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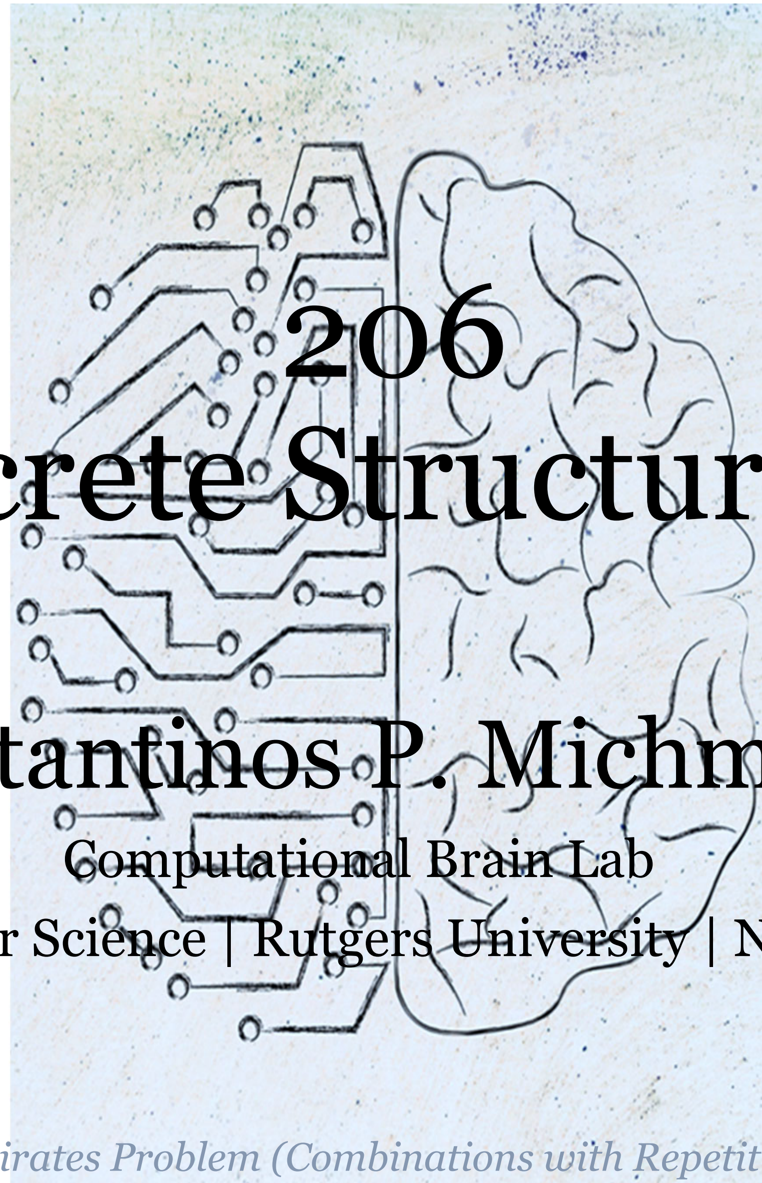

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Discrete Structures II

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Lecture 13 | Pirates Problem (Combinations with Repetitions) | Wednesday October 25th 2022

General Hint – Revisited

For each problem

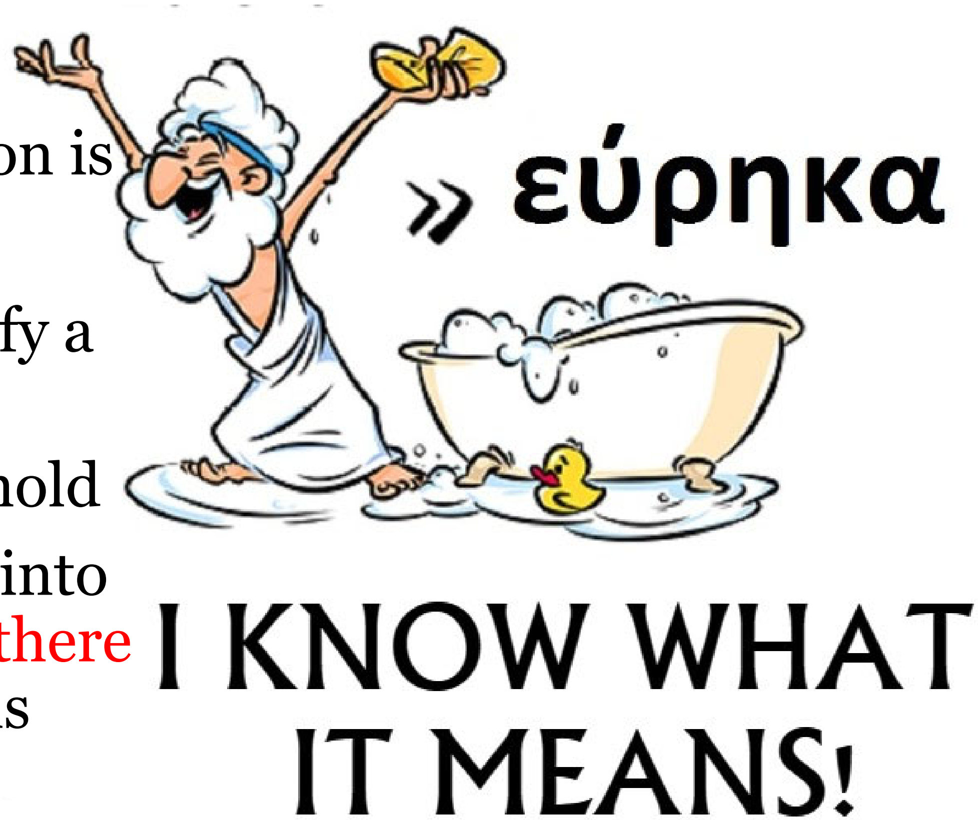
(1) Fully understand what the question is

(2) Fully understand what you know

(3) Based on the previous two, identify a method

(4) Make sure that the assumptions hold

(5) Turn the wording of the problem into the input to your method. Typically, there is a “key” thought that will unlock this part of the solution for you.



So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

So far...

Flowchart: Permutations/Combinations

n distinct elements

- Pick r out of n without repetitions
  - Order matters
    $$^{n}P_r = \frac{n!}{(n-r)!}$$
  - Order does not matter
    $$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$
- Pick r out of n with repetitions
  - Order matters
    $$n^r$$
  - Order does not matters
    $$\binom{n+r-1}{n-1}$$

So far...

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  1^{st} way:
  - No repetitions
  - Order matters $\underline{5} \times \underline{4} = 5 \cdot 4 = {}^5P_2$

So far...

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  2^{nd} way:
  - No repetitions
  - Order does not matter

$\rightarrow \binom{5}{2}$

So far…

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  3^{rd} way:

  - Repetitions allowed
  - Order matters

$\underline{5}\ \underline{5}$ answer $= 5^2$

So far…

Four Ways of Permuting/Choosing

- Choose 2 letters from {L,U,C,K,Y}

  4th way:

  - Repetitions allowed
  - Order does not matter

Partition Method

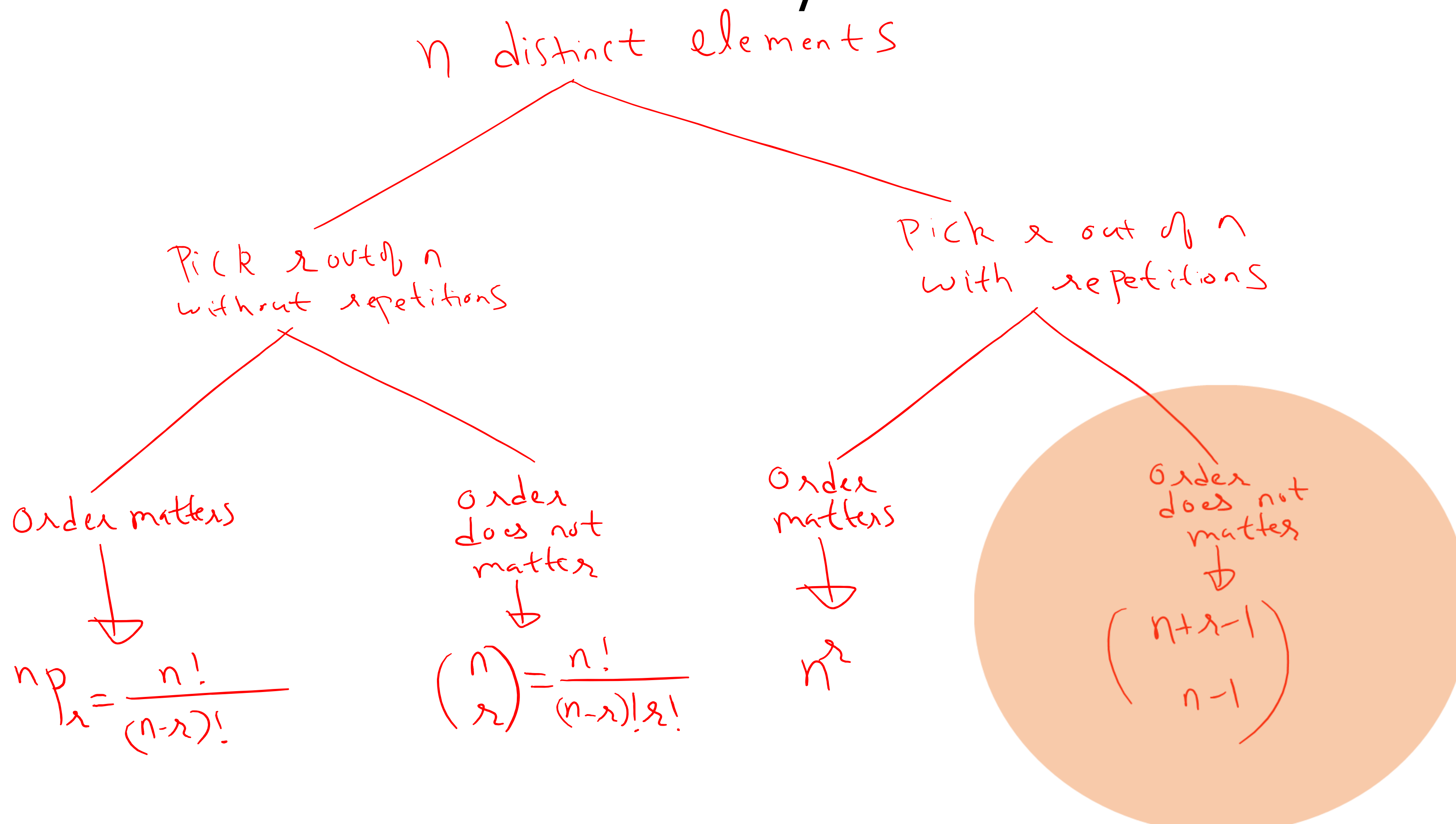
Case 1: There is a repetition $\rightarrow$ 5 ways

Case 2: There is no repetition $\rightarrow \binom{5}{2}$

answer $= \binom{5}{2} + 5$

So far...

Flowchart: Permutations/Combinations



Combinations with Repetition

- Want to pick 10 chocolates out of 4 different types. Can pick many of one type. How many ways?

(H) Hershey	(T) Twix	(M) M&M	(S) Snickers
2	2	4	2
1	1	1	7

Count all sequences of length 4 $\rightarrow (a,b,c,d)$

such that $a+b+c+d=10$

Combinations with Repetitions

- Want to pick 10 chocolates out of 4 different types. Can pick many of one type. How many ways?

Count all sequences of length 4, (a,b,c,d), $a+b+c+d=10$

H T M S

$(2,2,4,2) \longrightarrow \bullet\bullet \mid \bullet\bullet \mid \bullet\bullet\bullet\bullet \mid \bullet\bullet$

$(1,1,1,7) \longrightarrow \bullet \mid \bullet \mid \bullet \mid \bullet\bullet\bullet\bullet\bullet\bullet\bullet$

$(0,0,0,10) \longrightarrow \mid\mid\mid \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet$

$(3,3,3,1) \longleftarrow \bullet\bullet\bullet \mid \bullet\bullet\bullet \mid \bullet\bullet\bullet \mid \bullet$

Combinations with Repetitions

- Want to pick 10 chocolates out of 4 different types. Can pick many of one type. How many ways?

• • | • • • • | • • • | •

ways to choose = # arrangements of 10 dots and 3 vertical lines

So far...

Combinations with Repetitions

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot?

Count all sequences of (a,b,c,d,e) such that $a+b+c+d+e=20$

$a =$ # Pirate 1 gets
$b =$ # 2 gets
$c =$ # 3 gets
$d =$ 4
$e =$ 5 gets

$\cdot \mid \cdot \ \cdot \mid \cdot \ \cdot \mid \ \mid \cdot \ \cdot$

answer = all ways to arrange 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)} = \binom{24}{4}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_5 = 20$
  - $x_1, x_2, \ldots, x_5 \geq 0$

$(x_1, x_2, x_3, x_4, x_5)$ such that $\sum x_i = 20$

$\Rightarrow$ all arrangements of 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_k = n$
  - $x_1, x_2, \ldots, x_k \geq 0$

k-1 lines

→ all arrangements of n dots and k-1 lines

$$\rightarrow \frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1}$$

So far...

Combinations with Repetitions

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot?

Count all sequences of (a,b,c,d,e) such that $a+b+c+d+e=20$

$a=$ # pirate 1 gets

$b=$ # 2 gets

$c=$ # 3 gets

$d=$ 4

$e=$ 5 gets

$\cdot \mid \cdot\ \cdot \mid \cdot\ \cdot \mid\ \mid \cdot\ \cdot$

answer = all ways to arrange 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)} = \binom{24}{4}$$

So far...

Combinations - Adv'ced (with constraints)

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot when each must get at least 1 bar?

→ Give 1 bar to each

→ 5 Pirates, 15 bars

→ 15 dots, 4 lines

$$\text{ans } \text{wu} = \frac{19!}{(15!)(4!)} = \binom{19}{4}$$

So far...

Combinations

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot when each must get at least 2 bars?

→ Give 2 bars to everyone

→ 10 bars, 5 pirates

→ 10 dots, 4 lines

$$\text{answer} = \frac{14!}{10!\,4!}$$

Combinations

- k distinct pirates want to divide up n identical, indivisible bars of gold. How many ways to divide the loot when each must get at least r bars?

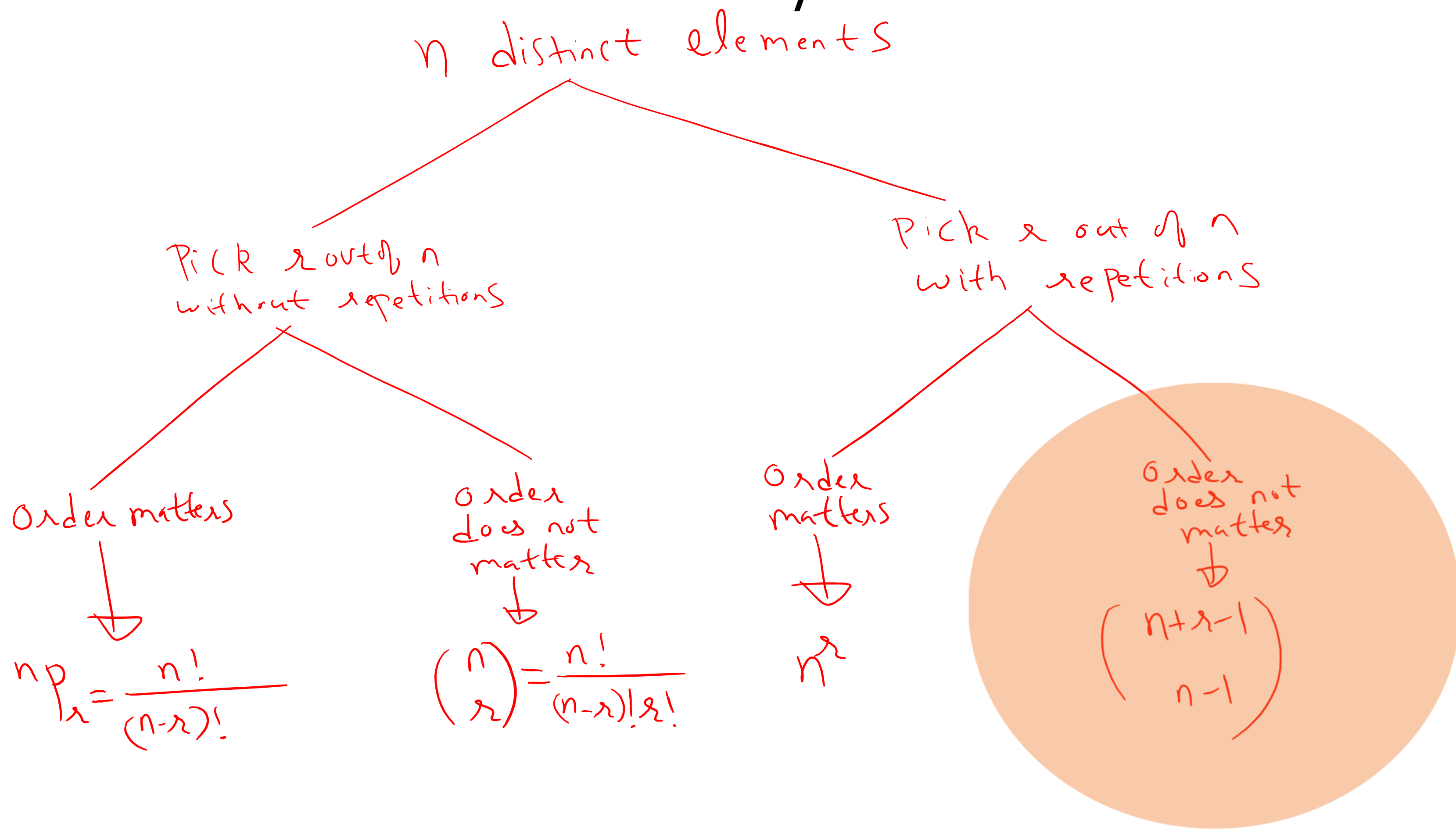
→ Give everyone r bars

→ $n-kr$ items, k pirates

→ $n-kr$ dots, $k-1$ lines

answer = $\binom{n-kr+k-1}{k-1}$

Flowchart: Permutations/Combinations



Flowchart: Permutations/Combinations

n identical elements

Divide into k pieces

$$\binom{n+k-1}{k-1}$$

Divide into k pieces with each piece having $\geq r$ elements

$$\binom{n-kr+k-1}{k-1}$$

Get your  in gear

- 5 pirates want to divide 20 identical bars of gold among them. How many ways to divide if each pirate wants at least 2 bars and no pirate can get more than 8 bars.

First give 2 bars to each pirate. We are left with 10 bars. No pirate can get more than 6 of them.

Difference method. Count all possible ways to divide 10 bars among 5 pirates and subtract the number of ways in which some pirate gets more than 6 bars.

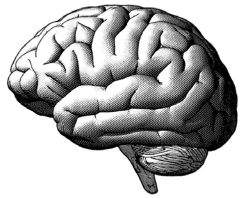
There are $\binom{14}{4}$ ways to divide 10 bars among 5 pirates.

Now, let's count the ways in which some pirate gets more than 6 bars.

Notice that only one pirate can get more than 6 bars. There are five cases:

- – All ways to distribute gold such that pirate 1 gets more than 6 bars.
- – All ways to distribute gold such that pirate 2 gets more than 6 bars.
- – All ways to distribute gold such that pirate 3 gets more than 6 bars.
- – All ways to distribute gold such that pirate 4 gets more than 6 bars.
- – All ways to distribute gold such that pirate 5 gets more than 6 bars.

For each case, the answer is $\binom{7}{4}$. In total there are $5 \cdot \binom{7}{4}$ ways to distribute such that some pirate gets more than 6. By the **difference method**, the final answer is $\binom{14}{4} - 5 \cdot \binom{7}{4}$.

Get your  in gear

- How many bit strings of length 8 either start with a 1 or end with a 00?

Solution: Use the **partition method**.
Let A1 = number of bit strings that start with 1 and end with 00.
Let A2 = number of strings that start with 1 and do not end with 00.
Let A3 = number of strings that start with 0 and end in 00.
We have $|A1| = 2^5$, $|A2| = 2^5 \cdot 3$, $|A3| = 2^5$.
Hence the total number of strings = $2^5 + 2^5 \cdot 3 + 2^5$.

Explanation for |A2|. In A2 we are counting all string that start with 1 and do not end with 00. Again using the **partition method**, we can divide the outcomes into 3 possible subsets: start with 1 and end with 01, start with 1 and end with 10, start with 1 and end with 11. In each cases, there are 2^5 choices for the remaining 5 elements.

Combinations – adv'ed

- If we roll 7 dice how many different outcomes if
  - Order matters → dice are diff color
  - Order does not matter → dice are all white

Combinations – adv'ed

- If we roll 7 dice how many different outcomes if
  - Order matters

answer → 6^7

Combinations – Adv'ced

- If we roll 7 dice how many different outcomes if
  - Order does not matter

Counting all sequences of (a, b, c, d, e, f)

such that $a+b+c+d+e+f = 7$

$a = \#$ times 1 appears

$b = \#$ times 2 appears

$\vdots$

$f = \#$ times 6 appears

$\cdot \,|\, \circ \,|\, \circ \,|\, \circ \,|\, \circ \,|\, \cdot$

7 dots

5 lines

answer $= \dfrac{12!}{7!\,5!} = \dbinom{12}{5}$