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206
Discrete Structures II

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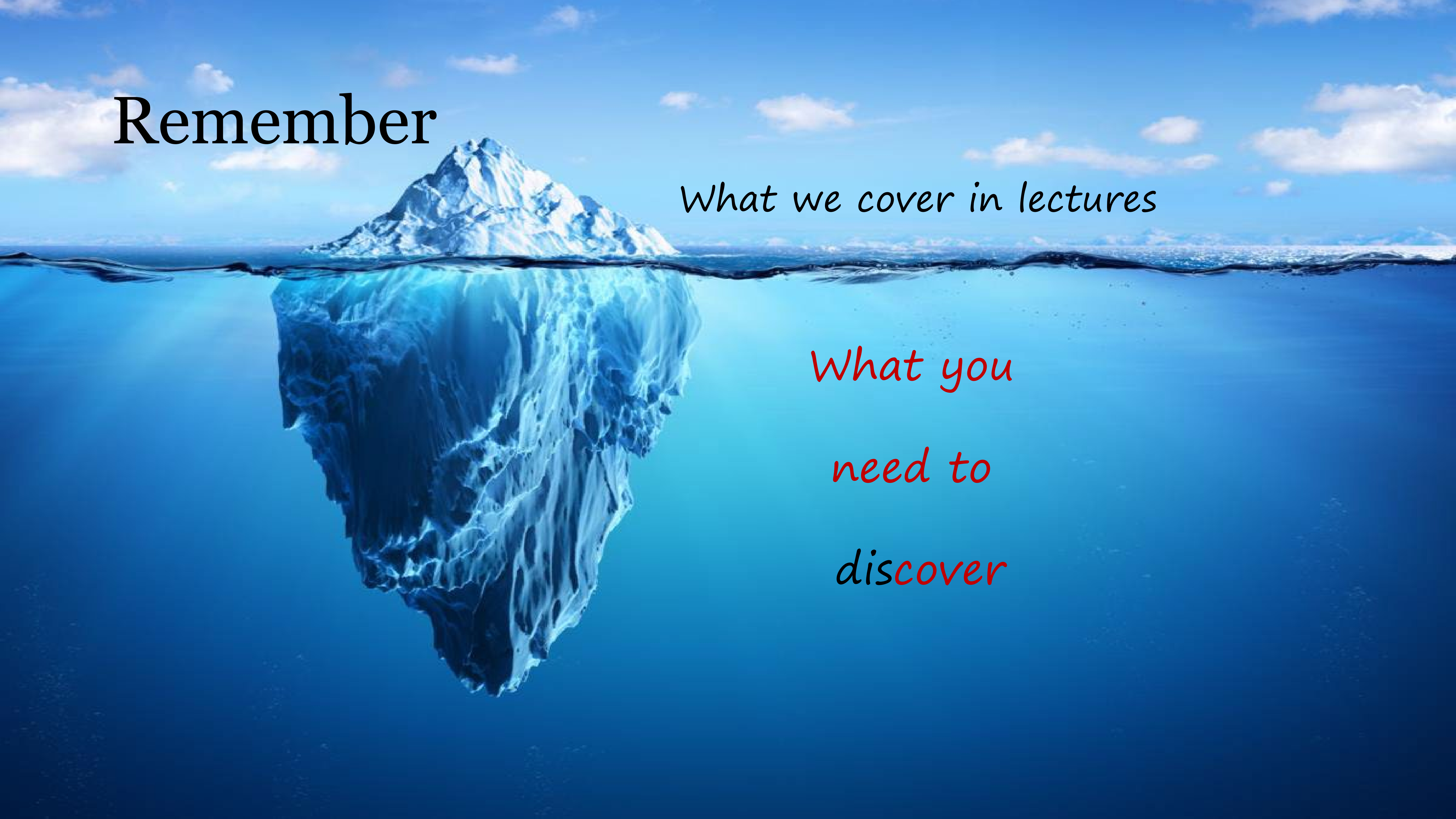
Lecture 3 | Sets – Functions - Proofs | Wednesday September 20th 2023


Remember
What we cover in lectures
What you
need to
discover

Reading for Quiz 1 (and beyond…)

Lecture 2	Recap and Basics of Counting	Chapters 1, 2 and 5 of Rosen
Lecture 3	Basics of Counting	Chapters 1, 2 and 5 of Rosen Chapter 15 of Lehman
Lecture 4	Basics of Counting	Chapters 6 of Rosen Chapter 15 of Lehman

Quiz 1 – Monday October 2 & **Wednesday** October 4

What we will cover today

- Recap
  - Sets - Set Operations – Venn Diagram
- Combinatorics Intro (Recap 205)
  - Functions
  - Proofs
  - Induction

Combinatorics

- The study of arrangements of objects
- Studied as long ago as the 17th century, when combinatorial questions arose in the study of gambling games
- Used to solve many different types of problems
  - Examples:

  Enumeration, the **counting of objects** ***with certain properties***

  1. Counting determines the complexity of algorithms
  2. Counting determines whether there are enough resources to solve a problem
  3. …

Used to solve many different types of problems

Enumeration, the **counting of objects** ***with certain properties***

Example:

1. Counting determines the complexity of algorithms
2. Counting determines whether there are enough resources to solve a problem.

Combinatorics

- Study of discrete structures
  - Counting structures of a given kind/size

```
function TARJAN(Node* node)
    node.visited ← true
    node.index ← indexCounter
    s.push(node)
    for all successor in node.successors do
        if !node.visited then TARJAN(successor)
        end if
        node.lowlink ← MIN(node.lowlink, successor.lowlink)
    end for
    if node.lowlink == node.index then
        repeat
            successor ← stack.pop()
        until successor == node
    end if
end function
```

What questions can you ask?

Combinatorics

Used to solve many different types of problems

Enumeration, the **counting of objects** ***with certain properties***

Example:

1. Counting determines the complexity of algorithms
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- Study of discrete structures
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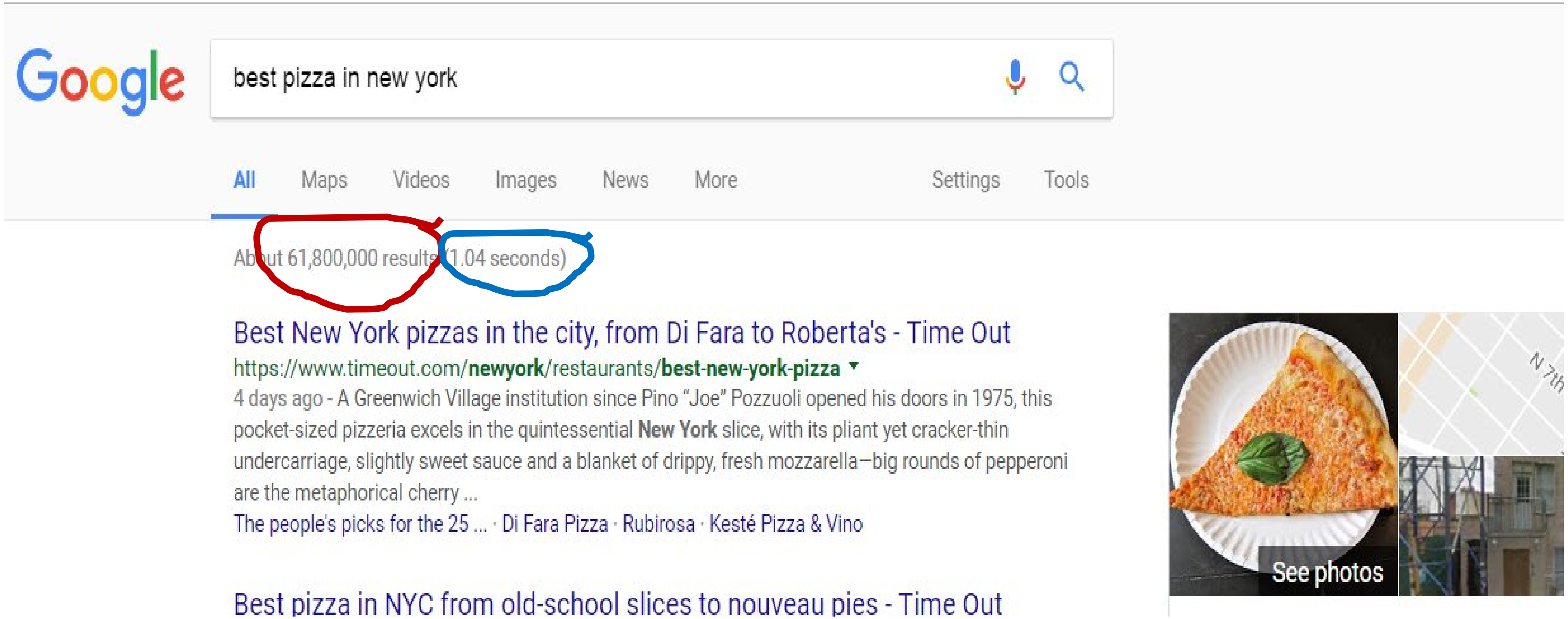
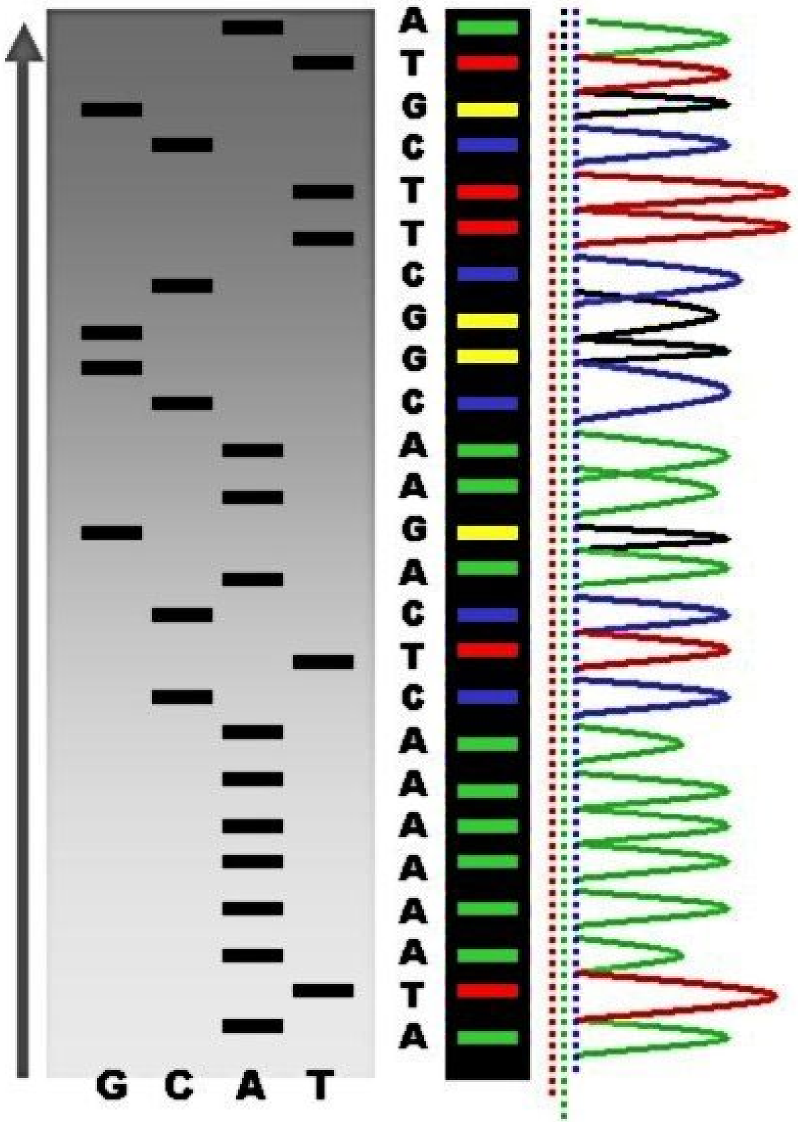
Complexity:
What is the runtime?

Resources:
What is the memory usage?

Combinatorics – Enumerating Example



Combinatorics



Recently, it has played a key role in

mathematical biology,

e.g., in sequencing DNA.

Rosen, 385

Combinatorics

- We will study the **basic rules of counting**
  - They can solve a tremendous variety of problems, such as:
    - Enumerate the **different telephone numbers** possible in the United States,
    - Enumerate the **allowable passwords** on a computer system,
    - Enumerate the different orders in which the runners in a race can finish
  - They can help us answer questions that seem hard: *What is the chance that among the 240 students in this class, we find 2 with the same birthday*?

- An important **combinatorial tool** is the *pigeonhole principle*: When objects are placed in boxes and there are more objects than boxes, then there is a box containing at least 2 objects.
  - E.g., we can use this principle to show that among a set of 15 or more students, at least 3 were born on the same day of the week

Combinatorics

Your Password:

- Must be different from your User ID
- Must contain 8 to 20 characters, including one letter and number
- May include one of the following characters: %, &, _, ?, #, =, -
- Your new password cannot have any spaces and will not be case sensitive.

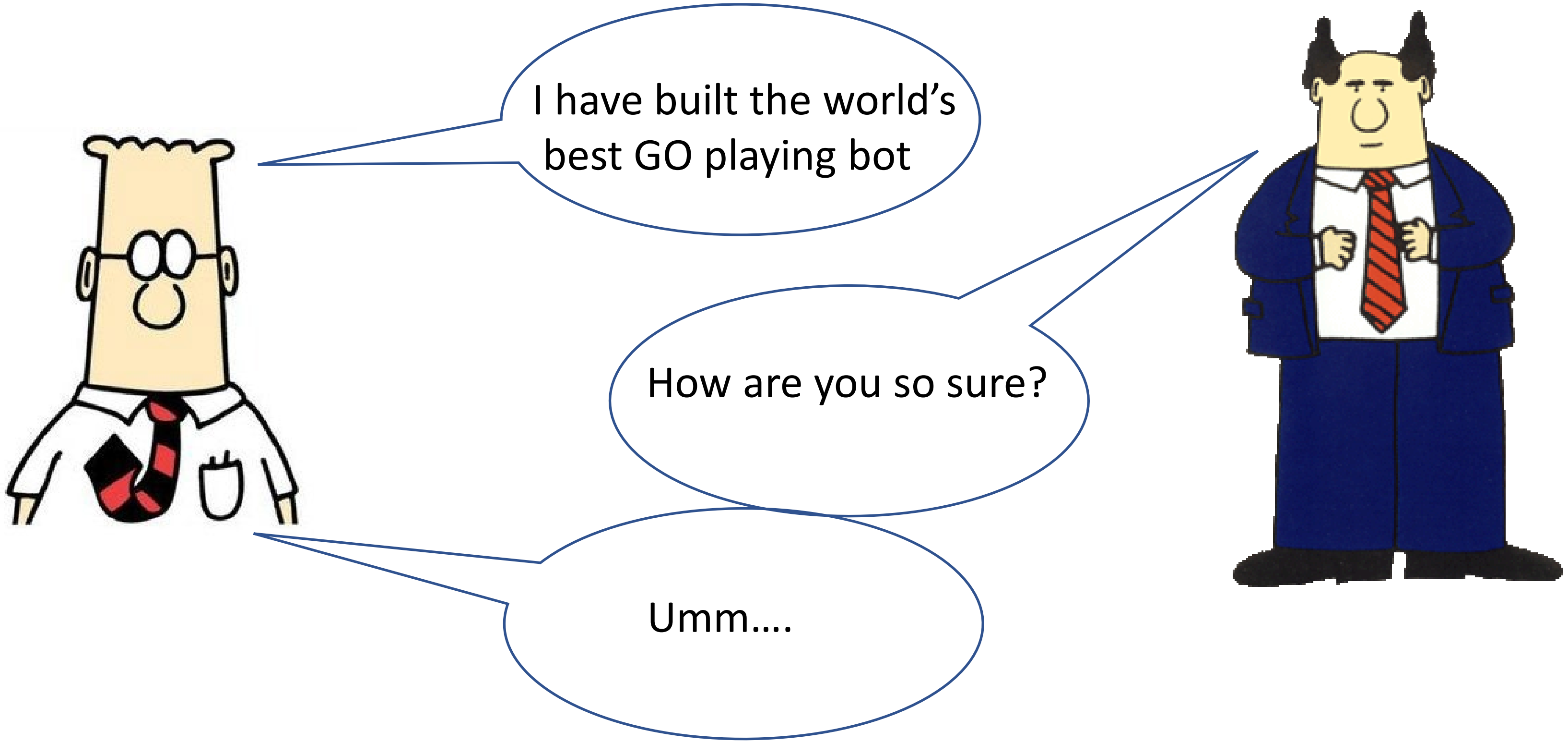
*REQUIRED FIELD

How many different passwords we can create?

Last Time

Combinatorics -> Probability Theory

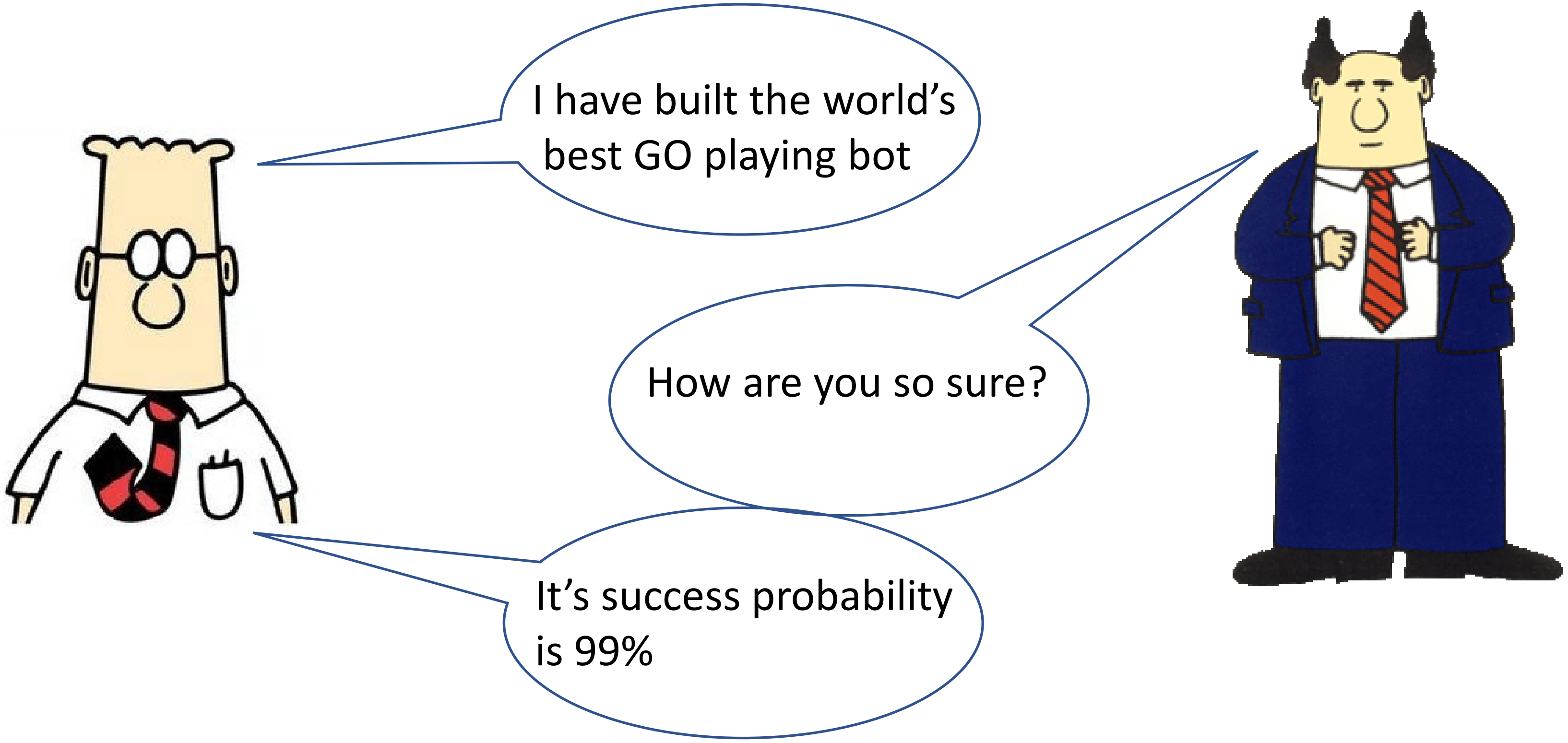
- Analysis of uncertain or random phenomena



Last Time

Probability Theory

- Analysis of uncertain or random phenomena



Last Time

Probability Theory

- Analysis of uncertain or random phenomena

Last Time

Probability Theory

- Analysis of uncertain or random phenomena

Last Time

Course Outline

- Part I
  - Recap of basics – sets, function, proofs, induction
  - Basic counting techniques
  - Pigeonhole principle
  - Generating functions
- Part II
  - Sample spaces and events
  - Basics of probability
  - Independence, conditional probability
  - Random variables, expectation, variance
  - Moment generating functions
- Part III
  - Graph Theory
  - Machine learning and statistical inference

Sets

- What is a *Set?*
  - A collection of objects which are called *elements*
  - Elements are objects that share the same property
- Examples
  - My followers on Twitter
  - The set of webpages for a given Google query
  - Collection of YouTube videos

Sets

- The order of elements is not significant, so $\{x, y\}$ and $\{y, x\}$ are the same set written two different ways.

- And what about $y = x$?
  - $\{x, x\} = \{x\}$

- The expression $e \in S$ asserts that e **is an element of** set S
  - E.g.,$32 \in S$ or $blue \notin S$

Sets – Common Sets

- What is a *Set?*
  - A collection of objects which are called *elements*.
- Some common sets in Math

curly braces

• $\emptyset$	Empty set	{}
• $\mathbb{N}$	Nonnegative integers	{0,1,2,3,…}
• $\mathbb{Z}$	Integers	{…,-2,-1,0,1,2,…}
• $\mathbb{Q}$	Rational numbers	{1/2, 16, -5/2}
• $\mathbb{R}$	Real numbers	$\{\pi, e, -9, \sqrt{2}\}$
• $\mathbb{C}$	Complex numbers	$\{i, 19/2, \sqrt{2}\text{-}2i\}$

A superscript "+" restricts a set to its positive elements; for example, $\mathbb{R}^+$ denotes the set of positive real numbers. Similarly, $\mathbb{Z}^-$ denotes the set of negative integers

Sets - Set Operations

Example
$X ::= \{1,2,3\}$
$Y ::= \{2,3,4\}$

- Union: $X \cup Y$
  - All elements present in X or Y or both.
- Intersection: $X \cap Y$
  - All elements present in *both* X and Y.
- Difference: $X \setminus Y$
  - All elements present in X but not in Y.
  - Not symmetric!
- Product: $X \times Y$
  - Collection of all tuples (a, b) where $a \in X$ and $b \in Y$.
- Size: $|X|$
  - Number of elements in X.

$X \cup Y = \{1,2,3,4\}$

$X \cap Y = \{2,3\}$

$X \setminus Y = \{1\}$

$Y \setminus X = \{4\}$

$X \times Y = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$

$|X| = 3$

$|Y| = 3$

Sets - Set Comparisons

- Subset: $X \subset Y$
  - Every element present in X is also present in Y.
  - X is not the same as Y.

$X = \{1\}, Y = \{1, 2, 3\}$

$X \subset Y$

- Superset: $X \supset Y$
  - Every element present in Y is also present in X.
  - X is not the same as Y.

- Note: There is a direct analogy between [$\subset$ and $<$] and [$\subseteq$ and $\leq$]

Last Time

Power Set

$X = \{1,2,3\}$

$Power(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

- Let X be a set.
- $Power(X)$ = set of all subsets of X

- E.g., $Power(\{1,2\}) = \{1\}, \{2\}, and\ \{1,2\}$

- Is this correct?
  - NO!
  - $Power(\{1,2\}) = \{1\}, \{2\}, \{1,2\}, \boldsymbol{and}\ \{\}$

- Generally, if A has n elements, then there are 2^n sets in $Power(A)$

Last Time

Set Builder Notation

- Often sets cannot be fully described by listing the elements explicitly or by taking unions, intersections, etc., of easily-described sets
- **Set builder notation** often comes to the rescue
- The idea is to define a set using a predicate; in particular, the set consists of all values that make the predicate true

Examples:

- $X = \{n \in \mathbb{N} : n \text{ is prime}\}$
- $Y = \{x \in \mathbb{R} : x^3 - 3x + 1 > 0\}$
- $Z = \{z \in YouTube\_videos : z \text{ is less than } 3 \text{ minutes long}\}$

Last Time

Exercise 1: Put everything together

$A = \{0,1,2\}$ $B = \{1,4,9\}$

- Let $A = \{n \in N : n^2 < 7\}$ and $B = \{1,4,9\}$

Find

- $A \cup B$ $A \cup B = \{0,1,2,4,9\}$
- $A \cap B$ $A \cap B = \{1\}$
- $A \times B$ $A \times B = \{(0,1),(0,4) \dots$
- $A \setminus B$ $A \setminus B = \{0,2\}$

Last Time

Venn Diagram

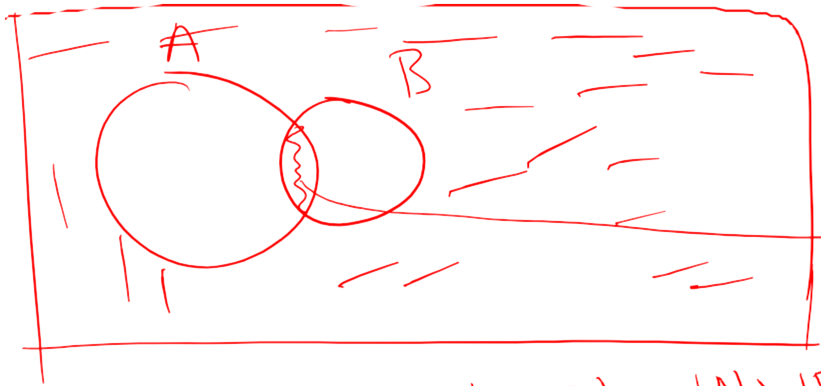
- Represent sets as circles and elements as points within it.
- Elegant way to capture relationships among sets.

Last Time

Exercise 2: Venn Diagram

- There are 131 students in CS 206.
- 100 like chocolate ice cream. 50 like vanilla ice cream.
- 20 like both chocolate and vanilla ice cream.
- Draw a Venn diagram to represent this.
- How many students do not like either flavor of ice cream.

A = chocolate
B = vanilla



$|A| = 100$
$|B| = 50$
$|A \cap B| = 20$

Want $131 - |A \cup B|$, $|A \cup B| = |A| + |B| - |A \cap B| = 130$

$\Rightarrow 131 - |A \cup B| = 1$

Functions

- What is a *function?*
  - A function *assigns* an element of one set to an element of another set
  - The **mapping** is done from one set, called *domain*, to another set, called *codomain*
  - Notation $f: A \mapsto B$
- Examples
  - $f: \mathbb{R} \mapsto \mathbb{R}$
  - $x \mapsto 4x^2$

  The familiar notation $f(a) = b$ indicates that f assigns the element $b \in B$ to a. Here b would be called the value of f at argument a
  - Example using a formula for b: $f(x) = 4x^2$

Functions - Example

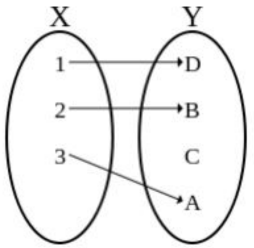
- **Algorithms are functions**
  - Example:
  - Let $X = set\ of\ all\ web\ pages$
  - PageRank: $X \mapsto \mathbb{R}$

Types of Functions

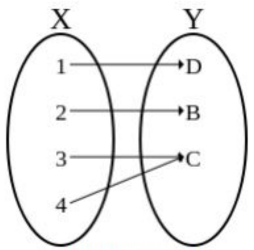
- Injection (one-to-one)
  - $f: X \mapsto Y$ is injective if each $x \in X$ is mapped to a *different* $y \in Y$.

  This function *preserves distinctness* as it never maps distinct elements of its domain to the same elements of its codomain.

- Subjection (onto)
  - $f: X \mapsto Y$ is subjective if each y $\in Y$, there exists $x \in X$ such that $f(x) = y$.

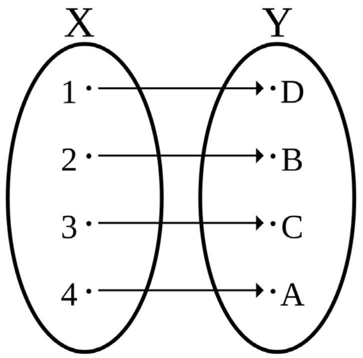


one-to-one (injective), but not onto

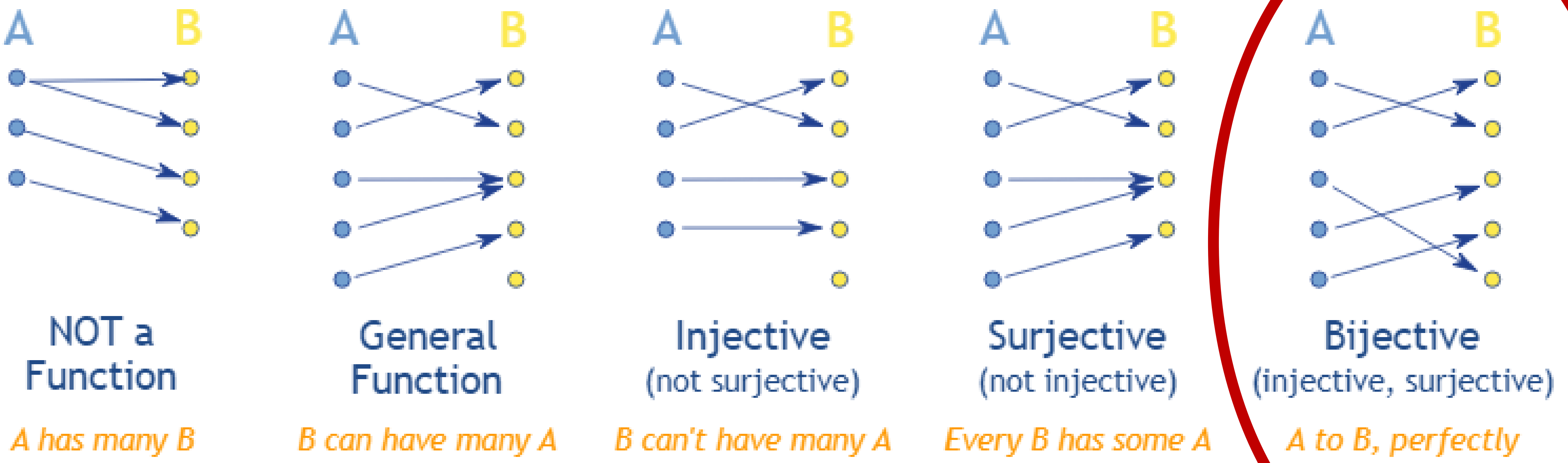


onto (surjective), but not one-to-one

- Bijection
  - $f: X \mapsto Y$ is a bijection if it is both one-to-one and onto.



Types of Functions



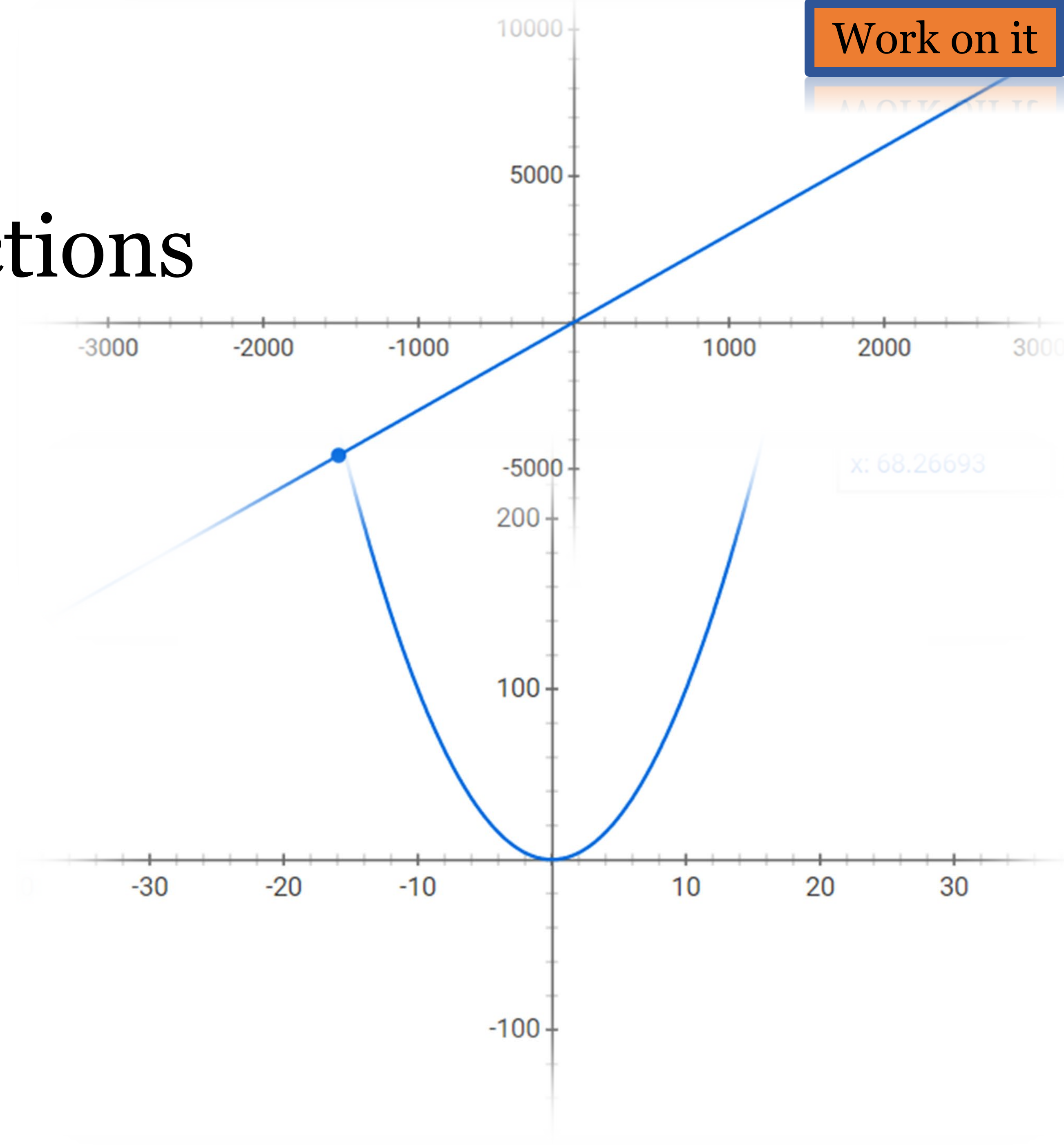
Why bijection is important to us? It allows us to count another set (codomain) and find the number of elements in another set (domain)

Work on it

Exercise 3: Types of Functions

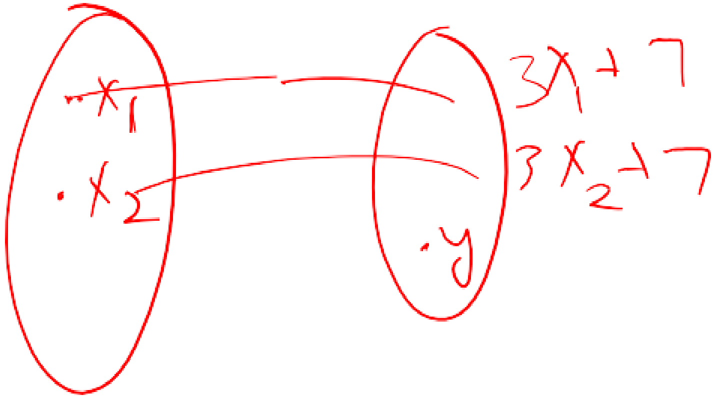
- $f: \mathbb{R} \mapsto \mathbb{R}, f(x) = 3x + 7$

- $f: \mathbb{R} \mapsto \mathbb{R}, f(x) = x^2$



Work on it

Exercise 3: Types of Functions



- $f: \mathbb{R} \mapsto \mathbb{R}$, $f(x) = 3x + 7$

— $f(x) = 3x+7$ is one-to-one

Since if x_1 and x_2 are different then $3x_1+7$ and $3x_2+7$ are also different

— $f(x) = 3x+7$ is onto since for any real value y, $f\left(\frac{y-7}{3}\right) = y$

Hence $f(x) = 3x+7$ is a bijection

Work on it

Exercise 3: Types of Functions

- $f: \mathbb{R} \mapsto \mathbb{R}, f(x) = x^2$

- $f(x) = x^2$ is not one-to-one since
  $f(2) = f(-2) = 4$

- $f(x) = x^2$ is not onto since for
  $y = -3$, no real value x exists such
  that $f(x) = -3$

What we will cover today

- Recap
  - ~~Sets - Set Operations~~
  - ~~Venn Diagram~~
- Combinatorics Intro (Recap 205)
  - ~~Functions~~
  - Proofs
  - Induction

Proofs

The first two Proofs we will learn:

- Direct Proof

- Proof by Contrapositive

Proofs

- A mathematical proof…
  - …of a **proposition** is a chain of logical deductions from axioms and previously proved statements.

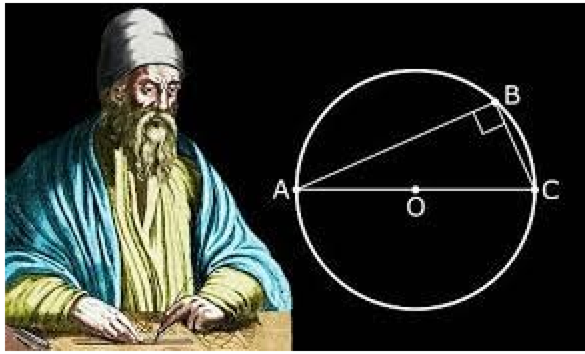
A ***prime*** is an integer greater than 1 that is not divisible by any other integer greater than 1, e.g., 2, 3, 5, 7, 11, . . .

- Proposition
  - A statement that is either *true* or *false*

e.g., Every even integer greater than 2 is the sum of two primes (Goldbach's Conjecture – remains unsolved since 1742…)

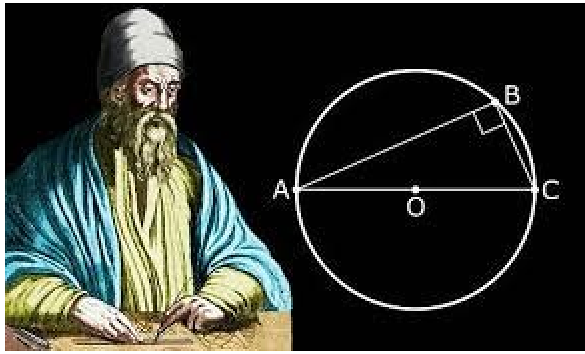
- Predicates
  - A proposition whose truth depends on the value of variables
  - e.g., $P(n)$ ::= "$n\ is\ a\ perfect\ square$" – $P(4)$ is true but $P(5)$ is false

Axioms



- A standard procedure for establishing truth in mathematics was invented by Euclid, a Greek mathematician working in Alexandria, circa 300 BC.
- He began with 5 assumptions about geometry, which seemed undeniable based on direct experience. (For example, "There is a straight line segment between every pair of points.)
- Propositions like these that are **simply accepted as true** are called axioms
- Starting from these axioms, Euclid established the truth of many additional propositions by providing "proofs."

Axioms



- Euclid's axiom-and-proof approach, now called the *axiomatic method*, remains the foundation for mathematics today.

- In fact, just a handful of axioms, called the axioms Zermelo-Frankel with Choice (ZFC), together with a few logical deduction rules, ***appear to be sufficient to derive essentially all of mathematics***

Logical Deductions (or Inference Rules)

Used to prove new propositions using previously proved ones

- $\dfrac{P, P \Rightarrow Q}{Q}$
  - If P is true and P implies Q, then Q is true.
- $\dfrac{P \Rightarrow Q, Q \Rightarrow R}{P \Rightarrow R}$
  - If P implies Q and Q implies R, then P implies R.
- $\dfrac{\neg P \Rightarrow \neg Q}{Q \Rightarrow P}$
  - If $\neg P$ implies $\neg Q$, then Q implies P

If we can prove this …

$$\frac{\boxed{\textit{antecedents}}}{\boxed{\textit{consequent}}}$$

…then this is true

Proving an Implication via Direct Proof

- To prove: $P \Rightarrow Q$
  - Assume that P is true.
  - Show that Q logically follows

Work on it

Direct Proof

- To prove: $P \Rightarrow Q$
  - Assume that P is true.
  - Show that Q logically follows

The sum of two even numbers is even.

Proof $x = 2m, y = 2n$

$x+y = 2m+2n$

$= 2(m+n)$

The product of two odd numbers is odd.

Proof $x = 2m+1, y = 2n+1$

$xy = (2m+1)(2n+1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn+m+n) + 1$

Work on it

Example of Proving an Implication

P   Q

- Theorem: $1 \leq x \leq 2 \Rightarrow x^2 - 3x + 2 \leq 0$

Assume $1 \leq x \leq 2$

Step1: $x^2 - 3x + 2 = (x-1)(x-2)$

Step2: $1 \leq x \Rightarrow (x-1) \geq 0$

Step3: $x \leq 2 \Rightarrow (x-2) \leq 0$

Step4: $(x-1) \geq 0, (x-2) \leq 0 \Rightarrow (x-1)(x-2) \leq 0$

Intuition: When x grows, 3x grows faster than x^2 in that range.

New Stuff

What we will cover today

- Recap
  - ~~Sets – Venn – Functions – Proofs (Direct)~~
- Combinatorics
  - ~~Proofs~~
    - ~~Direct~~
    - Contrapositive
    - Case Analysis
    - Contradiction
    - Induction
  - Counting
    - Partition Method
    - Difference Method