Lecture 1 - 4th Sept 2024

Statistics: the science of understanding data and making decisions in the face of variability and uncertainty.

Probability: A branch of mathematics concerned with describing and modeling uncertain events.

Preliminaries:

- **Experiment:** the process of obtaining an observed result of some phenomenon.
- **Trial:** the performance of an experiment.
- **Outcome:** the result of a single trial (attempt) of an experiment.
- **Event:** one or more outcomes of an experiment.
- **Probability:** the measure of how likely an event is.

Sample Space: the set of ALL possible distinct outcomes in a random experiment, denoted by S.

↳ note: one and only one of the outcomes occurs in any single trial of the experiment.

↳ example:

- Roll a six-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- Flip a coin: $S = \{\text{heads}, \text{tails}\}$
- Waiting time for a bus: $S = \{t \in \mathbb{R}, 0 \leq t \leq 10\}$

- a sample space is finite if it consists of a finite number of outcomes, say $\{a_1, a_2, \dots, a_N\}$

- a sample space can be a set of countable infinite outcomes, say $\{a_1, a_2, \dots\}$, where the outcomes can be put into a one-on-one correspondence with positive integers.

↳ Example: Suppose our experiment consists of tossing a coin until it lands on heads.

↳ the possible outcomes in the sample space are:

H, TH, TTH, TTTH, TTTTH, ...

∴, the outcomes are countable but infinite.

Discrete Sample Space: when a sample space is either finite or countably infinite. Else, it's non-discreet.

↳ Example:

- Roll a six-sided die: discreet
- Flip a coin: discreet
- Waiting time for a bus: non-discreet

Event: a subset of sample space S.

↳ we say A is an event if $A \subset S$.

↳ note: A and S are both sets!

↳ Example: consider tossing two coins.

a) what is the event of obtaining at least one head?

$S = \{HH, HT, TH, TT\}$

$\therefore A = \{HH, HT, TH\}$

b) What is the event of obtaining at least one head and at least one tail?

$A = \{HT, TH\}$.

Elementary / Simple Event: an event that contains only one outcome of the experiment. Eg, $A = \{a_1\}$.

Compound Event: an event made up of two or more simple events. Eg: $A = \{a_1, a_2\}$.

Set Notations and Terminology:

- A set is a collection of elements. Eg, $A = \{\text{apple}, \text{orange}\}$.
- The notation $x \in A$ or $x \notin A$ will mean that x is or is not an element of A.
- The empty / null set is $\emptyset$.
- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The notation $B \subset A$ means that B is a subset of A.
- Complement: $A^c = \{x \mid x \in S, x \notin A\} = \bar{A}$
- Two sets A and B are disjoint if $A \cap B = \emptyset$.

- The set $A \cap B^c$ is "A but not B"
- The set $A^c \cap B^c$ is "neither A nor B"

- If $A_1, \dots, A_n$ is a finite collection of sets,
  - $A_1 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i$ is "in all A_i"
  - $A_1 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$ is "in at least one A_i"

very similar to disjoint!

Mutually Exclusiveness: when two events A and B if $A \cap B = \emptyset$. If events are exclusive, they have no outcomes in common.

- Example: tossing two coins. A is the event "at least one head", and B is the event "both tails".
  - $S = \{HH, HT, TH, TT\}$.
    $A = \{HH, HT, TH\}$, $B = \{TT\}$.

    $A \cap B = \{HH, HT, TH\} \cap \{TT\} = \emptyset$!

    $\therefore$, A and B are mutually exclusive.

- Events $A_1, A_2, A_3, \dots$ are said to be mutually exclusive if they are pairwise mutually exclusive. That is,

$$A_i \cap A_j = \emptyset, \text{ wherever } i \neq j.$$

Probability Modeling: assigning a probability $P(A)$ to each event A. This probability measures how likely it is that A will happen when the experiment is conducted.

↳ think of $P(A)$ as a (set) function, whose domain is a collection of sets (events), and the range of which is a subset of real numbers.

↳ not all set functions are appropiate for assigning probabilities to event.

- Let $S = \{a_1, a_2, a_3, \ldots\}$ be a discrete sample space. We assign to each elementary event $A_i = \{a_i\}$ for $i = 1, 2, \ldots$, a number $P(A_i) = P(\{a_i\})$, such that

  - $0 \le P(A_i)$ and $\sum_{\text{all } i} P(A_i) = 1$

- We call $P(A)$ the probability of A, and call the set of probabilities $\{P(A_i), i = 1, 2, \ldots\}$ the probability distribution on S.

↳ **General Properties:**

- The null event (empty set) has a probability of 0. $\therefore$ $P(\emptyset) = 0$.
- If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$
- If $A_1, A_2, \ldots, A_k$ is a finite collection of pairwise mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k)$$

- $P(\bar{A}) = 1 - P(A)$

↳ Example: suppose a six-sided die is rolled once. If A is the event that an even number is obtained. What is $P(A)$?

- $S = \{1, 2, 3, 4, 5, 6\}$. Since each number is equally likely, $P(\{i\}) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

$A = \{2, 4, 6\}$. $\rightarrow P(A) = P(\{2\}) + P(\{4\}) + P(\{6\})$
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.

↳ Example: Toss a coin twice. What's the probability of getting exactly one head?

- $S = \{HH, HT, TH, TT\}$. Each option is uniformly likely.
- $A = \{HT, TH\} \rightarrow P(A) = P(\{HT\}) + P(\{TH\})$
$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Random Selection: when choosing an object from a finite set where each object has an equal chance of being selected.

Uniform Probability Model: consider a discrete finite sample space $S = \{a_1, a_2, \ldots, a_N\}$, where each simple event has a probability of $1/N$. This is a uniform distribution over the set $\{a_1, a_2, \ldots, a_N\}$.

↳ For a compound event A, we can calculate $P(A)$ by:

$$P(A) = \frac{n(A)}{N},$$

where $n(A)$ is the number of outcomes in A.

- The formula $P(A) = \frac{n(A)}{N}$ is classical probability.

(trivial) example: draw one card at random from a well-shuffled deck without jokers. What is the probability that the card is a spade?

↳ $S = \{A♡, 2♡, \ldots\}$.
$A_s = \{A♠, 2♠, \ldots, K♠\}$.

↳ $P(A_s) = \frac{n(A_s)}{N} = \frac{13}{52} = \frac{1}{4}$.

Example: suppose that P is a probability, and A and B are events such that A and B are mutually exclusive, and $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$. Is each statement true or false?

A) A and B are each non-empty

Since $P(\bar{A}) = 0.8$ and $P(\bar{B}) = 0.7$, we know that

$P(A) = 0.2$ and $P(B) = 0.3$.

we also know that $P(\emptyset) = 0$. Since $0.2 \neq 0 \neq 0.3$, we know A and B cannot be empty.
$\therefore$, the statement is true.

B) $P(A \cap B) = 0$

Since we're told that A and B are mutually exclusive, we know that $A \cap B = \emptyset$.

we also know that $P(\emptyset) = 0$, so the statement is true!

C) $P(A \cup B) = 0.4$

we know that $\overbrace{P(A \cup B)}^{\text{"the probability of A or B"}} = P(A) + P(B)$

$P(A) + P(B) = 0.2 + 0.3 = 0.5 \neq 0.4$.
$\therefore$, the statement is false!

Example: suppose two six-sided die are rolled where the outcomes are of the form (die 1, die 2). If A is the event that the first die is even, and B is the event that the sum of both rolls is 6, compute the event $C = \bar{A} \cap B$

↳ $A = \{(2,1), (2,2), \ldots, (4,1), (4,2), \ldots, (6,1), (6,2), \ldots, (6,6)\}$.

$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$\bar{A}$ means the event that A does not happen, so in other words, $\bar{A}$ is the event that the first die is odd.

$\therefore, \bar{A} = \{(1,1), \ldots, (1,6), (3,1), \ldots, (3,6), (5,1), \ldots, (5,6)\}$.

$\bar{A} \cap B$ is A AND B.

↳ $\therefore, C = \{(1,5), (3,3), (5,1)\}$.

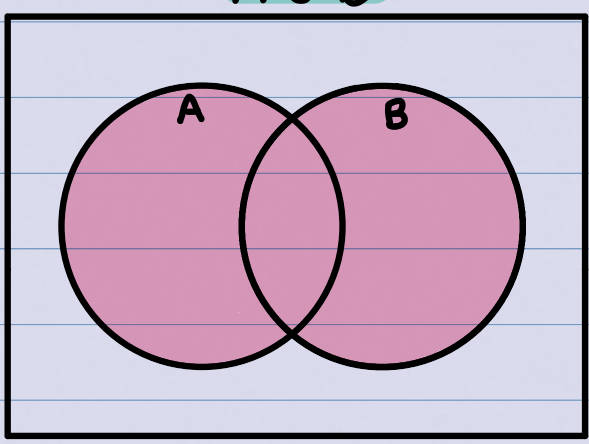
Venn Diagram: a tool to illustrate the relationship among sets (or events).

↳ made up of:

- a rectangle to represent the sample space S.
- Circles within the rectangle to illustrate the events.

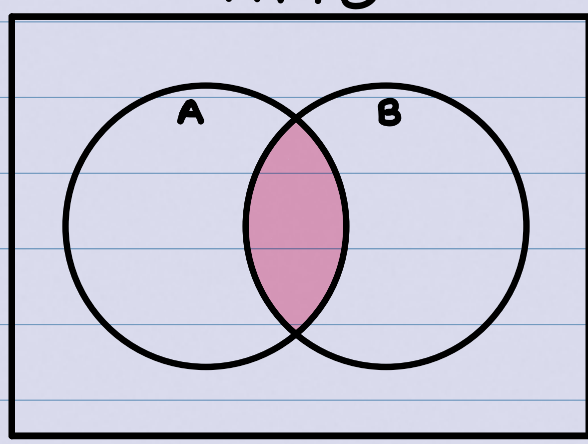
The Union and Intersection of Two Events

$A \cup B$



"A or B"

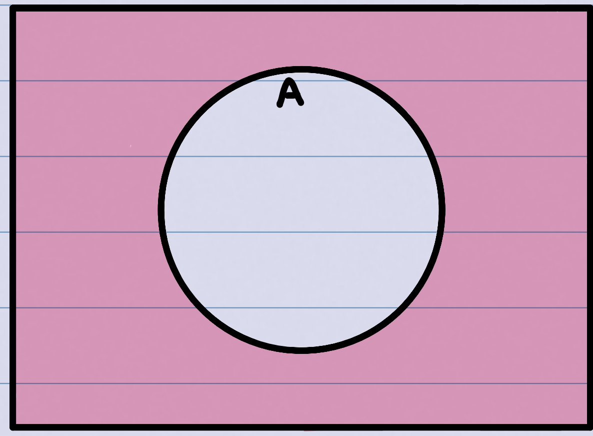
$A \cap B$



"A and B"

The Compliment of an Event:

$\bar{A}$



"S but not A"

De Morgan's Laws:

- $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Probability Rules

- **Probability of the Complement of an Event**

  ↳ $P(A) = 1 - P(\bar{A})$

bc mutually exclusive!

Proof: $A \cup \bar{A} = S$, so $P(A \cup \bar{A}) = P(A) + P(\bar{A})$

since $P(A \cup \bar{A}) = P(S) = 1$, $1 = P(A) + P(\bar{A})$!

Example: an experiment involves tossing a coin four times. The event A is "at least one head". Find the probability of A, ie, find $P(A)$.

- the only case where there is NOT at least one head is $\bar{A} = \{TTTT\}$.

Flipping a coin 4 times would result in 2^4 possible outcomes, and $\bar{A}$ is only 1 of these.

$\therefore$, $P(\bar{A}) = \frac{1}{16}$, so $\overbrace{P(A) = 1 - P(\bar{A})}^{\text{probability of complement!}} = 1 - \frac{1}{16} = \frac{15}{16}$.

Example: two ordinary dice are rolled. Find the probability that at least one lands on a 6.

$S = \{(1,1), \dots, (1,6), (2,1), \dots, (2,6), \dots, (6,1), \dots, (6,6)\}$.

A = the event of at least one six.
$\therefore$, $\bar{A}$ is the event that neither die is a six.

classical probability of each die to NOT be a six: $\frac{n(x)}{N}$.
where $n(x)$ is the number of outcomes that satisfy the roll not being a 6 – ie; $\{1,2,3,4,5\} \Rightarrow 5$, and N is the total number of outcomes in the set – ie; $\{1,2,3,4,5,6\} \Rightarrow 6$.

$$\therefore \frac{n(x)}{N} = \frac{5}{6}!$$

Since both dice are independent, the probability of both NOT being a six (aka $\bar{A}$) is $\frac{5}{6} \cdot \frac{5}{6}$.

$\therefore$, $P(\bar{A}) = \frac{25}{36}$, so $P(A) = 1 - \frac{25}{36} = \frac{11}{36}$!

Rules for Union:

- **Union of two events:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: At a university, 70% of the students own a car, 60% of students live on campus, and 50% both live on campus and own a car. If a student is chosen at random, what is the probability that the student does not own a car and does not live on campus?

- Let A be the event that a student has a car.
- Let B be the event that a student lives on campus.

we want: $P(\bar{A} \cap \bar{B})$.

↳ we know that $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

↳ $P(\bar{A} \cap \bar{B}) = 1 - P(\overline{\bar{A} \cap \bar{B}}) = 1 - P(A \cup B)$

from the union of two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$
\begin{aligned}
\therefore, P(\bar{A} \cap \bar{B}) &= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= 1 + P(A \cap B) - P(A) - P(B) \\
&= 1 + 0.5 - 0.7 - 0.6
\end{aligned}
$$

↳ $\therefore, P(\bar{A} \cap \bar{B}) = 0.2 = 20\%$.

Union of Three Events: for any 3 events A, B, and C:

↓

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Conditional Probability: determining the probability of some event A, while knowing that some related event B has occured.

Example: consider that a fair die has been rolled and you are asked to give the probability that it was a 5.

↳ $A = \{5\}$, $S = \{1, 2, 3, 4, 5, 6\}$.

$\therefore, P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}$.

However, if you're told that the dice has landed on an odd number, the odds change:

↳ $A = \{5\}$, $S = \{1, 3, 5\}$.

$\therefore, P(A) = \frac{n(A)}{n(S)} = \frac{1}{3}$.

Example: a box contains 100 microchips, some produced by Factory 1 and the rest by Factory 2. Among these microchips, some are defective, while others are good. Let A be the event of "obtaining a defective microchip" and B be the event of "the microchip was produced in Factory 2."

the number of microchips in each category:

	B	$\bar{B}$	totals
A	15	5	20
$\bar{A}$	45	35	80
totals	60	40	100

a) Find the probability of obtaining a defective microchip.

the total of A is 20, and there are 100 total microchips.

↳ $P(A) = \frac{n(A)}{N} = \frac{20}{100} = \frac{1}{5}$ (20%)

b) Given that a microchip is from factory 1, find the probability that it is defective.

the sample space S shrinks from all 100 microchips to only the 60 made by factory 1 (event B).

↳ ∴, $P(A) = \frac{n(A)}{N} = \frac{15}{60} = \frac{1}{4}$ (25%)

Conditional Probability mathematical definition:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad (P(B) > 0)$$

↳ "the probability of A given B"

- $0 \leq P(A \mid B) \leq 1$
- $P(A \mid B) = 1 - P(\bar{A} \mid B)$
- $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

pretty much the same as the regular union of two events

Multiplication Theorem of Probability: a way to compute the joint occurence of A and B:

$$P(A \cap B) = P(B) \cdot P(A \mid B) = P(A) \cdot P(B \mid A)$$

- **Law of Total Probability:** if $B_1, B_2, \ldots, B_k$ is a collection of mutually exclusive events and $B_1 \cup B_2 \cup \ldots \cup B_k = S$, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A \mid B_i)$$

Note: for a special case $k=2$:

only for $k=2$!!

$$P(A) = P(A \mid B) \cdot P(B) + P(A \mid \bar{B}) \cdot P(\bar{B})$$