

eg:  $\{a, b, c, b\}$  is not a set!

Sets: an unordered collection of distinct objects.

↳  $\{a, b, c\} = \{c, b, a\}$  is a set of size 3

↳  $\{0, 1, 2, \dots\} = \mathbb{N}$  is an infinite set

Set operations: let  $A, B$  be finite sets:

• Cartesian Product:  $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

• Size of Cartesian Product:  $|A \times B| = |A| \cdot |B|$

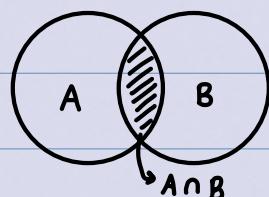
↳ eg: how many binary strings have length 4?

$$\underline{0/1} \quad \underline{0/1} \quad \underline{0/1} \quad \underline{0/1} \rightarrow 2 \times 2 \times 2 \times 2 = 2^4 = 16.$$

$$\# \text{ strings} = |\{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\}| = |\{0, 1\}| \times |\{0, 1\}| \times |\{0, 1\}| \times |\{0, 1\}| = 2 \times 2 \times 2 \times 2 = 16.$$

In general, the number of binary strings of length  $n$  is  $2^n$ .

• Union:  $A \cup B = \{x : x \in A \text{ or } x \in B\}$ .



• Size of Union:  $|A \cup B| = |A| + |B| - |A \cap B|$

↳ if  $|A \cap B| = 0$ , then  $|A \cup B| = |A| + |B|$ .

↳  $A \cup B$  with  $|A \cap B| = 0$  is a disjoint union.

↳ eg: how many binary strings of length 8 are there which begin with 001 or 1011?

let  $S$  be the set of all strings.

let  $A_1$  = set of length-8 binary strings that begin with 001.

let  $A_2$  = set of length-8 binary strings that begin with 1011.

then  $S = A_1 \cup A_2$  and  $A_1 \cap A_2 = \emptyset$ . So,  $|S| = |A_1| + |A_2| = 2^5 + 2^4 = 48$ .

## Counting Subsets

Permutations: a permutation of a set  $S$  is an ordered listing of the elements in  $S$ .

↳ eg: the permutations of  $\{a, b, c\}$  are abc, acb, bac, bca, cab, cba.

• the number of permutations of a set  $S$  of size  $n$  is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{2 \text{ choices}} \times \underline{1 \text{ choice}} = n!$$

Partial Permutations: a permutation of a subset of  $S$ .

• number of partial permutations of  $S$  (with  $|S|=n$ ) of size  $k$  is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{n-k+2 \text{ choices}} \times \underline{n-k+1 \text{ choices}}$$

$$\therefore n(n-1)(n-2)\dots(n-k+2)(n-k+1).$$

↳ note: this works even if  $k > n$ .

How many subsets of  $S$  (with  $|S|=n$ ) are there of size  $k$ ?

- number of partial permutations of  $S$  of size  $k$  is  $n(n-1)(n-2)\dots(n-k+1)$ .
- each size- $k$  subset of  $S$  has  $k!$  permutations.

$$\hookrightarrow \therefore \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

• Combinations: let  $0 \leq k \leq n$ . Then "n choose  $k$ " is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (\text{aka, binomial coefficient})$$

↳ note, if  $k > 1$ , then  $\binom{n}{k} = 0$ .

$$\text{eg: } \binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = \frac{1}{0!} = 1.$$

$$\text{eg: } \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{1}{0!} = 1.$$

$$\text{eg: } \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$$

eg:  $\binom{1999}{73} = \frac{1999 \times 1998 \times 1997 \times 1927}{73 \times 72 \times \dots \times 1} = \text{some positive integer.}$

**Binomial Theorem:** for any  $n \geq 1$ ,  $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k$

eg:  $(x+1)^3 = (x+1)(x+1)(x+1) = x^3 + 3x^2 + 3x + 1.$

$$\begin{array}{ccccccc} & \overset{\binom{3}{3}=1}{\nearrow} & \overset{\binom{3}{2}}{\nearrow} & \overset{\binom{3}{1}}{\nearrow} & \overset{\binom{3}{0}=1}{\nearrow} \\ & & & & & & \end{array}$$

## Combinatorial Proof of the Binomial Theorem:

$$(x+1)^n = (x+1)(x+1) \dots \overset{n \text{ copies of } (x+1)}{(x+1)}$$

how do we get an  $x^k$  term? by selecting  $x$  from  $k$  binomials and  $1$  from the other  $n-k$  binomials.  $\therefore$  the total number of ways is  $\binom{n}{k}$ .

Therefore,  $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k$ .  $\square$ .

## Combinatorial Proofs

- Procedure to prove  $M=N$ :

- 1) "Cleverly" choose a set  $S$

- 2) Count the number of elements in  $S$  in two ways:

- a)  $|S|=M$  and

- b)  $|S|=N$

- 3) Conclude that  $M=N$ .

Claim:  $2^n = \sum_{k=0}^n \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

↳ note: algebraically, we can prove this by substituting  $x=1$  into the binomial theorem.

Combinatorial proof:

- let  $S$  be the set of all subsets of  $\{1, 2, \dots, n\}$ .

- a) let  $S_i$  be the subsets in  $S$  of size  $i$ , where  $0 \leq i \leq n$ .

Then  $S = S_0 \cup S_1 \cup \dots \cup S_n$  (note, a disjoint union).

$$\therefore |S| = |S_0| + |S_1| + |S_2| + \dots + |S_n|$$

$$|S| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{k=0}^n \binom{n}{k}.$$

b) to choose a subset  $X$  of  $\{1, 2, \dots, n\}$ , we do:

- for each  $i$ ,  $1 \leq i \leq n$ , we include  $i$  in  $X$ , or not.

- Therefore, the total number of choices is  $2^n$ . Hence  $|S| = 2^n$ .

$$\therefore |S| = \sum_{k=0}^n \binom{n}{k} = 2^n. \quad \square.$$

Claim: let  $1 \leq k \leq n$ . Then  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$  [Pascal's Identity].

• Algebraic Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} + \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} = \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} + \frac{(n-1)! \cdot k}{k!(n-k)!}$$

$$= \frac{(n-1)![(n-k)+k]}{k!(n-k)!} = \frac{(n-1)! \cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \quad \square.$$

• Combinatorial Proof:

let  $S$  = set of all size- $k$  subsets of  $\{1, 2, 3, \dots, n\}$ .

then,  $|S| = \binom{n}{k}$  by definition.

let  $A$  = size- $k$  subsets that don't include  $n$ .

let  $B$  = size- $k$  subsets that includes  $n$ .

Then  $S = A \cup B$  is a disjoint union. So,  $|S| = |A| + |B|$ .

Now,  $|A| = \binom{n-1}{k}$ , and  $|B| = \binom{n-1}{k-1}$ . Thus,  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ .  $\square$ .

• Claim: for  $k, n \geq 0$ ,  $\binom{n+k}{n} = \sum_{i=0}^k \binom{n+i-1}{n-1} = \binom{n-1}{n-1} + \binom{n}{n-1} + \dots + \binom{n+k-1}{n-1}$

Combinatorial Proof:

$$\therefore |S| = \binom{n+k}{n}$$

• let  $S$  = the size- $n$  subsets of  $\{1, 2, \dots, n, n+1, \dots, n+k-1, n+k\}$ .

↳ note: the largest number in a size- $n$  subset is between  $n$  and  $n+k$ .

for each  $0 \leq i \leq k$ , let  $S_i$  = subset whose largest element is  $n+i$ .

Then,  $S$  = disjoint union  $S_0 \cup S_1 \cup \dots \cup S_k$ . So,  $|S| = \sum_{i=0}^k |S_i| = \sum_{i=0}^k \binom{n+i-1}{n-1}$ .  $\square$ .

Visually:

See that the LHS is size- $n$  subsets of  $n+k$  sets. So, we choose subsets of size  $n$  ( $k$  times):

Subsets:  $\{\underbrace{\quad}_{n \text{ size}}\} \cup \{\underbrace{\quad}_n\} \cup \dots \cup \{\underbrace{\quad}_n\} \cup \{\underbrace{\quad}_n\}$

$\cdot \{\underbrace{\quad}_n\} \cup \{\underbrace{\quad}_n\} \cup \dots \cup \{\underbrace{\quad}_n\} \cup \{\underbrace{\dots, n+k}_n\}$

↳ see that the last subset can be counted as  $\binom{n+k-1}{n-1}$ .

Claim: for  $0 \leq k \leq n$ ,  $\binom{n}{k} = \binom{n}{n-k}$

↳ eg:  $\binom{100}{98} = \binom{100}{2} = \frac{100 \times 99}{2}$ .

eg:  $n=4, k=1$  : size-1 subsets of  $\{1, 2, 3, 4\} = \{1\}, \{2\}, \{3\}, \{4\}$ .

size-3 subsets of  $\{1, 2, 3, 4\} = \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}$ .

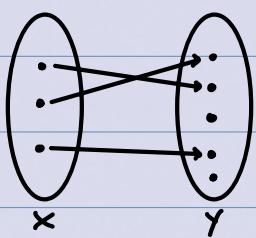
we can pair these sets by picking the size-3 subset which does not contain the element in the size-1 subset. ie:  $\{1\} \leftrightarrow \{2, 3, 4\}, \{2\} \leftrightarrow \{1, 3, 4\} \dots$

→ aka, a correspondence or a bijection!

Bijections: let  $f: X \rightarrow Y$  be a function.

•  $f$  is injective (1-1) if  $\forall x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ .

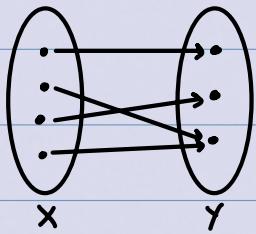
• Visually:



. see that  $|X| \leq |Y|$ .

•  $f$  is surjective (onto) if  $\forall y \in Y, \exists x \in X \text{ st } f(x) = y$ .

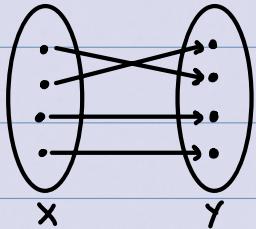
• Visually:



. see that  $|X| \geq |Y|$

•  $f$  is bijective if it is injective AND surjective.

• Visually:

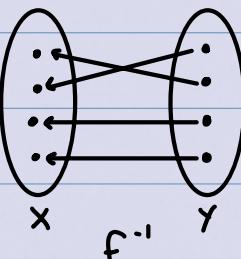
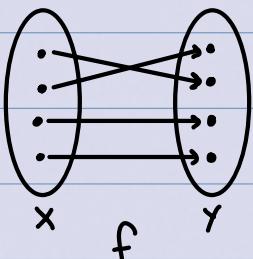


. see that  $|X| = |Y|$ .

• Theorem: let  $X, Y$  be finite sets. Suppose  $f: X \rightarrow Y$  is a bijection. Then,  $|X| = |Y|$ .

• Inverse function: the inverse of a function  $f: X \rightarrow Y$  is a function  $f^{-1}: Y \rightarrow X$  such that:

- 1) ( $f^{-1}$  reverses  $f$ )  $\Rightarrow \forall x \in X, f^{-1}(f(x)) = x$ .
- 2) ( $f$  reverses  $f^{-1}$ )  $\Rightarrow \forall y \in Y, f(f^{-1}(y)) = y$ .



• Theorem: let  $f: X \rightarrow Y$ . Then  $f$  is a bijection if and only if  $f$  has an inverse.

### Combinatorial Proofs Using Bijections that $M=N$ .

1) Select two sets  $X, Y$ , with  $|X|=M$  and  $|Y|=N$ .

2) Define  $f: X \rightarrow Y$ . ( $f$  is a bijection).

3) Define  $f^{-1}: Y \rightarrow X$ .

4) Prove that  $f^{-1}$  is the inverse function

↳ prove the two conditions of the inverse function definition are satisfied.

5) Conclude that  $M=N$ .

• A function  $f: X \rightarrow Y$  is well-defined if  $\forall x \in X, f(x) \in Y$ .

Claim: for  $0 \leq k \leq n$ ,  $\binom{n}{k} = \binom{n}{n-k}$  (continued)

let  $X$  be the set of all size- $k$  subsets of  $\{1, 2, \dots, n\}$ .

$$\therefore |X| = \binom{n}{k}.$$

let  $Y$  be the set of all size- $(n-k)$  subsets of  $\{1, 2, \dots, n\}$ .

$$\therefore |Y| = \binom{n}{n-k}.$$

Define  $f: X \rightarrow Y$  as follows:  $\forall S \in X, f(S) = S^c$  ( $S^c = \{1, 2, \dots, n\} - S$ ).

• See that  $f$  is well-defined, ie,  $f(S) = S^c \in Y$  since  $|S^c| = n-k$

Define  $f^{-1}: Y \rightarrow X$  as follows:  $\forall T \in Y, f^{-1}(T) = T^c$  ( $T^c = \{1, 2, \dots, n\} - T$ ).

• See that  $f^{-1}$  is well-defined, ie,  $f^{-1}(T) = T^c \in X$  since  $|T^c| = k$ .

now,  $\forall S \in X, f^{-1}(f(S)) = f^{-1}(S^c) = (S^c)^c = S$ .

$$\forall T \in Y, f(f^{-1}(T)) = f(T^c) = (T^c)^c = T.$$

$\therefore f$  is a bijection, and therefore  $|X|=|Y|$ , so  $\binom{n}{k} = \binom{n}{n-k}$ .

## Generating Series

↳ we'll encode solutions to counting problems as coefficients of a "generating series".

• eg: how many subsets of  $\{1, 2, 3\}$  have size  $n$ ,  $\forall n \ 0 \leq n \leq 3$ ?

let  $S = \text{all size-}n \text{ subsets of } \{1, 2, 3\} \ \forall n \ 0 \leq n \leq 3$ .

define weight function  $w: S \rightarrow \mathbb{N}$ , by  $w(\sigma) = |\sigma| \ \forall \sigma \in S$ .

$\sigma \in S$	$\{\emptyset\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$w(\sigma)$	0	1	1	1	2	2	2	3

associate each element  $\sigma \in S$  with the term  $x^{w(\sigma)}$ :

$$x^{w(\sigma)} \quad | \quad x \quad x \quad x \quad x^2 \quad x^2 \quad x^2 \quad x^3$$

the generating series of  $X$  with respect to  $w$  is the sum of the  $x^{w(\sigma)}$  terms:

$$\Phi_S^w(x) = 1 + 3x + 3x^2 + x^3 = (x+1)^3.$$

↑  
3 elements in  $S$  have weight 1  
↓  
3 elements in  $S$  have weight 2

$\therefore$  the number of size- $n$  subsets of  $\{1, 2, 3\}$  is the coefficient of  $x^n$  in  $\Phi_S^w(x)$ ,  $\forall 0 \leq n \leq 3$ .

(of any length!)

• eg: how many binary strings don't have 000 or 00111 as a substring?

• let  $S = \text{all binary strings with no 000 or 00111 as a substring}$ .

↳ we want to organize these strings by their length, so:

define  $w(\sigma)$  be the length of  $\sigma$  (where  $\sigma \in S$ ).

Definition: let  $S$  be a set.

- A function  $w: S \rightarrow \mathbb{N}$  is a weight function if for  $n \in \mathbb{N}$  there are only finitely many elements in  $S$  of weight  $n$ .
- The generating series for  $S$  with respect to  $w$  is  $\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)}$ .
- The coefficient of  $x^n$  in a  $\Phi_S^w(x)$  is denoted  $[x^n] \Phi_S^w(x)$ .  
↳ eg:  $[x^{73}] (x+1)^{97} = \binom{97}{73} \rightarrow \binom{97}{73} x^{73}$ .  
↳ result:  $[x^n] \Phi_S^w(x)$  is the number of elements in  $S$  of weight  $n$ .

Eg:  $S = \text{all subsets from } \{1, 2, \dots, n\}$ .

For  $\sigma \in S$ ,  $w(\sigma) = |\sigma|$ .

$$\text{Then, } \Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{i=0}^n \binom{n}{i} x^i = (x+1)^n$$

$$\text{So, } [x^i] \Phi_S^w(x) = \binom{n}{i}.$$

Eg:  $S = \text{set of all binary strings}$ .

for  $\sigma \in S$ , let  $w(\sigma) = \text{length of } \sigma$ .

$$\text{Then, } \Phi_S^w(x) = ? + ?x + ?x^2 + ?x^3 + \dots$$

$\nearrow \# \text{binary strings of length 1}$   
 $\nwarrow \# \text{binary strings of length 0}$        $\nearrow \# \text{binary strings of length 2}$ .

$$\therefore \Phi_S^w(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$$

## Formal Power Series (FPS)

• An FPS is an expression of the form  $A(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k \geq 0} a_k x^k$ , where  $a_i \in \mathbb{R}$ .

↳ we don't care about convergence or divergence - we only care about

the coefficients.

- So,  $\infty$  is called an "indeterminate".

## Operations on Formal Power Series

$$\left. \begin{aligned} \text{let } A(x) &= a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k \geq 0} a_k x^k \\ \text{let } B(x) &= b_0 + b_1 x + b_2 x^2 + \dots = \sum_{k \geq 0} b_k x^k. \end{aligned} \right\} \text{FPSs.}$$

$$1) (=): A(x) = B(x) \Leftrightarrow a_k = b_k \quad \forall k \geq 0.$$

$$2) (+): A(x) + B(x) = \sum_{k \geq 0} (a_k + b_k) x^k$$

$$2b) (-): A(x) - B(x) = \sum_{k \geq 0} (a_k - b_k) x^k$$

$\curvearrowleft = \sum_{i=0}^n a_i b_{n-i}$

$$3) (\times): A(x) \cdot B(x) = C(x) = \sum_{n \geq 0} C_n x^n, \text{ where } C_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

4) (Inversion): the inverse of a FPS  $A(x)$ , if it exists, is a new FPS  $B(x)$  such that  $A(x)B(x) = 1$ .

↳ we'll write  $B(x) = A(x)^{-1} = \frac{1}{A(x)}$ .

Example: Show that  $(1-x)^{-1} = 1+x+x^2+\dots = \sum_{k=0}^{\infty} x^k$ .

$$\text{let } P(x) = \text{RHS} = \sum_{k \geq 0} x^k.$$

$$\text{Then, } (1-x)P(x) = (1-x) \sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - x \sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - \sum_{k \geq 0} x^{k+1}$$

$$= \sum_{k \geq 0} x^k - \sum_{k \geq 1} x^k = 1 + \left( \sum_{k \geq 1} x^k - \sum_{k \geq 1} x^k \right) = 1 + 0 = 1.$$

$$1) (1-x)^{-1} = \frac{1}{1-x} = \sum_{k \geq 0} x^k \rightarrow [x^n](1-x)^{-1} = 1 \quad \forall n \geq 0.$$

↳ geometric series!

$$2) \text{partial geometric series: } 1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}.$$

↑ partial bc not infinite!  
bc if  $k > n, \binom{n}{k} = 0$  anyway.

$$3) \text{binomial series: } (x+1)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^n \binom{n}{k} x^k. \quad \therefore [x^k](x+1)^n = \binom{n}{k}.$$

$$4) \text{negative binomial series: } \forall n \geq 1, (1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k$$

↳ so  $[x^k](1-x)^{-n} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ .

Example: determine  $[x^n] \left( \frac{x}{1-2x} \right)^5$ .

$$[x^n] \left( \frac{x}{1-2x} \right)^5 = [x^n] x^5 (1-2x)^{-5} = [x^{n-5}] (1-2x)^{-5} \text{ for } n \geq 5.$$

$$\text{Then, see that } (1-x)^{-5} = \sum_{k \geq 0} a_k x^k. \text{ So, } (1-2x)^{-5} = \sum_{k \geq 0} a_k x^k 2^k.$$

$$= 2^{n-5} [x^{n-5}] (1-x)^{-5} = 2^{n-5} \binom{n-5+5-1}{5-1} = 2^{n-5} \binom{n-1}{4}.$$

Proving the negative binomial theorem  $(1-x)^{-n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k, n \geq 1$ .

Combinatorial Proof (sketch):

$$\begin{aligned} (1-x)^{-n} &= (1-x)^{-1} \cdot (1-x)^{-1} \cdot \dots \cdot (1-x)^{-1} \quad (\text{n times}). \\ &= (1+x+x^2+\dots) \cdot (1+x+x^2+\dots) \cdot \dots \cdot (1+x+x^2+\dots) \quad (\text{n times}) \end{aligned}$$

We get an  $x^k$  term by selecting  $x^{a_1}$  from the first series,  $x^{a_2}$  from the second series, ..., and  $x^{a_n}$  from the last series, where  $a_1, a_2, \dots, a_k$  are non-negative integers with  $a_1 + a_2 + \dots + a_n = k$ . Then, multiplying the  $x^{a_i}$  terms gives  $x^{a_1} x^{a_2} \dots x^{a_n} = x^{a_1 + a_2 + \dots + a_n} = x^k$ .

The number of ways of choosing the  $a_i$ 's is equal to the coefficient of  $x^k$  in  $(1-x)^{-n}$ .

Illustration by example:  $n=4$ ,  $k=13$ :

begin with a string of  $k=13$  0s and then insert  $n-1=3$  1s.

$\hookrightarrow \underbrace{000}_{a_1=3} \underbrace{10000}_{a_2=4} \underbrace{00}_{a_3=2} \underbrace{10}_{a_4=4} \underbrace{000}_{}$

$$\text{note that } a_1 + a_2 + a_3 + a_4 = 3 + 4 + 2 + 4 = 13!$$

This gives a bijection between binary strings of length  $k+n-1$  with exactly  $n-1$  ones, and non-negative integers  $a_1, a_2, \dots, a_n$  st  $\sum_{i=1}^n a_i = k$ .

$\therefore$ , the number of non-negative integers  $a_1, \dots, a_k$  with  $\sum_{i=1}^n a_i = k$  is equal to the number of binary strings of length  $k+n-1$  with exactly  $n-1$  ones, which is  $\binom{k+n-1}{n-1} = \binom{n+k-1}{k}$ .

$\therefore$ , the coefficient of  $x^k$  in  $(1-x)^{-n}$  is  $\binom{n+k-1}{k}$ .  $\square$ .

### Extracting Coefficients from an FPS:

let  $A(x) = \sum_{n \geq 0} a_n x^n$  and  $B(x) = \sum_{n \geq 0} b_n x^n$ .

$$1) [x^n] (A(x) \pm B(x)) = [x^n] A(x) + [x^n] B(x).$$

$$2) [x^n] (A(x) \cdot B(x)) = \sum_{i=0}^n a_i b_{n-i} = \sum_{i=0}^n ([x^i] A(x)) ([x^{n-i}] B(x))$$

$$3) [x^n] c A(x) = c [x^n] A(x), \text{ where } c \text{ is a constant.}$$

$$4) [x^n] x^l A(x) = 0 \text{ if } n < l, \text{ and } [x^n] x^l A(x) = [x^{n-l}] A(x) \text{ if } n \geq l.$$

$$5) [x^n] A(cx) = c^n [x^n] A(x), \text{ where } c \text{ is a constant}$$

6)  $[x^n] A(x^l) = 0$  if  $l \neq n$ , and  $[x^n] A(x^l) = [x^{n-l}] A(x)$  if  $l < n$ .

Example: determine  $[x^n] x^3(3x+1)^7$  where  $n \geq 3$ .

$$[x^n] x^3(3x+1)^7 = [x^{n-3}] (3x+1)^7 = 3^{n-3} [x^{n-3}] (x+1)^7 \\ = 3^{n-3} \binom{7}{n-3}$$

Example: determine  $[x^n] x^3(3x+1)^7(1-4x^2)^{-m}$ ,  $m \geq 1$ ,  $n \geq 3$ .

$$[x^n] x^3(3x+1)^7(1-4x^2)^{-m}$$

$$= [x^n] x^3 \left( \sum_{i \geq 0} \binom{7}{i} (3x)^i \right) \left( \sum_{j \geq 0} \binom{m+j-1}{j} (4x^2)^j \right)$$

$$= [x^n] \sum_{i \geq 0, j \geq 0} \binom{7}{i} \binom{m+j-1}{j} 3^i 4^j x^{3+i+2j}$$

$$= \sum_{i \geq 0, j \geq 0} \binom{7}{i} \binom{m+j-1}{j} 3^i 4^j, \text{ and only care where } 3+i+2j=n.$$

$$\hookrightarrow i=n-2j-3, \quad j = \frac{n-3-i}{2}$$

Substituting:

$$= \sum_{j \geq 0} \binom{7}{n-2j-3} \binom{m+j-1}{j} 3^{n-2j-3} 4^j$$

$$\text{Since } j = \frac{n-3-i}{2}, \quad j \leq \frac{n-3}{2} \text{ (as } i \geq 0\text{).} \quad \therefore j \leq \left\lfloor \frac{n-3}{2} \right\rfloor$$

$$\left\lfloor \frac{n-3}{2} \right\rfloor$$

$$= \sum_{j=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \binom{7}{n-2j-3} \binom{m+j-1}{j} 3^{n-2j-3} 4^j.$$

Example: how many length-64 binary strings have neither 000 nor 00111 as a substring?

• don't know how to do this for specifically length-64. So, let's generalize for length- $n$  and make a generating series.

• Roadmap:

- 1) Let  $S$  be a set we wish to count by weight
- 2) Decompose  $S$  into "simpler" sets using disjoint union and cartesian product
- 3) Determine the generating series for the simpler sets
- 4) Combine the generating series  $\Phi_{S_i}(x)$  to get  $\Phi_S(x)$ .
- 5) The number of elements of weight  $n$  in  $S$  is  $[x^n] \Phi_S(x)$ .

• Sum Lemma: let  $S = A \cup B$  be a disjoint union. Let  $w$  be a weight function  $S \rightarrow N$ .

Then,  $\Phi_S^w(x) = \Phi_A^w(x) + \Phi_B^w(x)$ .

↳ Proof:

$$\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in A \cup B} x^{w(\sigma)} = \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)} = \Phi_A^w(x) + \Phi_B^w(x).$$

• Product Lemma: let  $S = A \times B = \{(a, b) : a \in A, b \in B\}$ . Let  $u: A \rightarrow N$  be a weight function on  $A$ , and let  $v: B \rightarrow N$  be a weight function on  $B$ . Define the weight function  $w: S \rightarrow N$  as follows:

• If  $\sigma = (a, b) \in S$ ,  $w(\sigma) = u(a) + v(b)$ .

Then,  $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_B^v(x)$ .

↳ Proof:

$$\Phi_A^u(x) \cdot \Phi_B^v(x) = \left( \sum_{a \in A} x^{u(a)} \right) \left( \sum_{b \in B} x^{v(b)} \right) = \sum_{a \in A, b \in B} x^{u(a) + v(b)} = \sum_{\sigma \in S} x^{w(\sigma)} = \Phi_S^w(x).$$

eg: find the generating series of binary strings of length  $n$  where the weight of the string is the number of 1s in it.

Method 1:

let  $S = \text{all binary strings of length } n$ .

Then,  $\Phi_S^w(x) = \sum_{k \geq 0} \alpha_k x^k$ , where  $\alpha_k = \text{number of elements of weight } k$   
 $= \text{number of binary strings of length } n \text{ with } k \text{ 1s} = \binom{n}{k}$ .

$$\therefore \Phi_S^w(x) = \sum_{k \geq 0} \binom{n}{k} x^k = (1+x)^n$$

Method 2:

let  $A = \{0, 1\}$ . Define  $u(0) = 0$  and  $u(1) = 1$ .

$$\Phi_A^u(x) = x^{u(0)} + x^{u(1)} = x^0 + x^1 = 1 + x.$$

now, we use  $A$  to describe the more complicated set  $S$ :

$S = A \times A \times A \times \dots \times A$  ( $n$  times).

define  $w: S \rightarrow \mathbb{N}$  as follows:

$\sigma = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , then  $w(\sigma) = \# \text{ 1s in } \sigma = u(\alpha_1) + u(\alpha_2) + \dots + u(\alpha_n) = \sum_{i=0}^n u(\alpha_i)$ .

→ see that the weight of the complicated set  $S$  is the sum of the weight of the less complicated set  $A$ .

$\therefore$  by the product lemma,  $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_A^u(x) \cdots \Phi_A^u(x)$  ( $n$  times).

$$= [\Phi_A^u(x)]^n = (1+x)^n.$$

☞ note: order matters!!

Compositions: A composition of  $n \in \mathbb{N}$  is a sequence of positive integers  $(\alpha_1, \alpha_2, \dots, \alpha_k)$  that add to  $n$ .  $k = \# \text{ of parts}$ .

example: the compositions of  $n=4$  are:

$(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2), (1,3), (3,1), (4)$ .

Note:  $n=0$  has one composition,  $()$ .

How many  $k$ -part compositions of  $n$  are there?

Let  $S$  = the set of all compositions with  $k$  parts.

Define  $w: S \rightarrow \mathbb{N}$  as follows:

if  $\sigma = (\alpha_1, \alpha_2, \dots, \alpha_k)$ , then  $w(\sigma) = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_k$ .

Goal: find  $\Phi_S^w(x)$ , and the answer will be  $[x^n] \Phi_S^w(x)$ .

let  $P = \{1, 2, 3, 4, \dots\}$ .

then,  $S = P \times P \times P \times \dots \times P$  ( $k$  times)  $= P^k$  ↑ cartesian product of  $k$  copies of  $P$ .

define  $u: P \rightarrow \mathbb{N}$  by  $u(a) = a \quad \forall a \in P$ .

Then,  $w(\sigma) = \alpha_1 + \alpha_2 + \dots + \alpha_k = u(\alpha_1) + u(\alpha_2) + \dots + u(\alpha_k)$ .

by the product lemma,  $\Phi_S^w(x) = \Phi_P^u(x) \times \dots \times \Phi_P^u(x)$  ( $k$  times)  
 $= [\Phi_P^u(x)]^k$ .

Now,  $\Phi_P^u(x) = \sum_{a \in P} x^{u(a)} = x + x^2 + x^3 + x^4 + \dots = x(1+x+x^2+\dots) = x(1-x)^{-1}$ .

So,  $\Phi_S^w(x) = [\Phi_P^u(x)]^k = x^k (1-x)^{-k}$

$\therefore$  the number of  $k$ -part coefficients of  $n$  is  $[x^n] x^k (1-x)^{-k}$   
 $= [x^{n-k}] (1-x)^{-k} = \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ .

String Lemma: let  $S$  be a set with weight function  $w$ .

Suppose that  $S$  has no elements of weight 0. Then,

$$\Phi_{S^*}^{\omega^*}(x) = \frac{1}{1 - \Phi_S^\omega(x)}.$$

**Proof:** let  $S$  be a set and  $\omega: S \rightarrow N$  it's weight function.

let  $S^k$  denote  $S \times S \times S \times \dots \times S$  ( $k$  times) ( $S^0 = \{\emptyset\}$ ).

Define  $S^* = S' \cup S^2 \cup S^3 \cup \dots = \bigcup_{k \geq 0} S^k$  is a disjoint union.

Define  $\omega_k: S^k \rightarrow N$ : if  $(a_1, a_2, \dots, a_k) \in S^k$ , then  $\omega_k = \omega(a_1) + \dots + \omega(a_k)$ .

Define  $\omega^*: S^* \rightarrow N$ :  $\omega^*(\sigma) = \omega_k(\sigma)$  where  $\sigma \in S^k$ .

→ we assume that  $S$  has no elements of weight 0!!

Then,  $\Phi_{S^*}^{\omega^*}(x) = \sum_{k \geq 0} \Phi_{S^k}^{\omega_k}(x)$  by the sum lemma.

Since  $\Phi_{S^k}^{\omega_k}(x) = \prod_{i=1}^k \Phi_S^\omega(x) = [\Phi_S^\omega(x)]^k$ .

We have  $\Phi_{S^*}^{\omega^*}(x) = \sum_{k \geq 0} [\Phi_S^\omega(x)]^k = \frac{1}{1 - \Phi_S^\omega(x)}$ .

How many compositions of  $n$  are there where each part is  $\geq 2$ ?

let  $S = \text{all compositions where each part is } \geq 2$ .

let  $P = \{2, 3, 4, \dots\}$

Then,  $S = P^0 \cup P \cup P^2 \cup P^3 \dots$ . So,  $S = \bigcup_{k \geq 0} P^k$ , ∴  $S = P^*$ .

By the string lemma,  $\Phi_S(x) = \frac{1}{1 - \Phi_P(x)}$ .

$$\text{Now, } \Phi_P(x) = x^2 + x^3 + x^4 + \dots$$

$$= x^2(1 + x + x^2 + \dots)$$

$$= x^2(1-x)^{-1}$$

$$\therefore \Phi_S(x) = \frac{1}{1 - \frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}. \text{ So, } [x^n] \frac{1-x}{1-x-x^2}$$

A formal power series  $A(x)$  is rational if there exist polynomials  $P(x), Q(x)$  such that  $A(x) = P(x)/Q(x)$ .

- we extract coefficients from ration power series using recurrences and partial fractions.

How many compositions have an odd number of parts, where each part  $\equiv 1 \pmod{3}$ .

Let  $S =$  the set of all compositions with an odd number of parts, with each part  $\equiv 1 \pmod{3}$ .

Let  $P = \{1, 4, 7, 10, 13, \dots\}$  (ie,  $P = \{x : x \equiv 1 \pmod{3}\}$ ).

$$\begin{aligned} \text{Then, } S &= P^1 \cup P^3 \cup P^5 \cup P^7 \cup \dots && \rightarrow \text{since } A \times (B \cup C) = (A \times B) \cup (A \times C). \\ &= P \times (P^0 \cup P^2 \cup P^4 \cup P^6 \cup \dots) \\ &= P \times (P^2)^* \end{aligned}$$

$$\begin{aligned} \therefore \Phi_S(x) &= \Phi_P(x) \cdot \Phi_{(P^2)^*}(x) \quad \text{by the product lemma} \\ &= \Phi_P(x) \cdot \frac{1}{1 - \Phi_{P^2}(x)} \quad \text{by the string lemma} \end{aligned}$$

$$\therefore \Phi_P(x) = x + x^4 + x^7 + x^{10} + \dots = \frac{x}{1 - x^3}$$

$$\begin{aligned} \Phi_{P^2}(x) &= \Phi_P(x) \cdot \Phi_P(x) \quad \text{by the product lemma} \\ &= \frac{x^2}{(1-x^3)^2} \end{aligned}$$

$$\begin{aligned} \text{So, } \Phi_S(x) &= \frac{x}{1-x^3} \cdot \frac{1}{1 - \frac{x^2}{(1-x^3)^2}} = \frac{x}{1-x^3} \cdot \frac{(1-x^3)^2}{(1-x^3)^2 - x^2} = \frac{x - x^4}{1-x^2 - 2x^3 + x^6} \\ \therefore [x^n] \frac{x - x^4}{1-x^2 - 2x^3 + x^6} \end{aligned}$$

## Binary Strings

A binary string  $\sigma$  is a sequence of bits  $b_1, b_2, \dots, b_n$ , where  $b_i = 0$  or  $1$  for  $1 \leq i \leq n$ , and  $n$  is the length of  $\sigma$ .

- There's only one binary string of length 0: the empty string  $\epsilon$ .
- We will (almost) always use the length of a binary string as its weight!

If  $a = a_1, a_2, \dots, a_m$ ,  $b = b_1, b_2, \dots, b_n$  are binary strings, their concatenation is  $ab = a_1, a_2, \dots, a_m b_1, b_2, \dots, b_n$ .

↳ note the length of  $ab$  is  $m+n$ .

↳ also,  $\epsilon\sigma = \sigma$  ∀ strings  $\sigma$ .

$a$  is a substring of  $b$  if  $b = cad$  for some strings  $c$  and  $d$ .

→  → blocks

A block of a binary string is a non-empty maximal substring of all 0s or all 1s.

Regular Expressions: a method for generating/producing a set of binary strings.

Let  $A, B$  be binary strings.

1)  $A \cup B$  is the union of  $A$  and  $B$

↳ eg:  $A = \{00\}$ ,  $B = \{1, 11, 111\}$ .  $A \cup B = \{00, 1, 11, 111\}$  or  $00 \cup (1 \cup 11 \cup 111)$ .

2)  $AB = \{ab : a \in A, b \in B\}$  is the concatenation of  $A$  and  $B$

↳ eg:  $A = \{1, 11\}$ ,  $B = \{001\}$ .  $AB = \{1001, 11001\}$ , or  $(1 \cup 11)001$ .

3)  $A^* = \bigcup_{k \geq 0} A^k$ , where  $A^k = AAA\dots A$   <sup>$\geq k$  times</sup> and  $A^0 = \{\epsilon\}$

↳ eg:  $0^* \rightarrow$  all 0-strings of any length (including 0).

Examples:  $(00)^*$  = { $\epsilon$ , 00, 0000, 000000, ...}.

$0(00)^*$  ( $= (00)^*0$ ) = all odd-length strings of 0s

$(0 \cup 1)^*$  = all binary strings!

$(0 \cup 111)^*$  = all binary strings where each block of 1s has length a multiple of 3.

Ideally, we'd like to use the Sum, Product, and String lemmas to find the generating series for a set of strings produced by a regular expression. But you have to be careful!

eg: let  $A = \{1, 0\}$ ,  $B = \{1, 01\}$

Then  $AB = \{11, 101, 01, 001\}$  Same-ish  
 $A \times B = \{(1, 1), (1, 01), (0, 1), (0, 01)\}$ .

eg: let  $A = \{1, 10\}$ ,  $B = \{1, 01\}$

Then  $AB = \{11, 101, \cancel{101}, 1001\}$  not same!  
 $A \times B = \{(1, 1), (1, 01), (10, 1), (10, 01)\}$ .

So,  $|AB| \neq |A \times B|$ !! Issue when we're trying to count. Removing commas from  $A \times B$  might cause ambiguity.

A regular expression is unambiguous if every string generated by the expression can be generated in exactly one way.

↳ eg:  $(1 \cup 0)(1 \cup 01)$  is unambiguous.

$(1 \cup 10)(1 \cup 01)$  is ambiguous, since 101 can be generated in two ways: 101 and 10, 1.

Example: Is  $0^* \cup 1^*$  ambiguous or not?

↳ ambiguous, because the empty string  $\epsilon$  can be generated in two ways. (So,  $\Phi_{0^* \cup 1^*} \neq \Phi_{0^*} + \Phi_{1^*}$ ).

Example:  $S = (0^* 11 \cup 001^*)^*$ . Is this an ambiguous expression?

↳ ambiguous, since 0011 can be generated in two ways.

Let A, B be sets of strings.

- 1) Then  $A \cup B$  is unambiguous if the union is disjoint (ie  $A \cap B = \emptyset$ ).
- 2) Then  $AB$  is unambiguous if  $\forall \sigma \in AB$ , there's exactly one pair of strings  $a, b \in A \times B$  such that  $\sigma = ab$ .
- 3) Then  $A^*$  is unambiguous if  $A^R$  is unambiguous  $\forall R$ , and  $A^0, A^1, A^2, \dots$  are disjoint.

• If  $A \cup B$  is unambiguous, then  $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$ . SUM LEMMA

• If  $AB$  is unambiguous, then  $\Phi_{AB}(x) = \Phi_A(x) \cdot \Phi_B(x)$ . PRODUCT LEMMA

• If  $A^*$  is unambiguous, then  $\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$ . STRING LEMMA

## Counting Binary Strings

eg: find the generating series for all binary strings where

Weight = length:

1)  $\Phi_S(x) = \sum_{n=0}^{\infty} a_n x^n$ , where  $a_n = \# \text{ strings of length } n = 2^n$ .

$$= \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}.$$

2)  $(0 \cup 1)^*$  is an unambiguous expression for S.

So,  $\Phi_S(x) = \Phi_{(0 \cup 1)^*}(x) = \frac{1}{1 - \Phi_{(0 \cup 1)}(x)}$  by string lemma

$$= \frac{1}{1 - (x^0 + x^1)} = \frac{1}{1-2x}$$

3) Claim:  $1^* (00^* 11^*)^* 0^*$  is an unambiguous expression for S.

eg:  $\underbrace{0\ 0\ 0\ 0\ 0}_{1^*} \underbrace{1\ 1\ *}_{00^*} \underbrace{0\ 0\ 1}_{11^*} \underbrace{0\ 0\ 1}_{00^*} \underbrace{0\ 0\ 1}_{11^*} \underbrace{0\ 0\ 1}_{00^*} \underbrace{0\ 0\ 1}_{11^*} \underbrace{0\ 0\ 1}_{00^*}$

justification: the decomposition of a string into its blocks is unique.

$$\text{So, } \Phi_S(x) = \Phi_{1^*(00^*11^*)^*0^*}(x)$$

$$\cdot \Phi_{1^*}(x) = \frac{1}{1-\Phi_1(x)} = \frac{1}{1-x} = \Phi_{0^*}(x).$$

$$\begin{aligned} \cdot \Phi_{00^*11^*}(x) &= \Phi_0(x) \cdot \Phi_{0^*}(x) \cdot \Phi_1(x) \cdot \Phi_{1^*}(x) \\ &= x \cdot \frac{1}{1-x} \cdot x \cdot \frac{1}{1-x} = \frac{x^2}{(1-x)^2} \end{aligned}$$

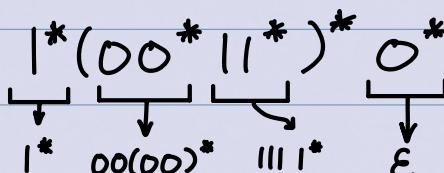
$$\text{So, } \Phi_{(00^*11^*)^*}(x) = \frac{1}{1-\Phi_{00^*11^*}(x)} = \frac{1}{1-\frac{x^2}{(1-x)^2}} = \frac{(1-x)^2}{(1-x)^2-x^2}$$

$$\begin{aligned} \therefore \Phi_{1^*(00^*11^*)^*0^*}(x) &= \Phi_{1^*}(x) \cdot \Phi_{(00^*11^*)^*}(x) \cdot \Phi_{0^*}(x) \\ &= \frac{1}{1-x} \cdot \frac{(1-x)^2}{(1-x)^2-x^2} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2-x^2} = \frac{1}{1-2x}. \end{aligned}$$

eg: find the generating series for all strings where every block of 0s has even length and must be followed by a block of at least 3 1s.

let  $S$  = set of all binary strings where every block of 0s has even length and is followed by a block of at least 3 1s.

idea: begin with  $1^*(00^*11^*)^*0^*$ . If we replace any portion of this block decomposition with an unambiguous expression for a non-empty subset of that portion, then the new expression is also unambiguous.



$\therefore 1^*(00(00)^*1111^*)^*$  is an unambiguous expression for S.

$$\begin{aligned}\Phi_S(x) &= \frac{1}{1-x} \cdot \frac{1}{1-(x^2 \cdot \frac{1}{1-x^2} \cdot x^3 \cdot \frac{1}{1-x})} = \frac{1}{(1-x)(1 - \frac{x^5}{(1-x^2)(1-x)})} \\ &= \frac{(1-x)(1-x^2)}{(1-x)((1-x^2)(1-x)-x^5)} = \frac{1-x^2}{1-x-x^2+x^3-x^5}\end{aligned}$$

Let A, B be sets such that  $B \subseteq A$ . Then,  $\Phi_{A \setminus B}(x) = \Phi_A(x) - \Phi_B(x)$ , since  $|A \setminus B| = |A| - |B|$ .

Eg: how many length-n binary strings don't have 111 as a substring?

$$(\epsilon \cup 1 \cup 11)(00^*(1 \cup 11 \cup \epsilon))^* 0^*$$

Eg: how many length-n binary strings don't have 110100 as a substring?

↳ we must use recursive decomposition here!!

## Recursive Decompositions

- express a set in terms of itself

Eg: find a recursive decomposition for A = all binary strings.

every string  $\sigma$  either begins with a 0 or a 1 followed by a binary string, except if  $\sigma = \epsilon$ .

So,  $A = \underbrace{\epsilon}_{\text{base case}} \cup \underbrace{0A \cup 1A}_{\text{recursive step}}$  is an unambiguous expression for A.

$$\therefore \Phi_A(x) = 1 + (2x \cdot \Phi_A(x)). \text{ Solving, we get } \Phi_A(x) = \frac{1}{1-2x}.$$

Eg: find a recursive decomposition which don't have 111 as a substring.

let  $A$  be the set of all strings which don't have  $111$  as a substring.

let  $\sigma \in A$ . Either  $\sigma$  has a  $0$  or it doesn't.

- If it doesn't, then  $\sigma \in \{\epsilon, 1, 11\}$ .

- If it does, then either  $\sigma$  begins with a  $0$ ,  $10$ , or  $110$ , followed by a string in  $A$ .

So,  $A = \underbrace{(\epsilon \cup 1 \cup 11)}_{\text{base case}} \cup \underbrace{(0 \cup 10 \cup 110) A}_{\text{recursive step}}$  is unambiguous.

$$\Phi_A(x) = (1+x+x^2) + (x+x^2+x^3) \Phi_A(x), \quad \text{so } \Phi_A(x) = \frac{1+x+x^2}{1-x-x^2-x^3}.$$

Eg: how many length-4 binary strings have neither  $000$  nor  $00111$  as a substring?

i) find a block decomposition for the set of all such strings.

ii) find  $\Phi_S(x) = \frac{1+x+x^2}{1-x-x^2-x^3+x^5}$

iii) answer is  $[x^4] \Phi_S(x)$ .

We'll see that the number of such strings of length  $n$  is given by the recurrence relation  $a_n - a_{n-1} - a_{n-2} - a_{n-3} + a_{n-5} = 0 \quad \forall n \geq 5$ , which comes from the denominator of  $\Phi_S(x)$ .

So,  $a_n = a_{n-1} + a_{n-2} + a_{n-3} - a_{n-5}$ , but we need initial conditions s.t.  $n \leq 4$ .

By inspection,  $a_0 = 1$ ,  $a_1 = 2$ ,  $a_2 = 4$ ,  $a_3 = 7$ , and  $a_4 = 13$ .

Then,  $a_5 = 13 + 7 + 4 - 1 = 23$ ,  $a_6 = 41$ ,  $a_7 = 73$ ,  $a_{64} = 13,076,512,262,747,676$ .

## Rational Power Series

(I) Rational series  $\rightarrow$  recurrence relation:

Eg: consider  $A(x) = \frac{P(x)}{Q(x)} = \frac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \sum_{n \geq 0} a_n x^n$ . Find a recurrence

relation for  $a_n$ .

↳ Strategy: Multiply both sides by  $Q(x)$  and equate coefficients of  $x^n$  on both sides:

$$(1 - 8x + 21x^2 - 18x^3) \sum_{n \geq 0} a_n x^n = 1 - 2x + 3x^2$$

$$\hookrightarrow \sum_{n \geq 0} a_n x^n - 8 \sum_{n \geq 0} a_n x^{n+1} + 21 \sum_{n \geq 0} a_n x^{n+2} - 18 \sum_{n \geq 0} a_n x^{n+3} = 1 - 2x + 3x^2$$

$$\hookrightarrow \sum_{n \geq 0} a_n x^n - 8 \sum_{n \geq 1} a_{n-1} x^n + 21 \sum_{n \geq 2} a_{n-2} x^n - 18 \sum_{n \geq 3} a_{n-3} x^n = 1 - 2x + 3x^2$$

extracting  $[x^n]$  of both sides:

$$n=0: a_0 = 1 \Rightarrow a_0 = 1$$

$$n=1: a_1 - 8a_0 = -2 \Rightarrow a_1 = 6$$

$$n=2: a_2 - 8a_1 + 21a_0 = 3 \Rightarrow a_2 = 30$$

$$n \geq 3: a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0.$$

∴ for  $n \geq 3$ , we have that  $a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n-3}$ .

↳ NOTE!!:  $Q(x)$  was  $1 - 8x + 21x^2 - 18x^3$ , so these coefficients will always match!

Let  $a_0, a_1, a_2, \dots$  be a sequence of complex numbers.

Let  $c_1, c_2, \dots, c_d$  be constants.

Then,  $a_n$  satisfies a (linear homogeneous) recurrence relation if

$$a_n + c_1 a_{n+1} + c_2 a_{n+2} + \dots + c_d a_{n+d} = 0 \quad \forall n \geq d.$$

↳ note:  $a_0, a_1, \dots, a_{d-1}$  are the initial conditions

Theorem: Let  $A(x) = P(x)/Q(x)$  be a rational power series

with  $Q(x) = 1 + c_1x + c_2x^2 + \dots + c_dx^d$  AND  $\deg(P) < \deg(Q)$

then the  $a_n$  satisfy a recurrence relation:

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_da_{n-d} \neq 0 \quad \forall n \geq d$$

with initial conditions  $a_0, a_1, a_2, \dots, a_{d-1}$ .

(II) recurrence relation  $\rightarrow$  rational series:

e.g.: consider  $a_n - 7a_{n-1} - 16a_{n-2} + 12a_{n-3} = 0$ , with  $a_0 = 1$ ,  $a_1 = 0$ , and  $a_2 = 2$ .

find  $P(x)$ ,  $Q(x)$  such that  $A(x) = \sum_{n \geq 0} a_n x^n = \frac{P(x)}{Q(x)}$ .

Strategy: multiply both sides of the recurrence relation by  $A(x)$  and sum for  $n \geq 3$ :

$$\hookrightarrow \sum_{n \geq 3} a_n x^n - 7 \sum_{n \geq 3} a_{n-1} x^n - 16 \sum_{n \geq 3} a_{n-2} x^n + 12 \sum_{n \geq 3} a_{n-3} x^n = 0$$

$$\sum_{n \geq 3} a_n x^n - 7x \sum_{n \geq 3} a_{n-1} x^{n-1} - 16x^2 \sum_{n \geq 3} a_{n-2} x^{n-2} + 12x^3 \sum_{n \geq 3} a_{n-3} x^{n-3} = 0$$

$$\sum_{n \geq 3} a_n x^n - 7x \sum_{n \geq 2} a_n x^n - 16x^2 \sum_{n \geq 1} a_n x^n + 12x^3 \sum_{n \geq 0} a_n x^n = 0$$

$$\therefore (A(x) - a_0 - a_1x - a_2x^2) - 7x(A(x) - a_0 - a_1x) - 16x^2(A(x) - a_0) + 12x^3A(x) = 0$$

$$\hookrightarrow A(x)[1 - 7x - 16x^2 + 12x^3] = a_0 + a_1x + a_2x^2 - 7x(a_0 + a_1x) - 16x^2a_0$$

Since  $a_0 = 1$ ,  $a_1 = 0$ , and  $a_2 = 2$ , we get:

$$A(x) = [1 - 7x - 16x^2 + 12x^3] = 1 - 7x - 14x^2, \quad \text{so:}$$

$$A(x) = \frac{1 - 7x - 14x^2}{1 - 7x - 16x^2 + 12x^3}.$$

Theorem: let  $\alpha_0, \alpha_1, \alpha_2, \dots$  be a sequence that satisfies a recurrence relation  $\alpha_n + C_1\alpha_{n-1} + C_2\alpha_{n-2} + \dots + C_d\alpha_{n-d} = 0 \quad \forall n \geq d$  with initial conditions  $\alpha_0, \alpha_1, \dots, \alpha_{d-1}$ . Let  $A(x) = \sum_{n \geq 0} \alpha_n x^n$ . Then,  $A(x) = \frac{P(x)}{Q(x)}$  where:  $Q(x) = 1 + C_1x + C_2x^2 + \dots + C_dx^d$  and  $\deg(P) < d$ .

### (III) Partial Fractions (rational series $\rightarrow$ "explicit formula")

eg: let  $A(x) = \frac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \frac{P(x)}{Q(x)} = \sum_{n \geq 0} \alpha_n x^n$ . Find an "explicit formula" for  $\alpha_n$ .

1) Factor  $Q(x)$  into a product of linear factors:

$$Q(x) = (1 - \lambda_1 x)^{d_1} (1 - \lambda_2 x)^{d_2} \dots (1 - \lambda_m x)^{d_m}$$

$$\text{we have } Q(x) = (1-2x)(1-3x)^2.$$

2) Partial Fractions:  $A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-3x} + \frac{C_3}{(1-3x)^2}$

where  $C_1, C_2, C_3$  are unique constants.

3) Find the constants  $(C_1, C_2, C_3)$ :

Multiply of sides of the partial Series expression by

$$Q(x) = (1-2x)(1-3x)^2$$

$$\hookrightarrow C_1(1-3x)^2 + C_2(1-2x)(1-3x) + C_3(1-2x) = 1-2x+3x^2$$

$$C_1(1-6x+9x^2) + C_2(1-5x+6x^2) + C_3(1-2x) = 1-2x+3x^2$$

equating coefficients of  $x^0, x^1, x^2$  on both sides:

$$x^0 \rightarrow C_1 + C_2 + C_3 = 1$$

$$x^1 \rightarrow -6C_1 - 5C_2 - 2C_3 = -2$$

$$x^2 \rightarrow 9C_1 + 6C_2 = 3$$

Solving the systems of linear equations, we get that

$$C_1 = 3, C_2 = -4, C_3 = 2.$$

$$\text{So, } A(x) = \frac{3}{1-2x} - \frac{4}{1-3x} + \frac{2}{(1-3x)^2}$$

4) Extract coefficients

$$\begin{aligned} a_n &= [x^n] A(x) = [x^n] \frac{3}{1-2x} - [x^n] \frac{4}{1-3x} + [x^n] \frac{2}{(1-3x)^2} \\ &= 3 \cdot 2^n - 4 \cdot 3^n + 2 [x^n] \frac{1}{(1-3x)^2} \\ &= 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n [x^n] \frac{1}{(1-x)^2} = 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \binom{n+2-1}{2-1} \\ &= 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \binom{n+1}{1} = 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \cdot (n+1) \\ &= 3 \cdot 2^n + (2n-2)3^n \end{aligned}$$

$$\therefore a_n = 3 \cdot 2^n + (2n-2)3^n \quad \forall n \geq 0.$$

Note: such a factorisation always exists if you allow  $\lambda_i$  to be complex, by the fundamental theorem of algebra!

↳ if you cannot factor  $Q(x)$ , then the method fails.

Notation:  $Q(x) = (1-\lambda_1 x)^{d_1} (1-\lambda_2 x)^{d_2} \dots (1-\lambda_m x)^{d_m}$ , where  $\lambda_i$  is an "inverse root" of  $Q(x)$ , and  $d_i$  is its multiplicity.

Eg: Suppose  $A(x) = \frac{1-3x^2+7x^3}{(1+4x)^2(1-3x)^4}$

↳  $Q(x)$  has two inverse roots,  $-4$  with multiplicity 2, and  $3$  with

multiplicity 4.

$$A(x) = \frac{C_1}{1+4x} + \frac{C_2}{(1+4x)^2} + \frac{C_3}{1-3x} + \frac{C_4}{(1-3x)^2} + \frac{C_5}{(1-3x)^3} + \frac{C_6}{(1-3x)^4}$$

polynomial of n in degree < 2, λ<sub>1</sub>      polynomial in n of degree < 4, λ<sub>2</sub>

$$\text{Then } a_n = [x^n] A(x) = (B + C_n)(-4)^n + (D + E_n + F_n^2 + G_n^3) \cdot 3^n$$

Theorem: let  $A(x) = \frac{P(x)}{Q(x)}$ , where  $\deg(P) < \deg(Q)$  and the constant term of  $Q(x)$  is 1. Factor  $Q(x)$  into a product of linear polynomials  $Q(x) = (1 - \lambda_1 x)^{d_1} \cdots (1 - \lambda_m x)^{d_m}$  where  $\lambda_i$  are the inverse roots of  $Q(x)$ , with multiplicities  $d_i$ . Then, there exist unique constants  $C_{ij}$ , where  $1 \leq i \leq m$ ,  $1 \leq j \leq d_i$  such that:

$$A(x) = \left( \frac{C_{1,1}}{1-\lambda_1 x} + \frac{C_{1,2}}{(1-\lambda_1 x)^2} + \frac{C_{1,d_1}}{(1-\lambda_1 x)^{d_1}} \right) + \cdots + \left( \frac{C_{m,1}}{1-\lambda_m x} + \frac{C_{m,2}}{(1-\lambda_m x)^2} + \cdots + \frac{C_{m,d_m}}{(1-\lambda_m x)^{d_m}} \right).$$

Moreover,  $[x^n] A(x) = p_1(n) \lambda_1^n + \cdots + p_m(n) \lambda_m^n \quad \forall n \geq 0$ , where  $p_i(n)$  is a polynomial of degree  $< d_i$ .

#### (IV) recurrence → explicit formula

e.g. given  $a_n - 6a_{n-1} + 12a_{n-2} - 10a_{n-3} + 3a_{n-4} = 0 \quad \forall n \geq 4$ , with  $a_0 = 5, a_1 = 11, a_2 = 23, a_3 = 49$ . Find an explicit formula for  $a_n$ .

let  $A(x) = \sum_{k \geq 0} a_k x^k = \frac{P(x)}{Q(x)}$ , directly from recurrence!

where  $Q(x) = 1 - 6x + 12x^2 - 10x^3 + 3x^4$   
 $= (1-3x)(1-x)^3$  given!

the inverse roots of  $Q(x)$  are  $\lambda_1 = 3$  and  $\lambda_2 = 1$  with multiplicities  $d_1 = 1$  and  $d_2 = 3$ , respectively.

$\therefore$ , an explicit formula for  $a_n$  is:  $a_n = (\text{polynomial in } n \text{ of degree } < d_1=1) 3^n + (\text{polynomial in } n \text{ of degree } < d_2=3) 1^n$

So,  $a_n = C \cdot 3^n + (D + E n + F n^2) \quad \forall n \geq 0.$

Now, we use the initial conditions to find the constants  $C, D, E$ , and  $F$ :

$$\cdot n=0: a_0 = 5 = C + D$$

$$\cdot n=1: a_1 = 11 = 3C + D + E + F$$

$$\cdot n=2: a_2 = 23 = 9C + D + 2E + 4F$$

$$\cdot n=3: a_3 = 49 = 27C + D + 3E + 9F$$

Solving, we get:  
 $C=1, D=4, E=3, F=1$

$$\therefore a_n = 3^n + (4 + 3n + n^2) \quad \forall n \geq 0.$$

**General Method:** given a recurrence relation

$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_d a_{n-d} \quad \forall n \geq d$  with initial conditions  $a_0, a_1, \dots, a_{d-1}$ , we find the explicit formula by:

- 1) Defining the denominator  $Q(x) = 1 + C_1 x + C_2 x^2 + \dots + C_d x^d$
- 2) Find the inverse roots  $\lambda_1, \lambda_2, \dots, \lambda_m$  and their multiplicities  $d_1, d_2, \dots, d_m$  of  $Q(x)$  by factoring  $Q(x)$  into a product of linear factors.
- 3) Set up the general form  $a_n = p_1(n) \lambda_1^n + \dots + p_m(n) \lambda_m^n$  for all  $n \geq 0$ , where  $p_i(n)$  has degree  $< d_i$  with unknown coefficients.
- 4) Use the initial conditions to determine these coefficients.

(V) explicit formula  $\rightarrow$  recurrence

eg: given  $a_n = (2n-3)(-1)^n + (2n^2+n-5)4^n \quad \forall n \geq 0$ . Find the recurrence for  $a_n$ .

$$\text{let } A(x) = \sum_{k \geq 0} a_k x^k = P(x)/Q(x).$$

Then,  $Q(x)$  must be a polynomial with inverse roots  $\lambda_1 = -1$  and  $\lambda_2 = 4$ , with multiplicities  $d_1 = 2$  and  $d_2 = 3$ . ↗  
↳ note, we get multiplicities by seeing degree of respective polynomials, plus one!

$$\text{So, } Q(x) = (1-\lambda_1 x)^{d_1} (1-\lambda_2 x)^{d_2} = (1+x)^2 (1-4x)^3.$$

expanding, we get:  $Q(x) = 1 - 10x + 25x^2 + 20x^3 - 80x^4 - 64x^5$ , so:

$$a_n - 10a_{n-1} + 25a_{n-2} + 20a_{n-3} - 80a_{n-4} - 64a_{n-5} = 0 \quad \forall n \geq 5.$$

let's find initial conditions by plugging  $n \in \{0, 1, 2, 3, 4\}$  into the given explicit formula:  $a_0 = -8, a_1 = -7, a_2 = 81, a_3 = 1021, a_4 = 7941$ .

**General Method:** given an explicit formula of the form  $a_n = p_1(n)\lambda_1^n + \dots + p_m(n)\lambda_m^n$  for all  $n \geq 0$ , where  $p_i(n)$  has degree  $< d_i$ , we get the recurrence relation by:

1) Define the denominator polynomial  $Q(x) = (1-\lambda_1 x)^{d_1} \dots (1-\lambda_m x)^{d_m}$

2) Compute  $Q(x) = 1 + C_1 x + C_2 x^2 + \dots + C_d x^d$

3) A recurrence relation for the  $a_n$  is then:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_d a_{n-d} = 0 \quad \forall n \geq d.$$

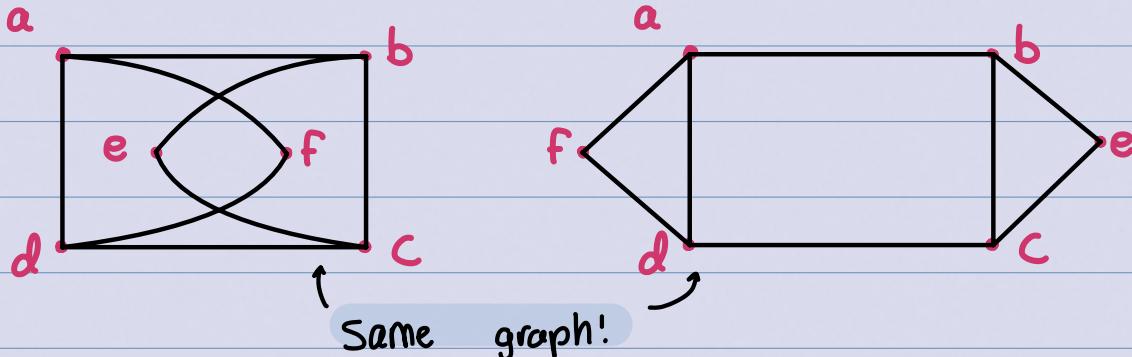
4) The initial conditions  $a_0, a_1, \dots, a_{d-1}$  can be determined from the explicit formula.

## Graph Theory

A graph has a finite set of vertices  $V(G)$ .

Vertices that are connected are paired in the Edge Set  $E(G)$ .

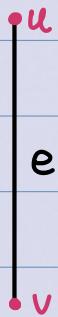
Example:  $V(G) = \{a, b, c, d, e, f\}$ ,  $E(G) = \{(a,b), (b,c), (c,d), (d,a), (a,f), (d,f), (b,e), (c,e)\}$



A graph  $G$  consists of a pair  $(V(G), E(G))$ , where  $V(G)$  is a finite non-empty set of objects called vertices, and  $E(G)$  is a set of unordered pairs of distinct vertices called edges.

Terminology: for an edge  $(u, v) \in E(G)$ , we say:

- $u$  is adjacent to  $v$
- $u$  is a neighbor of  $v$
- $e$  is incident with  $u$  and  $v$
- $u$  and  $v$  are the ends of  $e$ ,
- $e$  joins  $u$  and  $v$



→ we use the shorthand  $e=uv$  to represent the edge  $\{u, v\}$ .

→ note, if  $e=uv$ , then  $e=vu$ .

The degree of a vertex  $v$  in a graph  $G$  is the number of edges incident with  $v$  in  $G$ , denoted  $\deg(v)$ .

Handshaking Lemma: for any graph  $G$ ,  $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$



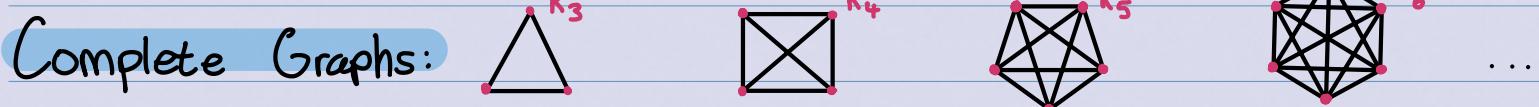
A path of length  $n$ :

- $P_n = (\{v_0, v_1, \dots, v_n\}, \{v_0v_1, v_1v_2, v_2v_3, \dots, v_{n-1}v_n\})$
- number of vertices is  $n+1$
- number of edges is  $n$



A cycle of length  $n$ :

- $C_n = (\{v_1, v_2, \dots, v_n\}, \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\})$
- number of vertices is  $n$
- number of edges is  $n$



A complete graph on  $n$  vertices:

- Every pair of vertices is adjacent
- $K_n = (\{v_1, v_2, \dots, v_n\}, \{v_iv_j : 1 \leq i \leq j \leq n\})$
- number of vertices is  $n$
- number of edges is  $\binom{n}{2} = \frac{n(n-1)}{2}$

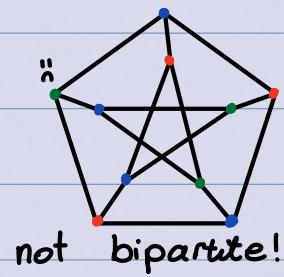
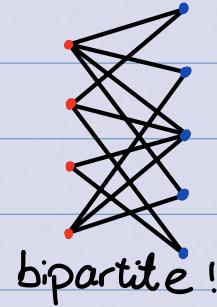
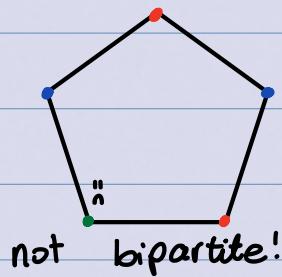
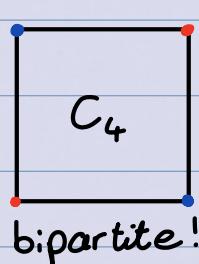
A graph is bipartite if there is a partition  $(A, B)$  of its vertices such that every edge joins a vertex in  $A$  with a vertex in  $B$ .

We can (informally) check if a graph is bipartite by coloring its

Vertices. If two adjacent vertices have the same color, then the graph is not bipartite!

$C_n$  is bipartite if and only if  $n$  is even.

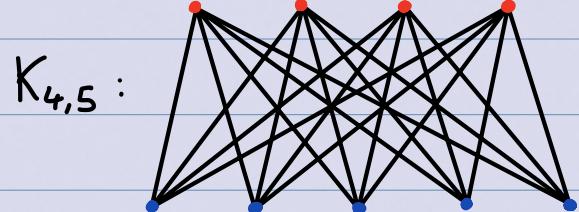
Moreover, a graph is bipartite if and only if it does not contain an odd cycle.



**Complete Bipartite Graph:** a bipartite graph with bipartition  $(A, B)$ , where  $|A|=m$ ,  $|B|=n$ , and every vertex in  $A$  is adjacent to every vertex in  $B$

$$\hookrightarrow K_{m,n} = (\{u_1, u_2, \dots, u_m\} \cup \{v_1, v_2, \dots, v_n\}, \{u_i v_j : 1 \leq i \leq m \leq j \leq n\}).$$

- number of vertices is  $m+n$
- number of edges is  $mn$



A graph is regular if every vertex has the same degree.

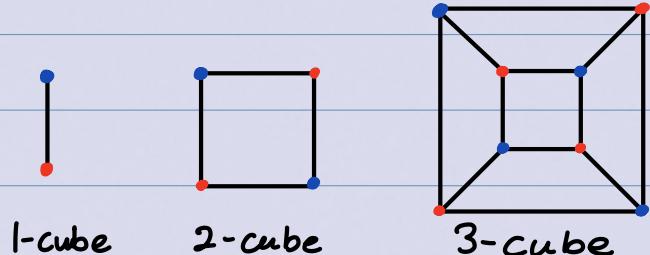
$\hookrightarrow$  more precisely, a graph is  $k$ -regular if every vertex has degree  $k$ .

- Paths  $\rightarrow P_0$  is 0-regular,  $P_1$  is 1-regular,  $P_n$  is not regular for  $n \geq 2$
- Cycles  $\rightarrow$  cycles are 2-regular
- The complete graph  $K_n \rightarrow (n-1)$ -regular
- The complete bipartite graph  $K_{m,n} \rightarrow m$ -regular and  $n$ -regular
- Number of edges in a  $k$ -regular graph with  $n$  vertices  $\rightarrow \frac{kn}{2}$ .

**n-Cubes (Hypercubes)**: graphs whose vertices are the length- $n$  binary strings, and two vertices are adjacent if and only if their binary strings differ in exactly one bit.

Properties of the  $n$ -cube:

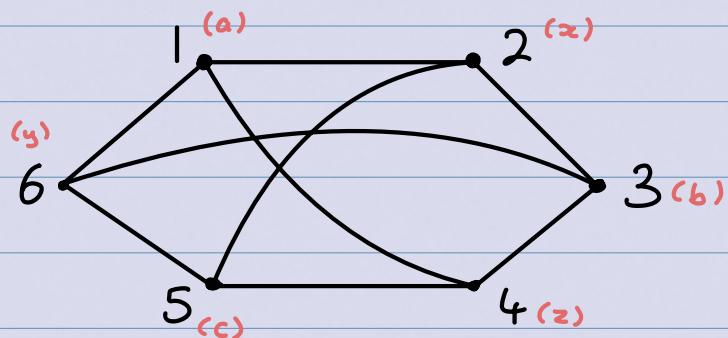
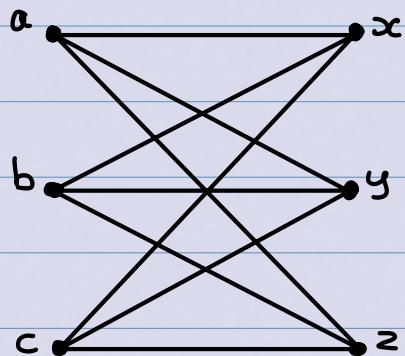
- number of vertices is  $2^n$
- $n$ -cubes are  $n$ -regular
- number of edges is  $n2^{n-1}$
- $n$ -cubes are bipartite



## Isomorphic Graphs

Informally: two graphs are **isomorphic** (ie, essentially the same) if one can be obtained from the other by "relabelling" vertices.

Eg: Are the following graphs isomorphic?

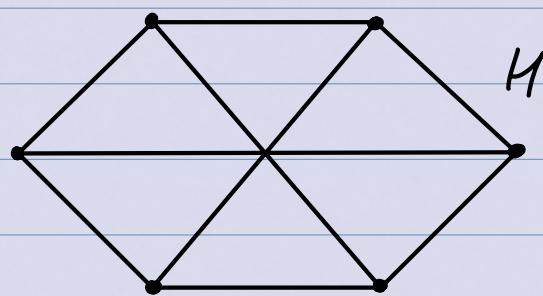
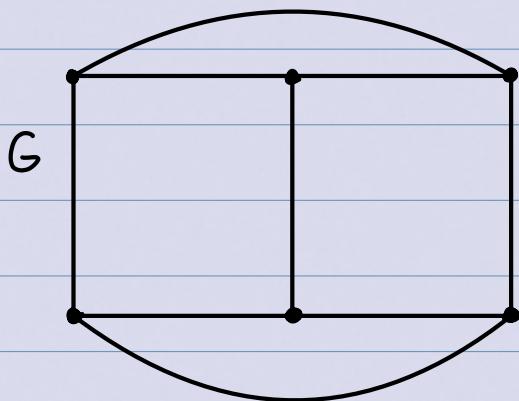


1) 1 2 3 4 5 6  
2) a x b z c y

Yes! we rename vertices as follows:

Formally, two graphs  $G$  and  $H$  are isomorphic if there is a bijection  $f: V(G) \rightarrow V(H)$  which preserves adjacency, ie,  $uv \in E(G)$  if and only if  $f(u)f(v) \in H$ .

Eg: are the following graphs isomorphic?



No! G has 3 mutually adjacent vertices (a triangle), but H doesn't. So, there doesn't exist a bijection from  $V(G)$  to  $V(H)$  that preserves adjacency.

## Graph Invariants

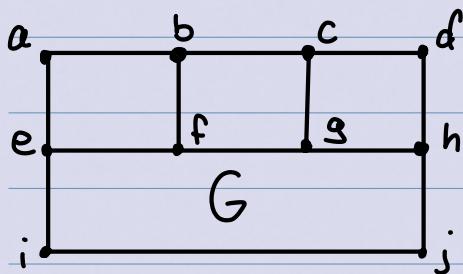
- A graph invariant is a property of a graph that does not change over time.
  - ↳ examples of possible graph invariants:
    - 1) number of vertices (isomorphic graphs have the same number of vertices)
    - 2) number of edges (isomorphic graphs have the same number of edges)
    - 3) the degree sequence: a list of all vertex degrees in non-decreasing sequence (isomorphic graphs have the same degree sequence)
    - 4) bipartiteness (isomorphic graphs are either both bipartite or both non-bipartite.)
    - 5) containment of a cycle of a certain length
    - 6) etc

Determining if two graphs are isomorphic

- to show that two graphs  $G$  and  $H$  are isomorphic, give a bijection from  $V(G)$  to  $V(H)$  that preserves adjacency.
- to show that two graphs  $G$  and  $H$  are not isomorphic, find a graph invariant that is different for  $G$  and  $H$  (eg, #vertices, #edges, degree sequence, etc), or find an adjacency structure that is in  $G$  but not in  $H$ .
- note: no efficient algorithm is known for determining whether two graphs are isomorphic or not!

## Fundamental Concepts in Graph Theory

### Walks and Paths



- A walk in  $G$  of length 5  $b, f, g, f, e, i$  is called a  $b, i$  walk
- $a, b, c, b, a$  is a closed  $a, a$  walk
- $b$  is a path (and a closed  $b, b$  walk)  
↳ length 0 walk

- Definition: let  $u, v \in V(G)$ . A  $u, v$ -walk in  $G$  is a sequence of vertices  $v_0, v_1, \dots, v_k$ , where  $v_0 = u$ ,  $v_k = v$ , and  $v_i, v_{i+1} \in E(G)$
- The length of the walk is  $k$   
↳  $\forall 0 \leq i \leq k+1$
  - The walk is closed if  $v_0 = v_k$  (end where we start)
  - A  $u, v$ -path in  $G$  is a  $u, v$ -walk with no repeated vertices (and no repeated edges).

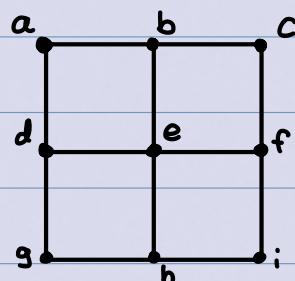
Theorem: let  $u, v \in V(G)$ . If there exists a  $u, v$ -walk in  $G$ , then there exists a  $u, v$ -path in  $G$ .

↳ Proof: let  $w = v_0, v_1, v_2, \dots, v_k$  be a  $u, v$  walk in  $G$  (so

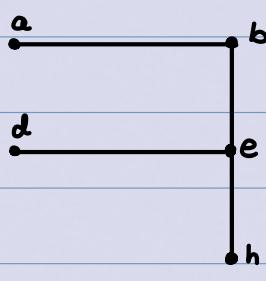
$V_0 = u$  and  $V_k = v$ ) of shortest length. Suppose there exists indices  $i$  and  $j$  with  $i < j$  and  $V_i = V_j$ . Then,  $\omega' = V_0, V_1, \dots, V_{j+1}, \dots, V_k$  is a  $u, v$ -walk in  $G$  that is shorter than  $\omega$ . So, this contradicts the choice of  $\omega$ . ∴ the vertices in  $\omega$  are distinct, and so  $\omega$  is a  $u, v$ -path in  $G$ .

↳ **Corollary:** let  $u, v, w \in V(G)$ . If there exists a  $u, v$ -path and a  $v, w$ -path in  $G$ , then there exists a  $u, w$ -path in  $G$ .

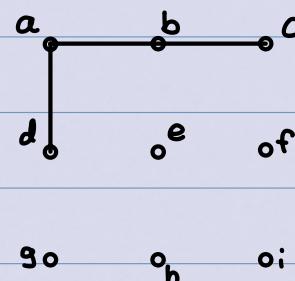
## Subgraphs



h o subgraph



subgraph of G



spanning subgraph

A graph  $H$  is a subgraph of  $G$  if:

- 1)  $V(H) \subseteq V(G)$

- 2)  $E(H) \subseteq E(G)$

- 3) each edge in  $E(H)$  has both ends in  $V(H)$

→ if  $V(H) = V(G)$  then  $H$  is a spanning subgraph of  $G$

note: a  $u, v$ -path in  $G$  is a subgraph of  $G$ .

**Cycles:** A cycle in a graph  $G$  is a closed walk  $V_0, V_1, V_2, \dots, V_k$  and where  $V_1, V_2, \dots, V_k$  are distinct and  $k \geq 2$  and  $V_0 = V_k$  (the length of the cycle is  $k$ ).

**Theorem:** let  $G$  be a graph where each vertex has degree  $\geq 2$ . Then  $G$  contains a cycle!

• proof: let  $P = V_0, V_1, V_2, \dots, V_k$  be the longest path in  $G$ .

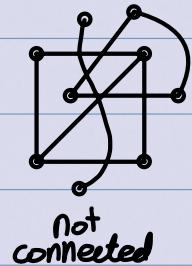
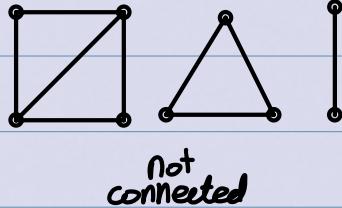
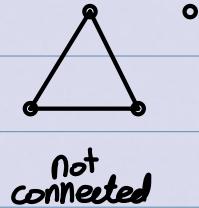
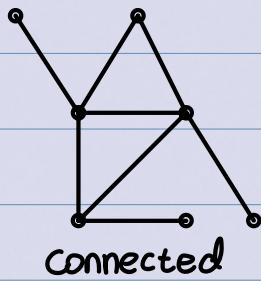
note that  $k \geq 2$ , since every vertex has at least 2, so any vertex  $V_i$  together with its two neighbors, gives a path of at least 2.

now,  $V_0$  is adjacent to  $V_1$ . Since  $\deg(V_0) \geq 2$ ,  $V_0$  is adjacent to at least one other vertex, say  $x \neq V_1$ . If  $x$  is not on path  $P$ , then  $x, V_0, \dots, V_k$  is a path that is longer than  $P$ , contradicting the choice of  $P$ .  $\therefore x$  is on  $P$ , so  $x = V_i$  for some  $2 \leq i \leq k$ . Then  $V_0, V_1, \dots, V_i = x, V_0$  is a cycle in  $G$  (of length  $\geq 3$ ).

## Connectedness + Components

A graph  $G$  is connected if  $\forall x, y \in V(G)$ , there exists an  $x, y$ -path in  $G$ .

eg:

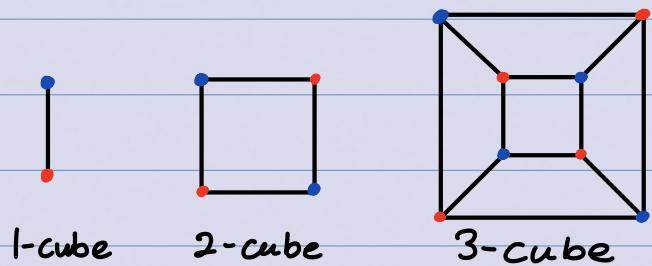


Theorem: let  $v \in V(G)$ . If there exists a  $v, w$ -path in  $G$   $\forall w \in V(G)$ , then  $G$  is connected.

proof: let  $x, y \in V(G)$ . By assumption, there is a  $v, x$ -path in  $G$ , so there's also a  $x, v$ -path in  $G$ . Also by assumption, there's a  $v, y$ -path in  $G$ . By the previous corollary, there exists an  $x, y$ -path in  $G$ .

$\therefore G$  is connected.

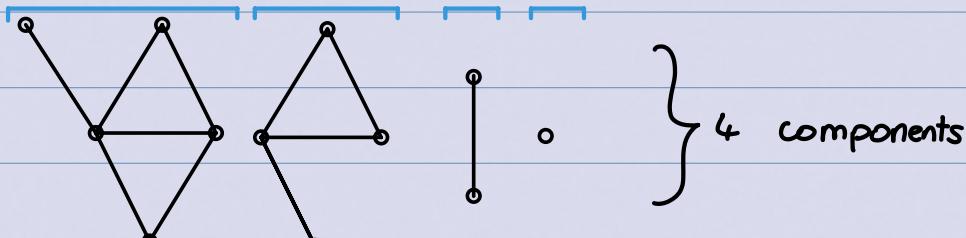
Eg: Show that the  $n$ -cube is connected.



consider  $V = 000000$ ,  $w = 010110$ .  
then,  $000000$ ,  $010000$ ,  $010100$ ,  $010110$  is  
a  $V, w$ -walk in the 6-cube.

formally, let  $V = \underbrace{000 \dots 00}_n$  and  $w = \{0, 1\}^n$ . Suppose  $w$  has exactly  $k$  1s ( $0 \leq k \leq n$ ) with 1s in positions  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . For each  $1 \leq j \leq k$ , define  $V_i$  to be a length- $n$  binary string with 1s in positions  $i_1, i_2, \dots, i_j$  and 0s elsewhere. Then  $V, V_1, V_2, \dots, w^{V_k}$  is a  $V, w$ -path in the  $n$ -cube.  $\therefore$  by the previous theorem, the  $n$ -cube is connected.

### Components



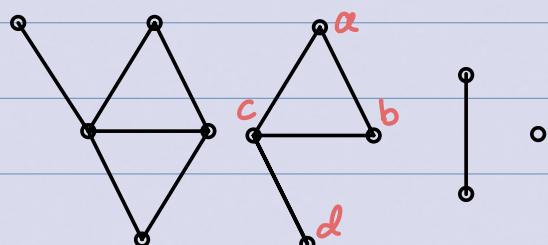
A component of a graph  $G$  is a maximal connected subgraph of  $G$ .

Observation: a graph is connected if and only if it has only one component!

### Cuts

Let  $X \subseteq V(G)$ . The cut, induced by  $X$ , is the set of all edges in  $E(G)$  with exactly one end in  $X$ .

Eg:

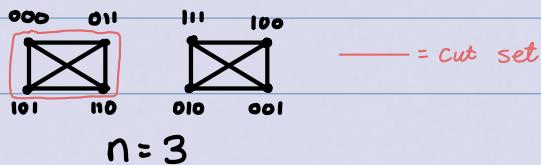
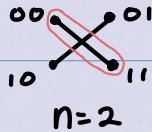


if  $X = \{a, b, c, d\}$ , then the cut induced by  $X$  is empty.

**Theorem:** a graph is connected if and only if for all non-empty proper subset  $X \subseteq V(G)$ , the cut induced by  $X$  is non-empty.  
 ↴ subset that cannot be empty

Eg: let  $n \geq 1$ . Define  $G_n$  to be a graph whose vertex set is the set of length- $n$  binary strings, with two vertices being adjacent iff they differ in exactly two bit positions.

examples:



Claim:  $G_n$  is not connected!

let  $X = \text{vertices in } G_n \text{ labelled with an even number of } 1\text{s}$ . Then,  $X$  is non-empty (since the string  $000\dots000 \in X$ ) and is a proper subset of  $V(G_n)$  (since  $1000\dots000 \notin X$ ). We'll argue that the cut induced by  $X$  is empty:

ie,  $y \in V(G) \setminus X$

let  $x \in X$  and  $y \notin X$  (but  $y \in V(G)$ ). Now  $x$  has an even number of 1s (by defn. of  $X$ ). Flipping one bit in  $x$  gives a vertex in  $V(G) \setminus X$  (not in  $X$ ). Flipping a second bit of this vertex gives a vertex with an even number of 1s, so it's in  $X$ .  $\therefore$ , there's no edge from a vertex in  $X$  to a vertex outside of  $X$ .

Therefore,  $G_n$  is not connected!