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206
Discrete Structures II

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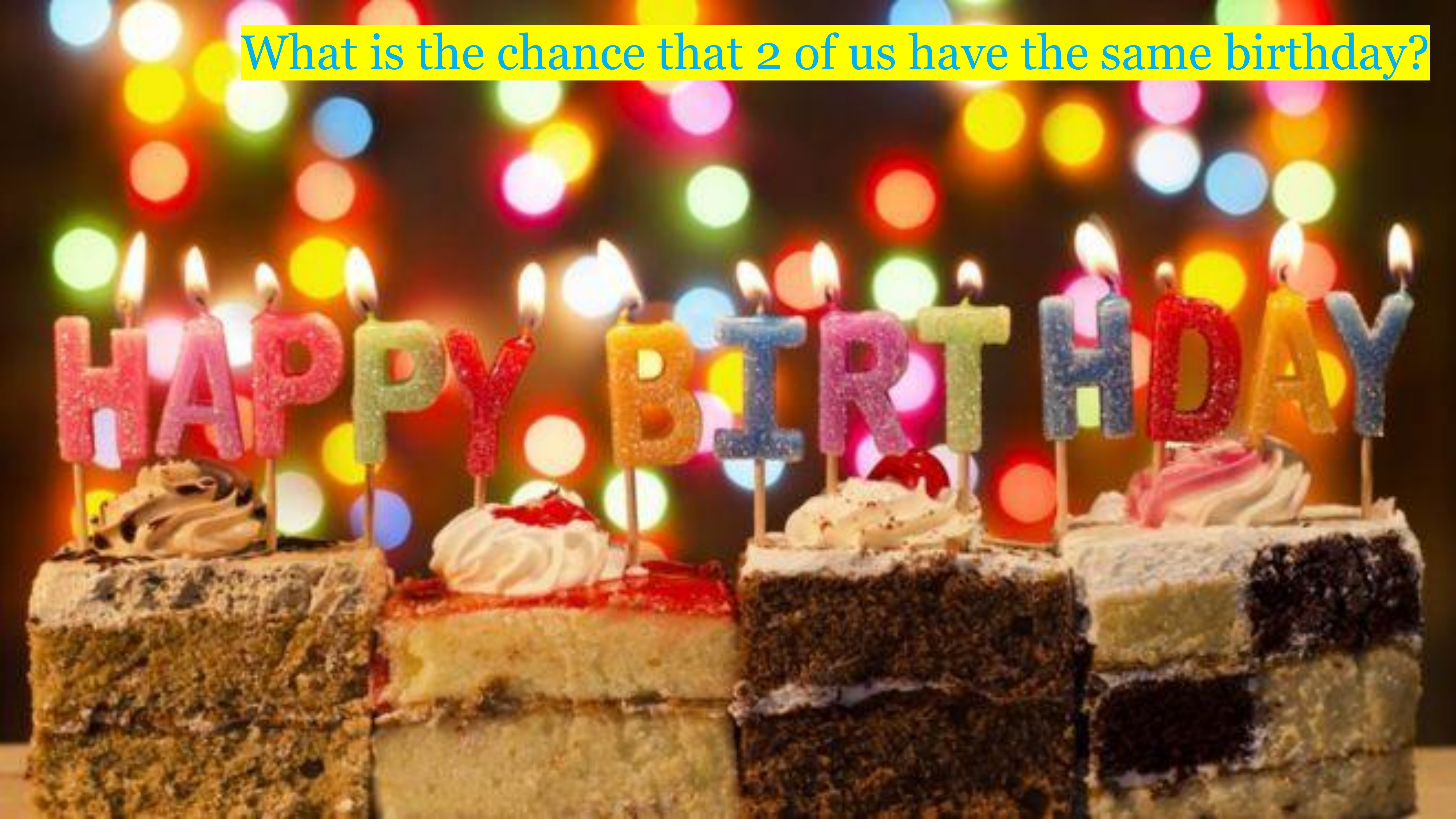
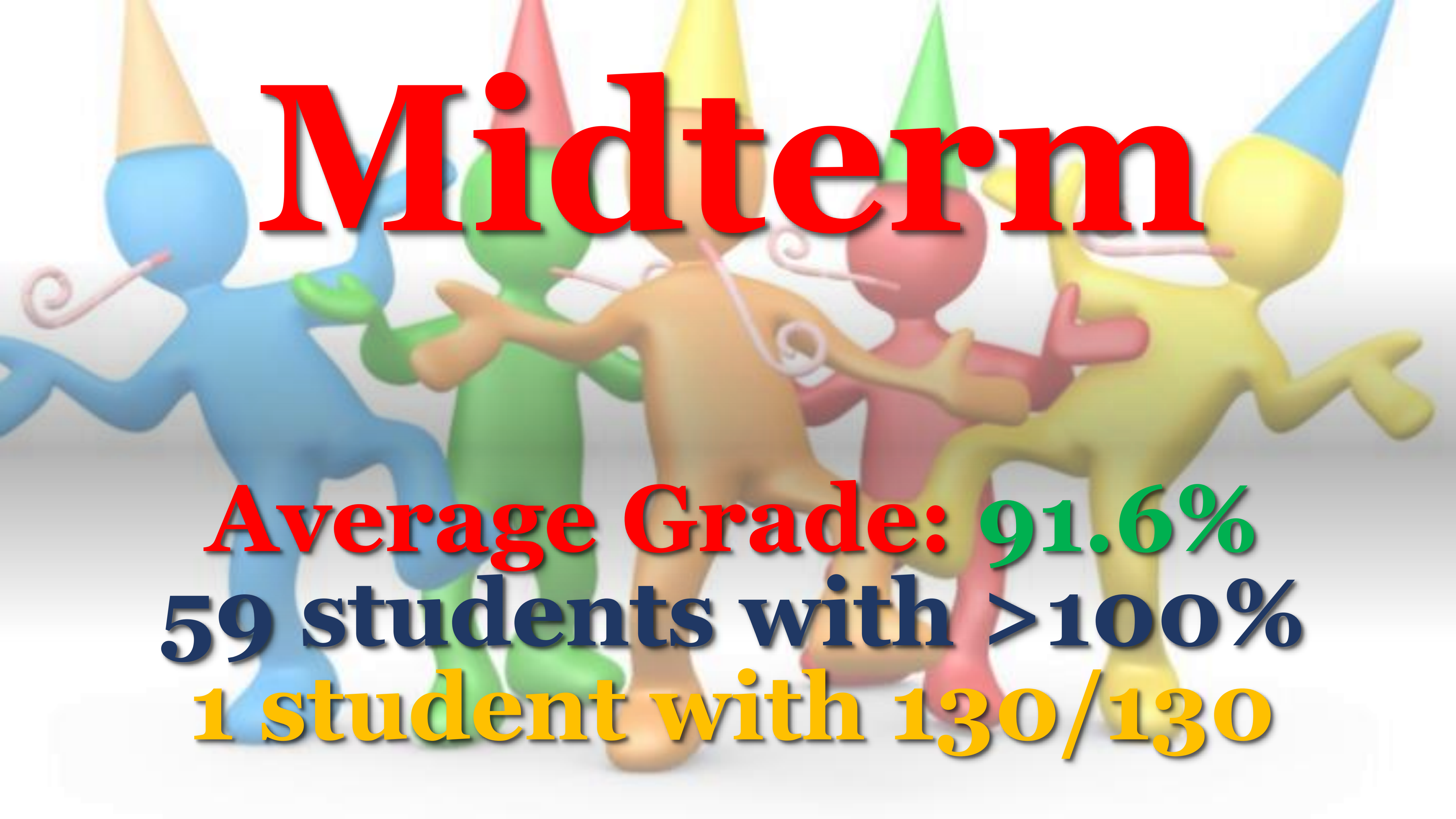
Lecture 20 | Conditional Probabilities| Monday November 27th 2023

What is the chance that 2 of us have the same birthday?




THE COMFORT ZONE
Safe and in control
COMFORT ZONE
Low risk Low reward
Low self-confidence
Find excuses
FEAR ZONE
Affected by others' opinions
Face challenges
Problem solve
LEARNING ZONE
Acquire new skills
Extend comfort zone
Find purpose
Live dreams
GROWTH ZONE
Set new goals
Conquer objectives



Midterm

Average Grade: 91.6%

59 students with >100%

1 student with 130/130

last class

Outline for this month

- Sample spaces and events
- Basics of probability
- Independence, conditional probability
- Random variables, expectation, variance

→ Basic building blocks

→ Intermediate

→ Advanced

Outline

- Sample space
- Axioms of probability
- Conditional probability
- Independence
- Bayes rule
- Random Variables
- Expectation

Quiz 5 (Lectures 19-20)
This Week

What Quiz 5 will cover

- Probability Definitions and Axioms

(Lectures 19-20)

- Know how to count a) sample space, and b) event space
  - Do not forget your abilities to count sets (pirates, difference, etc.)

- *Conditional Probabilities (The very basics)*

(Lectures 20-21)

Probability – so far…

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Simple Event
  - Any element of the sample space
- Compound Event
  - Subsets of the sample space
- Probability Distribution - Axioms

Events - Operations

- A' → Complement of A
- $A \cap B$ → intersection
- $A \cup B$ → union

Disjoint Events

- A and B are disjoint events if $A \cap B = \phi$

Roll 2 dice

— A: die 1 = 1, die 2 = 1

— B: die 1 = 2, die 2 = 2

$A \cap B = \phi$

— A: Sum of dice = 2 $\rightarrow$ (1,1)

— B: Sum of dice = 3 $\rightarrow$ (1,2), (2,1)

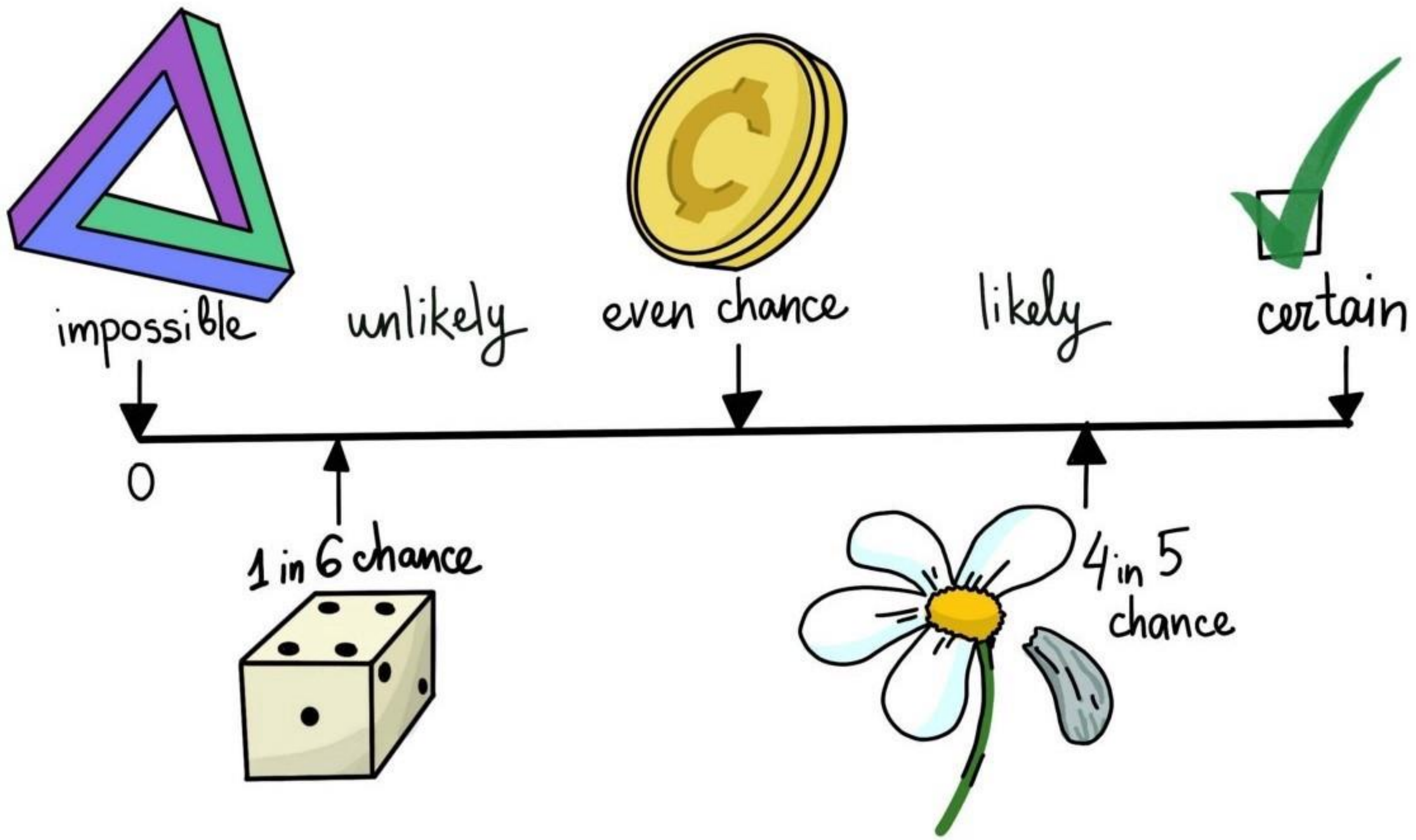
$A \cap B = \phi$

Probability

- Fix experiment and sample space Ω.

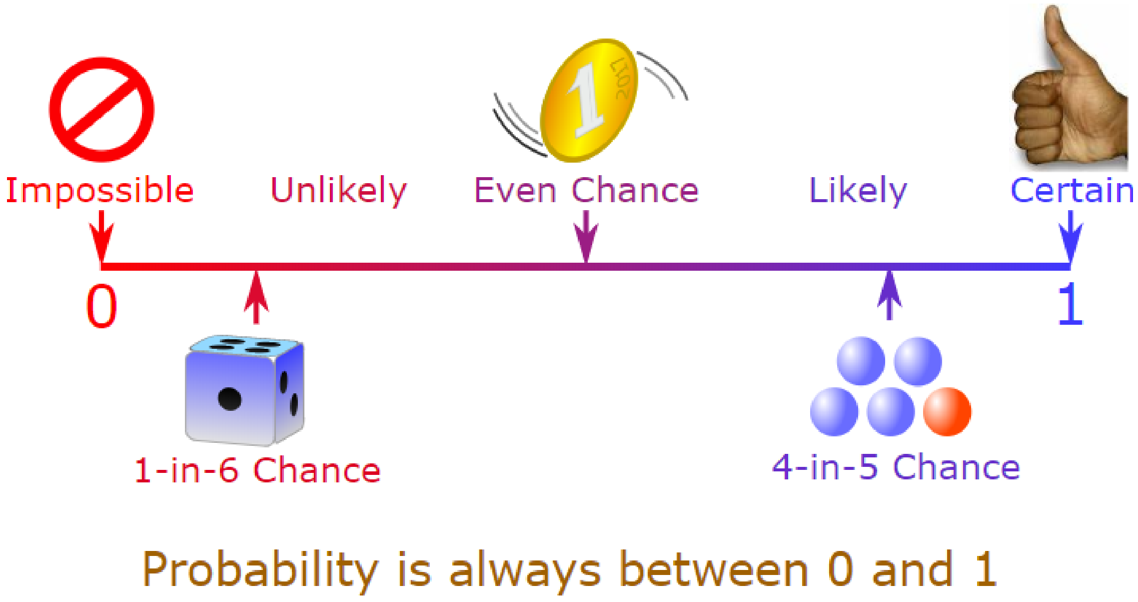
A probability distribution P assigns a number $P(A)$ to each event A.

- P needs to satisfy certain basic axioms.



Axioms of Probability

- $P(A) \geq 0$
- $P(\Omega) = 1$
- For a collection of disjoint events $A_1, A_2, \ldots$
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$



Equally Likely Outcomes

Consider experiment and a finite sample space Ω

- For every simple event $e \in \Omega$, assign $P(e) = \frac{1}{|\Omega|}$

- For every compound event A, assign $P(A) = \frac{|A|}{|\Omega|}$

- Then, P is a valid probability distribution.

Equally Likely Outcomes

Proof:

- $P(A) \geq 0$ since $|A| \geq 0$
- $P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$
- Let $A_1, A_2, \ldots$ be disjoint events. Then
  - $P(A_1 \cup A_2 \ldots) = \frac{|A_1 \cup A_2 \cup \cdots|}{|\Omega|}$

$$= \frac{|A_1|}{\Omega} + \frac{|A_2|}{\Omega} + \cdots = P(A_1) + P(A_2$$

- We have proved that all 3 axioms are true.

Inclusion/Exclusion for Probabilities

- $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
- Extends to more than 2 sets

Union Bound

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) \leq P(A) + P(B) \longrightarrow$ Union bound

$\longrightarrow$ Boole's inequality

Interpretation of Probability

- For a collection of disjoint events $A_1, A_2, \ldots$
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

→ sum rule for Probability

Interpretation of P(A)

If $P(A) = .6$

- If we repeat experiment N times (N is very large)
- Then the outcome will lie in A .6N of the times

Uniform Distribution (all outcomes equally likely)

- A fair coin is tossed 100 times. What is the probability that we get exactly 50 heads.

$\Omega \to \left\{ \begin{array}{c} (H, H, - - H) \\ (T, T, - - T) \\ \vdots \end{array} \right\}$, $|\Omega| = 2^{100}$

$A \to$ all outcomes with exactly 50 Heads, $|A| = \binom{100}{50}$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{100}{50}}{2^{100}}$$

Uniform Distribution (all outcomes equally likely)

- If we roll a white die and a black die (both fair), what is the probability that the sum is 7 or 11?

$A \to$ Sum is 7

$B \to$ Sum is 11

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = \frac{6}{36} + \frac{2}{36} = \frac{8}{36}$$

$|\Omega| = 36$

$|A| = |\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}| = 6$

$|B| = |\{(6,5),(5,6)\}| = 2$

Uniform Distribution (all outcomes equally likely)

- If we roll a white die and a black die (both fair), what is the probability that the sum is 7 or die 1 is more than 3?

$A \to$ sum is 7

$B \to$ die 1 more than 3

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

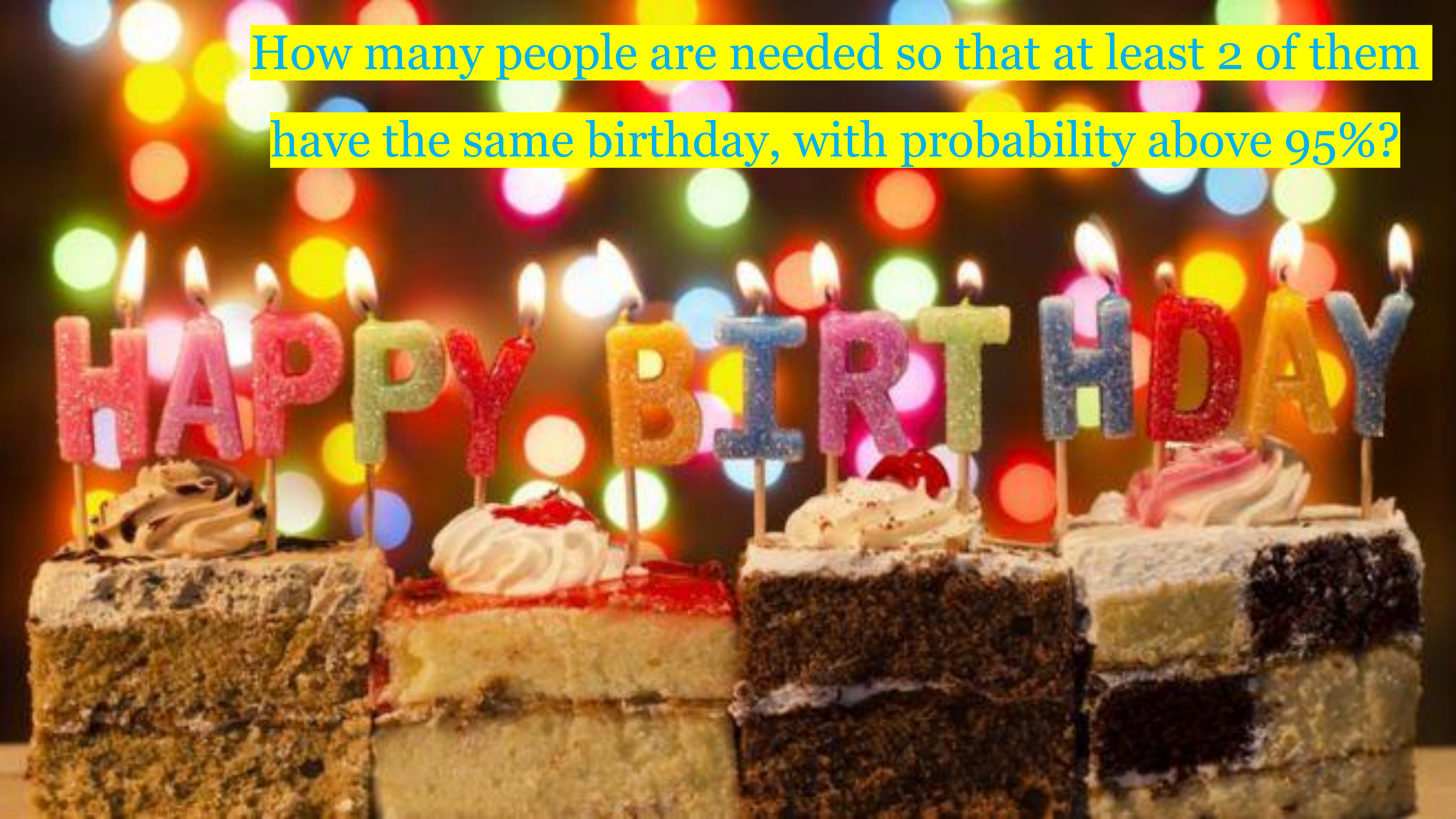
$$= \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|} = \frac{6}{36} + \frac{18}{36} - \frac{3}{36}$$

$|A| = 6$

$|B| = |\{(4,1) \ldots (4,6), (5,1) \ldots (5,6), (6,1) \ldots (6,6)\}| = 18$

$|A \cap B| = |\{(4,3), (5,2), (6,1)\}| = 3$

How many people are needed so that at least 2 of them have the same birthday, with probability above 95%?



Uniform Distribution

- 23 people are in a room. Suppose all birthdays are equally likely. What is the probability that two will have the same birthday?

Assume $\rightarrow$ 365 possible birth days

$\Omega \rightarrow$ all possible assignment of birthdays to 23 people

$|\Omega| = 365^{23}$

$A \rightarrow$ all outcomes where at least two have same birthday

Uniform Distribution

- 23 people are in a room. Suppose all birthdays are equally likely. What is the probability that two will have the same birthday?

$$P(A) = 1 - \frac{|B|}{|\Omega|}$$

B → all outcomes where no two have same birthday

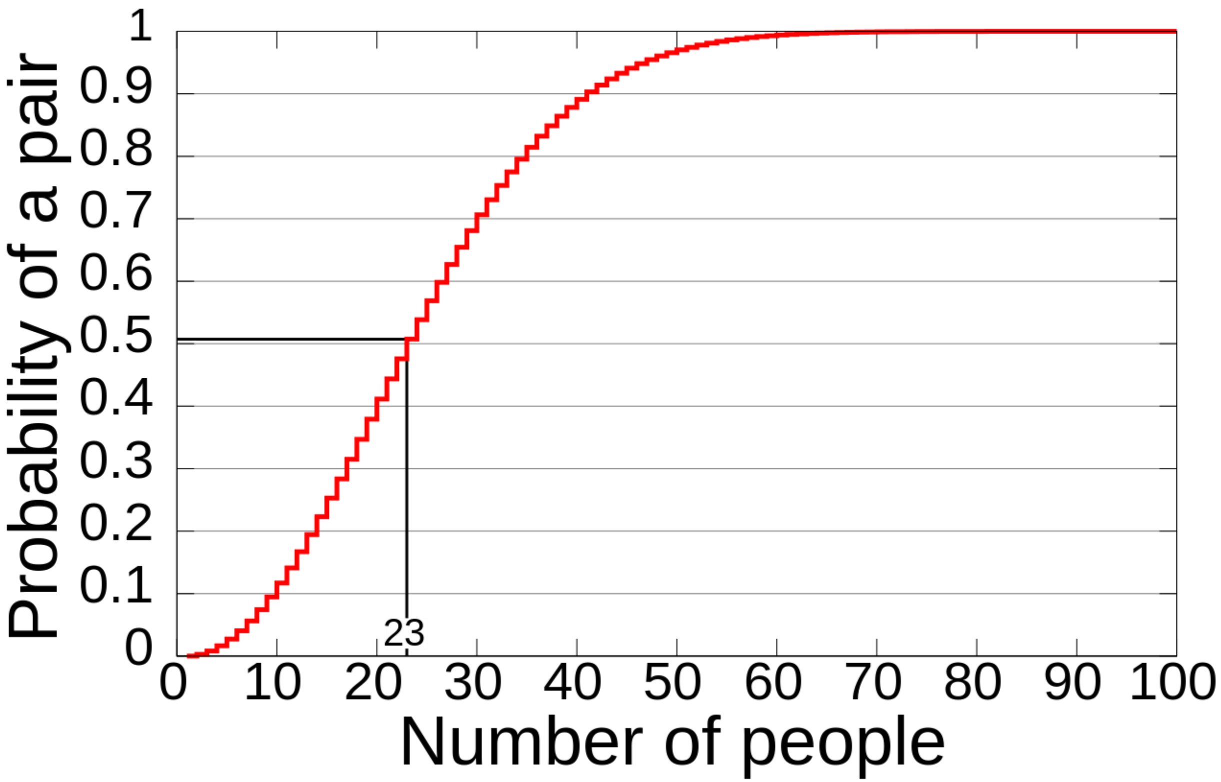
$$|B| = {}_{365}P_{23} = 365 \cdot 364 \cdot 363 - - \_\_$$

$$P(A) = 1 - \frac{365P_{23}}{(365)^{23}} \approx .5027$$

Uniform Distribution

- 23 people are in a room. Suppose all birthdays are equally likely. What is the probability that two will have the same birthday?

Birthday Paradox!!



Uniform Distribution

- 55% of adults consume coffee regularly, 45% consume soda and 70% regularly consume at least one of the two.
- What is probability that a randomly selected adult consumes both coffee and soda?

$A \rightarrow$ an adult consumes coffee regularly

$B \rightarrow$ an adult consumes soda regularly

$P(A) = .55, \; P(B) = .45, \; P(A \cup B) = .7$

Want: $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= .55 + .45 - .7$

Uniform Distribution

- 55% of adults consume coffee regularly, 45% consume soda and 70% regularly consume at least one of the two.
- What is the probability that a randomly selected individual doesn't consume either of the two.

$A \cup B$ = people who consume at least one of two

$(A \cup B)'$ = people who don't consume either

$P((A \cup B)') = 1 - P(A \cup B) = 1 - .7$

Uniform Distribution

identical identical identical

- A box contains six 40W bulbs, five 60W bulbs and four 75W bulbs. If bulbs are selected one by one in a random order, what is the probability that at least two bulbs must be selected in order to get one that is rated 75W?

$A \rightarrow$ at least 2 tries for seeing 75W

$A' \rightarrow$ see 75 w bulb on first try

$$P(A) = 1 - P(A') = 1 - \frac{|A'|}{|\Omega|}$$

$$|\Omega| = \frac{15!}{6!\,5!\,4!}, \quad |A'| = \frac{14!}{6!\,5!\,3!}$$

Conditional Probabilities - Example

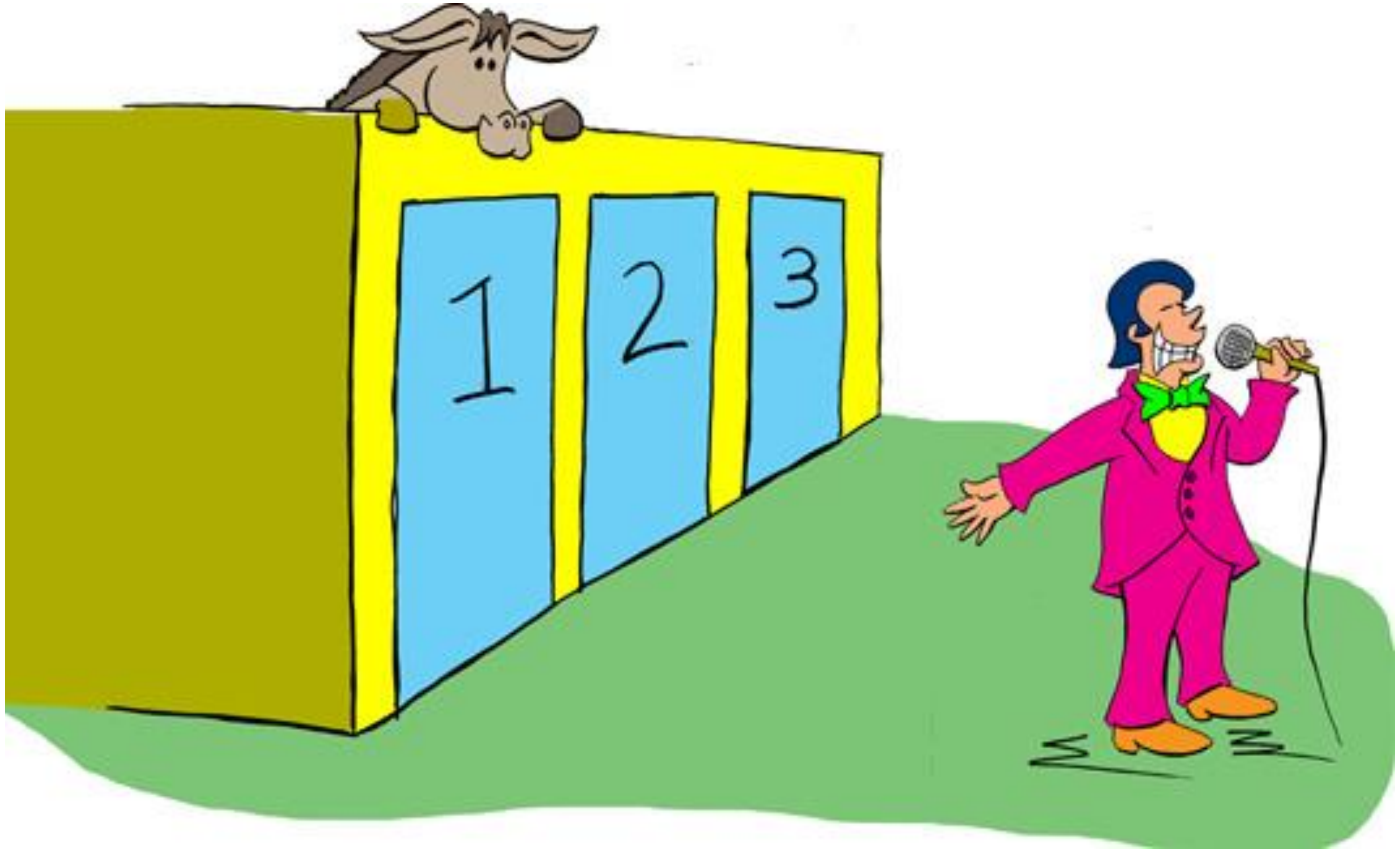
A = man survives

- A man went on an airplane ride.
- Unfortunately, he fell out. → P(A) = .1
- Fortunately, he had a parachute on. → P(A) = .7
- Unfortunately, the parachute did not open. → P(A) = .1
- Fortunately, there was a haystack below him, directly in the path of his fall. → P(A) = .5
- Unfortunately, there was a pitchfork sticking out of the top of the haystack.
- Fortunately, he missed the pitchfork. → P(A) = .1
- Unfortunately, he missed the haystack.


WARNING
DON'T
TRY THIS
AT HOME

Monty Hall Problem

Door 1 → G
Door 2 → G
Door 3 → Car




The New York Times

"All the News That's Fit to Print"

Quandary Created by Gain In Detecting Breast Cancer

New Methods Still Leave Doubts on What to Do

SAUDIS WOULD LIFT BOYCOTT IF ISRAEL HALTS SETTLEMENT

LEBANON SUPPORTS TALKS

States Seeking Aid on Budgets From Pensions

After 16 Years of War, Beirut Tentatively Reclaims Peace

U.N. Says Kurds Take Most Parts Of an Iraqi City

Behind Monty Hall's Doors: Puzzle, Debate and Answer?

Communities and Their Residents Age Gracefully

Monty Hall Problem

Door 1 → Goat
Door 2 → Goat
Door 3 → Car

- Announcer hides prize behind one of 3 doors. You select some door at random. Announcer opens one of others with no prize. You can decide to keep or switch.
- What to do?

$A \rightarrow$ I win If I switch

$B \rightarrow$ door with no prize is revealed

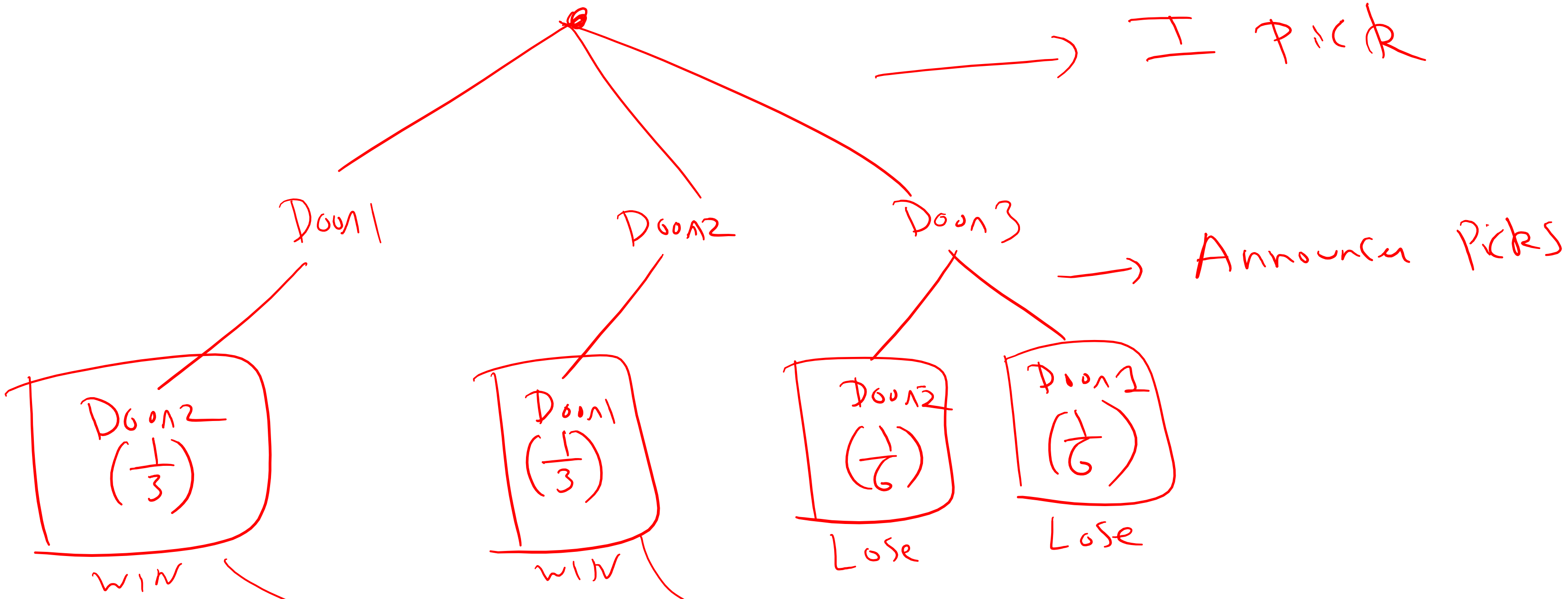
$P(A|B) > \frac{1}{2}$ or $< \frac{1}{2}$??

Will show $P(A|B) = \frac{2}{3}$

In this case $P(B) = 1$ [Because of rules of the game]

Monty Hall Problem

Door 1 → G
Door 2 → G
Door 3 → Car



$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$