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We are what we repeatedly do.
Excellence, then, is not an act, but a habit.
– Aristotle

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Discrete Structures II

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Average grade for Quiz 3

85

Quiz 4 – Next Week

- When
  - Monday 11/6 & Wednesday 11/8, during recitation
- What
  - Product rule (always handy - Week 4-5 Lectures)
  - Permutations
    - with and without constraints
    - with and without repetitions (Week 5 & Week 6 Lectures)
  - Combinations
    - With and without constraints
    - Without repetitions (Week 6 Lectures; pirates problem)

General Hint – Revisited

For each problem

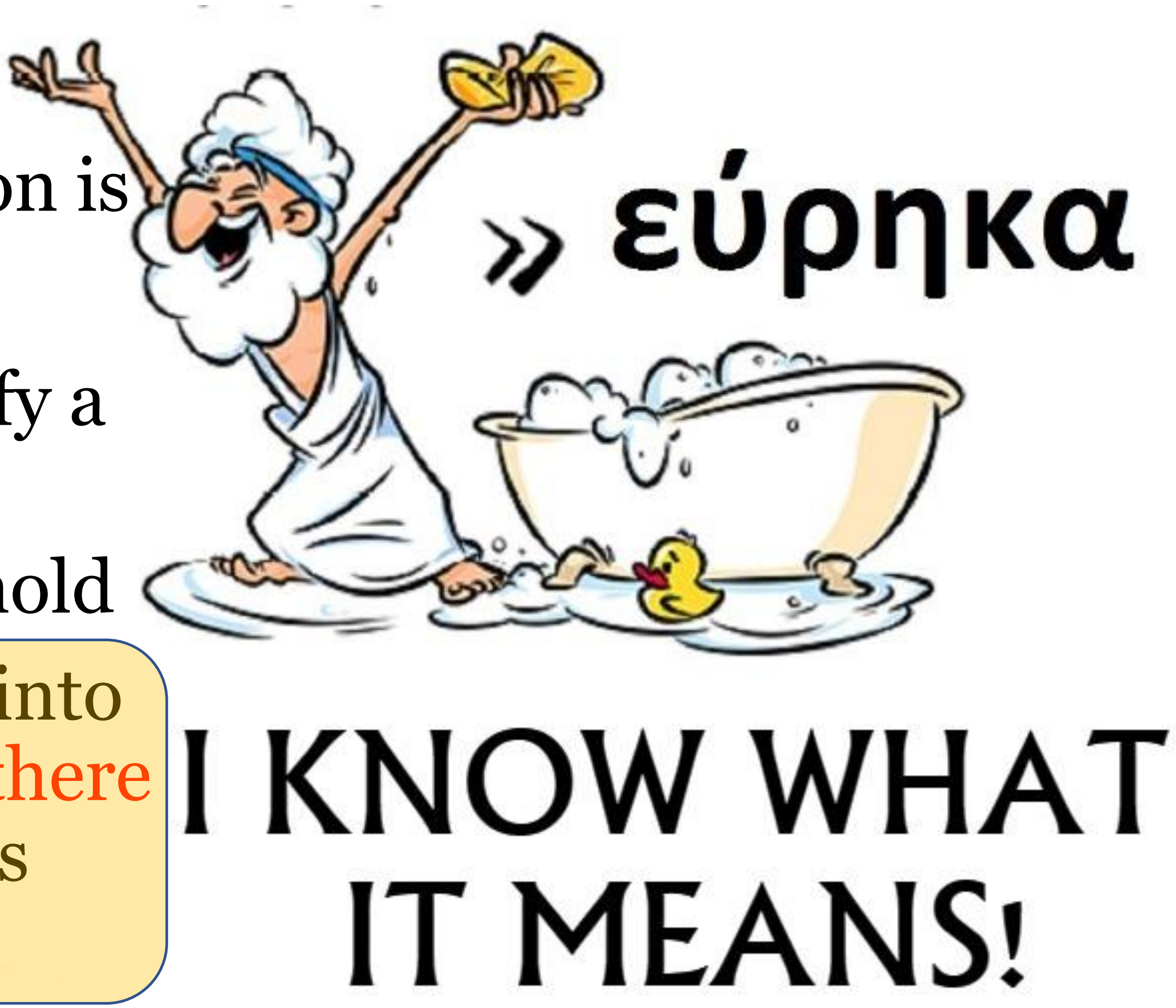
(1) Fully understand what the question is

(2) Fully understand what you know

(3) Based on the previous two, identify a method

(4) Make sure that the assumptions hold

(5) Turn the wording of the problem into the input to your method. Typically, there is a "key" thought that will unlock this part of the solution for you.



So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- ~~Permutation/Combinations~~
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

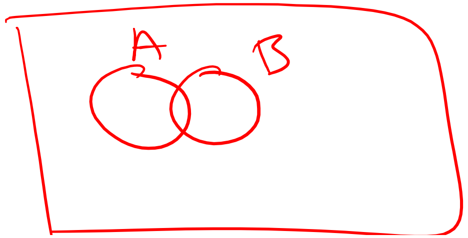
Can you solve this?

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

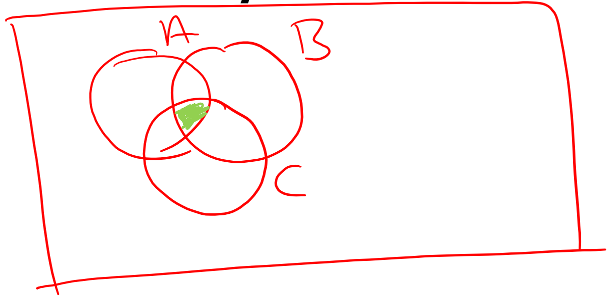
Inclusion/Exclusion

> Sum Rule:
> If A and B are disjoint sets, then $|A \cup B| = |A| + |B|$

- What if A and B are not disjoint? $|A \cup B| = ?$



Inclusion/Exclusion for 3 sets



$$|A \cup B \cup C|$$

$$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$|A \cup B \cup C| = \underbrace{|A| + |B| + |C|}_{\text{Include}} \underbrace{- |A \cap B| - |B \cap C| - |A \cap C|}_{\text{Exclude}} + \underbrace{|A \cap B \cap C|}_{\text{Include}}$$

Direct Proof

Inclusion/Exclusion for 3 sets

$|A \cup B \cup C|$, Let $X = B \cup C$

$= |A \cup X| = |A| + |X| - |A \cap X|$

$|X| = |B \cup C| = |B| + |C| - |B \cap C|$ → follows from formula for 2 sets

$|A \cap X| = |A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$ → Apply formula for 2 sets

$= |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|$

$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$

Hence,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Inclusion/Exclusion for 4 sets

$|A \cup B \cup C \cup D|$, $X = B \cup C \cup D$

$|A \cup X| = |A| + |X| - |A \cap X|$

$|X| = |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |C \cap D| - |B \cap D| + |B \cap C \cap D|$

$|A \cap X| = |A \cap (B \cup C \cup D)| = |(A \cap B) \cup (A \cap C) \cup (A \cap D)|$

use formula for 3 sets

$\vdots$

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
$+ |A \cap B \cap C| + |B \cap C \cap D| + |A \cap C \cap D| + |A \cap B \cap D|$
$- |A \cap B \cap C \cap D|$

General Inclusion/Exclusion

$$|A_1 \cup A_2 \cdots \cup A_n| = |A_1| + |A_2| + \cdots |A_n| \to n \text{ terms}$$

$$- |A_1 \cap A_2| - |A_2 \cap A_3| - \cdots \to \binom{n}{2} \text{ terms}$$

$$+ |A_1 \cap A_2 \cap A_3| + \cdots \to \binom{n}{3} \text{ terms}$$

$$- |A_1 \cap A_2 \cap A_3 \cap A_4| - \cdots \to \binom{n}{4} \text{ terms}$$

$$\vdots$$

$$(-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cdots \cap A_n| \to \binom{n}{n} = 1 \text{ term}$$

General Inclusion/Exclusion

- In the set S={1,2,.....100} how many multiples of 6 or 7?

$A =$ all multiples of 6

$B =$ all multiples of 7

Want $|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = |\{6, 12, 18, \ldots, 96\}| = 16$

$|B| = |\{7, 14, 21, \ldots, 98\}| = 14$

$|A \cap B| = |\{42, 84\}| = 2$

answer $= 16 + 14 - 2 = 28$

General Inclusion/Exclusion

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$? $(x, y, z) \geq 0$

A_1 = # solutions with $x \leq 3$

A_2 = # solutions with $y \leq 4$

Want: $|A_1 \cap A_2|$

General Inclusion/Exclusion

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$?

$x, y, z \geq 0$

$A_1 =$ # solutions with $x \leq 3$

$A_2 =$ # solutions with $y \leq 4$

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$|A_1| =$ (all solutions) $-$ (solutions with $x \geq 4$)

$= \binom{17}{2} - \binom{13}{2}$

General Inclusion/Exclusion

$x, y, z \geq 0$

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$?

$|A_2| =$ # solutions to $x+y+z=15$ and $y \leq 4$

$= (\text{all solutions}) - (\text{solutions with } y \geq 5)$

$= \binom{17}{2} - \binom{12}{2}$

$|A_1 \cup A_2| =$ # solutions with $x \leq 3$ or $y \leq 4$

$= (\text{all solutions}) - (\text{all solutions with } x \geq 4 \text{ and } y \geq 5)$

$= \binom{17}{2} - \binom{8}{2}$

General Inclusion/Exclusion

$x, y, z \geq 0$

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$?

Hence, $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$

$$= \binom{17}{2} - \binom{13}{2} + \binom{17}{2} - \binom{12}{2} - \binom{17}{2} + \binom{8}{2}$$

$$= 20$$

General Inclusion/Exclusion

- A group of 3 rabbits is playing outside their individual burrows when they are surprised by an eagle.
- Each rabbit escapes down a random hole. One rabbit per hole.
- How many ways to reorganize while avoiding their own hole.

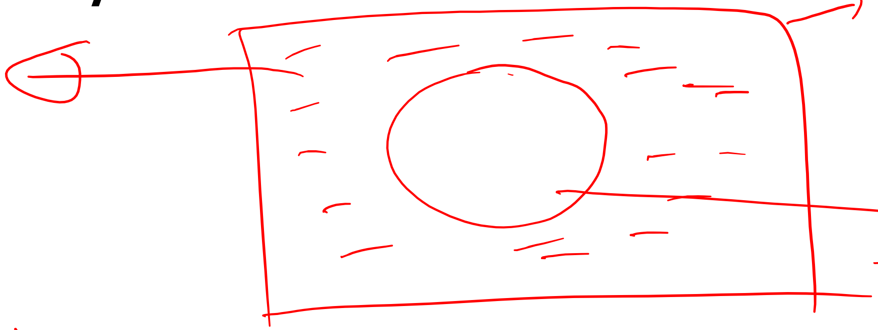
A_1 = all ways to reorganize such that rabbit 1 goes to its own hole

A_2 = all ways to " rabbit 2 goes to its own hole

A_3 = all ways to " rabbit 3 goes to its own hole

General Inclusion/Exclusion

What we want



all ways to reorganize

$A_1 \cup A_2 \cup A_3$ = all outcomes where some rabbit goes to its own hole

$$\text{answer} = \text{all ways} - |A_1 \cup A_2 \cup A_3|$$

$$= 3! - |A_1 \cup A_2 \cup A_3|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$$

$|A_1| = 2!$, $|A_2| = 2!$, $|A_3| = 2!$

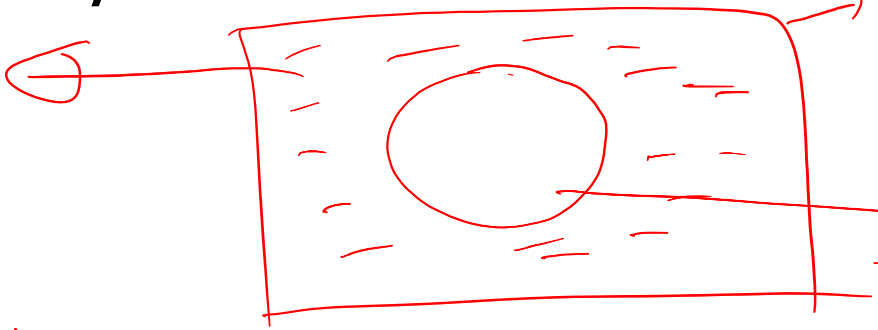
$|A_1 \cap A_2| = 1$, $|A_2 \cap A_3| = 1$, $|A_3 \cap A_1| = 1$, $|A_1 \cap A_2 \cap A_3| = 1$

$$|A_1 \cup A_2 \cup A_3| = 2+2+2-1-1-1+1 = 4$$

$$\text{answer} = 3! - 4 = 2$$

General Inclusion/Exclusion

What we want



all ways to reorganize

$A_1 \cup A_2 \cup A_3$ = all outcomes where some rabbit goes to its own hole

answer = all ways $- |A_1 \cup A_2 \cup A_3|$

$= 3! - |A_1 \cup A_2 \cup A_3|$

$|A_1| = 2!$, $|A_2| = 2!$, $|A_3| = 2!$

$|A_1 \cap A_2| = 1$, $|A_2 \cap A_3| = 1$, $|A_3 \cap A_1| = 1$

$|A_1 \cap A_2 \cap A_3| = 1$

Hence answer $= 3! - (2+2+2-1-1-1+1) = 2$

General Inclusion/Exclusion

- A group of n rabbits is playing outside their individual burrows when they are surprised by an eagle.
- Each rabbit escapes down a random hole. One rabbit per hole.
- How many ways to reorganize while avoiding their own hole.

$A_1 =$ all outcomes when rabbit 1 goes to own space

$\vdots$

$A_i =$ " rabbit i "

$\vdots$

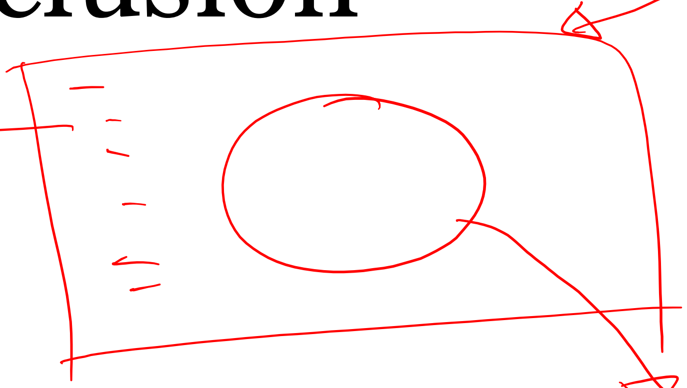
$A_n =$ " rabbit n "

General Inclusion/Exclusion

answer $= n! - |A_1 \cup \cdots A_n|$

$$
\begin{aligned}
|A_1 \cup A_2 \cdots \cup A_n| &= |A_1| + |A_2| + \cdots |A_n| \\
&\quad - |A_1 \cap A_2| - |A_2 \cap A_3| - \cdots \\
&\quad + |A_1 \cap A_2 \cap A_3| + \cdots \\
&\quad \vdots \\
&\quad (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|
\end{aligned}
$$

all ways



$A_1 \cup A_2 \cdots \cup A_n$ = all ways where someone goes to its own hole