Lecture 1 - 9th Jan

- **Tribolectric Effect** → when we rub two objects with each other, small amounts of charges are transferred from one object to the other
  ↳ only electrons get transferred, not protons!

- **Quantifying Electric Charge**
  ↳ **Atom**: nucleus with surrounding electron cloud
  ↳ nucleus has protons and electrons.
  ↳ **Charge**: e is the elementary charge and is quantized.
  - neutrons: $q_n = 0$.
  - electrons: $q_e = -e = -1.602 \cdot 10^{-19}$ C
  - protons: $q_p = e = 1.602 \cdot 10^{-19}$ C.

  ↳ **Mass**
  - protons: $m_p \approx m_n \approx 1.7 \times 10^{-27}$ kg.
  - electrons: $m_e \approx 9.1 \times 10^{-31}$ kg

Conservation of Electric Charge

- The net electric charge in an isolated system must remain constant.
- We cannot create or destroy charge
- But, we can distribute them to create particles and localized areas with varying charges.

- **Like charges repel and opposite charges attract!!!**

Behavior of Electric Charge:

- **Conductors:** charges can move freely
- **Insulators:** charges cannot move freely
- **Semiconductors:** conducts charge under specific conditions.
- **Superconductors:** no resistance to flow of charge.

unit vector describing direction of force
points from Q to q.

$$\vec{F}_{qQ} = k\frac{qQ}{r^2}\hat{r}_{qQ} \quad \text{and} \quad \vec{F}_{Qq} = k\frac{qQ}{r^2}\hat{r}_{Qq} = -\vec{F}_{qQ}$$

• $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

↳ where $k = \frac{1}{4\pi\varepsilon_0} \approx 8.99 \times 10^9 \ Nm^2/C^2$

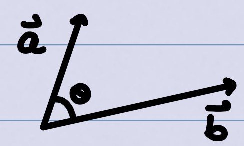
where $\varepsilon_0 \approx 8.854 \times 10^{-12} \ m^{-3}kg^{-1}s^4A^2$

- ε_0 is the permittivity of <u>free space</u>.

Dot Product

↳ **Example:** $(3\hat{i}+4\hat{j})\cdot(-2\hat{i}+3\hat{j}) = (3\cdot -2)+(3\cdot 4) = 6$

Generally, $\vec{a}\cdot\vec{b} = |a||b|\cos\theta$



Specifically for e&m, dot products will be important for equations like $dW = \vec{E}\cdot dr$, where $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$.

Coulomb's Law: $F_E = k\frac{q_1q_2}{r^2}$

↳ **Example:** find magnitude and direction of the force on q_3.

q_1: $-3\ \mu C$ at $(0, 2)$

q_2 : $7\ \mu C$ at $(0,-1)$
q_3 : $9\ \mu C$ at $(3,0)$.

① Find F on q_3 by q_1
② Find F on q_3 by q_2
③ add F_s together

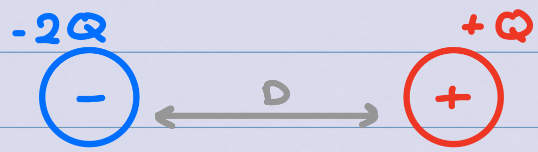
① $|F_{13}| = k\dfrac{(-3)(7)}{\sqrt{2^2+3^2}} = k\dfrac{-21}{\sqrt{13}}$ in $\dfrac{-3\hat{i}+2\hat{j}}{\sqrt{13}}$ direction

unit vectors

② $|F_{23}| = k\dfrac{(7)(9)}{\sqrt{1^2+3^2}} = k\dfrac{63}{\sqrt{10}}$ in $\dfrac{3\hat{i}+\hat{j}}{\sqrt{10}}$ direction

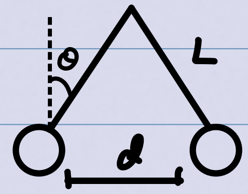
③ $\vec{F}_{net} = \dfrac{63k}{\sqrt{10}}\left(\dfrac{3\hat{i}+\hat{j}}{\sqrt{10}}\right) - \dfrac{21k}{\sqrt{13}}\left(\dfrac{2\hat{j}-3\hat{i}}{\sqrt{13}}\right)$

Conceptual Example: where would a positive charge experience a net 0 electrical force?

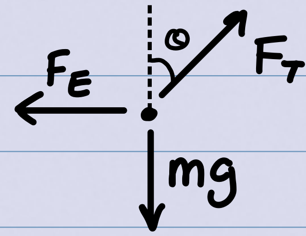


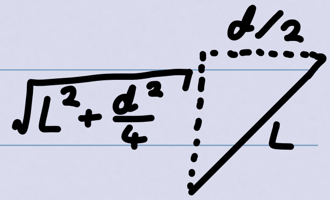
- it cannot be in the center as both forces point in the same direction
- on the left of the -2Q, the -2Q would overpower the +Q
- ∴, it would be on the right of the +Q!

Equilibrium Example (medium difficulty): two identical spheres are hung from strings (length L), they both have charge +q and can be treated as point charges. What's the equilibrium seperation?



left sphere:





$$\Sigma F_x = ma = 0 = F_T \sin\theta - F_E = 0 \rightarrow F_T \sin\theta = F_E$$
$$\Sigma F_y = ma = 0 = F_T \cos\theta - mg \rightarrow F_T = mg/\cos\theta$$

$$\therefore F_E = \frac{mg\sin\theta}{\cos\theta} = mg\tan\theta = k\frac{q^2}{r^2}$$

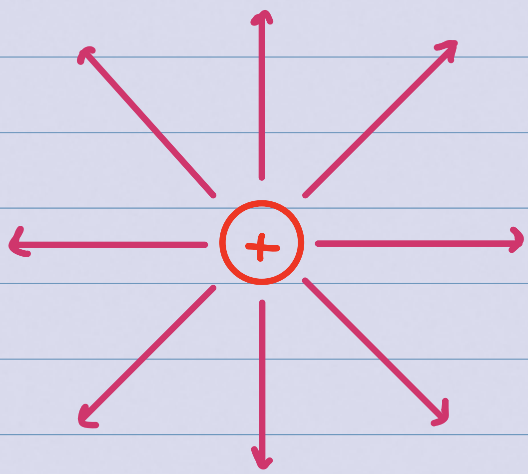
$$\tan\theta = \frac{O}{A} = \frac{d/2}{\sqrt{L^2+d^2/4}} \rightarrow k\frac{q^2}{d^2} = \frac{mgd}{2\sqrt{L^2+d^2/4}}$$

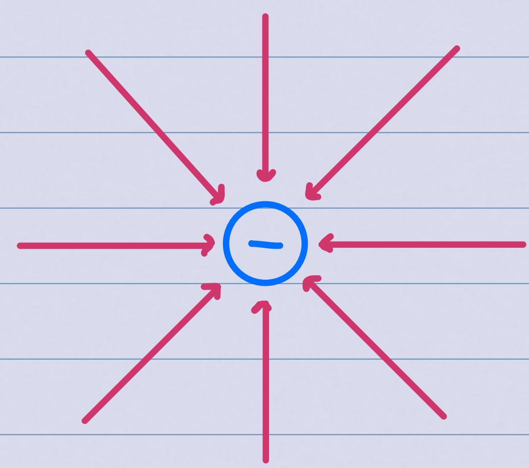
then solve for d!

Electric Fields: $\left(\vec{E} = \frac{\vec{F}}{q_0}\right)$

direction in which a positive charge would move

- Electric fields point radially out or in for point charges.





- Electric Fields tell charges how to move

$$\vec{E}(\vec{r}) = \lim_{q\to 0}\frac{\vec{F}_{qQ}}{q} = \lim_{q\to 0} k\frac{Q}{r^2}\hat{r} \longrightarrow \vec{E} = k\frac{Q}{r^2}\underbrace{\hat{r}}_{\text{radial!!}} = \frac{\vec{F}}{q} \quad \left[\frac{N}{C}\right]$$

Example: there is a +28pC charge a distance 0.24mm away from a -11pC charge. What is the electric field midway between the points?

$+Q = 28 \times 10^{-12}\,C$ $\qquad\qquad$ $-q = -11 \times 10^{-12}\,C$

$\vec{E}$ here?

$$\vec{E}_1 = k\frac{Q}{(\frac{d}{2})^2}\,\hat{x}, \qquad \vec{E}_2 = k\frac{q}{(\frac{d}{2})^2}\,\hat{x}$$

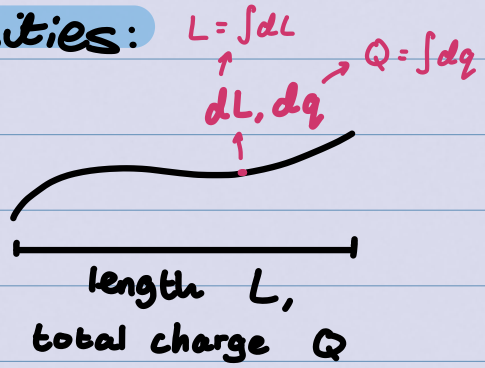
$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 = k\frac{Q}{(\frac{d}{2})^2} + k\frac{q}{(\frac{d}{2})^2}\,\hat{x}$$

$$= 2.44 \times 10^{-7}\,\hat{x}\ N/C.$$

$$\vec{E} = \int d\vec{E} = k\int \frac{1}{r^2}\,dq\,\hat{r}$$

Charge Densities:

1D - linear:



linear charge density!

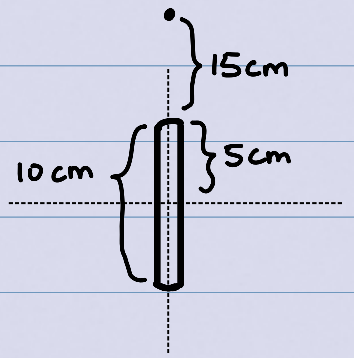
$$\therefore \frac{dQ}{dL} = \lambda \rightarrow dQ = \lambda\, dL$$

if λ is constant, $\lambda = \frac{Q}{L}$

$$\therefore \int dQ = \int \lambda\, dL$$

- Given a charge density, how do we find $\vec{E}$?
  - Choose coordinate system
  - Find dq
  - Find dE → vector, so consider components eg dE_x
  - Look at symmetry to try to cancel out components
  - Integrate!

Example: An electric field exists near a straight, thin rod of length = 10cm, uniformly charged at $\lambda = 3\ nC/cm$. Assume that the rod is placed along the y-axis, centered at the origin. Find the electric field d = 20cm above the rod (y = 15cm).



① cartesian

② $dQ = \lambda\, dL$

③ $d\vec{E} = k \frac{dQ}{r^2} \hat{r}$

$r = d - y \rightarrow r^2 = (d-y)^2, \quad \hat{r} = \hat{y}$

$dL = dy!$

$\hookrightarrow d\vec{E} = k \frac{\lambda\, dy}{(d-y)^2} \hat{y} \rightarrow \vec{E} = \lambda k \int_{-5}^{5} \frac{1}{(d-y)^2} dy\ \hat{y}$

$u = d - y \rightarrow du = -dy \rightarrow dy = -du \rightarrow \vec{E} = -\lambda k \int_{-5}^{5} \frac{1}{u^2} du\ \hat{y}$

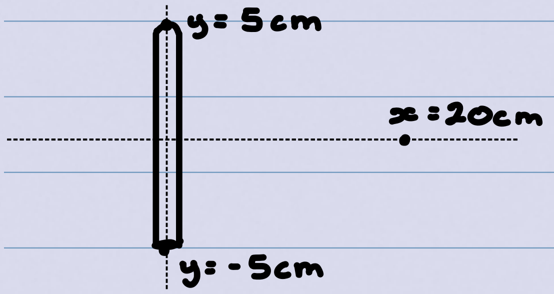
$\vec{E} = -\lambda k \left[-\frac{1}{u} \right]_{-5}^{5} \hat{y} \rightarrow E = \lambda k \left[\frac{1}{d-y} \right]_{-5}^{5}$

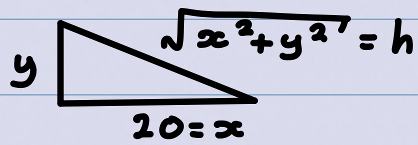
$\vec{E} = \lambda k \left[\frac{1}{d-5} - \frac{1}{d+5} \right] = \lambda k \left[\frac{1}{0.15} - \frac{1}{0.25} \right]$

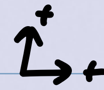
$\therefore \vec{E} = 135\ N/c!$

Example: An electric field exists near a straight, thin rod of length $L = 10cm$, with uniform $\lambda = 3\ nC/cm$. Assume that the rod is placed along the y-axis, centered at the origin. Find the electric field at a distance

20cm away from the rod on the x-axis ($x = 20$cm)







① cartesian

② $dq = \lambda dL$ ③ $dE = k\frac{dQ}{r^2}\hat{r}$

$$\therefore d\vec{E} = k\lambda \frac{1}{\sqrt{x^2+y^2}^2}\hat{r}\, dL$$

note: $\vec{E} \neq k\lambda\int\frac{1}{\sqrt{x^2+y^2}}dy\,\hat{r}$, as the direction of $\hat{r}$ keeps changing! ∴ don't pull it out!

$$\vec{E} = 2k\lambda\int_0^5 \frac{1}{x^2+y^2}\hat{r}\,dy$$

$$\vec{r} = x\hat{x} + y\hat{y} \rightarrow \hat{r} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$$

$$\therefore \vec{E} = 2k\lambda\int_0^5 \frac{1}{x^2+y^2}\cdot\frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}dy$$

$$\vec{E} = 2k\lambda\left(\int_0^5\frac{x}{(x^2+y^2)^{3/2}}dy\,\hat{x} + \int_0^5\frac{y}{(x^2+y^2)^{3/2}}dy\,\hat{y}\right)$$

goes to 0 as the y-components must cancel at point P!

$$\vec{E} = 2k\lambda\int_0^5\frac{x}{(x^2+y^2)^{3/2}}dy\,\hat{x}$$

→ trig-sub w/ $y = x\tan\theta$!

$$\therefore \vec{E} = \frac{k\lambda L}{x\sqrt{x^4(\frac{L}{2})^2}} = 65.4\ N/C\ \hat{x}$$

Example: An electric field exists near a straight, thin rod of infinite length, with uniform $\lambda = 3$ nC/cm. Assume that the rod is placed along the y-axis, centered at the origin. Find the electric field at a distance 20cm away from the rod on the x-axis ($x = 20$cm)

(

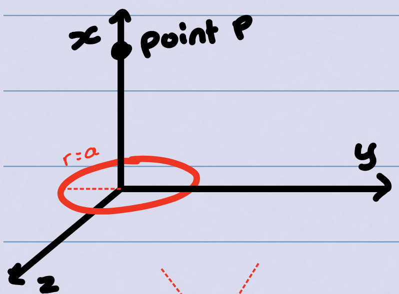
→ Same as last example, but L = infinite, not 20cm!

$$\vec{E} = \lim_{L\to\infty} \frac{k\lambda L}{x\sqrt{x^4\left(\frac{L}{2}\right)^2}}\hat{x} = \lim_{L\to\infty} \frac{k\lambda L}{x\left(\frac{L}{2}\right)\sqrt{1+\left(\frac{x}{L/2}\right)^2}}\hat{x}$$

can also sub in $k = \frac{1}{4\pi\varepsilon_0}$

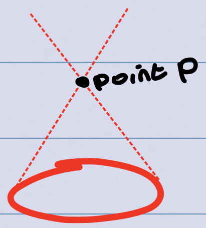
$$\rightarrow \vec{E} = \lim_{L\to\infty} \frac{2k\lambda}{x\sqrt{1+\left(\frac{x}{L/2}\right)^2}}\hat{x} \rightarrow \vec{E} = \frac{2k\lambda}{x\sqrt{1}}\hat{x} \rightarrow \vec{E} = \frac{2k\lambda}{x}\hat{x}$$

Example: A uniformly charged circular ring with linear charge density λ and radius a is in the y-z plane, centered at the origin. Find $\vec{E}$ at any point P on the x-axis.



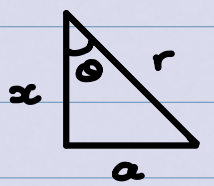
$$dQ = \lambda \underbrace{dL}_{\text{arc length}} = \lambda \underbrace{a\,d\phi}$$

$$\vec{E} = \int d\vec{E} \rightarrow d\vec{E} = \frac{k\,dq}{r^2}\hat{r} = d\vec{E}_x\hat{x} + d\vec{E}_y\hat{y} + d\vec{E}_z\hat{z}$$



→ $\vec{E}$ field cancels in the y and z-directions, so it only applies a force in the x-direction.

$$\therefore d\vec{E} = k\frac{dq}{r^2}\hat{r} = d\vec{E}_x\hat{x}\,!$$



$\rightarrow dE_x = dE\cos\theta = dE\,\frac{x}{r}$.

$\rightarrow r = \sqrt{a^2 + x^2}$

$$\hookrightarrow dE = d\vec{E}_x\hat{x} = k\frac{dq}{r^2}\hat{x}\cos\theta = k\frac{\lambda a\,d\phi}{\sqrt{a^2+x^2}^{\,2}}\frac{x}{\sqrt{a^2+x^2}}$$

$$\hookrightarrow \vec{E} = \int_0^{2\pi} \frac{xk\lambda a}{(a^2+x^2)^{3/2}}\,d\phi\,\hat{x} = \frac{xk\lambda a}{(a^2+x^2)^{3/2}}\int_0^{2\pi} d\phi\,\hat{x}$$

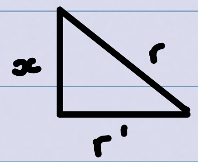
$$\therefore \vec{E} = \frac{2\pi k a\lambda x}{(a^2+x^2)^{3/2}}\hat{x}$$

Example: disk of charge (same as previous example, but disk, not ring, w/ surface density σ).

$$dq = \sigma \, dA, \quad A = \pi r^2, \quad dA = 2\pi r \, dr \rightarrow dq = 2\pi r \sigma \, dr$$

again, $\vec{E}$ will only be in x-direction!

$\hookrightarrow \therefore d\vec{E} = dE_x \hat{x} = k \dfrac{dq}{r^2} \cos\theta = k \dfrac{dq}{r^2} \dfrac{x}{r}$

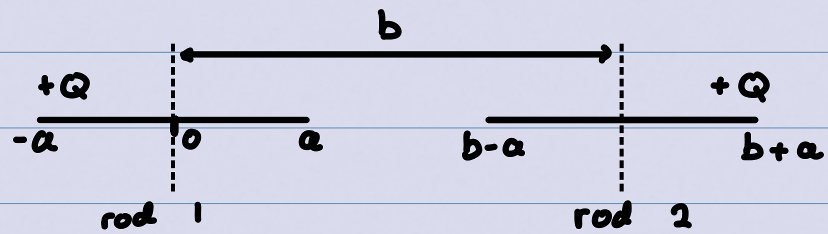


where $0 < r' < a$

$$\vec{E} = \int_0^a \frac{kx \, 2\pi\sigma r' dr'}{\left(\sqrt{x^2 + r'^2}\right)^3} \hat{x}$$

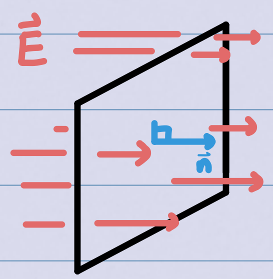
$$\hookrightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{x}{\sqrt{a^2 + x^2}}\right)\hat{x}$$

Example: find force on rod 2 from rod 1:

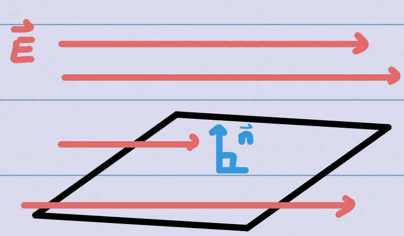


$$F_{2 \text{ on } 1} = \int_{b-a}^{b+a} \vec{E} \, dq = \int_{b-a}^{b+a} \vec{E} \, \lambda dx = \lambda k Q^2 \int_{b-a}^{b+a} \frac{1}{x^2} dx$$

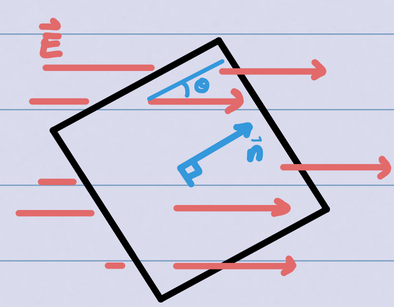
Electric Flux (ϕ): a measure of how many electric field lines pass through a given area.



$\vec{E} \parallel \hat{n}$ ($\theta = 0$)
maximum flux



$\vec{E} \perp \hat{n}$ ($\theta = 90°$)
minimum flux



$\vec{E}$ is at an angle θ to $\hat{n}$
some flux

Gauss' Law: $\phi_E = \oint \vec{E} \cdot d\vec{A}$

Since $\phi_E = \dfrac{q_{enclosed}}{\varepsilon_0}$, $\dfrac{q_{enc}}{\varepsilon_0} = \oint \vec{E} \cdot d\vec{A}$

Example: proving $E = k\frac{q}{r^2}\hat{r}$ using Gauss' Law:

$$\frac{q_{enc}}{\varepsilon_0} = \oint \vec{E} \cdot d\vec{A}$$

direction of $\vec{E}$ is $\hat{r}$! $\therefore d\vec{A} = d\vec{A}\,\hat{r}$.

since E is constant at all points on the sphere, as all points of the sphere are equidistant from the charge, we can pull $\vec{E}$ out of the integral.

$$\therefore \vec{E} \oint d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$\hookrightarrow \vec{E} A = q_{enc}(4\pi k)$$

$$\hookrightarrow \vec{E}(4\pi r^2) = q_{enc}(4\pi k)$$

$$\hookrightarrow \therefore, \vec{E} = k\frac{q_{enc}}{r^2} \; !$$

- Gauss' Law is most effective when you can pull $\vec{E}$ out of the integral.

Example: A charge Q is spread uniformly throughout a sphere of radius R. Find the electric field at all points (a) outside and (b) inside the sphere.

a) $\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$

$\vec{E} \oint d\vec{A} = \frac{Q}{\varepsilon_0}$

same as the point charge!

$\vec{E}\,(4\pi r^2) = 4\pi k Q \rightarrow \vec{E} = k\frac{Q}{r^2}$

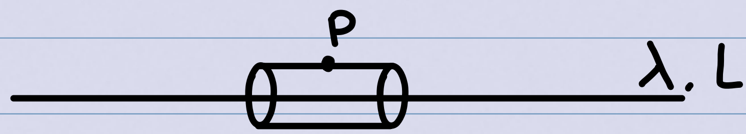
b) $\rho = \frac{Q}{V} \rightarrow \rho = \frac{Q_0}{\frac{4}{3}\pi R^3}$ $\therefore Q = \frac{Q_0}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3$

$\oint \vec{E} \cdot d\vec{A} = \vec{E} \int d\vec{A} = \vec{E}\,(4\pi r^2) = \frac{\frac{Q_0 r^3}{R^3}}{\varepsilon_0}$

$\vec{E}\,(4\pi r^2) = \frac{4\pi k Q_0 r^3}{R^3}$

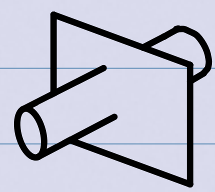
$\hookrightarrow \vec{E} = k\frac{Q_0 r}{R^3}\,\hat{r}$

For a line charge :



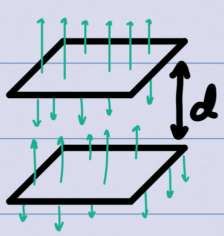
$\hookrightarrow \oint \vec{E}\, dA = \frac{Q}{\varepsilon_0} \rightarrow EA = \frac{\lambda L}{\varepsilon_0} \rightarrow E\,(2\pi r L) = \frac{\lambda L}{\varepsilon_0} \rightarrow E = \frac{\lambda}{2\pi r \varepsilon_0}$

For a plane of charge :



$\hookrightarrow \oint E\, dA = \frac{q}{\varepsilon_0} \rightarrow E\,(2A) = \frac{\sigma A}{\varepsilon_0} \rightarrow E = \frac{\sigma}{2\varepsilon_0}$

Example: find $\vec{E}$ above, between, and below two identical infinite and parallel planes of charge.



above/below : $\vec{E} = 2 \cdot \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$

between : $\vec{E}$ cancels, so $\vec{E} = 0$.

Conductors:

- Electric field inside a conductor is always 0!
  - ↳ all charges in a conductor are surface charges.
  - ↳ Charge density in a conductor is also always 0!

Work: done whenever a force acts over a certain distance.
↳ If you do positive work on a system, you lose that amount of energy and the system gains it.

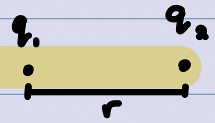
- Work is the dot product of force and displacement vectors → $W = \vec{F} \cdot \vec{r} = \int \vec{F} \cdot d\vec{r} = |\vec{F}||\vec{r}| \cos\theta$.

- $\Delta U_{AB} = U_B - U_A = -W_{AB}$

Electric Potential Energy (U): equal to the work we must do <u>against</u> $\vec{F}_E$ to move an object.

- rewriting: $\Delta U_{AB} = -W_{AB} = -\vec{F}_E \cdot \Delta\vec{r} = -q\vec{E} \cdot \Delta\vec{r} = -qE\Delta r\cos\theta$.

- $\Delta U = -W_{a \to b} = -q_0 \int_a^b \vec{E} \cdot d\vec{l}$, $\Delta U = k \frac{q_1 q_2}{r}$



Electric Potential Difference: $\Delta V = \frac{\Delta U}{q}$, $\Delta V_{AB} = -\int_A^B \vec{E} \cdot d\vec{l}$

Electric Potential: $V_p = k\frac{q}{r} = \int k \frac{dq}{r}$, $V_p = -\int_{\infty}^{P} \underbrace{\vec{E} \cdot d\vec{l}}_{\text{equivelent to } E\,dr}$

Steps to calculate V_p!

↳ Gaussian Symmetry?
↳ use Gauss' Law to find $\vec{E}$, then use $V_p = -\int_{\infty}^{r} \vec{E}\,dr$

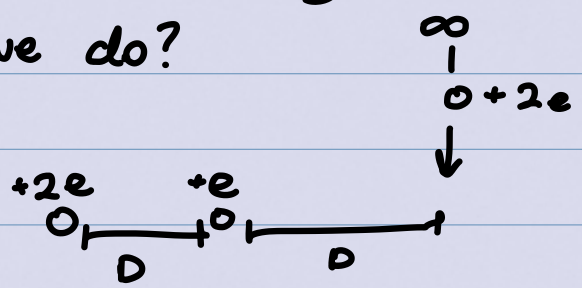
↳ No Gaussian Symmetry?

↳ Can the charge be broken up into many identical dq's?

↳ Use Gauss' Law as before!

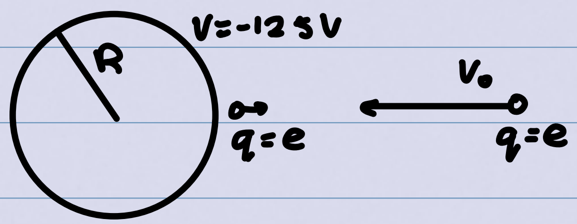
↳ If not, use superposition! Find $\vec{E} = \int \frac{k\,dq}{r^2}\hat{r}$, then $V_p = \int k\frac{dq}{r}$!

Example: we move a particle of charge $+2e$ from infinity to the x-axis. How much work do we do?



$D = 4\text{m}$

Example: A thin, spherical, conducting shell of radius R is mounted on an isolating support and charged to a potential of $-125\,V$. An electron is then fired directly toward the center of the shell, from point P at distance r from the center of the shell ($r \gg R$). What initial speed V_0 is needed for the electron to just reach the shell before reversing?



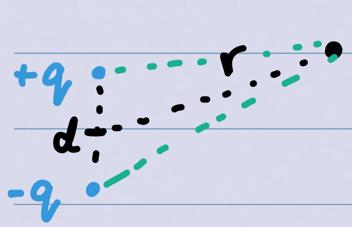
$$\Delta V = \frac{\Delta U}{q}$$

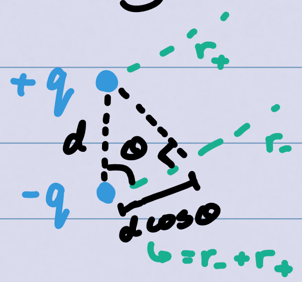
↳ $\Delta U = q\,\Delta V$

$$\frac{1}{2}mv_0^2 = q\,\Delta V$$

$\therefore V_0 = \sqrt{\frac{2q_e\Delta V}{m_e}}$ where $q_e = -1.602 \times 10^{-19}\,C$, $\Delta V = -125V$, $m_e = 9.11 \times 10^{-31}\,kg$.

Example: calculate the electric potential of a dipole anywhere in space



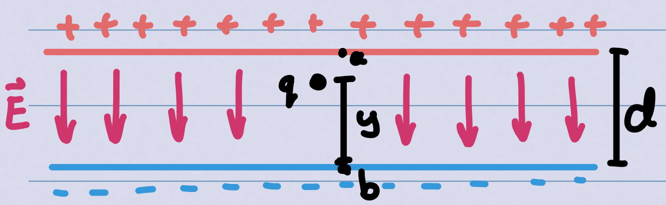


$$V_{tot} = V_+ + V_-$$
$$= k\frac{q}{r_+} + k\frac{-q}{r_-}$$
$$= kq\frac{(r_- + r_+)}{r_+ r_-}$$

since $r_+ \approx r_-$ at ∞, $r_+ r_- = r^2$.

$$\therefore V_{tot} = kq\frac{d\cos\theta}{r^2}$$

Example: Calculate the potential difference between two oppositely charged infinite sheets of charge, each with the same magnitude of charge density σ. The sheets are seperated by distance d.



$\vec{E} = \frac{\sigma}{2\varepsilon_0}$ between two sheets

$$V = -\int_0^d \vec{E} \cdot d\vec{y}$$

$$V = -\int_0^d |\vec{E}||d\vec{y}|\cos\theta$$

$\theta = 180°$

$$V = -\int_0^d \frac{\sigma}{2\varepsilon_0}\,dy\,(-1) = \int_0^d \frac{\sigma}{2\varepsilon_0}\,dy$$

Since both plates provide $\frac{\sigma}{2\varepsilon_0}$ in the same direction,

$$V = 2\int_0^d \frac{\sigma}{2\varepsilon_0}\,dy = \frac{\sigma}{\varepsilon_0}d = Ed.$$

Example: A loop of radius r has q charge

evenly spread on one quarter of the loop and $-Q$ charge evenly spread on the other three quarters of the loop. Calculate the electric potential at the center.

↳ $V_{total} = V_q + V_{-Q}$

$$V_q = k\int \frac{dq}{r} = k\int_0^{\frac{\pi}{2}} \frac{\lambda a \, d\theta}{a} = k\lambda \frac{\pi}{2} = k\left(\frac{q}{\pi r/2}\right)\frac{\pi}{2} = k\frac{q}{r}.$$

$$V_{-Q} = k\int \frac{dq}{r} = k\int_{\frac{\pi}{2}}^{2\pi} \lambda \, d\theta = k\lambda\frac{3\pi}{2} = k\left(\frac{-Q}{3\pi r/2}\right)\frac{3\pi}{2} = -k\frac{Q}{r}.$$

$$V_{total} = V_q + V_{-Q} = k\frac{q}{r} - k\frac{Q}{r}.$$

Equipotential lines: analogous to contour lines on a map

↳ the closer the equipotential lines are, the stronger the field and the higher the potential is.

- Since $\vec{E}$ and V are always perpendicular,

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

Capacitance: electric charge that must be added to the conducting body to increase it's electric potential by 1 V.

- A capacitor is formed when two conductors are seperated by an insulator or vacuum.

↳ It's a circuit element that stores electric energy.

↳ Can be filled with a dielectric.

- Each conductor has 0 net charge initially.

- Charging a capacitor means transferring electrons from one conductor to the other.

- $C = \frac{Q}{V}$ [F = Farads] $\quad C = \frac{K \varepsilon_0 A}{d}$

- General Steps to Calculate Capacitance:

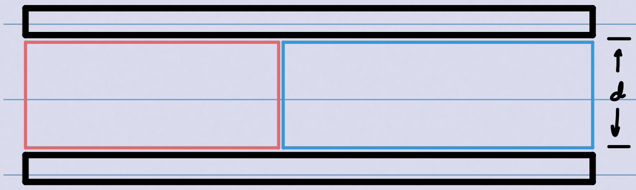
1) Pick coordinate system
2) Assume +Q / - Q charge on each conductor
3) Find $\vec{E}$ (Gauss' Law or otherwise)
   ↳ careful when there are multiple dielectrics!
4) Find $\Delta V_{2\to1}$ by going from -Q to +Q, so $C > 0$.
5) Find C using $C = \frac{Q}{\Delta V_{2\to1}}$.

- The work to charge a capacitor is equal to the total work done by the electric field on the charge when the capacitor discharges!
  ↳ we define the potential energy of an uncharged capacitor to be 0, so $W = U$!

- $U_e = \frac{1}{2} C V^2 = \frac{1}{2} Q V = \frac{Q^2}{2C}$ → potential energy stored in capacitor!
  ↳ aka, work required to charge capacitor!

In a parellel plate capacitor: $C = \frac{\varepsilon_0 A}{d}$!

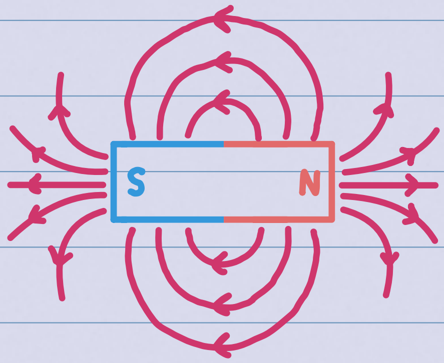
Example: Calculate the capacitance of the parellel plate capacitor below. Material 1 covers an area A_1 of the capacitor, and material 2 covers an area A_2 of the capacitor.



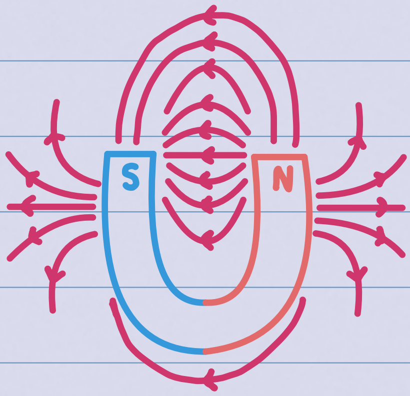
$$C_{tot} = C_1 + C_2$$
$$\frac{\varepsilon_1 A}{d} + \frac{\varepsilon_2 A_2}{d}$$

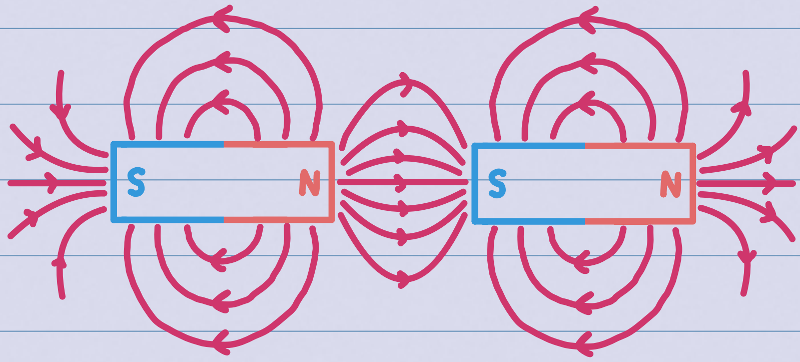
Magnetism

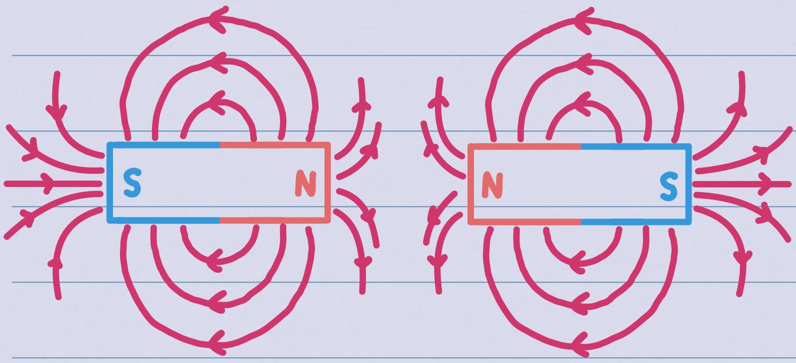
Magnetic Field Lines



- at each point, the field line is tangent to the magnetic field vector $\vec{B}$.
- Always point from N to S!
- Form loops inside the magnet!







- As with electric fields, opposite (poles, in this case) attract!
- Also, there cannot be any magnetic monopoles!

Magnetic Field ($\vec{B}$, [T, "Teslas"]).

↳ A moving charge or current creates a magnetic field in the surrounding space, in addition to the electric field!

↳ Exerts Force F_B on any moving charge or current in the $\vec{B}$ field.

$$\vec{F}_B = q\vec{v} \times \vec{B} = |q||v||B| \sin\theta \,!$$

a charge moving in parallel to a magnetic field experiences 0 magnetic force!

Gauss' Law for Magnetism: $\Phi = \oint \vec{B} \cdot d\vec{A} = 0$

Since there are no magnetic monopoles!

Example: An electron that has instantaneous velocity $\vec{v} = (2 \times 10^{-6})\hat{\imath} + (3 \times 10^{-6})\hat{\jmath}$ is moving through the uniform magnetic field defined as $\vec{B} = 0.03\hat{\imath} - 0.15\hat{\jmath}$.

a) Find the force on the electron due to $\vec{B}$.

b) What changes if it was a proton instead?

$F = q\vec{v} \times \vec{B}$
$= (-q)[(-\hat{x}+\hat{z}) \times (\hat{x}+\hat{y})]$
$= -q[0(-\hat{x}\times\hat{y}) + (\hat{z}\times\hat{x}) + (\hat{z}\times\hat{y})] = -q(-\hat{z}+\hat{y}-\hat{x}) = q(\hat{z}-\hat{y}+\hat{x})$

$\hat{x} \to \hat{y} \to \hat{z}$

$$\vec{F}_B = q\vec{V} \times \vec{B} \quad , \quad \vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} \, , \quad \vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$$

$$\vec{F}_B = q(V_x\hat{\imath} + V_y\hat{\jmath}) \times (B_x\hat{\imath} + B_y\hat{\jmath})$$

$$= q(V_xB_x\,\hat{\imath}\times\hat{\imath} + V_xB_y\,\hat{\imath}\times\hat{\jmath} + V_yB_x\,\hat{\jmath}\times\hat{\imath} + V_yB_y\,\hat{\jmath}\times\hat{\jmath})$$

$$= q(0 + V_xB_y\,\hat{\imath}\times\hat{\jmath} + V_yB_x\,\hat{\jmath}\times\hat{\imath} + 0)$$

since $\hat{\imath}\times\hat{\imath} = 0$ and $\hat{\jmath}\times\hat{\jmath} = 0$, bc $\sin 0 = 0$!

$$= q\left(V_x B_y \hat{k} + V_y B_x (-\hat{k})\right)$$

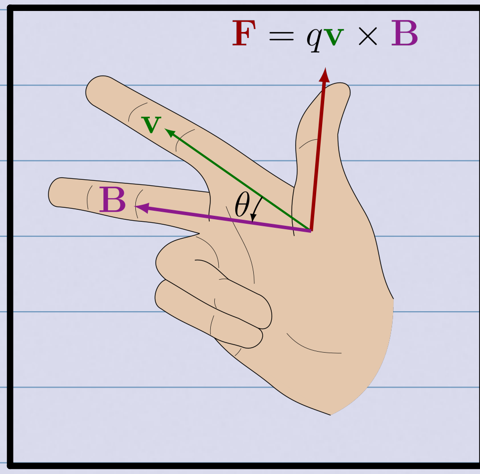
$$= q(V_x B_y - V_y B_x)\hat{k} \rightarrow \vec{F}_B = -6.25 \times 10^{-26}$$

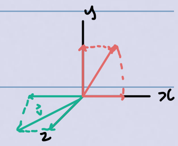
If it was a proton, q would have been positive, meaning $\vec{F}_B = 6.25 \times 10^{-26}$.

$\hat{x} \times \hat{y} = \hat{z}$
$\hat{y} \times \hat{z} = \hat{x}$
$\hat{z} \times \hat{x} = \hat{y}$

Right-Hand Rule:

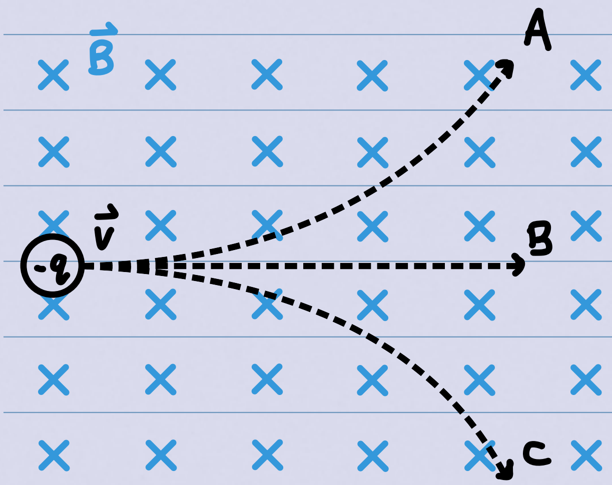
- $\vec{F}_B$ must always be perpendicular to $\vec{V}$ and $\vec{B}$!
- If q is negative, then F will be in the opposite direction!





$F(-\hat{z}) = qV(-\hat{x})\times$

Example: which path will the particle take?

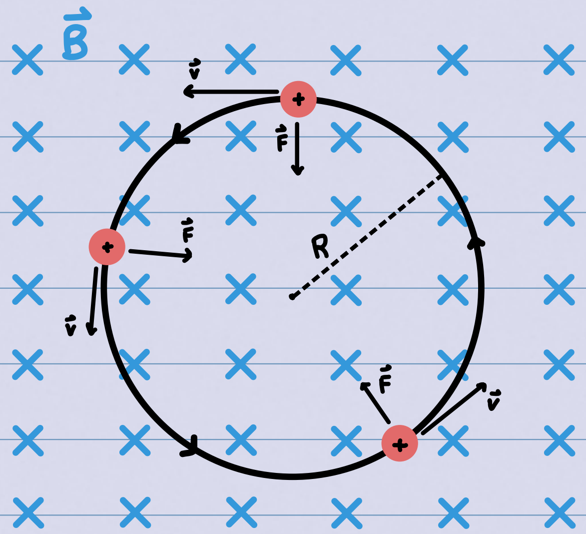


Using the right hand rule, we see that the index finger should point right, middle finger should point into the page, and thumb should point up.

However, since this is a negative charge, the direction of F will be flipped!

$\therefore$, $\vec{F}_B$ will point downwards, so the particle will take path C!

A charge moving at right angles to a uniform $\vec{B}$ field moves in a circle at a constant speed, because $\vec{F}$ and $\vec{v}$ are always perpendicular to each other!



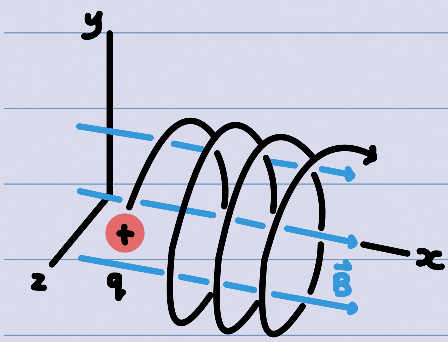
we can calculate the radius of this motion as follows:

$$\vec{F}_B = \vec{F}_c$$

$$\hookrightarrow qvB = m\frac{v^2}{R}$$

$$\hookrightarrow \therefore, \quad R = \frac{mv}{qB}$$

This particle's motion has components both parallel and perpendicular to the magnetic field, so it moves in a helical path:

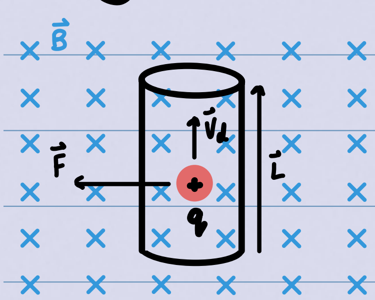


- Note, there is no work done by the $\vec{B}$ field as:

$$\vec{F}\cdot dr = q(\vec{v}\times\vec{B})\cdot dr = q(\vec{v}\times\vec{B})\cdot\vec{v}\,dt$$

since $\vec{v}\cdot\vec{v}=0$, $\vec{F}\cdot dr = 0$, so $W=0$!

Magnetic Force on Current-Carrying Wire



infinitesimal segment of wire

- $d\vec{F} = I\,d\vec{L}\times\vec{B}$

points in same direction as the current!

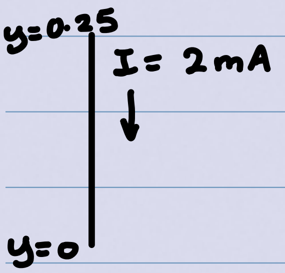
For a straight wire, $\vec{F} = I\vec{L}\times\vec{B}$

Example: A wire 50cm long carries a 0.5A current in the positive direction of an x-axis through a magnetic field $\vec{B} = (3\text{ mT})\hat{i} + (10\text{ mT})\hat{j}$. What is the magnetic force on the wire?

Since the wire is straight, $\vec{F} = I\vec{L} \times \vec{B}$

$$\vec{F} = (0.5)(0.5\hat{x}) \times (3\times10^{-3}\hat{x} + 10\times10^{-3}\hat{y})$$
$$= (0.25\cdot3\cdot10^{-3}\,\hat{x}\times\hat{x}) + (0.25\cdot10\cdot10^{-3}\,\hat{x}\times\hat{y})$$
$$= 0 + 0.25\cdot10\cdot10^{-3}\,\hat{z}$$
$$= 2.5\times10^{-2}\,\hat{z}\ \text{N}.$$

Example: A wire lying along a y-axis from $y=0$ to $y=0.25$m carries a current of 2mA in the negative y-direction. The wire lies in a nonuniform magnetic field $\vec{B} = 0.3y\,\hat{i} + 0.4y\,\hat{j}$. What is the magnetic force on the wire?



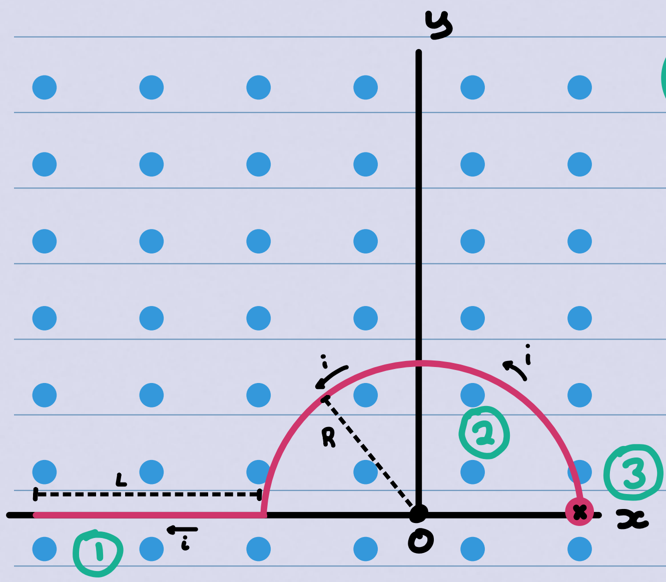
cannot use $\vec{F} = I\vec{L}\times\vec{B}$ because we have a non-uniform magnetic field!

$$d\vec{F} = I\,d\vec{L} \times \vec{B}$$
$$\hookrightarrow d\vec{F} = I(-dy\,\hat{y}) \times (B_x\hat{x} + B_y\hat{y})$$
$$\hookrightarrow d\vec{F} = I(-dy)(B_x\,\hat{y}\times\hat{x} + B_y\,\hat{y}\times\hat{y})$$

$$\therefore d\vec{F} = I(-dy)B_x(-\hat{z}) = I B_x\,dy\,\hat{z}$$

$$\therefore \vec{F} = \int_0^{0.25} (2\times10^{-3})(0.3y)\,dy\ \hat{z} = 1.875\times10^{-5}\,\hat{z}\ \text{N}.$$

Example: The magnetic field $\vec{B}$ is uniform and perpendicular to the plane of the figure, pointing out of the page. The conductor, carrying current to the left, has three segments: (1) a straight segment with length perpendicular to the plane of the figure, (2) a semicircle with radius R, and (3) another straight segment with length parallel to the plane of the figure. Find the total magnetic force on this conductor.



③: $\vec{F} = I\vec{L} \times \vec{B} = I|L||B|\sin\theta$

since $\theta = 180°$, $\vec{F} = 0$!

①: $\vec{F}$ will be upwards! Since the wire is straight, we can use $\vec{F} = I\vec{B} \times \vec{L}$!

But for ②: $|d\vec{F}| = IBdL$, so:

$dF_x = IBdL\cos\theta$, and $dF_y = IBdL\sin\theta$.

Using arc length to relate dL to $d\theta$: $dL = Rd\theta$.

↳ $dF_x = IBR\cos\theta\, d\theta$ and $dF_y = IBR\sin\theta\, d\theta$

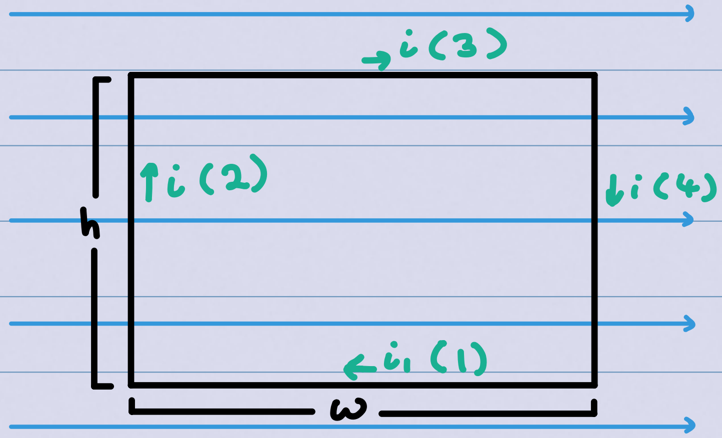
$\therefore F_x = \int_0^{\pi} IBR\cos\theta\, d\theta$, $\quad F_y = \int_0^{\pi} IBR\sin\theta\, d\theta$

↳ $F_x = 0$ $\quad$ ↳ $F_y = 2IBR$.

$F_{total} = F_1 + F_2 + F_3 \overset{\text{since } F_3=0}{=} F_1 + F_2$

$\therefore F_{total} = BIL + 2IBR = IB(L+2R)\,\hat{y}$.

Torque on a Current Loop



1 and 3 : $\sin\theta = 0$, so $\vec{F} = 0$!

2 : RHR says that $\vec{F}$ must be into the page

4 : RHR says that $\vec{F}$ must be out of the page

$\therefore$ only $\vec{F}$ is from 2 and 4

$|F_2| = |F_4|$, so there will be no net force!

However, there will be a net torque!

$\hookrightarrow \tau_2 = \vec{F} \times \vec{d} = Fd\sin\theta = Fd$

$\hookrightarrow \tau_2 = (BIh)\frac{w}{2} = \tau_4$

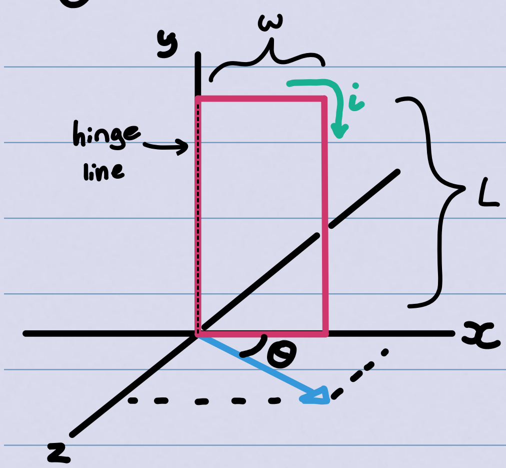
$\therefore \sum \tau = \tau_2 + \tau_4 = BIh\frac{w}{2} + BIh\frac{w}{2} = \vec{B}Ihw = \vec{B}I\vec{A}$!!

But, this assumes that the $\vec{B}$ field is perfectly parallel to the loop (ie, assuming $\vec{A}$ is perpendicular to $\vec{B}$).

To account for this, we must use another cross product! $\longrightarrow$ $\sum \tau = I\vec{B} \times \vec{A} = BIA\sin\theta$!

If we have N loops, then $\sum \tau = BIAN\sin\theta$!

Example: A rectangular 20-turn coil of wire, of dimensions 10 cm by 5 cm, carries a current of 0.1 A and is hinged along the long side on the y-axis. There is a magnetic field of magnitude 0.5 T acting at an angle $\theta = -30°$ from the x-axis. What is the torque acting on the coil about the hinge line?



- the long wire on the hinge line will not provide a torque, as it has no distance ($\tau = F \times d = F \times 0 = 0$).
- the short wires have current flowing in opposite directions, so their torques will cancel out.

Therefore, only the long wire on the right matters!

$\hookrightarrow$ $F = I\vec{L} \times \vec{B}$, where $\vec{L} = L(-\hat{y})$ and $\vec{B} = B_x\hat{x} + B_z\hat{z}$

$$\hookrightarrow F = IL(-\hat{y}) \times (B_x\hat{x} + B_z\hat{z})$$
$$= -ILB_x(\hat{y} \times \hat{x}) - ILB_z(\hat{y} \times \hat{z})$$
$$= -ILB_x(-\hat{z}) - ILB_z(\hat{x})$$
$$= ILB_x\hat{z} - ILB_z\hat{x}$$
$$\hookrightarrow \therefore, F = IL(B_x\hat{z} - B_z\hat{x})$$

however, the $\hat{x}$ component will not contribute to the torque because $\theta = 0$, so cross product $= 0$.

$$\hookrightarrow F = ILB_x\hat{z} = ILB\cos\theta\,\hat{z}$$

$\tau = \vec{r} \times \vec{F} \rightarrow |\tau| = rF\sin\theta$, where $r = w\hat{x}$.

$\rightarrow \vec{\tau} = \omega\hat{x} \times ILB\cos\theta\,\hat{z} = \omega ILB\cos\theta\,(\hat{x}\times\hat{z})$

$\vec{\tau} = \omega ILB\cos\theta\,(-\hat{y}) \rightarrow \vec{\tau} = -\omega ILB\cos\theta\,\hat{y}.$

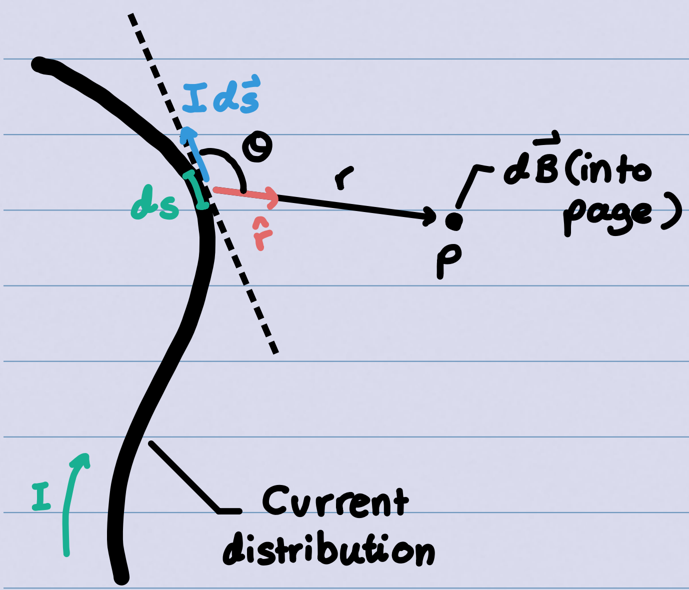
Since it's a 20-turn coil, $\vec{\tau} = -20\omega ILB\cos\theta\,\hat{y}.$

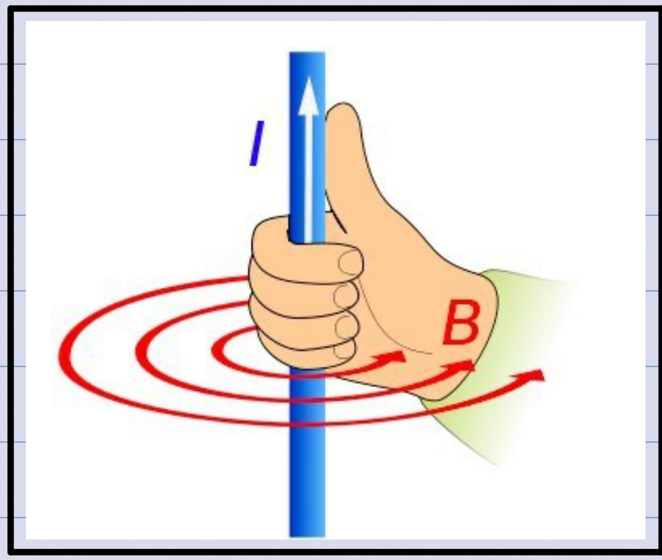
Biot-Savart Law: $d\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{I\,(d\vec{s}\times\hat{r})}{r^2}$

$d\vec{s}$ always points in the direction of current!

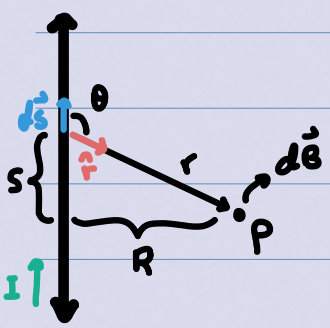
$\hat{r}$ points towards P

$\rightarrow$ where μ_0 is the permeability of free space and r is the distance from ds to P.





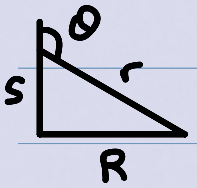
Example: Suppose an infinite wire carries a current I into the page. Calculate the magnitude of the magnetic field at a distance R away from the wire.



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{ds\times\hat{r}}{r^2}$$

since all $\vec{B}$ fields from wire point in direction of page,

$$B = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{|d\vec{s}\times\hat{r}|}{r^2} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{\sin\theta\, ds}{r^2} \qquad (|d\vec{s}| = ds)$$

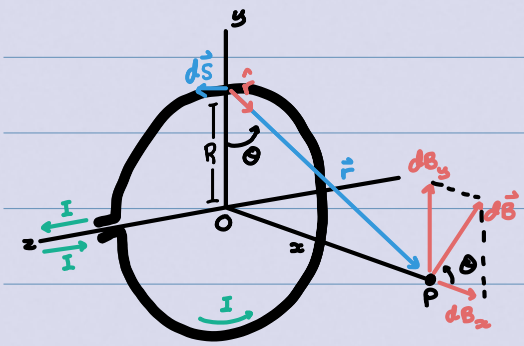


$\rightarrow r = \sqrt{s^2+R^2}$, and $\sin\theta = \sin(\pi-\theta) = \frac{R}{r} = \frac{R}{\sqrt{s^2+R^2}}$

$$\therefore B = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{R}{\sqrt{s^2+R^2}}\cdot\frac{1}{\left(\sqrt{s^2+R^2}\right)^2}\,ds = \frac{\mu_0 I R}{4\pi}\int_{-\infty}^{\infty}\frac{1}{(s^2+R^2)^{3/2}}\,ds$$

From the integral sheet, we see that $B = \frac{\mu_0 I}{2\pi R}$ into the page!

Example: Calculate the magnetic field a distance x along the axis of a circular loop of wire of radius R and carrying current I.



From symmetry, $B_y = B_z = 0$.

since ds is in y-z plane and $\hat{r}$ is in x-y plane, ds and r are always perpendicular!

$$|d\vec{s}\times\hat{r}| = ds\,|\hat{r}|\sin(90) = ds$$

$$dB_x = dB\cos\theta = \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{r^2}\,ds.$$

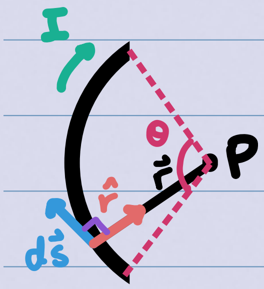
$\hookrightarrow B_x = \frac{\mu_0 I}{4\pi}\int\frac{\cos\theta}{r^2}\,ds$, where $\cos\theta = \frac{R}{\sqrt{R^2+x^2}} = \frac{R}{r}$

Since $ds = R\,d\theta$, $B_x = \frac{\mu_0 I}{4\pi}\frac{R}{\sqrt{R^2+x^2}}\frac{1}{\left(\sqrt{R^2+x^2}\right)^2}\int_0^{2\pi} R\,d\theta$

$$\therefore B_x = \frac{\mu_0 I}{4\pi}\frac{R^2}{(R^2+x^2)^{3/2}}\left[2\pi\right]$$

$$\hookrightarrow \therefore B_x = \frac{\mu_0 I R^2}{2(x^2+R^2)^{3/2}}\,\hat{x}$$

Example: find the magnetic field at point P which is a distance R away from an arc of wire carrying current I.



$$d\vec{B} = \frac{\mu_0 I}{4\pi}\,\frac{d\vec{s}\times\hat{r}}{R^2} = \frac{\mu_0 I}{4\pi}\,\frac{\overbrace{|ds|}^{|ds| = ds}\,\overbrace{|\hat{r}|}^{|\hat{r}|=1}\sin\phi}{R^2}$$

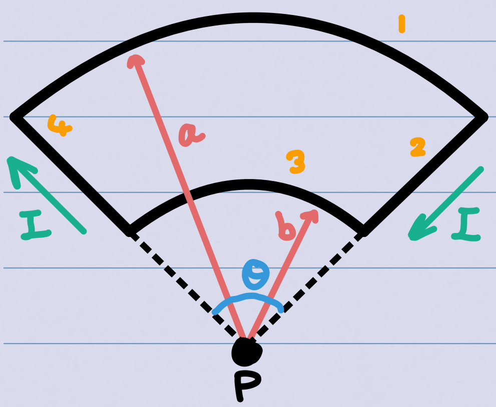
$\phi = 90°$ since $d\vec{s}$ and $\hat{r}$ are always perp.!

$$\hookrightarrow dB = \frac{\mu_0 I}{4\pi}\,\frac{ds}{R^2}, \text{ where } \overbrace{ds = R\,d\theta}^{\text{arc length}}$$

$$\therefore dB = \frac{\mu_0 I}{4\pi}\,\frac{R\,d\theta}{R^2} = \frac{\mu_0 I}{4\pi R}\,d\theta$$

$$\hookrightarrow B = \frac{\mu_0 I}{4\pi R}\int_0^{\theta} d\theta = \frac{\mu_0 I \theta}{4\pi R} \text{ into the page!}$$

Example: two circular arcs have radii $a = 13.5$ cm and $b = 10.7$ cm, subtend angle $\theta = 74°$, carry current $I = 0.411$ A, and share the same center of curvature P. Find the net magnetic field at point A.



wires 2 and 4 contribute no magnetic field to point P, as $d\vec{s}\times\hat{r} = |ds|\sin 0 = 0$!

$\therefore$, only magnetic field comes from wires 1 and 3.

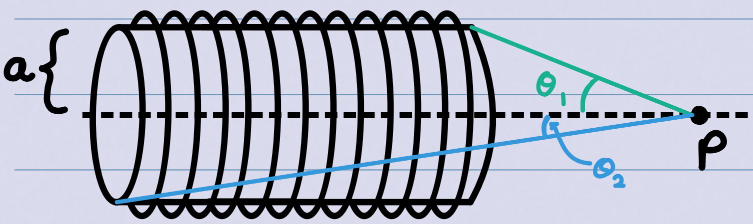
$\hookrightarrow \vec{B} = \vec{B}_1 + \vec{B}_3$, but B_1 is into the page whereas B_3 is out of the page.

$|B_1| = \frac{\mu_0 I \theta}{4\pi a}$, and $|B_3| = \frac{\mu_0 I \theta}{4\pi b}$ → proven in last example

↳ $\vec{B} = \vec{B}_3 - \vec{B}_1 = \frac{\mu_0 I \theta}{4\pi b} - \frac{\mu_0 I \theta}{4\pi a}$ → bc $a > b$, so $\frac{1}{b} > \frac{1}{a}$!

↳ $\therefore$, $\vec{B} = \frac{\mu_0 I \theta}{4\pi}\left(\frac{1}{b} - \frac{1}{a}\right)$ out of the page → $\approx 1.07 \times 10^{-7}$

Example: find the magnetic field at point P on the axis of a tightly wound solenoid consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I. Consider turns to be circular.



for a single loop a distance x away, we know that $|\vec{B}| = \frac{\mu_0 I' a^2}{2(a^2 + x^2)^{3/2}}$

↳ proved in older example!

however, in a solenoid, the current is: $I' = In\,dx$.

↳ current in a single loop!

$\therefore$, $|B_x| = \int \frac{\mu_0 I n a^2 \, dx}{2(a^2 + x^2)^{3/2}}$ → we must do a trig-sub!

let $x = a\tan\theta \rightarrow dx = a\sec^2\theta \, d\theta$

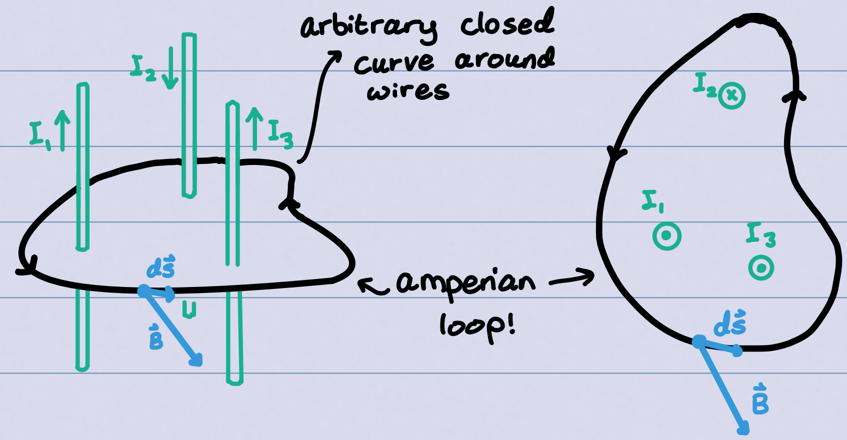
↳ $(a^2 + x^2)^{3/2} = a^3(1 + \tan^2\theta)^{3/2} = a^3 \sec^2\theta$

↳ $\int_{\theta_1}^{\theta_2} \frac{\mu_0 a^2 n I (a\sec^2\theta)}{2a^3\sec^3\theta} d\theta = \frac{\mu_0 n I}{2}\int_{\theta_1}^{\theta_2} \frac{1}{\sec\theta} d\theta$

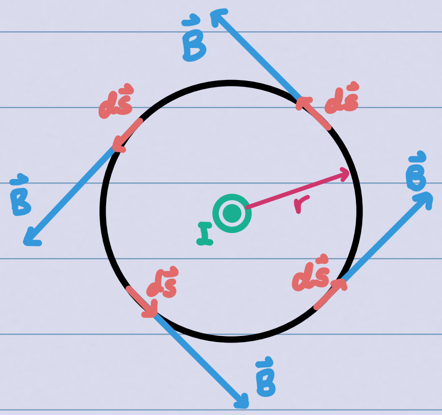
$\therefore |\vec{B}_x| = \frac{\mu_0 n I}{2}\left[\sin\theta_2 - \sin\theta_1\right]$

Ampere's Law : $\oint \vec{B} \cdot ds = \mu_0 I_{enc}$

Gauss' Law equivelant for magnetism!

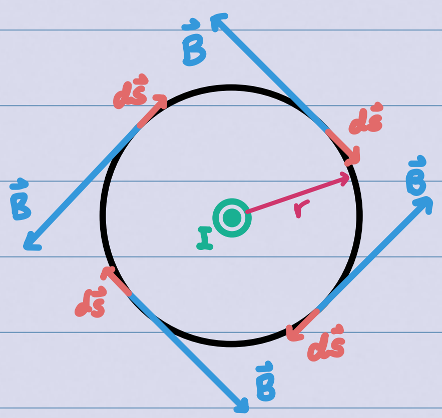


→ curl your right hand fingers around the path of integration. Your thumb points in the direction of positive current !!



→ since thumb points in the same direction as I when you curl your right hand in the direction of integration, we can simply say:

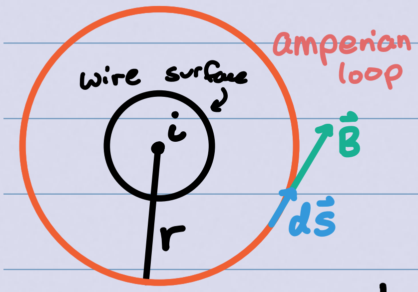
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I \, !$$



→ Since thumb points in opposite direction as I when you curl your right hand in the direction of integration, we must negate I :

$$\oint \vec{B} \cdot d\vec{s} = -\mu_0 I \, !$$

Example: Consider an infinite wire carrying current I. Using Ampere's Law, calculate the strength of the magnetic field a distance R away from the wire.



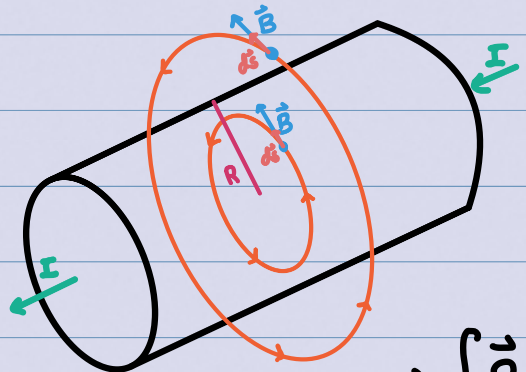
$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$$

↳ $B(s) = \mu_0 i$

↳ $B(2\pi r) = \mu_0 i$

↳ $\therefore$, $|B| = \dfrac{\mu_0 i}{2\pi r}$ → same result, but much faster than Biot-Savart!

Example: a cylindrical conductor with radius R carries a current I. The current is uniformly distributed over the cross-sectional area of the conductor. Find the magnetic field as a function of the distance r from the conductor axis for both $r < R$ and $r > R$. The current density is $J = ar$.



for $r > R$, our amperian loop will be enclosing all of the current, $I_{enc} = I$.

$$\rightarrow \oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc} \rightarrow B\oint d\vec{s} = \mu_0 I \rightarrow B(2\pi r) = \mu_0 I.$$

$\therefore$, for $r > R$: $B = \dfrac{\mu_0 I}{2\pi r}$ → note: the conductor acts exactly like a wire for $r > R$!!

for $r < R$: our amperian loop will not enclose all of the current. $\therefore$, we must solve for I_{enc}:

$$J = \frac{I_{enc}}{A} \rightarrow I_{enc} = JA \rightarrow I_{enc} = \int_0^r J\, dA. \rightarrow I_{enc} = a\int_0^R r\, dA$$

$$A = \pi r^2 \rightarrow dA = 2\pi r\, dr.$$

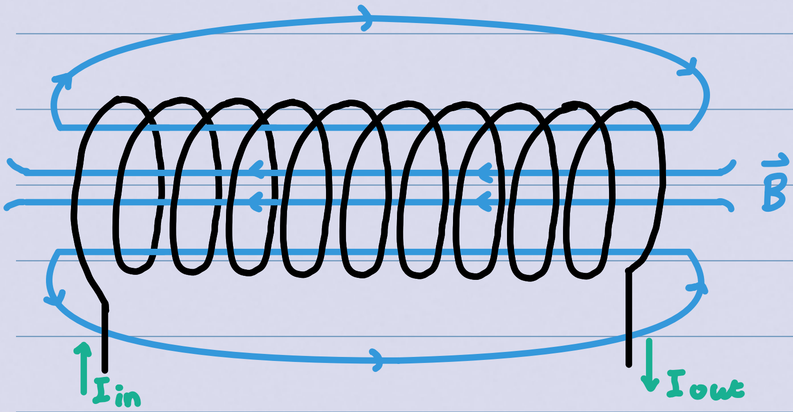
$\hookrightarrow I_{enc} = a\int_0^r r(2\pi r)\,dr = 2\pi a\int_0^r r^2\,dr = 2\pi a\left[\frac{r^3}{3}\right]_0^r$

$\therefore I_{enc} = \frac{2\pi a r^3}{3}$

$\hookrightarrow \oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc} \rightarrow B\oint ds = \frac{2}{3}\mu_0 \pi a r^3$

$\hookrightarrow B(2\pi r) = \frac{2}{3}\mu_0\pi a r^3 \rightarrow B = \frac{1}{3}\mu_0 a r^2 \quad (r < R)$

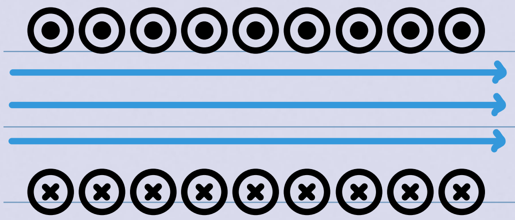
Solenoids: a type of electromagnet formed by a helical coil of wire whose length is significantly greater than diameter.



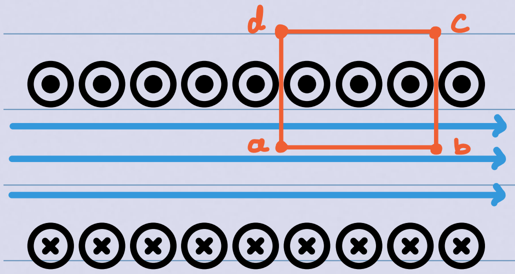
→ the $\vec{B}$ fields above and below the solenoid are very weak, so we assume they are 0!

→ wrap your fingers (right hand) in the direction of current flow in the loops. Your thumb points in the direction of the magnetic field inside the solenoid!

Example: Calculate the magnetic field inside a very long solenoid consisting of N turns, length L, and carrying current I.



To use Ampere's law, we must decide on an appropriate Amperian Loop.



↳ orange rectangle is amperian loop!!

$$\oint \vec{B}\cdot d\vec{s} = \oint_a^b \vec{B}_1 \cdot d\vec{s} + \int_b^c \vec{B}\cdot d\vec{s} + \oint_c^d \vec{B}\cdot d\vec{s} + \oint_d^a \vec{B}\cdot d\vec{s}$$

we see that $\oint_b^c \vec{B}\cdot d\vec{s} = \oint_d^a \vec{B}\cdot d\vec{s} = 0$, as $d\vec{s}$ is perpendicular to $\vec{B}$. Therefore, $\vec{B}\cdot d\vec{s} = |B||ds|\cos\theta = |B||ds|\cos 90 = 0$!

since the $\vec{B}$ field outside of a solenoid is approximated to 0, $\oint_c^d \vec{B}\cdot d\vec{s}$ is also 0:

$\therefore$, only $\oint_a^b \vec{B}\cdot d\vec{s}$ contributes to $\oint \vec{B}\cdot ds$.

$$\hookrightarrow \oint \vec{B}\cdot d\vec{s} = \oint_a^b \vec{B}\cdot d\vec{s} = B[s]_a^b = B(b-a).$$

$$I_{enc} = I\,\underbrace{\frac{N}{L}(b-a)}$$ expression for number of turns in the b-a length

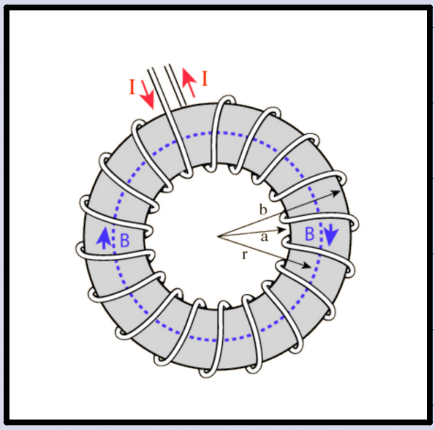
$$\therefore, \oint B\cdot d\vec{s} = B(b-a) = \mu_0 I \frac{N}{L}(b-a)$$

$$\hookrightarrow B = \mu_0 I \frac{N}{L} = \mu_0 I n \;!$$

the magnetic field inside a solenoid is:
$|\vec{B}| = nI\mu_0 = \frac{N}{L} I \mu_0$!

Toroids: a hollow circular ring that has many coils of wire around it that are wound so close to each other such that there is negligible space between turns of wire.

↳ AKA, a solenoid that's bent into a donut shape!



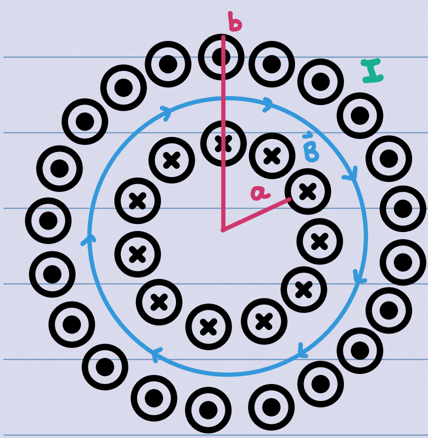
→ there is no magnetic field outside of a toroid!

→ the magnetic field within the toroid is given by the same equation as a solenoid!

$n = \frac{N}{L} = \frac{N}{2\pi R}$

↳ $|B_{toroid}| = \frac{N}{L} I \mu_0 = \frac{N I \mu_0}{2\pi R}$!

Example: you have a toroidial solenoid consisting of N turns of wire carrying current I. The inner radius of the toroid is a, while the outer radius of the toroid is b. Calculate the magnetic field everywhere.



an amperian loop with $r < a$ would enclose no charge, so $I_{enc} = 0$, so $B = 0$!

an amperian loop with $r > b$ would enclose equal positive and negative charge. ⟶ $I_{enc} = 0$!

∴, the only $\vec{B}$ is when $a < r < b$!

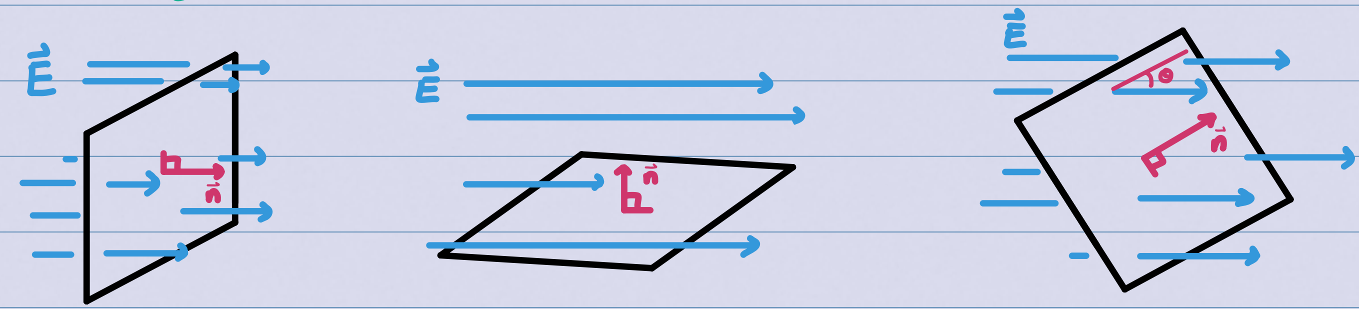
↳ $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \rightarrow B(2\pi r) = \mu_0 I N \rightarrow B = \frac{\mu_0 I N}{2\pi r}$!

↳ $r < a: B = 0, \quad a < r < b: \quad B = \frac{\mu_0 I N}{2\pi r}, \quad r > b: B = 0.$

Induction and Inductance: the tendency of an electrical conductor to oppose a change in the current flowing through it.

- $\phi_B = \int \vec{B} \cdot d\vec{A} \quad [Wb]$ → note: $\oint \vec{B} \cdot d\vec{A} = 0 \neq \int \vec{B} \cdot d\vec{A}$!

  ↳ magnetic flux



$\phi_B = BA \qquad \phi_B = 0 \qquad \phi_B = BA\cos\theta$

Electromotive Force (EMF): the amount of energy transferred to system (per unit charge)

* EMF is not actually a force!!!

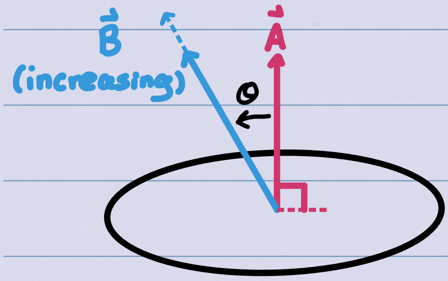
$$\varepsilon = -\frac{d\phi_B}{dt} = -N\frac{d\phi_B}{dt} \quad [V]$$

→ for N turns

negative, as it's opposing a change in flux ϕ_B.

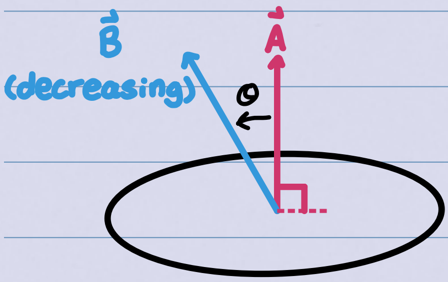
finding the direction of EMF:

1) choose direction of $\vec{A}$
2) determine sign of changing flux → ie, $\frac{d\phi_B}{dt} > 0$ or < 0?
3) determine sign of EMF
4) put thumb (right hand) along $\vec{A}$. If:
   EMF > 0: fingers curl in same direction as EMF
   EMF < 0: fingers curl in opposite direction as EMF



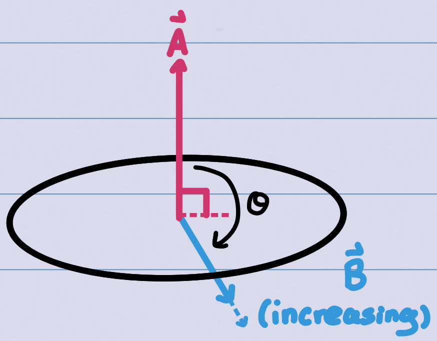
→ ϕ_B is positive and increasing as $\vec{B}$ is increasing. ∴, $\frac{d\phi_B}{dt} > 0$.
∴, EMF is negative!

↳ fingers curl in opposite direction to EMF, so induced EMF is clockwise!



→ ϕ_B is positive but decreasing as $\vec{B}$ decreasing. ∴ $\frac{d\phi_B}{dt} < 0$.
∴, EMF is positive!

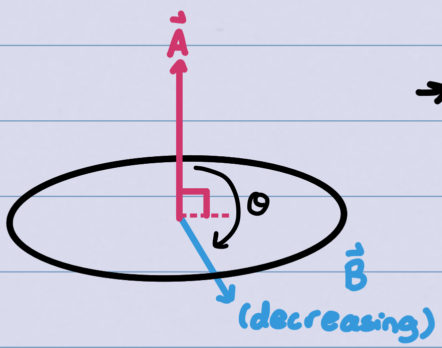
↳ fingers curl in same direction as EMF, so induced EMF is counterclockwise!



ie, becoming more negative.

→ ϕ_B is negative and decreasing as $\vec{B}$ is increasing. ∴ $\frac{d\phi_B}{dt} < 0$.
∴, EMF is positive!

↳ fingers curl in same direction as EMF, so induced EMF is counterclockwise!

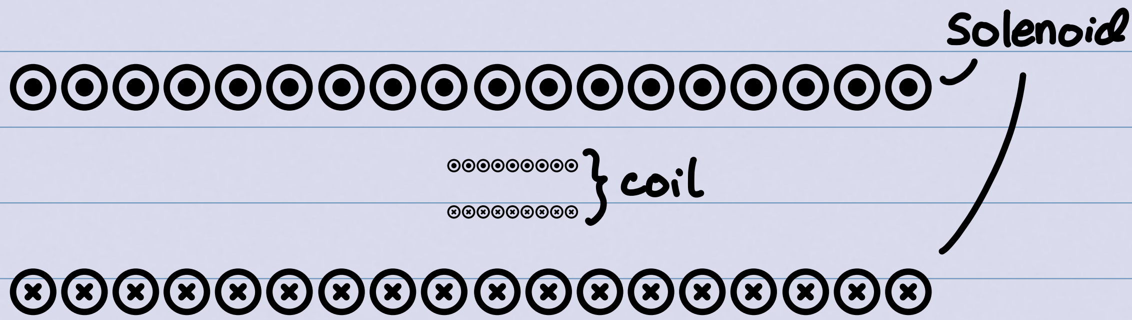


ie, becoming less negative

→ ϕ_B is negative but increasing as $\vec{B}$ is decreasing. ∴ $\frac{d\phi_B}{dt} > 0$
∴, EMF is negative!

↳ fingers curl in opposite direction as EMF, so induced EMF is clockwise!

Example: a long solenoid has 220 turns/cm and carries a current $I = 1.5\,A$ with a diameter $D_1 = 3.2\,cm$. At its center, we place a 130-turn closely packed coil of diameter $D_2 = 2.1\,cm$. The current in the solenoid is reduced to 0 at a steady rate in 25 milliseconds. What is the magnitude of the EMF induced in the coil while the current in the solenoid is changing?



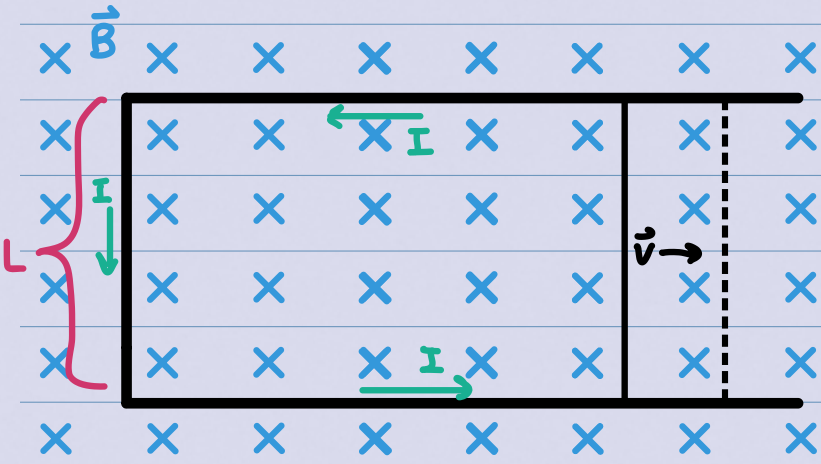
we know that the magnetic field due to a solenoid is $B = \mu_0 n i$.

$$\phi_B = \int \vec{B} \cdot d\vec{A} = \vec{B} \int d\vec{A} = \vec{B}\vec{A} = \mu_0 n I A = \mu_0 n I (\pi D_2^2)$$

$$\mathcal{E} = -N \frac{d\phi_B}{dt} \rightarrow |\mathcal{E}| = N \frac{d\phi_B}{dt} = 130 \left| \frac{0 - \mu_0 n I \pi D_2^2}{25 \cdot 10^{-3}} \right|$$

$$\hookrightarrow |\mathcal{E}| = 5200\, \mu_0 n I \pi D_2^2 .$$

Example: there is a U-shaped conductor in a uniform magnetic field perpendicular to the plane of the figure and directed into the page. We lay a metal rod with length L across the two arms of the conductor, forming a circuit, and move it to the right with constant velocity $\vec{v}$. Find the resulting induced EMF.



we see that $|\phi_B|$ is increasing, as $|\phi_B| = BA$, and A is increasing. Let $\vec{A}$ point into the page, so ϕ_B is positive and increasing.

$\phi_B = \int \vec{B} \cdot d\vec{A} = BA(t)$. → where $A(t)$ is increasing!

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -B\frac{dA(t)}{dt} \qquad A(t) = L \cdot w(t)$$

→ Area = length · width (t)

$$\hookrightarrow \mathcal{E} = -BL\frac{dw}{dt} = -BLv.$$

Since $\mathcal{E}$ is negative, fingers curl in the opposite direction to the induced EMF.

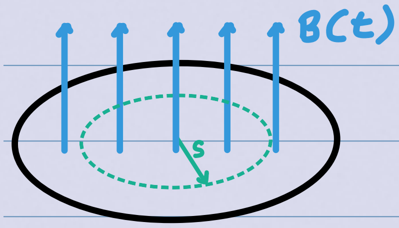
∴, the induced EMF is BLv in the same direction as I is flowing (counterclockwise).

Faraday's Law: $\oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A} = \mathcal{E}$

↳ "A changing magnetic flux induces an electric field/EMF"

→ AKA, $\oint \vec{E} \cdot d\vec{l} = \mathcal{E} = -\frac{d}{dt}(\phi_B) = -\frac{d}{dt}\left(\int \vec{B}\, d\vec{A} \cos\theta\right)$

Example: A uniform magnetic field $B(t)$, pointing straight up, fills a circular region. If B is changing with time, what is the induced electric field?



$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\phi_m}{dt}, \quad \phi_m = \int \vec{B}\cdot d\vec{A}$$

$$\downarrow \qquad\qquad \downarrow$$

$$E(2\pi s) = -\frac{d\phi_m}{dt}, \quad \phi_m = B(t)\pi s^2$$

$$\hookrightarrow \; 2\pi s E = -\frac{dB}{dt}\pi s^2 \rightarrow |E| = -\frac{s}{2}\frac{dB}{dt} \quad (s < R)$$

Self-Inductance: if the current I in the coil is changing, the changing flux through the coil induces an emf in the coil.

$$\hookrightarrow \; L = \frac{N\phi_B}{I}$$

ϕ_B → single loop flux

→ $N = 1$ for single loop

Example: A long solenoid of cross section A carries a constant current I through n turns per length. Calculate the self-inductance per unit length through the center of the solenoid. Now, suppose I begins to change with time (ie, $I = I(t)$). Calculate it's self-induced EMF.

$$\phi_B = \int \vec{B}\cdot d\vec{A} = BA = \mu_0 I n A.$$

l → length, L → inductance

$$N = nl, \quad L = \frac{N\phi_B}{I} \rightarrow L = \frac{nl\phi_B}{I} = \frac{nl(\mu_0 I n A)}{I} = \mu_0 n^2 l A$$

$$\therefore, \quad \frac{L}{l} = \frac{\mu_0 n^2 l A}{l} = \mu_0 n^2 A$$

Now, let's suppose $I = I(t)$ changes with time. Calculate it's self-induced EMF.

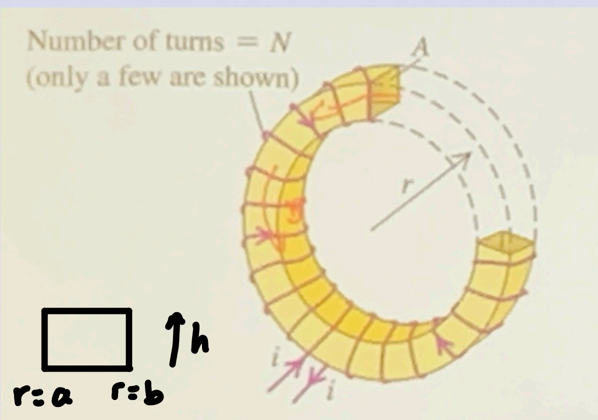
flux for one loop

$$\mathcal{E} = -\frac{d\phi_m^{tot}}{dt} = -N\frac{d\phi_m}{dt}, \quad \text{and} \quad L = \frac{N\phi_B}{I} \rightarrow LI = N\phi_B$$

$$LI = N\phi_B \rightarrow L\frac{dI}{dt} = N\frac{d\phi_B}{dt}.$$

$$\therefore, \quad \mathcal{E} = -N\frac{d\phi_m}{dt} = -L\frac{dI}{dt}$$

Example: determine the self-inductance of a toroidial solenoid with cross-section A closely wound with N turns of wire on a non-magnetic core.



$r=a$ $r=b$ h

$A = (b-a)h$
$= rh$

$$L = \frac{N\phi}{I}, \quad B_{toroid} = \frac{\mu_0 NI}{2\pi r}$$

$$\phi = \int \vec{B}\cdot d\vec{A}$$

$$= \int B\cdot(2h\,dr) = 2h\int_a^b \frac{\mu_0 NI}{2\pi r}\,dr$$

$$\hookrightarrow \frac{\mu_0 h N I}{\pi}\int_a^b \frac{1}{r}\,dr = \frac{\mu_0 h N I \ln\left(\frac{b}{a}\right)}{\pi}$$

Mutual Inductance:

$$\mathcal{E}_1 = -M\frac{dI_2}{dt}$$

$$M = \frac{N_2\phi_2}{I_1} = \frac{N_1\phi_1}{I_2}, \quad \mathcal{E}_2 = -M\frac{dI_1}{dt}$$

lowkey no time for this, just hoping it doesn't come ♡