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Character is simply habit long continued
-Plutarch

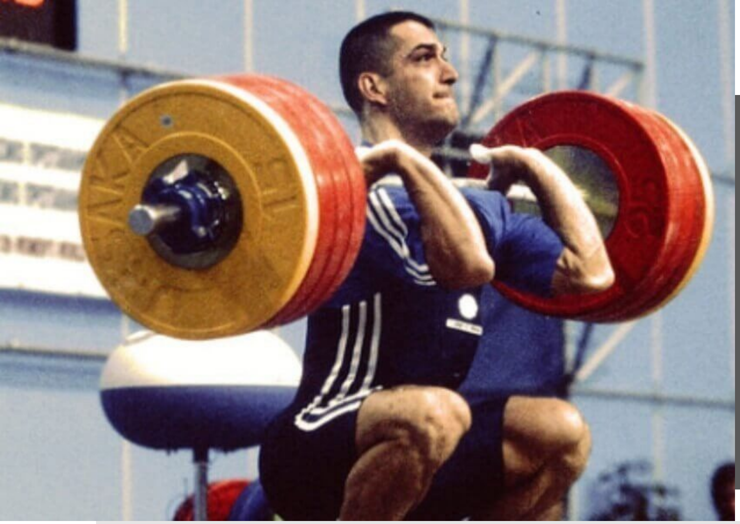
206
Discrete Structures II

Konstantinos P. Michmizos

Computational Brain Lab

Computer Science | Rutgers University | NJ, USA

Lecture 9 | Permutations - Combinations | Wednesday October 11th 2023



Quiz 1 - Statistics

Sections	#Students	Mean	Median	Range
Section - 1	52	90.25	91	[34,120]
Section - 2	51	85.76	90	[28,115]
Section - 3	46	89.47	99	[20,120]
Whole Class	149	89	93	[20,120]

Quiz 2 and 3

Quiz 2 Product Rule

Quiz 3 Perms/Comb

Quiz 4
Midterm and
Quiz 5

~~Quiz 4 Perms/Comb~~

Midterm

Quiz 5 Probabilities

Quiz 6 and 7

Quiz 6 Probabilities

Quiz 7 Probabilities

So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Lehman, 475-480

Product Rule $A \times B$

- Elements of $A \times B$ are ordered pairs:

$$A \times B = \{(x,y) : x \in A, y \in B\}$$

Product Rule:
$|A \times B| = |A| \cdot |B|$

- To create $A \times B$ elements, we choose one element from A and also one element from B.

e.g., If there are 4 types of coffee

{espresso, americano, cappuccino, latte}

and 3 types of sugar

{raw sugar, white sugar, and brown sugar}

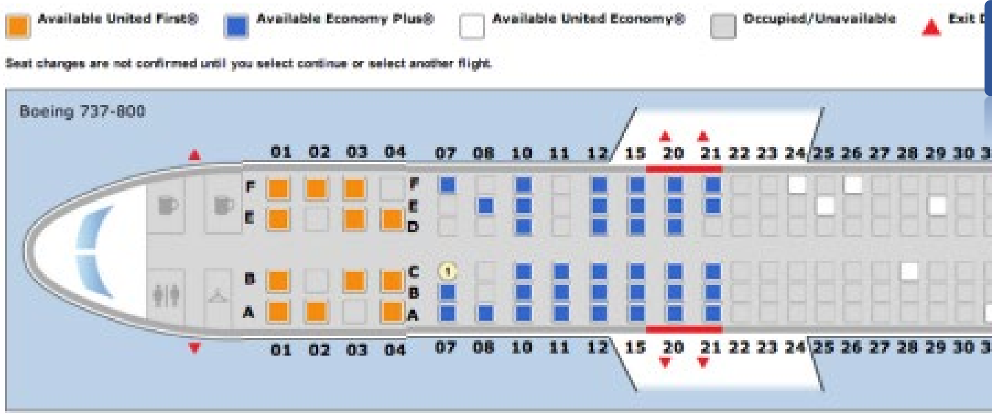
then there are $12 = 4 \times 3$ ways to make a coffee.

Last Time...

Generalized Product Rule – Order is important

- Suppose every object of a set S, can be constructed by a sequence of n choices with P_1 possibilities for the first choice, P_2 possibilities for the second choice, and so on
- **IF**
  - Each sequence of choices constructs an object in S.
  - No two different sequences **create the same** object
- THEN
  - $|S| = P_1 \times P_2 \times \cdots P_n$

Generalized Product Rule



Last Time...

- How many ways to assign 100 passengers to 100 seats?

Let $P1, -- P100$ be the passengers.

100 choices for seat of $P1$

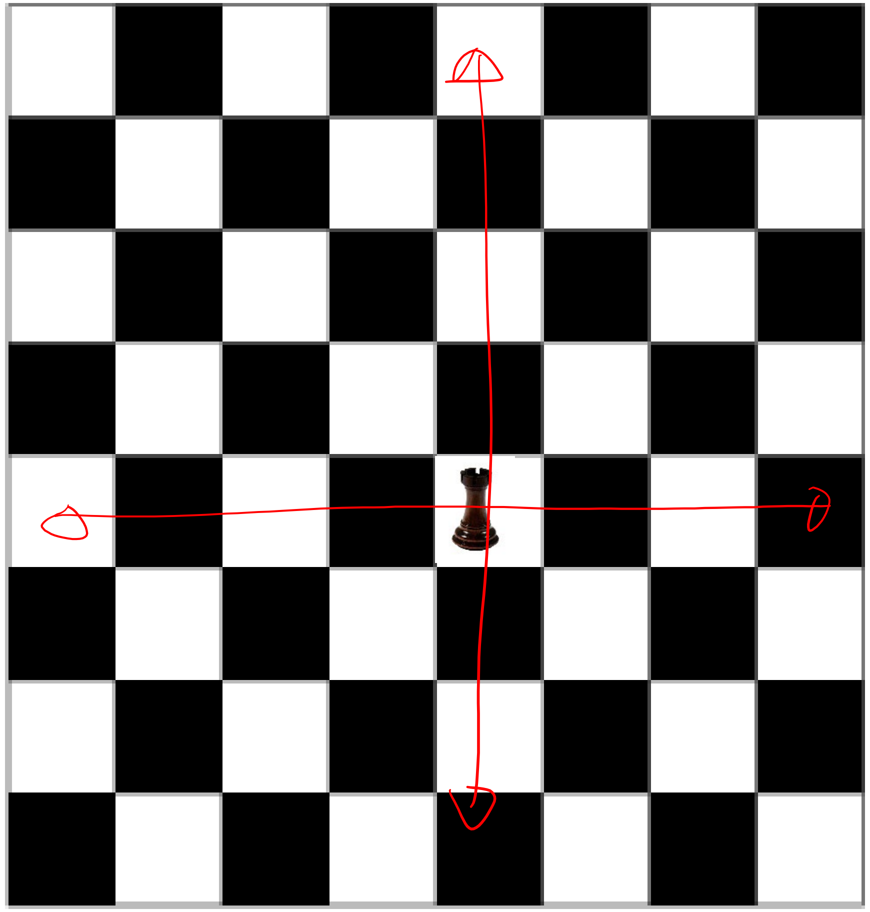
99 choices for seat of $P2$

⋮

1 choices for seat of $P100$

$\Rightarrow$ answer = $100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$

Generalized Product Rule



- Given two rooks labeled 1 and 2
- How many ways to place them so that they don't threaten each other?

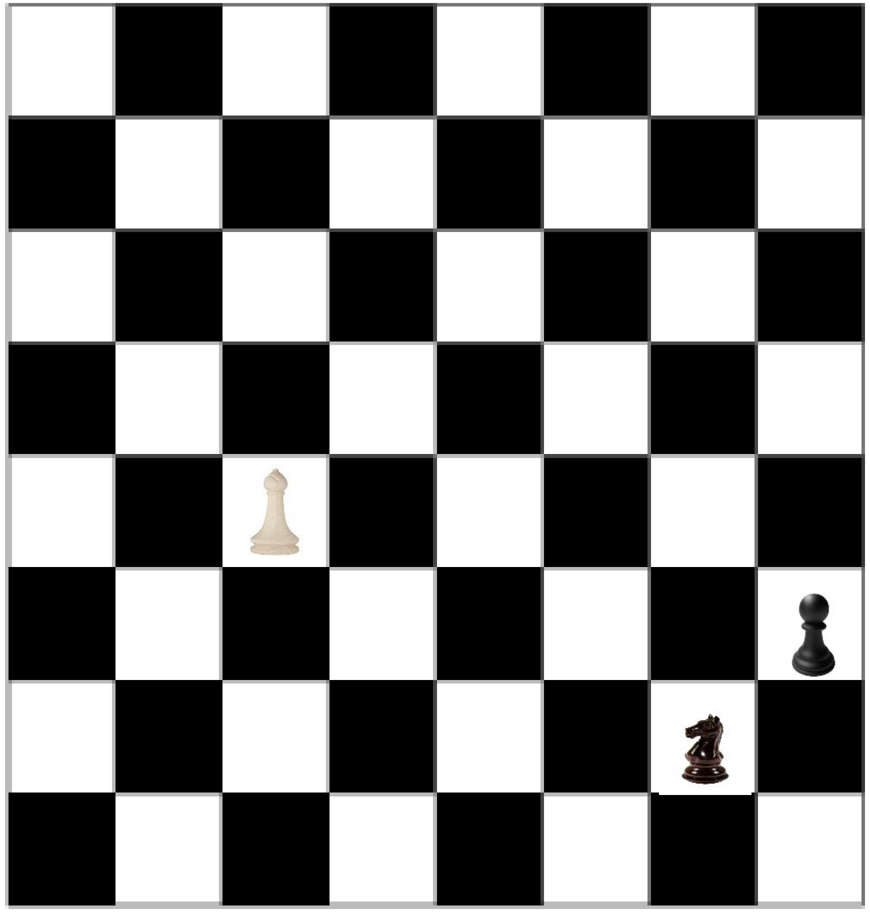
— 64 choices for first rook

— (64-15) choices for second rook

answer (64 - 49)

Generalized Product Rule

Potential Pitfall????



YES! If we had two (interchangeable) knights!!!

How many ways to place a knight, bishop, and pawn so that no two share a row or column?

Pick row for bishop → 8
Pick column for bishop → 8
Pick row for knight → 7 ways
Pick col for knight → 7 ways
Pick row for pawn → 6 ways
Pick col for pawn → 6 ways

ans = $8^2 7^2 6^2$

New stuff...

Exercise

- An IP address is a string of 32 bits. It begins with a network number (netid) followed by a host number (hostid).
  - There are three forms of addresses.
  - Class A addresses consists of 0, followed by a 7-bit netid and a 24-bit hostid.
  - Class B addresses consists of 10, followed by a 14-bit netid and a 16-bit hostid.
  - Class C addresses consists of 110, followed by a 21-bit netid and a 8-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

How many IP addresses are there?

Exercise

- Class A addresses consists of 0, followed by a 7-bit netid and a 24-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

$\underline{0}$ $\underset{\text{7 bits netid}}{\underline{\qquad\qquad\qquad}}$ $\underset{\text{24 bits hostid}}{\underline{\qquad\qquad\qquad\qquad\qquad}}$

$|A| = (\#\text{ choices of netid}) \cdot (\#\text{ choices for hostid})$

$= (2^7 - 1)(2^{24} - 2)$

Exercise

- Class B addresses consists of 10, followed by a 14-bit netid and a 16-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

$$\underline{1}\ \underline{0}\ \underbrace{\qquad\qquad\qquad}_{14\text{ bit netid}}\ \underbrace{\qquad\qquad\qquad}_{16\text{ bit hostid}}$$

$$|B| = (2^{14})(2^{16}-2)$$

Exercise

- Class C addresses consists of 110, followed by a 21-bit netid and a 8-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

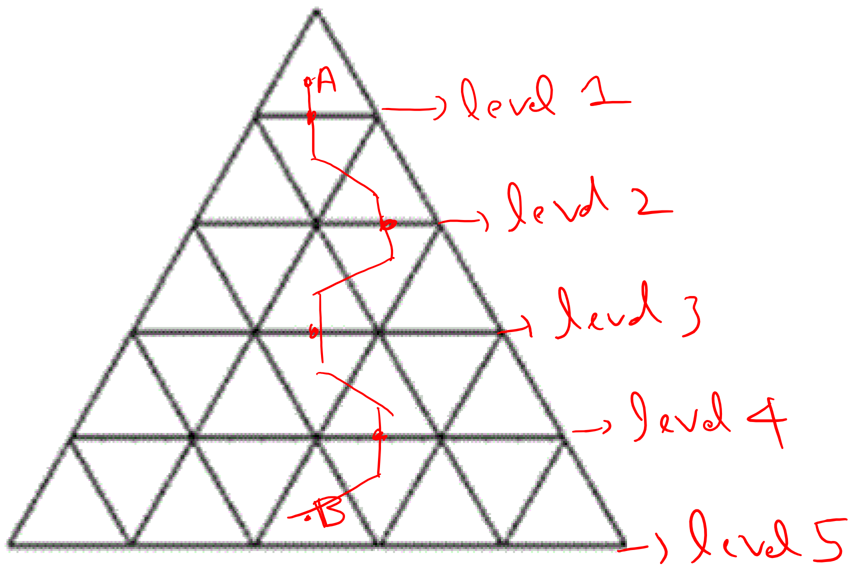
$$1\ 1\ 0\ \underbrace{\qquad\qquad\qquad}_{21 \text{ bit netid}}\ \underbrace{\qquad\qquad}_{8 \text{ bit hostid}}$$

$$|C| = 2^{21} \cdot (2^8 - 2)$$

answer $= |A| + |B| + |C|$

Exercise

Bijection Method

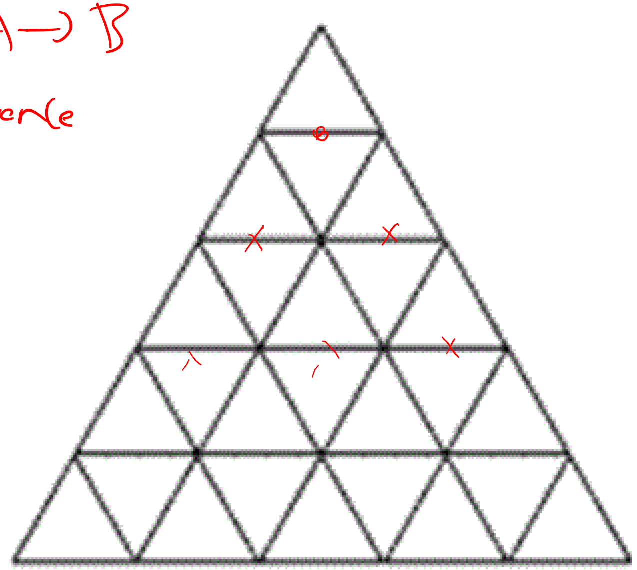


- Consider an equilateral triangle of side length 5, divided into unit length triangle. How many paths from point A to B?
  - Adjacent triangles in a valid path have to share a common edge.
  - A path can never go upwards or revisit a triangle.

How many paths from A to B?

Exercise

Each path from $A \to B$ has an exit sequence $(1, 2, 3, 3)$

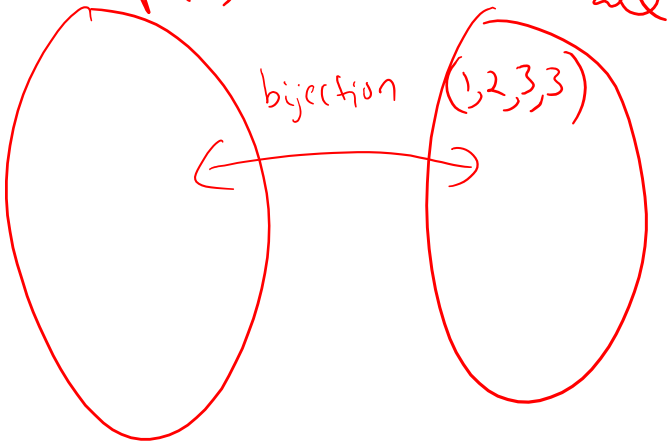


all valid exit sequences $= 1 \cdot 2 \cdot 3 \cdot 4$

$= 24$

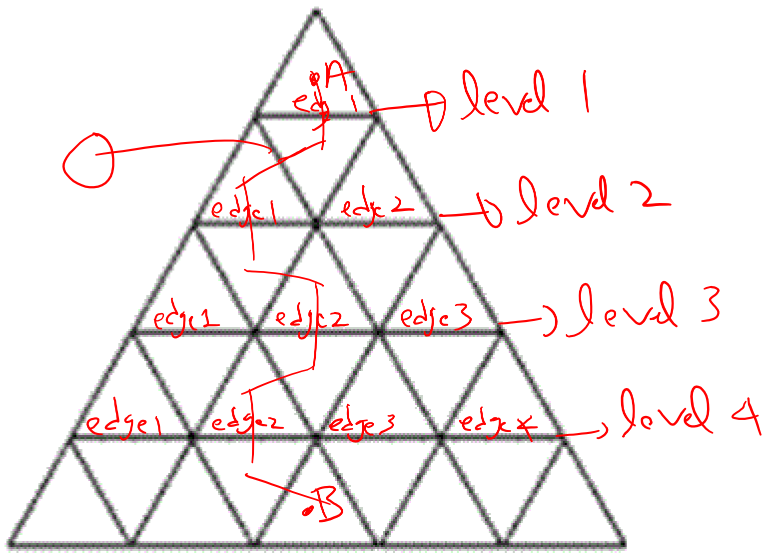
all valid paths

all exit sequences



Exercise

exit sequence = (1, 1, 2, 2)



Each path has an exit sequence and vice versa

- Consider an equilateral triangle of side length 5, divided into unit length triangle. How many paths from point A to B?
  - Adjacent triangles in a valid path have to share a common edge.
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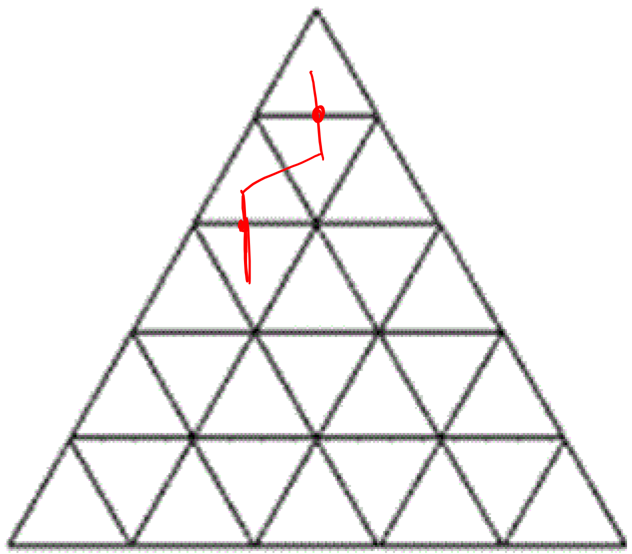
Exercise



valid paths = # exit sequences

= 1·2·3·4 = 24 paths

If n levels
answer = 1·2·3·4 ··· (n-1) = (n-1)!

Permutations and Combinations

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.

- Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected **does not** matter.

Difference between Permutation and Combinations

- The **difference between combinations and permutations** is ordering.

- With **permutations** we care about the order of the elements, whereas with **combinations** we don't.

- Examples:
  - Permutation: locker "combo" is 12345; Cellphone PIN is 5432
  - Combination: 5 students from a 180-student audience

Find 4-digit Permutations

of the numbers 2,3,4,5

______ ______ ______ ______

Find 4-digit Permutations

of the numbers 2,3,4,5

___ ___ ___ ___

The first digit can be any of the 4 numbers

4 ___ ___ ___

Find 4-digit Permutations

of the numbers 2,3,4,5

$\underline{\ 4\ }\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$

Now there are 3 options left for the second blank

$\underline{\ 4\ } \bullet \underline{\ 3\ }\ \underline{\quad}\ \underline{\quad}$

Find 4-digit Permutations

of the numbers 2,3,4,5

For the third position, we have two numbers left

$$\underline{4} \cdot \underline{3} \cdot \underline{2} \quad \underline{}$$

There is one number left for the last position

$$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$$

Find 4-digit Permutations

of the numbers 2,3,4,5

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Permutations with Repetition

____ ____ ____ ____

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5 …
- … but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

Permutations with Repetition

$$\underline{4} \cdot \underline{4} \cdot \underline{4} \cdot \underline{4} = 4^4 = 256$$

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5 …
- … but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

Choosing a subset (*a.k.a. Combinations*)

_____ _____ _____ _____ _____

- *How many different 5-card hands can be made from a standard deck of cards?**
- In this problem the order is irrelevant since it doesn't matter what order we select the cards.
- We'll begin with five lines to represent our 5-card hand.

** 52 cards in a standard deck*

Choosing a subset

$$\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48}$$

311,875,200 *permutations*

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** 52 cards in a standard deck*

- **That's permutations, not combinations**
- To fix this we need to divide by the number of hands that are different permutations but the same combination.

Choosing a subset

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

- That's permutations, not combinations.
- To fix this we need to divide by the number of hands that are different permutations but the same combination.

This is the same as saying

how many ways

can I arrange 5 cards?

Choosing a subset

$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

- So the number of five-card hands combinations is:

Rewriting with Factorials

$$\frac{52!}{47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}{\cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}$$

- With a little ingenuity we can rewrite the above calculation using factorials.
- We know 52! = 52•51•50•…•3•2•1, but we only need the products of the integers from 52 to 48. How can we isolate just those integers?
- We'd like to divide out all the integers except those from 48 to 52. To do this divide by 47! since it's the product of the integers from 47 to 1.

From an example to the formulas

Rewriting with Factorials

$$\frac{52!}{5!\,47!}$$

- Make sure to divide by **5!** to get rid of the extra permutations:

There we go!

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

Combinations Formula

$$\binom{n}{k} = C^n_k = {}_nC_k$$

- Different Annotations

Permutations Formula

$$P^n_k = \frac{n!}{(n-k)!}$$

- The formula for permutations is similar to the combinations formula, except we needn't divide out the permutations, so we can remove k! from the denominator: