



206

# Discrete Structures II

Konstantinos P. Michmizos

Computational Brain Lab

Computer Science | Rutgers University | NJ, USA

# This is the... **Last 4 Lectures**

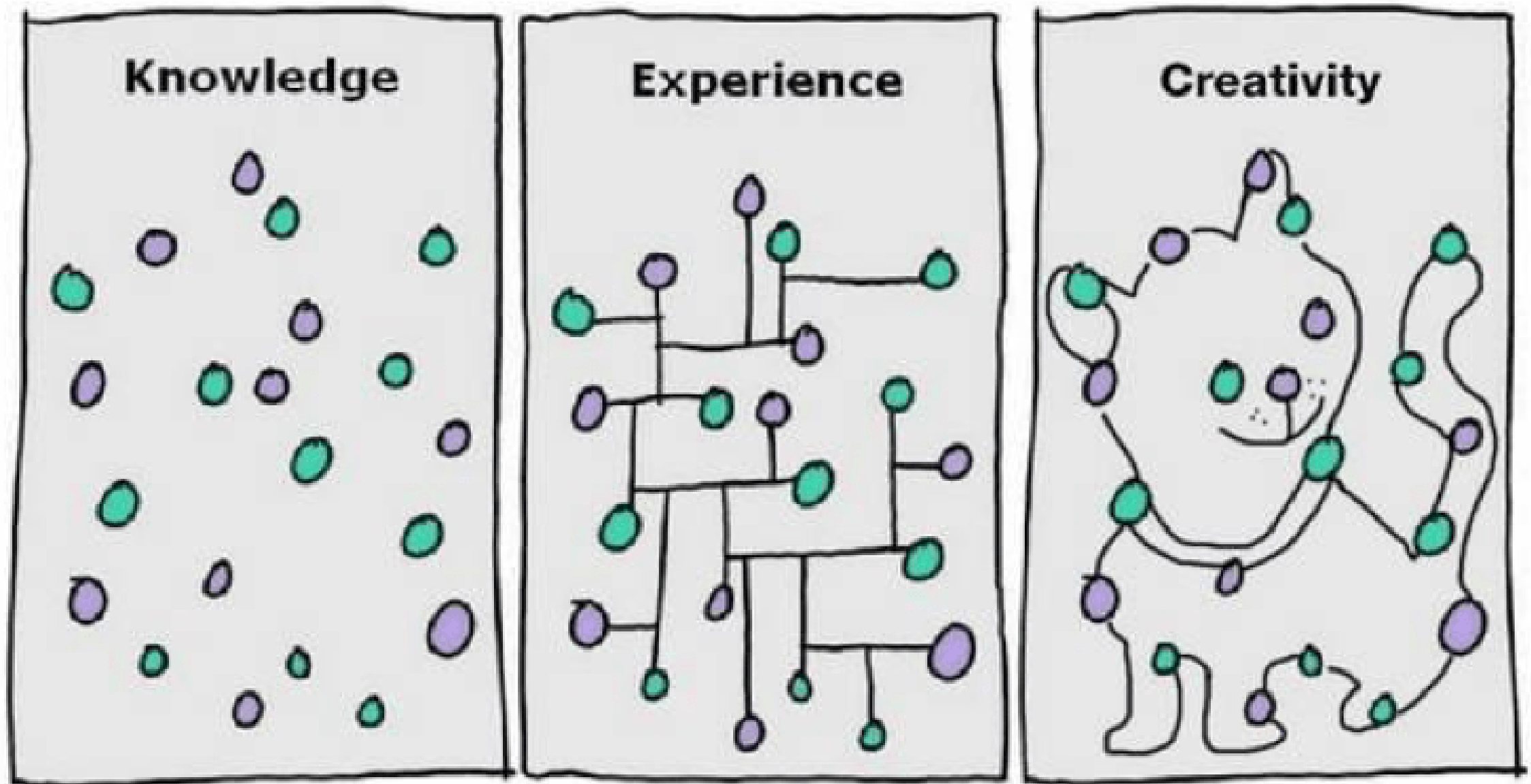
## Part 1: Counting

- Proofs/induction
- Sum Rule
- Partition Method
- Difference Method
- Bijection Method
- Product Rule
- Generalized product rule
- Pigeonhole Principle
- Inclusion/Exclusion
- Combinatorial proofs, binomial coefficients



## Part 2: Probability

- Sample space
- Axioms of probability
- Conditional probability
- Independence
- Bayes rule
- Random Variables
- Expectation and Linearity
- Variance

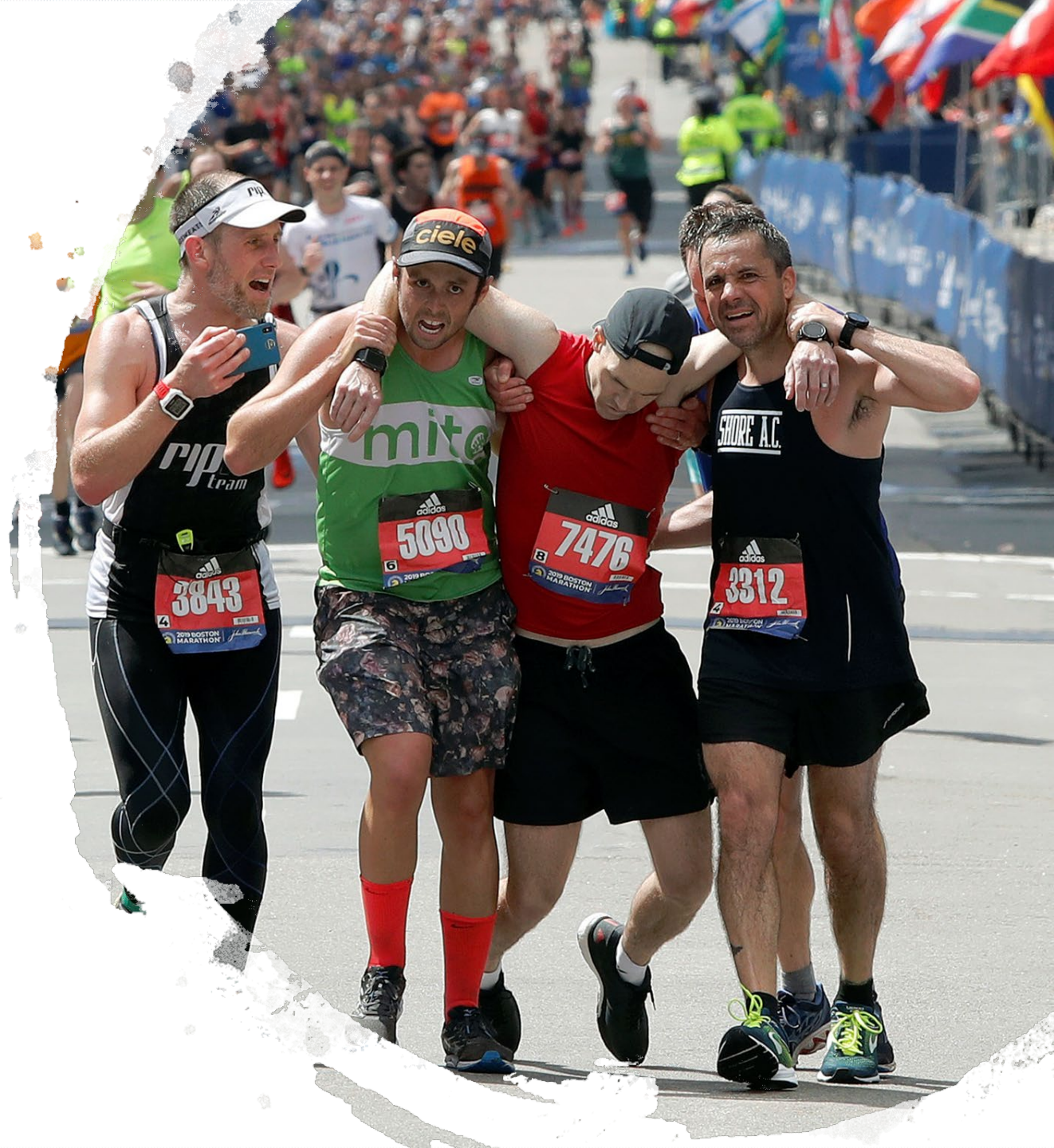


Lectures   Recitations/Extra Problems   Quizzes   Midterm/Final   Real-Life!



# Announcements

- Quiz 5 is running
- Quiz 6 → This week
  - Conditional Probability
  - **Bayes rule**





$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$



# Conditional Probabilities


- $P(B|A)$  means “Probability of event B **given** event A”

In other words, event A has already happened, now what is the chance of event B?

*"Probability Of"*                      *"Given"*

$$P(\text{A and B}) = P(\text{A}) \times P(\text{B} | \text{A})$$

*Event A*   *Event B*



*"Probability of **event A and event B** equals  
the probability of **event A** times the probability of **event B given event A**"*

$$P(\text{B} | \text{A}) = \frac{P(\text{A and B})}{P(\text{A})}$$

*"The probability of **event B given event A** equals  
the probability of **event A and event B** divided by the probability of **event A**"*

# Independence

- A and B are independent events if  $P(A|B) = P(A)$

$$P(A|B) = P(A) , \quad A \text{ and } B \text{ are independent}$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$$

# Conditional Probabilities – Example #1

Two fair coins are flipped.

- $A = \{\text{first coin is } H\}$ ,
- $B = \{\text{second coin is } H\}$ .

Are  $A$  and  $B$  independent?

$$\Omega = \{(H,H), (T,T), (H,T), (T,H)\}$$
$$P(A|B) = P(A), \quad P(A) = \frac{|A|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$$

$$P(A|B): \text{ new sample space } B = \{(H,H), (T,H)\}$$

$$P(A|B) = \frac{1}{2} = P(A)$$



# Conditional Probabilities – Example #2

Two fair coins are flipped.

- $A = \{\text{first coin is } H\}$
- $B = \text{two coins have different outcomes.}$

Are  $A$  and  $B$  independent?

$$P(A) = \frac{1}{2}$$

$$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$$

$$B = \{(H,T), (T,H)\}$$

$$P(A|B) = \frac{1}{2} = P(A)$$

# Independence extends to 3 events

- $A_1, A_2, A_3$  are independent events if  $P(A_1), P(A_2), P(A_3)$  does not change by knowing any subset of the other.

$$P(A_1|A_2) = P(A_1)$$

$$P(A_1|A_3) = P(A_1)$$

$$P(A_1|A_2, A_3) = P(A_1)$$

$$\Rightarrow P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

⋮

# Independence extends to n events

- $A_1, A_2, A_3, \dots, A_n$  are independent events if  $P(A_i)$  does not change by knowing any subset of the other.



# Independence for n events

- $A_1, A_2, A_3, \dots, A_n$  are independent events if for all  $k = 2, 3, \dots, n$ , and for all indices  $i_1, i_2, \dots, i_k$
- $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k})$

# Conditional Probability

- Direct enumeration of sample space
- Tree based enumeration
- Direct use of formula
- Independence
- **Bayes Rule**

$$\underline{\underline{P(A|B)}}$$

Suppose computing  $P(B|A)$  is much easier

# Bayes Rule

An internet search for "movie automatic shoe laces" brings up "Back to the future"

- Has the search engine watched the movie?

No, but it knows from lots of other searches what people are **probably** looking for.

- And it calculates that probability using Bayes' Theorem.


Google

movie automatic shoe laces


All Shopping Videos Images News More Settings Tools

About 13,100,000 results (0.79 seconds)


See movie automatic shoe laces Sponsored



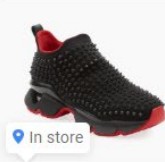
HICKIES Tie-Free Laces H2 Black  
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Nike Fly.By Low II Men's Basketball...  
\$48.75  
Kohl's  
★★★★★ (221)



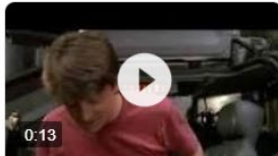
Christian Louboutin Spike Sock Donna...  
\$1,295.00  
Neiman Marcus  
★★★★★ (27)

Fox. In the **film**, Marty and Dr. Emmett "Doc" Brown travel to the future where, in 2015, **shoes** have power **laces**. A small number of fans got their hands on some working Nike Mag **shoes** with power **laces** in 2016. Jul 2, 2018


'Back to the Future: Part II' Film-Worn Sneaker Sells for Nearly ...  
<https://www.hollywoodreporter.com/heat-vision/back-future-part-ii-film-...>

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YouTube - Oct 21, 2015



Back to the Future 2 - Power Laces [Movie Clip] English (1989)  
Pro Movie Kino  
YouTube - Oct 26, 2017





Search bar containing the text "meaning of idk" with a microphone icon and a search icon.



- All
- Images
- News
- Videos
- Shopping
- More
- Settings
- Tools

About 24,000,000 results (0.73 seconds)

**Idk** is an abbreviation of the phrase I don't know. **Idk** is most commonly used in informal communication, such as text messaging. There are no formal rules about the capitalization of words like **idk**.

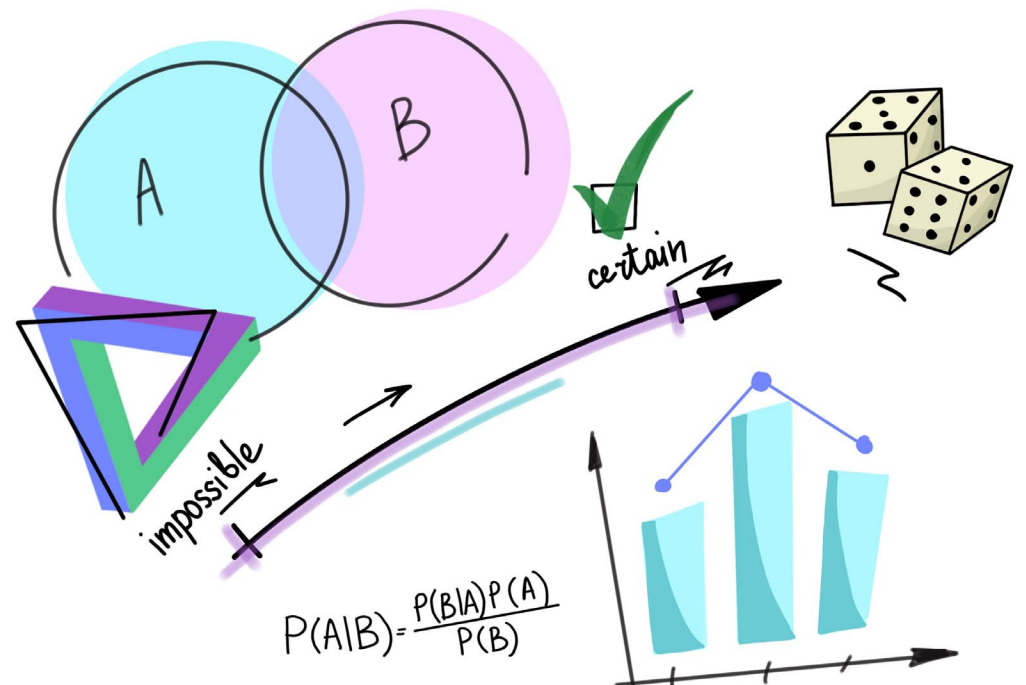


What Does Idk Mean? | Grammarly  
<https://www.grammarly.com/blog/idk-meaning>

About Featured Snippets Feedback

People also ask

*How does Google do this?*



Previously:  $P(A|B) = \frac{P(A \cap B)}{P(B)}$

∴  $P(B|A) = \frac{P(B \cap A)}{P(A)}$

⇒  $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

# Bayes Rule

$P(A|B) \rightarrow$  want

$P(B|A) \rightarrow$  easily compute

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow (i)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \rightarrow (ii)$$

$$\Rightarrow P(B) P(A|B) = P(A) P(B|A)$$

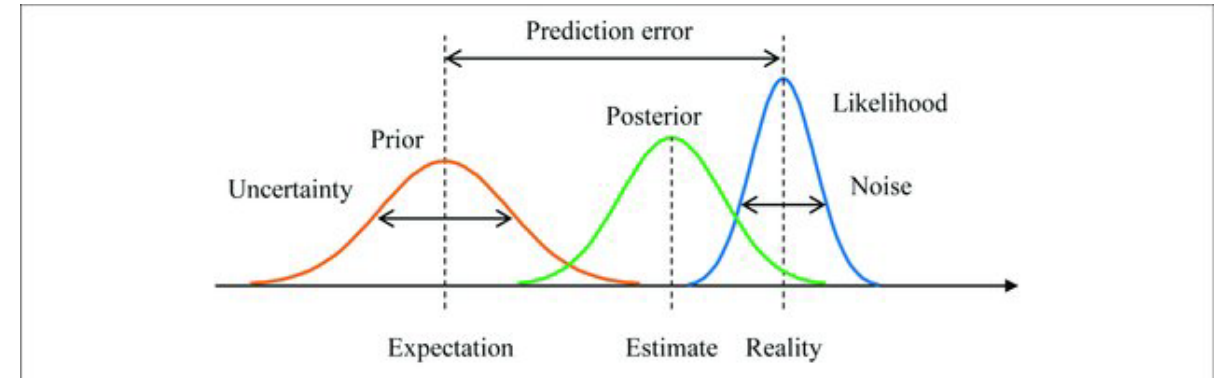
$$\Rightarrow \boxed{P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)}} \rightarrow \text{Bayes Rule}$$



# Bayesian Inference\*

$$\begin{array}{c} \text{Posterior} \\ \downarrow \\ P(A|B) \end{array} = \frac{\begin{array}{c} \text{Likelihood} \\ \downarrow \\ P(B|A) \end{array} * \begin{array}{c} \text{Prior} \\ \downarrow \\ P(A) \end{array}}{\begin{array}{c} P(B) \\ \uparrow \\ \text{Evidence} \end{array}}$$

\* *Inference = Educated Guess*



- Bayesian inference with a **prior distribution**, a **posterior distribution**, and a **likelihood function**.
- The prediction error is the difference between the **prior expectation** and the **peak of the likelihood function** (i.e., reality).
- **Uncertainty** is the variance of the prior. **Noise** is the variance of the likelihood function.

# Example #1 – Bayes rule

- What is the probability of a couple having two girls given that they have at least one girl?

- $P(2G | \text{at least } 1G) =$   $\{GG, GB, BB, BG\}$

$$= \frac{P(\text{at least } 1G \mid 2G) P(2G)}{P(\text{at least } 1G)} =$$

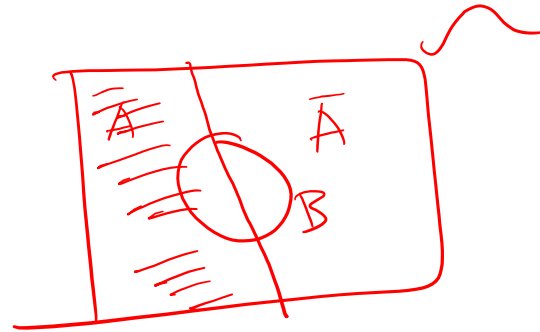
$$= \frac{1 \cdot 1/4}{3/4} = 1/3$$

# Law of Total Probability

- Sometimes conditional probability are used to easily compute (un)conditional probabilities.

$P(B)$

Let  $A$  be  
an event



Suppose  $P(B|A)$  and  $P(B|\bar{A})$  is easy

$$\begin{aligned} P(B) &= P(A) \cdot P(B|A) + P(\bar{A}) \cdot P(B|\bar{A}) \\ &= P(A) \cdot \frac{P(B \cap A)}{P(A)} + P(\bar{A}) \cdot \frac{P(B \cap \bar{A})}{P(\bar{A})} \\ &= P(B \cap A) + P(B \cap \bar{A}) \end{aligned}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

# Example #2 – Bayes rule



- If I randomly draw a cyan ball from two buckets that I can randomly choose from, B1 and B2, what is the probability of getting the cyan ball from the 1<sup>st</sup> bucket?

- Bayes Rule:  $P(B1 | A) = \frac{P(A | B1) P(B1)}{P(A)} = ???$

A photograph of a whiteboard with the Bayes' theorem formula written in blue marker: 
$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}$$



# Example #2 – Bayes rule



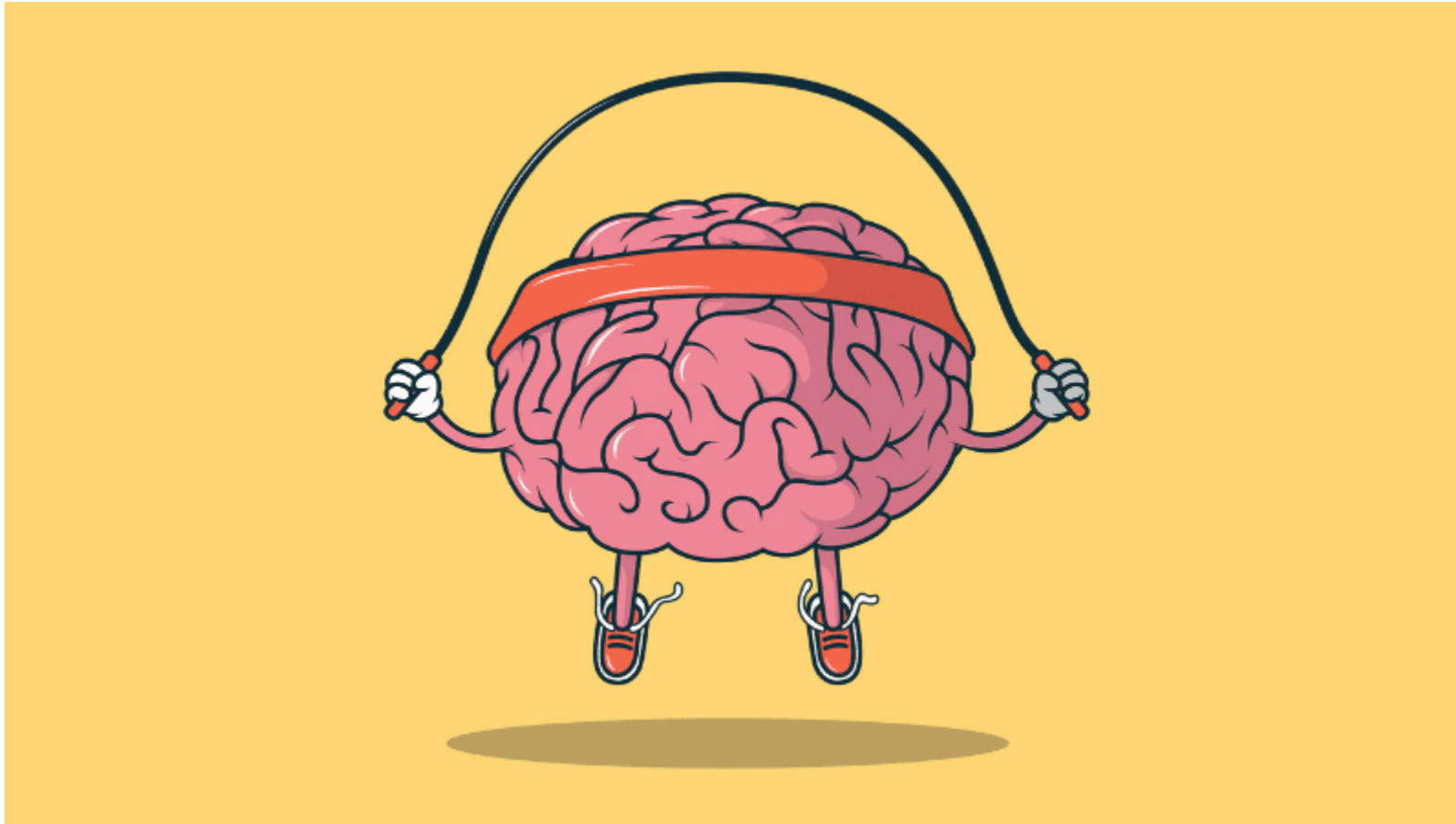
- If I randomly draw a **cyan ball** from two buckets that I can randomly choose from, B1 and B2, what is the probability of getting the **cyan ball** from the 1<sup>st</sup> bucket?
- $P(\mathbf{B1}) = P(B2) = 0.5$
- **A: select a cyan ball**
- $P(\mathbf{A} \mid \mathbf{B1}) = 0.5$
- $P(\mathbf{A} \mid B2) = 0.333$
- $P(A)=? \rightarrow$  Remember: B1 and B2 are disjoint! ! !  
[**SUM RULE**  $\rightarrow$  Disjoint sets  $\rightarrow$ ]  $P(A) = P(A \cap \mathbf{B1}) + P(A \cap B2) =$   
 $= P(A \mid B1) P(B) + P(A \mid B2) P(B2) = 0.5 \times 0.5 + 0.333 \times 0.5 = 5/12$

## Example #2 – Bayes rule



- If I randomly draw a **cyan ball** from two buckets that I can randomly choose from, B1 and B2, what is the probability of getting the **cyan ball** from the 1<sup>st</sup> bucket?
- Bayes Rule:  $P(B1 | A) = \frac{P(A | B1) P(B1)}{P(A)} = \frac{0.5 \times 0.5}{5/12} = 3/5.$
- *Does the result make sense???*

# Brain Break – 1 min



CTAAR survey

<https://sirs.rutgers.edu/blue>





# CTAAR Survey



Section 1



Section 2



Section 3

# Example #3 – Bayes rule

- 4 cards are drawn from a randomly shuffled deck of 52 cards.  
What is the probability that at least 2 Aces are drawn, given that at least one card is an Ace?

A → at least 2 Aces

B → at least 1 Ace

$$P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)} = \frac{P(A)}{P(B)}$$

$$P(A) = \frac{|A|}{|U|} = \frac{\binom{52}{4} - \binom{48}{4} - 4\binom{48}{3}}{\binom{52}{4}}$$

$$P(B) = \frac{|B|}{|U|} = \frac{\binom{52}{4} - \binom{48}{4}}{\binom{52}{4}}$$

# Example #4 - Bayes Rule

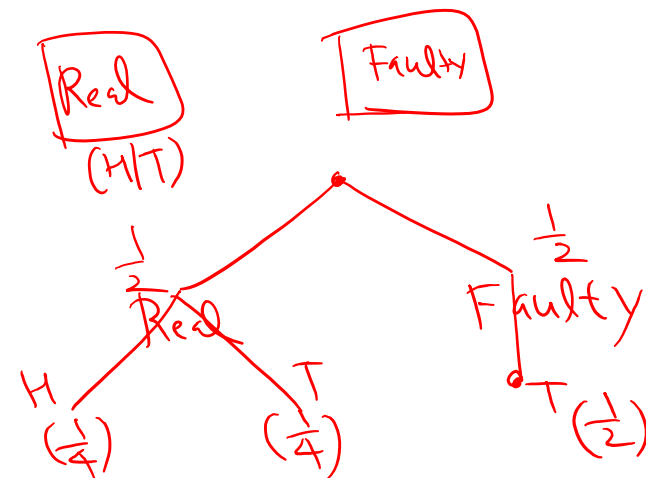
- James has two coins in his hand. One is a real coin and the second one is a faulty one with Tales on both sides. He blind folds himself choose a random coin and tosses it in the air. The coin falls down with Tale facing upwards.

What is the probability that this is the faulty coin?

$B \rightarrow$  outcome is Tale  
 $A \rightarrow$  coin chosen was faulty

$$P(A|B) = P(B|A) \frac{P(A)}{P(B)} = \frac{P(A)}{P(B)}$$

$$P(A) = \frac{1}{2} \quad , \quad P(B) = \frac{3}{4}$$

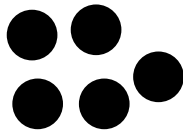


# Law of Total Probability – Example #2

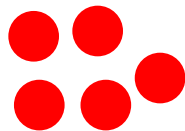


- You've been captured by pirates on an island.
- Need to play the following game to survive

100 black rocks



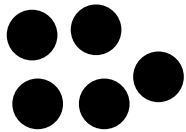
100 red rocks



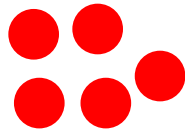
- Divide the rocks among two bags as you wish.
- Toss a fair coin and depending on the outcome draw a rock at random from the corresponding bag.
- If rock is black you win!

# Law of Total Probability – Example #2

100 black rocks



100 red rocks



$x$  Black rocks  
 $y$  Red rocks



$100-x$  Black rocks  
 $100-y$  Red rocks

$P(\text{black rock is selected})$

$$P(\text{black rock} | \text{Coin} = H) = \frac{x}{x+y}$$

$$P(\text{black rock} | \text{Coin} = T) = \frac{100-x}{200-x-y}$$

$$\begin{aligned} P(\text{black rock is selected}) &= P(\text{Coin} = H) P(\text{Black rock} | H) + P(\text{Coin} = T) P(\text{Black} | T) \\ &= \left(\frac{1}{2}\right) \left(\frac{x}{x+y}\right) + \left(\frac{1}{2}\right) \left(\frac{100-x}{200-x-y}\right) \end{aligned}$$



# Bayes Rule – One more example

*Find the probability for “when there is smoke, there is fire”*

Example: If dangerous fires are rare (1%) but smoke is fairly common (10%) due to barbecues, and 90% of dangerous fires make smoke then:

$$\begin{aligned} P(\text{Fire}|\text{Smoke}) &= \frac{P(\text{Fire}) P(\text{Smoke}|\text{Fire})}{P(\text{Smoke})} \\ &= \frac{1\% \times 90\%}{10\%} \\ &= 9\% \end{aligned}$$

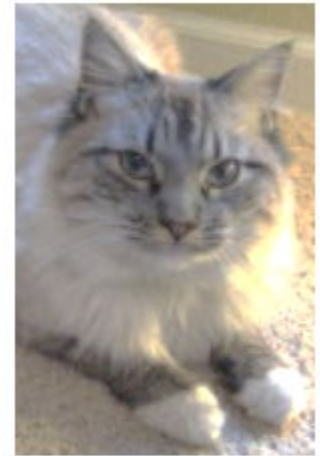
So the "Probability of dangerous Fire when there is Smoke" is 9%

# Bayes Rule – Yet another example

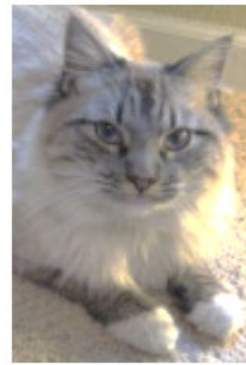
## Example: Allergy or Not?

Hunter says she is itchy. There is a test for Allergy to Cats, but this test is not always right:

- For people that **really do** have the allergy, the test says "Yes" **80%** of the time
- For people that **do not** have the allergy, the test says "Yes" **10%** of the time ("false positive")



If 1% of the population have the allergy, and **Hunter's test says "Yes"**, what are the chances that Hunter really has the allergy?



# Bayes Rule – Yet another example

$$P(\text{Allergy}|\text{Yes}) = \frac{P(\text{Allergy}) P(\text{Yes}|\text{Allergy})}{P(\text{Yes})}$$

$P(\text{Allergy})$  is Prob of Allergy = 1%

$P(\text{Yes}|\text{Allergy})$  is Prob of test saying "Yes" for people with allergy = 80%

$P(\text{Yes})$  is Prob of test saying "Yes" (to anyone) = ??%

- We **don't know** what the **general** chance of the test saying "Yes" is but we can calculate it by adding up those **with**, and those **without** the allergy:
  - 1% have the allergy, and the test says "Yes" to 80% of them
  - 99% do **not** have the allergy and the test says "Yes" to 10% of them
- $P(\text{Yes}) = 1\% \times 80\% + 99\% \times 10\% = 10.7\%$  of the population.

$$P(\text{Allergy}|\text{Yes}) = \text{about } 7\%$$