# Lecture 1 - 4th Sept 2024

**Statistics**: the science of understanding data and making decisions in the face of variability and uncertainty.

**Probability**: A branch of mathematics concerned with describing and modeling uncertain events.

## Preliminaries:

- **Experiment**: the process of obtaining an observed result of some phenomenon.
- **Trial**: the performance of an experiment.
- **Outcome**: the result of a single trial (attempt) of an experiment.
- **Event**: one or more outcomes of an experiment.
- **Probability**: the measure of how likely an event is.

**Sample Space**: the set of ALL possible distinct outcomes in a random experiment, denoted by $S$.
↳ note: one and only one of the outcomes occurs in any single trial of the experiment.
↳ example:
- Roll a six-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- Flip a coin: $S = \{\text{heads}, \text{tails}\}$
- Waiting time for a bus: $S = \{t \in \mathbb{R}, 0 \leq t \leq 10\}$

- a sample space is finite if it consists of a finite number of outcomes, say $\{a_1, a_2, \ldots, a_N\}$

- a sample space can be a set of countable infinite outcomes, say $\{a_1, a_2, \ldots\}$, where the outcomes can be put into a one-on-one correspondence with positive integers.
  - ↳ Example: Suppose our experiment consists of tossing a coin until it lands on heads.
    - ↳ the possible outcomes in the sample space are:

$$H, \; TH, \; TTH, \; TTTH, \; TTTTH, \; \ldots$$

$\therefore$, the outcomes are countable but infinite.

Discrete Sample Space: when a sample space is either finite or countably infinite. Else, it's non-discreet.

↳ Example:

- Roll a six-sided die: discreet
- Flip a coin: discreet
- Waiting time for a bus: non-discreet

Event: a subset of sample space S.

↳ we say A is an event if $A \subset S$.

↳ note: A and S are both sets!

↳ Example: consider tossing two coins.

a) what is the event of obtaining at least one head?

$S = \{HH, HT, TH, TT\}$

$\therefore A = \{HH, HT, TH\}$

b) What is the event of obtaining at least one head and at least one tail?

$A = \{HT, TH\}$.

**Elementary / Simple Event:** an event that contains only one outcome of the experiment. Eg, $A = \{a_1\}$.

**Compound Event:** an event made up of two or more simple events. Eg: $A = \{a_1, a_2\}$.

**Set Notations and Terminology:**

- A set is a collection of elements. Eg, $A = \{\text{apple, orange}\}$.
- The notation $x \in A$ or $x \notin A$ will mean that $x$ is or is not an element of $A$.
- The empty / null set is $\emptyset$.
- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The notation $B \subset A$ means that $B$ is a subset of $A$.
- Complement: $A^c = \{x \mid x \in S, x \notin A\} = \bar{A}$
- Two sets $A$ and $B$ are disjoint if $A \cap B = \emptyset$.

- The set $A \cap B^c$ is "A but not B"
- The set $A^c \cap B^c$ is "neither A nor B"

- If $A_1, \ldots, A_n$ is a finite collection of sets,
  - $\to A_1 \cap \ldots \cap A_n = \bigcap_{i=1}^n A_i$ is "in all $A_i$"
  - $\to A_1 \cup \ldots \cup A_n = \bigcup_{i=1}^n A_i$ is "in at least one $A_i$"

very similar to disjoint!

**Mutually Exclusiveness:** when two events A and B if $A \cap B = \emptyset$. If events are exclusive, they have no outcomes in common.

↳ Example: tossing two coins. A is the event "at least one head", and B is the event "both tails".

↳ $S = \{HH, HT, TH, TT\}$.
$A = \{HH, HT, TH\}$, $B = \{TT\}$.

$$A \cap B = \{HH, HT, TH\} \cap \{TT\} = \emptyset \text{ !}$$

$\therefore$, A and B are mutually exclusive.

↳ Events $A_1, A_2, A_3, \ldots$ are said to be mutually exclusive if they are pairwise mutually exclusive. That is,

$$A_i \cap A_j = \emptyset, \text{ wherever } i \neq j.$$

**Probability Modeling**: assigning a probability $P(A)$ to each event $A$. This probability measures how likely it is that $A$ will happen when the experiment is conducted.

↳ think of $P(A)$ as a (set) function, whose domain is a collection of sets (events), and the range of which is a subset of real numbers.

↳ not all set functions are appropiate for assigning probabilities to event.

- Let $S = \{a_1, a_2, a_3, \ldots\}$ be a discrete sample space. We assign to each elementary event $A_i = \{a_i\}$ for $i = 1, 2, \ldots$, a number $P(A_i) = P(\{a_i\})$, such that

  - $0 \leq P(A_i)$ and $\sum_{\text{all } i} P(A_i) = 1$

- We call $P(A)$ the probability of $A$, and call the set of probabilities $\{P(A_i), i = 1, 2, \ldots\}$ the probability distribution on $S$.

↳ **General Properties**:

- The null event (empty set) has a probability of 0. $\therefore$, $P(\emptyset) = 0$.
- If $A$ and $B$ are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$
- If $A_1, A_2, \ldots, A_k$ is a finite collection of pairwise mutually exclusive events, then

$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k)$

- $P(\bar{A}) = 1 - P(A)$

↳ Example: suppose a six-sided die is rolled once. If A is the event that an even number is obtained. What is $P(A)$?

- $S = \{1, 2, 3, 4, 5, 6\}$. Since each number is equally likely, $P(\{i\}) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

  $A = \{2, 4, 6\}$. → $\therefore P(A) = \frac{1}{2}$.

↳ Example: Toss a coin twice. What's the probability of getting exactly one head?

- $S = \{HH, HT, TH, TT\}$. Each option is uniformly likely.
- $A = \{HT, TH\}$.

  ↳ $\therefore P(A) = \frac{1}{2}$.

Random Selection: when choosing an object from a finite set where each object has an equal chance of being selected.

Uniform Probability Model: consider a discrete finite sample space $S = \{a_1, a_2, \ldots, a_N\}$, where each simple event has a probability of $1/N$. This is a uniform distribution over the set $\{a_1, a_2, \ldots, a_N\}$.

↳ For a compound event A, we can calculate P(A) by:

$$P(A) = \frac{n(A)}{N},$$

where n(A) is the number of outcomes in A.

- The formula $P(A) = \frac{n(A)}{N}$ is classical probability.

(trivial) example: draw one card at random from a well-shuffled deck without jokers. What is the probability that the card is a spade?

↳ $S = \{A\heartsuit, 2\heartsuit, \ldots\}$.

$A_s = \{A\spadesuit, 2\spadesuit, \ldots, K\spadesuit\}$.

↳ $P(A_s) = \frac{n(A_s)}{N} = \frac{13}{52} = \frac{1}{4}$.

Example: suppose that P is a probability, and A and B are events such that A and B are mutually exclusive, and $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$. Is each statement true or false?

A) A and B are each non-empty

since $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$, we know that

$P(A) = 0.2$ and $P(B) = 0.3$.

we also know that $P(\emptyset) = 0$. Since $0.2 \neq 0 \neq 0.3$, we know A and B cannot be empty.
$\therefore$, the statement is true.

B) $P(A \cap B) = 0$

Since we're told that A and B are mutually exclusive, we know that $A \cap B = \emptyset$.

we also know that $P(\emptyset) = 0$, so the statement is true!

C) $P(A \cup B) = 0.4$

we know that $\overbrace{P(A \cup B)}^{\text{"the probability of A or B"}} = P(A) + P(B)$

$P(A) + P(B) = 0.2 + 0.3 = 0.5 \neq 0.4$.
$\therefore$, the statement is false!

Example: suppose two six-sided die are rolled where the outcomes are of the form (die 1, die 2). If A is the event that the first die is even, and B is the event that the sum of both rolls is 6, compute the event $C = \bar{A} \cap B$

↳ $A = \{(2,1), (2,2), \ldots, (4,1), (4,2), \ldots, (6,1), (6,2), \ldots, (6,6)\}$.

$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$\bar{A}$ means the event that A does not happen, so in other words, $\bar{A}$ is the event that the first die is odd.

$\therefore, \bar{A} = \{(1,1), \ldots, (1,6), (3,1), \ldots, (3,6), (5,1), \ldots, (5,6)\}.$

$\bar{A} \cap B$ is A AND B.

↳ $\therefore, C = \{(1,5), (3,3), (5,1)\}.$