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Good luck on your Quiz#1

206
Discrete Structures II

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Lecture 6 | Counting | Monday October 2nd 2023

What is wrong?

(1)	Let $a = b$	given.
(2)	$a^2 = ab$	multiply both sides by a.
(3)	$a^2 + a^2 = a^2 + ab$	adding a^2 to both sides.
(4)	$2a^2 = a^2 + ab$	grouping the terms on the left.
(5)	$2a^2 - 2ab = a^2 + ab - 2ab$	subtracting 2ab from both sides.
(6)	$2a^2 - 2ab = a^2 - ab$	grouping terms on the right.
(7)	$2(a^2 - ab) = 1(a^2 - ab)$	pulling out the factor of 2 on the left.
(8)	$2 = 1$	dividing both sides by $a^2 - ab$

What we will cover today

Combinatorics

- Recap
  - Counting (Partition, Difference, Product)
- Today
  - Counting
    - Product Rule
    - Bijection Rule
- Next
  - Permutations/Combinations
  - Pigeonhole Principle

Course Outline

- Part I
  - Recap of basics – sets, function, proofs, induction
  - Basic counting techniques
  - Pigeonhole principle
  - Generating functions
- Part II
  - Sample spaces and events
  - Basics of probability
  - Independence, conditional probability
  - Random variables, expectation, variance
  - Moment generating functions
- Part III
  - Graph Theory
  - Machine learning and statistical inference

Counting

Counting

- Basic Question: What is the size of a given set?

- Easy when the set is explicitly defined.
  - $X = \{1,2,3,4\}$, what is $|X|$?

- Tricky when set is implicit or a defined via set operations.
  - How many ways to get flush in the game of poker?
  - How many ways to assign time slots to courses at Rutgers?
  - How many operations before my algorithm terminates?

So far...

Counting

- In the next few lectures
  - Fundamental tools and techniques for counting
  - Sum Rule
  - Product Rule
  - Difference Method
  - Bijection Method
  - Permutations/Combinations
  - Inclusion Exclusion
  - Binomial/Multinomial coefficients

Fundamental Blocks

→ Intermediate

→ Advanced

Sum Rule

- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- How many students are there in total in both sections?

Sum Rule:
If A and B are **disjoint** sets, then $|A \cup B| = |A| + |B|$

A = all students in section 5
B = all students in section 6

$|A \cup B| = |A| + |B| = 60 + 71 = 131$

So far...

Example: Sum Rule

- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- There are 80 students in section 1 of 206.
- There are 80 students in section 2 of 206.
- How many students are there in total?

$60 + 71 + 80 + 80 = 291$

So far...

Example: Sum Rule

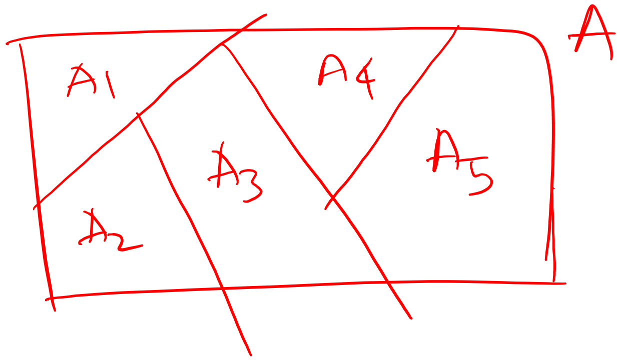
- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- There are 80 students in section 1 of 206.
- There are 80 students in section 2 of 206.
- How many students are there in total?

Sum Rule:
If $A_1, A_2, \ldots A_n$ are **disjoint** sets, then
$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$

Partition Method – How to

- To find the size of a set A,
  - Partition it into a union of disjoint sets $A_1, A_2, \ldots, A_n$
  - Use sum rule

- Example: How many students are there in total in 206?



Partition Method

- To find the size of a set A,
  - Partition it into a union of disjoint sets $A_1, A_2, \dots, A_n$
  - Use sum rule

- If I roll a white and black die, how many possible outcomes do I see?

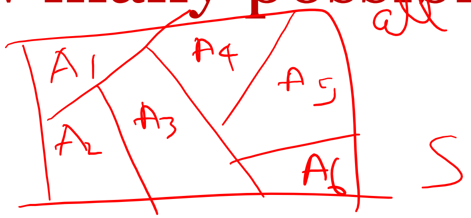
$$S = \left\{ \begin{array}{llll} (1,1), & (1,2), & \dots & (1,6) \\ (2,1), & (2,2), & \dots & (2,6) \\ \vdots & & & \\ (6,1), & (6,2), & \dots & (6,6) \end{array} \right\}$$

$$|S| = 36$$

Partition Method

- If I roll a white and black die, how many possible outcomes do I see?

all outcomes



$A_1 =$ all outcomes with black die $= 1$

$A_2 =$ all outcomes with black die $= 2$

$\vdots$

$A_6 =$ all outcomes with black die $= 6$

$|S| = |A_1| + |A_2| + \cdots + |A_6|$

$= 6 \cdot 6 = 36$

Partition Method

- Possible outcomes where white and black die have different values?

S = all possible outcomes

A_1 = all outcomes with black die = 1

A_2 = black die = 2

⋮

A_6 = black die = 6

$|A_1| = 5$, $|A_2| = 5$,

$|S| = 5+5+5+ \cdots 5 = 30$

So far...

Partition Method

- Possible outcomes where white die has a larger value then the black die?

$S =$ all possible outcomes

$A_1 =$ all outcomes with black die $= 1$

$\vdots$

$A_6 =$ black die $= 6$

$|A_1| = 5, \ |A_2| = 4, \ |A_3| = 3$

$|A_4| = 2, \ |A_5| = 1, \ |A_6| = 6$

$$|S| = 5+4+3+2+1 = \frac{5(5+1)}{2} = 15$$

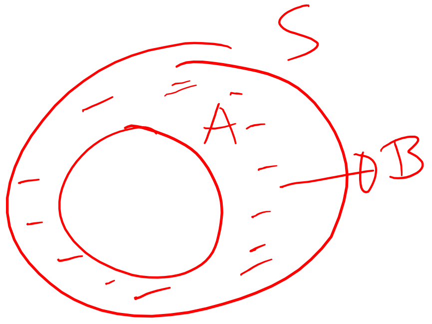
New Stuff

Difference Method

- To find the size of a set A,
  - Find a larger set S such that $S = A \cup B$ and
  - A and B are disjoint.
  - $|A| = |S| - |B|$

Want: $|A|$

Find S that contains A



$B = S \setminus A$

$|S| = |A| + |B|$

$\Rightarrow |A| = |S| - |B|$

New Stuff

Difference Method

- To find the size of a set A,
  - Find a larger set S such that $S = A \cup B$ and
  - A and B are disjoint.
  - $|A| = |S| - |B|$

- Possible outcomes where white and black die have different values?

  - Find S with all possible outcomes $|S|$=36
  - Subtract B with the same values |B|=6
  - $|A| = |S| - |B| = 36 - 6 = 30$

Difference Method vs. Partition Method

- Possible outcomes where white and black die have different values?

S = all possible outcomes

A_1 = all outcomes with black die = 1

A_2 = black die = 2

⋮

A_6 = black die = 6

$|A_1| = 5$, $|A_2| = 5$,

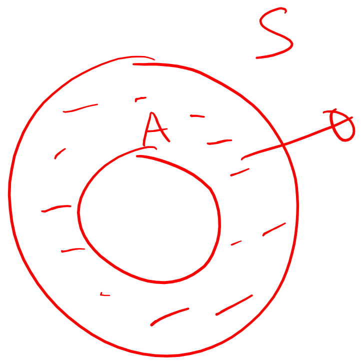
$|S| = 5 + 5 + 5 + \cdots 5 = 30$

Simplicity of Difference Method

- Possible outcomes where white and black die have different values?

A = all outcomes where white die $\neq$ black die

S = all outcomes, $|S| = 36$



$B = S \setminus A$

= all outcomes where white die = black die

$|B| = 6$

$\Rightarrow |A| = 36 - 6 = 30$

Lehman, 475-480

Product Rule

Product Rule:
$|A \times B| = |A| \cdot |B|$

- True even if A and B are not disjoint
- Useful when counting elements of a set involves dealing with tuples, sequences or a series of choices.

Insight: The Product Rule gives us how many different elements are possible

Insight #2: The multiplication finds all the possible "matches" across sets

Product Method

- If I roll a white and black die, how many possible outcomes do I see?

A = all outcomes of black die
B = all outcomes of white die

all outcomes = $|A \times B| = |A| \cdot |B| = 36$

Question: Can you make the above question not solvable with the product rule?

We are getting ready to leave behind us our ability to count elements and start developing skills that help us count sets without explicitly counting their elements

New stuff...

Product Rule

Product Rule:

$|A_1 \times A_2 \times \cdots A_n| = |A_1| \cdot |A_2| \dots |A_n|$

Product Rule

- A restaurant has a menu with 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts.
  - How many ways to choose a complete meal?

$A =$ all possible complete meals

$= \{ (\text{App}, \text{Entree}, \text{Salad}, \text{Dessert}) \; \vdots \}$

$|A| = 5 \times 6 \times 3 \times 7$

5 choices for Appetizers
6 " " Entree
3 " " Salad
7 " " Dessert

Product Rule

- A restaurant has a menu with 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts.
  - How many ways to choose a meal if I'm allowed to skip some (or all) the courses?

A = { (App, Entree, Salad, Dessert)
(App)
(Entree)
⋮ }

Step 1: Make all elements the same length by including a null option. For ex: (Entree) becomes (null, Entree, null, null)

Step 2: 6 choices for Appetizer, 7 for Entree, 4 for Salad, 8 Dessert

Answer = $6 \times 7 \times 4 \times 8$

Exercise: Counting Passwords…

- You are signing up for an account on FlixBiz.com. The password has the following requirements
  - The password must be 6 to 8 characters long.
  - Each password is an uppercase letter or digit.
  - Each password must contain **at least** one digit.

Partition Method

Q: How many possible passwords?

$A_6 \rightarrow$ all passwords with length 6

$A_7 \rightarrow$,, 7

$A_8 \rightarrow$,, 8

all passwords $= |A_6| + |A_7| + |A_8|$

Hint (or …When to think of Partition Method)

- When you are asked to count something that exists in easy-to-count ways (e.g., between 2 and 4), consider dividing the problem to the enumerable cases and then use the Partition Method

  - Note that if the different cases are too many (e.g., 100), then most probably the intention of the exercise is not to stress your patience mechanisms…

Exercise: Counting Passwords…

- You are signing up for an account on FlixBiz.com. The password has the following requirements
  - The password must be 6 to 8 characters long.
  - Each password is an uppercase letter or digit.
  - Each password must contain **at least** one digit.

Partition Method

Q: How many possible passwords?

$A_6 \rightarrow$ all passwords with length 6

$A_7 \rightarrow$ " 7

$A_8 \rightarrow$ " 8

all passwords $= |A_6| + |A_7| + |A_8|$

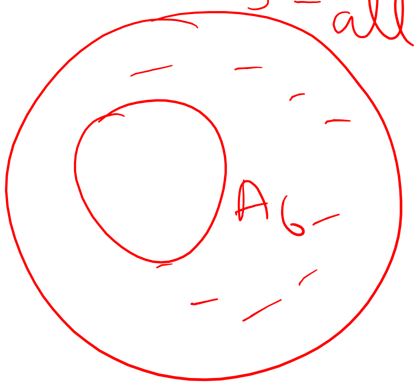
Exercise: Counting Passwords…

A_6 = all valid passwords of length 6

S = all passwords of length 6

$B = S \setminus A_6$

B = all passwords of length 6 with no digits



Difference Method

=>

Find Contrapositive

(see Hint on next slide)

$|A_6| = |S| - |B|$

$|S| = 36^6$

$|B| = 26^6$

$\Rightarrow |A_6| = 36^6 - 26^6$

Partition Method

$|A_7| = 36^7 - 26^7$

$|A_8| = 36^8 - 26^8$

Hint: When to use Difference Method

When you are asked to count something that exists in

"at least" one case, consider counting the opposite

(that is "no" case)

Which means: You need to be able to find the

"contrapositive argument".