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Any fool can know. The point is to understand — Albert Einstein

206
Discrete Structures II

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Course Outline

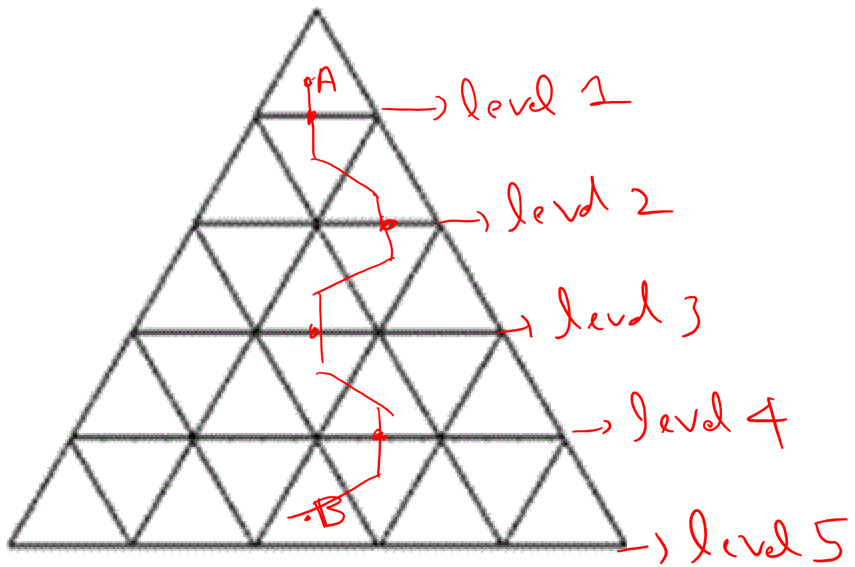
- Part I
  - Recap of basics – sets, function, proofs, induction
  - Basic counting techniques
  - Pigeonhole principle
  - Generating functions
- Part II
  - Sample spaces and events
  - Basics of probability
  - Independence, conditional probability
  - Random variables, expectation, variance
  - Moment generating functions
- Part III
  - Graph Theory
  - Machine learning and statistical inference

So Far

- ~~Sets / Functions~~
- ~~Proofs~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Revisiting an Exercise

Bijection Method

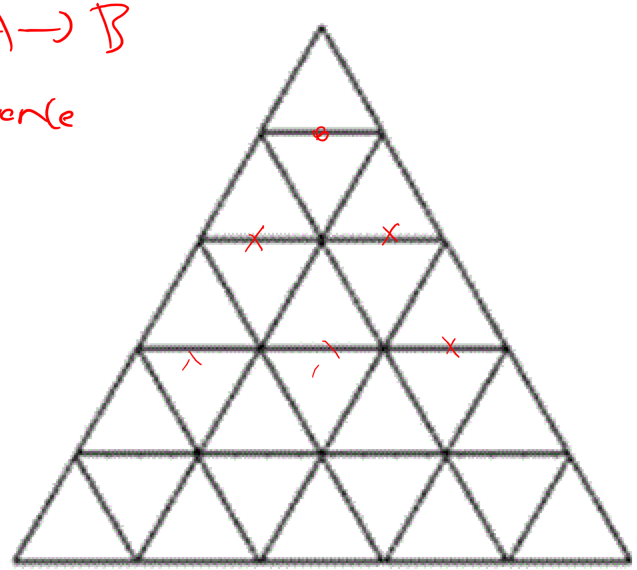


- Consider an equilateral triangle of side length 5, divided into unit length triangle. How many paths from point A to B?
  - Adjacent triangles in a valid path have to share a common edge.
  - A path can never go upwards or revisit a triangle.

How many paths from A to B?

Revisiting an Exercise

Each path from A → B has an exit sequence $(\underline{1}, \underline{2}, \underline{3}, \underline{3})$



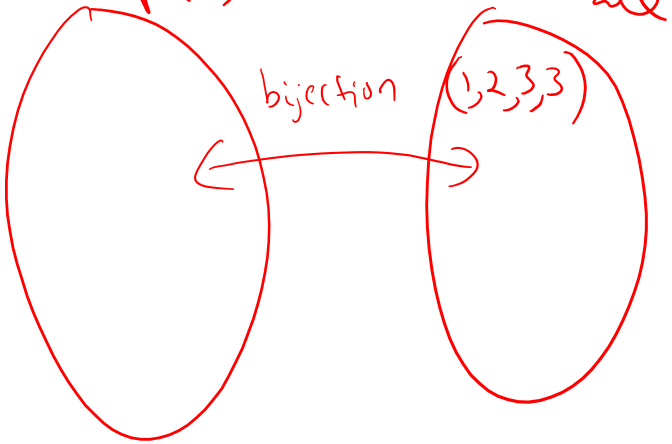
all valid exit sequences

$= 1 \cdot 2 \cdot 3 \cdot 4$

$= 24$

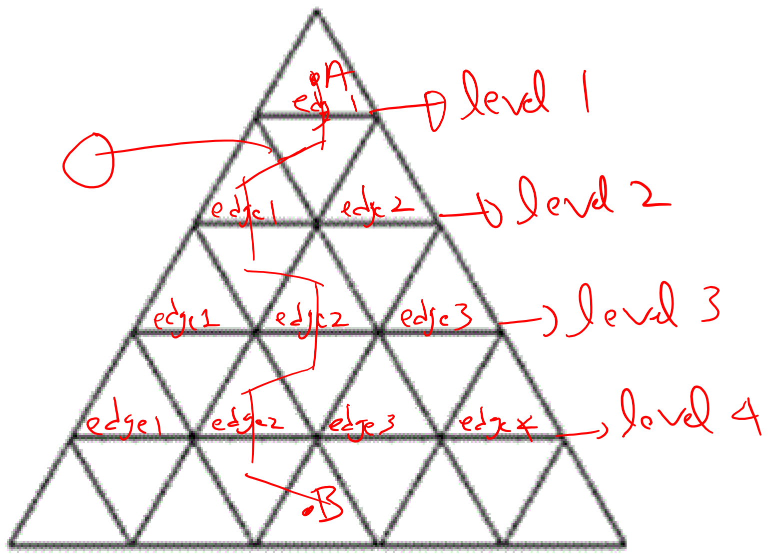
all valid paths

all exit sequences



Revisiting an Exercise

exit sequence = (1, 1, 2, 2)



Each path has an exit sequence and vice versa

- Consider an equilateral triangle of side length 5, divided into unit length triangle. How many paths from point A to B?
  - Adjacent triangles in a valid path have to share a common edge.
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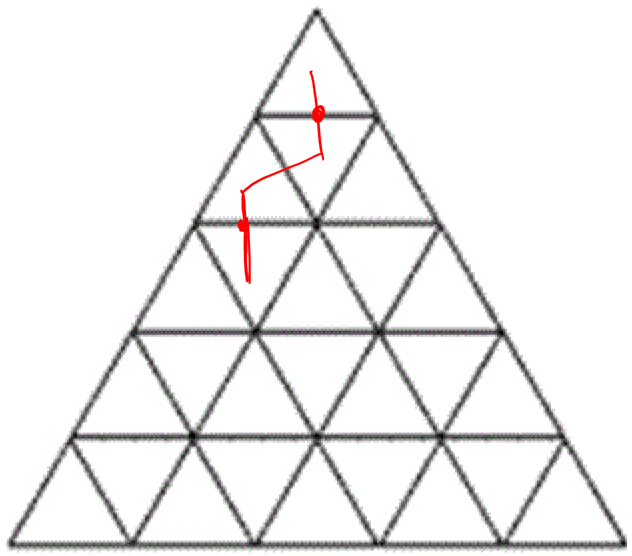
Revisiting an Exercise



\# valid paths = # exit sequences

$= 1 \cdot 2 \cdot 3 \cdot 4 = 24$ paths

If n levels answer $= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) = (n-1)!$

Product Rule

Recap

- If one event can occur in m ways, a second event in n ways and a third event in r, then the three events can occur in $m \times n \times r$ ways.

- Example

Erin has 5 tops, 6 skirts and 4 caps from which to choose an outfit. In how many ways can she select one top, one skirt and one cap?

Solution: Ways = $5 \times 6 \times 4$

Product Rule – with Repetition

If one event with n outcomes occurs r times with repetition allowed, then the number of ordered arrangements is n^r

- Example

What is the number of arrangements if a die is rolled

(a) 2 times? 6×6

(b) 3 times? $6 \times 6 \times 6$

(c) r times? $6 \times 6 \times 6 \times 6 \times \ldots\ldots = 6^r$

Product Rule – Adv'ed Repetition Problems

- How many different car number plates are possible with 3 letters followed by 3 digits?

Solution: $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 263 \times 103$

- How many of these number plates begin with ABC

Solution: $1 \times 1 \times 1 \times 10 \times 10 \times 10 = 10^3$

- In how many ways can 6 people be arranged in a row?

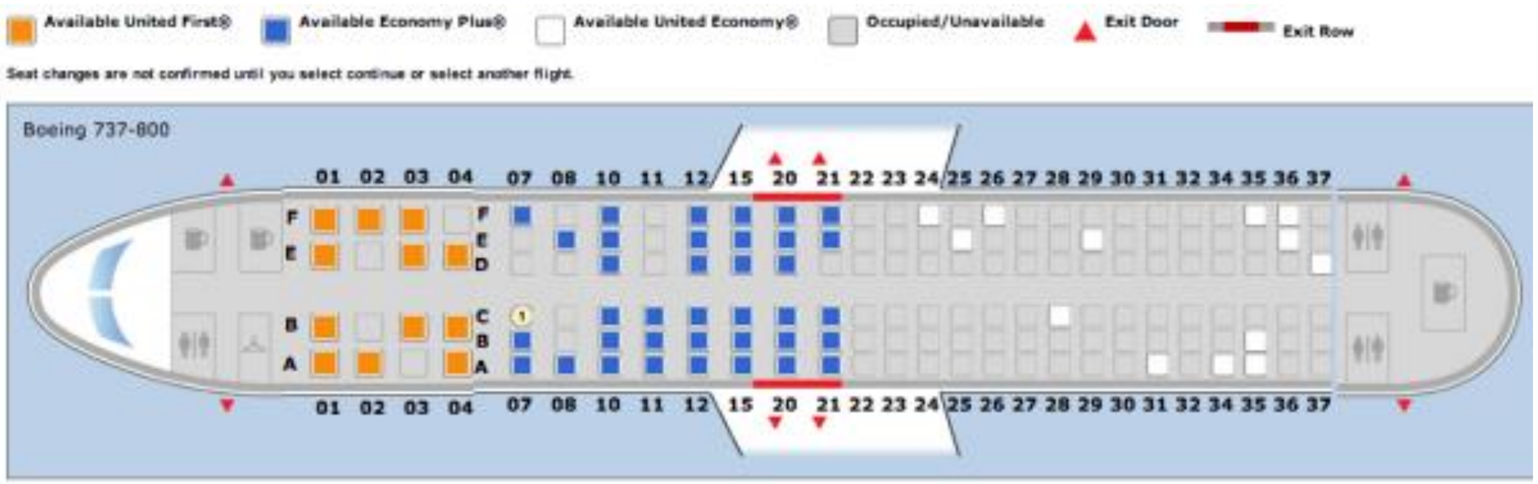
Solution : 6 x 5 x 4 x 3 x 2 x 1 = 6!

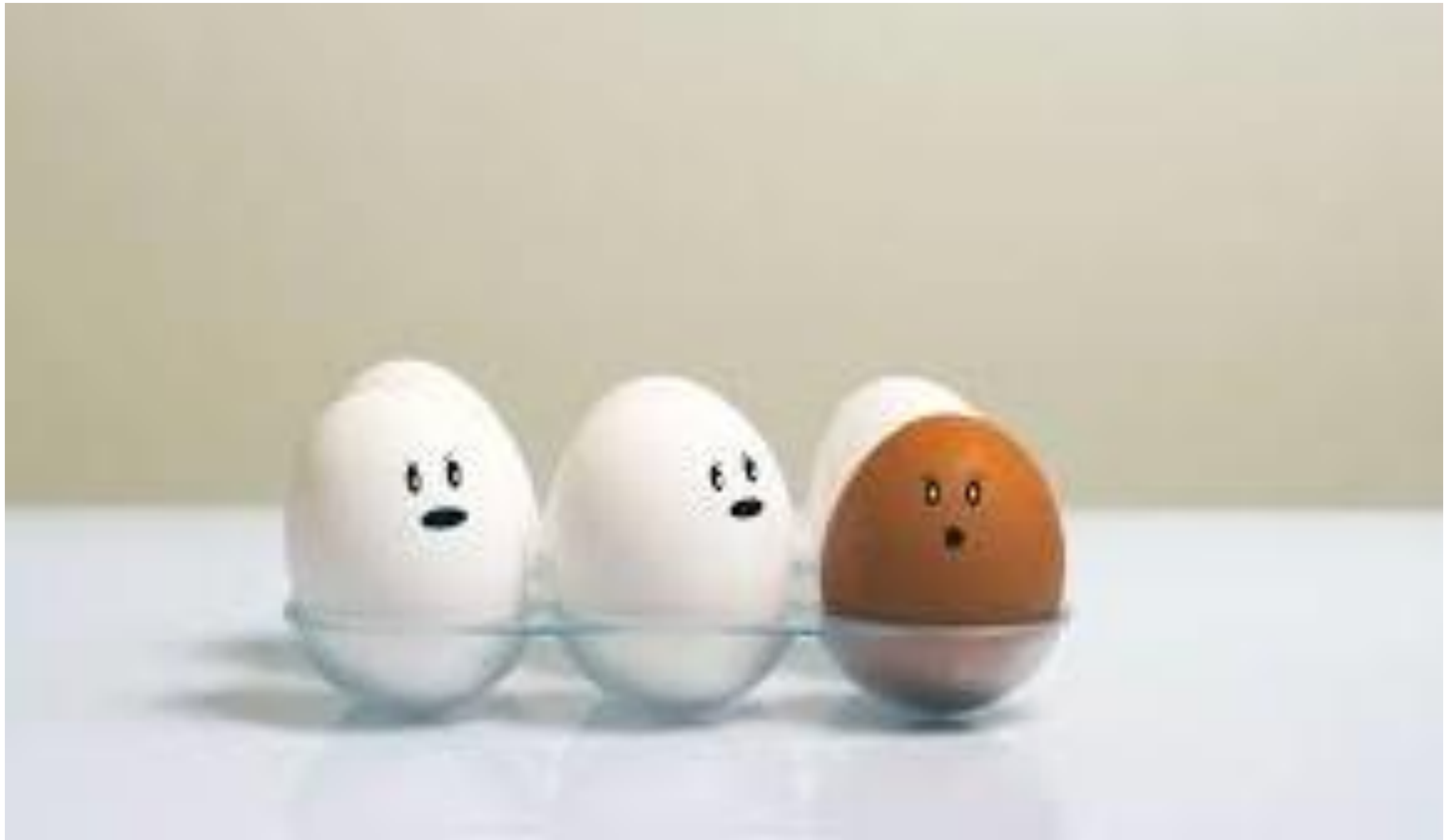
- How many arrangements are possible if only 3 of them are chosen?

Solution: 6 x 5 x 4 = 120

Permutations – Question Example

- How many ways to assign 100 passengers to 20 first class seats?

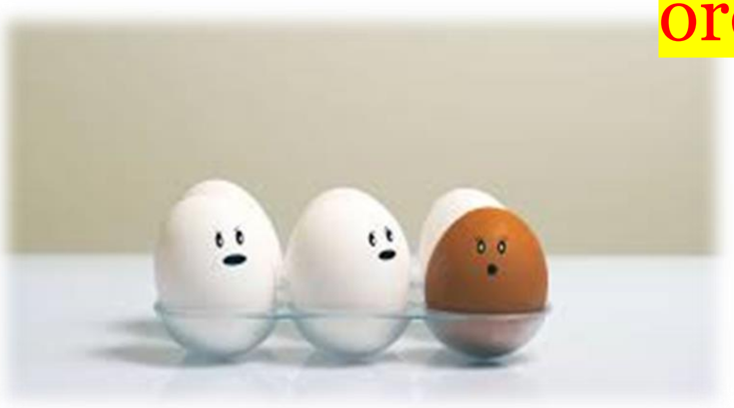




Permutations vs. Combinations

It's all about different elements folks…

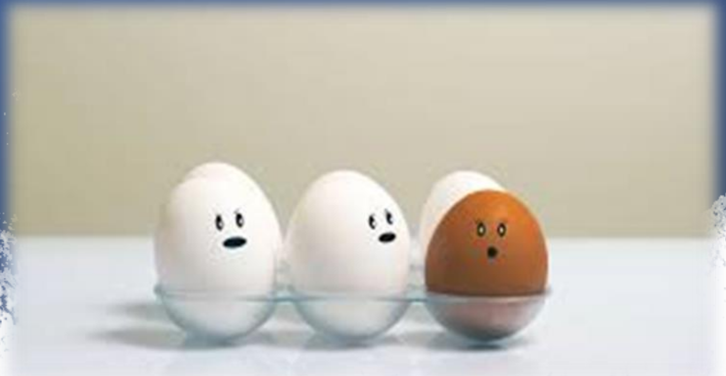
Permutations and Combinations



- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.

- Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected **does not** matter.

Difference between Permutations and Combinations



The **difference between combinations and permutations** is in

ordering

- With **permutations** we care about the order of the elements, whereas with **combinations** we don't care.

Examples:

- Permutation: Find a locker "combo" is 12345; Cellphone PIN is 5432
- Combination: Pick 5 students from a 180-student audience

Find 4-digit Permutations

of the numbers 2,3,4,5

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Permutations with Repetition

____ ____ ____ ____

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5
- but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

Permutations with Repetition

$$\underline{4} \cdot \underline{4} \cdot \underline{4} \cdot \underline{4} = 4^4 = 256$$

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5
- but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

Choosing a subset (a.k.a. Combinations)

_____ _____ _____ _____ _____

- *How many different 5-card hands can be made from a standard deck of cards?*
- In this problem the order is irrelevant since it doesn't matter what order we pick the cards.
- We'll begin with five lines to represent our 5-card hand.

Choosing a subset

$$\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48}$$

311,875,200 *permutations*

- *How many* ***<u>different</u>*** *5-card hands can be made from a standard deck of cards?*
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Choosing a subset

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- *How many different 5-card hands can be made from a standard deck of cards?*
- In this problem the order is irrelevant since it doesn't matter what order we select the cards.
- We'll begin with five lines to represent our 5-card hand.
- **That's permutations, not combinations**
- To fix this we need to **divide by the number of hands that are different permutations but the same combination**

Choosing a subset

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

- That's permutations, not combinations.
- To fix this we need to divide by the number of hands that are different permutations but the same combination.

- This is the same as saying *how many different ways can I arrange 5 cards?*

Why not subtraction?

We actually do repeated subtraction, a method that subtracts the equal number of items from a group, also known as division.

Choosing a subset - Combinations

$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

- So the number of five-card hands combinations is:

Rewriting with Factorials

$$\frac{52!}{47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}{\cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}$$

- With a little ingenuity we can rewrite the above calculation using factorials.
- We know 52! = 52•51•50•…•3•2•1, but we only need the products of the integers from 52 to 48. How can we isolate just those integers?
- We'd like to divide out all the integers except those from 48 to 52. To do this divide by 47! since it's the product of the integers from 47 to 1.

Rewriting with Factorials

$$\frac{52!}{5!47!}$$

- Make sure to divide by **5!** to get rid of the extra permutations:

There we go!

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

Combinations Formula

$$\binom{n}{k} = C_k^n = {}_nC_k$$

- Different Annotations

Permutations Formula

$$P_k^n = \frac{n!}{(n-k)!}$$

- The formula for permutations is similar to the combinations formula, except we needn’t divide out the permutations, so we can remove k! from the denominator:

Take a Break

2 min

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=1$, choose 1 out of n elements, n ways

easy... order is not important here

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=2$, Choose 2 out of n elements

Let $A =$ all ways to choose 2 out of n elements

Let $B =$ all ways to Permute 2 out of n elements

Order is important

We know $|B| = {}^nP_2 = n(n-1)$

How to go from $|B|$ to $|A|$?

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=2$, Choose 2 out of n elements

Associated choice sequence for Counting $|B|$

↓

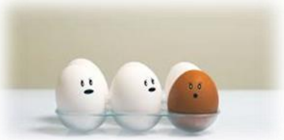
Pick 1st element

Pick 2nd element

Hence $n(n-1)$ ways

Order is important

Consider a pair $\{e_1,e_2\}$ in A. In B $\{e_1,e_2\}$ is counted twice. Either e_1 can be picked as 1st element and e_2 as second, or vice versa



Combinations – Permutations Better explained

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$r=2$, Choose 2 out of n elements

Associated choice sequence for Counting $|B|$

Pick 1st element

Pick 2nd element

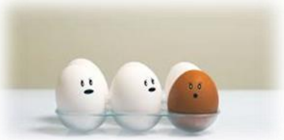
Hence $n(n-1)$ ways

A

$\{e_1, e_2\}$

Order is important

In other words, for each element of A there are 2 choices sequences in B that generate the same outcome.



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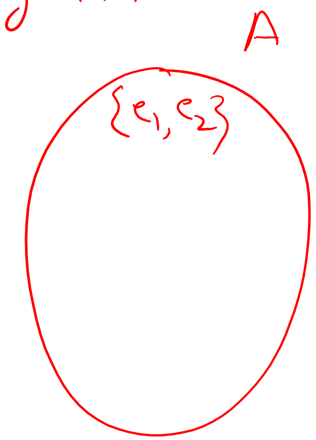
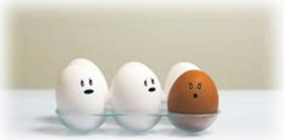


Order is important

Hence, $|A| = \frac{|B|}{2} = \frac{n(n-1)}{2} = \binom{n}{2} \leftarrow$ n choose 2



Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=3$, choose 3 out of n elements

A = all ways to choose 3 out of n elements

B = all ways to permute 3 out of n elements

$|B| = n(n-1)(n-2)$

choice sequence:

Pick 1st element

Pick 2nd element

Pick 3rd element

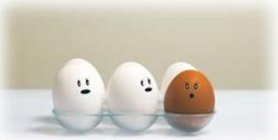
3! ways to reach

A

$\{e_1, e_2, e_3\}$

Order is important

Every element in A can be reached via 3! choice sequences in B



Combinations – Permutations Better explained

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choice sequence:

pick 1st element

pick 2nd element

pick 3rd element

3! ways to reach

A

$\{e_1, e_2, e_3\}$

Order is important

Hence, $|A| = \dfrac{|B|}{3!} = \dfrac{n(n-1)(n-2)}{3!} = \binom{n}{3}$

Permutations

- A permutation of n objects is an ordering of the objects.
- The number of permutations of n distinct elements
  $$= n \cdot (n-1) \cdot (n-2) \cdots (1) = n!$$

Permutations

- A permutation of n objects is an ordering of the objects.

- How many different permutations of a deck of 52 cards?

answer = $52 \cdot 51 \cdot 50 \cdot \cdots \cdot 1 = 52!$

Permutations

- How many ways to assign 100 passengers to 100 seats?

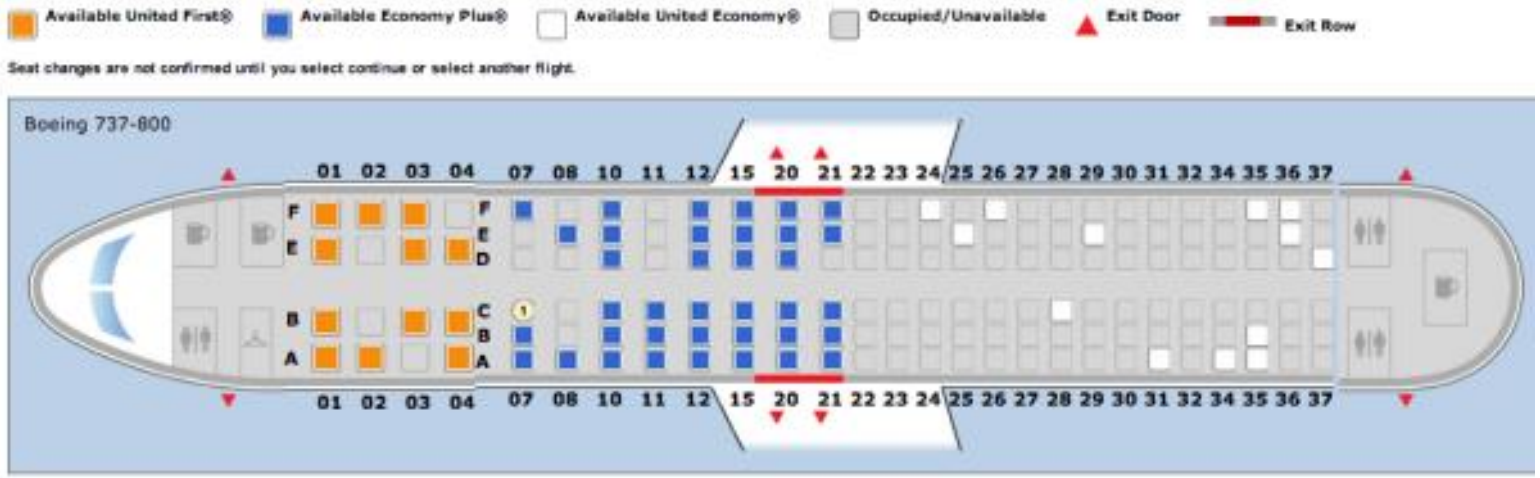
answer = $100!$

Permuting r out of n objects

- How many ways to assign 100 passengers to 20 first class seats?

$$\underset{S1}{\underline{100}}\ \underset{S2}{\underline{99}}\ \underline{98}\ \underline{\quad}\ \underline{\quad}\ \underset{S20}{\underline{81}}$$

answer $= (100 \cdot 99 \cdot 98 \cdot \ \cdots \cdot 81) = \dfrac{100!}{80!}$



Permutations Formula – One more time..

- Permuting r out of n distinct objects. $\quad {}^{n}P_{r} =$

$$\underset{P1}{\underline{n}} \quad \underset{P2}{\underline{n-1}} \quad \underline{n-2} \quad \underline{\quad} \qquad \underset{Pr}{\underline{n-r+1}}$$

$$\text{answer} = n \cdot (n-1) \cdot (n-2) \cdot \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

Repetitions

- Have n colors. Want to paint k tiles. How many ways?
  - Can reuse colors any number of times.

Tile1 — n choices

Tile2 — n choices

Tile k — n choices

answer $= n^k$

Permutations - Formulas

- Permuting r out of n distinct objects.
  - Without repetition $\rightarrow$ ${}^{n}P_{r} = \frac{n!}{(n-r)!}$
  - With repetition $\rightarrow$ n^{r}

Exercise

- How many sequences of 7 letters are there?

__ __ __ __ __ __ __

26   26

choices

answer $= 26^7$

Exercise

- If 10 horses race, how many orderings of the top 3 finishers are there?

$$\frac{10!}{7!} = {}^{10}P_3$$

Permutations

- Distinctly ordered sets are called permutations (arrangements). The number of permutations of n objects taken k at a time is given by:

$$P^n_k = \frac{n!}{(n-k)!}$$

N = number of objects
K = number of positions

Permutations Formula

$$P^{n}_{k} = \frac{n!}{(n-k)!}$$

- The formula for permutations is similar to the combinations formula, except we needn't divide out the permutations, so we remove k! from the denominator:

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left