Recitation 9 (Recap): K-nomial theorem, Pascal's Triangle, and Combinatorial Proofs

The binomial theorem

- Recall the following identities from highschool:
- $(x + y)^2 = x^2 + 2xy + y^2$
- $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x + y)^4 = x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + y^4$

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- Is there a pattern here? Can we easily generate the coefficients?
  - (Some of you might already know **how**, but we doubt that you know **why**)

$(x+y)^5$

- $(x+y)^5 = (x+y) \cdot (x+y) \cdot (x+y) \cdot (x+y) \cdot (x+y)$

- What is the coefficient of x^2y^3?

$(x+y)^5$

- $(x+y)^5 = (x+y)\cdot(x+y)\cdot\ (x+y)\cdot(x+y)\cdot(x+y)$

- What is the coefficient of x^2y^3?
- There are $2^5 = 32$ terms total (many combine, eg $xxyyy,\ xyxyy$ are both of form x^2y^3).
- How many of those terms have 2 ‘x’s and 3 ‘y’s?

$(x+y)^5$

- $(x+y)^5 = (x+y)\cdot(x+y)\cdot\ (x+y)\cdot(x+y)\cdot(x+y)$

$xxyyy,\quad xyxyy,\quad xyyxy,\quad xyyyx,$

$yxxyy,\quad yxyxy,\quad yxyyx,$

$yyxxy,\quad yyxyx,$

$yyyxx$

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All terms of form x^2y^3

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$$\left.\begin{array}{llll} xxyyy, & xyxyy, & xyyxy, & xyyyx, \\ yxxyy, & yxyxy, & yxyyx, & \\ yyxxy, & yyxyx, & & \\ yyyxx & & & \end{array}\right\} \text{All terms of form } x^2y^3$$

- This is just choosing 2 slots out of 5 to put the ‘x’s in.
- There are $\binom{5}{2} = 10$ ways of doing this.

You do this now

- What is the coefficient of x^3y^4 in $(x+y)^7$?

You do this **now**

- What is the coefficient of x^3y^4 in $(x+y)^7$?

$$\frac{7!}{3!\cdot 4!}=\binom{7}{3}$$

$(x + y)^n$

- We now generalize the previous results:
- $(x + y)^n = (x + y) \cdot (x + y) \cdot \ldots \cdot \; (x + y)$

- Co-efficient of $x^r y^{n-r}$ = # of ways to select r 'x's from n slots = $\binom{n}{r}$

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- Binomial Theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r}\, x^r y^{n-r}$$

How to find the coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

- Approach #1: Compute **directly** via formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Problem: Large intermediary numbers, **even if $\boldsymbol{n}, \boldsymbol{r}$ and $\binom{\boldsymbol{n}}{\boldsymbol{r}}$ are relatively small!**
  - Example: $\binom{20}{10} = \frac{20!}{10! \cdot 10!} = \frac{1 \times 2 \times \cdots \times 10 \times 11 \times 12 \times \cdots \times 20}{(1 \times 2 \times \cdots \times 10) \cdot (1 \times 2 \times \cdots \times 10)}$

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    Not too large!
  - Is our computer **smart enough** to cancel out the stuff in green?
    - Not every computer is!

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    - But assuming that ours is, we still have to compute $11 \times 12 \times \cdots \times 20$, which is **quite large, even though the final result is small!**

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  - Is our computer **smart enough** to cancel out the stuff in green?
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    - But assuming that ours is, we still have to compute $11 \times 12 \times \cdots \times 20$, which is **quite large.**
- Can we do better?

Another combinatorial identity

$$(\forall n, r \in \mathbb{N}^{\geq 1})\left[\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}\right]$$

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- How can we prove this?

Induction | Direct | Contradiction | Something else

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- How can we prove this? Induction Direct Contradiction Something else
  - Induction
  - Direct
  - Something else: We'll do now.

A combinatorial proof of $\binom{n}{r} = \boxed{\binom{n-1}{r-1} + \binom{n-1}{r}}$

- LHS: #ways to pick r people from a set of n people.
- RHS: Focus on one person, call him $Jason$.
  - If we pick $Jason$, then we are left with $n-1$ people to decide if we want to pick or not, from which we now have to pick $r-1$ people **(first term of RHS)**
  - OR, if we don't pick $Jason$, we are left with $n-1$ people to decide if we want to pick or not, yet still r people that we need to pick (second term of RHS).

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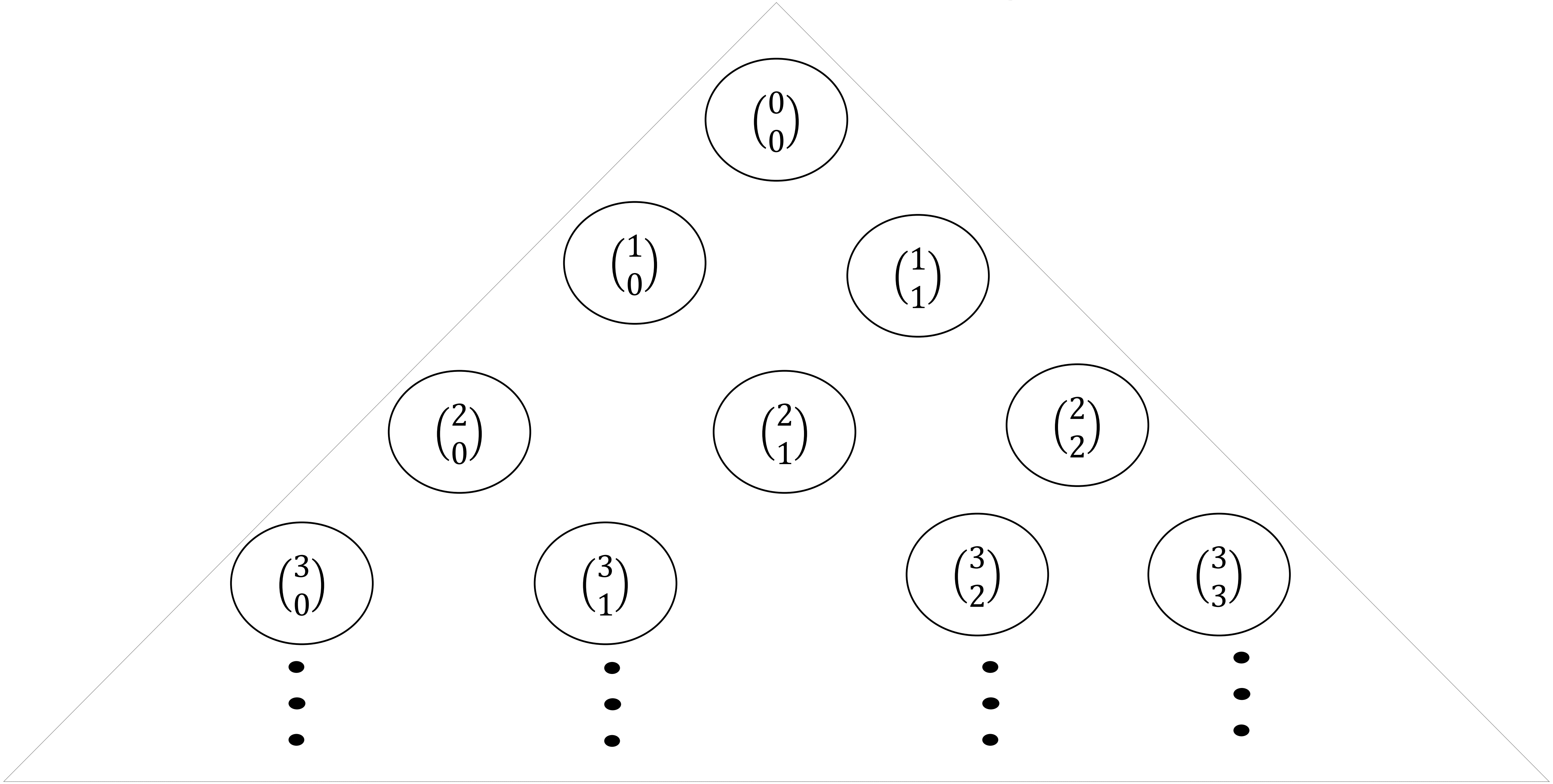
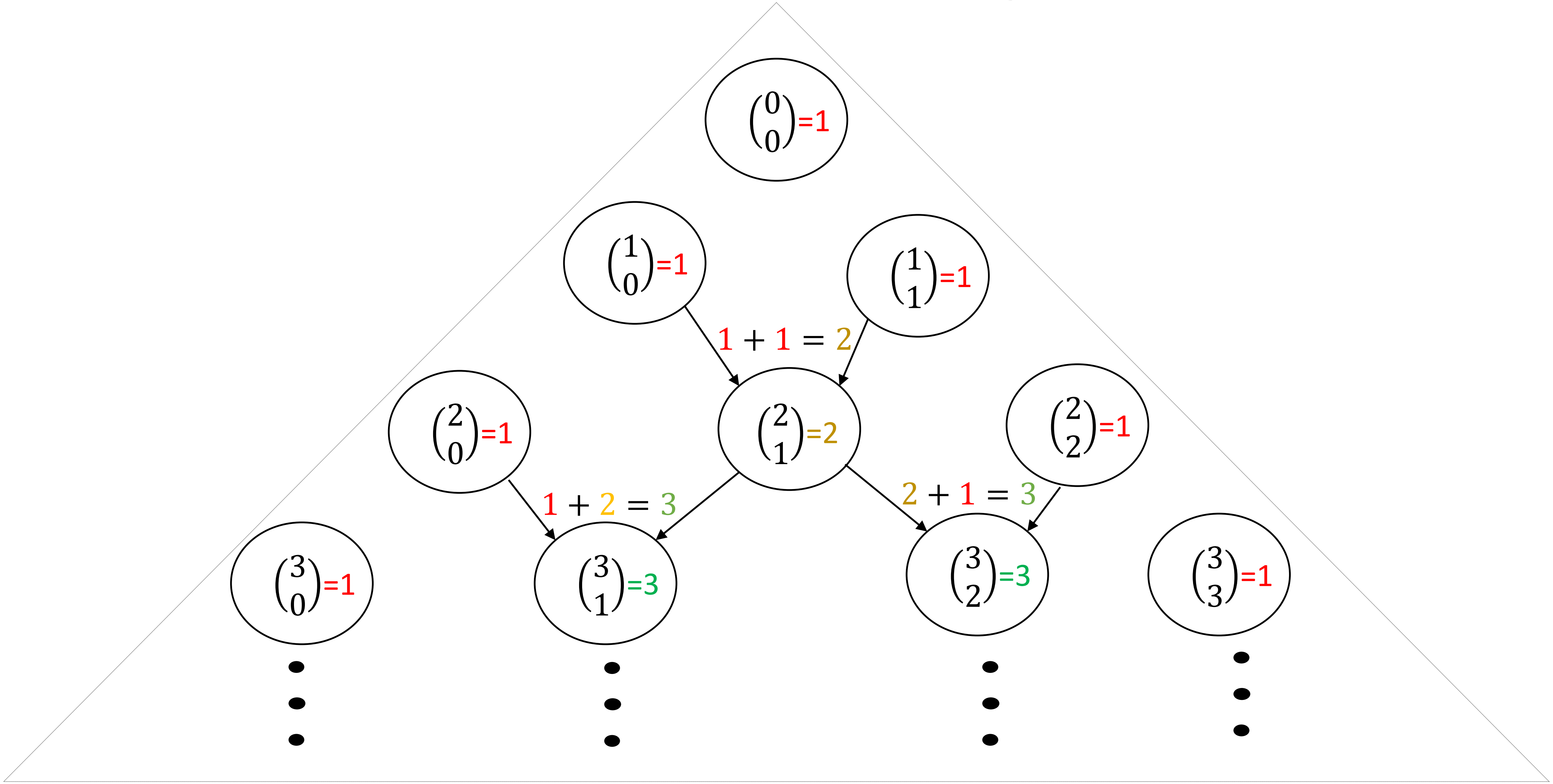
  - OR, if we don't pick $Jason$, we are left with $n-1$ people to decide if we want to pick or not, yet still r people that we need to pick (second term of RHS).
- This is a **combinatorial proof**!
- A combinatorial proof is a type of proof where we show two quantities are equal because they solve the same problem.

Pascal’s Triangle



Pascal's Triangle



Upshot

- Use combinatorial identity

generate Pascal's triangle

generate binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

use in the expansion of $(x+y)^n$

Efficiency of Pascal's triangle

- We avoid the intermediary large numbers problem
- i^{th} level of triangle gives us all coefficients $\binom{i}{0}, \binom{i}{1}, \dots, \binom{i}{i}$
- Compute the value of every node as the sum of its two parents
  - Note that the diagonal "edges" of the triangle always 1.

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- How many permutations of $x^a y^b$ are there?

$$\frac{(a + b)!}{a! \cdot b!}$$

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$$\frac{(a+b)!}{a! \cdot b!} = \frac{n!}{a! \cdot (n-a)!} = \binom{n}{a}$$

An exercise for you to do **now**

- Expand $(x + y + z)^2$

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$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

Trinomial theorem

- $(x+y+z)^5 = (x+y+z)\cdot(x+y+z)\cdot(x+y+z)\cdot(x+y+z)\cdot(x+y+z)$

- The expansion will have terms of form

$$x^a y^b z^c, \text{ where } a+b+c=5$$

- What should the coefficients be?

Trinomial theorem

$x^a y^b z^c$, where $a + b + c = 5$

- Once again, let's view $x^a y^b z^c$ as a string.
- #permutations of this string =

$$\frac{(a+b+c)!}{a! \cdot b! \cdot c!}$$

Trinomial theorem

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- Once again, let's view $x^a y^b z^c$ as a string.
- #permutations of this string =

$$\frac{(a+b+c)!}{a! \cdot b! \cdot c!} = \frac{5!}{a! \cdot b! \cdot c!}$$

Trinomial theorem

$$(x+y+z)^n = \sum_{\substack{a+b+c=n \\ 0\le a,b,c\le n}} \frac{n!}{a!\,b!\,c!} x^a y^b z^c$$

k-nomial theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 0\le a_1,a_2,\ldots,a_k\le n}} \frac{n!}{a_1!\,a_2!\ldots a_k!} x_1^{a_1} x_2^{a_2} \ldots x_k{}^{a_k}$$

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$$\Leftrightarrow \left(\sum_{i=1}^{k} x_i\right)^n = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 0\le a_1,a_2,\ldots,a_k\le n}} \frac{n!}{\prod_{i=1}^{k} a_i!} \prod_{i=1}^{k} x_i^{a_i}$$

Exercise

What is the coefficient of x^9 in the expansion of $(x+1)^{14} + x^3(x+2)^{15}$?

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$$\binom{14}{9} + \binom{15}{6}2^9.$$

Exercise

Prove the binomial identity $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

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Let's do a "pizza proof" again. We need to find a question about pizza toppings which has 2^n as the answer. How about this: If a pizza joint offers n toppings, how many pizzas can you build using any number of toppings from no toppings to all toppings, using each topping at most once?

On one hand, the answer is 2^n. For each topping you can say "yes" or "no," so you have two choices for each topping.

On the other hand, divide the possible pizzas into disjoint groups: the pizzas with no toppings, the pizzas with one topping, the pizzas with two toppings, etc. If we want no toppings, there is only one pizza like that (the empty pizza, if you will) but it would be better to think of that number as $\binom{n}{0}$ since we choose 0 of the n toppings. How many pizzas have 1 topping? We need to choose 1 of the n toppings, so $\binom{n}{1}$. We have:

- Pizzas with 0 toppings: $\binom{n}{0}$
- Pizzas with 1 topping: $\binom{n}{1}$
- Pizzas with 2 toppings: $\binom{n}{2}$
- $\vdots$
- Pizzas with n toppings: $\binom{n}{n}$.

The total number of possible pizzas will be the sum of these, which is exactly the left-hand side of the identity we are trying to prove.

Exercise

Prove the binomial identity $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

- Expand the binomial $(x+y)^n$:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x \cdot y^{n-1} + \binom{n}{n}y^n.$$

- Let $x = 1$ and $y = 1$. We get:

$$(1+1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n-1}1 \cdot 1^{n-1} + \binom{n}{n}1^n.$$

- Of course this simplifies to:

$$(2)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Exercise

Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

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Proof. Question: You have a large container filled with ping-pong balls, all with a different number on them. You must select k of the balls, putting two of them in a jar and the others in a box. How many ways can you do this?

Answer 1: First select 2 of the n balls to put in the jar. Then select $k-2$ of the remaining $n-2$ balls to put in the box. The first task can be completed in $\binom{n}{2}$ different ways, the second task in $\binom{n-2}{k-2}$ ways. Thus there are $\binom{n}{2}\binom{n-2}{k-2}$ ways to select the balls.

Answer 2: First select k balls from the n in the container. Then pick 2 of the k balls you picked to put in the jar, placing the remaining $k-2$ in the box. The first task can be completed in $\binom{n}{k}$ ways, the second task in $\binom{k}{2}$ ways. Thus there are $\binom{n}{k}\binom{k}{2}$ ways to select the balls.

Since both answers count the same thing, they must be equal and the identity is established.