CS206 Quiz 2 **Oct 16, 2023** **Section 1**

Name: ____________________________

NetID: ___________(Please **PRINT**)

Section No.: ___________

1. (10%) In Quiz 1, you were asked to solve the following problem for extra credits, using the sum rule. Now, it is time to solve the same problem, using the product rule:

   In a state far away from here, there must be 3 characters in a car plate that may include only uppercase letters. All letters are accepted except "A"s. Use the **product rule** to calculate how many car plates are available in that state.
   Note 1: The English alphabet has 26 letters;
   Note 2: Yes, there could be a car plate with all letters being the same, for the sake of this problem...

   Solution: Each place has $26-1=25$ options, as each plate may be starting with one of the 25 different letters. So the answer is $25 \times 25 \times 25 = 25^3$ car plates.

2. (20%) True/False (Give a 1-sentence explanation)

   (a) For finite sets X and Y, the number of possible binary relations (x, y) where $x \in X$ and $y \in Y$, is $|X| \cdot |Y|$.

   Solution: True, since every element of X can map onto any other element of Y, total number of mappings is product of elements in X and Y.

   (b) The number of nine digit numbers that start with the digit 3 is greater than the number of nine digit numbers in which no digit is repeated. (Note: A 9-digit number cannot start with 0.)

   Solution: True. There are 10^8 nine digit numbers that start with 3 (one choice for the first digit and 10 choices for all other digits) and $9 \cdot 9 \cdot 8 \cdot 7 \cdots 3 \cdot 2$ nine digit numbers in which no digits are repeated (nine choices for the first digit since it can't start with 0, nine choices for the second digit since you can't repeat the first digit, and so on). It is straightforward to confirm that the first product is larger.

3. (20%) Telephone area codes in the United States and Canada used to consist of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9.

   (a) How many area codes were possible?

   Solution: $8 \times 2 \times 9 = 144$. The size of set from which first digit is drawn is $|D_1| = 8$, the second digit is drawn from a $|D_2| = 2$ size set, the third digit is drawn from $|D_3| = 9$ size set. Since we are trying to find all the chains of maps $D_1 \to D_2 \to D_3$, we apply the product rule.

   (b) How many area codes starting with a 4 were possible?

   Solution: $2 \times 9 = 18$. The logic is same as above, except now we fix the first digit to 4. This means that the $|D_1| = 1$

4. (20%) Ten diplomats, each from a different country, are lined up in two rows of five for a photo. How many arrangements are possible if the Saudi and Turkish diplomats refuse to be in the same row as the Russian and Syrian diplomats? In other words we have the following constraints

   (a) The Saudi and Russian diplomats can't be in the same row.
   (b) The Saudi and Syrian diplomats can't be in the same row.
   (c) The Turkish and Russian diplomats can't be in the same row.
   (d) The Turkish and Syrian diplomats can't be in the same row.

   Solution: 1. Pick a row for Saudi and Turkish diplomats and place them in that row. This can be done in $2 \cdot 5 \cdot 4$ ways.
   2. Place the Russian and Syrian diplomats in the other row. This can be done in $5 \cdot 4$ ways.
   3. Place the remaining 6 diplomats in the remaining 6 positions. This can be done in 6! ways.
   Hence, total ways equals $2 \cdot 5 \cdot 4 \cdot 5 \cdot 4 \cdot 6! = 800 \cdot 6!$.

5. (30%) A, B and C are three sets such that $|A| = 30$, $|B| = 50$ and $|C| = 100$. In each of the cases below, either compute $|A \cup B \cup C|$ or explain why there's not enough information.

   (a) A is contained inside B and B is contained inside C.

   Solution: In this case we have

   $$|A \cup B \cup C| = |C| = 100.$$

   (b) A and B are disjoint and are both contained inside C.

Solution: Again we have $|A \cup B \cup C| = |C| = 100$.

(c) A and B are disjoint and $\overline{A}, B$ are both contained inside C.

Solution: In this case we don't have enough information.

6. (20% - extra credits) Let a set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Use the **product rule** to answer the following:

(a) How many subsets of X contain the element x_1?

Solution: We pick whether or not x_1 is included in the subset. We do the same for x_2, x_3, ..., x_7. Therefore, each of the x's will either have 2 options (included or not included), or 1 option according to the constraints set by the particular step of the problem. Note: There are no possibilities for 'MUST be' and 'must NOT be'. They are prescribed. So the only elements that contribute to the total number of subsets are the ones we still have not decided 'Include or not include it to the set?', which have two possibilities. Here x_2, x_3, ..., x_7 can or cannot be included in the creation of the subsets, so we have $2^6 = 64$ subsets.

(b) How many subsets of X contain the elements x_2 **and** x_3, **and do not** contain x_6?

Solution: Same as before. But this time x_2, x_3 and x_6 have no variability to contribute (see above). The variability (number of options) is coming from the remaining 7-3 = 4 elements (that is, the x_1, x_4, x_5, x_6). That is, $2^4 = 16$ subsets.

GOOD LUCK!

CS206 Quiz 2 **Oct 18, 2023** **B**

Name: ____________________________

NetID: ___________(Please **PRINT**)

Section No.: ___________

1. (10%) In Quiz 1, you were asked to solve the following problem for extra credits, using the sum rule. Now, it is time to solve the same problem, using the product rule:

   In a state far away from here, there must be 3 characters in a car plate that may include only uppercase letters. Use the **product rule** to calculate how many car plates are available in that state.
   Note 1: The English alphabet has 26 letters;
   Note 2: Yes, there could be a car plate with all letters the same, for the sake of this problem...

   Solution: Each place has 26 options. So the answer is $26 \times 26 \times 26 = 26^3$ car plates.

2. (20%) True/False (Give a 1-sentence explanation)

   (a) For finite set X, the number of possible binary relations (x_1, x_2) where $x_1 \in X$ and $x_2 \in X$, is $|X|^2$.

   Solution: TRUE, since every element of X can map onto any other element of X including itself.

   (b) The number of five-digit numbers that end with the digit 0 is greater than the number of five-digit numbers in which no digit is repeated. (Note: The "00000" is a valid 5-digit number for this problem.)

   Solution: False. There are 10^4 five-digit numbers that end with 0 (one choice for the last digit and 10 choices for all other digits) and $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ five-digit numbers in which no digits are repeated (ten choices for the first digit since it can start with 0, nine choices for the second digit since you can't repeat the first digit, and so on). It is straightforward to confirm that the first product is larger.

3. (20%) For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9.

(a) How many area codes were possible?

Solution: $8 * 2 * 9 = 144$. The size of set from which first digit is drawn is $|D_1| = 8$, second digit is drawn is $|D_2| = 2$,third digit is drawn is $|D_3| = 9$. Since we are trying to find all the chains of maps $D_1 \to D_2 \to D_3$, we apply product rule.

(b) How many area codes starting with a 9 were possible?

Solution: $2 * 9 = 18$. The logic is same as above, except now we fix the first digit to 9. This means that the $|D_1| = 1$

4. (20%) Ten diplomats, each from a different country, are lined up in two rows of five for a photo. How many arrangements are possible if the Saudi and Turkish diplomats refuse to be in the same row as the Russian and Syrian diplomats? In other words we have the following constraints

(a) The Saudi and Russian diplomats can't be in the same row.
(b) The Saudi and Syrian diplomats can't be in the same row.
(c) The Turkish and Russian diplomats can't be in the same row.
(d) The Turkish and Syrian diplomats can't be in the same row.

Solution: 1. Pick a row for Saudi and Turkish diplomats and place them in that row. This can be done in $2 \cdot 5 \cdot 4$ ways.
2. Place the Russian and Syrian diplomats in the other row. This can be done in $5 \cdot 4$ ways.
3. Place the remaining 6 diplomats in the remaining 6 positions. This can be done in 6! ways.
Hence, total ways equals $2 \cdot 5 \cdot 4 \cdot 5 \cdot 4 \cdot 6! = 800 \cdot 6!$.

5. (30%) A, B and C are three sets such that $|A| = 30$, $|B| = 50$ and $|C| = 100$. In each of the cases below, either compute $|A \cup B \cup C|$ or explain why there's not enough information.

(a) A is contained inside B and B is contained inside C.

Solution: In this case we have

$$|A \cup B \cup C| = |C| = 100.$$

(b) A and B are disjoint and are both contained inside C.

Solution: Again we have $|A \cup B \cup C| = |C| = 100$.

(c) A and B are disjoint and $\overline{A}, B$ are both contained inside C.

Solution: In this case we don't have enough information.

6. (20% - extra credits) Given the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$. Use the **product rule** to answer the following:

   (a) How many subsets of X do not contain elements x_1 and x_2?

   Solution: We approach this solution using the product rule. We pick whether or not x_1 is included in the subset. We do the same for x_2, x_3, ..., x_7. Therefore, each of the x's will either have 2 options (included or not included), or 1 option according to the constraints set by the particular step of the problem. Note: There are no possibilities for 'MUST be' and 'must NOT be'. They are prescribed. So the only elements that contribute to the total number of subsets are the ones we still have not decided 'Include or not include it to the set?', which have two possibilities. Here x_3, ..., x_8 can or cannot be included in the creation of the subsets, so we have $2^6 = 64$ subsets.

   (b) How many subsets of X contain element x_1, **and do not** contain x_8?

   Solution: Same as before. This time the two elements are fixed while the rest are not. Here x_2, ..., x_7 can or cannot be included in the creation of the subsets, so, again, we have $2^6 = 64$ subsets.

GOOD LUCK!

CS206 Quiz 2 **Oct 18, 2023** **Section 3**

Name: ______________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

1. (10%) In Quiz 1, you were asked to solve the following problem for extra credits, using the sum rule. Now, it is time to solve the same problem, using the product rule:

   In a state far away from here, there must be 4 characters in a car plate that may include only digits. Use the **product rule** to calculate how many car plates are available in that state.
   Note 1: There are 10 digits;
   Note 2: Yes, there could be a valid car plate with 0000, for the sake of this problem...

   Solution: Each place has 10 options, so the answer is $10 \times 10 \times 10 \times 10 = 10^4$ car plates.

2. (20%) True/False (Give a 1-sentence explanation)

   (a) For finite set X, the number of possible binary relations (x_1, x_2) where $x_1 \in X$ and $x_2 \in X$, is $|X|^2$.

   Solution: TRUE, since every element of X can map onto any other element of X including itself.

   (b) The number of five-digit numbers that end with the digit 0 is greater than the number of five-digit numbers in which no digit is repeated. (Note: The "00000" is a valid 5-digit number for this problem.)

   Solution: True. There are 10^4 five-digit numbers that end with 0 (one choice for the last digit and 10 choices for all other digits) and $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ five-digit numbers in which no digits are repeated (ten choices for the first digit since it can start with 0, nine choices for the second digit since you can't repeat the first digit, and so on). It is straightforward to confirm that the first product is larger.

3. (20%) For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9.

(a) How many area codes were possible?

Solution: $8 * 2 * 9 = 144$. The size of set from which first digit is drawn is $|D_1| = 8$, second digit is drawn is $|D_2| = 2$,third digit is drawn is $|D_3| = 9$. Since we are trying to find all the chains of maps $D_1 \to D_2 \to D_3$, we apply product rule.

(b) How many area codes starting with an 8 were possible?

Solution: $2 * 9 = 18$. The logic is same as above, except now we fix the first digit to 8. This means that the $|D_1| = 1$

4. (20%) Ten diplomats, each from a different country, are lined up in two rows of five for a photo. How many arrangements are possible if the Saudi and Turkish diplomats refuse to be in the same row as the Russian and Syrian diplomats? In other words we have the following constraints

(a) The Saudi and Russian diplomats can't be in the same row.
(b) The Saudi and Syrian diplomats can't be in the same row.
(c) The Turkish and Russian diplomats can't be in the same row.
(d) The Turkish and Syrian diplomats can't be in the same row.

Solution: 1. Pick a row for Saudi and Turkish diplomats and place them in that row. This can be done in $2 \cdot 5 \cdot 4$ ways.
2. Place the Russian and Syrian diplomats in the other row. This can be done in $5 \cdot 4$ ways.
3. Place the remaining 6 diplomats in the remaining 6 positions. This can be done in 6! ways.
Hence, total ways equals $2 \cdot 5 \cdot 4 \cdot 5 \cdot 4 \cdot 6! = 800 \cdot 6!$.

5. (30%) A, B and C are three sets such that $|A| = 30$, $|B| = 50$ and $|C| = 100$. In each of the cases below, either compute $|A \cup B \cup C|$ or explain why there's not enough information.

(a) A is contained inside B and B is contained inside C.

Solution: In this case we have

$$|A \cup B \cup C| = |C| = 100.$$

(b) A and B are disjoint and are both contained inside C.

Solution: Again we have $|A \cup B \cup C| = |C| = 100$.

(c) A and B are disjoint and $\overline{A}, B$ are both contained inside C.

Solution: In this case we don't have enough information.

6. (20% - extra credits) Given the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$. Use the **product rule** to answer the following:

   (a) How many subsets of X do not contain elements x_5 and x_6?

   Solution: We pick whether or not x_1 is included in the subset. We do the same for x_2, x_3, ..., x_7. Therefore, each of the x's will either have 2 options (included or not included), or 1 option according to the constraints set by the particular step of the problem. Note: There are no possibilities for 'MUST be' and 'must NOT be'. They are prescribed. So the only elements that contribute to the total number of subsets are the ones we still have not decided 'Include or not include it to the set?', which have two possibilities. Here x_1, ..., x_4, x_7, x_8 can or cannot be included in the creation of the subsets, so we have $2^6 = 64$ subsets.

   (b) How many subsets of X contain element x_8, **and do not** contain x_1?

   Solution: Same as before. This time the two elements are fixed while the rest are not. Here x_2, ..., x_7 can or cannot be included in the creation of the subsets, so, again, we have $2^6 = 64$ subsets.

GOOD LUCK!