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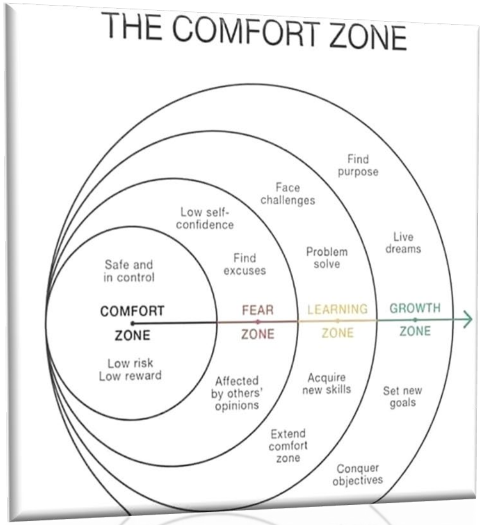

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Discrete Structures II

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Lecture 23 | Bayes Rule + Bayes Rule ☺ | Wednesday December 6th 2023

This is the... Last 3 Lectures

Part 1: Counting

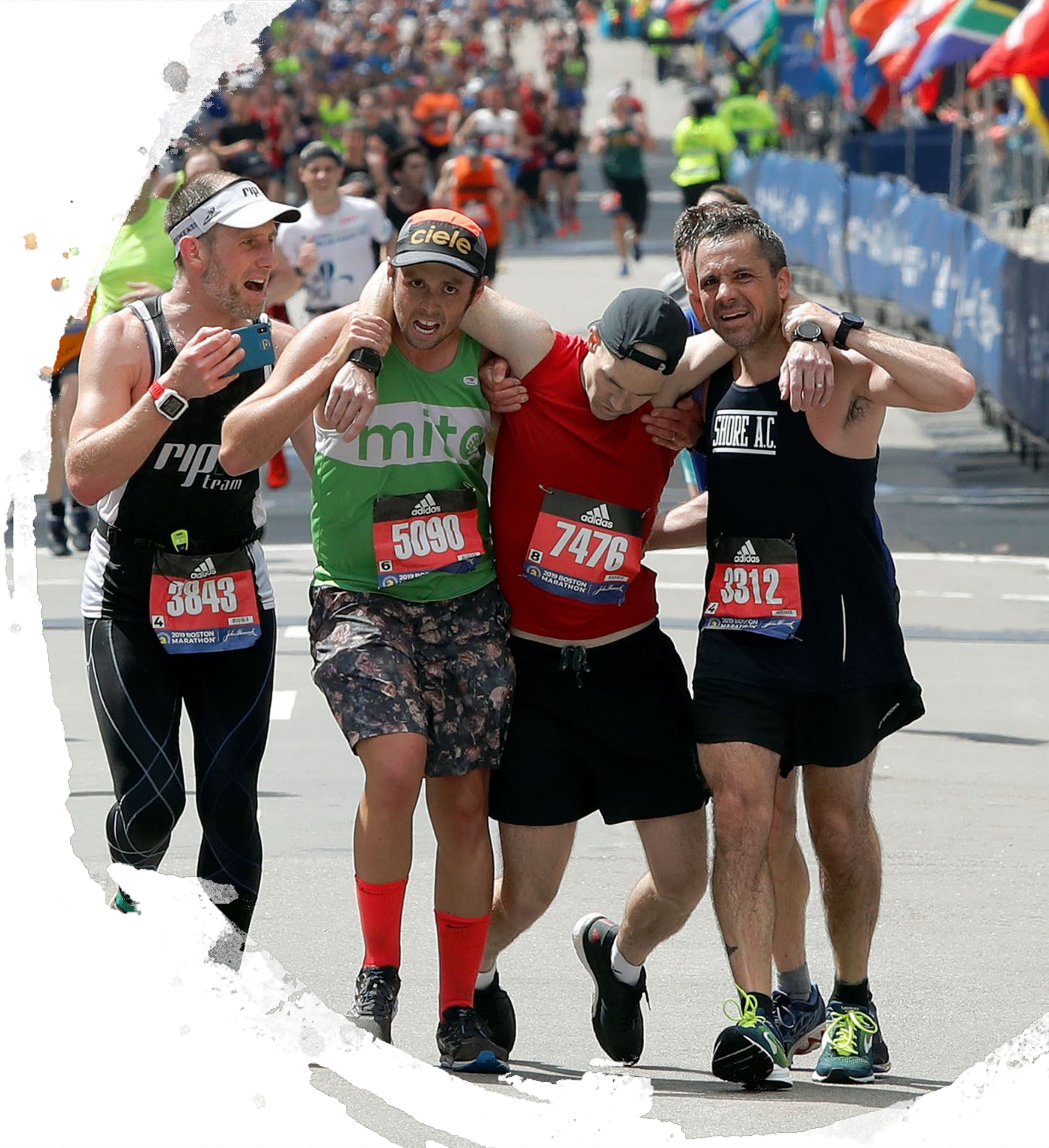
- Proofs/induction
- Sum Rule
- Partition Method
- Difference Method
- Bijection Method
- Product Rule
- Generalized product rule
- Pigeonhole Principle
- Inclusion/Exclusion
- Combinatorial proofs, binomial coefficients

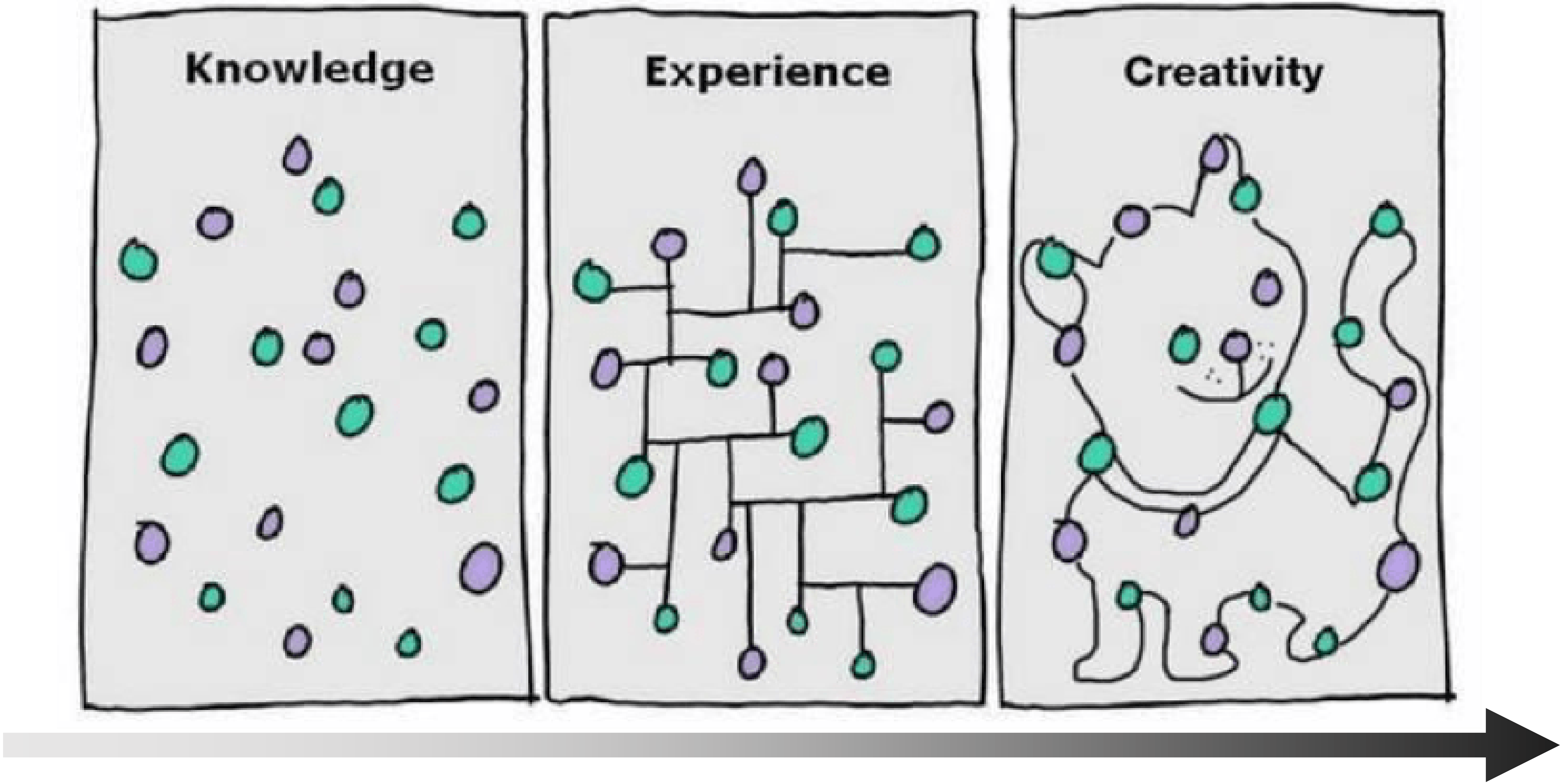
Part 2: Probability

- Sample space
- Axioms of probability
- Conditional probability
- Independence
- Bayes rule
- Random Variables
- Expectation and Linearity
- Variance

Announcements

- Final Quiz next week
- 20% extra credits
  - In total, 120 extra credits that count as 1.2 quizzes
  - Dropping the lowest-grade Quiz





Lectures Quizzes Midterm Final Real-Life!

Conditional Probabilities

- P(B|A) means “Probability of event B **given** event A"

In other words, event A has already happened, now what is the chance of event B?

"Probability Of" *"Given"*

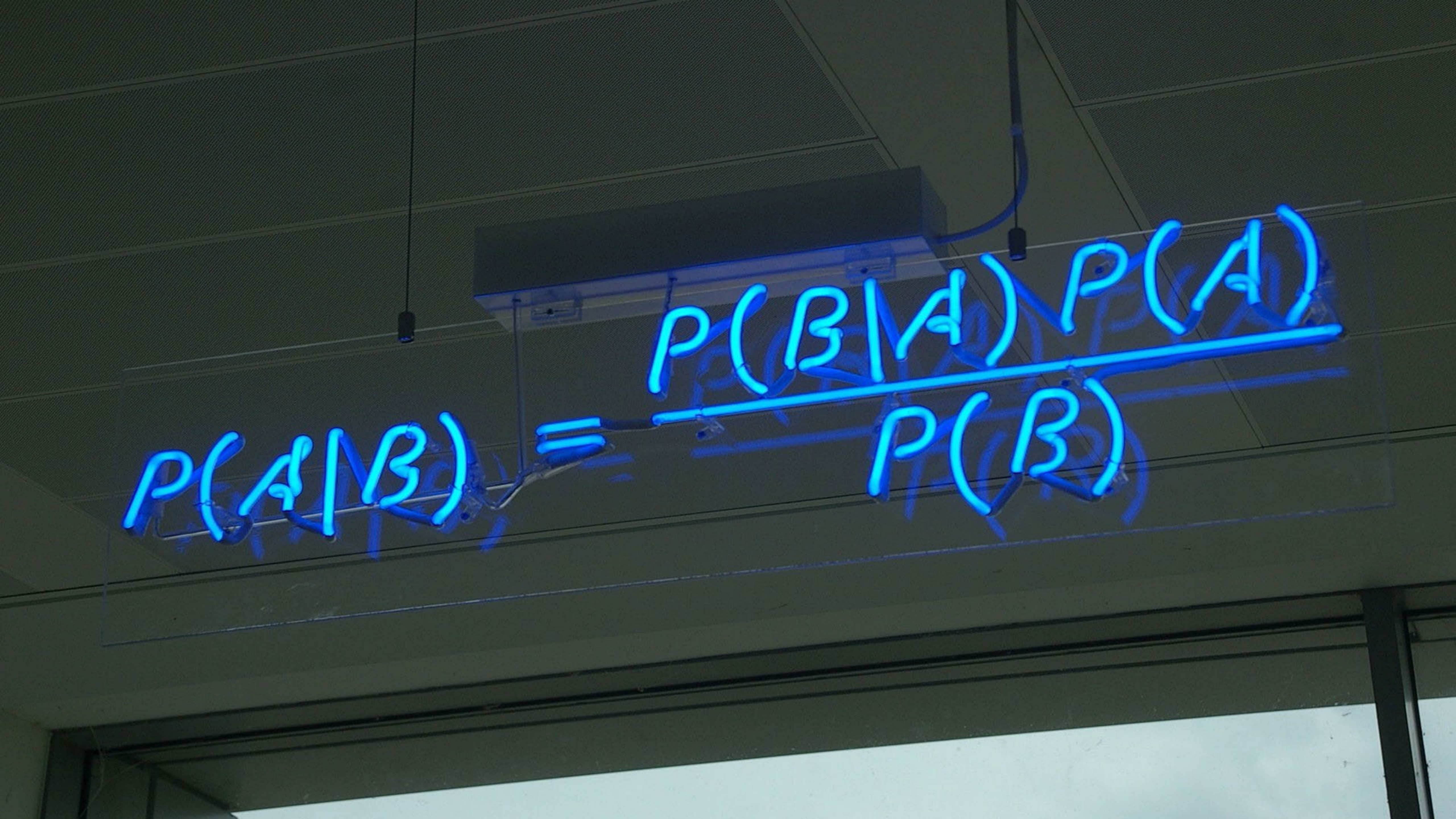
$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

Event A *Event B*

"Probability of ***event A and event B*** equals
the probability of ***event A*** times the probability of ***event B given event A***"

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

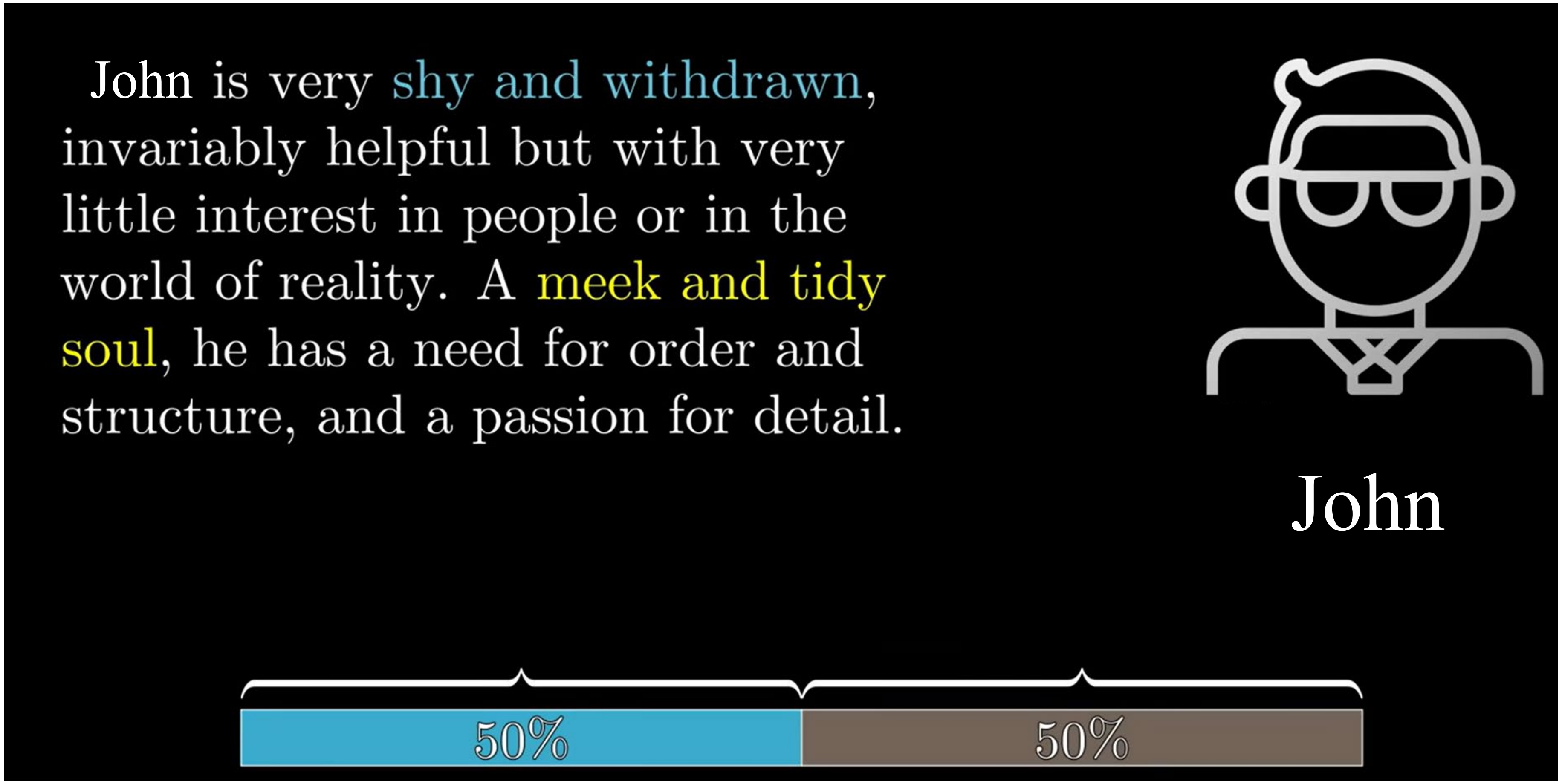
"The probability of ***event B given event A*** equals
the probability of ***event A and event B*** divided by the probability of ***event A***


$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Bayes Rule – 3 questions to answer

- What is it saying?
- Why is it true?
- When should we use it?
  - Rationality is not about knowing facts, it is about knowing which facts are relevant.

Is John a librarian or a farmer?



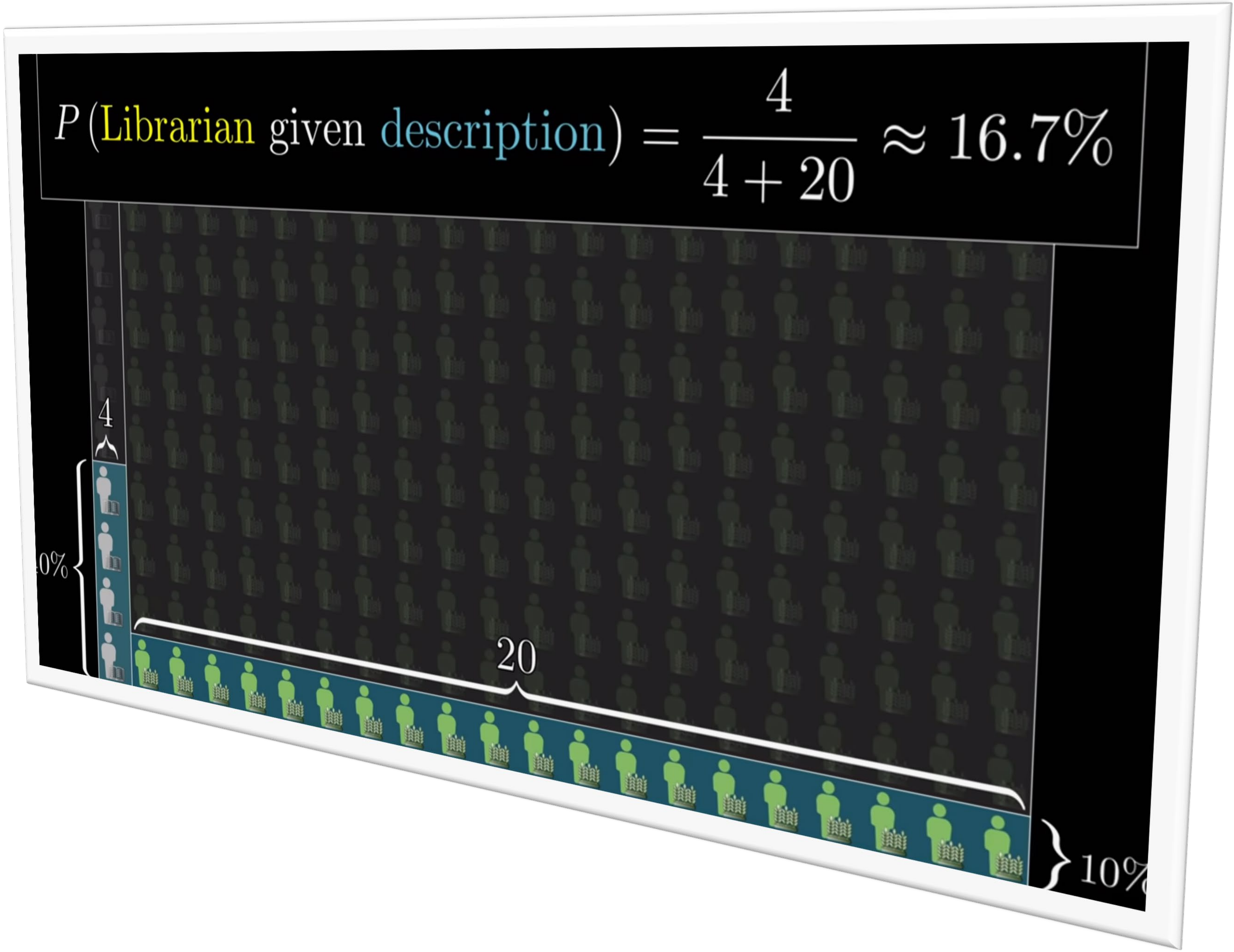
Daniel Kahneman & Amos Tversky (Nobel Prize 2002)

Is John a librarian or a farmer?

Daniel Kahneman & Amos Tversky (Nobel Prize 2002)

Is John a librarian or a farmer?

- We are not supposed to know details like the ratio farmers / librarian, or the stereotype of a librarian
- We are supposed to consider the ratio farmers / librarian, and recognizing which fact is relevant (or not)



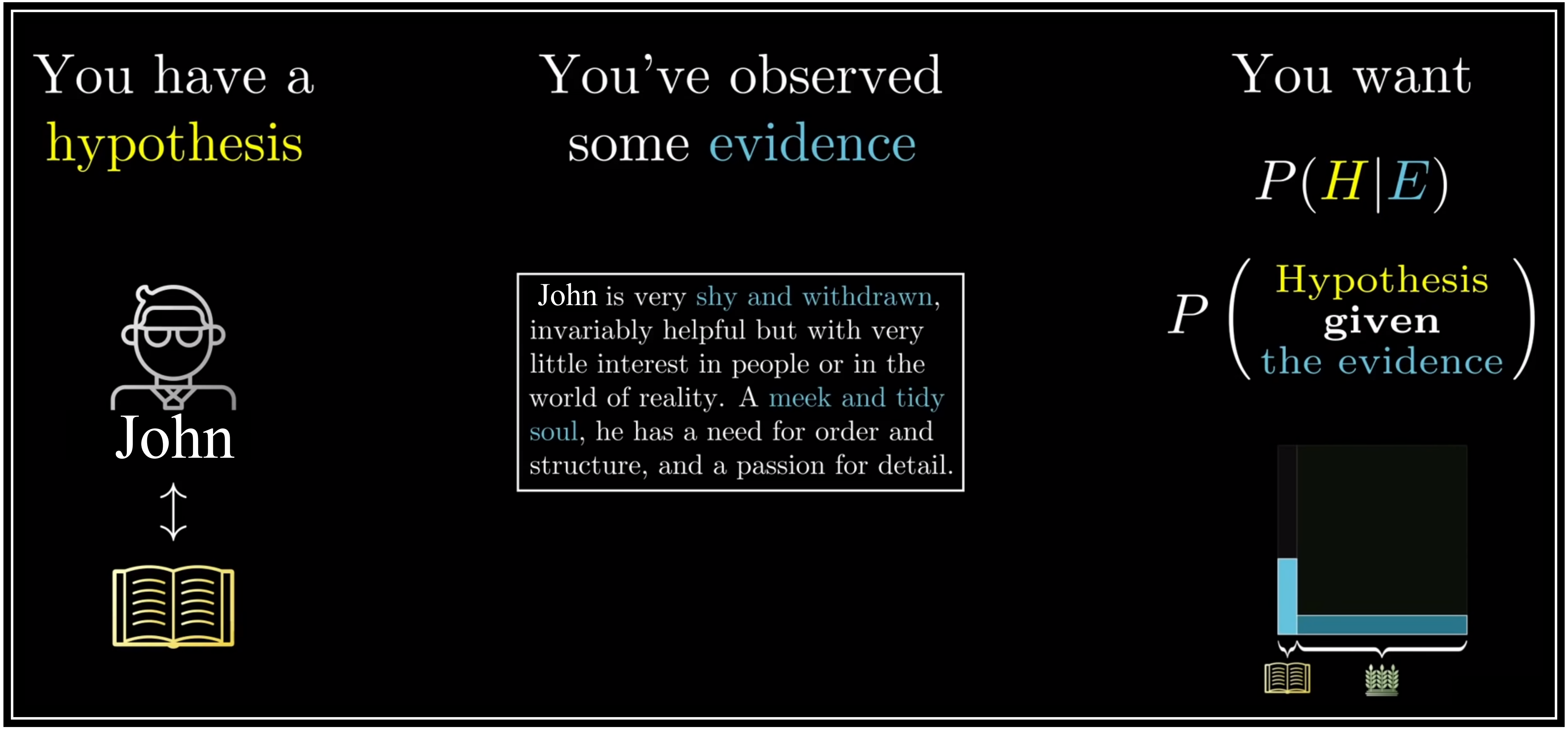
The key idea behind Bayes Theorem

- Quantify and systematize the idea of changing beliefs

- New evidence should not completely determine our beliefs in a vacuum

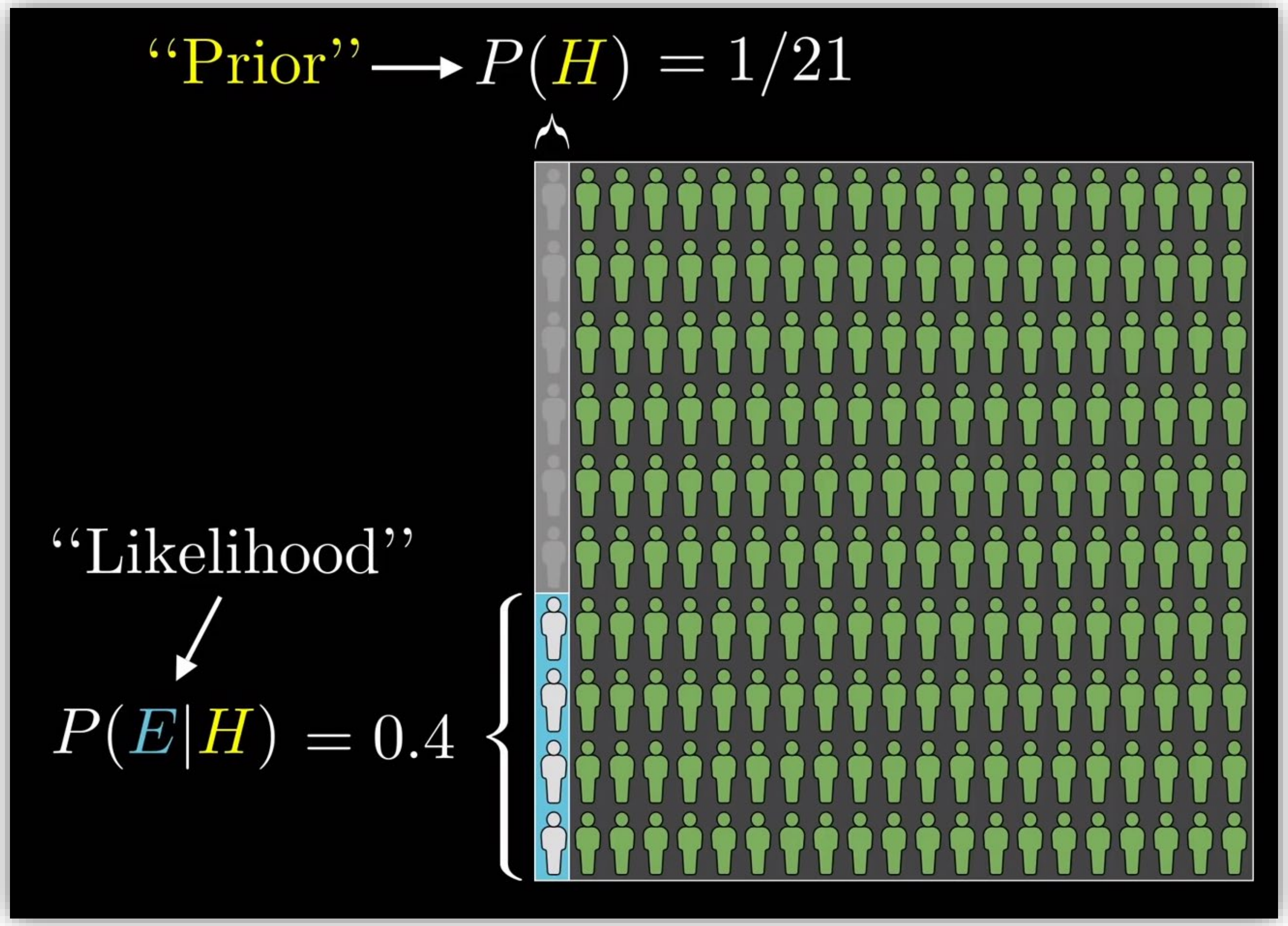
- New evidence updates prior beliefs
  - In other words, new evidence restricts the sample space
    from "All Possibilities" to "All Possibilities fitting the evidence"
  - In this example, we updated our belief that John is a farmer
    from P=(1/21) to P=(4/24)

When to use Bayes Theorem



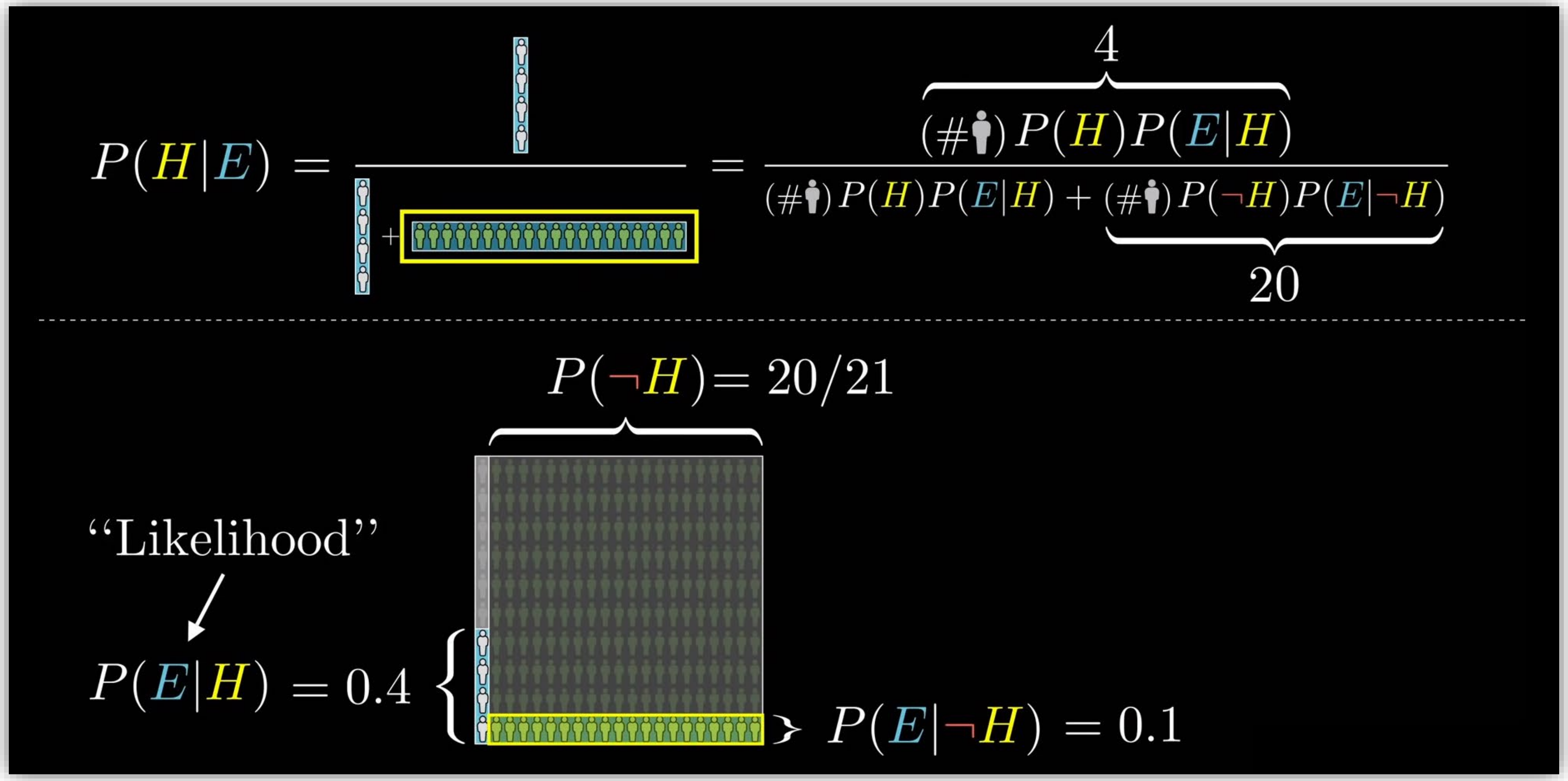
When to use Bayes Theorem – P(H|E)=?

The Probability that the Hypothesis holds before considering any evidence



The proportion of librarians that fit the description: The Probability that we would see the evidence **given** that the Hypothesis is true

When to use Bayes Theorem



When to use Bayes Theorem

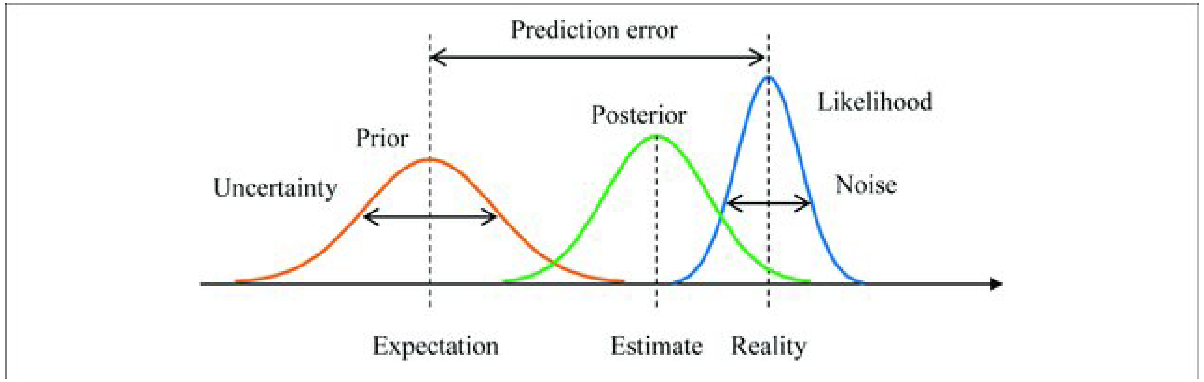
Bayes' theorem

$$P(H|E) = \boxed{\frac{P(H)P(E|H)}{P(E)}} = \frac{P(H)P(E|H)}{P(H)P(E|H) + P(\neg H)P(E|\neg H)}$$

Bayesian Inference*

$$\overbrace{P(A|B)}^{\text{Posterior}} = \frac{\overbrace{P(B|A)}^{\text{Likelihood}} * \overbrace{P(A)}^{\text{Prior}}}{\underbrace{P(B)}_{\text{Evidence}}}$$

* *Inference = Educated guessing*



- Bayesian inference with a prior distribution, a posterior distribution, and a likelihood function.
- The prediction error is the difference between the prior expectation and the peak of the likelihood function (i.e., reality).
- Uncertainty is the variance of the prior. Noise is the variance of the likelihood function.

Bayes Rule – One more example

Find the probability for "when there is smoke, there is fire"

Example: If dangerous fires are rare (1%) but smoke is fairly common (10%) due to barbecues, and 90% of dangerous fires make smoke then:

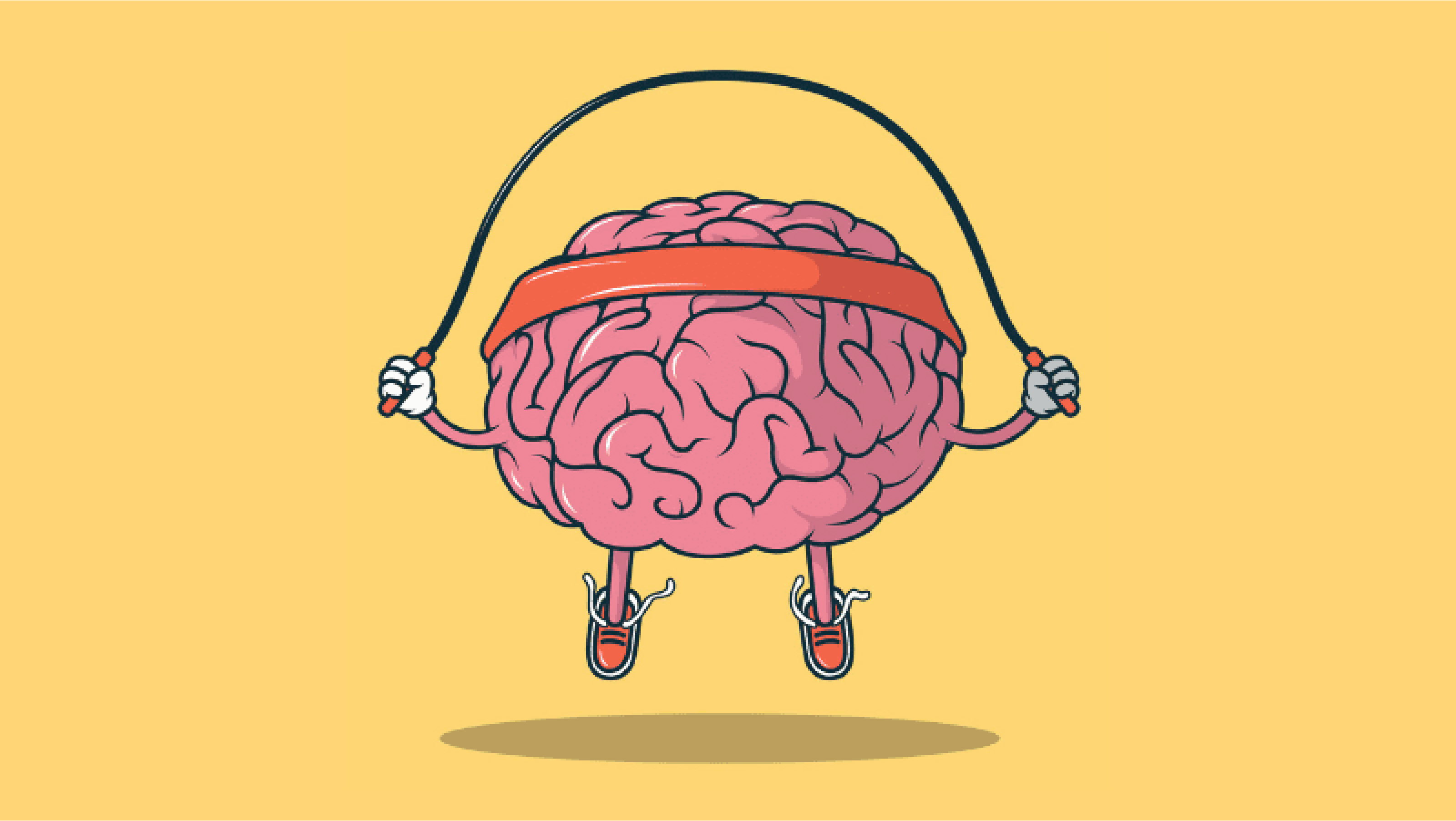
$$P(\text{Fire}|\text{Smoke}) = \frac{P(\text{Fire})\,P(\text{Smoke}|\text{Fire})}{P(\text{Smoke})}$$

$$= \frac{1\% \times 90\%}{10\%}$$

$$= 9\%$$

So the "Probability of dangerous Fire when there is Smoke" is 9%

Brain Break – 1 min



CTAAR survey

https://sirs.rutgers.edu/blue

CTAAR Survey

Section 1

Section 2

Section 3

Example #3 – Bayes rule

- 4 cards are drawn from a randomly shuffled deck of 52 cards. What is the probability that at least 2 Aces are drawn, given that at least one card is an Ace?

A → at least 2 Aces

B → at least 1 Ace

$$P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)} = \frac{P(A)}{P(B)}$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{52}{4} - \binom{48}{4} - 4\binom{48}{3}}{\binom{52}{4}}$$

$$P(B) = \frac{|B|}{|\Omega|} = \frac{\binom{52}{4} - \binom{48}{4}}{\binom{52}{4}}$$

Example #4 - Bayes Rule

- Seif has two coins in his hand. One is a real coin and the second one is a faulty one with Tales on both sides. He blind folds himself, chooses a random coin and tosses it in the air. The coin falls down with Tale facing upwards.

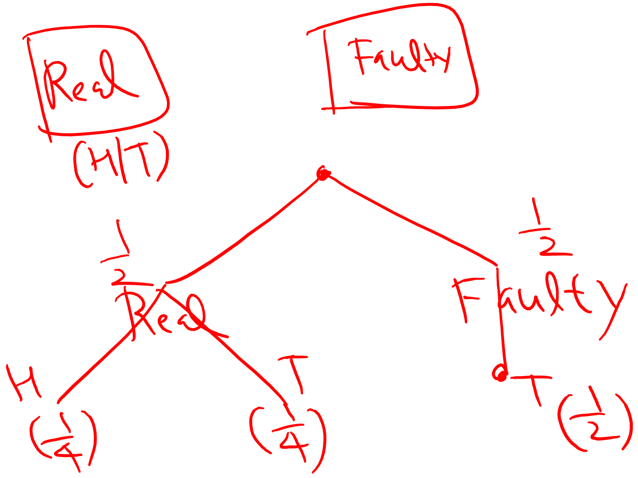
  What is the probability that this is the faulty coin?

$B \rightarrow$ outcome is Tale

$A \rightarrow$ coin chosen was faulty

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(A)}{P(B)}$$

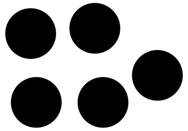
$P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$



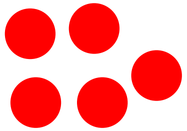
Law of Total Probability – Example #2

- You've been captured by pirates on an island.
- Need to play the following game to survive

100 black rocks



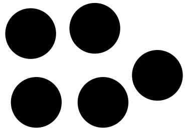
100 red rocks



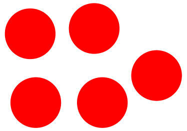
- Divide the rocks among two bags as you wish.
- Toss a fair coin and depending on the outcome draw a rock at random from the corresponding bag.
- If rock is black you win!

Law of Total Probability – Example #2

100 black rocks



100 red rocks



X Black rocks
Y Red rocks

100−X Black rocks
100−Y Red rocks

$$P(\text{black rock is selected})$$

$$P(\text{black rock} \mid \text{Coin}=H) = \frac{X}{X+Y}$$

$$P(\text{black rock} \mid \text{Coin}=T) = \frac{100-X}{200-X-Y}$$

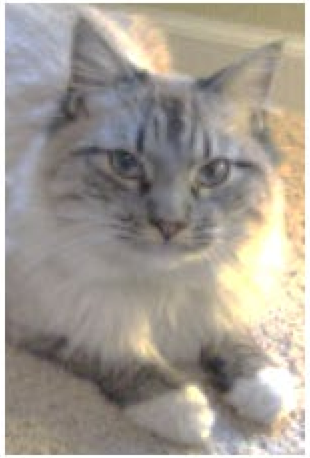
$$P(\text{black rock is selected}) = P(\text{Coin}=H)\,P(\text{Black rock} \mid H) + P(\text{Coin}=T)\,P(\text{Black} \mid T)$$

$$= \left(\frac{1}{2}\right)\left(\frac{X}{X+Y}\right) + \left(\frac{1}{2}\right)\left(\frac{100-X}{200-X-Y}\right)$$

Bayes Rule – Yet another example

Example: Allergy or Not?

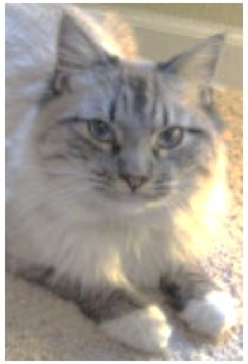
Hunter says she is itchy. There is a test for Allergy to Cats, but this test is not always right:



- For people that **really do** have the allergy, the test says "Yes" **80%** of the time
- For people that **do not** have the allergy, the test says "Yes" **10%** of the time ("false positive")

If 1% of the population have the allergy, and **Hunter's test says "Yes"**, what are the chances that Hunter really has the allergy?

Bayes Rule – Yet another example



$$P(\text{Allergy}|\text{Yes}) = \frac{P(\text{Allergy})\ P(\text{Yes}|\text{Allergy})}{P(\text{Yes})}$$

P(Allergy) is Prob of Allergy = 1%

P(Yes|Allergy) is Prob of test saying "Yes" for people with allergy = 80%

P(Yes) is Prob of test saying "Yes" (to anyone) = ??%

- We **don't know** what the **general** chance of the test saying "Yes" is but we can calculate it by adding up those **with**, and those **without** the allergy:
- 1% have the allergy, and the test says "Yes" to 80% of them
- 99% do **not** have the allergy and the test says "Yes" to 10% of them

P(Yes) = 1% × 80% + 99% × 10% = 10.7% of the population.

P(Allergy|Yes) = about **7%**