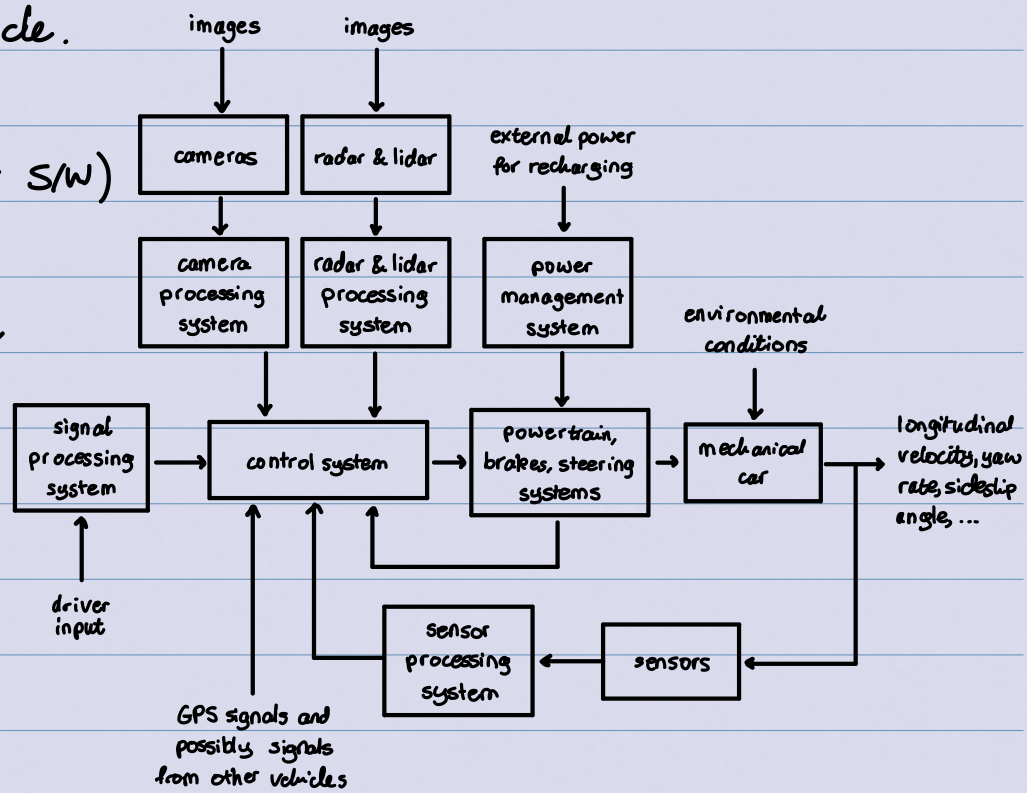
Motivating Example: design of software and hardware used to control an electric vehicle.

boxes = systems (H/W or S/W)

arrows = signals

⇒ this is an example of a block diagram



Can decompose this into 3 types of tasks:

1. Modelling
   - we need some way to describe how systems process to generate outputs
2. Analysis
   - we need tools to determine and study the behavior of the various systems
   - eg. is the control system stable? is it fast or slow? How does it respond in windy conditions?
3. Design
   - we need to have a systematic way to create and tune the various systems (control system, image processing system, etc)

Signal: a function of one or more independent variables, generally containing information about the behavior of some phenomenon of interest.

⇒ Example: a signal may represent a force, a torque, an angle, a speed, a stock price, available SSD memory, etc.

- We will deal with only the situation where there is one independent variable, namely time:
  - If time is varying consistently, it's a continuous-time signal.
    ↳ we denote time by t and continuous-time signals as $x(t), u(t), y(t)$, etc
  - If time jumps from one value to the next, it's a discrete-time signal.
    ↳ we denote time by k and discrete-time signals as $x[k], u[k], y[k]$, etc

System: a device, process, or algorithm that takes one or more input signals and generates one or more output signals.

⇒ Example: each of the blocks in the electric vehicle system, a rocket, a heart, a phone, a planet, etc

It's traditional to denote a generic input signal by u (ie, either $u(t)$ or $u[k]$) and a generic output signal by y (ie, either $y(t)$ or $y[k]$)

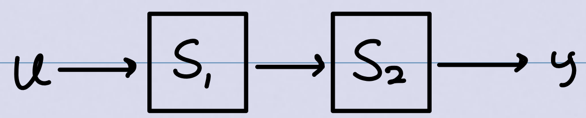
Systems that have one input signal and one output signal are called single-input single-output (SISO). Systems that have multiple inputs and multiple outputs are called multi-input multi-output (MIMO).

The output of the system is also called the response of the system.

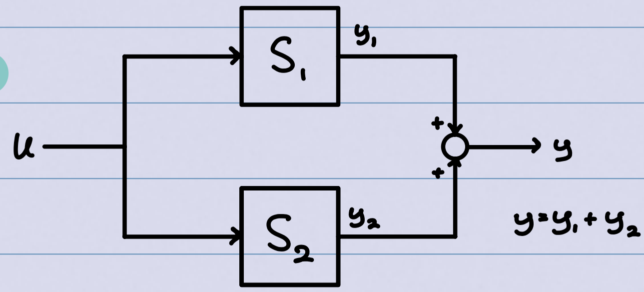
If both the input signal(s) and output signal(s) are continuous-time signals, then we say the system is a continuous-time system. Similarly, if both are discrete-time signals, then we say the system is a discrete-time system. Any other combination results in a hybrid system.

In a block diagram, blocks can be connected in series (aka a cascade connection) or in parallel (with the help of a summer):

series:



parallel:



Differential Equation: any math equation that, in contrast to a purely algebraic equation, includes the derivatives of one or more dependent variable with respect to one or more independent variables.

Ordinary Differential Equation (ODE): a differential equation with only one independent variable.

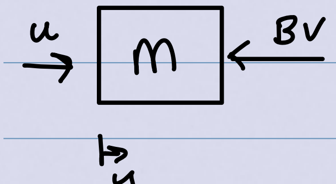
Partial Differential Equation (PDE): a differential equation with more than one independent variable.

Order of a Differential Equation: the order of the highest derivative in the equation.

⇒ Example: are the following algebraic, ODEs, or PDEs?

- $\frac{d^3y}{dt^3} + 4y = \frac{du}{dt} + 2u$ — ODE, 3rd order
- $F = ma$ — algebraic
- $F = m\frac{d^2y}{dt^2}$ — ODE, 2nd order
- $\dddot{y} + 2(2 - \dot{y}^3)\ddot{y} + 4y = u$ — ODE, 3rd order
- $\frac{\partial y(x,t)}{\partial t} - K\frac{\partial y^2(x,t)}{\partial^2 x} = u(x,t)$ — PDE, 2nd order

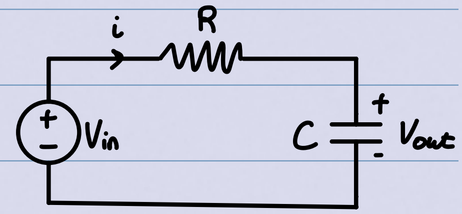
Example: consider the dynamics of a vehicle moving in a straight line. The system is affected mainly by the force applied by the engine and air resistance (friction). Let u = input force due to engine and v = output velocity ($= \dot{y}$)



$$F = ma \rightarrow u - Bv = m\ddot{y}$$
$$\rightarrow u - Bv = m\dot{v}$$
$$\rightarrow u = m\dot{v} + Bv$$

$\therefore$, 1^{st} order ODE

Example: consider this RC circuit. We desire to know the dynamic relationship between the output voltage V_{out} and the input voltage V_{in}. Let i be the current in the loop.

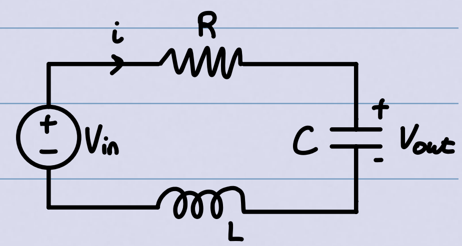


$$Ri - V_{in} + V_{out} = 0, \quad \text{and} \quad i = C\frac{dV_{out}}{dt}.$$

$$\hookrightarrow V_{out} + RC\frac{dV_{out}}{dt} = V_{in}$$

$\therefore$, 1^{st} order ODE

Example: Same as previous example, but now with an inductor included.



$$Ri - V_{in} + V_{out} + L\frac{di}{dt} = 0 \quad \text{and} \quad i = C\frac{dV_{out}}{dt}$$

$$\hookrightarrow V_{out} + RC\frac{dV_{out}}{dt} + LC\frac{d^2V_{out}}{dt^2} = V_{in}$$

$\therefore$, 2^{nd} order ODE

Static / Memoryless System: at each time instant, each possible output doesn't depend on any value of the input except perhaps for the input at the same time instant.

↳ else, the system is said to be Dynamic or to have Memory.

Example : a resistor ($V(t) = i(t)R$) is a static system

Example : a capacitor ($C\frac{dv(t)}{dt} = i(t)$ or $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$) is a dynamic system.

Causal / Non-Anticipative System: at each time instant, each possible output does not depend on the future values of the input.

↳ else, the system is said to be non-causal or acausal.

Example : the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k-1]$ is causal.

Example: the discrete-time system $y[k] = u[k] + 2u[k] + 3u[k+1]$ is noncausal.

Example : the system $y(t) = K\int_{-\infty}^{t} u(\tau)d\tau$ is causal

⇒ A static system is causal!

Linear System: satisfies the superposition property, that is, for any input signals u_1 (w/ output y_1) and u_2 (w/ output y_2) and any constants a_1 and a_2, a response to the input signal $u = a_1u_1 + a_2u_2$ is $y = a_1y_1 + a_2y_2$.

↳ Else, the system is said to be nonlinear.

Homogeneity Property: for any input signal u_1 (w/ output y_1) and any constant a_1, a response to the input signal $u = a_1 u_1$ is $y = a_1 y_1$.

Additivity Property: for any input signals u_1 (w/ output y_1) and u_2 (w/ output y_2), a response to the input signal $u = u_1 + u_2$ is $y = y_1 + y_2$.

A system satisfies the superposition property (ie, the system is linear) if and only if it satisfies both the homogeneity property and the additivity property.

Proof:

($\Rightarrow$) If superposition holds:

- set $a_1 = a_2 = 1$ to conclude additivity holds
- set $a_2 = 0$ to conclude homogeneity holds

($\Leftarrow$) If both homogeneity and additivity hold, then let $y_1 = S(u_1)$ and $y_2 = S(u_2)$. Then a response to input $a_1 u_1 + a_2 u_2$ is:

$S(a_1 u_1 + a_2 u_2)$

$= S(a_1 u_1) + S(a_2 u_2)$ by additivity

$= a_1 S(u_1) + a_2 S(u_2)$ by homogeneity

$= a_1 y_1 + a_2 y_2$.

$\therefore$ superposition property is satisfied.

$\Rightarrow$ it's often faster to check for both homogeneity and additivity instead of superposition directly!

Example: are the following systems linear or nonlinear?

a) $y(t) = K u(t)$

- Satisfies superposition:

  Apply input u_1 to get output $y_1 = Ku_1$

  Apply input u_2 to get output $y_2 = Ku_2$

  Apply input $a_1u_1 + a_2u_2$ to get output $K(a_1u_1 + a_2u_2)$

  $= a_1(Ku_1) + a_2(Ku_2) = a_1y_1 + a_2y_2$

$\therefore$, the system is linear.

b) $y(t) = Ku(t) + 1$

- Fails both homogeneity and additivity. Eg:

  Apply input u_1 to get output $y_1 = Ku_1 + 1$

  Apply input u_2 to get output $y_2 = Ku_2 + 1$

  Apply input $u_1 + u_2$ to get output $K(u_1 + u_2) + 1 \neq y_1 + y_2$.

$\therefore$, the system is nonlinear.

c) $y(t) = au(t) + bu^2(t)$

- Fails additivity. Eg:

  Apply input u_1 to get output $y_1 = au_1 + bu_1^2$

  Apply input u_2 to get output $y_2 = au_2 + bu_2^2$

  Apply input $u_1 + u_2$ to get output $a(u_1 + u_2) + b(u_1 + u_2)^2 \neq y_1 + y_2$

$\therefore$, the system is nonlinear (if $b \neq 0$)

d) $y(t) = \sin(u(t))$

- Fails homogeneity. Eg:

  Apply input u_1 to get output $y_1 = \sin(u_1)$

  Apply input a_1u_1 to get output $y = \sin(a_1u_1) \neq a_1u_1$

$\therefore$, the system is nonlinear

e) $y(t) = \sin(t)u(t)$

· Satisfies superposition!

Apply input u_1 to get output $y_1(t) = \sin(t)\,u_1(t)$

Apply input u_2 to get output $y_2(t) = \sin(t)\,u_2(t)$

Apply input $a_1u_1 + a_2u_2$ to get output

$y(t) = \sin(t)(a_1u_1(t) + a_2u_2(t)) = a_1y_1(t) + a_2y_2(t)$.

$\therefore$, the system is linear!

f) $y(t) = 5u(t) + u(t)y(t)$

Solving for y: $y = 5u + uy \rightarrow y = \dfrac{5u}{1-u}$

Fails homogeneity:

Apply input u_1 to get output $y_1 = \dfrac{5u_1}{1-u_1}$ $(u_1 \neq 1)$

Apply input a_1u_1 to get output $y = \dfrac{5a_1u_1}{1-a_1u_1} \neq a_1y_1$

$\therefore$ the system is nonlinear!

g) $y(t) = K\dfrac{du(t)}{dt}$

· Satisfies superposition:

Apply input u_1 to get output $y_1 = K\dfrac{du_1}{dt}$

Apply input u_2 to get output $y_2 = K\dfrac{du_2}{dt}$

Apply input $a_1u_1 + a_2u_2$ to get output $y = K\frac{d}{dt}(a_1u_1 + a_2u_2)$

$\therefore$ the system is linear! $= a_1y_1 + a_2y_2$

h) $y(t) = K\displaystyle\int_{-\infty}^{t} u(\tau)\,d\tau$

· Satisfies superposition!

Apply input u_1 to get output $y_1 = K\int_{-\infty}^{t} u_1(\tau)\,d\tau$

Apply input u_2 to get output $y_2 = K\int_{-\infty}^{t} u_2(\tau)\,d\tau$

Apply input $a_1u_1 + a_2u_2$ to get output $y = K\int_{-\infty}^{t} a_1u_1(\tau) + a_2u_2(\tau)\,d\tau$

$\therefore$, the system is linear! $= a_1y_1 + a_2y_2$

i) $M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + K_s y(t) = u(t)$ (mass-spring damper system)

· Satisfies superposition

Apply input u_1 to get output y_1 satisfying $M\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_s y_1 = u_1$ ①

Apply input u_2 to get output y_2 satisfying $M\frac{d^2y_2}{dt^2} + B\frac{dy_2}{dt} + K_s y_2 = u_2$ ②

Compute $a_1 \cdot ① + a_2 \cdot ②$:

$$M\frac{d^2}{dt^2}(a_1y_1 + a_2y_2) + B\frac{d}{dt}(a_1y_1 + a_2y_2) + K_s(a_1y_1 + a_2y_2) = a_1u_1 + a_2u_2$$

$\therefore$, $a_1y_1 + a_2y_2$ is a solution to the ODE when $u = a_1u_1 + a_2u_2$.

So, the system is linear!

j) $\frac{dP(t)}{dt} = aP(t) + bP^2(t) + u(t)$

· Fails homogeneity.

Apply input u_1 to get output satisfying $\frac{dP_1}{dt} = aP_1 + bP_1^2 + u_1$ ①

Suppose the system satisfies homogeneity. Then, $\frac{da_1P_1}{dt} = a\alpha_1P_1 + b(a_1P_1)^2 + u_1$

$\Rightarrow \frac{dP_1}{dt} = aP_1 + ba_1P_1^2 + u_1$ ②

If ① and ② both hold, then so does ①-②:

$0 = bP_1^2 - b\alpha_1P_1^2 \Rightarrow b(a_1 - 1)P_1^2 = 0$ ③

But ③ doesn't hold in general (except if $b=0$ or $\alpha_1 = 1$ or $P_1 = 0$ which are not of interest).

$\therefore$, the system does not satisfy homogeneity by contradiction, and therefore, the system is nonlinear!

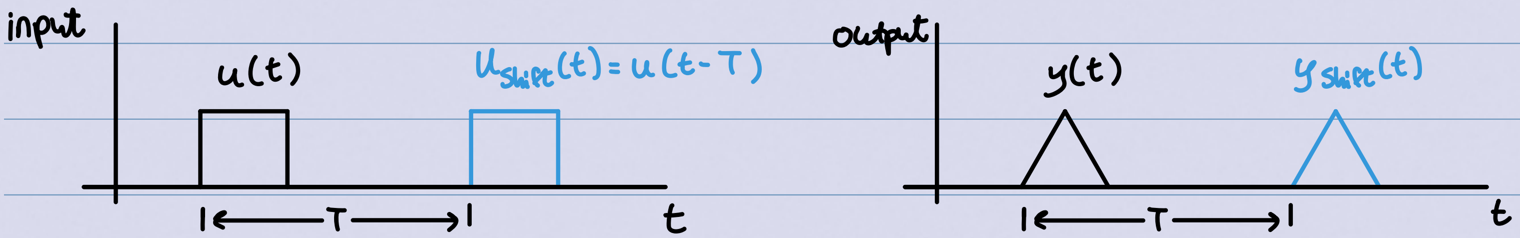
Time-Invariant Systems vs Time-Varying Systems

A system is said to be time-invariant if a time shift in the input signal always causes the same time shift (but no other distortion) in the output signal. More formally:

Assume input u is applied to the system with an associated output y. For a constant T (with $-\infty < T < \infty$), let $u_{shift}(t) = u(t-T)$ denote the shifted output. The system is said to be time-invariant if, for all u and all T, there exists an output associated with the input u_{shift}, denoted y_{shift}, such that:

$$y_{shift}(t) = y(t-T) \text{ for } -\infty < T < \infty.$$

A system that is not time-invariant is said to be time-varying.



↳ system is time-invariant if $y_{shift}(T) = y(t-T)$.

Example: indicate if the following systems are time-invariant or time-varying:

a) $y(t) = Ku(t)$

Apply input u to get output $y(t) = Ku(t)$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t)$

$\therefore$, the system is time-invariant! $= Ku(t-T) = y(t-T)$

b) $y(t) = Ku(t) + 1$

Apply input u to get output $y(t) = Ku(t) + 1$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t) + 1$

$\therefore$, the system is time-invariant! $= K(u(t-T)) + 1 = y(t-T)$

c) $y(t) = Ku(t) + t$

Apply input u to get output $y(t) = Ku(t) + t$

Apply input u_{shift} to get output $y_{shift}(t) = K u_{shift}(t) + t$

$\therefore$, the system is time-varying. $= Ku(t-T)+t \neq Ku(t-T)+t-T = y(t-T)$

d) $y(t) = \sin(t)u(t)$

Apply input u to get output $y(t) = \sin(t)u(t)$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(t)u_{shift}(t)$

$\therefore$, the system is time-varying.

$$= \sin(t)u(t-T) \neq \sin(t-T)u(t-T) = y(t-T)$$

e) $y(t) = \sin(u(t))$

Apply input u to get output $y(t) = \sin(u(t))$

Apply input u_{shift} to get output $y_{shift}(t) = \sin(u_{shift}(t))$

$\therefore$, the system is time-invariant. $= \sin(u(t-T)) = y(t-T)$.

f) $y(t) = K\dfrac{du(t)}{dt}$

Apply input u to get output $y(t) = K\dfrac{du(t)}{dt}$

Apply input u_{shift} to get output $y_{shift}(t) = K\dfrac{du_{shift}(t)}{dt}$

$\therefore$, the system is time-invariant.

$$= K\frac{du(t-T)}{dt} = y(t-T).$$

g) $y(t) = K\displaystyle\int_{-\infty}^{t} u(\tau)\,d\tau$

Apply input u to get output $y(t) = K\displaystyle\int_{-\infty}^{t} u(\tau)\,d\tau$

Apply input u_{shift} to get output $y_{shift}(t) = K\displaystyle\int_{-\infty}^{t} u_{shift}(\tau)\,d\tau$

$\therefore$, the system is time-invariant!

$$= K\int_{-\infty}^{t} u(t-\tau)\,d\tau = K\int_{-\infty}^{t-T} u(\tau)\,d\tau$$

$$= y(t-T)$$

h) $M\dfrac{d^2y(t)}{dt^2} + B\dfrac{dy(t)}{dt} + K_s y(t) = u(t)$

Apply input u to get output y satisfying:

$$M\frac{d^2y(t)}{dt}\bigg|_t + B\frac{dy}{dt}\bigg|_t + K_s y(t) = u(t) \qquad (1)$$

Apply input $u_{shift}(t)$ to get output y_{shift} satisfying:

$$M\frac{d^2y_{shift}(t)}{dt}\bigg|_t + B\frac{dy_{shift}}{dt}\bigg|_t + K_s y_{shift}(t) = u_{shift}(t) = u(t-T)$$

let $x = t-T \rightarrow dx = dt$

$$\Rightarrow M \frac{d^2 y_{shift}(t)}{dx^2}\bigg|_{x+T} + B \frac{dy_{shift}}{dx}\bigg|_{x+T} + K_s\, y_{shift}(t) = u(x) \qquad (2)$$

(1) and (2) represent the same ODE with the same input.
Therefore, the outputs match: $y_{shift}(x+T) = y(x)$

$$\Leftrightarrow y_{shift}(t) = y(t-T).$$

$\therefore$, the system is time-invariant.