Recitation 7

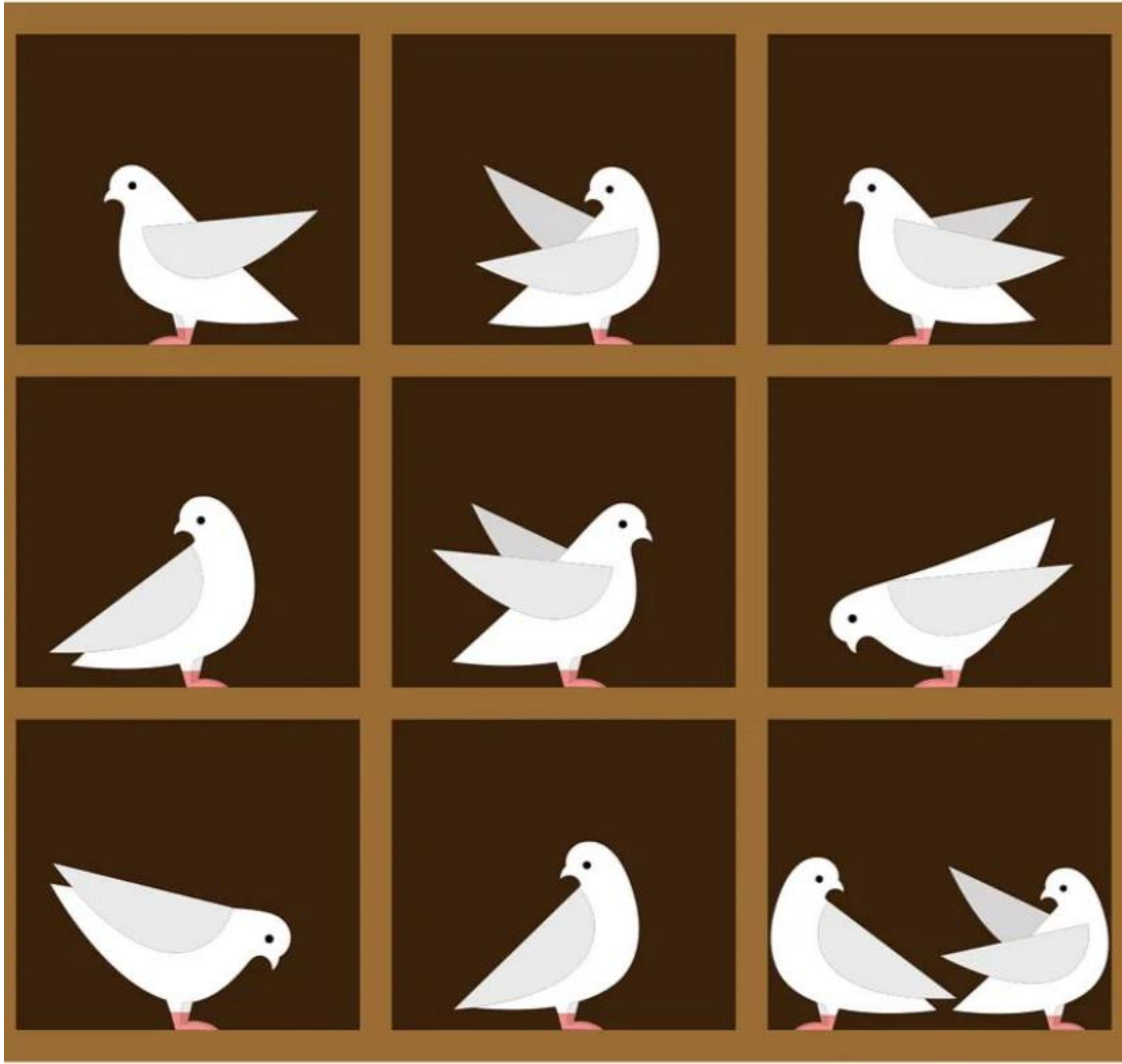
Overview

Pigeonhole Principle

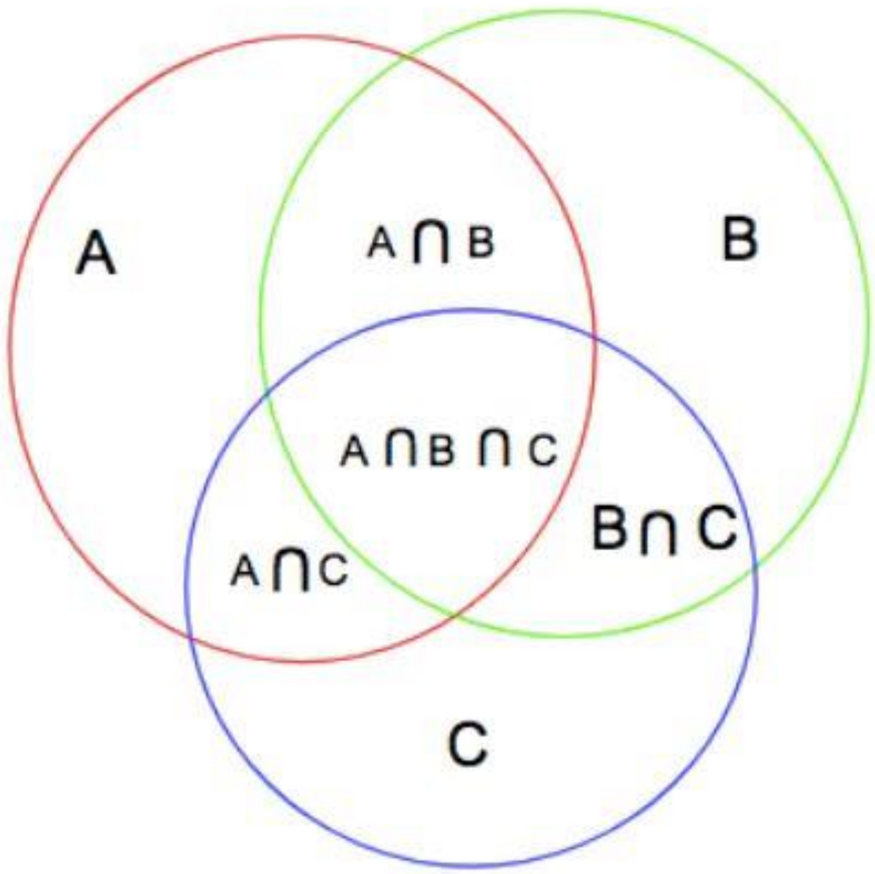
Inclusion-Exclusion Principle

Pigeonhole Principle

If you put n items
into m containers and $n > m$, then
at least one container must
contain more than one item.



Inclusion-Exclusion Principle



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

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After grabbing 7 socks, worst case scenario, I have grabbed a sock of each color. Thus, after grabbing one more sock, it has to match up with one of the previous socks so after grabbing 8 socks I am guaranteed to have a pair.

1. b)

What if there are likewise colored pairs of gloves in there and I cannot tell the difference between gloves and socks and I want a matching set?

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After grabbing 21 objects, it is possible that I have grabbed 3 items for each color and hence have gotten no sets yet. But the 22nd thing I grab must complete one of these 7 sets so after 22 items, I am guaranteed to have a matching set.

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Solution: There are $10 \cdot 10^6 + 1$ different number of lines of code you can write. So, there exists a number of line of codes with at least $\lceil 300 \cdot 10^6/(10^6 + 1) \rceil = 30$ people.

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Solution: Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from the generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements. In the former case, suppose that B, C, and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise, B, C, and D form a set of three mutual enemies. The proof in the latter case, when there are three or more enemies of A, proceeds in a similar manner. ◀

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Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade. ◀

1. Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?

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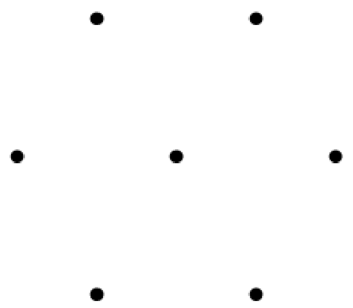
Solution:

Solution. Let M, S, and C denote the sets of students who study math, science and computing respectively and let U be the entire set of 18 students. Then $|M| = 7, |S| = 10$, and $|C| = 10$. Also, we have $|MS| = 3, |MC| = 4$, and $|SC| = 5$, where, $|x|$ denotes the number of elements of the set x and juxtoposition of sets means intersection. Finally, $|MCS| = 1$. Then

$$|U|-(|M|+|S|+|C|-|MS|-|MC|-|SC|+|MSC|) = \overline{MSC}| = 18-(27-12+1) = 2.$$

5. How many equilateral triangles have at least two vertices in the hexagonal lattice shown?



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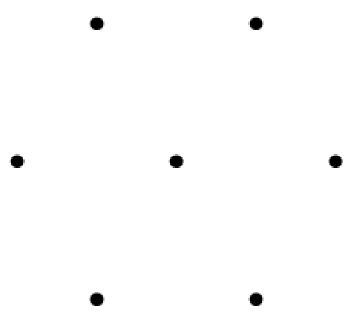

Solution: Solution. There are 6 "area 1" triangles that have the center point as a vertex and 6 more area 1 triangles that don't. There are 4 triangle with area 3, two of whose vertices are in the set $\{B, D, F\}$ and two others with vertices in the set $\{A, C, E\}$. Finally there are 6 triangles of area 4 (two each with edges AD, BE, and CF. Thus the total is $12 + 8 + 6 = 26$.