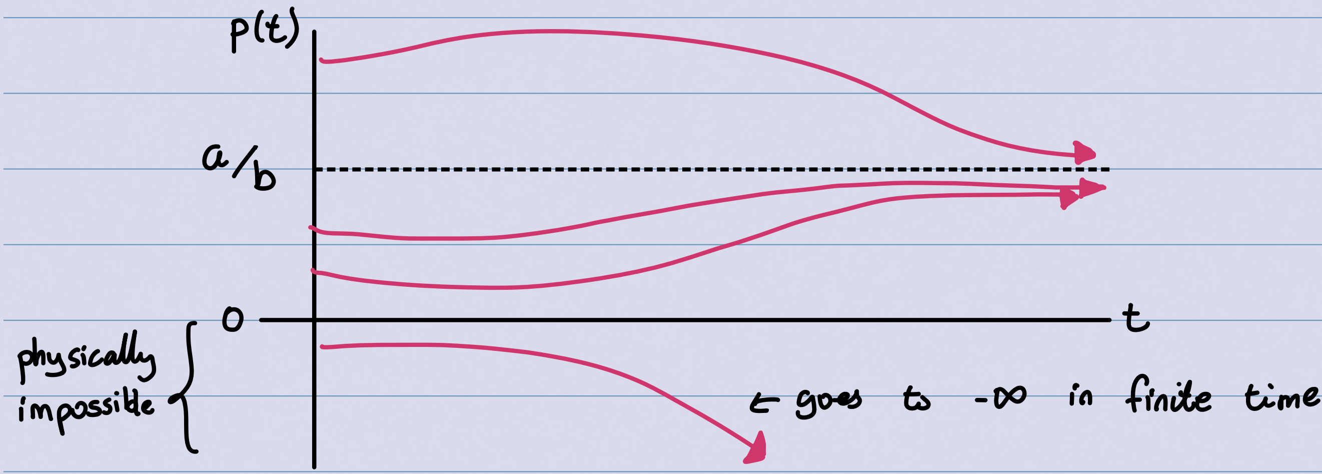
Example: consider the logistic equation with $u=0$, $\frac{dP}{dt} = P(a-bP)$

we can determine the sign of the derivative for different values of P:

P value	sign of dP/dt	
$P<0$	$-$	→ this case isn't possible
$P=0$	0	
$0<P<a/b$	$+$	
$P=a/b$	0	
$P>a/b$	$-$	

Using this information, we can sketch on an axis the phase portrait of the ODE:

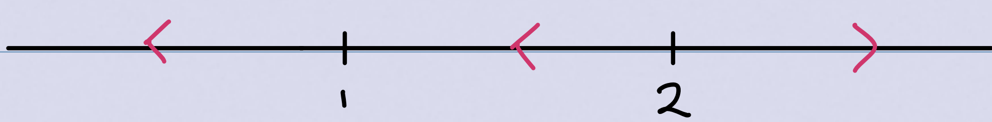
So, we can expect the solution P(t) to look something like:



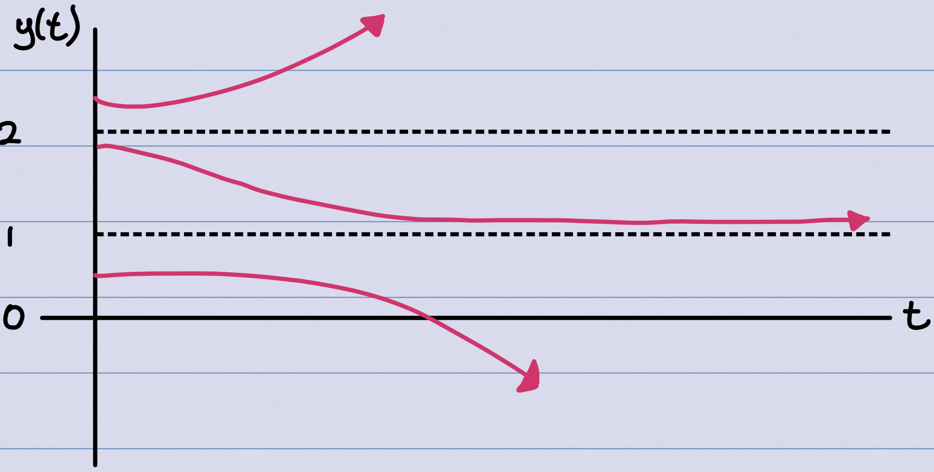
→ we expect it to always converge to a/b in steady-state

Example: consider the following ODE $\frac{dy}{dt} = (y-1)^2(y-2)$ w/ $y(0)=y_0$. Use the phase portrait approach to characterize the solutions.

y value	sign of dy/dt
$y<1$	$-$
$y=1$	0
$1<y<2$	$-$
$y=2$	0
$y>2$	$+$



the expected solution for various values of y_0:



Approach 2: Separation of Variables

Phase portrait method is simple to apply and usually insightful. However, if we want more detail, we need quantitative analysis.

Separation of Variables works for any first-order ODE that can be written (for zero u) in the form: $\frac{dy}{dt} = f(t) \cdot g(y)$.
An ODE in this form is said to be separatable.

↳ for nonzero u, the structure is: $\frac{dy}{dt} = f(t,u) \cdot g(y)$ or $\underbrace{\frac{dy}{dt} = f(t,u) \cdot g(y,u)}$ only if u is constant

This form is nice since we can rearrange into a form that can often be solved explicitly via integration.

Example: which of the following can be written in separable form?

a) $\frac{dy}{dt} = e^t y$ yes! already in separate form

b) $t\frac{dy}{dt} = ye^{y+2t}$ $\Rightarrow \frac{dy}{dt} = \left(\frac{e^{2t}}{t}\right)(ye^y)$ yes!

c) $\frac{dy}{dt} = y + e^t$ not separable

Example: a) use separation of variables to determine the general solution to $\frac{dy}{dt} = e^t y$. b) what is the particular solution associated with the initial condition $y(0) = 2$?

a) $\frac{dy}{dt} = e^t y$!! can split up dy/dx

$\Rightarrow \frac{dy}{y} = e^t dt \Rightarrow \int \frac{1}{y} dy = \int e^t dt \Rightarrow \ln|y| = e^t + C_1 \Rightarrow |y| = e^{e^t + C_1}$ if $y \neq 0$

$\therefore, y(t) = C_2 e^{e^t}$

special case $y(t) = 0$: this is a valid solution to the ODE, and the general solution addresses it when $C_2 = 0$. so C_2 can be anything!

$\therefore, y(t) = C_2 e^{e^t} \quad (-\infty < t < \infty, -\infty < C_2 < \infty)$

b) $y(0) = 2 \Rightarrow C_2 e^{e^0} = 2 \Rightarrow C_2 e^1 = 2 \Rightarrow C_2 = 2e^{-1}$

$\therefore, y(t) = 2e^{e^t - 1}, \quad t \geq 0.$

Example: a) consider the logistic equation, simplified with $b, u = 0$: $\frac{dP(t)}{dt} = aP(t)$. Use the separation of variables method to determine the general solution. b) what is the particular solution associated with the initial condition $P(0) = P_0$?

a) $\frac{dP}{dt} = aP(t)$

$\Rightarrow \int \frac{1}{P} dP = \int a\, dt \Rightarrow \ln|P| = at + C_1 \Rightarrow |P| = e^{at + C_1} \Rightarrow P(t) = C_2 e^{at}$

special case: $P(t) = 0 \Rightarrow$ a solution to the ODE and the general solution includes this case when $c_2 = 0$.

$\therefore P(t) = c_2 e^{at}$ $(-\infty < t < \infty$, c_2 arbitrary constant$)$

b) $P(0) = P_0$: $P_0 = c_2 e^{a(0)} = c_2$.

$\therefore P(t) = P_0 e^{at}, \quad t \geq 0$.

Approach 3: Exact Differential approach

(Review) Schwartz's Theorem: consider a function $\phi(t,y)$ whose second-order (partial) derivatives $\frac{\partial^2\phi}{\partial t^2}, \frac{\partial^2\phi}{\partial y^2}, \frac{\partial^2\phi}{\partial t\partial y}, \frac{\partial^2\phi}{\partial y\partial y}$ all exist and are continuous. Then it's a fact that:

$$\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial^2\phi}{\partial y\partial t}.$$

In addition, the differential: $d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial y}dy$ exists.

Example: consider $\phi(t,y) = t + 2y + e^{ty}$.

All the partial derivatives exist and are continuous:

$$\frac{\partial\phi}{\partial t} = 1 + ye^{ty} \qquad \frac{\partial\phi}{\partial y} = 2 + te^{ty}$$

$$\hookrightarrow \frac{\partial^2\phi}{\partial y\partial t} = e^{ty} + tye^{ty} \qquad \hookrightarrow \frac{\partial\phi}{\partial t\partial y} = e^{ty} + tye^{ty}$$

$$\therefore \frac{\partial^2\phi}{\partial y\partial t} = \frac{\partial\phi^2}{\partial t\partial y}!$$

& the differential of $\phi(t,y)$ is $d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial y}dy = (1+ye^{ty})dt + (2+te^{ty})dy$

Consider a first-order ODE of the form (for zero u): $\frac{dy}{dt} = -\frac{M(t,y)}{N(t,y)}$

and rewrite it as $M(t,y)dt + N(t,y)dy = 0$.

If there is a function $\phi(t,y)$ with continuous second-order partial derivatives whose differentials happen to match the above equation, that is, $d\phi(t,y) = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial y}dy = M(t,y)dt + N(t,y)dy$, then we say that $M(t,y)dt + N(t,y)dy$ is an exact differential.

If $M(t,y)dt + N(t,y)dy$ is an exact differential, then $M(t,y)dt + N(t,y)dy = 0$ can be written equivalently as $d\phi(t,y) = 0$.

We conclude that the family of solutions is (for arbitrary constant c): $\phi(t,y) = c$.

Example: use the exact differential approach to find all solutions to the ODE $\frac{dy}{dt} = -\left(\frac{1+ye^{ty}}{2+te^{ty}}\right)$

$$(1+ye^{ty})dt + (2+te^{ty})dy = 0$$

from the previous example, we saw that $(1+ye^{ty})dt + (2+te^{ty})dy$ is an exact differential with $\phi(t,y) = t + 2y + e^{ty}$.

$\therefore$, general solution of the ODE is $t + 2y + e^{ty} = c$ $(-\infty < t < \infty)$

↳ can't solve for y → an "implicit" solution

Theorem: Assume $M(t,y)$ and $N(t,y)$ are both continuous with continuous first-order partial derivatives. Then $M(t,y)dt + N(t,y)dy$ is an exact differential if and only if $\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t}$.

↳ Proof:

(⇒) if $M(t,y)dt + N(t,y)dy$ is an exact differential, then (by definition) $\exists$ a function $\phi(t,y)$ st. $d\phi(t,y) = M(t,y)dt + N(t,y)dy$ with $\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial^2\phi}{\partial y\partial t}$.

But, $\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial^2\phi}{\partial y\partial t}$ is equivalent to $\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t}$ since:

$$\frac{\partial^2\phi}{\partial t\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial t}(N(t,y)) \quad \text{and} \quad \frac{\partial^2\phi}{\partial y\partial t} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) = \frac{\partial}{\partial y}(M(t,y)).$$

$\therefore$, ⇒ proved.

(⇐) suppose that $\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t}$ holds. we must construct a function $\phi(t,y)$ such that $d\phi(t,y) = M(t,y)dt + N(t,y)dy$. This is hard so it'll be illustrated in the following examples:

Example) which of the following are exact differentials?

a) $ydt + tdy$ $\quad \frac{\partial(y)}{\partial y} = \frac{\partial(t)}{\partial t} = 1$ ⇒ yes!

b) $ydt - tdy$ $\quad \frac{\partial}{\partial y}(y) \neq \frac{\partial}{\partial y}(-t)$! ⇒ no

c) $-y\sin(t)dt + \cos(t)dy$ $\quad \frac{\partial}{\partial y}(-y\sin t) = \frac{\partial}{\partial t}(\cos t)) = -\sin t$ ⇒ yes!

d) $t^2dt + (1+y+y^2)dy$ $\quad \frac{\partial}{\partial y}(t^2) = \frac{\partial}{\partial t}(1+y+y^2) = 0$ ⇒ yes!

Example: use the exact differentials approach to find the solution to $\frac{dy(t)}{dt} + u(t)y(t) = 0$ for the initial condition $y(1) = 4$ with $u(t) = 1/t$ for $t \geq 1$.

$\frac{dy}{dt} + \frac{1}{t}y = 0 \Rightarrow y\,dt + t\,dy = 0$, (so $M = y$ and $N = t$).

See that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} = 1$, so $y\,dt + t\,dy$ is an exact differential.

Therefore, there exists a $\phi(t,y)$ such that $d\phi = y\,dt + t\,dy$:

don't assume $+C$, just do $+f(t)$!!

Step 1: $\frac{\partial \phi}{\partial y} = t \Rightarrow \phi(t,y) = ty + f(t)$, arbitrary function $f(t)$

Step 2: $\frac{\partial \phi}{\partial t} = y \Rightarrow \frac{\partial}{\partial t}(ty + f(t)) = y \Rightarrow y + f'(t) = y \Rightarrow f'(t) = 0$.
Since the derivative of $f(t) = 0$, $f(t)$ must be a constant, say C_1
$\therefore$, $f(t) = C_1$.
So, $\phi(t,y) = ty + C_1$.

$\therefore$, the general solution is $\phi(t,y) = C$, ie $ty = C_2$ or $y(t) = \frac{C_2}{t}$

now force $y(1) = 4$ to get $C_2 = 4$. $\therefore$, $y(t) = 4/t$, $t \geq 1$.

Example: use the exact differential approach for $\frac{dP}{dt} = aP$.

$dP - aP\,dt = 0$ $(M = 1, N = aP)$.

but, $\frac{\partial M}{\partial t} = 0 \neq \frac{\partial N}{\partial t} = aP'(t)$.

we can fix this by rearranging to $\underbrace{\frac{1}{P}}_{M}dP - \underbrace{a}_{N}dt = 0$,
so $\frac{\partial M}{\partial t} = 0 = \frac{\partial N}{\partial t}$! Now we must find $\phi(t,y)$ st $d\phi = \frac{1}{P}dP - a\,dt$.

Step 1: $\frac{\partial \phi}{\partial P} = 1/P \Rightarrow \int \partial\phi = \int 1/P\, dP \Rightarrow \phi = \ln|P| + f(t)$

Step 2: $\frac{\partial \phi}{\partial t} = -a \Rightarrow \frac{\partial}{\partial t}(\ln|p| + f(t)) = -a \Rightarrow f'(t) = -a$
so, $f(t) = -at + C_1$.   P is function of t?

$\therefore$, $\phi = \ln|P| - at + C_1$. so, general solution is $\phi(t,y) = C$,
ie, $\ln|P| - at = C_1$.

remove absolute value

↳ so, $|P(t)| = e^{at+C_1} = C_2 e^{at}$. $\therefore$, $P(t) = C_3 e^{at}$.

now we force $P(0) = P_0$ to get $P_0 = C_3(1) = C_3$.

therefore, $P(t) = P_0 e^{at}$, $t \geq 0$.

Example: use the exact differential approach to find the solution to $\frac{dy}{dt} = \frac{-ty^2}{2+t^2y}$ with initial condition $y(0) = 3$.

rewrite as $ty^2 dt + (2+t^2y)dy = 0$. $(M = ty^2, \; N = 2+t^2y)$

$\frac{\partial}{\partial y}(ty^2) = 2ty = \frac{\partial}{\partial t}(2+t^2y)$! So, we must find $\phi(t,y)$ such that $d\phi(t,y) = ty^2 dt + (2+t^2y)dy$

Step 1: $\frac{\partial \phi}{\partial t} = ty^2 \Rightarrow \int \partial\phi = y^2 \int t\,dt \Rightarrow \phi(t,y) = \frac{y^2t^2}{2} + f(y)$

step 2: $\frac{\partial \phi}{\partial y} = 2+t^2y \Rightarrow \frac{\partial}{\partial y}(\frac{1}{2}y^2t^2 + f(y)) = 2+t^2y \Rightarrow t^2y + f'(y) = 2+t^2y$
$\Rightarrow f'(y) = 2 \Rightarrow f(y) = 2y + C_1$.

$\therefore$, $\phi(t,y) = \frac{1}{2}t^2y^2 + 2y + C_1$.

the family of solutions to the ODE is $\frac{1}{2}t^2y^2 + 2y = C_2$.

force $y(0)=3$ to get $6=C_2$.

$$\therefore,\ y(t)=\begin{cases} \frac{2}{t^2}\sqrt{1+3t^2}-1, & t>0 \\ 3, & t=0 \end{cases}$$

Use of Integrating Factors

Sometimes a re-arrangement is needed to make an ODE solveable by the exact differential approach.

Suppose that $M(t,y)dt + N(t,y)dy = 0$ is NOT an exact differential. We modify this function by multiplying each side by a yet-to-be-determined $\mu(t,y)$: $\mu(t,y)M(t,y)dt + \mu(t,y)N(t,y)dy = 0$

we know this is an exact differential if and only if:

$$\frac{\partial \mu(t,y)M(t,y)}{\partial y} = \frac{\partial \mu(t,y)N(t,y)}{\partial t}$$

or: $$\mu(t,y)\frac{\partial M(t,y)}{\partial y} + \frac{\partial \mu(t,y)}{\partial y}M(t,y) = \mu(t,y)\frac{\partial N(t,y)}{\partial t} + \frac{\partial \mu(t,y)}{\partial t}N(t,y)$$

This is super messy, but we can solve for $\mu(t,y)$ under some following special cases:

- suppose that we require that $\mu(t,y)$ depends only on t. then it simplifies to an ODE:

$$\frac{d\mu(t)}{dt} = \Delta_1(t,y)\mu(t) \quad \text{where} \quad \Delta_1(t,y) = \frac{\left(\frac{\partial M(t,y)}{\partial y} - \frac{\partial N(t,y)}{\partial t}\right)}{N(t,y)}$$

if $\Delta_1(t,y)$ also happens to depend only on t, then we can use separation of variables to solve for $\mu(t)$: $\mu(t) = Ce^{\int \Delta_1(t)dt}$

can use $c=1$

- or, suppose $\mu(t,y)$ depends only on y. Then the messy equation simplifies to:

$$\frac{\partial \mu(y)}{\partial y} = \Delta_2(t,y) \quad \text{where} \quad \Delta_2(t,y) = \frac{\left(\frac{\partial N(t,y)}{\partial t} - \frac{\partial M(t,y)}{\partial y}\right)}{M(t,y)}.$$

if $\Delta_2(t,y)$ also happens to depend only on y, we can then use separation of variables to solve for $\mu(y)$: $\mu(y) = Ce^{\int \Delta_2(y)\,dy}$ ↳ can use $C=1$

Example: use the exact differential approach to find the solution to $6ty\,dt + (4y + 9t^2)\,dy = 0$ with $y(0) = 1$.

let $ty = M$ & $4y + 9t^2 = N$.

then $\frac{\partial M}{\partial t} = y \neq \frac{\partial N}{\partial y} = 4$. so let's see if the special cases apply:

$$\Delta_1(t,y) = \frac{\left(\frac{\partial M(t,y)}{\partial y} - \frac{\partial N(t,y)}{\partial t}\right)}{N(t,y)} = \frac{6t - 18t}{4y + 9t^2} \times \text{ depends on both } t \,\&\, y$$

$$\Delta_2(t,y) = \frac{\left(\frac{\partial N(t,y)}{\partial t} - \frac{\partial M(t,y)}{\partial y}\right)}{M(t,y)} = \frac{18t - 6t}{6ty} = \frac{12t}{6ty} = \frac{2}{y} \checkmark \text{ depends only on } y!$$

so we can use the following integrating factor: $\mu(y) = e^{\int \Delta_2(y)\,dy}$

$$\mu(y) = e^{\int 2/y\,dy} = e^{2\ln|y|} = e^{\ln y^2} = y^2.$$

the modified ODE is: $6ty^3\,dt + (4y^3 + 9t^2y^2)\,dy = 0$

↳ $\frac{\partial}{\partial y}(6ty^3) = 18ty^2 = \frac{\partial}{\partial t}(4y^3 + 9t^2y^2) \checkmark$

Step 1: $\frac{\partial \phi}{\partial t} = 6ty^3 \Rightarrow \phi(t,y) = 6y^3 \int t\,dt \Rightarrow \phi(t,y) = 3t^2y^3 + f(y)$

Step 2: $\frac{\partial \phi}{\partial y} = 4y^3 + 9t^2y^2 \Rightarrow \frac{\partial}{\partial y}(3t^2y^3 + f(y)) = 4y^3 + 9t^2y^2$

$\Rightarrow 9t^2y^2 + f'(y) = 4y^3 + 9t^2y^2 \Rightarrow f'(y) = 4y^3 \Rightarrow f(y) = y^4 + C_1$

$\therefore$, $\phi(t,y) = 3t^2y^3 + y^4 + C_1$.

Solution is $3t^2y^3 + y^4 = C_2$. Apply $y(0)=1$ to get $C_2 = 1$.

$\therefore$, $3t^2y^3 + y^4 = 1$.

Solving Constant-Coefficient Linear ODEs

If we specialize to the case where the ODE is linear and time-invariant, there are 3 methods that can solve any linear constant-coefficient ODE of any order (we will only do the last method)

A general n^{th}-order constant coefficient linear ODE, with input $u(t)$ and output $y(t)$ has the following form (for $a_n \neq 0$):

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \dots + b_1 \frac{du}{dt} + b_0 u$$

We're usually interested in the associated IVP for time $t \geq 0$ with the initial conditions: $y(0) = y_0,\ y'(0) = y_1,\ y''(0) = y_2, \dots, y^{(n-1)}(0) = y_{n-1}$.

Notes:

- we only deal with situations where $n \geq m$
- there is no loss of generality in stating the initial conditions

at time $t_0 = 0$ (because it's time-invariant!)

The presence of initial conditions somewhat complicates the type of linearity and time-invariance analysis that we did previously.

Linearity: the IVP is linear if and only if the initial conditions $y_0, \dots, y_{n-1}$ are all zero. However, we will now allow for the possibility of nonzero initial conditions throughout.

Time-invariance: the presence of initial conditions slightly affects how we analyse time-invariance - the time at which the ICs are assessed must be shifted in the same manner that the input signal is shifted.

Theorem: Consider the IVP above. There exists a solution defined for all $t \geq 0$, and the solution is unique!

Also, for constant-coefficient linear ODEs, the system can be decomposed into the sum of two responses, one due to the initial conditions and the other due to the input signal:

$$y(t) = (\text{response due to initial conditions}) + (\text{response due to input signal})$$

Laplace Transform Method

Fundamental Idea: ODEs and IVPs are challenging because of the presence of derivatives, so we seek a method of transforming the original time-domain IVP into a new domain where the IVP is represented purely by algebraic equations. We call the new domain the Laplace domain or the s-domain. $\therefore$, Calculations with derivatives now become routine algebraic calculations!

Major Benefits of the Laplace Method

- Systematic and Straightforward!
- Can readily handle inputs that are discontinuous or that contain impulses
- Clearly distinguishes between the effect of the initial conditions and that of the input signal.

Example: use the Laplace Transform Methods to find the solution to:

solve $\ddot{y} + 3\dot{y} + 2y = a$ subject to $y(0) = y_0$ and $\dot{y}(0) = 0$.

Step 1: apply the Laplace transform of each side of the ODE to map the problem to the s-domain:

$$\left(s^2 Y(s) - s y_0\right) + 3\left(sY(s) - y_0\right) + 2Y(s) = \frac{a}{s}$$

↳ $Y(s)$ denotes the Laplace transform of $y(t)$.

Step 2: solve algebraically for $Y(s)$:

$$Y(s) = \left[\frac{1}{s(s^2+3s+2)}\right] a + \left[\frac{s+3}{s^2+3s+2}\right] y_0$$

$$= \left[\frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2}\right] a + \left[\frac{2}{s+1} - \frac{1}{s+2}\right] y_0$$

Step 3: apply the inverse Laplace transform to map back to time:

$$y(t) = \left(0.5 - e^{-t} + 0.5e^{-2t}\right) a + \left(2e^{-t} - e^{-2t}\right) y_0 \quad \text{for} \quad t \geq 0.$$

The Laplace transform of a signal $f(t)$ is denoted equivalently by

$$\mathcal{L}\{f(t)\}, \quad \text{or} \quad \mathcal{L}\{f\}, \quad \text{or} \quad F(s)$$

The inverse Laplace transform of $F(s)$ recovers $f(t)$. We write

$\mathcal{L}^{-1}\{F(s)\}$

time domain $\xrightarrow[\quad]{\text{Laplace Transform } F(s)=\mathcal{L}\{f(t)\}}$ Laplace domain

time domain $\xleftarrow[\text{inverse Laplace transform } f(t)=\mathcal{L}^{-1}\{F(s)\}]{\quad}$ Laplace domain

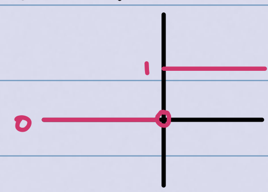
the Laplace transform of a signal $f(t)$ defined for $t \geq 0$ is the following function of $s \in \mathbb{C}$:

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \quad \left(= \lim_{\substack{T\to\infty \\ \varepsilon\to 0^+}} \int_{-\varepsilon}^{T} f(t)\,e^{-st}\,dt\right)$$

Region of Convergence (ROC) of the Laplace transform: the set of all s values for which the above integral converges

↳ Depending on the signal $f(t)$, the ROC may be the empty set, the entire complex plane, or something in between.

Example: the unit step (aka Heaviside function) is defined to be the signal: $u_{step}(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$. Determine it's Laplace transform.



$$\mathcal{L}\{u_{step}(t)\} = \int_{0^-}^{\infty} u_{step}(t)e^{-st}\,dt$$

$$= \lim_{T\to\infty} \int_0^T 1\cdot e^{-st}\,dt = \lim_{T\to\infty} -\frac{1}{s}e^{-st}\Big|_0^T$$

$$= \lim_{T\to\infty} -\frac{1}{s}\left(e^{-sT}-1\right) = \frac{1}{s} - \frac{1}{s}\lim_{T\to\infty} e^{-sT}$$

Analysing $\lim_{T\to\infty} e^{-sT}$: let $s = \sigma + j\omega$.

$$\therefore \lim_{T\to\infty} e^{-sT} = \lim_{T\to\infty} e^{-(\sigma+j\omega)T} = \lim_{T\to\infty} e^{-\sigma T} e^{-j\omega T}.$$

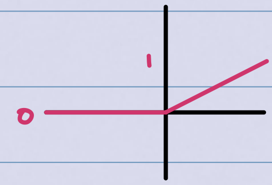
if $\sigma < 0$: $|e^{-\sigma T} e^{-j\omega t}| = e^{-\sigma T} \to \infty$ ∴ limit DNE

if $\sigma = 0$: $e^{-\sigma T} e^{-j\omega t} = e^{-j\omega T}$ a complex # that rotates as $T \uparrow$ ∴ limit DNE

if $\sigma > 0$: $|e^{-\sigma T} e^{-j\omega T}| = e^{-\sigma T} \to 0$ as $T \to \infty$ ∴ limit converges

so, $\mathcal{L}\{u_{step}(t)\} = \frac{1}{s}$ and the ROC is the half-plane $\{s : Re(s) > 0\}$.

Example: determine the Laplace transform of the unit ramp, defined to be $u_{ramp}(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$. Note that $u_{ramp}(t) = t\, u_{step}(t)$.



$$\mathcal{L}\{u_{ramp}(t)\} = \int_{0^-}^{\infty} u_{ramp}(t) e^{-st}\, dt = \int_{0^-}^{\infty} t e^{-st}\, dt$$

$$= \lim_{T\to\infty} \int_0^T t e^{-st}\, dt = \lim_{T\to\infty} \left(-\frac{te^{-st}}{s} \Big|_{0^-}^{T} + \frac{1}{s} \int_{0^-}^{T} e^{-st}\, dt \right)$$

← by integration by parts! $u = t$ $dv = e^{-st}$ $du = 1$ $v = -\frac{1}{s} e^{-st}$

$$= \lim_{T\to\infty} \left(-\frac{te^{-st}}{s} \Big|_0^T - \frac{e^{-st}}{s^2} \Big|_{0^-}^{T} \right) = \lim_{T\to\infty} \left(-\frac{Te^{-sT}}{s} - 0 - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right)$$

$$= \frac{1}{s^2} - \frac{1}{s^2} \lim_{T\to\infty} (sT + 1) e^{-sT}$$

analysing $\lim_{T\to\infty} (sT+1) e^{-sT}$: set $s = \sigma + j\omega$

$$\lim_{T\to\infty} (sT+1) e^{-sT} = \lim_{T\to\infty} (\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}$$

if $\sigma < 0$, the limit DNE since $|(\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}| = |\sigma T + j\omega T + 1| e^{-\sigma T} \geq e^{-\sigma T} \to \infty$ as $T \to \infty$

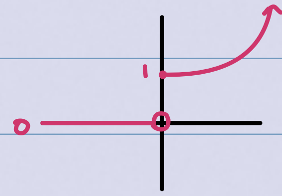
if $\sigma = 0$, the limit DNE since $|(\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}| = |j\omega T + 1| \to \infty$ as $T \to \infty$

if $\sigma > 0$, the limit is 0 since $|(\sigma T + j\omega T + 1) e^{-\sigma T} e^{-j\omega T}| = |\sigma T + j\omega T + 1| e^{-\sigma T}$

$\leq (|\sigma T + j\omega T + 1|) e^{-\sigma T} = T|\sigma + j\omega| e^{-\sigma T} + e^{-\sigma T} \to 0$ as $T \to \infty$

so, $\mathcal{L}\{u_{ramp}(t)\} = \frac{1}{s^2}$ and the ROC is the half-plane $\{s : Re(s) > 0\}$

Example: determine the Laplace transform of the exponential function $f(t) = e^{at} u_{step}(t)$. Allow for the possibility that a is a complex number.



$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st}\, dt = \lim_{T\to\infty} \int_0^T e^{(a-s)t}\, dt$$

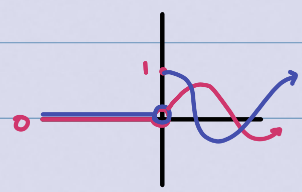
$$= \lim_{T\to\infty} \frac{1}{a-s} e^{(a-s)t} \Big|_0^T = \lim_{T\to\infty} \left(\frac{e^{(a-s)T}}{a-s} - \frac{1}{a-s} \right)$$

$$= \frac{1}{s-a} - \frac{1}{s-a} \lim_{T\to\infty} e^{(a-s)T}$$

similar analysis to the unit step example (unit step is identical, but with $a=0$)

so, $\mathcal{L}\{f(t)\} = \frac{1}{s-a}$ and the ROC is $\{s: Re(s) > Re(a)\}$.

Example: determine the LT of the signals $f(t) = \sin(at) u_{step}(t)$ and $f(t) = \cos(at) u_{step}(t)$.



$$\mathcal{L}\{\cos(at) u_{step}(at)\} = \int_{0^-}^{\infty} \cos(at) e^{-st}\, dt$$

$$= \lim_{T\to\infty} \int_0^T \cos(at) e^{-st}\, dt \xrightarrow{\text{from table of integrals}} = \lim_{T\to\infty} \left(\frac{a\sin(at) - s\cos(at)}{s^2 + a^2} e^{-st} \right) \Big|_{0^-}^{T}$$

$$= \lim_{T\to\infty} \left(\frac{a\sin(aT) - s\cos(aT)}{s^2 + a^2} e^{-sT} \right) - \left(\frac{-s}{s^2 + a^2} \right)$$

$$= \frac{s}{s^2 + a^2} + \frac{1}{s^2 + a^2} \lim_{T\to\infty} [\underbrace{a\sin(aT) - s\cos(aT)}_{\text{let this be } z}] e^{-sT}$$

let $s = \sigma + j\omega$. $\therefore$, $\lim_{T\to\infty} z \cdot e^{-(\sigma + j\omega)T} = \lim_{T\to\infty} z \cdot e^{-\sigma T} e^{-j\omega T}$

$z \cdot e^{-\sigma T} e^{-j\omega T}$ is a complex number with magnitude $|z| \cdot e^{-\sigma T}$.

if $\sigma < 0$: limit DNE since $|z|$ varies with T but doesn't converge to 0 and $e^{-\sigma T} \to \infty$ as $T \to \infty$.

if $\sigma = 0$: limit DNE since $|z|$ varies with T but doesn't converge to 0 and $e^{-\sigma T} = 1$ and $e^{-j\omega T}$ has magnitude 1 with rotating phase.

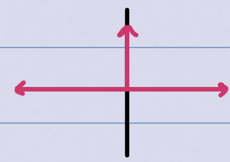
if $\sigma > 0$: limit is 0 since $|z|$ is a bounded function of T (in fact, $|z| \le a + |s|$) and $e^{-\sigma T} \to 0$ as $T \to \infty$.

So, $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$ and the ROC is the half-plane $\{s : Re(s) > 0\}$

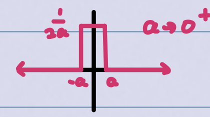
Similarly, $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$ and the ROC is also the half-plane $\{s : Re(s) > 0\}$.

The Impulse Function $\delta(t)$:

loosely: $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$



better: $\delta_a(t) = \begin{cases} 1/2a, & -a \le t \le a \\ 0, & \text{otherwise} \end{cases}$



note: the area under the curve of $\delta_a(t)$ is 1 for all $a > 0$.

we rely heavily on the following facts:

1. $\int_{-\infty}^{\infty} \delta_a(t)\,dt = 1$
2. $\int_{-\infty}^{\infty} \delta_a(t)\,g(t)\,dt = g(0)$
3. $\int_{-\infty}^{\infty} \delta(t-t_0)\,g(t)\,dt = g(t_0)$
4. $\int_{-\infty}^{\infty} \delta(t)\,g(t-t_0)\,dt = g(-t_0)$

the sifting property of the impulse function!

Example: $\int_{-\infty}^{\infty} \delta(t)(t+2)\,dt = 2$.

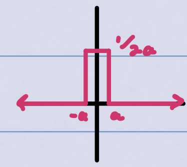
Example: $\int_{-\infty}^{\infty} \delta(t-5)(t^2)\,dt = 5^2 = 25$.

Also, $\delta(t)$ is kind of the derivative of $u_{step}(t)$: $\frac{d}{dt} u_{step}(t) = \delta(t)$.

Example: find the Laplace transform of the impulse function.

we need to handle the lower limit carefully. Let's use $t = -\varepsilon$ (for small $\varepsilon > 0$) instead of the vague "0^-" notation, and consider only a small enough that $0 < a < \varepsilon$.

$$\mathcal{L}\{f_a(t)\} = \int_{-\varepsilon}^{\infty} f_a(t) e^{-st}\, dt = \int_{-a}^{a} \frac{1}{2a} e^{-st}\, dt$$



$$= \frac{1}{2a}\left(-\frac{e^{-st}}{s}\right)\Big|_{-a}^{a} = \frac{e^{as} - e^{-as}}{2as}$$

let's now take the limit $a \to 0^+$ as follows:

$$\mathcal{L}\{\delta(t)\} = \lim_{a \to 0^+} \mathcal{L}\{f_a(t)\} = \lim_{a \to 0^+} \frac{e^{as} - e^{-as}}{2as} \overset{L'H}{=} \lim_{a \to 0^+} \frac{se^{as} + se^{-as}}{2s}$$

$$= \frac{2s}{2s} = 1$$

Alternatively, the result quickly follows from the sifting property of the impulse function:

LT of $\delta(t)$: $\int_{0^-}^{\infty} \delta(t) e^{-st}\, dt = e^{-st}\Big|_{t=0} = 1$ //.

$\therefore$, $\mathcal{L}(\delta_a(t)) = 1$ and the ROC is the entire S-plane!

Laplace Transform Pairs Table

Time Domain	S-Domain, $F(s) = \mathcal{L}\{f(t)\}$
1. $u_{step}(t)$	$\frac{1}{s}$

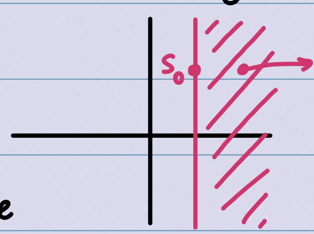
2. $u_{ramp}(t)$ (ie, $t\,u_{step}(t)$)	$\frac{1}{s^2}$
3. $e^{at}u_{step}(t)$	$\frac{1}{s-a}$
4. $\sin(at)\,u_{step}(t)$	$\frac{a}{s^2+a^2}$
5. $\cos(at)\,u_{step}(t)$	$\frac{s}{s^2+a^2}$
6. $\delta(t)$	1
7. $t^n u_{step}(t)$ for $n \geq 1$	$\frac{n!}{s^{n+1}}$
8. $\sin^2(at)\,u_{step}(t)$	$\frac{2a^2}{s(s^2+4a^2)}$
9. $\cos^2(at)\,u_{step}(t)$	$\frac{s^2+2a^2}{s(s^2+4a^2)}$
10. $t\sin(at)\,u_{step}(t)$	$\frac{2as}{(s^2+a^2)^2}$
11. $t\cos(at)\,u_{step}(t)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
12. $[\sin(at) - at\cos(at)]\,u_{step}(t)$	$\frac{2a^3}{(s^2+a^2)^2}$
13. $e^{bt}\sin(at)\,u_{step}(t)$	$\frac{a}{(s-b)^2+a^2}$
14. $e^{bt}\cos(at)\,u_{step}(t)$	$\frac{s-b}{(s-b)^2+a^2}$
15. $te^{at}u_{step}(t)$	$\frac{1}{(s-a)^2}$
16. $t^2e^{at}u_{step}(t)$	$\frac{2}{(s-a)^3}$
17. $t^ne^{at}u_{step}(t)$ for $n \geq 1$	$\frac{n!}{(s-a)^{n+1}}$

memorise these!

Two important results of the ROC

1. A non-empty ROC must be either a half-plane or a full-plane

Theorem: If the Laplace Transform $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$ converges at point $s = s_0$, then necessarily it also converges in the open half-plane $\{s : Re(s) > Re(s_0)\}$.



all these points are guaranteed to be in the ROC!

↳ notes: this theorem implies that the ROC is either the full complex plane or a right half-plane, but it doesn't tell us anything about whether or not the "edge" of a half-plane ROC is included in the ROC.

↳ fortunately, whether or not the "edge" is included in the ROC has no implications for solving IVPs ;)

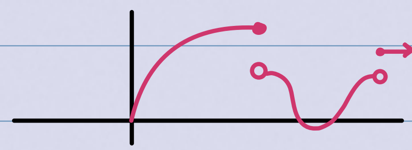
this is good news for us since all the calculations we'll be doing in the Laplace Domain are guaranteed to work out for a particular problem if the intersection of the various ROCs includes a right half-plane.

2. Conditions for the ROC to be non-empty

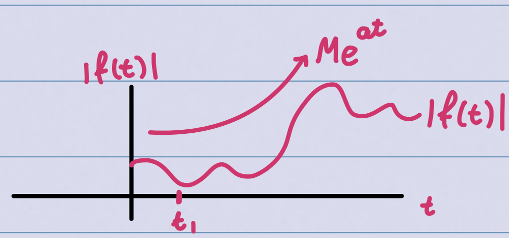
Result 1 is general in that it makes no assumptions about signal $f(t)$ beyond the existence of at least one point in the ROC, but it's limited because it can't be used to tell us whether a signal has non-empty ROC in the first place, which is something we definitely need to know. The following theorem provides sufficient conditions for the ROC to be non-empty.

Theorem: The Laplace Transform of $f(t)$ has non-empty ROC if the signal satisfies the following two conditions:

1. $f(t)$ is piecewise-continuous for $t \geq 0$, which means that, on every interval, $a \leq t \leq b$ with $0 \leq a < b$, the function is continuous except possibly for a finite number of finite jumps.

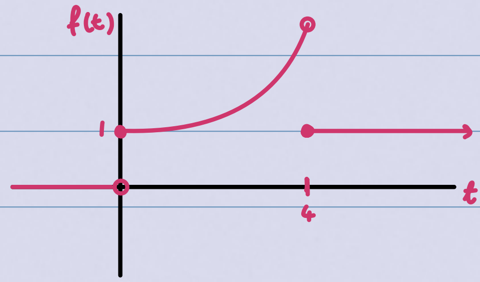


2. $f(t)$ is of exponential order a (for real number a), which means that there exists $M > 0$ and $t_1 > 0$ such that $|f(t)| \leq Me^{at}$ $\forall\, t > t_1$.



↳ we say that a signal that is both piecewise continuous for $t \geq 0$ and of exponential order is an acceptable signal. For an acceptable signal of exponential order a, the ROC includes the half-plane $\{ s : Re(s) > a \}$.

Example: determine if $f(t) = \begin{cases} 0, & t < 0 \\ e^t, & 0 \le t < 4 \\ 1, & t \ge 4 \end{cases}$ is acceptable. If it is, compute it's Laplace Transform and confirm that the ROC is nonempty.



- $f(t)$ is piecewise continuous
- $f(t)$ is of exponential order 0 (ie $a=0$)

↳ ∴, $f(t)$ is acceptable

Computing it's Laplace Transform:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^4 e^t e^{-st}\,dt + \int_4^\infty 1 \cdot e^{-st}\,dt$$

$$= \int_0^4 e^{(1-s)t}\,dt + \lim_{T\to\infty} \int_4^T e^{-st}\,dt$$

$$= \frac{1}{1-s} e^{(1-s)t} \Big|_0^4 + \lim_{T\to\infty} \left(-\frac{1}{s}\right) e^{-st} \Big|_4^T$$

$$= \frac{1-e^4 e^{-4s}}{s-1} + \frac{e^{-4s}}{s} - \frac{1}{s} \lim_{T\to\infty} e^{-sT}$$

∴, $F(s) = \dfrac{1-e^4 e^{-4s}}{s-1} + \dfrac{e^{-4s}}{s}$ with ROC $= \{ s : Re(s) > 0 \}$

↳ note: technically, $F(s)$ isn't defined at $s=1$. so we can better write it as $F(s) = \begin{cases} \frac{1-s^4 e^{-4s}}{s-1} + \frac{e^{-4s}}{s} & \text{if } s \neq 1 \\ 4 + e^{-4} & \text{if } s = 1 \end{cases}$ ROC $= \{ s : Re(s) > 0 \}$

Proof of the theorem: Assume $f(t)$ is acceptable. Then, it is piecewise-continuous and of exponential order a. Break up $\mathcal{L}\{f(t)\}$:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\,dt = \int_0^{t_1} f(t)e^{-st}\,dt + \int_{t_1}^\infty f(t)e^{-st}\,dt$$

The first integral on the right exists for all values of s, since $f(t)$ is piecewise-continuous. The second integral on the right, which is improper, converges for any s value with $s = \sigma + j\omega$ and $\sigma > a$ since:

since $f(t)$ is of exponential order a

$$\left| \int_{t_1}^{\infty} f(t) e^{-st} dt \right| \leq \int_{t_1}^{\infty} |f(t) e^{-st}| dt = \int_{t_1}^{\infty} |f(t)| |e^{-st}| dt \leq \int_{t_1}^{\infty} M e^{at} |e^{-st}| dt$$

$s = \sigma + \omega j \Rightarrow |e^{-st}| = |e^{-\sigma t} e^{-j\omega t}| = e^{-\sigma t}$

$$= \int_{t_1}^{\infty} M e^{at} e^{-\sigma t} dt = \int_{t_1}^{\infty} M e^{(a-\sigma)t} dt = \lim_{T \to \infty} \int_{t_1}^{T} M e^{(a-\sigma)t} dt = \lim_{T \to \infty} \frac{M}{\sigma - a} \left[e^{-(\sigma - a)t_1} - e^{-(\sigma - a)T} \right]$$

$$= \frac{M}{\sigma - a} \left[e^{-(\sigma - a)t_1} \right] \text{ since } \sigma - a > 0 \quad < \infty.$$

$\therefore$, $\mathcal{L}\{f(t)\}$ exists and the ROC includes the half-plane $\{s : Re(s) > a\}$

Notes:

- All piecewise-continuous signals that are bounded are acceptable!
  (A signal is bounded if $\exists$ a constant c s.t. $|f(t)| \leq c \ \forall t$.
- Signals that are weighted sums or products of acceptable signals are themselves acceptable
- All signals in the table above are acceptable, other than the impulse function!

To construct an unacceptable signal, we need to think of signals that "blow up" faster than an exponential, have singularities, or that have weird discontinuities. Here are a few examples:

$f(t) = e^{t^2} u_{step}(t)$, $f(t) = \begin{cases} 1/t, & t \neq 0 \\ 0, & t = 0 \end{cases}$, $f(t) = \begin{cases} 1/(t-1)^2, & t \neq 1 \\ 0, & t = 1 \end{cases}$

$f(t) = \begin{cases} \text{round}(\sin(1/t)), & t \neq 0 \\ 0, & t = 0 \end{cases}$, and $f(t) = \begin{cases} 1 & \text{if } t \text{ is rational} \\ 0 & \text{if } t \text{ is irrational} \end{cases}$

For the rest of the course, we will assume that all input signals are acceptable! The only exceptions are impulse function inputs.

Conclusion: since we're restricting ourselves to acceptable signals (and sometimes the impulse function, which has a full-plane ROC), we're guaranteed by result 2 that all ROCs will be non-empty and they will include a right half-plane. ∴, all the calculations we do in the s-domain will work out correctly!!

↳ ∴, for the rest of the course, we won't specify (or even think about) the ROC for any Laplace transform :)

Two Fundamental Properties of the Laplace Transform

Property 1: Linearity of the Laplace Transform

For any acceptable signals $f(t)$ and $g(t)$ and any scalars a and b, $\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = aF(s)+bG(s)$.

↳ Derivation: $\mathcal{L}\{af(t)+bg(t)\} = \int_0^\infty (af(t)+bg(t))e^{-st}\,dt$

$$= a\int_0^\infty f(t)e^{-st}\,dt + b\int_0^\infty g(t)e^{-st}\,dt$$

$$= aF(s) + bG(s).$$

see table of LT's above

Examples of Linearity of Laplace Transforms:

a) $u(t) = 6u_{step}(t)$ → $U(s) = 6/s$

b) $f(t) = (e^{-2t} - 10e^{6t})u_{step}(t)$: $F(s) = \frac{1}{s+2} - \frac{10}{s-6}$

c) $y(t) = (4\sin(t) + t + 3)u_{step}(t)$: $Y(s) = \frac{4}{s^2+1} + \frac{1}{s^2} + \frac{3}{s}$.

Example: $\mathcal{L}\{\cos(at)u_{step}(t)\} = \mathcal{L}\left\{\left(\frac{e^{jat}+e^{-jat}}{2}\right)u_{step}(t)\right\}$ exploit linearity

$$= \frac{1}{2}\mathcal{L}\{e^{jat}u_{step}(t)\} + \frac{1}{2}\mathcal{L}\{e^{-jat}u_{step}(t)\}$$

$$= \frac{1}{2}\left(\frac{1}{s-ja}\right) + \frac{1}{2}\left(\frac{1}{s+ja}\right) = \frac{1}{2}\,\frac{(s+ja)+(s-ja)}{(s-ja)(s+ja)}$$

$$= \frac{s}{s^2+a^2}.$$

→ note: can similarly compute $\mathcal{L}\{\sin(at)\,u_{step}(t)\}$.

Property 2: Laplace Transform of a Derivative

Let $f(t)$ be an acceptable signal. If $f(t)$ is differentiable and $f'(t)$ is acceptable, then $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0^-)$ → $f(0^-) = \lim_{t\to 0^-} f(t)$

$$= sF(s) - f(0^-).$$

If $f(t)$ is twice differentiable and $f''(t)$ is acceptable, then

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s f(0^-) - f'(0^-) = s^2F(s) - sf(0^-) - f'(0^-)$$

If $f(t)$ is thrice differentiable and $f'''(t)$ is acceptable, then

$$\mathcal{L}\{f'''(t)\} = s^3\,\mathcal{L}\{f(t)\} - s^2 f(0^-) - s f'(0^-) - f''(0^-)$$

$$= s^3F(s) - s^2 f(0^-) - s f'(0^-) - f''(0^-).$$

The pattern continues for higher-order derivatives!

↳ Multiplication by s in the s-domain is equivalent to differentiation in the time domain. ∴, when working in the s-domain, we often call s a "differentiator"

↳ Derivation of Property 2:

Use Integration by Parts to compute:

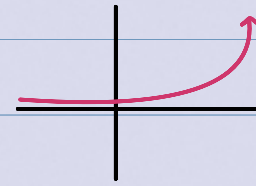
$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt = \lim_{T\to\infty}\int_0^T f'(t)e^{-st}\,dt$$

$$= \lim_{T\to\infty}\left(f(t)e^{-st}\Big|_0^T - \int_0^T f(t)(-s)e^{-st}\,dt\right) = \lim_{T\to\infty} f(T)e^{-sT} - f(0^-) + s\int_{0^-}^\infty f(t)e^{-st}\,dt$$

$$= s\,\mathcal{L}\{f(t)\} - f(0^-).$$

Example: verify that property 2 holds for $f(t) = t^2 u_{step}(t)$

$f(t) = t^2 u_{step}(t) \Rightarrow \mathcal{L}\{f(t)\} = \frac{2}{s^3}$ ↗ from table

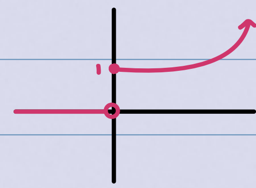


$$f'(t) = \frac{d}{dt}\left(t^2 u_{step}(t)\right)$$
$$= t^2\delta(t) + 2t\,u_{step}(t)$$
$$= 2t\,u_{step}(t) \quad \underset{=0\ \forall t}{t^2\delta(t)} \quad \Rightarrow \quad \mathcal{L}\{f'(t)\} = 2/s^2$$

$\therefore$, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$ holds!

Example: verify that property 2 holds for $f(t) = e^{2t} u_{step}(t)$.

we have to watch that the jump at $t=0$ is handled correctly.

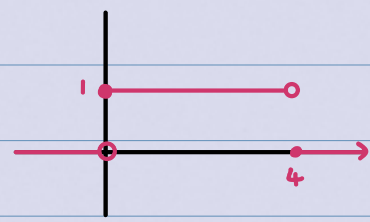


$$f(t) = e^{2t} u_{step}(t) \quad \Rightarrow \quad \mathcal{L}\{f(t)\} = 1/(s-2)$$
$$f'(t) = e^{2t}\delta(t) + 2e^{2t}u_{step}(t)$$
$$= \delta(t) + 2e^{2t}u_{step}(t) \quad \Rightarrow \quad \mathcal{L}\{f'(t)\} = 1 + 2/(s-2)$$

$\therefore$, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$!

Example: verify that property 2 holds for $f(t) = u_{step}(t) - u_{step}(t-4)$.

$f(t) = u_{step}(t) - u_{step}(t-4)$



↳ $\mathcal{L}\{f(t)\} = \frac{1}{s} - \int_0^\infty u_{step}(t-4)e^{-st}\,dt$

$$= \frac{1}{s} - \int_4^\infty e^{-st}\,dt = \frac{1}{s} + \left[\frac{1}{s}e^{-st}\right]_4^\infty = \frac{1}{s} + \frac{1}{s}\left[\lim_{T\to\infty} e^{-sT} - e^{-4s}\right]$$

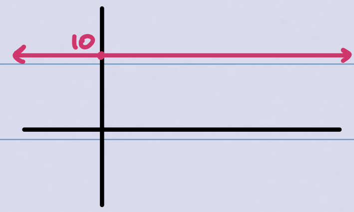
$= 1/s - e^{-4s}/s$ for s with $\mathrm{Re}(s) > 0$.

$$f'(t) = \delta(t) - \delta(t-4) \Rightarrow \mathcal{L}\{f'(t)\} = 1 - \int_0^\infty \delta(t-4)e^{-st}\,dt = 1 - e^{-4s}$$

sifting property

$\therefore$, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$ holds!

Example: verify directly that property 2 holds for $f(t) = 10$



$f(t) = 10 \rightarrow \mathcal{L}\{f(t)\} = 10/s$

$f'(t) = 0 \rightarrow \mathcal{L}\{f(t)\} = 0$

$0 = s(10/s) - 10$ ✓

$\therefore$, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$ holds!

We can now fully map any constant-coefficient linear ODE to the s-domain! Following examples will help us become familiar.

Example: Determine the Laplace transform of the following signals, all for $t \geq 0$.

a) $f'(t)$ with $f(0^-) = 0$ $\Rightarrow sF(s) - 0 = sF(s)$.

b) $\ddot{y}(t)$ with $y(0^-) = \dot{y}(0^-) = 0$ $\Rightarrow s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s)$

c) $4\ddot{y}(t) - 10\dot{y}(t) + 3y(t)$ with $y(0^-) = \dot{y}(0^-) = 0$ $\Rightarrow 4s^2Y(s) - 10sY(s) + 3Y(s)$

d) $f'(t)$ with $f(0^-) = 10$ $\Rightarrow sF(s) - 10$

e) $\ddot{y}(t) + e^{-t}u_{step}(t)$ with $y(0^-) = 4$ and $\dot{y}(0^-) = -3$ $\Rightarrow s^2Y(s) - 4s + 3 + \frac{1}{s+1}$

f) $4\ddot{y}(t) - 10\dot{y}(t) + 3u_{step}(t)$ with $y(0^-) = 2$ and $\dot{y}(0^-) = 7$

$\hookrightarrow 4(s^2 Y(s) - 2s - 7) - 10(sY(s) - 2) + 3/s$

g) $\dddot{y}(t) + 4\ddot{y}(t) - 10\dot{y}(t) + 3y(t)$ with $y(0^-) = 2$, $\dot{y}(0^-) = 7$, $\ddot{y}(0^-) = 1$

$\hookrightarrow (s^3 Y(s) - 2s^2 - 7s - 1) + 4(s^2 Y(s) - 2s - 7) - 10(sY(s) - 2) + 3Y(s)$

h) $4\ddot{y}(t) - 8\dot{y}(t) + 2\sin(t)y(t) + 5y^2(t)$ with $y(0^-) = \dot{y}(0^-) = 0$

$\hookrightarrow 4s^2 Y(s) - 8sY(s) + 2\mathcal{L}\{\sin(t)y(t)\} + 5\mathcal{L}\{y^2(t)\}$

can't do anything with these!

Example: $\frac{dT}{dt} = k(T_{oven} - T)$. Suppose a pizza, initially at room temp (22°C), is put into an oven that has been pre-warmed to 180°C. We're interested in how the temperature of the pizza varies over time. Assume the thermal coefficient k is 0.01.

a) write the IVP in the time domain using $u = T_{oven}$ and $y = T$:

$$\frac{dy}{dt} = 0.01(u - y) \quad \text{with} \quad y(0^-) = 22.$$

plugging in $u(t) = 180\,u_{step}(t)$

$$\hookrightarrow \frac{dy}{dt} = 1.8\,u_{step}(t) - 0.01y(t), \quad \text{with} \quad y(0^-) = 22 \quad \text{for} \quad t \geq 0.$$

b) map the IVP over to the s-domain and solve for $Y(s)$.

$$sY(s) - y(0^-) = \frac{1.8}{s} - 0.01\,Y(s)$$

$\hookrightarrow sY(s) - 22 = 1.8/s - 0.01Y(s)$

$\hookrightarrow (s + 0.01)Y(s) = 1.8/s + 22 \rightarrow Y(s) = \frac{1}{s + 0.01} \cdot \frac{1.8 + 22s}{s}$

$\therefore, \; Y(s) = \dfrac{22(s+0.0818)}{s(s+0.01)}$ is the solution to the IVP in the s-domain!

Example: consider the mass-spring-damper system ODE: $M\ddot{y} + B\dot{y} + K_s y = u$. Assume $M = B = K_s = 1$. Map the ODE to the Laplace domain in the following cases and then solve for $Y(s)$:

a) $u(t) = 0$ with $y(0^-) = 5$, $\dot{y}(0^-) = -2$.

$$(s^2 Y(s) - 5s + 2) + (sY(s) - 5) + Y(s) = 0$$

$$Y(s)(s^2 + s + 1) = 5s - 2 + 5$$

$$\hookrightarrow Y(s) = \frac{5s+3}{s^2+s+1}$$

b) $u(t) = 4u_{step}(t)$ with $y(0^-) = \dot{y}(0^-) = 0$

$$s^2 Y(s) + sY(s) + Y(s) = u(s) = 4/s.$$

$$Y(s)(s^2 + s + 1) = 4/s \Rightarrow Y(s) = \frac{4}{s(s^2+s+1)}.$$

c) $u(t) = \sin(3t)u_{step}(t)$ with $y(0^-) = -1$, $\dot{y}(0) = 0$

$$(s^2 Y(s) + s) + (sY(s) + 1) + Y(s) = u(s) = \frac{3}{s^2+9}$$

$$\hookrightarrow Y(s)(s^2 + s + 1) = \frac{3}{s^2+9} - s - 1$$

$$\hookrightarrow Y(s) = \frac{3}{(s^2+9)(s^2+s+1)} - \frac{s+1}{s^2+s+1}$$

models landing a rocket

Example: $(M+m)\ddot{x} + mL\ddot{\theta} = u$ & $\ddot{x} + L\ddot{\theta} - g\theta = 0$. Assume the

following initial conditions: $x(0^-)=0$, $\dot{x}(0^-)=0$, $\Theta(0^-)=0$, $\dot{\Theta}(0^-)=0$.
Our goal is to study how the rocket responds to a small object that accidentally smashes into it's side during launch. Determine, in the s-domain, an expression for $\Theta(s)$.

let's model the smashing object as an impulse force of magnitude $a>0$: $u(t) = a\,\delta(t)$.

take Laplace transforms of both ODEs:

$$(M+m)s^2X(s) + mLs^2\Theta(s) = a \Rightarrow ①$$

$$s^2X(s) + Ls^2\Theta(s) - g\Theta(s) = 0 \Rightarrow ②$$

solving for $X(s)$ in ②: $X(s) = \dfrac{(g-Ls^2)}{s^2}\Theta(s)$

plugging into ①: $(m+M)s^2\dfrac{(g-Ls^2)}{s^2}\Theta(s) + mLs^2\Theta(s) = a$

$\therefore$, $\Theta(s) = \dfrac{a}{(M+m)g - mLs^2}$! now, how do we determine $\Theta(t)$ from $\Theta(s)$?

The Inverse Laplace Transform

The inverse Laplace transform is used to recover $f(t)$ from a given $F(s)$, that is, $f(t) = \mathcal{L}^{-1}\{F(s)\}$. How do we do this?

- if we have access to the Laplace transform table, just look up the transform and write down $f(t)$ by inspection!
- if it's not in the table, usually we do partial fraction expansion to decompose $F(s)$ into a sum of terms that appear in the table! → Note: exploiting the fact that the Laplace transform is linear.
- or can use Mellin's inverse formula: $f(t) = \dfrac{1}{2\pi j}\displaystyle\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds$. ignore this!

Examples:

a) $\mathcal{L}^{-1}\{1/s\} = u_{step}(t)$.

b) $\mathcal{L}^{-1}\left\{\frac{10}{s-5}\right\} = 10e^{5t}u_{step}(t)$

c) $\mathcal{L}^{-1}\left\{4 + \frac{1}{s+1}\right\} = 4\delta(t) + e^{-t}u_{step}(t)$.

d) $\mathcal{L}^{-1}\left\{\frac{6}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{3 \cdot \frac{2}{s(s^2+4)}\right\} = 3\sin^2(t)u_{step}(t)$ → table #8.

Example: determine the inverse Laplace transform in each case

a) $Y(s) = \frac{10}{(s+2)(s+4)}$

↳ by PFE, $Y(s) = \frac{5}{s+2} - \frac{5}{s+4}$.

$\therefore$, $y(t) = (5e^{-2t} - 5e^{-4t})u_{step}(t)$.

b) $F(s) = \frac{s+4}{s(s+6)}$

↳ by PFE, $F(s) = \frac{2}{3}\frac{1}{s} + \frac{1}{3}\frac{1}{s+6}$.

$\therefore$, $f(t) = \left(\frac{2}{3} + \frac{1}{3}e^{-6t}\right)u_{step}(t)$.

c) $H(s) = \frac{10}{(s^2+2)(s+4)}$

↳ by PFE, $H(s) = -\frac{5}{9}\frac{s}{s^2+2} + \frac{20}{9}\frac{1}{s^2+2} + \frac{5}{9}\frac{1}{s^2+4}$

$H(s) = -\frac{5}{9}\frac{s}{s^2+2} + \frac{20}{9\sqrt{2}}\frac{\sqrt{2}}{s^2+2} + \frac{5}{9}\frac{1}{s^2+4}$

$\therefore,\ h(t) = \left(-\frac{5}{9}\cos(\sqrt{2}t) + \frac{20}{9\sqrt{2}}\sin(\sqrt{2}t) + \frac{5}{9}e^{-4t}\right)u_{step}(t).$

d) $Y(s) = \dfrac{1}{(s+1)^2(s+2)^2}$

$\hookrightarrow Y(s) = \frac{2}{s+1} - \frac{1}{(s+1)^2} - \frac{2}{s+2} - \frac{1}{(s+2)^2}$ by PFE

#15 on table

$\therefore,\ y(t) = (2e^{-t} - te^{-t} - 2e^{-2t} - te^{-2t})\,u_{step}(t).$

e) $H(s) = \dfrac{2(s+1)^2}{s^2+2s} = \dfrac{2(s+1)^2}{s(s+2)}$

$\hookrightarrow H(s) = 2 + \frac{1}{s} - \frac{1}{s+2}$ by PFE.

$\therefore,\ h(t) = 2\delta(t) + (1 - e^{-2t})\,u_{step}(t).$

f) $F(s) = \dfrac{1}{(s+2)(s^2+2s+5)}$

$\hookrightarrow F(s) = \frac{1}{5}\left(\frac{1}{s+2}\right) - \frac{1}{5}\left(\frac{1}{s^2+2s+5}\right).$

completing the square, $F(s) = \frac{1}{5}\left(\frac{1}{s+2}\right) - \frac{1}{5}\left(\frac{s}{(s+1)^2+4}\right)$

$\hookrightarrow$ so, $F(s) = \frac{1}{5}\left(\frac{1}{s+2}\right) - \frac{1}{5}\left(\frac{s+1}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4}\right)$

#14 #13

$\therefore,\ f(t) = \left(\frac{1}{5}e^{-2t} - \frac{1}{5}e^{-t}\cos(2t) + \frac{1}{10}e^{-t}\sin 2t\right)u_{step}(t).$

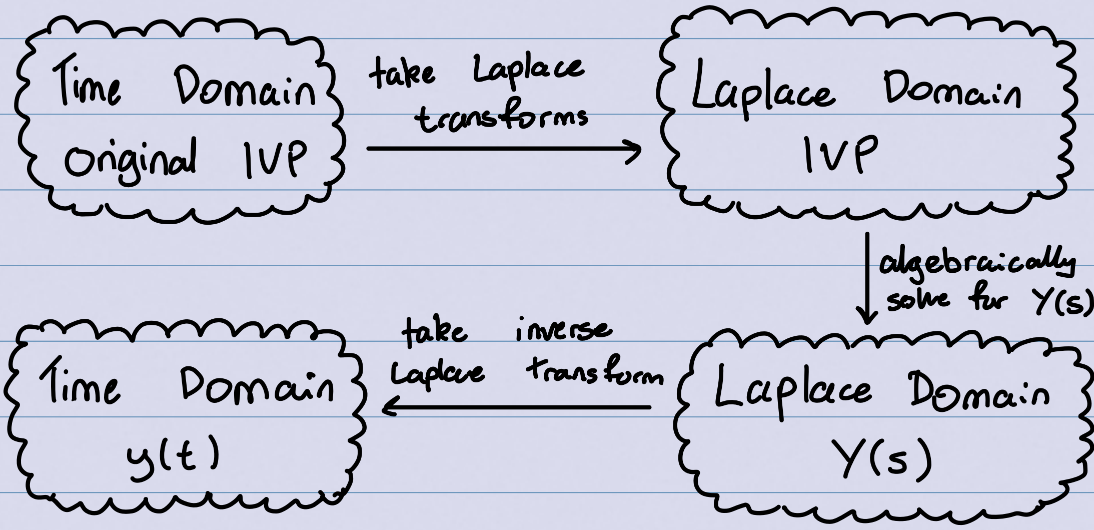
Using Laplace Transforms to Solve IVPs

Recall that an IVP for a general constant-coefficient linear ODE has the form $a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0y = b_m\frac{d^mu}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \dots + b_1\frac{du}{dt} + b_0u$

for $t \geq 0$, $a_n \neq 0$, and $n \geq m$, subject to $y(0^-) = y_0$, $y'(0^-) = y_1, \ldots, y^{(n-1)}(0^-) = y_{n-1}$.

↳ we assume that u is acceptable.

- Unless otherwise stated, we assume $u(t) = 0$ for all $t < 0$, implying that the "initial conditions" associated with the input are zero (but we can easily relax this assumption if needed.



Example: we previously had $Y(s) = \dfrac{22(s+0.0818)}{s(s+0.01)}$. Determine $y(t)$.

by PFE, $Y(s) = \dfrac{22(s+0.818)}{s(s+0.01)} = \dfrac{180}{s} - \dfrac{158}{s+0.01}$.

$\therefore$, $y(t) = (180 - 158e^{-0.01t})\, u_{step}(t)$.

Example: we also previously had $Y(s) = \dfrac{3}{(s^2+9)(s^2+s+1)} - \dfrac{s+1}{s^2+s+1}$. find $y(t)$.

$$Y(s) = \frac{-s^3 - s^2 - 9s - 6}{(s^2+9)(s^2+s+1)}$$

$$= \frac{-0.0411s - 0.329}{s^2+9} + \frac{-0.959s - 0.630}{s^2+s+1}$$ PFE!

$$= \frac{-0.0411s}{s^2+9} - \frac{0.329}{s^2+9} + \frac{-0.959s - 0.630}{(s+0.5)^2 + 0.75}$$

$$= \frac{-0.0411s}{s^2+9} - \frac{0.110 \times 3}{s^2+9} + \frac{-0.959(s+0.5) - 0.174 \times 0.866}{(s+0.5)^2 + 0.866^2}$$

$$= \frac{-0.0411s}{s^2+9} - \frac{0.110\times 3}{s^2+9} - \frac{0.959(s+0.5)}{(s+0.5)^2+0.866^2} - \frac{0.174\times 0.866}{(s+0.5)^2+0.866^2}.$$

now, taking the inverse Laplace transforms by inspection:

$$y(t) = \left(-0.0411\cos(3t) - 0.11\sin(3t) - 0.959e^{-0.5t}\cos(0.866t) - 0.174e^{-0.5t}\sin(0.866t)\right)u_{step}(t)$$

#5 #4 #14 #13

Example: another previous example resulted in $\Theta(s) = \frac{a}{(M-m)g - MLs^2}$. find $\Theta(t)$.

let $p = \sqrt{\left(\frac{M+m}{ML}\right)g}$ be a constant.

then, $\Theta(s) = -\frac{a}{ML}\frac{1}{(s^2-p^2)} = -\frac{a}{ML}\frac{1}{(s-p)(s+p)} = -\frac{a}{ML}\left(\frac{1}{2p}\frac{1}{(s-p)} - \frac{1}{2p}\frac{1}{(s+p)}\right)$ PFE

$$= -\frac{a}{2MLp}\left(\frac{1}{s-p} - \frac{1}{s+p}\right)$$

$\therefore$, by inspection, $\Theta(t) = -\frac{a}{2MLp}\left(e^{pt} - e^{-pt}\right)u_{step}(t)$.

Example: solve $\ddot{y} + 3\dot{y} = u$ where $u(t) = u_{step}(t)$, $y(0^-) = 0$, and $\dot{y}(t) = 5$.

Step 1: take Laplace transforms

$\mathcal{L}\{u_{step}(t)\} = 1/s$

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + 3(sY(s) - y(0)) = \frac{1}{s}$$

Step 2: solve for $Y(s)$:

$$s^2Y(s) - 0 - 5 + 3sY(s) - 0 = 1/s$$

$$s^2Y(s) + 3sY(s) - 5 = 1/s$$

$$Y(s)(s^2+3s) = \frac{1}{s} + 5 = \frac{5s+1}{s}. \quad \therefore, Y(s) = \frac{5s+1}{s(s^2+3s)}.$$

Step 3: take inverse Laplace transform

$$Y(s) = \frac{5s+1}{s(s^2+3s)} = \frac{5s+1}{s^2(s+3)} = \frac{14}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{14}{9}\frac{1}{s+3} \text{ by PFE.}$$

$\therefore$, by inspection, $y(t) = \left(\frac{14}{9} + \frac{1}{3}t - \frac{14}{9}e^{-3t}\right)u_{step}(t)$.

↳ this is the step response of the system

Example: Similarly, solve $\ddot{y} + 3\dot{y} = u$ where $u(t) = u_{step}(t-10)$, $y(10^-)=0$, $\dot{y}(10^-)=5$.

exploiting time-invariance: $y(t) = \left(\frac{14}{9} + \frac{1}{3}(t-10) - \frac{14}{9}e^{-3(t-10)}\right)u_{step}(t-10)$.

General Structure of the IVP Solution

↗ page 118 for details!

After taking the Laplace transform of the general IVP ODE and solving for $Y(s)$, we get:

$$Y(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}U(s) + \frac{s^2(a_3 y_0) + s(a_3 y_1 + a_2 y_0) + (a_3 y_2 + a_2 y_1 + a_1 y_0)}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

↳ or: $Y(s) = \frac{P(s)}{Q(s)}U(s) + \frac{R(s)}{Q(s)}$.

where:

- $P(s)$ and $Q(s)$ are polynomials that depend only on the ODE coefficients.
- $R(s)$ is a polynomial that depends on the initial conditions and some of the ODE coefficients.
  - ↳ Note: $R(s)$ is the only term that depends on the initial conditions. If the initial conditions are all 0, then $R(s) = 0 \ \forall s$!
- $U(s)$ is the only term that depends on the input signal
- $Y(s)$ is the only term that depends on the output signal.

(Important) Theorem: In the s-domain, the solution to an IVP for a constant-coefficient linear ODE has the structure:

$$Y(s) = \underbrace{\frac{P(s)}{Q(s)}U(s)}_{\text{term 1}} + \underbrace{\frac{R(s)}{Q(s)}}_{\text{term 2}}$$

- term 1 shows the effect of the input on the response. It is called the 'input response' (aka the 'zero state response').
- term 2 shows the effect of the initial conditions on the response. It is called the initial-condition response (aka the zero-input response).

term 1 is especially important: the ratio $H(s) = \frac{P(s)}{Q(s)}$ is called the transfer function of the ODE.

↳ the transfer function depends only on the coefficients of the ODE, not on the input signal and not on the initial conditions.

Example: derive both terms as well as the transfer function for the mass-spring-damper system $M\ddot{y} + B\dot{y} + K_s y = u$, $y(0^-) = y_0$, $\dot{y}(0^-) = y_1$

taking Laplace transforms:

$$M(s^2Y(s) - sy(0) - \dot{y}(0)) + B(sY(s) - y(0)) + K_s Y(s) = u(s).$$

$$M(s^2Y(s) - sy_0 - y_1) + B(sY(s) - y_0) + K_s Y(s) = u(s)$$

$$Y(s)(Ms^2 + Bs + K_s) = u(s) + Msy_0 + My_1 + By_0$$

$$\hookrightarrow Y(s) = \overbrace{\frac{1}{Ms^2+Bs+K_s}}^{\text{transfer function}} u(s) + \frac{(Ms+B)y_0 + My_1}{Ms^2+Bs+K_s}$$

$\therefore$, term $1 = \frac{1}{Ms^2+Bs+K_s}u(s)$, term 2: $\frac{(Ms+B)y_0+My_1}{Ms^2+Bs+K_s}$

↱ ignore initial conditions when finding transfer functions

Examples: find the transfer function for each of these:

a) $10\dot{y} = u \Rightarrow 10sY(s) = u(s) \Rightarrow Y(s) = \frac{1}{10s}u(s) \Rightarrow TF = \frac{1}{10s}$.

b) $\dddot{y} - 6\dot{y} = u + \dot{u}$ $\Rightarrow$ $s^3 Y(s) - 6sY(s) = (s+1)U(s)$

$$Y(s)(s^3 - 6) = (s+1)U(s) \Rightarrow Y(s) = \frac{s+1}{s^3-6}U(s).$$

$\therefore$, $TF = \dfrac{s+1}{s^3-6}$

c) $LC\ddot{V}_{out} + RC\dot{V}_{out} + V_{out} = V_{in}$

$$\Rightarrow V_{out}(s)(LCs^2 + RCs + 1) = V_{in}(s) \Rightarrow V_{out}(s) = \frac{1}{LCs^2 + RCs + 1}V_{in}(s).$$

$\therefore$, $TF = \dfrac{1}{LCs^3 + RCs + 1}$

d) $\dot{y} + \sin(t)y = u$ $\Rightarrow$ the $\sin(t)$ term makes this time-varying!

$\therefore$, there's no transfer function!

e) the system whose step response (assuming zero initial conditions) is $y(t) = (10 - 10e^{-t})u_{step}(t)$.

step response means "response when input is the unit step function" $\Rightarrow$ $\therefore$, $u = u_{step}(t)$, so $U(s) = 1/s$.

$$Y(s) = 10/s - 10/s+1 = \frac{10}{s(s+1)}.$$

$\therefore$, $TF = \dfrac{Y(s)}{U(s)} = \dfrac{\frac{10}{s(s+1)}}{1/s} = \dfrac{10}{s+1}$

$\rightarrow$ for a real-world example, check pages 121-122.