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206
Discrete Structures II

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Lecture 14 | Pirates Problem & Inclusion-Exclusion | Monday October 30th 2023

Quiz 2 – Stats

	# Submissions	Mean	Range
Section - 1	52	63.3	[10, 120]
Section - 2	53	60.0	[20, 100]
Section - 3	48	64.5	[5, 120]
Whole Class	153	63.0	[5, 120]

Quiz 3 – Stats pending

	# Submissions	Mean	Range
Section - 1			
Section - 2			
Section - 3			
Whole Class		~85	

Quiz 4 – Next Week

- When
  - Monday 11/7 & Wednesday 11/9, during recitation
- What
  - Product rule (always handy - Week 4-5 Lectures)
  - Permutations
    - with and without constraints
    - with and without repetitions (Week 5 & Week 6 Lectures)
  - Combinations
    - With and without constraints
    - Without repetitions (Week 6 Lectures; pirates problem)

General Hint – Revisited

For each problem

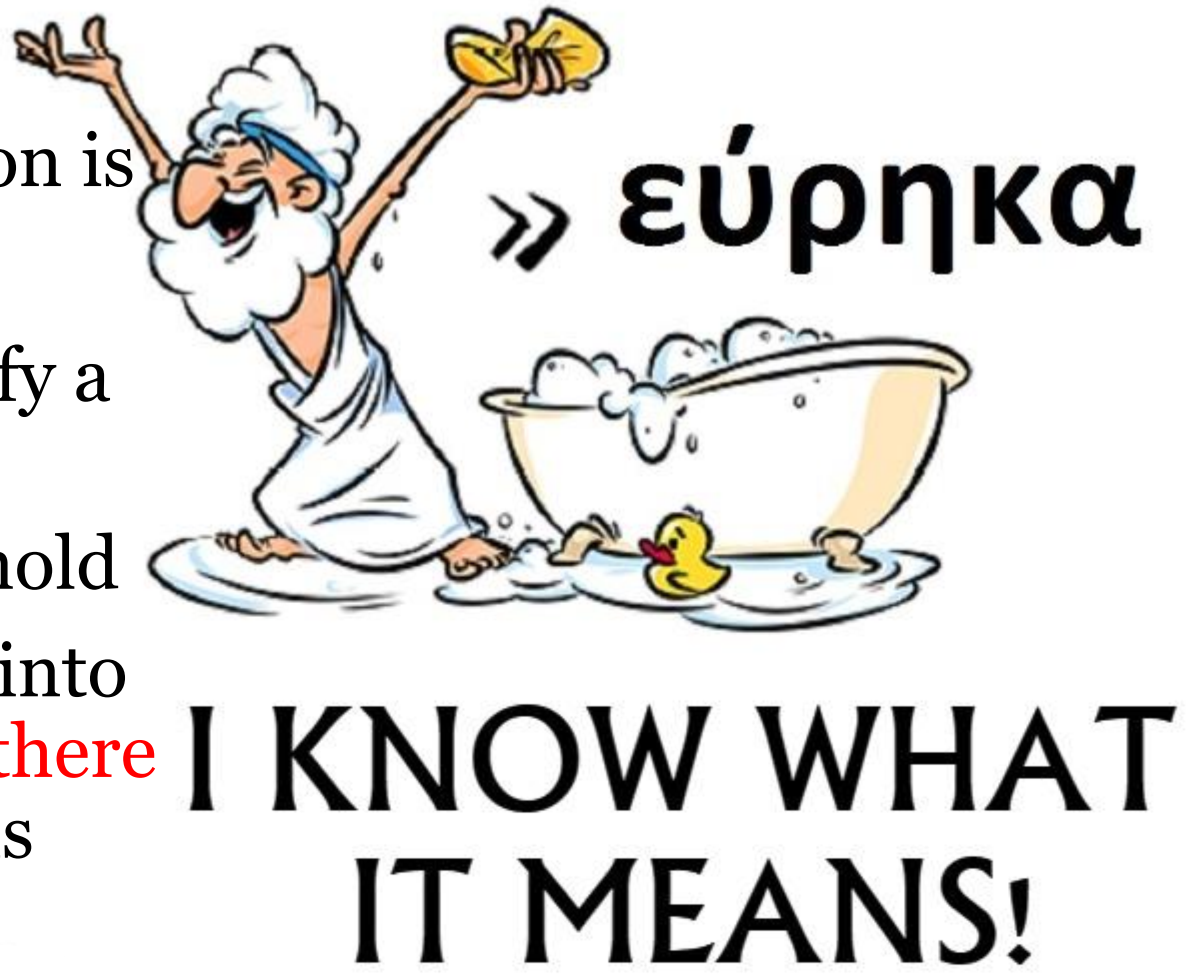
(1) Fully understand what the question is

(2) Fully understand what you know

(3) Based on the previous two, identify a method

(4) Make sure that the assumptions hold

(5) Turn the wording of the problem into the input to your method. Typically, there is a “key” thought that will unlock this part of the solution for you.

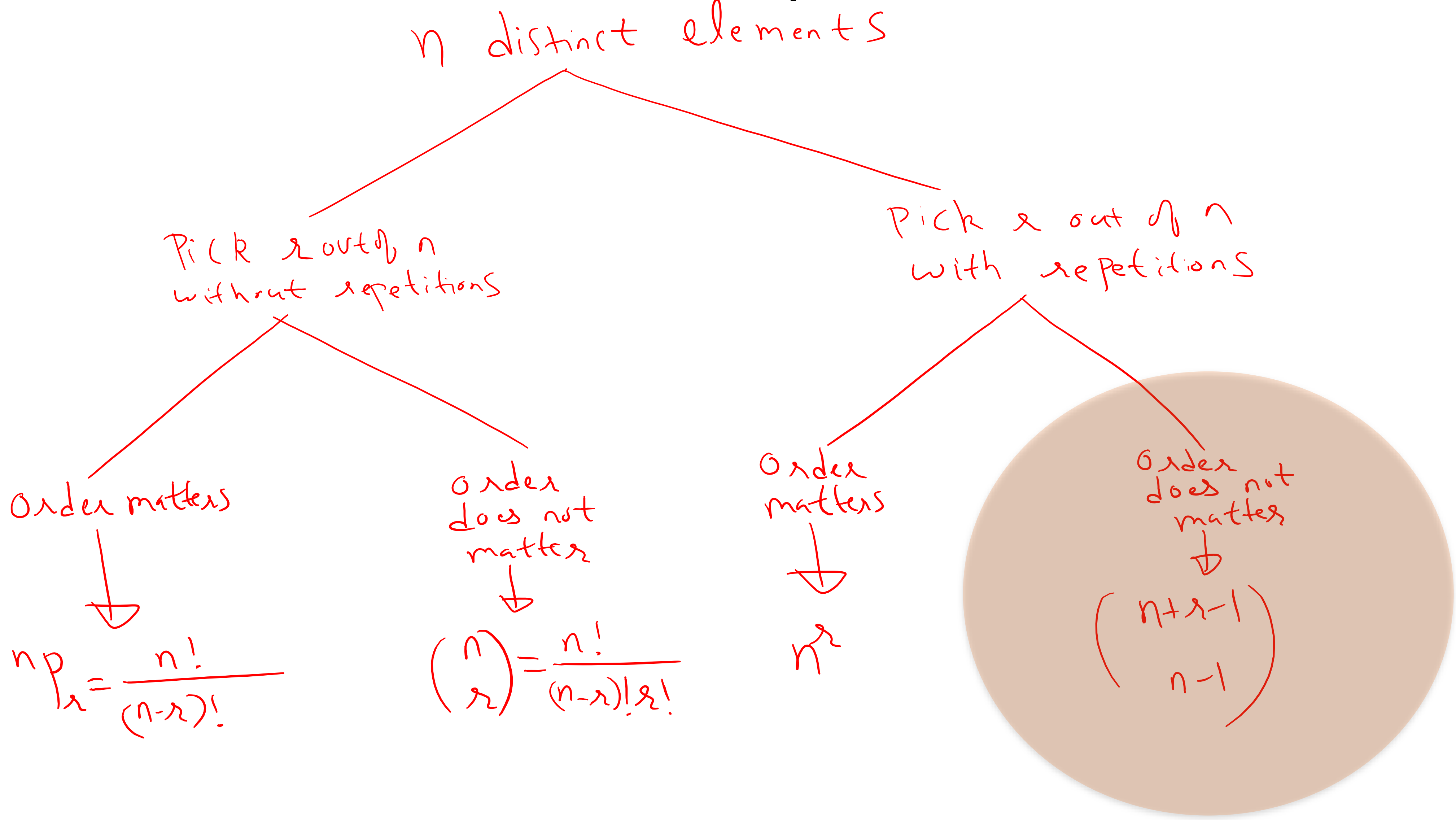


So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- ~~Permutation/Combinations~~
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

So far...

Flowchart: Permutations/Combinations



So far...

Combinations with Repetitions

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot?

Count all sequences of (a,b,c,d,e) such that $a+b+c+d+e=20$

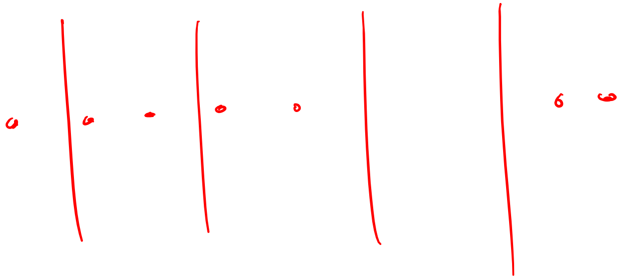
a = # Pirate 1 gets

b = # 2 gets

c = # 3 gets

d = 4

e = 5 gets



answer = all ways to arrange 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)} = \binom{24}{4}$$

So far...

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_5 = 20$
  - $x_1, x_2, \ldots, x_5 \geq 0$

$(x_1, x_2, x_3, x_4, x_5)$ such that $\sum x_i = 20$

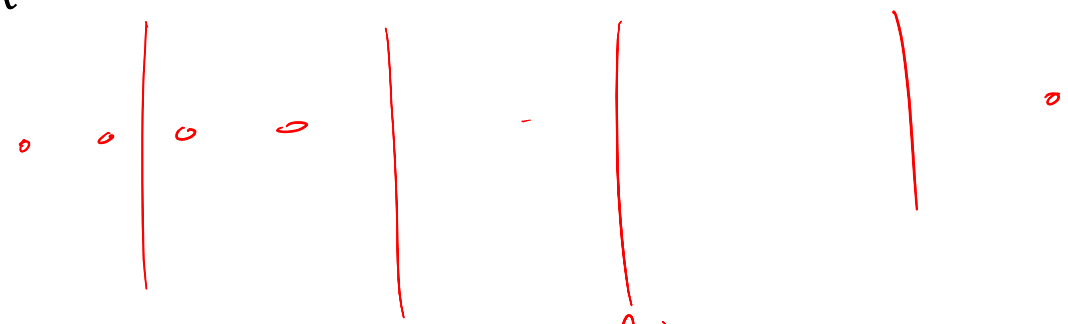
$\Rightarrow$ all arrangements of 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)}$$

So far...

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_k = n$
  - $x_1, x_2, \ldots, x_k \geq 0$



K-1 lines

→ all arrangements of n dots and K-1 lines

$$\rightarrow \frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1}$$

Combinations - Adv'ced (with constraints)

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot when each must get at least 1 bar?

→ Give 1 bar to each

→ 5 pirates, 15 bars

→ 15 dots, 4 lines

$$\text{ans now} = \frac{19!}{(15!)(4!)} = \binom{19}{4}$$

So far...

Combinations

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot when each must get at least 2 bars?

→ Give 2 bars to everyone

→ 10 bars, 5 pirates

10 dots, 4 lines

→

answer = $\frac{14!}{10!\,4!}$

Combinations

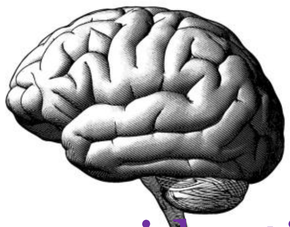
- k distinct pirates want to divide up n identical, indivisible bars of gold. How many ways to divide the loot when each must get at least r bars?

→ Give everyone r bars

→ n−kr items, k pirates

→ n−kr dots, k−1 lines

answer = $\binom{n-kr+k-1}{k-1}$

Get your  in gear

- 5 pirates want to divide 20 identical bars of gold among them. How many ways to divide if each pirate wants at least 2 bars and no pirate can get more than 8 bars.

First give 2 bars to each pirate. We are left with 10 bars. No pirate can get more than 6 of them.

Difference method. Count all possible ways to divide 10 bars among 5 pirates and subtract the number of ways in which some pirate gets more than 6 bars.

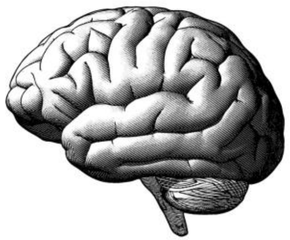
There are $\binom{14}{4}$ ways to divide 10 bars among 5 pirates.

Now, let's count the ways in which some pirate gets more than 6 bars.

Notice that only one pirate can get more than 6 bars. There are five cases:

- – All ways to distribute gold such that pirate 1 gets more than 6 bars.
- – All ways to distribute gold such that pirate 2 gets more than 6 bars.
- – All ways to distribute gold such that pirate 3 gets more than 6 bars.
- – All ways to distribute gold such that pirate 4 gets more than 6 bars.
- – All ways to distribute gold such that pirate 5 gets more than 6 bars.

For each case, the answer is $\binom{7}{4}$. In total there are $5 \cdot \binom{7}{4}$ ways to distribute such that some pirate gets more than 6. By the **difference method**, the final answer is $\binom{14}{4} - 5 \cdot \binom{7}{4}$.

Get your  in gear

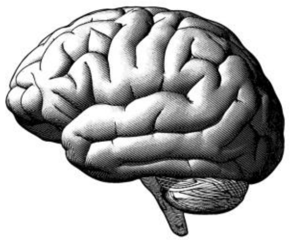
- How many ways to seat 6 boys and 8 girls in a row such that no two boys are seated next to each other.

— First seat 8 girls

— There are 9 empty places next to them

— Pick 6 out of 9 and seat the boys

answer = $(8!)\binom{9}{6}(6!)$

Get your  in gear

- How many bit strings of length 8 either start with a 1 or end with a 00?

Solution: Use the **partition method**.
Let A1 = number of bit strings that start with 1 and end with 00.
Let A2 = number of strings that start with 1 and do not end with 00.
Let A3 = number of strings that start with 0 and end in 00.
We have $|A1| = 2^5$, $|A2| = 2^5 \cdot 3$, $|A3| = 2^5$.
Hence the total number of strings $= 2^5 + 2^5 \cdot 3 + 2^5$.

Explanation for |A2|. In A2 we are counting all string that start with 1 and do not end with 00. Again using the **partition method**, we can divide the outcomes into 3 possible subsets: start with 1 and end with 01, start with 1 and end with 10, start with 1 and end with 11. In each cases, there are 25 choices for the remaining 5 elements.

1. (15%) How many 8 letter words contain only vowels?

Hint: There are 5 vowels

Solution: 5^8. There are 5 vowels. Hence, for each letter there are 5 choices and these are repeatable, $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$.

How many 8 letter words contain only consonants?

Solution: 21^8. Same as before. There are 21 consonants. Hence, for each letter there are 21 choices.

How many 8 letter words contain both vowels and consonants?

Solution: DIFFERENCE METHOD: $26^8 - 5^8 - 21^8$. Let S be the set of all 8-letter words, S_1 be the set of all 8-letter words containing only vowels, and S_2 be the set of all 8-letters words containing only consonants. Then by the difference method, the number of 8-letter words that contain both vowels and consonants is $|S| - |S_1| - |S_2| = 26^8 - 5^8 - 21^8$.

2. (10%) How many ways to divide 10 identical Hershey bars among 3 kids?

Solution: See slides 33–45 of lecture 7. This is the same as number of non-negative solutions to $x_1 + x_2 + x_3 = 10$. And the same as counting the ways for arranging 3-1 = 2 dividers within the 10 dots, as in lecture. Hence, $\binom{12}{2}$.

How many ways to divide a subset of 10 identical Hershey bars among 3 kids when you can decide to keep some of the bars for yourself.

Solution: See slides 33–45 of lecture 7. This is the same as number of non-negative solutions to $x_1 + x_2 + x_3 + x_4 = 10$. Hence, $\binom{10+3}{3}$.

3. (30%) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$ where x_1, x_2, x_3, x_4 are non-negative integers?

> **Solution:** See slides 33–45 of lecture 7. We use 3 dividers to divide 20 dots, and we have $\binom{23}{3}$ ways to place them within the 20 dots.

In each of the following cases, how many solutions are there to the above equation, when, in addition:

(a) $x_1 \geq 1$.

> **Solution:** $\binom{22}{3}$. We give 1 (golden bar) out of 20 to x_1 and then we re-do everything we did before.

(b) $x_i \geq 2$ for $i = 1, 2, \ldots, 4$.

> **Solution:** $\binom{15}{3}$. We give 2 to each of the 4 "pirates". We are left with 12. We still have 3 dividers.

(c) $0 \le x_1 \le 10$.

Solution: Difference Method: Total $\binom{23}{3}-$ Opposite Set $\binom{12}{3}$. For opposite set consider $x_1 \ge 11$, so we give to "$pirate_1$" 11 golden bars and then we re-do what we know to do. After giving 11 bars to x_1, we now have 9 golden bars to give away and still 3 dividers.

(d) $0 \le x_1 \le 3$, $1 \le x_2 \le 4$, $x_3 \ge 15$.

Solution: Since $x_3 \ge 15$ and $x_2 \ge 1$, we give to x_3 15 bars and to x_2 1 bar. We are left with just 4 bars. So we are now looking at the number of solutions to $x_1 + x_2 + x_3 + x_4 = 20 - 15 - 1 = 4$ with $0 \le x_1 \le 3$ and $0 \le x_2 \le 3$. Difference Method: The number of solutions is the total number of solutions minus the special cases that do not hold based on our two constraints. That is, $\binom{4+3}{3} - 1 - 1$. The two "1"'s are for the two cases that need to be excluded, when $x_1 = 4$ and $x_2 = 4$, respectively.