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Strive not to be a success, but rather to be of value — Albert Einstein

206
Discrete Structures II

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Lecture 12 | Permutations, Combinations & The Almighty Pirates Problem | Monday October 23rd 2023

Preview…

k distinct pirates want to divide up n identical, indivisible bars of gold. How many ways to divide the loot when each must get at least r bars?

Preview...

How many integer solutions to the following equation?

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_1, x_2, \ldots, x_k \geq 0$$

Quiz 3 – When and What?

- When
  - This week - during recitation
- What will cover
  - Sum/Product rules (Week 3 & Week 4 Lectures)
  - Permutations with and without repetitions (up to last week's lecture)

BTW - Have you seen the Extra Problems?

Extra_Problems_1_Sum and Product Rules.pdf

Extra_Problems_2_Combinations_Permutations.pdf

So Far

- ~~Sets / Functions~~
- ~~Proofs~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

So far...

Last Class

- Permutations
- Combinations

Today

- Nothing

General Hint

For each problem

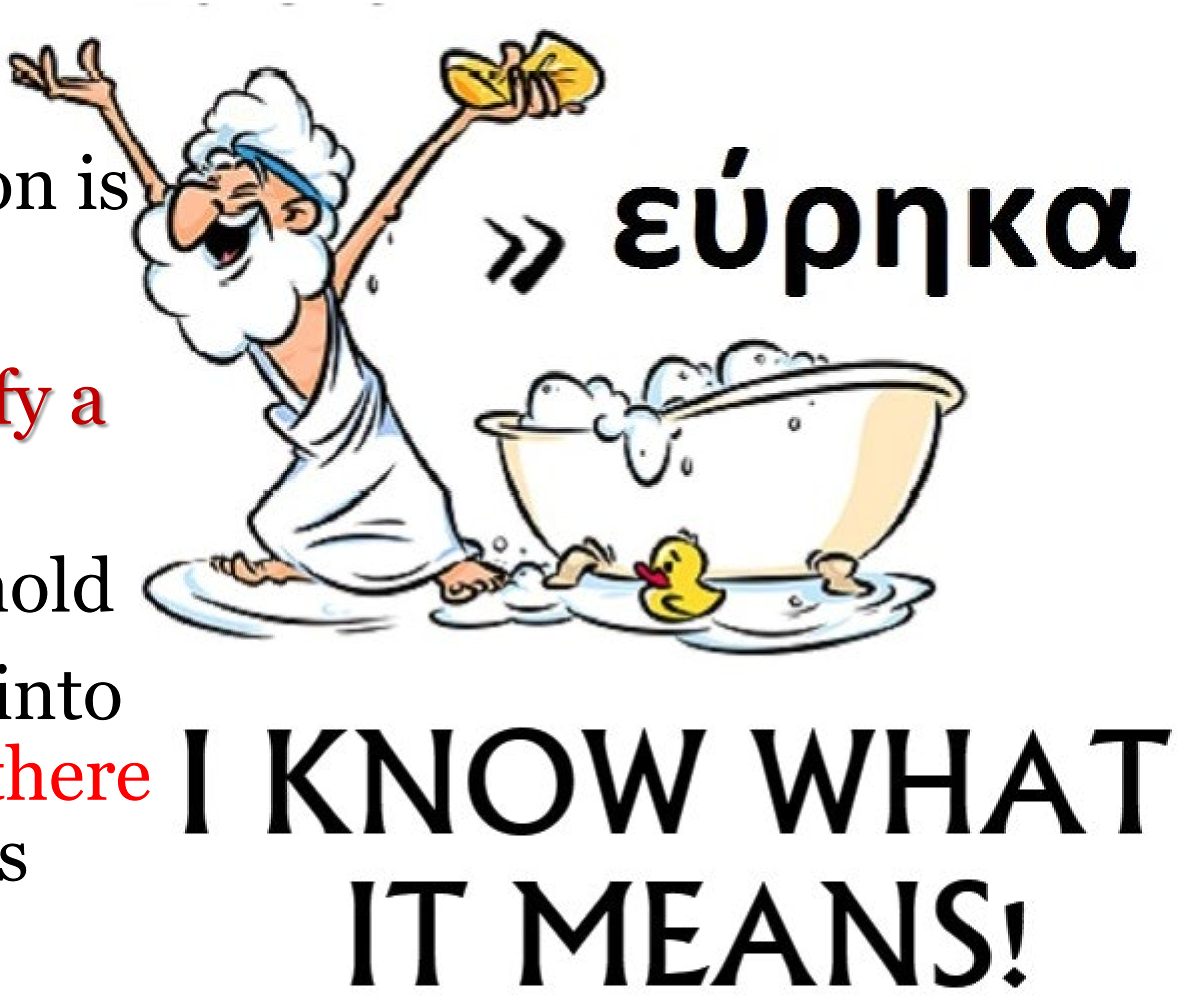
(1) Fully understand what the question is

(2) Fully understand what you know

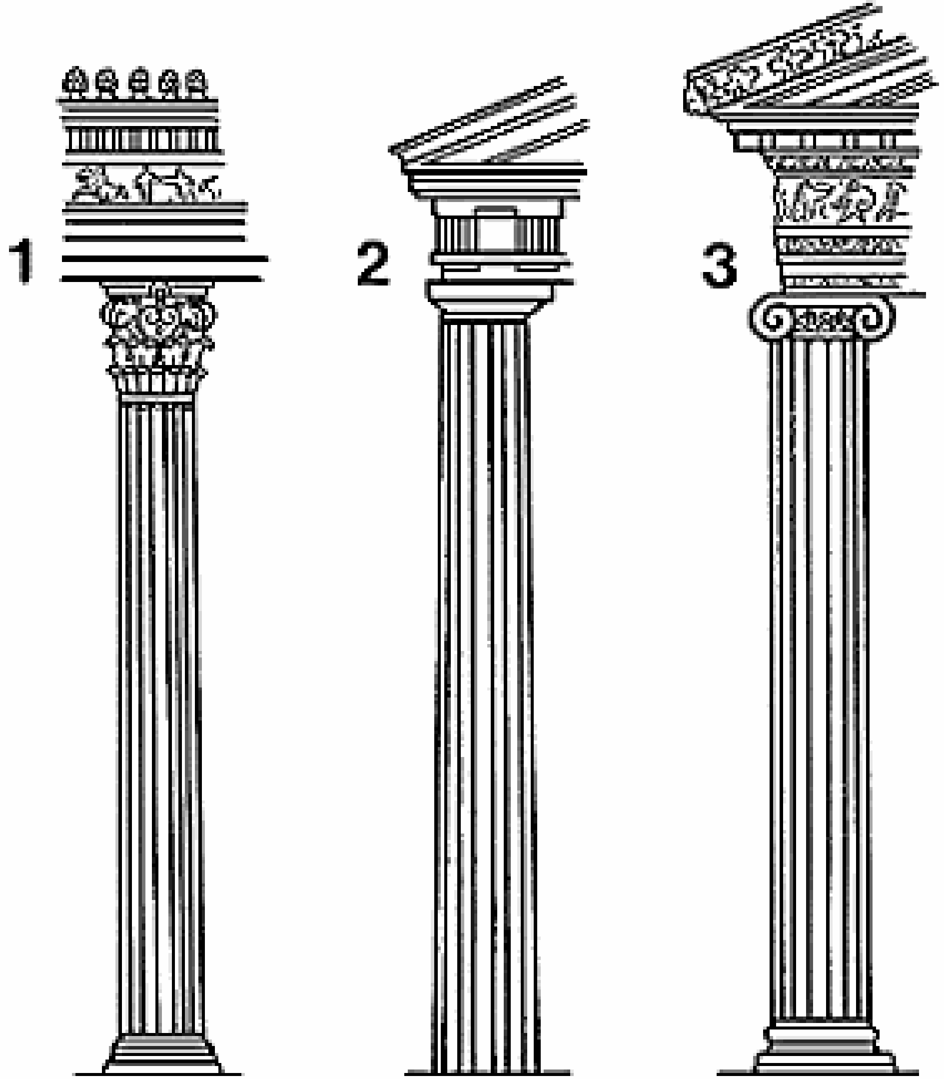
(3) Based on the previous two, identify a method

(4) Make sure that the assumptions hold

(5) Turn the wording of the problem into the input to your method. Typically, there is a “key” thought that will unlock this part of the solution for you.



Product Rule



order is important

So far...

Permutations

Permutation

Permutation of Full Set

$$ {}_{n}P_{k} = \frac{n!}{(n-k)!} $$

Size of Set

Selected

Permutation of Left Behind Set

- Distinctly ordered sets are called permutations (arrangements). The number of permutations of n distinct objects taken k at a time is given by:

$$ P^{n}_{k} = \frac{n!}{(n-k)!} $$

n = number of distinct objects

k = number of positions

So far...

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

So far...

Permutations without Repetitions

A maths debating team consists of 4 speakers.

- In how many ways can all 4 speakers be arranged in a row for a photo?

Solution : **$4 \times 3 \times 2 \times 1 = 4!$ or $^4P_4$**

- How many ways can the captain and vice-captain be chosen?

Solution : **$4 \times 3 = 12$ or $^4P_2$**

Permutations without Repetitions

<u>A flutter on the horses</u>

There are 7 horses in a race.

- In how many different orders can the horses finish?

Solution : $7x6x5x4x3x2x1 = 7!$ or ${}^7P_7$

- How many trifectas (1^{st}, 2^{nd} and 3^{rd}) are possible?

Solution : $7x6x5 = 210$ or ${}^7P_3$

Permutations with Restrictions

In how many ways can 5 boys and 4 girls be arranged on a bench if

- there are no restrictions?

  Solution : 9! or $^{9}P_{9}$

- boys and girls alternate?

  Solution : A boy will be on each end

BGBGBGBGB = $5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$

$= 5! \times 4!$ or $^{5}P_{5} \times {}^{4}P_{4}$

Permutations with Restrictions

In how many ways can 5 boys and 4 girls be arranged on a bench if

- boys and girls are in separate groups?

Solution : Boys & Girls or Girls & Boys

$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$

or ${}^5P_5 \times {}^4P_4 \times 2$

d) Anne and Jim wish to stay together?

Solution : (AJ) _ _ _ _ _ _ _

$= 2 \times 8!$ or $2 \times {}^8P_8$

So far...

Permutations with Repetitions

How many permutations of the word **PARRAMATTA** are possible?

Solution :

P

A A A A

R R

M

T T

10 letters but note repetition (4 A's, 2 R's, 2 T's)

$$\text{No. of arrangements} = \frac{10!}{4!\,2!\,2!}$$

$$= 37\,800$$

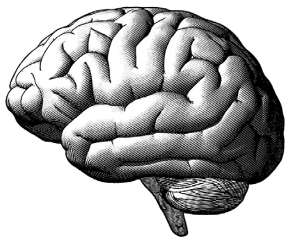
So far…

Permutations with Repetitions

If we have **n** elements of which x are alike of one kind, y are alike of another kind, z are alike of another kind, ………… then the **number of ordered selections or permutations** is given by:

$$\boxed{\frac{n!}{x!\,y!\,z!}}$$

So far...

Get your  in gear

- How many different numbers can you make from the digits 11122337?

Solution: $8! / (3!\, 2!\, 2!)$

So far...

Permutations with Restrictions

How many arrangements of the letters of REMAND are possible if:

- there are no restrictions?

Solution : $^{6}P_{6} = 720$ or $6!$

- they begin with RE?

Solution : R E _ _ _ _ $= {}^{4}P_{4} = 24$ or $4!$

- they do not begin with RE?

Solution : Total – (b) = $6! - 4! = 696$

So far...

Permutations with Restrictions

How many arrangements of the letters of REMAND are possible if:

- they have RE together in order?

Solution : **(RE) _ _ _ _ = $^5P_5$ = 120 or 5!**

- they have REM together in any order?

Solution : **(REM) _ _ _ = $^3P_3 \times {}^4P_4$ = 144**

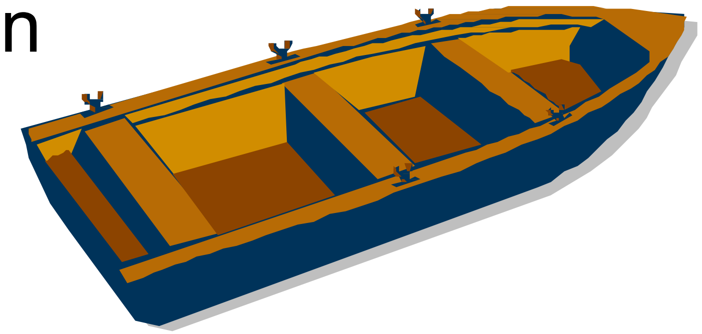
- R, E and M are not to be together?

Solution : **Total – (e) = 6! – 144 = 576**

New Stuff...

Permutations with Restrictions

There are 6 boys who enter a boat with 8 seats, 4 on each side. In how many ways can



- they sit anywhere? **Solution :** $^{8}P_{6}$

- two boys A and B sit on the port side and another boy W sit on the starboard side?

Permutation

Permutation of Full Set

$$_{n}P_{k} = \frac{n!}{(n-k)!}$$

Size of Set

Selected

Permutation of Left Behind Set

Solution : $A \,\&\, B = {}^{4}P_{2}$

$W = {}^{4}P_{1}$

$\text{Others} = {}^{5}P_{3}$

$\text{Total} = {}^{4}P_{2} \times {}^{4}P_{1} \times {}^{5}P_{3}$

Permutations with Restrictions

From the digits 2, 3, 4, 5, 6

- how many numbers greater than 4,000 can be formed?

Solution : **5 digits (any) = $^{5}P_{5}$**

4 digits (must start with digit $\geq$ 4) = $^{3}P_{1} \times {}^{4}P_{3}$

Total = $^{5}P_{5} + {}^{3}P_{1} \times {}^{4}P_{3}$

- how many 4 digit numbers would be even?

Even (ends with 2, 4 or 6) = _ _ _ $^{3}P_{1}$

$= {}^{5}P_{3} \times {}^{3}P_{1}$

Summary of Formulas

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  1st way:

  - No repetitions
  - Order matters

$\underline{5} \times \underline{4} = 5 \cdot 4 = {}_5P_2$

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  2nd way:

  $\rightarrow \binom{5}{2}$

  - No repetitions
  - Order does not matter

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  3rd way:

  - Repetitions allowed
  - Order matters

$\underline{5}\ \underline{5}$

answer $= 5^2$

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  4th way:

  - Repetitions allowed
  - Order does not matter

Partition Method

Case 1: There is a repetition $\to$ 5 ways

Case 2: There is no repetition $\to \binom{5}{2}$

answer $= \binom{5}{2} + 5$

Flowchart: Permutations/Combination

n distinct elements

- Pick r out of n without repetitions
  - Order matters
    $$^{n}P_r = \frac{n!}{(n-r)!}$$
  - Order does not matter
    $$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$
- Pick r out of n with repetitions
  - Order matters
    $$n^r$$
  - Order does not matter
    $$\binom{n+r-1}{n-1}$$

Preview…

k distinct pirates want to divide up n identical, indivisible bars of gold. How many ways to divide the loot when each must get at least r bars?

Preview...

How many integer solutions to the following equation?

$$x_1 + x_2 + \cdots + x_k = n$$
$$x_1, x_2, \ldots, x_k \geq 0$$

Take a Break

5 min

Pirates Problems

- Extra problems will soon become available on canvas
  - On Advanced Counting – Pirates Problem

Now: Advanced Counting

- Choosing r out of n distinct elements in no specific order.
  - With repetition

Combinations with Repetition

- Want to pick 10 chocolates out of 4 different types. Can pick many of one type. How many ways?

(H) Hershey	(T) Twix	(M) M&M	(S) Snickers
2	2	4	2
1	1	1	7

Count all sequences of length 4 → (a,b,c,d)
such that $a+b+c+d=10$

Combinations with Repetitions

- Want to pick 10 chocolates out of 4 different types. Can pick many of one type. How many ways?

Count all sequences of length 4, (a,b,c,d), $a+b+c+d=10$

H T M S

$(2,2,4,2) \longrightarrow$ ∘ ∘ | ∘ ∘ | ∘ ∘ ∘ ∘ | ∘ ∘

$(1,1,1,7) \longrightarrow$ ∘ | ∘ | ∘ | ∘ ∘ ∘ ∘ ∘ ∘ ∘

$(0,0,0,10) \longrightarrow$ | | | ∘ ∘ ∘ ∘ ∘ ∘ ∘ ∘ ∘ ∘

$(3,3,3,1) \longleftarrow$ ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘

Combinations with Repetitions

- Want to pick 10 chocolates out of 4 different types. Can pick many of one type. How many ways?

• • | • • • • | • • • | •

\# ways to choose = # arrangements of 10 dots and 3 vertical lines

Combinations with Repetitions

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot?

Count all sequences of (a,b,c,d,e) such that $a+b+c+d+e=20$

$a =$ # Pirate 1 gets
$b =$ # 2 gets
$c =$ # 3 gets
$d =$ 4
$e =$ 5 gets

∘ | ∘ ∘ | ∘ ∘ | | ∘ ∘

answer = all ways to arrange 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)} = \binom{24}{4}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_5 = 20$
  - $x_1, x_2, \ldots, x_5 \geq 0$

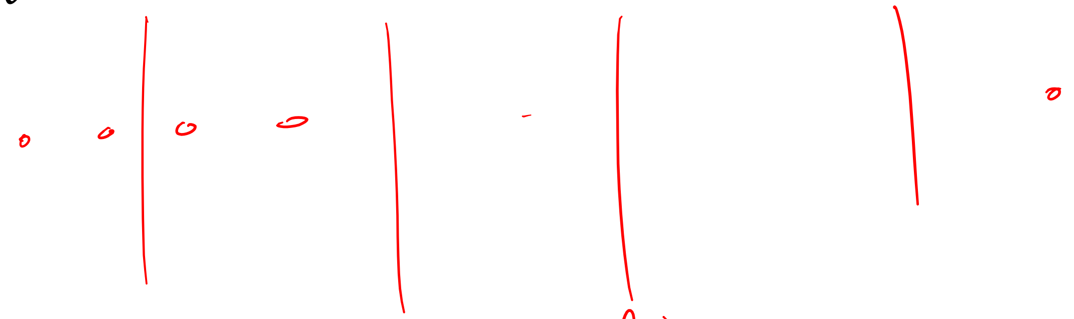
$(x_1, x_2, x_3, x_4, x_5)$ such that $\sum x_i = 20$

$\Rightarrow$ all arrangements of 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_k = n$
  - $x_1, x_2, \dots, x_k \geq 0$



k-1 lines

→ all arrangements of n dots and k-1 lines

$$\rightarrow \frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1}$$

Combinations - Adv’ced (with constraints)

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot when each must get at least 1 bar?

→ Give 1 bar to each

→ 5 pirates, 15 bars

→ 15 dots, 4 lines

$$\text{ans now} = \frac{19!}{(15!)(4!)} = \binom{19}{4}$$

Combinations

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot when each must get at least 2 bars?

→ Give 2 bars to everyone

→ 10 bars, 5 pirates

→ 10 dots, 4 lines

$$\text{answer} = \frac{14!}{10!\,4!}$$

Combinations

- k distinct pirates want to divide up n identical, indivisible bars of gold. How many ways to divide the loot when each must get at least r bars?

→ Give everyone r bars

→ $n-kr$ items, k pirates

→ $n-kr$ dots, $k-1$ lines

answer $= \binom{n-kr+k-1}{k-1}$

Flowchart: Permutations/Combinations

n distinct elements

Pick r out of n without repetitions

Order matters

$$^nP_r = \frac{n!}{(n-r)!}$$

Order does not matter

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Pick r out of n with repetitions

Order matters

$$n^r$$

Order does not matter

$$\binom{n+r-1}{n-1}$$

Flowchart: Permutations/Combinations

n identical elements

Divide into k pieces

$$\binom{n+k-1}{k-1}$$

Divide into k pieces with each piece having $\geq r$ elements

$$\binom{n-kr+k-1}{k-1}$$