CS206 Quiz 6 **Dec 7, 2023** **A**

Name: ____________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

1. (40%) Which of the following are true statements? Briefly explain.

   (a) $P(-A|B) = 1 - P(A|B)$

   Solution: True

   (b) $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

   Solution: True

   (c) $P(A|B) = \frac{P(B \bigcap A)}{P(B)}$

   Solution: True

   (d) The probability of flipping an unbiased coin and getting 2 heads in a row is $P(\{H, H\}) = 0.5 + 0.5 = 1.0$

   Solution: False

2. A toy car company produces 2 types of model cars; 40% are sedans and 60% are SUVs. Additionally, each type of car can be either red or blue. Sedans are 50% red and SUVs are 75% red. After the production process, cars randomly partitioned into boxes, with 100 cars per box. You bought one of these boxes.

   (a) (10%) Draw the probability tree for the car model production. Label the branch probabilities involved. Don't calculate leaf probabilities

   Solution: From root 2 branches with nodes: 0.4 sedan node and 0.6 SUV node. Sedan node has 2 branches with nodes: 0.5 blue color node and 0.5 red color node. SUV has 2 branches with nodes: 0.75 red color and 0.25 blue color.

(b) (10%) What is the probability that you choose a blue car from the box?

Solution: Let B be the event that the car you choose is blue and C be the event that the car type you choose is a sedan. Then, $P(B) = p(B|C)P(C) + p(B|C')P(C') = (0.5 \times 0.4) + (0.25 \times 0.6) = 0.35$

(c) (10%) What is the probability that the blue car that you choose in part (b) is an SUV?

Solution: By bayes theorem and part b, we have $P(C'|B) = \frac{P(B|C')P(C')}{P(B)} = \frac{0.25\times0.6}{0.35} = 0.429$

3. A test for a rare medical disease has a probability of 0.95 to positively classify a person being diseased. Also, it has a probability of 0.1 to positively classify a non-diseased person. There is a probability of 0.005 for any given person to have the disease. Given an arbitrary person, what is the probability that:

Solution: Let C =person is classified positively, D =person is diseased,W =person is classified incorrectly

(a) (10%) The test classification will be positive? (Hint: Calculate the unconditional probability using conditional probabilities)

Solution: $P(C) = P(C|S)P(D) + P(C|-D)P(-D) = (0.95 \times 0.005) + (0.1 \times 0.995) = 0.10425$

(b) (10%) The person is diseased, given a positive classification?

Solution: $P(D|C) = \frac{P(C|D)P(D)}{P(C|S)P(D)+P(C|-D)P(-D)} = \frac{0.95\times0.005}{(0.95\times0.005)+(0.1\times0.995)} = 0.0455$

(c) (10%) The person is not diseased, given a negative classification?

Solution: $P(-D|-C) = \frac{P(-C|-D)P(-D)}{P(-C)} = \frac{0.9\times0.995}{1-0.10425} = 0.9997$

(d) (10%) The person is classified incorrectly?

Solution: $P(W) = P(C\bigcap -D) + P(-C\bigcap D) = P(C|-D)P(-D) + P(-C|D)P(D) = (0.1 \times 0.995) + (0.05 \times 0.005) = 0.09975$

4. (10%) A box contains three coins: two regular coins and one fake two-headed coin ($P(H) = 1$). You pick a coin at random and toss it. What is the probability that it lands heads up?

Solution: 2/3

5. Suppose that there are two slot machines, one of which pays out 10% of the time and the other pays out 20% of the time. Unfortunately, you have no idea which is which. You randomly choose a machine and put in a quarter.

   (a) (10%) If you don't get a jackpot, what is the chance that you chose the machine that pays out 20% of the time?

Solution: $P(S^C|J^C) = \frac{0.8*0.5}{0.8*0.5+0.9*0.5} = 8/17 = 0.471$

(b) (10%) If you had instead gotten a jackpot, what would be the chance that you chose the one that pays out 20% of the time?

Solution: $P(S^C|J) = \frac{0.2*0.5}{0.1*0.5+0.2*0.5} = 2/3 = 0.667$

CS206 Quiz 6 **Dec 8, 2023** **B**

Name: ______________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

1. (40%) Which of the following are true statements? Explain briefly.

   (a) $P(-A|B) = P(A|B) - 1$

   Solution: False

   (b) $P(B|A)P(A) = P(A|B)P(B)$

   Solution: True

   (c) $P(A|B) = \frac{P(B)}{P(B \bigcap A)}$

   Solution: False

   (d) The probability of flipping an unbiased coin and getting 2 heads in a row is $P(\{H, H\}) = 0.5 \times 0.5 = 0.25$

   Solution: True

2. A toy car company produces 2 types of model cars; 40% are sedans and 60% are SUVs. Additionally, each type of car can be either red or blue. Sedans are 50% red and SUVs are 75% red. After the production process, cars randomly partitioned into boxes, with 100 cars per box. You bought one of these boxes.

   (a) (10%) Draw the probability tree for the car model production. Label the branch probabilities involved. Don't calculate leaf probabilities.

   Solution: From root 2 branches with nodes: 0.4 sedan node and 0.6 SUV node. Sedan node has 2 branches with nodes: 0.5 blue color node and 0.5 red color node. SUV has 2 branches with nodes: 0.75 red color and 0.25 blue color.

(b) (10%) What is the probability that you choose a blue car from the box?

Solution: Let B be the event that the car you choose is blue and C be the event that the car type you choose is a sedan. Then, $P(B) = p(B|C)P(C) + p(B|C')P(C') = (0.5 \times 0.4) + (0.25 \times 0.6) = 0.35$

(c) (10%) What is the probability that the blue car that you choose in part (b) is an SUV?

Solution: By bayes theorem and part b, we have $P(C'|B) = \frac{P(B|C')P(C')}{P(B)} = \frac{0.25 \times 0.6}{0.35} = 0.429$

3. A test for a rare medical disease has a probability of 0.9 to positively classify a person suffering with the disease. Also, it has a probability of 0.15 to positively classify a non-diseased person. There is a probability of 0.01 for any given person to have the disease. Given an arbitrary person, what is the probability that:

Solution: Let C =person is classified positively, D =person is diseased,W =person is classified incorrectly

(a) (10%) The test classification will be positive?

Solution: $P(C) = P(C|S)P(D) + P(C|-D)P(-D) = (0.9 \times 0.01) + (0.15 \times 0.99) = 0.1575$

(b) (10%) The person is diseased, given a positive classification?

Solution: $P(D|C) = \frac{P(C|D)P(D)}{P(C|S)P(D)+P(C|-D)P(-D)} = \frac{0.9\times 0.01}{(0.9\times 0.01)+(0.15\times 0.99)} = 0.0571$

(c) (10%) The person is not diseased, given a negative classification?

Solution: $P(-D|-C) = \frac{P(-C|-D)P(-D)}{P(-C)} = \frac{0.85 \times 0.99}{1-0.1575} = 0.9988$

(d) (10%) The person is classified incorrectly?

Solution: $P(W) = P(C \bigcap -D) + P(-C \bigcap D) = P(C|-D)P(-D) + P(-C|D)P(D) = (0.15 \times 0.99) + (0.1 \times 0.01) = 0.1495$

4. (10%) A box contains three coins: two regular coins and one fake two-headed coin ($P(H) = 1$). You pick a coin at random and toss it. What is the probability that it lands heads up? (Hint: Calculate the unconditional probability using conditional probabilities)

Solution: Let C1 be the event that you choose a regular coin, and let C2 be the event that you choose the two-headed coin. $P(H) = P(H|C1)P(C1) + P(H|C2)P(C2) = 1/2 * 2/3 + 1 * 1/3 = 2/3$

5. Suppose that there are two slot machines, one of which pays out 10% of the time and the other pays out 20% of the time. Unfortunately, you have no idea which is which. You randomly choose a machine and put in a quarter.

(a) (10%) If you don't get a jackpot, what is the chance that you chose the machine that pays out 20% of the time?

Solution: $P(S^C|J^C) = \frac{0.8*0.5}{0.8*0.5+0.9*0.5} = 8/17 = 0.471$

(b) (10%) If you had instead gotten a jackpot, what would be the chance that you chose the one that pays out 20% of the time?

Solution: $P(S^C|J) = \frac{0.2*0.5}{0.1*0.5+0.2*0.5} = 2/3 = 0.667$

6. (extra - 20%) 4 fair dice, each of a different color are rolled. Let X denote the number of distinct values observed. For example if the outcome of the dice were (2,3,5,1) then there are 4 distinct values. If the outcome was (2,2,3,3) then there are two distinct values observed. Compute $E[X]$.

   [Hint: Notice that X can take values from 1 to 4. Hence, use the formula $E[X] = \sum_{k=1}^{4} kP(X = k)$.]

Solution: We have

$$E[X] = 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) + 4 \cdot P(X=4).$$

Next,

$$\begin{aligned}
P(X=1) &= P(\text{ all dice have same value }) = 6(\frac{1}{6})^4 \\
P(X=2) &= P(\text{ two distinct values }) \\
&= P(\text{ two dice roll one value and the other two roll a different value}) \\
&+ P(\text{ three dice roll one value and the other one rolls a different value}) \\
&= \frac{1}{2}\binom{4}{2} \cdot 6 \cdot 5 \cdot (\frac{1}{6})^4 + \binom{4}{3} \cdot 6 \cdot 5 \cdot (\frac{1}{6})^4 \\
P(X=3) &= P(\text{ two dice have same value and the other two have two other values}) \\
&= \binom{4}{2} \cdot 6 \cdot 5 \cdot 4(\frac{1}{6})^4 \\
P(X=4) &= P(\text{ all dice have different values }) = 4!\binom{6}{4}(\frac{1}{6})^4.
\end{aligned}$$

Alternate Solution: For each $i = 1$ to 6 define a Bernoulli random variable X_i such that

$$X_i = \begin{cases} 1, & \text{at least one dice rolled } i \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned}
X &= \sum_{i=1}^{6} X_i \text{ and} \\
E[X] &= \sum_{i=1}^{6} E[X_i] \text{ [by linearity of expectation] }.
\end{aligned}$$

Finally we have for each i

$$\begin{aligned}
E[X_i] &= P(X_i = 1) \\
&= P(\text{ at least one dice rolled } i) \\
&= 1 - P(\text{ no dice rolled } i) \\
&= 1 - (\frac{5}{6})^4.
\end{aligned}$$