CS206 Quiz 1 **Oct 2, 2023** **Section 1**

Name: ______________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

1. (10%) True/False

   (a) Let $\mathcal{P}(S)$ be the power set of a set S. If S is a finite set with n elements, i.e., $|S| = n$ and $n \in \mathbb{N}$, then $|\mathcal{P}(S)| = 2^n - 1$.

   Solution: FALSE. The correct answer is 2^n.

   (b) The contrapositive statement of "if you get math jokes, you don't have friends" is "if you have friends, you don't get math jokes."

   Solution: TRUE.

2. (20%) Using the method of **proof by contrapositive**, prove the following theorem:

$$\text{if } r \text{ is irrational, then } r^{\frac{1}{5}} \text{ is irrational.}$$

Solution: The contrapositive statement is "if $r^{\frac{1}{5}}$ is rational, then r is rational." To prove this, let $r^{\frac{1}{5}} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. The we have $r = (r^{\frac{1}{5}})^5 = (\frac{a}{b})^5 = \frac{a^5}{b^5}$. And we know $a^5 \in \mathbb{Z}$ and $b^5 \in \mathbb{Z}$ because $a, b \in \mathbb{Z}$, hence $r \in \mathbb{Q}$. ■

3. (20%) Using the method of **direct proof**, prove the following theorem:

$$\forall n \in \mathbb{Z},\ \text{if } n \text{ is an even number, then } n^2 + 3n \text{ is an even number.}$$

Solution: Let $n = 2x$ and $x \in \mathbb{Z}$, then we have the following proof: $\mathrm{n}^2+3n = (2x)^2 + 3(2x)$
$= 4x^2 + 6x$
$= 2(2x^2+3x). And we also know that \mathrm{x} \in \mathbb{Z} \Rightarrow 2x^2 \in \mathbb{Z} \wedge 3x \in \mathbb{Z} \Rightarrow 2x^2+3x \in \mathbb{Z}$, so $2(2x^2 + 3x)$ is an even number.

4. (20%) Using the method of **contradiction**, prove the following theorem:

$$\text{there exists no combination of } x, y \in \mathbb{Z} \text{ such that } 36y + 12x = 3.$$

Solution: Assume the statement P: "there exists at least one pair of $x, y \in \mathbb{Z}$ such that $36y + 12x = 3$" is true. Then we get $3y + x = \frac{3}{12} = \frac{1}{4}$. Since $3y + x$ is an integer, it can never equal $\frac{1}{4}$. Due to this contradiction, the statement P is false, so the original theorem is true. ■

5. (10%) For any set A, let $\mathcal{P}(A)$ be its power set. Let $\emptyset$ denote the empty set.

   (a) Write down all the elements of $\mathcal{P}(\{10, 20, 30\})$.

   Solution:

   $$\mathcal{P}(\{10, 20, 30\}) = \{\emptyset, \{10\}, \{20\}, \{30\}, \{10, 20\}, \{10, 30\}, \{20, 30\}, \{10, 20, 30\}\}.$$

   (b) How many elements are there in $\mathcal{P}(\{1, 2, 3, 4, 5, 6\})$?

   Solution: $|\mathcal{P}(1, 2, 3, 4, 5, 6)| = 2^6 = 64$.

6. (20%) There are 120 students in a school, 50 don't play sports, 70 don't play music, and 30 don't do either. How many students do both?

   Solution: "50 students don't play sports" $\Rightarrow$ "$120 - 50 = 70$ student play sports" "70 students don't play music" $\Rightarrow$ "$120 - 70 = 50$ students play music" Let x be the number of students that do both. Then we have the equation that $120 = (70 + 50 - x) + 30$ and hence $x = 30$. (In conclusion, there are $70 - 30 = 40$ students exclusively play sports, $50 - 30 = 20$ students exclusively play music, 30 students do both, and 30 do neither.)

7. (Extra Credits - 10%) Let n be an integer. Prove by **contradiction** that if n^2 is even, then n must be even.

   Solution: Negation Statement: If n^2 is even, then there exists at least one n that is not even, and therefore it is odd. So we can write $n = 2k + 1$ and we can also write the expression for $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$. Therefore, n^2 is odd. BUT n^2 is even. So, our assumption must not be true, and n must be even.

8. (Extra Credits - 10%) In a state far away from here, there must be 3 characters in a car plate that may include only uppercase letters. All letters are accepted except "A"s. Use the **sum rule** to calculate how many car plates are available in that state.

   Note 1: The English alphabet has 26 letters;

   Note 2: Yes, there could be a car plate with all letters being the same, for the sake of this problem...

   Solution: We create 25 subgroups, each starting with one of the 25 different letters (any letter is valid, but the'A'). For each one of these subgroups, we can have 25×25 options, so the answer is $25 \times 25 \times 25 = 25^3$ car plates.

GOOD LUCK!

CS206 Quiz 1 **Oct 4, 2023** **Section 2**

Name: ______________________________

NetID: _____________(Please **PRINT**)

Section No.: _____________

1. (10%) True/False

   (a) Let $\mathcal{P}(S)$ be the power set of a set S. If S is a finite set with n elements, i.e., $|S| = n$ and $n \in \mathbb{N}$, then $|\mathcal{P}(S)| = 2^n + 1$.

   Solution: FALSE. The correct answer is 2^n.

   (b) The negation statement of "no pigs eat mud" is "there exists one pig that eats mud."

   Solution: CORRECT.'

2. (20%) Using the method of **proof by contrapositive**, prove the following theorem:

   $$\text{if } r \text{ is irrational, then } r^{\frac{1}{3}} \text{ is irrational.}$$

   Solution: The contrapositive statement is "if $r^{\frac{1}{3}}$ is rational, then r is rational." To prove this, let $r^{\frac{1}{3}} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. The we have $r = (r^{\frac{1}{3}})^3 = (\frac{a}{b})^3 = \frac{a^3}{b^3}$. And we know $a^3 \in \mathbb{Z}$ and $b^3 \in \mathbb{Z}$ because $a, b \in \mathbb{Z}$, hence $r \in \mathbb{Q}$. ■

3. (20%) Using the method of **direct proof**, prove the following theorem:

   $$\forall n \in \mathbb{Z},\ \text{if } n \text{ is an even number, then } n^2 + 3n + 1 \text{ is an odd number.}$$

   Solution: Let $n = 2x$ and $x \in \mathbb{Z}$, then we have the following proof:

   $$\begin{aligned} n^2 + 3n + 1 &= (2x)^2 + 3(2x) + 1 \\ &= 4x^2 + 6x + 1 \\ &= 2(2x^2 + 3x) + 1 \end{aligned}$$

   And we also know that $x \in \mathbb{Z} \Rightarrow 2x^2 \in \mathbb{Z} \wedge 3x \in \mathbb{Z} \Rightarrow 2x^2 + 3x \in \mathbb{Z}$, so $2(2x^2 + 3x) + 1$ is an odd number.■

4. (20%) Using the method of **contradiction**, prove the following theorem:

   $$\text{there exists no combination of } x, y \in \mathbb{Z} \text{ such that } 24y + 12x = 1.$$

Solution: Assume the statement P: "there exists at least one pair of $x, y \in \mathbb{Z}$ such that $24y + 12x = 1$" is true. Then we get $2y + x = \frac{1}{12}$. Since $2y + x$ is an integer, it can never equal $\frac{1}{12}$. Due to this contradiction, the statement P is false, so the original theorem is true. ■

5. (10%) For any set A, let $\mathcal{P}(A)$ be its power set. Let $\emptyset$ denote the empty set.

   (a) Write down all the elements of $\mathcal{P}(\{10, 12, 24\})$.

   Solution:

   $\mathcal{P}(\{10, 12, 24\}) = \{\emptyset, \{10\}, \{12\}, \{24\}, \{10, 12\}, \{10, 24\}, \{12, 24\}, \{10, 12, 24\}\}$.

   (b) How many elements are there in $\mathcal{P}(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\})$?

   Solution: $|\mathcal{P}(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)| = 2^{10} = 1024$.

6. (20%) There are 200 students in a class, 100 like don't like coffee, 80 don't like tea, and 30 don't like either. How many students like both tea and coffee?

   Solution: "100 students don't like coffee" $\Rightarrow$ "$200 - 100 = 100$ student like coffee" "80 students don't like tea" $\Rightarrow$ "$200 - 80 = 120$ students like tea" Let x be the number of students that like both. Then we have the equation that $200 = (100 + 120 - x) + 30$ and hence $x = 50$. (In conclusion, there are $100 - 50 = 50$ students who like only coffee, $120 - 50 = 70$ students who like tea, 50 students like both, and 30 like neither coffee nor tea.)

7. (Extra Credits - 10%) Let n be an integer. Prove by **contradiction** that if n^2 is even, then n must be even.

   Solution: Negation Statement: If n^2 is even, then there exists at least one n that is not even, and therefore it is odd. So we can write $n = 2k + 1$ and we can also write the expression for $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$. Therefore, n^2 is odd. BUT n^2 is even. So, our assumption must not be true, and n must be even.

8. (Extra Credits - 10%) In a state far away from here, there must be 3 characters in a car plate that may include only uppercase letters. Use the **sum rule** to calculate how many car plates are available in that state.
   Note 1: The English alphabet has 26 letters;
   Note 2: Yes, there could be a car plate with all letters the same, for the sake of this problem...

   Solution: We create 26 subgroups, each starting with one of the 26 different letters. For each one of these subgroups, we can have 26×26 options, so the answer is $26 \times 26 \times 26 = 26^3$ car plates.

GOOD LUCK!

CS206 Quiz 1 **Oct 4, 2023** **Section 3**

Name: ____________________________

NetID: ___________(Please **PRINT**)

Section No.: ___________

1. (10%) True/False

   (a) Let $\mathcal{P}(S)$ be the power set of a set S. If S is a finite set with n elements, i.e., $|S| = n$ and $n \in \mathbb{N}$, then $|\mathcal{P}(S)| = 2^n + 1$.

   Solution: FALSE. The correct answer is 2^n.

   (b) The contrapositive statement of "if you get math jokes, you don't have friends" is "if you don't get math jokes, you have friends."

   Solution: FALSE. The correct contrapositive statement is "if you have friends, you don't get math jokes."

2. (20%) Using the method of **proof by contrapositive**, prove the following theorem:

$$\text{if } 5x - 7 \text{ is odd, then } x \text{ is even.}$$

Solution: The contrapositive statement is "suppose x is odd, then $5x - 7$ is even". Since x is assumed to be odd, $x = 2k + 1$ for some $k \in \mathbb{Z}. But 5$x-7=5(2k+1)-7=10k-2=2(5k-1), where 5k-1$\in \mathbb{Z}. This shows that 5$x-7$is even.$■

3. (20%) Using the method of **direct proof**, prove the following theorem:

$$\forall n \in \mathbb{Z}, \text{ if } n \text{ is an even number, then } n^2 + 3n + 1 \text{ is an odd number.}$$

Solution: Let $n = 2x$ and $x \in \mathbb{Z}$, then we have the following proof:

$$\begin{aligned} n^2 + 3n + 1 &= (2x)^2 + 3(2x) + 1 \\ &= 4x^2 + 6x + 1 \\ &= 2(2x^2 + 3x) + 1 \end{aligned}$$

And we also know that $x \in \mathbb{Z} \Rightarrow 2x^2 \in \mathbb{Z} \wedge 3x \in \mathbb{Z} \Rightarrow 2x^2 + 3x \in \mathbb{Z}$, so $2(2x^2 + 3x) + 1$ is an odd number.■

4. (20%) Using the method of **contradiction**, prove the following theorem:

$$\text{there exists no combination of } x, y \in \mathbb{Z} \text{ such that } 24y + 12x = 1.$$

Solution: Assume the statement P: "there exists at least one pair of $x, y \in \mathbb{Z}$ such that $24y + 12x = 1$" is true. Then we get $2y + x = \frac{1}{12}$. Since $2y + x$ is an integer, it can never equal $\frac{1}{12}$. Due to this contradiction, the statement P is false, so the original theorem is true. ■

5. (10%) For any set A, let $\mathcal{P}(A)$ be its power set. Let $\emptyset$ denote the empty set.

   (a) Write down all the elements of $\mathcal{P}(\{0, 2, 4\})$.

   Solution:

   $$\mathcal{P}(\{0,2,4\}) = \{\emptyset, \{0\}, \{2\}, \{4\}, \{0,2\}, \{0,4\}, \{2,4\}, \{0,2,4\}\}.$$

   (b) How many elements are there in $\mathcal{P}(\{1, 2, 3, 4, 5, 6, 7, 8\})$?

   Solution: $|\mathcal{P}(1, 2, 3, 4, 5, 6, 7, 8)| = 2^8 = 256$.

6. (20%) There are 120 kids in a party, 50 don't like hamburgers, 70 don't like hot dogs, and 30 don't like either. How many kids like both?

   Solution: "50 kids don't like hamburgers" $\Rightarrow$ "$120 - 50 = 70$ kids like hamburgers" "70 kids don't like hot dogs" $\Rightarrow$ "$120 - 70 = 50$ kids like hot dogs" Let x be the number of kids that like both. Then we have the equation that $120 = (70 + 50 - x) + 30$ and hence $x = 30$. (In conclusion, there are $70 - 30 = 40$ kids who exclusively like hamburgers, $50 - 30 = 20$ kids who exclusively like hot dogs, 30 kids like both, and 30 like neither.)

7. (Extra Credits - 10%) Let n be an integer. Prove by **contradiction** that if n^2 is even, then n must be even.

   Solution: Negation Statement: If n^2 is even, then there exists at least one n that is not even, and therefore it is odd. So we can write $n = 2k + 1$ and we can also write the expression for $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$. Therefore, n^2 is odd. BUT n^2 is even. So, our assumption must not be true, and n must be even.

8. (Extra Credits - 10%) In a state far away from here, there must be 4 characters in a car plate that may include only digits. Use the **sum rule** to calculate how many car plates are available in that state.
   Note 1: There are 10 digits;
   Note 2: Yes, there could be a valid car plate with 0000, for the sake of this problem...

   Solution: We create 10 subgroups, each starting with one of the 10 different digits. For each one of these subgroups, we can have $10 \times 10 \times 10$ options, so the answer is $10 \times 10 \times 10 \times 10 = 10^4$ car plates.

GOOD LUCK!