# Recitation 4

# Inductive Proofs

Given some property P(n), an inductive proof:

- proves P(0) is true as a base case;
- proves that if P(k) is true, then P(k+1) must be true as well; and
- concludes that P(n) is true for any natural number n.

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Let P(n) be the mathematical statement $11^n - 6$ is divisible by 5.

Base Case: When n = 1 we have $11^1 - 6 = 5$ which is divisible by 5. So P(1) is correct.

Induction hypothesis: Assume that P(x) is correct for some positive integer x. That means $11^x - 6$ is divisible by 5 and hence $11^x - 6 = 5m$ for some integer m. So $11^x = 5m+6$

# Prove by induction that $11^n - 6$ is divisible by 5 for every positive integer n

Induction step: We will now show that P(x+ 1) is correct.

$$11^{x+1} - 6 = (11 \times 11^x) - 6$$
$$= 11(5m+6) - 6$$
$$= 11(5m) + 66 - 6$$
$$= 5(11m) + 60$$
$$= 5(11m+12).$$

Hence by mathematical induction P(n) is correct for all positive integers n.

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Let P(n) be the mathematical statement $2^n > 2n$.

Base Case: When n = 3 we have $2^3 = 8 > 6 = 2\times3$. So P(3) is correct.

Induction hypothesis: Assume that P(x) is correct for some positive integer k. That means that $2^x > 2x$.

Induction step: We will now show that P(x +1) is correct. $2^{x+1} = 2\times2^x > 2\times2x$ by the induction hypothesis, and $2\times2x > 2(x+1)$. So P(x +1) is correct. Hence by mathematical induction P(n) is correct for all positive integers n >2.

# Counting Continued

- **Sum Rule**
- **Product Rule**
- **Difference Method**
- **Bijection Method**

# Problem 1

Electing Heads of State

*N* = {**A**lan, **B**ill, **C**athy, **D**avid, **E**velyn}.

No one can hold more than one office, list and count the different ways the club could elect each group of officers. (David and Cathy are not natural-born US citizens) *Note: Citizens who are not natural-born citizens cannot be elected president but they can serve as speaker of the house or senate majority leader.

1. How many ways can you elect the speaker of the House of representatives and the senate majority leader?

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AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED; 20 ways 5*(5-1)=20

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AB, AE, BA, BE, CA, CB, CE, DA, DB, DE, EA, EB;
12 ways = Two distinct sets: pick A,B,E as speaker (3*2), pick C, D as speaker (2*3) = 6+6

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3+4+5+6+5+4=27

1-[4...6], 2-(3-6), 3-(2-6), 4-(1-6), 5-(1-5), 6-(1-4)

# Problem 3

Construct a tree diagram showing all possible results when three fair coins are tossed. Then list the ways of getting each result.

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2. more than two heads
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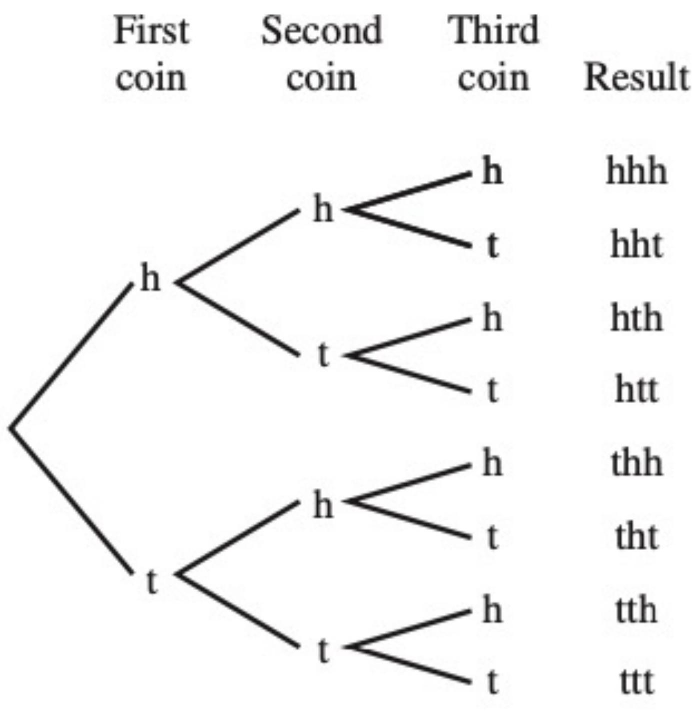

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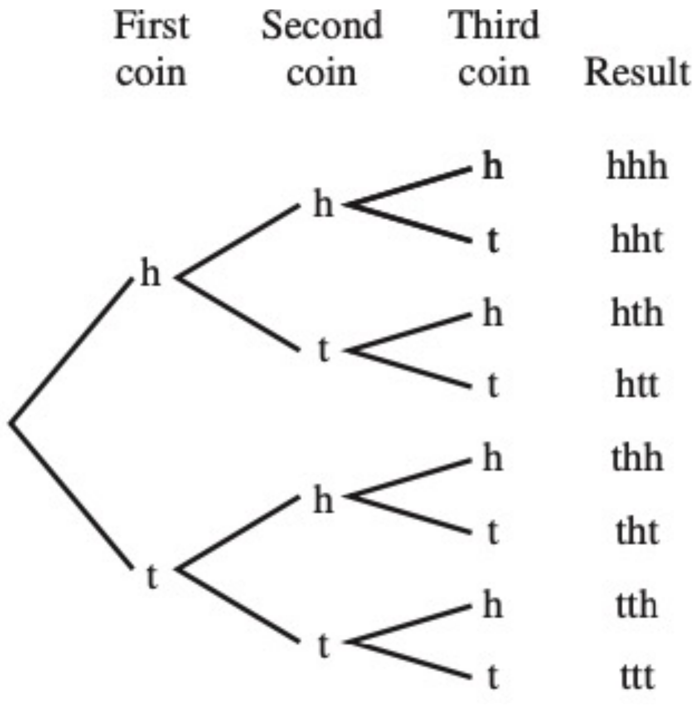

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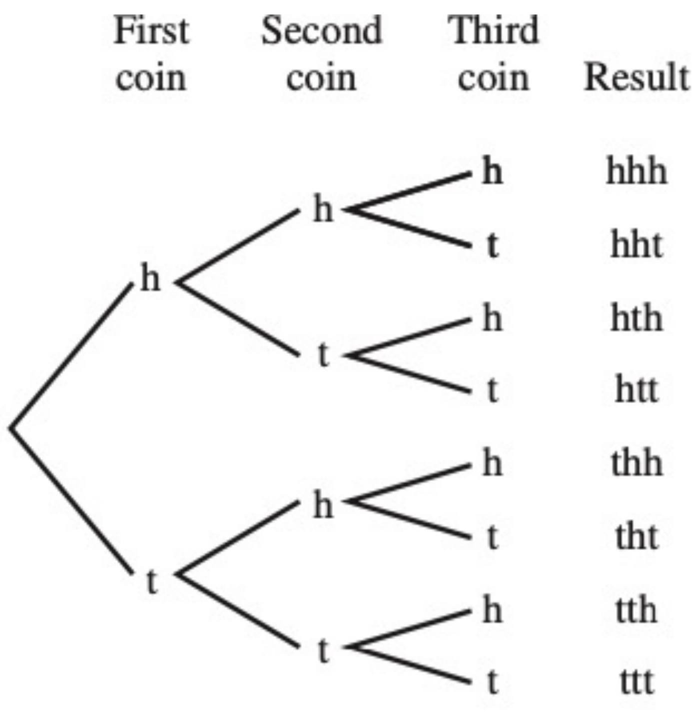

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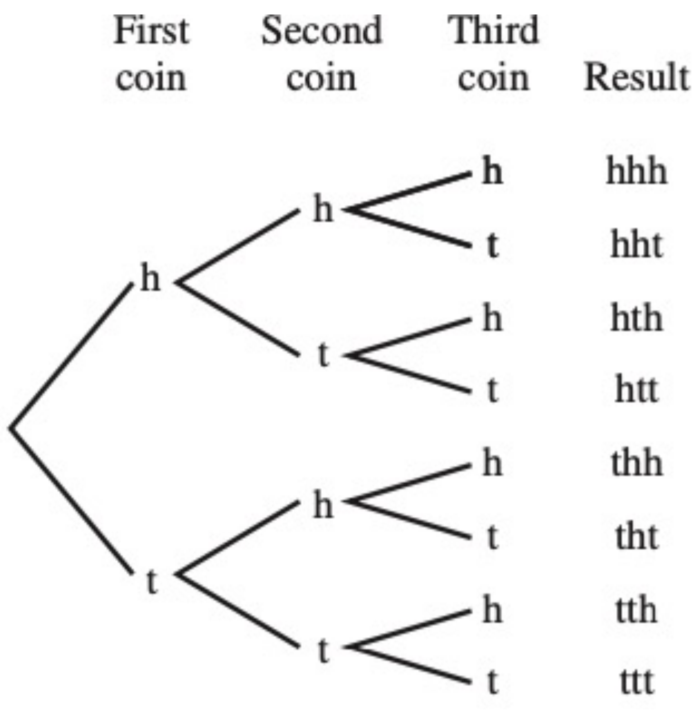

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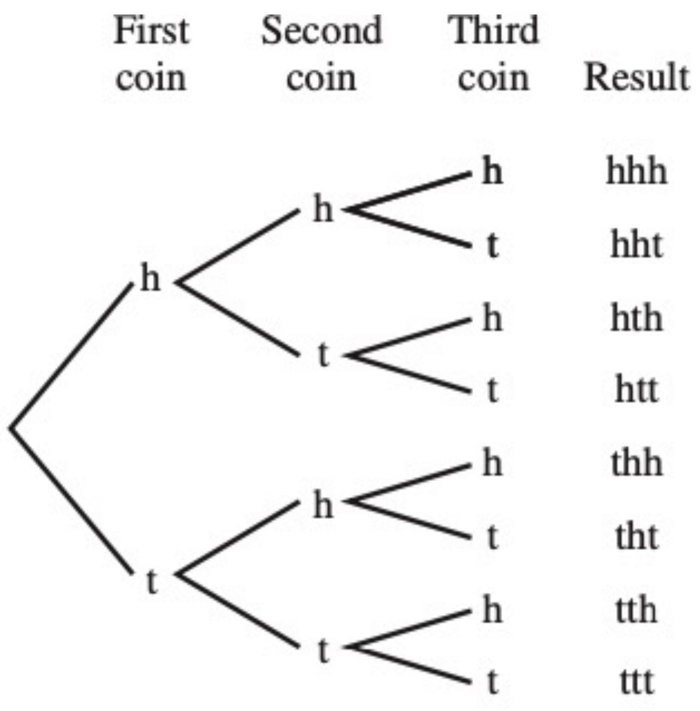

# Problem 4

A group of twenty people, consisting of ten married couples, is randomly seated in a row of twenty seats.

(a) Suppose that one of the couples is Alice and Bob. What is the number of possible arrangements that Alice and Bob are seated next to each other?

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Solution: $2! * 19!$

19! represents arrangements of all individuals plus AB as one individual. 2! is the interchanging of A and B.

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(b) What is the number of possible arrangements that everyone is seated next to their spouse (i.e. that every couple is seated together)?

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Solution: $2^{10} * 10!$

10! is the number of arrangements of the couples. $2^{10} = (2!)^{10}$ accounts for additional arrangements if you are counting the different ordering of each couple.

# Problem 5

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Solution: This is equal to the number of subsets $S_0 \subseteq S = \{x_1, x_2, \dots, x_{20}\}$. There are $|P(S)| = 2^{20}$ total subsets, where

(a) 20 of which have size 1 — $\{x_1\}, \{x_2\}, \dots, \{x_{20}\}$;
(b) and 1 which has size 0 — an empty study group $\emptyset$.

So the answer is $2^{20} - 20 - 1$.

# Problem 6

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SOLUTION: Using the inclusion-exclusion principle for 2 sets, $|A \cup B| = |A| + |B| - |A \cap B|$, we start by defining A as the set of positive integers not exceeding 1000 divisible by 7, B the set of positive integers not exceeding 1000 divisible by 11, and $A \cap B$ the set of positive integers not exceeding 1000 divisible by both 7 AND 11.

Knowing the size of these three sets, we can calculate $A \cup B$ which is the set of positive integers not exceeding 1000 divisible by either 7 OR 11. The sizes of the sets are as follows:
|A| = 1000 // 7 = 142
|B| = 1000 // 11 = 90
|A ∩ B| = 1000 // (7*11) = 12

Therefore, by inclusion exclusion principle, we get $|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$