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206
Discrete Structures II

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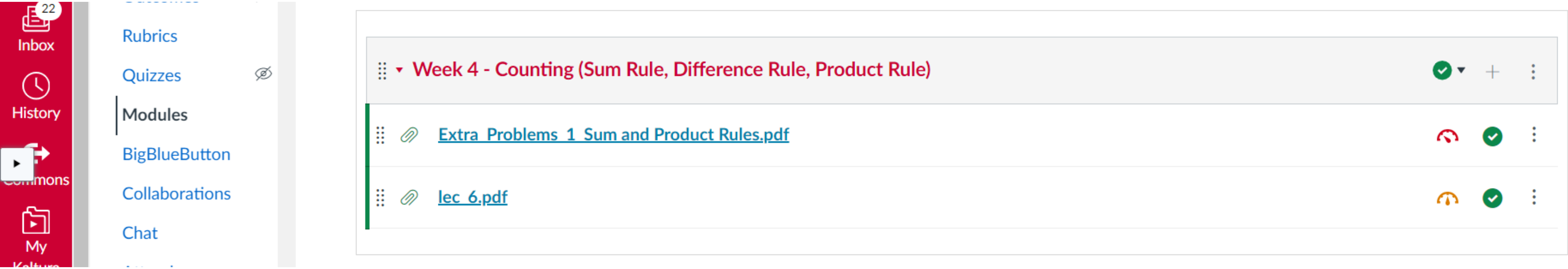
Extra Problems – Set 1



What we will cover today

Combinatorics

- Recap
  - Counting (Partition, Difference, Product)
- Today
  - Counting
    - Product Rule
    - Bijection Rule
- Next
  - Permutations/Combinations
  - Pigeonhole Principle

Course Outline

- Part I
  - Recap of basics – sets, function, proofs, induction
  - Basic counting techniques
  - Pigeonhole principle
  - Generating functions
- Part II
  - Sample spaces and events
  - Basics of probability
  - Independence, conditional probability
  - Random variables, expectation, variance
  - Moment generating functions
- Part III
  - Graph Theory
  - Machine learning and statistical inference

Counting

Counting

- In the next few lectures
  - Fundamental tools and techniques for counting
  - Sum Rule
  - Product Rule
  - Difference Method
  - Bijection Method
  - Permutations/Combinations
  - Inclusion Exclusion
  - Binomial/Multinomial coefficients

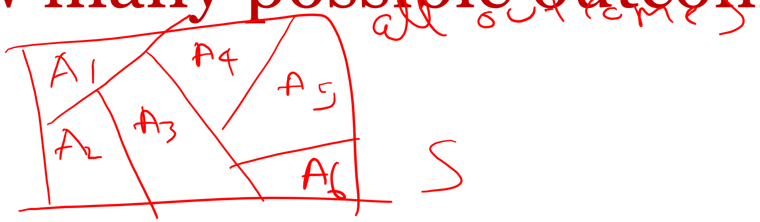
Fundamental Blocks

Intermediate

Advanced

Partition Method

- If I roll a white and black die, how many possible outcomes do I see?



A_1 = all outcomes with black die = 1

A_2 = all outcomes with black die = 2

$\vdots$

A_6 = all outcomes with black die = 6

$$|S| = |A_1| + |A_2| + \cdots + |A_6|$$
$$= 6 \cdot 6 = 36$$

Difference Method

- To find the size of a set A,
  - Find a larger set S such that $S = A \cup B$ and
  - A and B are disjoint.
  - $|A| = |S| - |B|$

- Possible outcomes where white and black die have different values?
  - Find S with all possible outcomes $|S|$=36
  - Subtract B with the same values |B|=6
  - $|A| = |S| - |B| = 36 - 6 = 30$

Partition Method

- Possible outcomes where white and black die have different values?

S = all possible outcomes

A_1 = all outcomes with black die = 1

A_2 = black die = 2

⋮

A_6 = black die = 6

$|A_1| = 5$, $|A_2| = 5$,

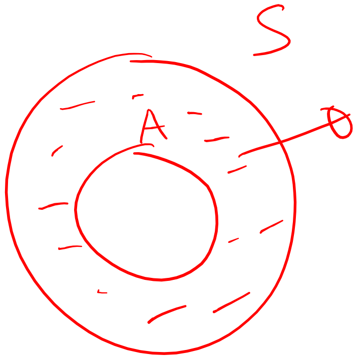
$|S| = 5 + 5 + 5 + \cdots 5 = 30$

...or we can use the Difference Method

- Possible outcomes where white and black die have different values?

$A =$ all outcomes where white die $\neq$ black die

$S =$ all outcomes, $|S| = 36$



$B = S \setminus A$
$=$ all outcomes where white die $=$ black die

$|B| = 6$

$\Rightarrow |A| = 36 - 6 = 30$

Product Rule

Product Rule:
$|A \times B| = |A| \cdot |B|$

- True even if A and B are not disjoint
- Useful when counting elements of a set involves dealing with tuples, sequences or a series of choices.

Insight: The Product Rule gives us how many different elements are possible

Insight #2: The multiplication finds all the possible "matches" across sets

Product Method

- If I roll a white and black die, how many possible outcomes do I see?

$A =$ all outcomes of black die

$B =$ all outcomes of white die

all outcomes $= |A \times B| = |A| \cdot |B| = 36$

Question: Can you make the above question not solvable with the product rule?

Remember: Now we are leaving behind us our ability to count elements and start developing skills that help us count sets without explicitly counting their elements

Product Rule

Product Rule:

$|A_1 \times A_2 \times \cdots A_n| = |A_1| \cdot |A_2| \ldots |A_n|$

Product Rule

- A restaurant has a menu with 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts.
  - How many ways to choose a complete meal?

A = all possible complete meals

= { (App, Entree, Salad, Dessert)
⋮
}

$|A| = 5 \times 6 \times 3 \times 7$

5 choices for Appetizers
6 " " Entre
3 " " Salad
7 " " Dessert

Last Time...

Product Rule

- A restaurant has a menu with 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts.
  - How many ways to choose a meal if I'm allowed to skip some (or all) the courses?

$A = \{ (\text{APP}, \text{Entree}, \text{Salad}, \text{Dessert})$
(APP)
(Entree)
$\vdots \}$

Step 1: Make all elements the same length by including a null option. For ex: (Entree) becomes (null, Entree, null, null)

Step 2: 6 choices for Appetizer, 7 for Entree, 4 for Salad, 8 Dessert

Answer = $6 \times 7 \times 4 \times 8$

Last Time…

Exercise: Counting Passwords…

- You are signing up for an account on FlixBiz.com. The password has the following requirements
  - The password must be 6 to 8 characters long.
  - Each password is an uppercase letter or digit.
  - Each password must contain **at least** one digit.

Partition Method

Q: How many possible passwords?

$A_6 \rightarrow$ all passwords with length 6

$A_7 \rightarrow$ " 7

$A_8 \rightarrow$ " 8

all passwords $= |A_6| + |A_7| + |A_8|$

Hint (or …When to think of Partition Method)

- When you are asked to count something that exists in easy-to-count ways (e.g., between 2 and 4), consider dividing the problem to the enumerable cases and then use the Partition Method

  - Note that if the different cases are too many (e.g., 100), then most probably the intention of the exercise is not to stress your patience mechanisms…

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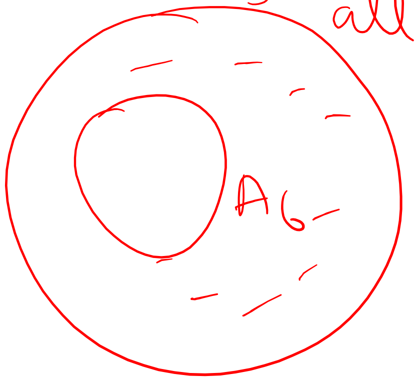
all passwords $= |A_6| + |A_7| + |A_8|$

Exercise: Counting Passwords...

$A_6 =$ all valid passwords of length 6

S = all passwords of length 6

$B = S \setminus A_6$



B = all passwords of length 6 with no digits

Difference Method

=>

Find Contrapositive

(see Hint on next slide)

$|A_6| = |S| - |B|$

$|S| = 36^6$

$|B| = 26^6$

$\Rightarrow |A_6| = 36^6 - 26^6$

Partition Method

$|A_7| = 36^7 - 26^7$

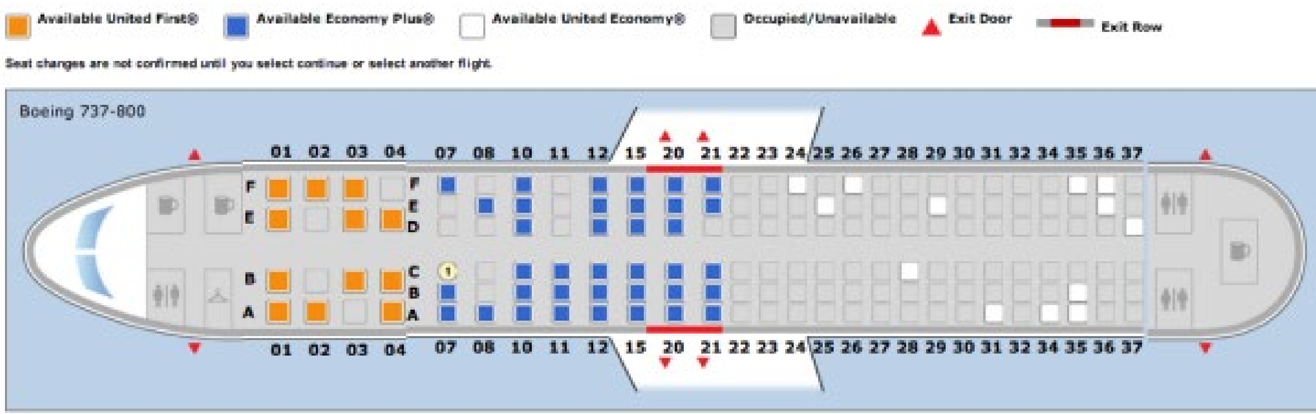
$|A_8| = 36^8 - 26^8$

Hint: When to use Difference Method

When you are asked to count something that exists in "at least" one place, consider counting the opposite (that is "nowhere")

Which means: You need to be able to find the "contrapositive argument".

Generalized Product Rule



- How many ways to assign 100 passengers to 100 seats?

Let $P1, \ldots, P100$ be the passengers.

100 choices for seat of $P1$
99 choices for seat of $P2$
⋮
1 choices for seat of $P100$

$\Rightarrow$ answer $= 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$

Generalized Product Rule – Order is important

- Suppose every object of a set S, can be constructed by a sequence of n choices with P_1 possibilities for the first choice, P_2 possibilities for the second choice, and so on
- **IF**
  - Each sequence of choices constructs an object in S.
  - No two different sequences **create the same** object
- THEN
  - $|S| = P_1 \times P_2 \times \cdots P_n$