

eg: $\{a, b, c, b\}$ is not a set!

Sets: an unordered collection of distinct objects.

↳ $\{a, b, c\} = \{c, b, a\}$ is a set of size 3

↳ $\{0, 1, 2, \dots\} = \mathbb{N}$ is an infinite set

Set operations: let A, B be finite sets:

• Cartesian Product: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

• Size of Cartesian Product: $|A \times B| = |A| \cdot |B|$

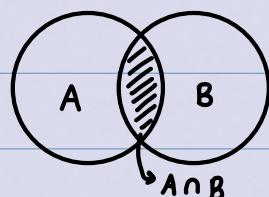
↳ eg: how many binary strings have length 4?

$$\underline{0/1} \quad \underline{0/1} \quad \underline{0/1} \quad \underline{0/1} \rightarrow 2 \times 2 \times 2 \times 2 = 2^4 = 16.$$

$$\# \text{ strings} = |\{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\}| = |\{0, 1\}| \times |\{0, 1\}| \times |\{0, 1\}| \times |\{0, 1\}| = 2 \times 2 \times 2 \times 2 = 16.$$

In general, the number of binary strings of length n is 2^n .

• Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.



• Size of Union: $|A \cup B| = |A| + |B| - |A \cap B|$

↳ if $|A \cap B| = 0$, then $|A \cup B| = |A| + |B|$.

↳ $A \cup B$ with $|A \cap B| = 0$ is a disjoint union.

↳ eg: how many binary strings of length 8 are there which begin with 001 or 1011?

let S be the set of all strings.

let A_1 = set of length-8 binary strings that begin with 001.

let A_2 = set of length-8 binary strings that begin with 1011.

then $S = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. So, $|S| = |A_1| + |A_2| = 2^5 + 2^4 = 48$.

Counting Subsets

Permutations: a permutation of a set S is an ordered listing of the elements in S .

↳ eg: the permutations of $\{a, b, c\}$ are abc, acb, bac, bca, cab, cba.

• the number of permutations of a set S of size n is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{2 \text{ choices}} \times \underline{1 \text{ choice}} = n!$$

Partial Permutations: a permutation of a subset of S .

• number of partial permutations of S (with $|S|=n$) of size k is:

$$\underline{n \text{ choices}} \times \underline{n-1 \text{ choices}} \times \underline{n-2 \text{ choices}} \times \dots \times \underline{n-k+2 \text{ choices}} \times \underline{n-k+1 \text{ choices}}$$

$$\therefore n(n-1)(n-2)\dots(n-k+2)(n-k+1).$$

↳ note: this works even if $k > n$.

How many subsets of S (with $|S|=n$) are there of size k ?

- number of partial permutations of S of size k is $n(n-1)(n-2)\dots(n-k+1)$.
- each size- k subset of S has $k!$ permutations.

↳ $\therefore \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

• Combinations: let $0 \leq k \leq n$. Then "n choose k " is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (\text{aka, binomial coefficient})$$

↳ note, if $k > 1$, then $\binom{n}{k} = 0$.

eg: $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = \frac{1}{0!} = 1$.

eg: $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{1}{0!} = 1$.

eg: $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$.

eg: $\binom{1999}{73} = \frac{1999 \times 1998 \times 1997 \times 1927}{73 \times 72 \times \dots \times 1} = \text{some positive integer.}$

Binomial Theorem: for any $n \geq 1$, $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k$

eg: $(x+1)^3 = (x+1)(x+1)(x+1) = x^3 + 3x^2 + 3x + 1.$

$$\begin{array}{ccccccc} & \overset{\binom{3}{3}=1}{\nearrow} & \overset{\binom{3}{2}}{\nearrow} & \overset{\binom{3}{1}}{\nearrow} & \overset{\binom{3}{0}=1}{\nearrow} \\ & & & & & & \end{array}$$

Combinatorial Proof of the Binomial Theorem:

$$(x+1)^n = (x+1)(x+1) \dots \overset{n \text{ copies of } (x+1)}{(x+1)}$$

how do we get an x^k term? by selecting x from k binomials and 1 from the other $n-k$ binomials. \therefore the total number of ways is $\binom{n}{k}$.

Therefore, $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k$. \square .

Combinatorial Proofs

- Procedure to prove $M=N$:

- 1) "Cleverly" choose a set S

- 2) Count the number of elements in S in two ways:

- a) $|S|=M$ and

- b) $|S|=N$

- 3) Conclude that $M=N$.

Claim: $2^n = \sum_{k=0}^n \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

↳ note: algebraically, we can prove this by substituting $x=1$ into the binomial theorem.

Combinatorial proof:

- let S be the set of all subsets of $\{1, 2, \dots, n\}$.

- a) let S_i be the subsets in S of size i , where $0 \leq i \leq n$.

Then $S = S_0 \cup S_1 \cup \dots \cup S_n$ (note, a disjoint union).

$$\therefore |S| = |S_0| + |S_1| + |S_2| + \dots + |S_n|$$

$$|S| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{k=0}^n \binom{n}{k}.$$

b) to choose a subset X of $\{1, 2, \dots, n\}$, we do:

- for each i , $1 \leq i \leq n$, we include i in X , or not.

- Therefore, the total number of choices is 2^n . Hence $|S| = 2^n$.

$$\therefore |S| = \sum_{k=0}^n \binom{n}{k} = 2^n. \quad \square.$$

Claim: let $1 \leq k \leq n$. Then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ [Pascal's Identity].

• Algebraic Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} + \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} = \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} + \frac{(n-1)! \cdot k}{k!(n-k)!}$$

$$= \frac{(n-1)![(n-k)+k]}{k!(n-k)!} = \frac{(n-1)! \cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \quad \square.$$

• Combinatorial Proof:

let S = set of all size- k subsets of $\{1, 2, 3, \dots, n\}$.

then, $|S| = \binom{n}{k}$ by definition.

let A = size- k subsets that don't include n .

let B = size- k subsets that includes n .

Then $S = A \cup B$ is a disjoint union. So, $|S| = |A| + |B|$.

Now, $|A| = \binom{n-1}{k}$, and $|B| = \binom{n-1}{k-1}$. Thus, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. \square .

• Claim: for $k, n \geq 0$, $\binom{n+k}{n} = \sum_{i=0}^k \binom{n+i-1}{n-1} = \binom{n-1}{n-1} + \binom{n}{n-1} + \dots + \binom{n+k-1}{n-1}$

Combinatorial Proof:

$$\therefore |S| = \binom{n+k}{n}$$

• let S = the size- n subsets of $\{1, 2, \dots, n, n+1, \dots, n+k-1, n+k\}$.

↳ note: the largest number in a size- n subset is between n and $n+k$.

for each $0 \leq i \leq k$, let S_i = subset whose largest element is $n+i$.

Then, S = disjoint union $S_0 \cup S_1 \cup \dots \cup S_k$. So, $|S| = \sum_{i=0}^k |S_i| = \sum_{i=0}^k \binom{n+i-1}{n-1}$. \square .

Visually:

See that the LHS is size- n subsets of $n+k$ sets. So, we choose subsets of size n (k times):

Subsets: $\{\underbrace{\quad}_{n \text{ size}}\} \cup \{\underbrace{\quad}_n\} \cup \dots \cup \{\underbrace{\quad}_n\} \cup \{\underbrace{\quad}_n\}$

$\cdot \{\underbrace{\quad}_n\} \cup \{\underbrace{\quad}_n\} \cup \dots \cup \{\underbrace{\quad}_n\} \cup \{\underbrace{\dots, n+k}_n\}$

↳ see that the last subset can be counted as $\binom{n+k-1}{n-1}$.

Claim: for $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$

↳ eg: $\binom{100}{98} = \binom{100}{2} = \frac{100 \times 99}{2}$.

eg: $n=4, k=1$: size-1 subsets of $\{1, 2, 3, 4\} = \{1\}, \{2\}, \{3\}, \{4\}$.

size-3 subsets of $\{1, 2, 3, 4\} = \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}$.

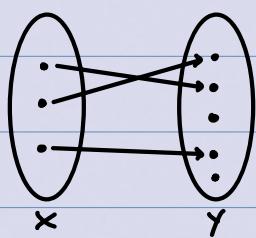
we can pair these sets by picking the size-3 subset which does not contain the element in the size-1 subset. ie: $\{1\} \leftrightarrow \{2, 3, 4\}, \{2\} \leftrightarrow \{1, 3, 4\} \dots$

→ aka, a correspondence or a bijection!

Bijections: let $f: X \rightarrow Y$ be a function.

• f is injective (1-1) if $\forall x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

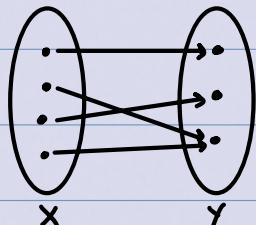
• Visually:



. see that $|X| \leq |Y|$.

• f is surjective (onto) if $\forall y \in Y, \exists x \in X \text{ st } f(x) = y$.

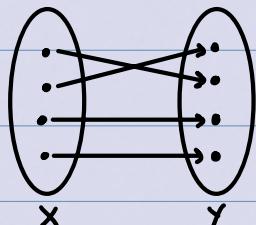
• Visually:



. see that $|X| \geq |Y|$

• f is bijective if it is injective AND surjective.

• Visually:

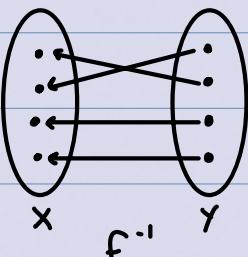
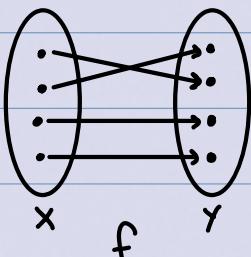


. see that $|X| = |Y|$.

• Theorem: let X, Y be finite sets. Suppose $f: X \rightarrow Y$ is a bijection. Then, $|X| = |Y|$.

• Inverse function: the inverse of a function $f: X \rightarrow Y$ is a function $f^{-1}: Y \rightarrow X$ such that:

- 1) (f^{-1} reverses f) $\Rightarrow \forall x \in X, f^{-1}(f(x)) = x$.
- 2) (f reverses f^{-1}) $\Rightarrow \forall y \in Y, f(f^{-1}(y)) = y$.



• Theorem: let $f: X \rightarrow Y$. Then f is a bijection if and only if f has an inverse.

Combinatorial Proofs Using Bijections that $M=N$.

1) Select two sets X, Y , with $|X|=M$ and $|Y|=N$.

2) Define $f: X \rightarrow Y$. (f is a bijection).

3) Define $f^{-1}: Y \rightarrow X$.

4) Prove that f^{-1} is the inverse function

↳ prove the two conditions of the inverse function definition are satisfied.

5) Conclude that $M=N$.

• A function $f: X \rightarrow Y$ is well-defined if $\forall x \in X, f(x) \in Y$.

Claim: for $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$ (continued)

let X be the set of all size- k subsets of $\{1, 2, \dots, n\}$.

$$\therefore |X| = \binom{n}{k}.$$

let Y be the set of all size- $(n-k)$ subsets of $\{1, 2, \dots, n\}$.

$$\therefore |Y| = \binom{n}{n-k}.$$

Define $f: X \rightarrow Y$ as follows: $\forall S \in X, f(S) = S^c$ ($S^c = \{1, 2, \dots, n\} - S$).

• See that f is well-defined, ie, $f(S) = S^c \in Y$ since $|S^c| = n-k$

Define $f^{-1}: Y \rightarrow X$ as follows: $\forall T \in Y, f^{-1}(T) = T^c$ ($T^c = \{1, 2, \dots, n\} - T$).

• See that f^{-1} is well-defined, ie, $f^{-1}(T) = T^c \in X$ since $|T^c| = k$.

now, $\forall S \in X, f^{-1}(f(S)) = f^{-1}(S^c) = (S^c)^c = S$.

$$\forall T \in Y, f(f^{-1}(T)) = f(T^c) = (T^c)^c = T.$$

$\therefore f$ is a bijection, and therefore $|X|=|Y|$, so $\binom{n}{k} = \binom{n}{n-k}$.

Generating Series

↳ we'll encode solutions to counting problems as coefficients of a "generating series".

• eg: how many subsets of $\{1, 2, 3\}$ have size n , $\forall n \ 0 \leq n \leq 3$?

let $S = \text{all size-}n \text{ subsets of } \{1, 2, 3\} \ \forall n \ 0 \leq n \leq 3$.

define weight function $w: S \rightarrow \mathbb{N}$, by $w(\sigma) = |\sigma| \ \forall \sigma \in S$.

$\sigma \in S$	$\{\emptyset\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$w(\sigma)$	0	1	1	1	2	2	2	3

associate each element $\sigma \in S$ with the term $x^{w(\sigma)}$:

$$x^{w(\sigma)} \quad | \quad x \quad x \quad x \quad x^2 \quad x^2 \quad x^2 \quad x^3$$

the generating series of X with respect to w is the sum of the $x^{w(\sigma)}$ terms:

$$\Phi_S^w(x) = 1 + 3x + 3x^2 + x^3 = (x+1)^3.$$

↑
3 elements in S have weight 1
↓
3 elements in S have weight 2

\therefore the number of size- n subsets of $\{1, 2, 3\}$ is the coefficient of x^n in $\Phi_S^w(x)$, $\forall 0 \leq n \leq 3$.

(of any length!)

• eg: how many binary strings don't have 000 or 00111 as a substring?

• let $S = \text{all binary strings with no 000 or 00111 as a substring}$.

↳ we want to organize these strings by their length, so:

define $w(\sigma)$ be the length of σ (where $\sigma \in S$).

Definition: let S be a set.

- A function $w: S \rightarrow \mathbb{N}$ is a weight function if for $n \in \mathbb{N}$ there are only finitely many elements in S of weight n .
- The generating series for S with respect to w is $\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)}$.
- The coefficient of x^n in a $\Phi_S^w(x)$ is denoted $[x^n] \Phi_S^w(x)$.
↳ eg: $[x^{73}] (x+1)^{97} = \binom{97}{73} \rightarrow \binom{97}{73} x^{73}$.
↳ result: $[x^n] \Phi_S^w(x)$ is the number of elements in S of weight n .

Eg: $S = \text{all subsets from } \{1, 2, \dots, n\}$.

For $\sigma \in S$, $w(\sigma) = |\sigma|$.

$$\text{Then, } \Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{i=0}^n \binom{n}{i} x^i = (x+1)^n$$

$$\text{So, } [x^i] \Phi_S^w(x) = \binom{n}{i}.$$

Eg: $S = \text{set of all binary strings}$.

for $\sigma \in S$, let $w(\sigma) = \text{length of } \sigma$.

$$\text{Then, } \Phi_S^w(x) = ? + ?x + ?x^2 + ?x^3 + \dots$$

$\nearrow \# \text{binary strings of length 1}$
 $\nwarrow \# \text{binary strings of length 0}$ $\nearrow \# \text{binary strings of length 2}$.

$$\therefore \Phi_S^w(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$$

Formal Power Series (FPS)

• An FPS is an expression of the form $A(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k \geq 0} a_k x^k$, where $a_i \in \mathbb{R}$.

↳ we don't care about convergence or divergence - we only care about

the coefficients.

- So, ∞ is called an "indeterminate".

Operations on Formal Power Series

$$\left. \begin{aligned} \text{let } A(x) &= a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k \geq 0} a_k x^k \\ \text{let } B(x) &= b_0 + b_1 x + b_2 x^2 + \dots = \sum_{k \geq 0} b_k x^k. \end{aligned} \right\} \text{FPSs.}$$

$$1) (=): A(x) = B(x) \Leftrightarrow a_k = b_k \quad \forall k \geq 0.$$

$$2) (+): A(x) + B(x) = \sum_{k \geq 0} (a_k + b_k) x^k$$

$$2b) (-): A(x) - B(x) = \sum_{k \geq 0} (a_k - b_k) x^k$$

$\curvearrowleft = \sum_{i=0}^n a_i b_{n-i}$

$$3) (\times): A(x) \cdot B(x) = C(x) = \sum_{n \geq 0} C_n x^n, \text{ where } C_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

4) (Inversion): the inverse of a FPS $A(x)$, if it exists, is a new FPS $B(x)$ such that $A(x)B(x) = 1$.

↳ we'll write $B(x) = A(x)^{-1} = \frac{1}{A(x)}$.

Example: Show that $(1-x)^{-1} = 1+x+x^2+\dots = \sum_{k \geq 0} x^k$.

$$\text{let } P(x) = \text{RHS} = \sum_{k \geq 0} x^k.$$

$$\text{Then, } (1-x)P(x) = (1-x) \sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - x \sum_{k \geq 0} x^k = \sum_{k \geq 0} x^k - \sum_{k \geq 0} x^{k+1}$$

$$= \sum_{k \geq 0} x^k - \sum_{k \geq 1} x^k = 1 + \left(\sum_{k \geq 1} x^k - \sum_{k \geq 1} x^k \right) = 1 + 0 = 1.$$

$$1) (1-x)^{-1} = \frac{1}{1-x} = \sum_{k \geq 0} x^k \rightarrow [x^n](1-x)^{-1} = 1 \quad \forall n \geq 0.$$

↳ geometric series!

$$2) \text{partial geometric series: } 1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}.$$

↑ partial bc not infinite!
bc if $k > n, \binom{n}{k} = 0$ anyway.

$$3) \text{binomial series: } (x+1)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^n \binom{n}{k} x^k. \quad \therefore [x^k](x+1)^n = \binom{n}{k}.$$

$$4) \text{negative binomial series: } \forall n \geq 1, (1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k$$

↳ so $[x^k](1-x)^{-n} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

Example: determine $[x^n] \left(\frac{x}{1-2x} \right)^5$.

$$[x^n] \left(\frac{x}{1-2x} \right)^5 = [x^n] x^5 (1-2x)^{-5} = [x^{n-5}] (1-2x)^{-5} \text{ for } n \geq 5.$$

$$\text{Then, see that } (1-x)^{-5} = \sum_{k \geq 0} a_k x^k. \text{ So, } (1-2x)^{-5} = \sum_{k \geq 0} a_k x^k 2^k.$$

$$= 2^{n-5} [x^{n-5}] (1-x)^{-5} = 2^{n-5} \binom{n-5+5-1}{5-1} = 2^{n-5} \binom{n-1}{4}.$$

Proving the negative binomial theorem $(1-x)^{-n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k, n \geq 1$.

Combinatorial Proof (sketch):

$$\begin{aligned} (1-x)^{-n} &= (1-x)^{-1} \cdot (1-x)^{-1} \cdot \dots \cdot (1-x)^{-1} \quad (\text{n times}). \\ &= (1+x+x^2+\dots) \cdot (1+x+x^2+\dots) \cdot \dots \cdot (1+x+x^2+\dots) \quad (\text{n times}) \end{aligned}$$

We get an x^k term by selecting x^{a_1} from the first series, x^{a_2} from the second series, ..., and x^{a_n} from the last series, where a_1, a_2, \dots, a_k are non-negative integers with $a_1 + a_2 + \dots + a_n = k$. Then, multiplying the x^{a_i} terms gives $x^{a_1} x^{a_2} \dots x^{a_n} = x^{a_1 + a_2 + \dots + a_n} = x^k$.

The number of ways of choosing the a_i 's is equal to the coefficient of x^k in $(1-x)^{-n}$.

Illustration by example: $n=4$, $k=13$:

begin with a string of $k=13$ 0s and then insert $n-1=3$ 1s.

$\hookrightarrow \underbrace{000}_{a_1=3} \underbrace{10000}_{a_2=4} \underbrace{00}_{a_3=2} \underbrace{10}_{a_4=4} \underbrace{000}_{}$

$$\text{note that } a_1 + a_2 + a_3 + a_4 = 3 + 4 + 2 + 4 = 13!$$

This gives a bijection between binary strings of length $k+n-1$ with exactly $n-1$ ones, and non-negative integers a_1, a_2, \dots, a_n st $\sum_{i=1}^n a_i = k$.

\therefore , the number of non-negative integers a_1, \dots, a_k with $\sum_{i=1}^n a_i = k$ is equal to the number of binary strings of length $k+n-1$ with exactly $n-1$ ones, which is $\binom{k+n-1}{n-1} = \binom{n+k-1}{k}$.

\therefore , the coefficient of x^k in $(1-x)^{-n}$ is $\binom{n+k-1}{k}$. \square .

Extracting Coefficients from an FPS:

let $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$.

$$1) [x^n] (A(x) \pm B(x)) = [x^n] A(x) + [x^n] B(x).$$

$$2) [x^n] (A(x) \cdot B(x)) = \sum_{i=0}^n a_i b_{n-i} = \sum_{i=0}^n ([x^i] A(x)) ([x^{n-i}] B(x))$$

$$3) [x^n] c A(x) = c [x^n] A(x), \text{ where } c \text{ is a constant.}$$

$$4) [x^n] x^l A(x) = 0 \text{ if } n < l, \text{ and } [x^n] x^l A(x) = [x^{n-l}] A(x) \text{ if } n \geq l.$$

$$5) [x^n] A(cx) = c^n [x^n] A(x), \text{ where } c \text{ is a constant}$$

6) $[x^n] A(x^l) = 0$ if $l \neq n$, and $[x^n] A(x^l) = [x^{n-l}] A(x)$ if $l \mid n$.

Example: determine $[x^n] x^3(3x+1)^7$ where $n \geq 3$.

$$[x^n] x^3(3x+1)^7 = [x^{n-3}] (3x+1)^7 = 3^{n-3} [x^{n-3}] (x+1)^7 \\ = 3^{n-3} \binom{7}{n-3}$$

Example: determine $[x^n] x^3(3x+1)^7(1-4x^2)^{-m}$, $m \geq 1$, $n \geq 3$.

$$[x^n] x^3(3x+1)^7(1-4x^2)^{-m}$$

$$= [x^n] x^3 \left(\sum_{i \geq 0} \binom{7}{i} (3x)^i \right) \left(\sum_{j \geq 0} \binom{m+j-1}{j} (4x^2)^j \right)$$

$$= [x^n] \sum_{i \geq 0, j \geq 0} \binom{7}{i} \binom{m+j-1}{j} 3^i 4^j x^{3+i+2j}$$

$$= \sum_{i \geq 0, j \geq 0} \binom{7}{i} \binom{m+j-1}{j} 3^i 4^j, \text{ and only care where } 3+i+2j=n.$$

$$\hookrightarrow i=n-2j-3, \quad j=\frac{n-3-i}{2}$$

Substituting:

$$= \sum_{j \geq 0} \binom{7}{n-2j-3} \binom{m+j-1}{j} 3^{n-2j-3} 4^j$$

$$\text{Since } j=\frac{n-3-i}{2}, \quad j \leq \frac{n-3}{2} \text{ (as } i \geq 0\text{). } \therefore j \leq \lfloor \frac{n-3}{2} \rfloor$$

$$\lfloor \frac{n-3}{2} \rfloor$$

$$= \sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{7}{n-2j-3} \binom{m+j-1}{j} 3^{n-2j-3} 4^j.$$

Example: how many length-64 binary strings have neither 000 nor 00111 as a substring?

• don't know how to do this for specifically length-64. So, let's generalize for length- n and make a generating series.

• Roadmap:

- 1) Let S be a set we wish to count by weight
- 2) Decompose S into "simpler" sets using disjoint union and cartesian product
- 3) Determine the generating series for the simpler sets
- 4) Combine the generating series $\Phi_{S_i}(x)$ to get $\Phi_S(x)$.
- 5) The number of elements of weight n in S is $[x^n] \Phi_S(x)$.

• Sum Lemma: let $S = A \cup B$ be a disjoint union. Let w be a weight function $S \rightarrow N$.

Then, $\Phi_S^w(x) = \Phi_A^w(x) + \Phi_B^w(x)$.

↳ Proof:

$$\Phi_S^w(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in A \cup B} x^{w(\sigma)} = \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)} = \Phi_A^w(x) + \Phi_B^w(x).$$

• Product Lemma: let $S = A \times B = \{(a, b) : a \in A, b \in B\}$. Let $u: A \rightarrow N$ be a weight function on A , and let $v: B \rightarrow N$ be a weight function on B . Define the weight function $w: S \rightarrow N$ as follows:

• If $\sigma = (a, b) \in S$, $w(\sigma) = u(a) + v(b)$.

Then, $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_B^v(x)$.

↳ Proof:

$$\Phi_A^u(x) \cdot \Phi_B^v(x) = \left(\sum_{a \in A} x^{u(a)} \right) \left(\sum_{b \in B} x^{v(b)} \right) = \sum_{a \in A, b \in B} x^{u(a) + v(b)} = \sum_{\sigma \in S} x^{w(\sigma)} = \Phi_S^w(x).$$

eg: find the generating series of binary strings of length n where the weight of the string is the number of 1s in it.

Method 1:

let $S = \text{all binary strings of length } n$.

Then, $\Phi_S^w(x) = \sum_{k \geq 0} \alpha_k x^k$, where $\alpha_k = \text{number of elements of weight } k$
 $= \text{number of binary strings of length } n \text{ with } k \text{ 1s} = \binom{n}{k}$.

$$\therefore \Phi_S^w(x) = \sum_{k \geq 0} \binom{n}{k} x^k = (1+x)^n$$

Method 2:

let $A = \{0, 1\}$. Define $u(0) = 0$ and $u(1) = 1$.

$$\Phi_A^u(x) = x^{u(0)} + x^{u(1)} = x^0 + x^1 = 1 + x.$$

now, we use A to describe the more complicated set S :

$S = A \times A \times A \times \dots \times A$ (n times).

define $w: S \rightarrow \mathbb{N}$ as follows:

$\sigma = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then $w(\sigma) = \# \text{ 1s in } \sigma = u(\alpha_1) + u(\alpha_2) + \dots + u(\alpha_n) = \sum_{i=0}^n u(\alpha_i)$.

→ see that the weight of the complicated set S is the sum of the weight of the less complicated set A .

\therefore by the product lemma, $\Phi_S^w(x) = \Phi_A^u(x) \cdot \Phi_A^u(x) \cdots \Phi_A^u(x)$ (n times).

$$= [\Phi_A^u(x)]^n = (1+x)^n.$$

☞ note: order matters!!

Compositions: A composition of $n \in \mathbb{N}$ is a sequence of positive integers $(\alpha_1, \alpha_2, \dots, \alpha_k)$ that add to n . $k = \# \text{ of parts}$.

example: the compositions of $n=4$ are:

$(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2), (1,3), (3,1), (4)$.

Note: $n=0$ has one composition, $()$.

How many k -part compositions of n are there?

Let S = the set of all compositions with k parts.

Define $w: S \rightarrow \mathbb{N}$ as follows:

if $\sigma = (\alpha_1, \alpha_2, \dots, \alpha_k)$, then $w(\sigma) = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_k$.

Goal: find $\Phi_S^w(x)$, and the answer will be $[x^n] \Phi_S^w(x)$.

let $P = \{1, 2, 3, 4, \dots\}$.

then, $S = P \times P \times P \times \dots \times P$ (k times) $= P^k$ ↑ cartesian product of k copies of P .

define $u: P \rightarrow \mathbb{N}$ by $u(a) = a \quad \forall a \in P$.

Then, $w(\sigma) = \alpha_1 + \alpha_2 + \dots + \alpha_k = u(\alpha_1) + u(\alpha_2) + \dots + u(\alpha_k)$.

by the product lemma, $\Phi_S^w(x) = \Phi_P^u(x) \times \dots \times \Phi_P^u(x)$ (k times)
 $= [\Phi_P^u(x)]^k$.

Now, $\Phi_P^u(x) = \sum_{a \in P} x^{u(a)} = x + x^2 + x^3 + x^4 + \dots = x(1+x+x^2+\dots) = x(1-x)^{-1}$.

So, $\Phi_S^w(x) = [\Phi_P^u(x)]^k = x^k (1-x)^{-k}$

\therefore the number of k -part coefficients of n is $[x^n] x^k (1-x)^{-k}$
 $= [x^{n-k}] (1-x)^{-k} = \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$.

String Lemma: let S be a set with weight function w .

Suppose that S has no elements of weight 0. Then,

$$\Phi_{S^*}^{\omega^*}(x) = \frac{1}{1 - \Phi_S^\omega(x)}.$$

Proof: let S be a set and $\omega: S \rightarrow N$ it's weight function.

let S^k denote $S \times S \times S \times \dots \times S$ (k times) ($S^0 = \{\emptyset\}$).

Define $S^* = S' \cup S^2 \cup S^3 \cup \dots = \bigcup_{k \geq 0} S^k$ is a disjoint union.

Define $\omega_k: S^k \rightarrow N$: if $(a_1, a_2, \dots, a_k) \in S^k$, then $\omega_k = \omega(a_1) + \dots + \omega(a_k)$.

Define $\omega^*: S^* \rightarrow N$: $\omega^*(\sigma) = \omega_k(\sigma)$ where $\sigma \in S^k$.

→ we assume that S has no elements of weight 0!!

Then, $\Phi_{S^*}^{\omega^*}(x) = \sum_{k \geq 0} \Phi_{S^k}^{\omega_k}(x)$ by the sum lemma.

Since $\Phi_{S^k}^{\omega_k}(x) = \prod_{i=1}^k \Phi_S^\omega(x) = [\Phi_S^\omega(x)]^k$.

We have $\Phi_{S^*}^{\omega^*}(x) = \sum_{k \geq 0} [\Phi_S^\omega(x)]^k = \frac{1}{1 - \Phi_S^\omega(x)}$.

How many compositions of n are there where each part is ≥ 2 ?

let $S = \text{all compositions where each part is } \geq 2$.

let $P = \{2, 3, 4, \dots\}$

Then, $S = P^0 \cup P \cup P^2 \cup P^3 \dots$. So, $S = \bigcup_{k \geq 0} P^k$, ∴ $S = P^*$.

By the string lemma, $\Phi_S(x) = \frac{1}{1 - \Phi_P(x)}$.

$$\text{Now, } \Phi_P(x) = x^2 + x^3 + x^4 + \dots$$

$$= x^2(1 + x + x^2 + \dots)$$

$$= x^2(1-x)^{-1}$$

$$\therefore \Phi_S(x) = \frac{1}{1 - \frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}. \text{ So, } [x^n] \frac{1-x}{1-x-x^2}$$

A formal power series $A(x)$ is rational if there exist polynomials $P(x), Q(x)$ such that $A(x) = P(x)/Q(x)$.

- we extract coefficients from ration power series using recurrences and partial fractions.

How many compositions have an odd number of parts, where each part $\equiv 1 \pmod{3}$.

Let S = the set of all compositions with an odd number of parts, with each part $\equiv 1 \pmod{3}$.

Let $P = \{1, 4, 7, 10, 13, \dots\}$ (ie, $P = \{x : x \equiv 1 \pmod{3}\}$).

$$\begin{aligned} \text{Then, } S &= P^1 \cup P^3 \cup P^5 \cup P^7 \cup \dots && \rightarrow \text{since } A \times (B \cup C) = (A \times B) \cup (A \times C). \\ &= P \times (P^0 \cup P^2 \cup P^4 \cup P^6 \cup \dots) \\ &= P \times (P^2)^* \end{aligned}$$

$$\begin{aligned} \therefore \Phi_S(x) &= \Phi_P(x) \cdot \Phi_{(P^2)^*}(x) \quad \text{by the product lemma} \\ &= \Phi_P(x) \cdot \frac{1}{1 - \Phi_{P^2}(x)} \quad \text{by the string lemma} \end{aligned}$$

$$\therefore \Phi_P(x) = x + x^4 + x^7 + x^{10} + \dots = \frac{x}{1 - x^3}$$

$$\begin{aligned} \Phi_{P^2}(x) &= \Phi_P(x) \cdot \Phi_P(x) \quad \text{by the product lemma} \\ &= \frac{x^2}{(1-x^3)^2} \end{aligned}$$

$$\begin{aligned} \text{So, } \Phi_S(x) &= \frac{x}{1-x^3} \cdot \frac{1}{1 - \frac{x^2}{(1-x^3)^2}} = \frac{x}{1-x^3} \cdot \frac{(1-x^3)^2}{(1-x^3)^2 - x^2} = \frac{x - x^4}{1-x^2 - 2x^3 + x^6} \\ \therefore [x^n] \frac{x - x^4}{1-x^2 - 2x^3 + x^6} \end{aligned}$$

Binary Strings

A binary string σ is a sequence of bits b_1, b_2, \dots, b_n , where $b_i = 0$ or 1 for $1 \leq i \leq n$, and n is the length of σ .

- There's only one binary string of length 0: the empty string ϵ .
- We will (almost) always use the length of a binary string as its weight!

If $a = a_1, a_2, \dots, a_m$, $b = b_1, b_2, \dots, b_n$ are binary strings, their concatenation is $ab = a_1, a_2, \dots, a_m b_1, b_2, \dots, b_n$.

↳ note the length of ab is $m+n$.

↳ also, $\epsilon\sigma = \sigma$ ∀ strings σ .

a is a substring of b if $b = cad$ for some strings c and d .

→  → blocks

A block of a binary string is a non-empty maximal substring of all 0s or all 1s.

Regular Expressions: a method for generating/producing a set of binary strings.

Let A, B be binary strings.

1) $A \cup B$ is the union of A and B

↳ eg: $A = \{00\}$, $B = \{1, 11, 111\}$. $A \cup B = \{00, 1, 11, 111\}$ or $00 \cup (1 \cup 11 \cup 111)$.

2) $AB = \{ab : a \in A, b \in B\}$ is the concatenation of A and B

↳ eg: $A = \{1, 11\}$, $B = \{001\}$. $AB = \{1001, 11001\}$, or $(1 \cup 11)001$.

3) $A^* = \bigcup_{k \geq 0} A^k$, where $A^k = AAA\dots A$ ^{$\geq k$ times} and $A^0 = \{\epsilon\}$

↳ eg: $0^* \rightarrow$ all 0-strings of any length (including 0).

Examples: $(00)^*$ = { ϵ , 00, 0000, 000000, ...}.

$0(00)^*$ ($= (00)^*0$) = all odd-length strings of 0s

$(0 \cup 1)^*$ = all binary strings!

$(0 \cup 111)^*$ = all binary strings where each block of 1s has length a multiple of 3.

Ideally, we'd like to use the Sum, Product, and String lemmas to find the generating series for a set of strings produced by a regular expression. But you have to be careful!

eg: let $A = \{1, 0\}$, $B = \{1, 01\}$

Then $AB = \{11, 101, 01, 001\}$ Same-ish
 $A \times B = \{(1, 1), (1, 01), (0, 1), (0, 01)\}$.

eg: let $A = \{1, 10\}$, $B = \{1, 01\}$

Then $AB = \{11, 101, \cancel{101}, 1001\}$ not same!
 $A \times B = \{(1, 1), (1, 01), (10, 1), (10, 01)\}$.

So, $|AB| \neq |A \times B|$!! Issue when we're trying to count. Removing commas from $A \times B$ might cause ambiguity.

A regular expression is unambiguous if every string generated by the expression can be generated in exactly one way.

↳ eg: $(1 \cup 0)(1 \cup 01)$ is unambiguous.

$(1 \cup 10)(1 \cup 01)$ is ambiguous, since 101 can be generated in two ways: 101 and 10, 1.

Example: Is $0^* \cup 1^*$ ambiguous or not?

↳ ambiguous, because the empty string ϵ can be generated in two ways. (So, $\Phi_{0^* \cup 1^*} \neq \Phi_{0^*} + \Phi_{1^*}$).

Example: $S = (0^* 11 \cup 001^*)^*$. Is this an ambiguous expression?

↳ ambiguous, since 0011 can be generated in two ways.

Let A, B be sets of strings.

- 1) Then $A \cup B$ is unambiguous if the union is disjoint (ie $A \cap B = \emptyset$).
- 2) Then AB is unambiguous if $\forall \sigma \in AB$, there's exactly one pair of strings $a, b \in A \times B$ such that $\sigma = ab$.
- 3) Then A^* is unambiguous if A^R is unambiguous $\forall R$, and A^0, A^1, A^2, \dots are disjoint.

- If $A \cup B$ is unambiguous, then $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$. SUM LEMMA
- If AB is unambiguous, then $\Phi_{AB}(x) = \Phi_A(x) \cdot \Phi_B(x)$. PRODUCT LEMMA
- If A^* is unambiguous, then $\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$. STRING LEMMA

Counting Binary Strings

eg: find the generating series for all binary strings where

Weight = length:

1) $\Phi_S(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n = \# \text{ strings of length } n = 2^n$.

$$= \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}.$$

2) $(0 \cup 1)^*$ is an unambiguous expression for S.

So, $\Phi_S(x) = \Phi_{(0 \cup 1)^*}(x) = \frac{1}{1 - \Phi_{(0 \cup 1)}(x)}$ by string lemma

$$= \frac{1}{1 - (x^0 + x^1)} = \frac{1}{1-2x}$$

3) Claim: $1^* (00^* 11^*)^* 0^*$ is an unambiguous expression for S.

eg: $\underbrace{0\ 0\ 0\ 0\ 0}_{1^*} \underbrace{1\ 1\ *}_{00^*} \underbrace{0\ 0\ 1}_{11^*} \underbrace{0\ 0\ 1}_{00^*} \underbrace{0\ 0\ 1}_{11^*} \underbrace{0\ 0\ 1}_{00^*}$

justification: the decomposition of a string into its blocks is unique.

$$\text{So, } \Phi_S(x) = \Phi_{1^*(00^*11^*)^*0^*}(x)$$

$$\cdot \Phi_{1^*}(x) = \frac{1}{1-\Phi_1(x)} = \frac{1}{1-x} = \Phi_{0^*}(x).$$

$$\begin{aligned} \cdot \Phi_{00^*11^*}(x) &= \Phi_0(x) \cdot \Phi_{0^*}(x) \cdot \Phi_1(x) \cdot \Phi_{1^*}(x) \\ &= x \cdot \frac{1}{1-x} \cdot x \cdot \frac{1}{1-x} = \frac{x^2}{(1-x)^2} \end{aligned}$$

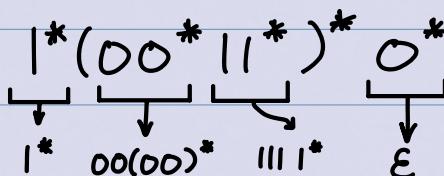
$$\text{So, } \Phi_{(00^*11^*)^*}(x) = \frac{1}{1-\Phi_{00^*11^*}(x)} = \frac{1}{1-\frac{x^2}{(1-x)^2}} = \frac{(1-x)^2}{(1-x)^2-x^2}$$

$$\begin{aligned} \therefore \Phi_{1^*(00^*11^*)^*0^*}(x) &= \Phi_{1^*}(x) \cdot \Phi_{(00^*11^*)^*}(x) \cdot \Phi_{0^*}(x) \\ &= \frac{1}{1-x} \cdot \frac{(1-x)^2}{(1-x)^2-x^2} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2-x^2} = \frac{1}{1-2x}. \end{aligned}$$

eg: find the generating series for all strings where every block of 0s has even length and must be followed by a block of at least 3 1s.

let S = set of all binary strings where every block of 0s has even length and is followed by a block of at least 3 1s.

idea: begin with $1^*(00^*11^*)^*0^*$. If we replace any portion of this block decomposition with an unambiguous expression for a non-empty subset of that portion, then the new expression is also unambiguous.



$\therefore 1^*(00(00)^*1111^*)^*$ is an unambiguous expression for S.

$$\begin{aligned}\Phi_S(x) &= \frac{1}{1-x} \cdot \frac{1}{1-(x^2 \cdot \frac{1}{1-x^2} \cdot x^3 \cdot \frac{1}{1-x})} = \frac{1}{(1-x)(1 - \frac{x^5}{(1-x^2)(1-x)})} \\ &= \frac{(1-x)(1-x^2)}{(1-x)((1-x^2)(1-x)-x^5)} = \frac{1-x^2}{1-x-x^2+x^3-x^5}\end{aligned}$$

Let A, B be sets such that $B \subseteq A$. Then, $\Phi_{A \setminus B}(x) = \Phi_A(x) - \Phi_B(x)$, since $|A \setminus B| = |A| - |B|$.

Eg: how many length-n binary strings don't have 111 as a substring?

$$(\epsilon \cup 1 \cup 11)(00^*(1 \cup 11 \cup \epsilon))^* 0^*$$

Eg: how many length-n binary strings don't have 110100 as a substring?

↳ we must use recursive decomposition here!!

Recursive Decompositions

- express a set in terms of itself

Eg: find a recursive decomposition for A = all binary strings.

every string σ either begins with a 0 or a 1 followed by a binary string, except if $\sigma = \epsilon$.

So, $A = \underbrace{\epsilon}_{\text{base case}} \cup \underbrace{0A \cup 1A}_{\text{recursive step}}$ is an unambiguous expression for A.

$$\therefore \Phi_A(x) = 1 + (2x \cdot \Phi_A(x)). \text{ Solving, we get } \Phi_A(x) = \frac{1}{1-2x}.$$

Eg: find a recursive decomposition which don't have 111 as a substring.

let A be the set of all strings which don't have 111 as a substring.

let $\sigma \in A$. Either σ has a 0 or it doesn't.

- If it doesn't, then $\sigma \in \{\epsilon, 1, 11\}$.

- If it does, then either σ begins with a 0 , 10 , or 110 , followed by a string in A .

So, $A = \underbrace{(\epsilon \cup 1 \cup 11)}_{\text{base case}} \cup \underbrace{(0 \cup 10 \cup 110) A}_{\text{recursive step}}$ is unambiguous.

$$\Phi_A(x) = (1+x+x^2) + (x+x^2+x^3) \Phi_A(x), \quad \text{so } \Phi_A(x) = \frac{1+x+x^2}{1-x-x^2-x^3}.$$

Eg: how many length-4 binary strings have neither 000 nor 00111 as a substring?

i) find a block decomposition for the set of all such strings.

ii) find $\Phi_S(x) = \frac{1+x+x^2}{1-x-x^2-x^3+x^5}$

iii) answer is $[x^4] \Phi_S(x)$.

We'll see that the number of such strings of length n is given by the recurrence relation $a_n - a_{n-1} - a_{n-2} - a_{n-3} + a_{n-5} = 0 \quad \forall n \geq 5$, which comes from the denominator of $\Phi_S(x)$.

So, $a_n = a_{n-1} + a_{n-2} + a_{n-3} - a_{n-5}$, but we need initial conditions s.t. $n \leq 4$.

By inspection, $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 7$, and $a_4 = 13$.

Then, $a_5 = 13 + 7 + 4 - 1 = 23$, $a_6 = 41$, $a_7 = 73$, $a_{64} = 13,076,512,262,747,676$.

Rational Power Series

(I) Rational series \rightarrow recurrence relation:

Eg: consider $A(x) = \frac{P(x)}{Q(x)} = \frac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \sum_{n \geq 0} a_n x^n$. Find a recurrence

relation for a_n .

↳ Strategy: Multiply both sides by $Q(x)$ and equate coefficients of x^n on both sides:

$$(1 - 8x + 21x^2 - 18x^3) \sum_{n \geq 0} a_n x^n = 1 - 2x + 3x^2$$

$$\hookrightarrow \sum_{n \geq 0} a_n x^n - 8 \sum_{n \geq 0} a_n x^{n+1} + 21 \sum_{n \geq 0} a_n x^{n+2} - 18 \sum_{n \geq 0} a_n x^{n+3} = 1 - 2x + 3x^2$$

$$\hookrightarrow \sum_{n \geq 0} a_n x^n - 8 \sum_{n \geq 1} a_{n-1} x^n + 21 \sum_{n \geq 2} a_{n-2} x^n - 18 \sum_{n \geq 3} a_{n-3} x^n = 1 - 2x + 3x^2$$

extracting $[x^n]$ of both sides:

$$n=0: a_0 = 1 \Rightarrow a_0 = 1$$

$$n=1: a_1 - 8a_0 = -2 \Rightarrow a_1 = 6$$

$$n=2: a_2 - 8a_1 + 21a_0 = 3 \Rightarrow a_2 = 30$$

$$n \geq 3: a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0.$$

∴ for $n \geq 3$, we have that $a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n-3}$.

↳ NOTE!!: $Q(x)$ was $1 - 8x + 21x^2 - 18x^3$, so these coefficients will always match!

Let a_0, a_1, a_2, \dots be a sequence of complex numbers.

Let c_1, c_2, \dots, c_d be constants.

Then, a_n satisfies a (linear homogeneous) recurrence relation if

$$a_n + c_1 a_{n+1} + c_2 a_{n+2} + \dots + c_d a_{n+d} = 0 \quad \forall n \geq d.$$

↳ note: a_0, a_1, \dots, a_{d-1} are the initial conditions

Theorem: Let $A(x) = P(x)/Q(x)$ be a rational power series

with $Q(x) = 1 + c_1x + c_2x^2 + \dots + c_dx^d$ AND $\deg(P) < \deg(Q)$

then the a_n satisfy a recurrence relation:

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_da_{n-d} \neq 0 \quad \forall n \geq d$$

with initial conditions $a_0, a_1, a_2, \dots, a_{d-1}$.

(II) recurrence relation \rightarrow rational series:

e.g.: consider $a_n - 7a_{n-1} - 16a_{n-2} + 12a_{n-3} = 0$, with $a_0 = 1$, $a_1 = 0$, and $a_2 = 2$.

find $P(x)$, $Q(x)$ such that $A(x) = \sum_{n \geq 0} a_n x^n = \frac{P(x)}{Q(x)}$.

Strategy: multiply both sides of the recurrence relation by $A(x)$ and sum for $n \geq 3$:

$$\hookrightarrow \sum_{n \geq 3} a_n x^n - 7 \sum_{n \geq 3} a_{n-1} x^n - 16 \sum_{n \geq 3} a_{n-2} x^n + 12 \sum_{n \geq 3} a_{n-3} x^n = 0$$

$$\sum_{n \geq 3} a_n x^n - 7x \sum_{n \geq 3} a_{n-1} x^{n-1} - 16x^2 \sum_{n \geq 3} a_{n-2} x^{n-2} + 12x^3 \sum_{n \geq 3} a_{n-3} x^{n-3} = 0$$

$$\sum_{n \geq 3} a_n x^n - 7x \sum_{n \geq 2} a_n x^n - 16x^2 \sum_{n \geq 1} a_n x^n + 12x^3 \sum_{n \geq 0} a_n x^n = 0$$

$$\therefore (A(x) - a_0 - a_1x - a_2x^2) - 7x(A(x) - a_0 - a_1x) - 16x^2(A(x) - a_0) + 12x^3A(x) = 0$$

$$\hookrightarrow A(x)[1 - 7x - 16x^2 + 12x^3] = a_0 + a_1x + a_2x^2 - 7x(a_0 + a_1x) - 16x^2a_0$$

Since $a_0 = 1$, $a_1 = 0$, and $a_2 = 2$, we get:

$$A(x) = [1 - 7x - 16x^2 + 12x^3] = 1 - 7x - 14x^2, \quad \text{so:}$$

$$A(x) = \frac{1 - 7x - 14x^2}{1 - 7x - 16x^2 + 12x^3}.$$

Theorem: let $\alpha_0, \alpha_1, \alpha_2, \dots$ be a sequence that satisfies a recurrence relation $\alpha_n + C_1\alpha_{n-1} + C_2\alpha_{n-2} + \dots + C_d\alpha_{n-d} = 0 \quad \forall n \geq d$ with initial conditions $\alpha_0, \alpha_1, \dots, \alpha_{d-1}$. Let $A(x) = \sum_{n \geq 0} \alpha_n x^n$. Then, $A(x) = \frac{P(x)}{Q(x)}$ where: $Q(x) = 1 + C_1x + C_2x^2 + \dots + C_dx^d$ and $\deg(P) < d$.

(III) Partial Fractions (rational series \rightarrow "explicit formula")

eg: let $A(x) = \frac{1-2x+3x^2}{1-8x+21x^2-18x^3} = \frac{P(x)}{Q(x)} = \sum_{n \geq 0} \alpha_n x^n$. Find an "explicit formula" for α_n .

1) Factor $Q(x)$ into a product of linear factors:

$$Q(x) = (1 - \lambda_1 x)^{d_1} (1 - \lambda_2 x)^{d_2} \dots (1 - \lambda_m x)^{d_m}$$

$$\text{we have } Q(x) = (1-2x)(1-3x)^2.$$

2) Partial Fractions: $A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-3x} + \frac{C_3}{(1-3x)^2}$

where C_1, C_2, C_3 are unique constants.

3) Find the constants (C_1, C_2, C_3) :

Multiply of sides of the partial Series expression by

$$Q(x) = (1-2x)(1-3x)^2$$

$$\hookrightarrow C_1(1-3x)^2 + C_2(1-2x)(1-3x) + C_3(1-2x) = 1-2x+3x^2$$

$$C_1(1-6x+9x^2) + C_2(1-5x+6x^2) + C_3(1-2x) = 1-2x+3x^2$$

equating coefficients of x^0, x^1, x^2 on both sides:

$$x^0 \rightarrow C_1 + C_2 + C_3 = 1$$

$$x^1 \rightarrow -6C_1 - 5C_2 - 2C_3 = -2$$

$$x^2 \rightarrow 9C_1 + 6C_2 = 3$$

Solving the systems of linear equations, we get that

$$C_1 = 3, C_2 = -4, C_3 = 2.$$

$$\text{So, } A(x) = \frac{3}{1-2x} - \frac{4}{1-3x} + \frac{2}{(1-3x)^2}$$

4) Extract coefficients

$$\begin{aligned} a_n &= [x^n] A(x) = [x^n] \frac{3}{1-2x} - [x^n] \frac{4}{1-3x} + [x^n] \frac{2}{(1-3x)^2} \\ &= 3 \cdot 2^n - 4 \cdot 3^n + 2 [x^n] \frac{1}{(1-3x)^2} \\ &= 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n [x^n] \frac{1}{(1-x)^2} = 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \binom{n+2-1}{2-1} \\ &= 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \binom{n+1}{1} = 3 \cdot 2^n - 4 \cdot 3^n + 2 \cdot 3^n \cdot (n+1) \\ &= 3 \cdot 2^n + (2n-2)3^n \end{aligned}$$

$$\therefore a_n = 3 \cdot 2^n + (2n-2)3^n \quad \forall n \geq 0.$$

Note: such a factorisation always exists if you allow λ_i to be complex, by the fundamental theorem of algebra!

↳ if you cannot factor $Q(x)$, then the method fails.

Notation: $Q(x) = (1-\lambda_1 x)^{d_1} (1-\lambda_2 x)^{d_2} \dots (1-\lambda_m x)^{d_m}$, where λ_i is an "inverse root" of $Q(x)$, and d_i is its multiplicity.

Eg: Suppose $A(x) = \frac{1-3x^2+7x^3}{(1+4x)^2(1-3x)^4}$

↳ $Q(x)$ has two inverse roots, -4 with multiplicity 2, and 3 with

multiplicity 4.

$$A(x) = \frac{C_1}{1+4x} + \frac{C_2}{(1+4x)^2} + \frac{C_3}{1-3x} + \frac{C_4}{(1-3x)^2} + \frac{C_5}{(1-3x)^3} + \frac{C_6}{(1-3x)^4}$$

polynomial of n in degree < 2, λ₁ polynomial in n of degree < 4, λ₂

$$\text{Then } a_n = [x^n] A(x) = (B + C_n)(-4)^n + (D + E_n + F_n^2 + G_n^3) \cdot 3^n$$

Theorem: let $A(x) = \frac{P(x)}{Q(x)}$, where $\deg(P) < \deg(Q)$ and the constant term of $Q(x)$ is 1. Factor $Q(x)$ into a product of linear polynomials $Q(x) = (1 - \lambda_1 x)^{d_1} \cdots (1 - \lambda_m x)^{d_m}$ where λ_i are the inverse roots of $Q(x)$, with multiplicities d_i . Then, there exist unique constants C_{ij} , where $1 \leq i \leq m$, $1 \leq j \leq d_i$ such that:

$$A(x) = \left(\frac{C_{1,1}}{1-\lambda_1 x} + \frac{C_{1,2}}{(1-\lambda_1 x)^2} + \frac{C_{1,d_1}}{(1-\lambda_1 x)^{d_1}} \right) + \cdots + \left(\frac{C_{m,1}}{1-\lambda_m x} + \frac{C_{m,2}}{(1-\lambda_m x)^2} + \cdots + \frac{C_{m,d_m}}{(1-\lambda_m x)^{d_m}} \right).$$

Moreover, $[x^n] A(x) = p_1(n) \lambda_1^n + \cdots + p_m(n) \lambda_m^n \quad \forall n \geq 0$, where $p_i(n)$ is a polynomial of degree $< d_i$.

(IV) recurrence → explicit formula

e.g. given $a_n - 6a_{n-1} + 12a_{n-2} - 10a_{n-3} + 3a_{n-4} = 0 \quad \forall n \geq 4$, with $a_0 = 5, a_1 = 11, a_2 = 23, a_3 = 49$. Find an explicit formula for a_n .

let $A(x) = \sum_{k \geq 0} a_k x^k = \frac{P(x)}{Q(x)}$, directly from recurrence!

where $Q(x) = 1 - 6x + 12x^2 - 10x^3 + 3x^4$
 $= (1-3x)(1-x)^3$ given!

the inverse roots of $Q(x)$ are $\lambda_1 = 3$ and $\lambda_2 = 1$ with multiplicities $d_1 = 1$ and $d_2 = 3$, respectively.

\therefore , an explicit formula for a_n is: $a_n = (\text{polynomial in } n \text{ of degree } < d_1=1) 3^n + (\text{polynomial in } n \text{ of degree } < d_2=3) 1^n$

So, $a_n = C \cdot 3^n + (D + E n + F n^2) \quad \forall n \geq 0.$

Now, we use the initial conditions to find the constants C, D, E , and F :

$$\cdot n=0: a_0 = 5 = C + D$$

$$\cdot n=1: a_1 = 11 = 3C + D + E + F$$

$$\cdot n=2: a_2 = 23 = 9C + D + 2E + 4F$$

$$\cdot n=3: a_3 = 49 = 27C + D + 3E + 9F$$

Solving, we get:
 $C=1, D=4, E=3, F=1$

$$\therefore a_n = 3^n + (4 + 3n + n^2) \quad \forall n \geq 0.$$

General Method: given a recurrence relation

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_d a_{n-d} \quad \forall n \geq d \text{ with initial}$$

conditions a_0, a_1, \dots, a_{d-1} , we find the explicit formula by:

- 1) Defining the denominator $Q(x) = 1 + C_1 x + C_2 x^2 + \dots + C_d x^d$
- 2) Find the inverse roots $\lambda_1, \lambda_2, \dots, \lambda_m$ and their multiplicities d_1, d_2, \dots, d_m of $Q(x)$ by factoring $Q(x)$ into a product of linear factors.
- 3) Set up the general form $a_n = p_1(n) \lambda_1^n + \dots + p_m(n) \lambda_m^n$ for all $n \geq 0$, where $p_i(n)$ has degree $< d_i$ with unknown coefficients.
- 4) Use the initial conditions to determine these coefficients.

(V) explicit formula \rightarrow recurrence

eg: given $a_n = (2n-3)(-1)^n + (2n^2+n-5)4^n \quad \forall n \geq 0$. Find the recurrence for a_n .

$$\text{let } A(x) = \sum_{k=0}^{\infty} a_k x^k = P(x)/Q(x).$$

Then, $Q(x)$ must be a polynomial with inverse roots $\lambda_1 = -1$ and $\lambda_2 = 4$, with multiplicities $d_1 = 2$ and $d_2 = 3$. ↗
↳ note, we get multiplicities by seeing degree of respective polynomials, plus one!

$$\text{So, } Q(x) = (1-\lambda_1 x)^{d_1} (1-\lambda_2 x)^{d_2} = (1+x)^2 (1-4x)^3.$$

expanding, we get: $Q(x) = 1 - 10x + 25x^2 + 20x^3 - 80x^4 - 64x^5$, so:

$$a_n - 10a_{n-1} + 25a_{n-2} + 20a_{n-3} - 80a_{n-4} - 64a_{n-5} = 0 \quad \forall n \geq 5.$$

let's find initial conditions by plugging $n \in \{0, 1, 2, 3, 4\}$ into the given explicit formula: $a_0 = -8, a_1 = -7, a_2 = 81, a_3 = 1021, a_4 = 7941$.

General Method: given an explicit formula of the form $a_n = p_1(n)\lambda_1^n + \dots + p_m(n)\lambda_m^n$ for all $n \geq 0$, where $p_i(n)$ has degree $< d_i$, we get the recurrence relation by:

- 1) Define the denominator polynomial $Q(x) = (1-\lambda_1 x)^{d_1} \dots (1-\lambda_m x)^{d_m}$
- 2) Compute $Q(x) = 1 + C_1 x + C_2 x^2 + \dots + C_d x^d$
- 3) A recurrence relation for the a_n is then:
 $a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_d a_{n-d} = 0 \quad \forall n \geq d$
- 4) The initial conditions a_0, a_1, \dots, a_{d-1} can be determined from the explicit formula.