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206
Discrete Structures II

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Lecture 19 | Probabilities| Monday November 20th 2023

How was the midterm?


THE COMFORT ZONE
Safe and in control
COMFORT ZONE
Low risk Low reward
Low self-confidence
Find excuses
FEAR ZONE
Affected by others' opinions
Face challenges
Problem solve
LEARNING ZONE
Acquire new skills
Extend comfort zone
Find purpose
Live dreams
GROWTH ZONE
Set new goals
Conquer objectives

Outline for this month

- Sample spaces and events
- Basics of probability
- Independence, conditional probability
- Random variables, expectation, variance

→ Basic building blocks

→ Intermediate

→ Advanced

Outline

- Sample space
- Axioms of probability
- Conditional probability
- Independence
- Bayes rule
- Random Variables
- Expectation

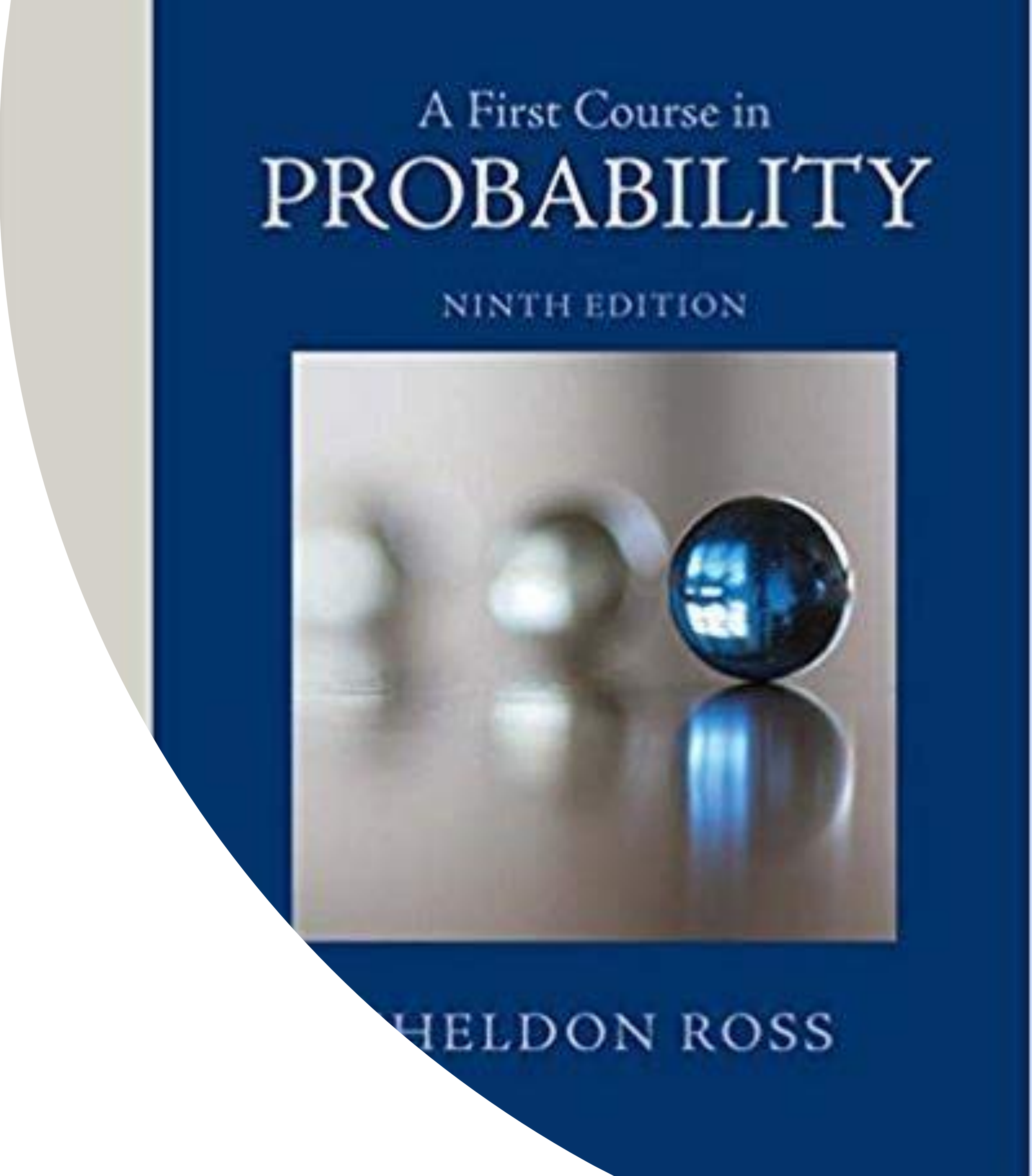
Quiz 5 (Lectures 19-21)
Wed 11/29 & Mon 12/4

Quiz 6 (Lectures 21-23)
Wed 12/6 & Mon 12/11

Textbook #1

A First Course in Probability

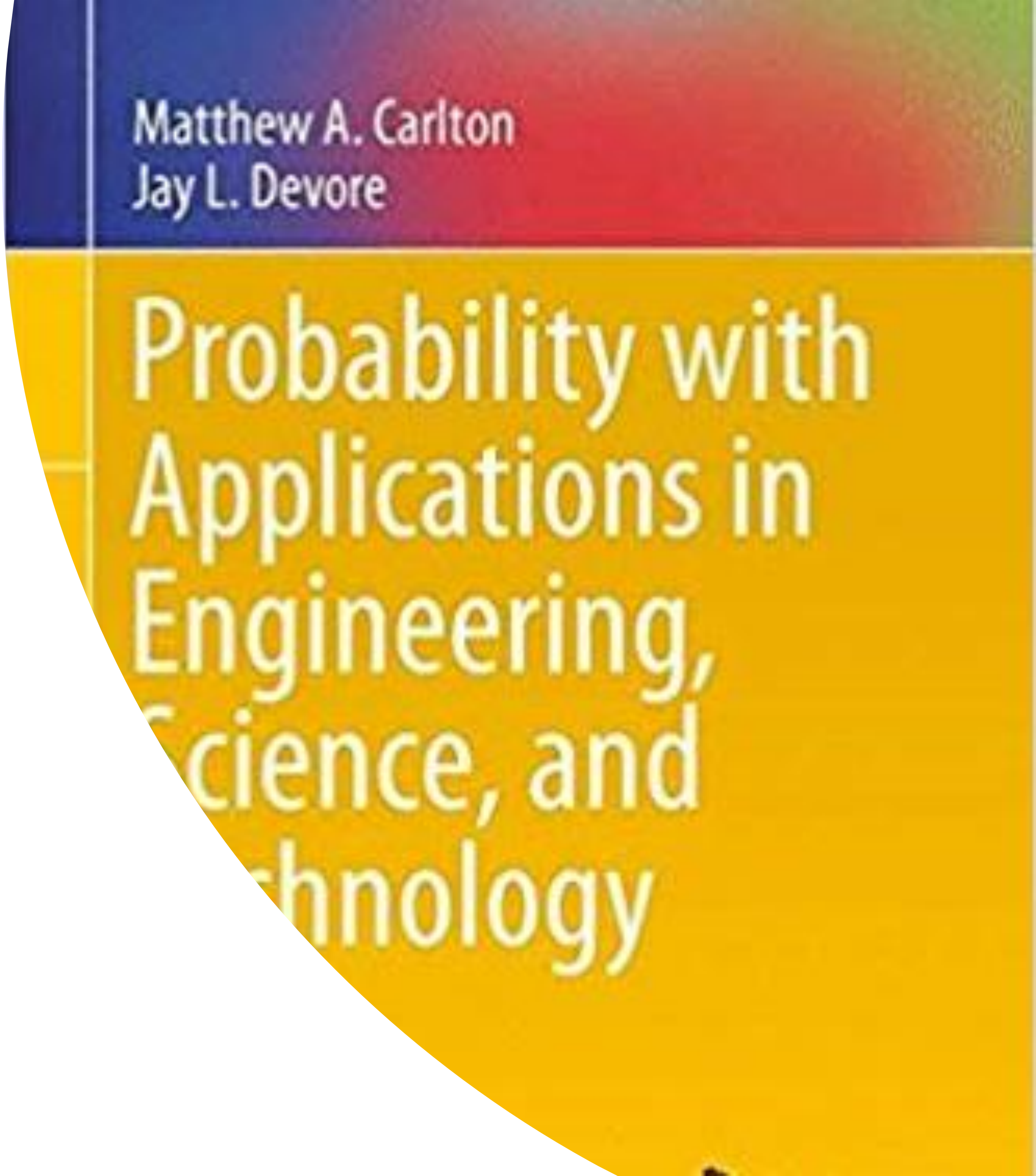
- S. Ross
- any edition



Textbook #2

Probability with Applications in Engineering, Science, and Technology

- M. A. Carlton and J. L. Devore
- Available for free through university library website.



Today – Probabilities

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Event
  - Outcome(s) that you are interested in understanding

Probabilities

- Experiment
  - Toss a fair coin 10 times

Consider an experiment as a random event whose outcome *is not predictable with certainty*.

Probabilities

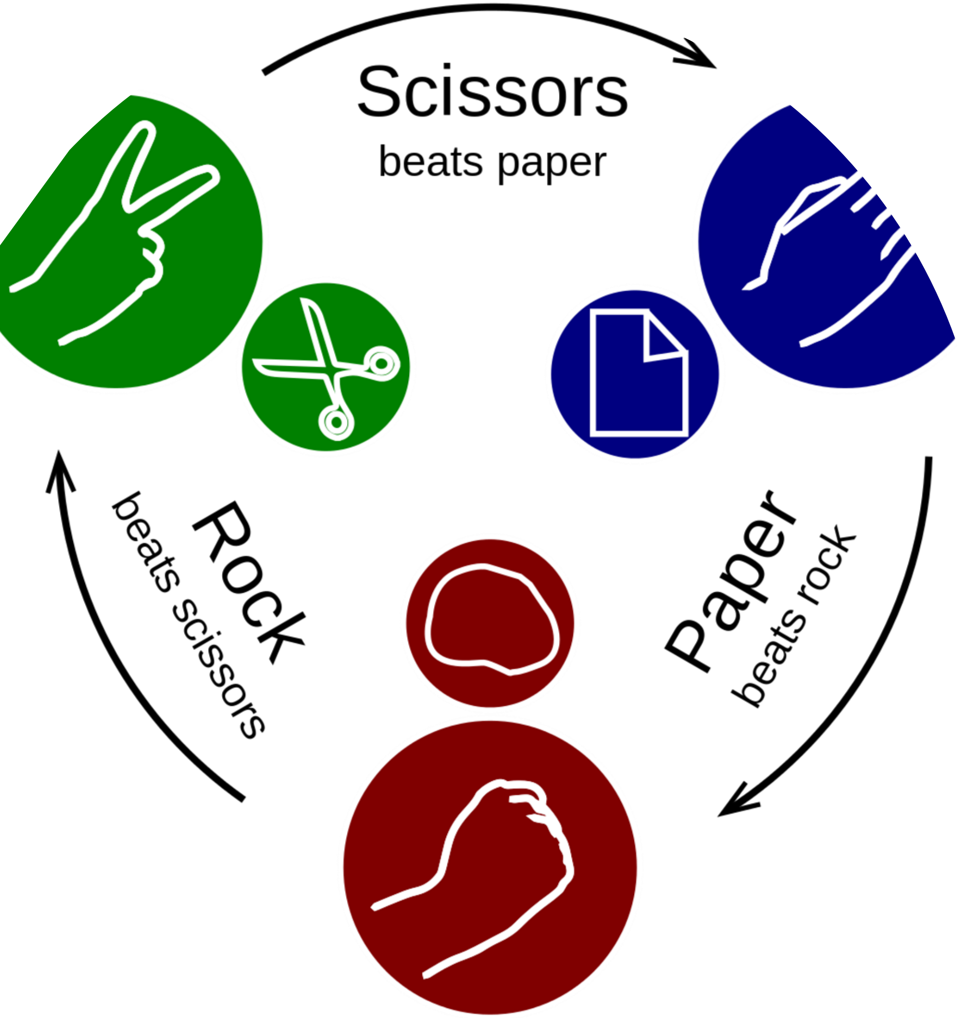
- Study of random/uncertain phenomena.
- Origins in gambling.
  - Pascal invented probability theory to come up with gambling strategies.
- Two dice are rolled 4 times. If (6,6) shows up I win. Else I lose. Should I play?

Probability's origin – The problem of points

…or the problem of the division of the stakes

- Two players are playing a best of 5 game
  - After 3 games, player 1 is leading 2 to 1
  - A fight breaks out and game cannot be finished
  - How should the prize money be divided?

led Blaise Pascal to the first explicit reasoning about what today is known as an *expected value*



Probability: Today's Applications

- Weather prediction
- Stock market prediction
- Inventory management
- Studying behavior of a virus
- Understanding rational behavior in economics
- Understanding the chance of failure of algorithms
- Cryptography
- Machine Learning

Probabilities – Real-life Example

- US population is ~350 million.
- Want to figure out if majority prefer Biden or Trump.

Probabilities – Real-life Example

independent of
350 million

Theorem:
Poll a random sample of 2000 people. Then, with probability > .99,

% preferring Biden over Trump = % in sample $\pm$ 2%

Probabilities

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment

Probability - Sample Space

Consider an experiment whose outcome *is not predictable with certainty*.

- However, although the outcome of the experiment will not be known in advance, let us suppose that *the set of all possible outcomes is known*.
- This set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S.

Probability – Sample Space Examples

- If the outcome of an experiment consists of the determination of the gender of a newborn child, then

$$S = \{g, b\}$$

- If the experiment consists of flipping two coins, then the sample space consists of the following four points,

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

- If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, and 7, then

$$S = \{\text{all } 7! \text{ Permutations of } (1,2,3,4,5,6,7)\}$$

Probability – Sample Space Examples

- Toss a coin 10 times

$$\Omega = \left\{ \begin{array}{l} (H, H, H, \ldots, H) \\ (H, T, \ldots \quad) \\ \\ (T, T, \ldots \ldots T) \end{array} \right\}, \quad |\Omega| = 2^{10}$$

Probability – Sample Space Examples

- Roll two dice

$$\Omega = \left\{ \begin{matrix} (1,1), (1,2), \cdots (1,6) \\ (2,1), (2,2), \cdots (2,6) \\ \vdots \\ (6,1), (6,2) \cdots (6,6) \end{matrix} \right\}, \quad |\Omega| = 6 \cdot 6$$

Probability – Sample Space Examples

Experiment : people vote

$$\Omega = \left\{ \begin{array}{l} (H, H, \text{—} H, T, \text{—} \text{—} T) \\ (T, T, \text{—} \text{—} \text{—} T) \\ (H, H, H, T, \text{—} T) \\ \vdots \end{array} \right\}, \quad |\Omega| = 2^{350 \times 10^6}$$

Probability – Sample Space Examples

- Toss a coin until you see a H

$$\Omega = \{H, TH, TTH, TTTH, \ldots\}$$

$$|\Omega| = \infty$$

Probability

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Event
  - Outcome(s) that you are interested in understanding (or counting…)

Probability – Events Examples

Probability is the likelihood that an **event** will occur.

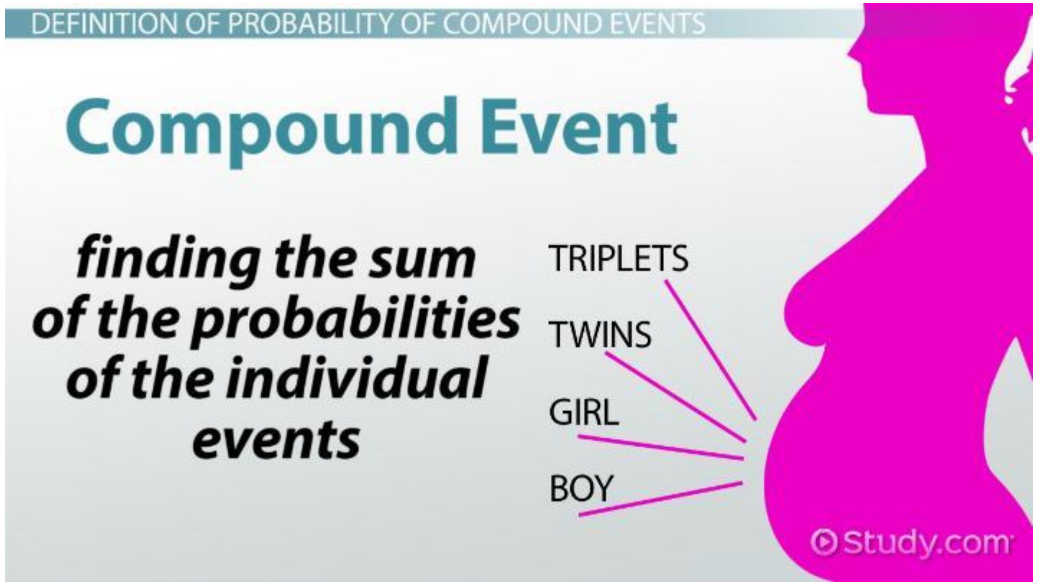
- Toss a coin 10 times

Any subset of Ω is an Event.

Events – Simple Event

- A **simple event** is an **event** where all possible outcomes are equally likely to occur.
- So the **probability** of **simple events** will have all possible outcomes equally likely to happen or occur.
- E.g., when you toss a coin, there are two possible outcomes – heads or tails, and the **probability** of heads or tails is equal.

Events - Compound Event



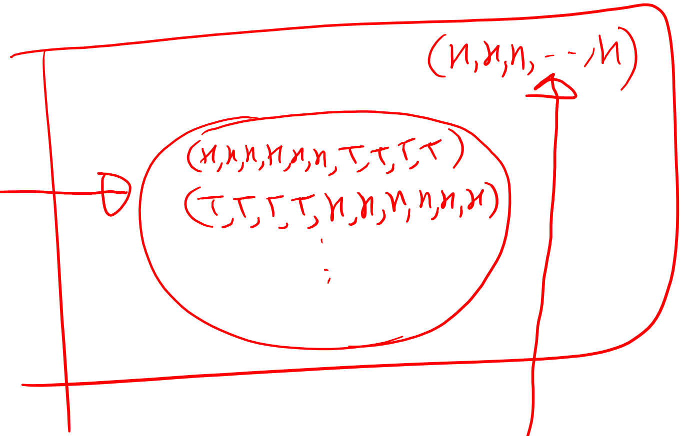
- A **compound event** is one in which there is more than one possible outcome.
- Determining the **probability** of a **compound event** involves finding the sum of the **probabilities** of the individual **events** and, if necessary, removing any overlapping **probabilities**.

Probability – Events Examples

- Toss a coin 10 times

Compound Event

Event 1: H appears 6 times



Event 2: only see Heads

Simple Event

Probability – Events Examples

- Roll two dice

$$\Omega = \begin{Bmatrix} (1,1),(1,2), \ldots (1,6) \\ \vdots \\ (6,1),(6,2), \ldots (6,6) \end{Bmatrix}$$

Simple Event : first die = 6, Second die = 5 → (6,5)

Compound Event : first die equals second die → (1,1), (2,2), (3,3), ... (6,6)

Probability – Events Examples

- Toss a coin until you see a H.

Simple Event: Get an H on first try.

Compound Event: Don't get an H on first try.

Events - Operations

- A' → Complement of A
- $A \cap B$ → intersection
- $A \cup B$ → union

Disjoint Events

- A and B are disjoint events if $A \cap B = \phi$

Roll 2 dice

— A: die 1 = 1, die 2 = 1 $\quad A \cap B = \phi$

— B: die 1 = 2, die 2 = 2

— A: Sum of dice = 2 → (1,1) $\quad A \cap B = \phi$

— B: Sum of dice = 3 → (1,2), (2,1)

Probability

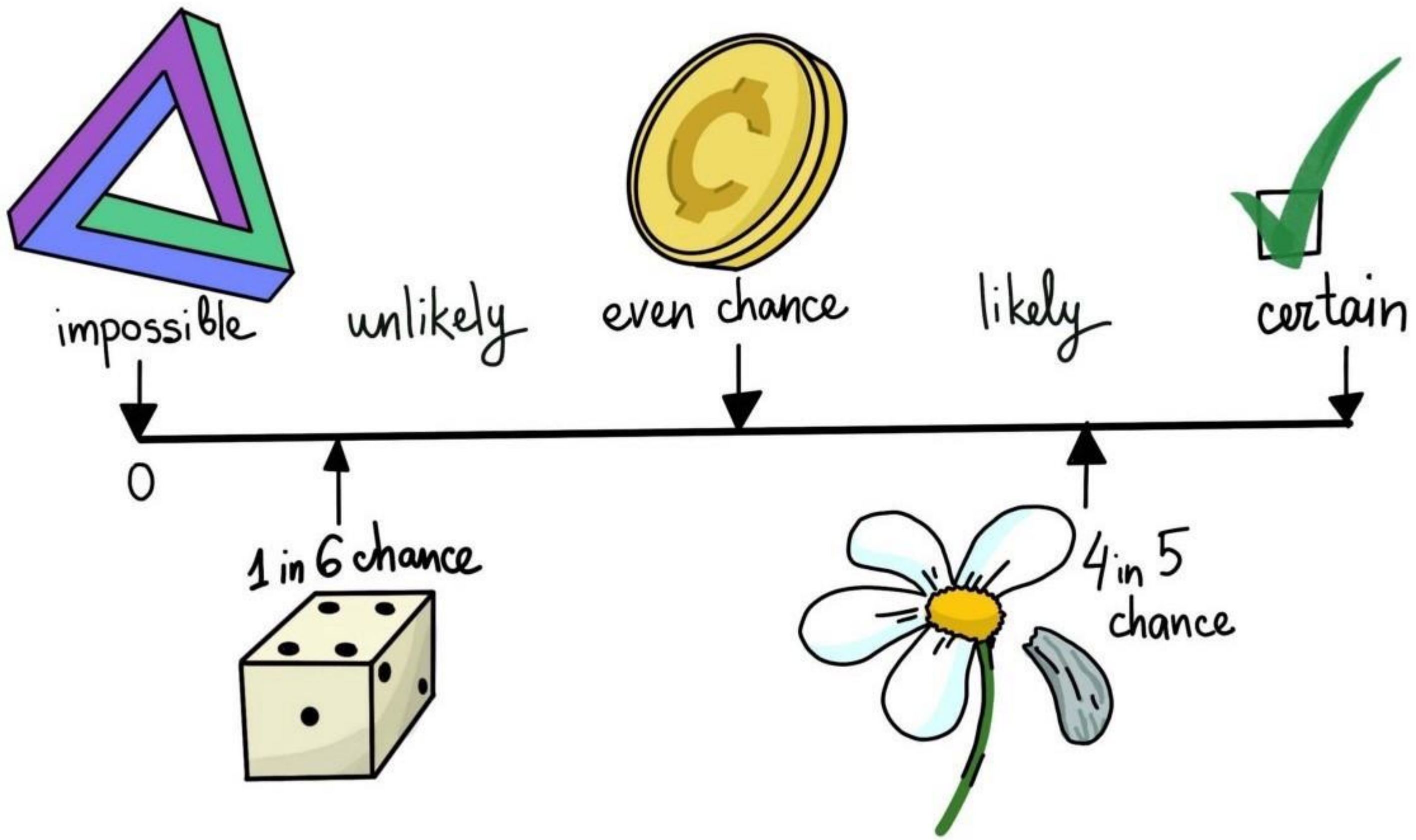
- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Simple Event
  - Any element of the sample space
- Compound Event
  - Subsets of the sample space
- Probability Distribution - Axioms

Probability

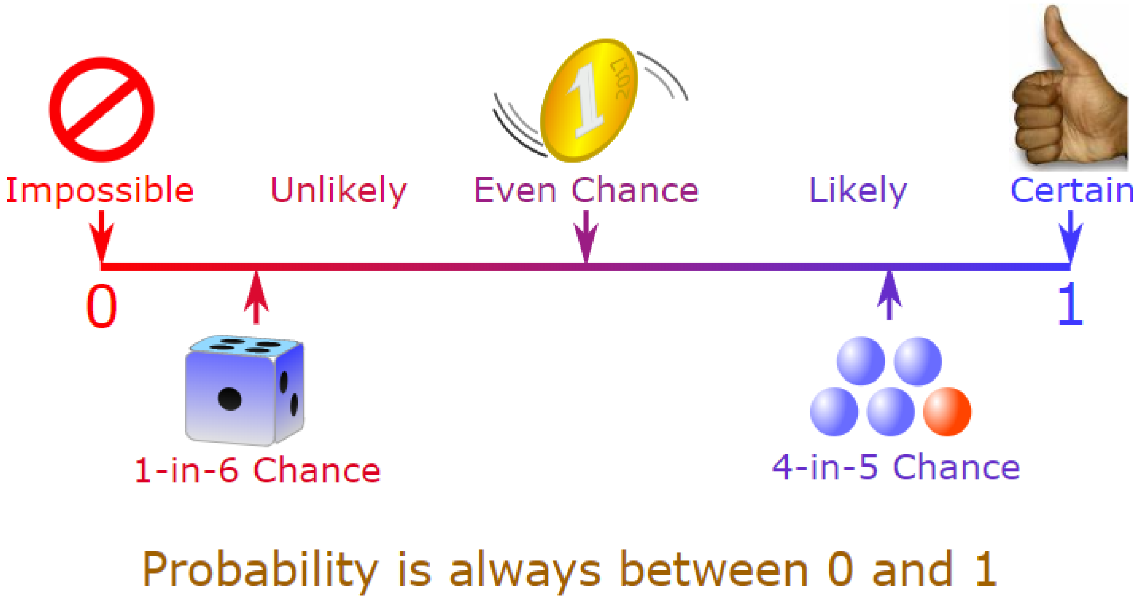
- Fix experiment and sample space Ω.

A probability distribution P assigns a number $P(A)$ to each event A.

- P needs to satisfy certain basic axioms.



Axioms of Probability



- $P(A) \geq 0$
- $P(\Omega) = 1$
- For a collection of disjoint events $A_1, A_2, \ldots$
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

Equally Likely Outcomes

Consider experiment and a finite sample space Ω

- For every simple event $e \in \Omega$, assign $P(e) = \frac{1}{|\Omega|}$
- For every compound event A, assign $P(A) = \frac{|A|}{|\Omega|}$

- Then, P is a valid probability distribution.

(Proof on next slide)

Equally Likely Outcomes

- Proof:
- $P(A) \geq 0$ since $|A| \geq 0$
- $P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$
- Let $A_1, A_2, \ldots$ be disjoint events. Then
- $P(A_1 \cup A_2 \ldots) = \frac{|A_1 \cup A_2 \cup \cdots|}{|\Omega|}$

$$= \frac{|A_1|}{\Omega} + \frac{|A_2|}{\Omega} + \cdots = P(A_1) + P(A_2) \ldots$$

- We have proved that all 3 axioms are true.

Probability – Calculate it

- Toss a coin.

$\Omega = \{H, T\}$

For equally likely outcome

$P(H) = \frac{1}{2}$

$P(T) = \frac{1}{2}$

$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{2}$

Probability

- Roll two dice. For any compound event A, of size |A| …

$|\Omega| = 36$

For equally likely outcomes

$P(A) = \frac{|A|}{36}$

More Implications – Prove it!

- $P(A') = 1 - P(A)$

— A and A' are disjoint

$\Rightarrow P(A) + P(A') = P(A \cup A') = P(\Omega) = 1$

More Implications – Prove it!

- $P(\mathrm{A}) \leq 1$

$$P(A) + P(A') = 1$$

$$\Rightarrow \quad P(A) \leq 1$$

$$P(A') \leq 1$$

More Implications

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

– Inclusion/Exclusion for Probabilities

– Extends to more than 2 sets

Union Bound

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) \leq P(A) + P(B)$ $\longrightarrow$ Union bound
$\longrightarrow$ Boole's inequality

Interpretation of Probability

- For a collection of disjoint events $A_1, A_2, \ldots$ → Sum rule for Probability
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

Interpretation of P(A)

If $P(A) = .6$

- If we repeat experiment N times (N is very large)
- Then the outcome will lie in A, .6N of the times

Uniform Distribution

- If we roll a ~~fair~~ unfair die, what is the probability that the result is an even number?

Given $P(1) = \frac{1}{6}$, $P(2) = \frac{1}{6}$, $P(3) = \frac{1}{6}$

$P(4) = P(5) = \frac{1}{4}$, $P(6) = 0$

$A = \{2, 4, 6\}$, want $P(A)$

By sum Rule $P(A) = P(2) + P(4) + P(6)$

$= \frac{1}{6} + \frac{1}{4} + 0 = \frac{5}{12}$