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Strive not to be a success, but rather to be of value — Albert Einstein

206
Discrete Structures II

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Lecture 11 | Permutations, Combinations & The Almighty Pirates Problem | Wednesday October 18th 2023

Preview…

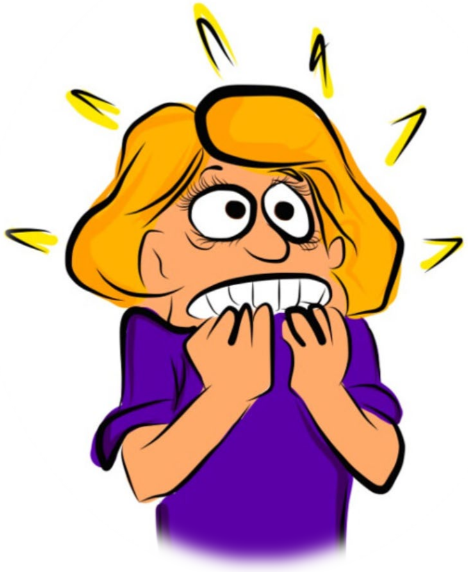
k distinct pirates want to divide up n identical, indivisible bars of gold. How many ways to divide the loot when each must get at least r bars?

Preview...

How many integer solutions to the following equation?

$$x_1 + x_2 + \cdots + x_k = n$$
$$x_1, x_2, \ldots, x_k \geq 0$$

Quiz 3 – When and What?



- When
  - Monday 10/23 and Wednesday 10/25 during recitation
- What will cover
  - Sum/Product rules
  - Permutations with and without repetitions (Up to this lecture)

BTW - Have you seen the Extra Problems?

Extra_Problems_1_Sum and Product Rules.pdf

Extra_Problems_2_Combinations_Permutations.pdf

So Far

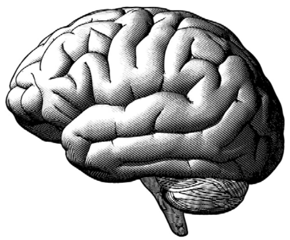
- ~~Sets / Functions~~
- ~~Proofs~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Last Class

- Permutations
- Combinations

Today

- Nothing

Get your  in gear

- Prove that 531!472! is a divisor of 1003!

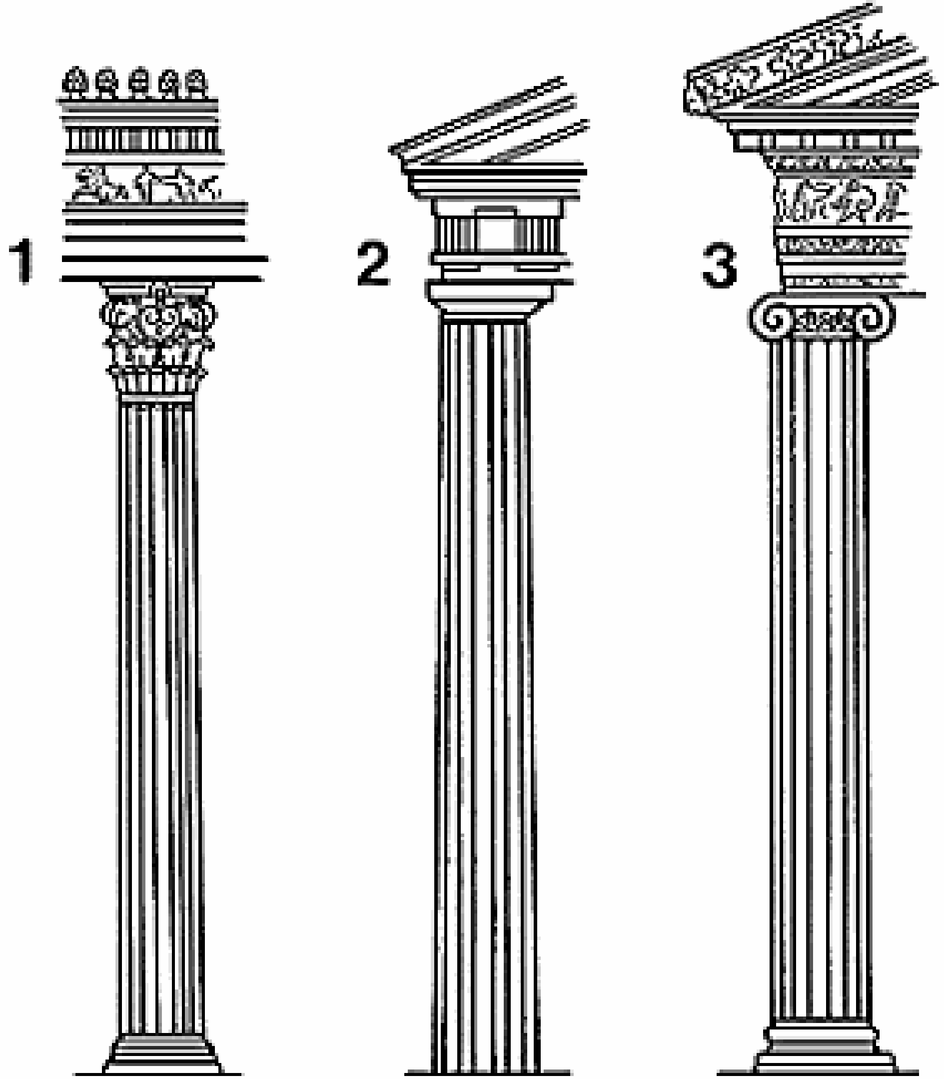
Solution: $\frac{1003!}{531!472!} = \binom{1003}{531} \rightarrow integer$

Why *integer*?

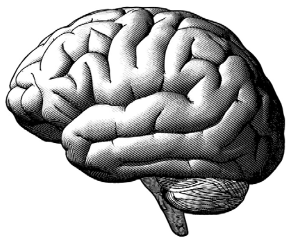
Well… There are *integer* ways we can arrange things

So far...

Product Rule



order is important

Get your  in gear

- How many length n binary strings are there in which 011 occurs starting at the 4th position?

_ _ _ 0 1 1 _ _ _

Partition Method

If $n < 6$ answer is 0

For $n > 6$ → For the three places before 011 there are 7 valid outcomes (excluding 011)

Difference Method

→ places after 011 have 2 choices each

Product Rule

answer → $7 \cdot 2^{n-6}$

So far...

Permutations

- Distinctly ordered sets are called permutations (arrangements). The number of permutations of n distinct objects taken k at a time is given by:

$$P^{n}_{k} = \frac{n!}{(n-k)!}$$

n = number of distinct objects
k = number of positions

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

So far...

Permutations Formula – Remember!

$$P_k^n = \frac{n!}{(n-k)!}$$

The formula for permutations is similar to the combinations formula, except we needn't divide out the permutations, so we remove k! from the denominator

So far...

Permutations without Repetitions

A maths debating team consists of 4 speakers.

- In how many ways can all 4 speakers be arranged in a row for a photo?

Solution : **$4 \times 3 \times 2 \times 1 = 4!$ or $^4P_4$**

- How many ways can the captain and vice-captain be chosen?

Solution : **$4 \times 3 = 12$ or $^4P_2$**

So far…

Permutations without Repetitions

<u>A flutter on the horses</u>

There are 7 horses in a race.

- In how many different orders can the horses finish?

Solution : $\mathbf{7x6x5x4x3x2x1 = 7!\ or\ {}^{7}P_{7}}$

- How many trifectas (1^{st}, 2^{nd} and 3^{rd}) are possible?

Solution : $\mathbf{7x6x5 = 210\ or\ {}^{7}P_{3}}$

Permutations with Restrictions

In how many ways can 5 boys and 4 girls be arranged on a bench if

- there are no restrictions?

Solution : 9! or $^9P_9$

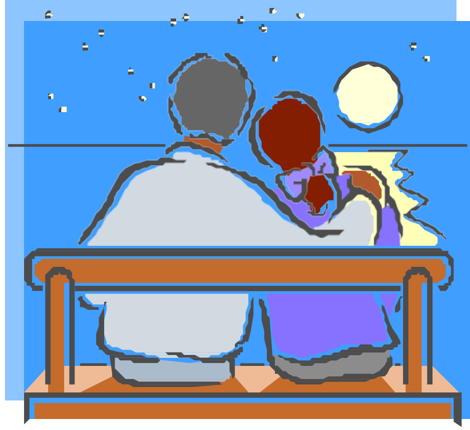
- boys and girls alternate?

Solution : A boy will be on each end

$$BGBGBGBGB = 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$$

$$= 5! \times 4! \text{ or } {}^5P_5 \times {}^4P_4$$

Permutations with Restrictions



In how many ways can 5 boys and 4 girls be arranged on a bench if

- boys and girls are in separate groups?

Solution : Boys & Girls or Girls & Boys

$$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$$

$$\text{or } {}^5P_5 \times {}^4P_4 \times 2$$

d) Anne and Jim wish to stay together?

Solution : (AJ) _ _ _ _ _ _ _

$$= 2 \times 8! \text{ or } 2 \times {}^8P_8$$

Permutations with Repetitions

How many permutations of the word **PARRAMATTA** are possible?

Solution :

P

A A A A

R R

M

T T

10 letters but note repetition (4 A's, 2 R's, 2 T's)

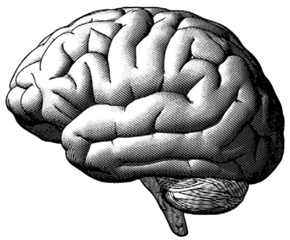
$$\text{No. of arrangements} = \frac{10!}{4!\,2!\,2!}$$

$$= 37\,800$$

Permutations with Repetitions

If we have **n** elements of which ***x*** are alike of one kind, ***y*** are alike of another kind, ***z*** are alike of another kind, ... then the **number of ordered selections or permutations** is given by:

$$\boxed{\frac{n!}{x!\,y!\,z!}}$$

Get your  in gear

- How many different numbers can you make from the digits 11122337?

Solution: 8! / (3! 2! 2!)

Permutations with Restrictions

How many arrangements of the letters of REMAND are possible if:

- there are no restrictions?

  Solution : ${}^{6}P_{6} = 720$ **or 6!**

- they begin with RE?

  Solution : **R E _ _ _ _ = $^{4}P_{4} =$ 24 or 4!**

- they do not begin with RE?

  Solution : Total – (b) = 6! – 4! = 696

Permutations with Restrictions

How many arrangements of the letters of REMAND are possible if:

- they have RE together in order?

Solution : **(RE) _ _ _ _ = $^5P_5$ = 120 or 5!**

- they have REM together in any order?

Solution : **(REM) _ _ _ = $^3P_3 \times {}^4P_4$ = 144**

- R, E and M are not to be together?

Solution : **Total – (e) = 6! – 144 = 576**