Recitation 11

Midterm

1. (a) (5 pts) Build the Pascal's Triangle up to the 5^{th} row.

   Solution:
   1
   1,1
   1,2,1
   1,3,3,1
   1,4,6,4,1
   1,5,10,10,5,1

   (b) (5 pts) Find the expansion of $(8x-2y)^3$ using the Pascal's Triangle.

   Solution: $1(8x)^3 + 3(8x)^2(-2y) + 3(8x)(-2y)^2 + 1(-2y)^3 = 512x^3 - 384x^2y + 96xy^2 - 8y^3$

   (c) (5 pts) Find the coefficient of $x^{20}y^5$ in $(8x-2y)^{25}$, using the binomial formula.

   Solution: Coefficient $= 25C5 \times (8^{20}) \times (-2^5)$

Midterm

2. (5 *pts*) How many 10-digit sequences can be made with the digits 1, 1, 1, 1, 2, 2, 3, 3, 8, 9?

Solution: 10!/4!2!2!

3. (10 *pts*) How many natural numbers, n, end with 5, where $0 < n < 500$?
Hint: Consider using first the sum and then the product rules.

Solution:

- For one-digit number: 1
- For two-digit numbers: 9 (15, 25, 35, 45,..95)
- For three-digit numbers: 4*10*1 = 40

[Sum Rule] Total number $= 1 + 9 + 40 = 50$

Midterm

4. (10 *pts*) In a dart-throwing game, a player throws a dart 5 times. To win, you need to get all 5 darts inside the bullseye. In how many ways you can lose?
Hint: Consider counting all the outcomes that one can get, to lose the game.

Solution: Sum method. The player will lose if he fails either in 1 or 2 or 3 or 4 or 5 efforts. Since the order of failed efforts is not important, there are 5C1 + 5C2 + 5C3 + 5C4 + 5C5 = 31 ways.

5. (10 *pts*) Using the **Pigeonhole Principle**, show that in every set of 100 integers, there exist two integers whose difference is a multiple of 47.

Solution: There are 47 possible remainders from $0, 1, \ldots, 46$ when divided by 47. Hence, among any 100 integers, there must exist a and b that have the same remainder. Then $a - b$ will be a multiple of 47.

Midterm

6. (10 pts) A painter needs to paint a house with 20 rooms. The available colors are: **b**lack, **w**hite, **r**ed, and **c**yan.

   (a) How many choices does the painter have, if there are no constraints in choosing the colors?

   (b) If the **b** color can be used only when at least 2 rooms are colored in **b**, how many choices are there?

   (c) If the painter can use **c** in at most 5 rooms, how many choices are there?

   (d) How many choices does the painter have if we decide to paint only half of the rooms?

Solution:

- Pirates problem with no constraints: $23C3$
- $(20+2)C2$ (not using **b** at all) + $(18+3)C3$ (color at least 2 rooms with **b**) $= 231 + 1330 = 1561$ **OR** $23C3$ (total possible number) - $(19+3-1)C(3-1)$(paint only 1 **b** room) $1771 - 210 = 1561$
- $23C3 - 17C3$
- $13C3$

Midterm

7. We permute the letters of the word MISSISSIPPI.

   (a) (5 *pts*) How many words (valid or non-valid English) can we create?

   Solution: $11!/(4!4!2!) = 34650$

   (b) (5 *pts*) If we list all the words that we created in part a in alphabetic order, how many words exist before the first word that starts with M?

   Solution: Writing the letters of the word MISSISSIPPI in alphabetic order, we get IIIIMPPSSSS. The total number of letters in this word is 11. Since only I appears before M, the number of words before the first word stating with M is equal to the number of words where an I is fixed in the first place. We can arrange the remaining 10 letters as $10!/(4!3!2!)$.

8. (10 *pts*) How many ways are there to arrange 20 books on a bookcase with 3 shelves? Assume, as in real life, that books are distinguishable and that the order of the books on each shelf matters.

   Solution: To put 20 books on 3 shelves, you can first put the books in order and then decide how many go on each shelf. There are $P(20,20) = 20!$ ways to put the books in order and, by stars and bars, $\binom{20+3-1}{20}$ ways to choose how many books go on each shelf. So the final answer is $20!\binom{22}{20}$.

Midterm

9. (10 *pts*) Using the **product rule**, prove that the number of different ways in which $2n$ students can be paired up is $(2n-1)(2n-3)(2n-5)...1$.

Solution: Number the students from 1 to $2n$ and let's look at them in order. Student 1 has $(2n-1)$ choices to pair up with someone. Then, the next unpaired student has $(2n-3)$ choices. The next unpaired student after these two has $(2n-5)$ choices and so on. Finally, two students will be left and they will have only one choice to pair up with each other.

10. (10 *pts*) In a survey of 1000 patients with a genetic disease, 150 patients are found to carry gene-X1, 225 patients carry gene-X2, and 100 patients carry both gene-X1 and gene-X2. How many patients do not carry either gene-X1 or gene-X2?

Solution: Let T be the set of patients who carry gene-X1 and C be the set of patients who carry gene-X2. Here, n(T) = 150, n(C) = 225 and n(C $\cap T$) = 100. n(C $\cup T) = n(C) + n(T) - n(C \cap T) = 150 + 225 - 100 = 275$. Number of patients that carry either gene-X1 or gene-X2 is 275. Number of patients that do not carry neither gene-X1 nor gene-X2 is $1000 - 275 = 725$.

Midterm

$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \ldots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}$.

Solution: We define the counting problem as follows: How many 5-element subsets are there of the set $1, 2, \ldots, n+3$? The answer to that question is the RHS of the identity we want to prove, that is $\binom{n+3}{5}$. To get the LHS, we break up the same question into cases by a) putting the 5 elements in order and b) creating subgroups based on what the middle (third smallest) element of the 5 element subset is. The smallest this could be is a 3. In that case, we have $\binom{2}{2}$ choices for the numbers below it, and $\binom{n}{2}$ choices for the numbers above it. If the middle number is 4 there are $\binom{3}{2}$ choices for the bottom two numbers and $\binom{n-1}{2}$ choices for the top two numbers. If the middle number is 5, then there are $\binom{4}{2}$ choices for the bottom two numbers and $\binom{n-1}{2}$ choices for the top two numbers. An so on, all the way up to the largest the middle number could be, which is $n+1$. In that case there are $\binom{n}{2}$ choices for the bottom two numbers and $\binom{2}{2}$ choices for the top number. Thus the number of 5 element subsets is $\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \ldots + \binom{n}{2}\binom{2}{2}$.

Midterm

12. (10 *pts*) In lecture, we showed that the number of ways to divide 16 pieces of gold (G) amongst 5 pirates is $\binom{20}{4}$ by counting all strings with 16 G and 4 dividers, /. The following alternate method is suggested to count such strings: Write down 16 G. Each of the 4 / can go to 17 places, for a total of 17^4 possibilities. The /'s are indistinguishable, so their ordering does not matter. Thus, we divide by 4! since we could have placed the 4 /'s in any order. So the answer must be $17^4/4!$. However, this is certainly not correct as it is not equal to our answer $\binom{20}{4}$, and in fact it is not even an integer! Explain what is wrong with this approach. Does it overcount or undercount? Why? *Hint: Applying your methods requires adhering to their assumptions.*

Solution: The method is undercounting. If the 4 '/s go into different positions, then they are being overcounted by 4!. However, if say two of them go into the same position and two go to different positions, then such a configuration is counted only $\binom{4}{2}2$ times and not 4! times. Hence, one cannot uniformly divide by 4! as not every configuration is being counted the same number of times.

Midterm

(5 *pts*) Prove by **contradiction**: If $x = 2$, then $3x - 5 \neq 10$.

Solution: The negation is "if $x = 2$, then $3x - 5 = 10$.
We solve for x the equation $3x - 5 = 10$.

$$3x - 5 = 10$$
$$x = 3$$

Hence, we found that $x = 3$, which contradicts with the original statement that $x = 2$.

14. (5 *pts*) Prove by **contrapositive**: For all integers a and b, if a+b is odd, then a is odd *or* b is odd. (Hint: How do you negate the "*or*"?)

Solution: The contrapositive is "if a is even and b is even then a+b is even" Using the definition of parity, let a = 2x and b = 2y for some integers x and y. Now, a+b = 2(x+y). Using closure of integers under addition, we can let x+y = c for some integer c. We now have that a+b = 2c for some integer c, which is even by definition of parity. Hence, we found that for all integers a and b, if a+b is odd, then a is odd or b is odd by contrapositive.

Bayes Rule

The probability of event A, given that event B has subsequently occurred, is

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{[P(A) \cdot P(B|A)] + [P(\overline{A}) \cdot P(B|\overline{A})]}$$

Question 8

A certain candy similar to Skittles is manufactured with the following properties: 30% of sweet ones and 70% of pieces are sour. Each candy piece is colored wither red or blue (but not both).If a candy piece is sweet, then it is colored blue in 80%, and red in 20%. If it is sour, 80% is in red and 20% is in blue.

The candy pieces are mixed together and randomly before they are sold.

(a) If you choose a piece at random from the jar, what is the probability that you choose a blue piece?

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P = 0.3*0.8+0.7*0.2 = 0.38

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(b) Given that the piece you chose is blue, what is probability that the piece is sour?

We want p(sour|blue) = p(sour ∩blue)/p(blue), but we cannot find p(sour ∩blue)/p(blue) in a direct way.

But we can expend it more, p(sour ∩blue) = p(blue|sour)*p(sour).

So, p(sour|blue) = p(blue|sour)*p(sour)/p(blue).

=0.2*0.7/0.38 = 0.3684…

Question 9

- Suppose we have a deck of four cards: A-spade, 2-spade, A-heart, 2-heart. After being dealt two random cards, facing down, the dealer tells us that we have at least one ace in our hand.

(a) What is the probability that our hand has both aces? In other words, what is P(Two aces| At least one ace)?

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$$P(2\,A \mid A) = P(A \mid 2A)\,P(2A) \,/\, P(A) = 1*(1/6) \,/\, (5/6) = 1/5$$

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2. What if the dealer tells us we have the A-spade in our hand, what is P(Two aces| A-spade)?

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P(2A | A-spade) = P(A-spade | 2A) P(2A) / P(A-spade) = 1*(1/6) / (3/6) = 1/3

Question 10

A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% are sick with measles. Let F stand for an event of a child being sick with flu and M stand for an event of a child being sick with measles. Assume for simplicity that $F \cup M = \Omega$, i.e., that there no other maladies in that neighborhood.

A well-known symptom of measles is a rash (the event of having which we denote R). Assume that the probability of having a rash if one has measles is $P(R \mid M) = 0.95$. However, occasionally children with flu also develop rash, and the probability of having a rash if one has flu is $P(R \mid F) = 0.08$.

Upon examining the child, the doctor finds a rash. What is the probability that the child has measles?

$$P(M|R) = P(R|M)\,P(M) / P(R) = P(R|M)\,P(M) / (P(R,M) + P(R,F))$$
$$= P(R|M)\,P(M) / (P(R|M)\,P(M) + P(R|F)\,P(F))$$

Question 11

- Three bags contain 3 red, 7 black; 8 red, 2 black, and 4 red & 6 black balls respectively. 1 of the bags is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the third bag.

Question 11

- Let E1, E2, E3 and A are the events defined as follows.
  E1 = First bag is chosen
  E2 = Second bag is chosen
  E3 = Third bag is chosen
  A = Ball drawn is red
- Since there are three bags and one of the bags is chosen at random, so P (E1) = P(E2) = P(E3) = 1 / 3
  If E1 has already occurred, then first bag has been chosen which contains 3 red and 7 black balls. The probability of drawing 1 red ball from it is 3/10. So, $P\ (A/E_1) = 3/10$, similarly $P(A/E_2) = 8/10$, and $P(A/E_3) = 4/10$. We are required to find $P(E_3/A)$ i.e. given that the ball drawn is red, what is the probability that the ball is drawn from the third bag by Baye's rule.

Question 11

$$= \frac{\frac{1}{3}\times\frac{4}{10}}{\frac{1}{3}\times\frac{3}{10}+\frac{1}{3}\times\frac{8}{10}+\frac{1}{3}\times\frac{4}{10}} = \frac{4}{15}.$$

Independent Events

- Two events, A and B, are independent if the outcome of A does not affect the outcome of B.

- In many cases, you will see the term, "With replacement".

- As we study a few probability problems, you will see how "replacement" allows the events to be independent of each other.

Question 1

A coin is tossed and a six-sided die is rolled. Find the probability of getting a head on the coin and a 6 on the die.

A coin is tossed and a six-sided die is rolled. Find the probability of getting a head on the coin and a 6 on the die.

These two events (the coin and die) are independent events because the flipping of the coin does not affect rolling the die. The events are independent of each other.

Solution:

Let's find the probability of each independent event:

$P(\text{heads}) = \frac{1}{2}$ There is only 1 head on a coin. There are two total outcomes (heads or tails)

$P(6) = \frac{1}{6}$ There is only one 6 on a die. There are 6 total outcomes on a die (1,2,3,4,5,6)

To find the probability of two or more independent events that occur in sequence, find the probability of each event separately, and then multiply the answers.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Now, let's apply our new rule:

$$P(\text{heads and a } 6) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Question 2

- A jar of marbles contains 4 blue marbles, 5 red marbles, 1 green marble, and 2 black marbles.
- A marble is chosen at random from the jar. After replacing it, a second marble is chosen.
- Find the probability for the following:
- P(green and red)

- Notice that the problem states that a marble is chosen and then replaced. Because the marble is replaced, the two events are independent of each other.
- When the second marble is chosen, the jar contains the same exact marbles as it did for the first pick.
- Therefore, we can classify these two events as independent events. This allows us to use our probability formula of:
- P(A and B) = P(A) • P(B)

$$P(\text{green and red}) = \frac{1}{12} \bullet \frac{5}{12} = \frac{5}{144}$$