combra.cs.rutgers.edu

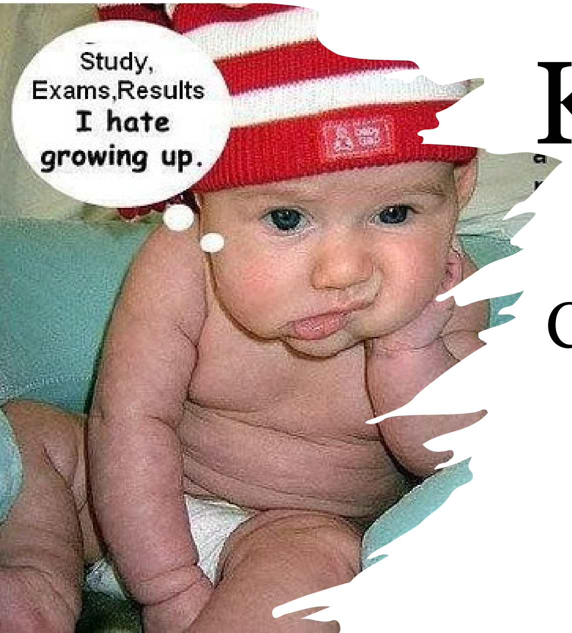
Quiz 5 this week
Good luck!

# 206
# Discrete Structures II


Konstantinos P. Michmizos

Computational Brain Lab

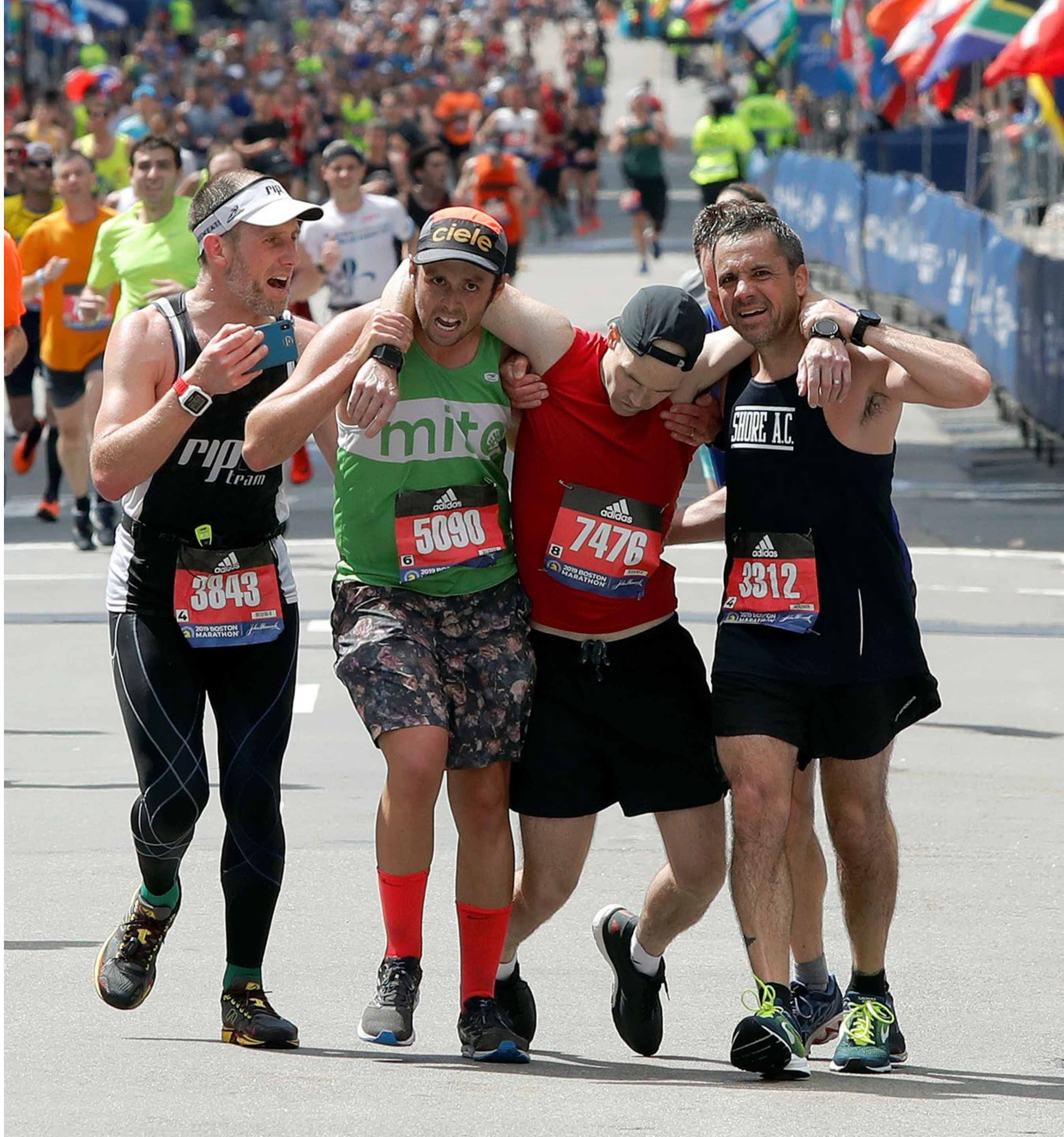
Computer Science | Rutgers University | NJ, USA

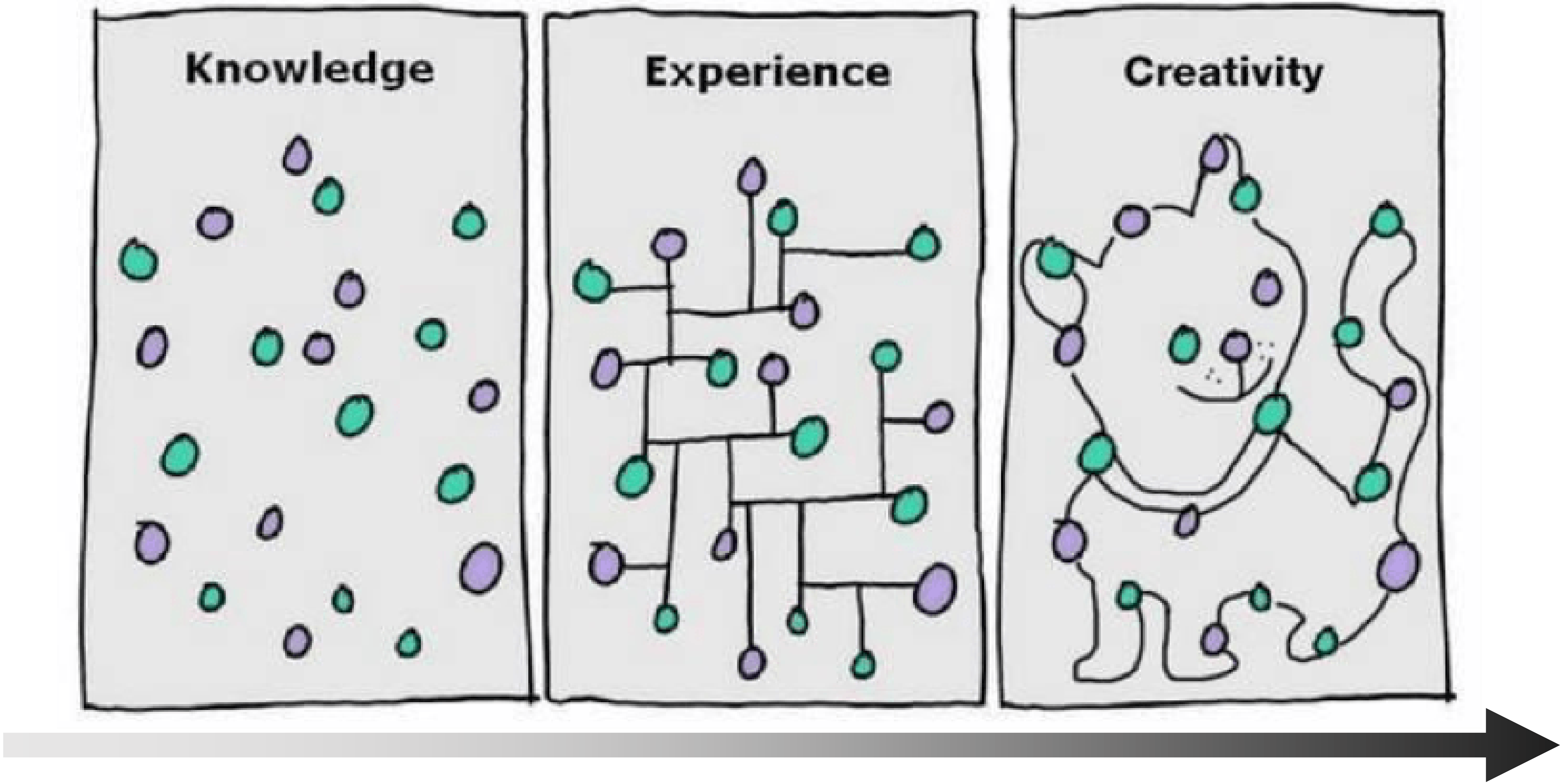




*Lecture 21 | Conditional Probabilities| Wednesday November 29th 2023*

# Announcements

- Quiz 5 running this week
- Quiz 6 → Next week






Lectures Recitations/Extra Problems Quizzes Midterm/Final Real-Life!

# Quiz 5 this week

- During recitation time (Wednesday 11/29 and Monday 12/4)
- 7 problems | 30 minutes
- 120 points (20% extra credits!)
- No need to answer all questions (but you should at least try)
- Start from what you know better
- Typically, the higher the points, the more difficult the problem is
    - Problem Difficulty = synthesize multiple methods
- Don’t Panic! ! !

# CTAAR survey

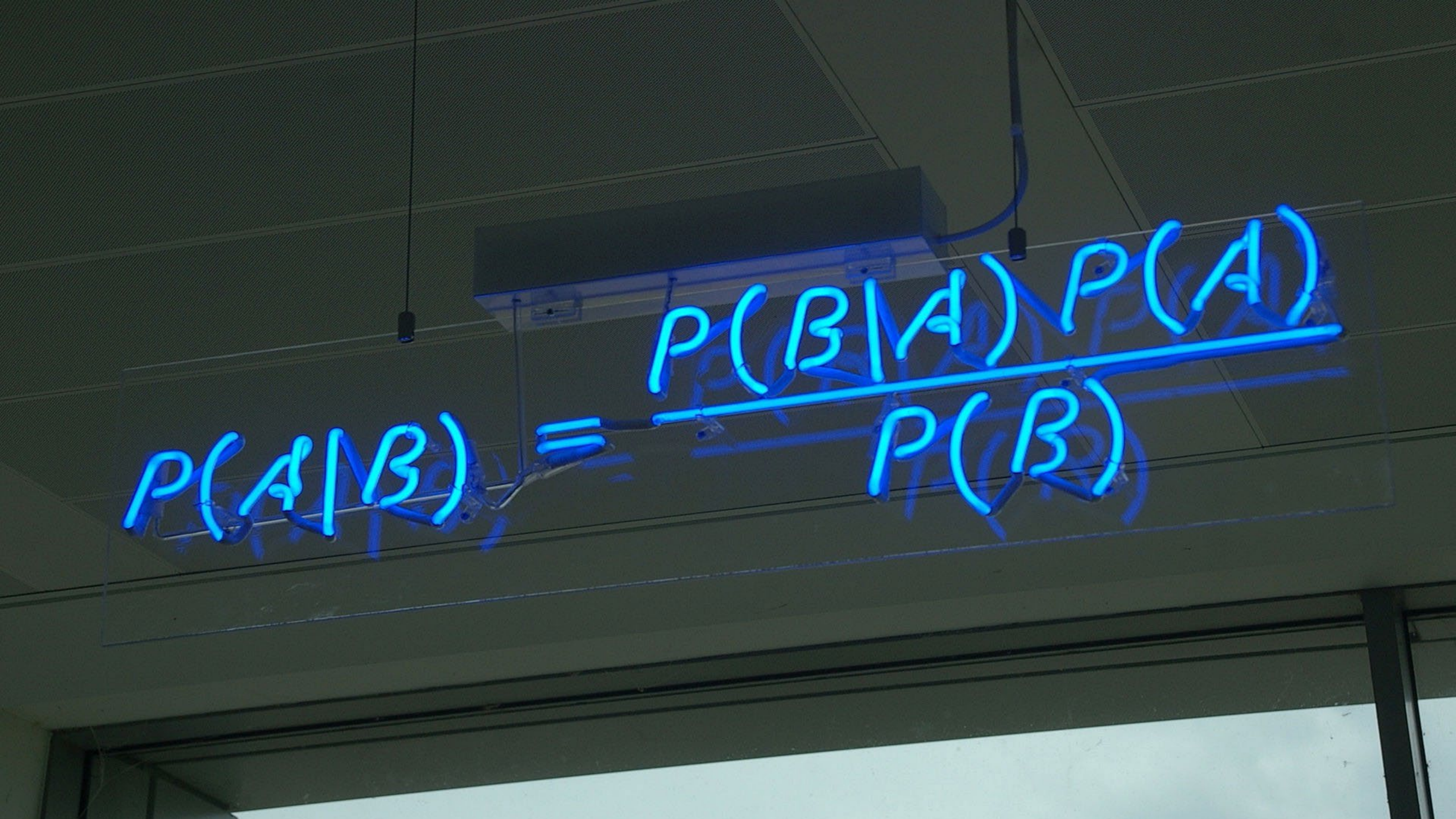
https://sirs.rutgers.edu/blue

# CTAAR Survey

Section 1

Section 2

Section 3


$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$

# Conditional Probability - Example

- One bag has two silver coins, another has two gold coins, and the third has one of each.
- One bag is selected at random. One coin from it is selected at random.
- It turns out to be gold What is the probability that the other coin is gold?

last class

# Outline for this month

- Sample spaces and events
- Basics of probability
- Independence, conditional probability
- Random variables, expectation, variance

→ Basic building blocks

→ Intermediate

→ Advanced

# So Far, Last Lecture, and Today/This week

- Sample space - Events
- Axioms of probability

- Conditional probability
- Independence

- Bayes rule

last class

# Probability – so far…

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Simple Event
  - Any element of the sample space
- Compound Event
  - Subsets of the sample space
- Probability Distribution - Axioms

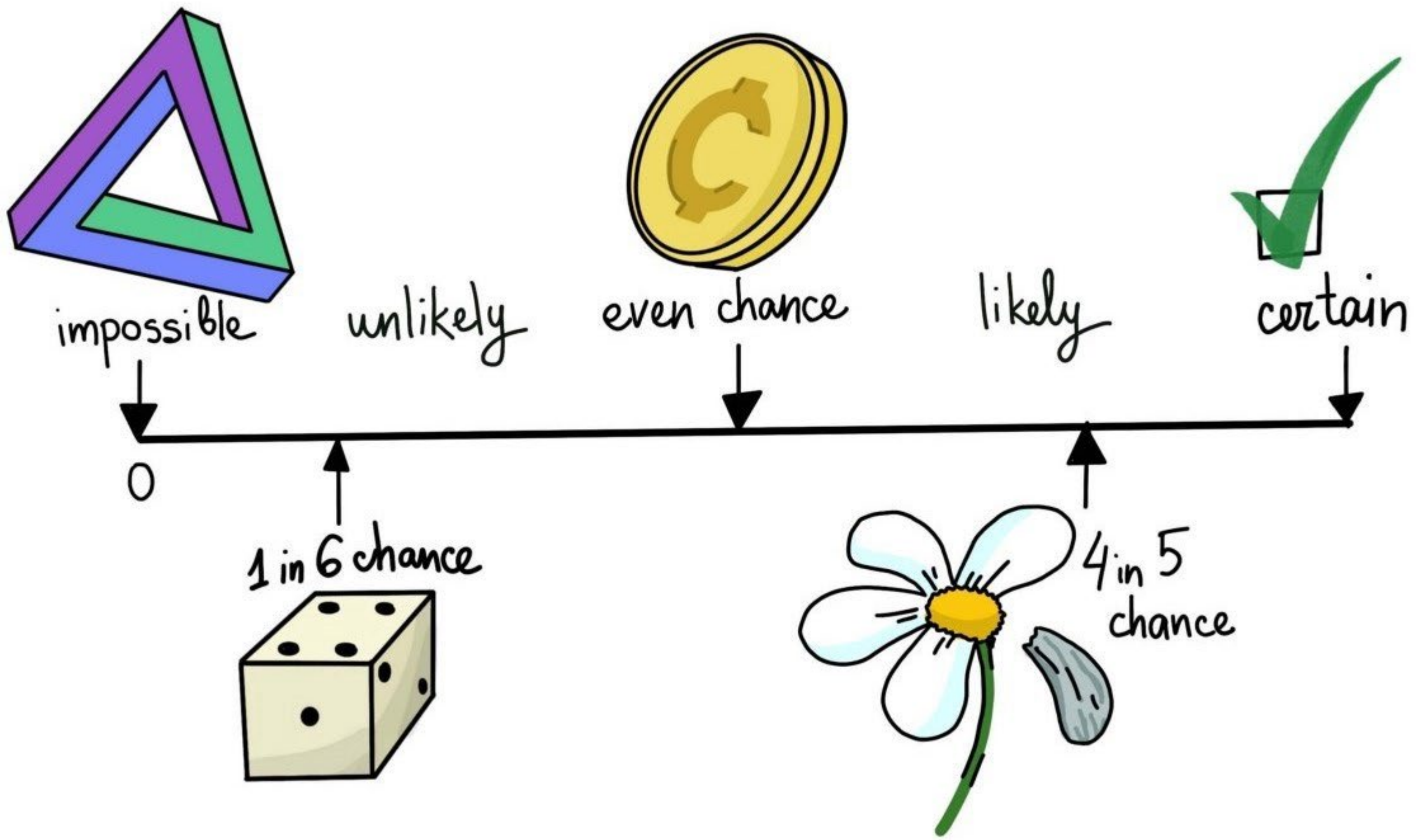
last class

# Probability

- Fix experiment and sample space $\Omega$.

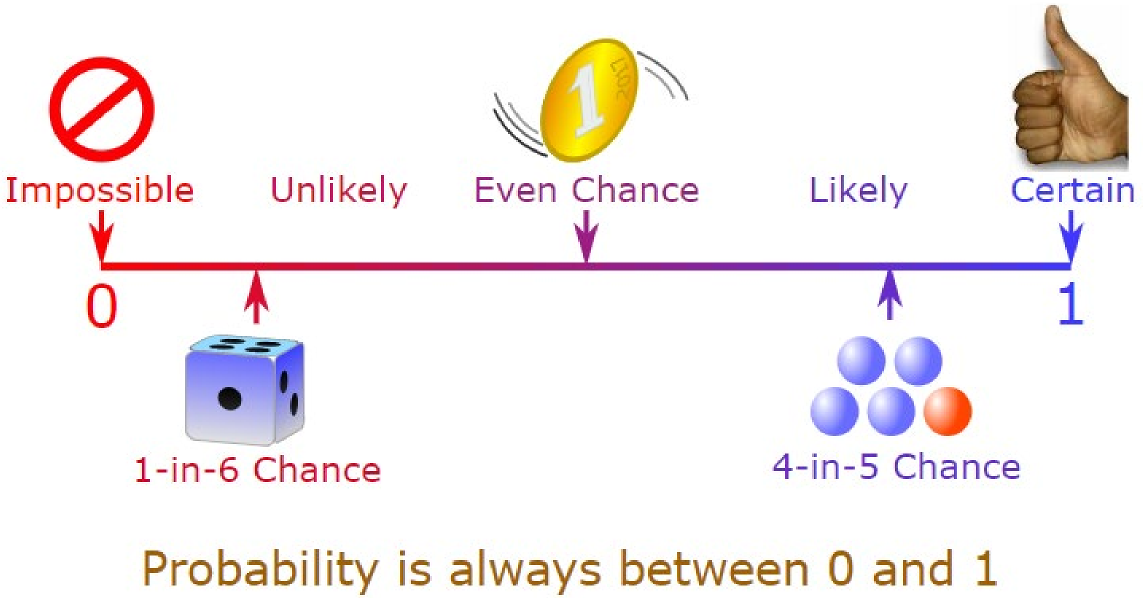
A probability distribution $P$ assigns a number $P(A)$ to each event $A$.

- $P$ needs to satisfy certain basic axioms.



last class

# Axioms of Probability




- $P(A) \geq 0$
- $P(\Omega) = 1$
- For a collection of disjoint events $A_1, A_2, \ldots$
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

last class

# Inclusion/Exclusion for Probabilities

- $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Extends to more than 2 sets

last class

# Union Bound

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) \leq P(A) + P(B)$ $\longrightarrow$ Union bound

$\longrightarrow$ Boole's inequality

last class

# Interpretation of Probability

- For a collection of disjoint events $A_1, A_2, \ldots$ — sum rule for Probability
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

Interpretation of $P(A)$

If $P(A) = .6$

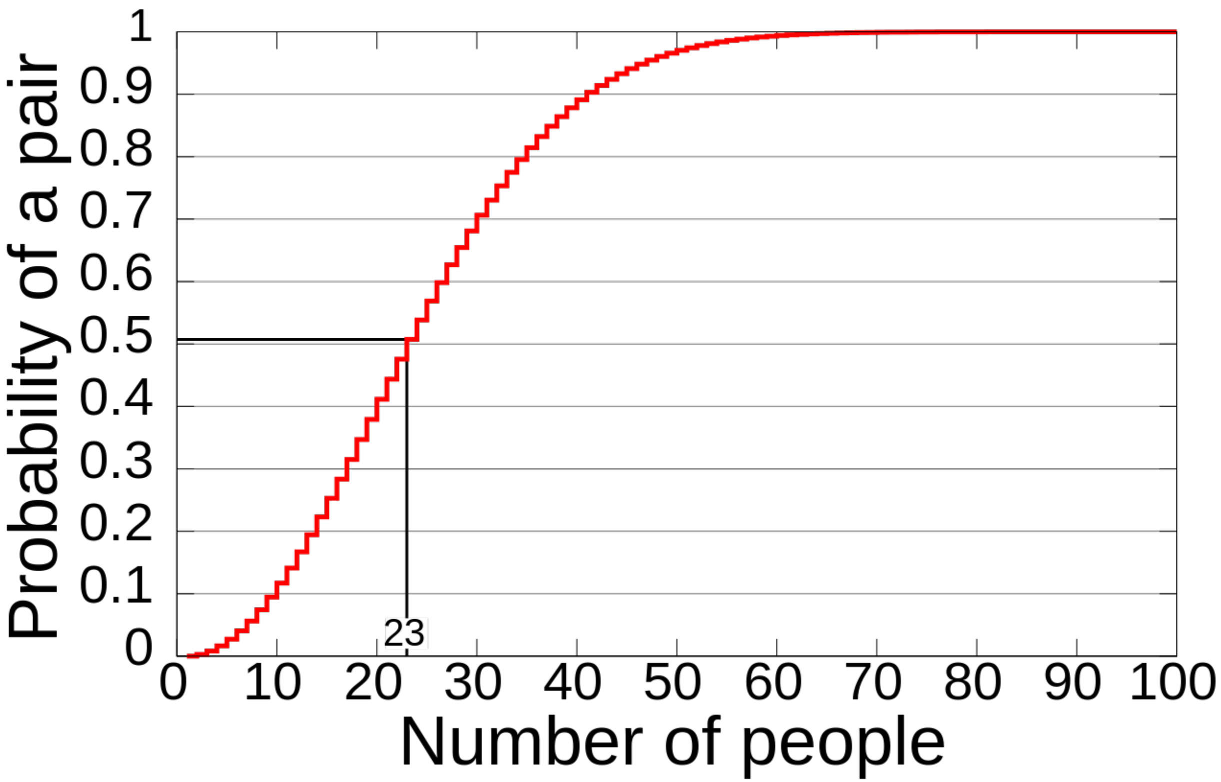
- If we repeat experiment $N$ times ($N$ is very large)
- Then the out come will lie in $A$ $.6N$ of the times

last class

# Uniform Distribution

- 23 people are in a room. Suppose all birthdays are equally likely. What is the probability that two will have the same birthday?

Birthday Paradox!!



last class

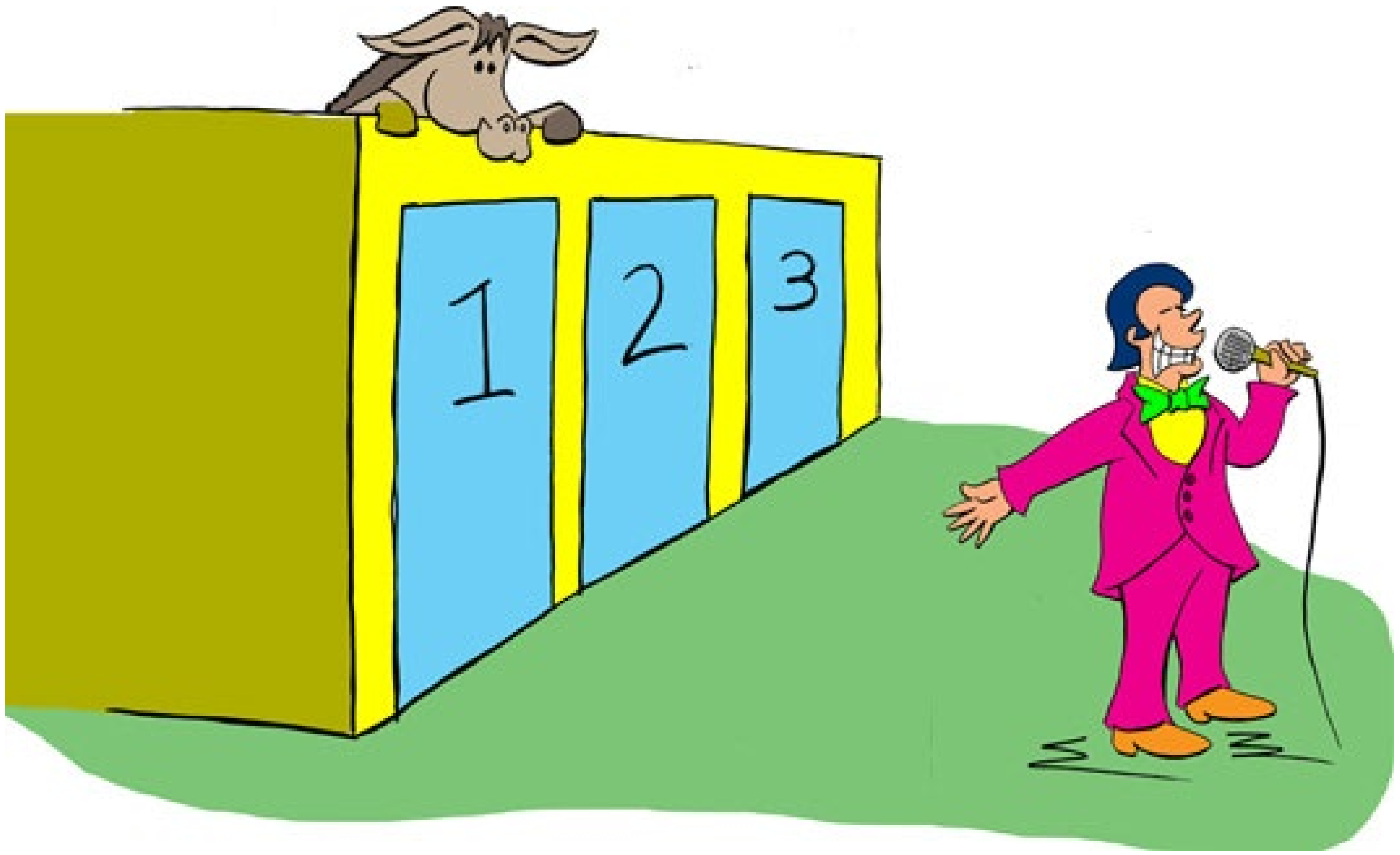
# Conditional Probabilities - Example

A= man survives

- A man went on an airplane ride. $P(A) = .1$
- Unfortunately, he fell out.
- Fortunately, he had a parachute on. $P(A) = .7$
- Unfortunately, the parachute did not open. $P(A) = .1$
- Fortunately, there was a haystack below him, directly in the path of his fall. $P(A) = .5$
- Unfortunately, there was a pitchfork sticking out of the top of the haystack.
- Fortunately, he missed the pitchfork. $P(A) = .1$
- Unfortunately, he missed the haystack.

# Monty Hall Problem

Door 1 → G
Door 2 → G
Door 3 → Car




The New York Times

Quandary Created by Gain In Detecting Breast Cancer

New Methods Still Leave Doubts on What to Do

SAUDIS WOULD LIFT BOYCOTT IF ISRAEL HALTS SETTLEMENT

LEBANON SUPPORTS TALKS

States Seeking Aid on Budgets From Pensions

After 16 Years of War, Beirut Tentatively Reclaims Peace

U.N. Says Kurds Take Most Parts Of an Iraqi City

Split on Redistricting

Communities and Their Residents Age Gracefully

Behind Monty Hall's Doors: Puzzle, Debate and Answer?

# Conditional Probability – 3 ways to solve

1. Direct enumeration of sample space

2. Tree based enumeration

3. Direct use of formula

# 1. Direct Enumeration

- Consider a family with 2 children. Given that 1 of the children is a boy, what is the probability that both children are boys?

$A \to$ both are boys

$B \to$ at least one is a boy

$\Omega = \{(B,B), (G,G), (B,G), (G,B)\}$

$B \to \{(B,B), (B,G), (G,B)\}$

$P(A|B) = \frac{1}{3}$

# Direct Enumeration (Conditional Probability)

- Consider a family with 2 children. Given that 1 of the children is a boy, what is the probability that both children are boys?

$\longrightarrow$ children are equally likely

$\Omega = \{(B,B), (B,G), (G,B), (G,G)\}$

$B \longrightarrow$ one of them is a boy $\longrightarrow \{(B,B), (B,G), (G,B)\}$

$A \longrightarrow$ both of them are boys

$P(A|B) = \frac{1}{3}$

$P(A) = \frac{1}{4}$

# 2. Tree Enumeration

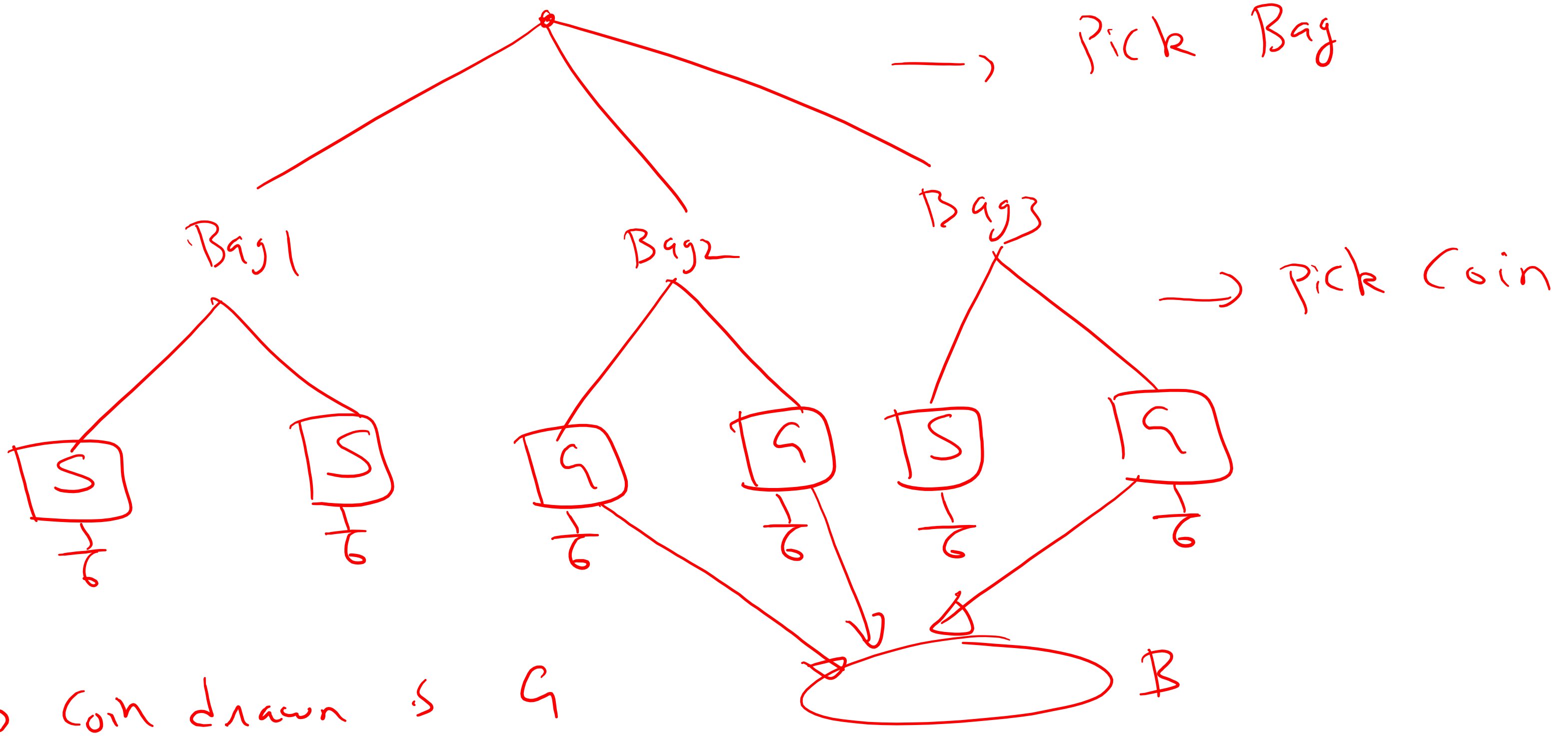
- One bag has two silver coins, another has two gold coins, and the third has one of each.
- One bag is selected at random. One coin from it is selected at random.
- It turns out to be gold. What is the probability that the other coin is gold?

$A \rightarrow$ other coin in bag is G

$B \rightarrow$ Coin Drawn is G

$P(A|B)$

# 2. Tree Enumeration



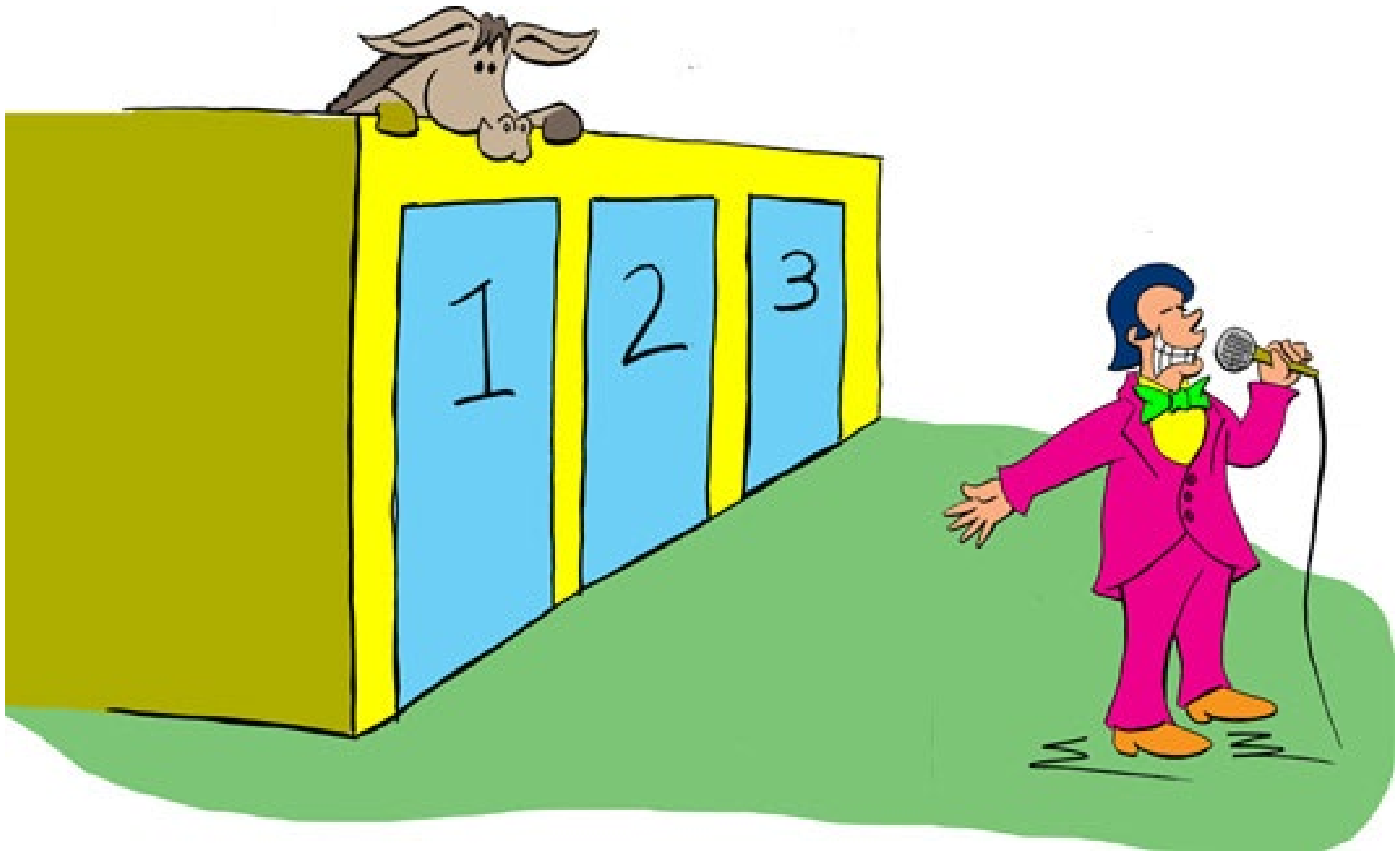

B → Coin drawn is G

A → other coin is also G

$$P(A|B) = \frac{2}{3}$$

# Monty Hall Problem

Door 1 → G
Door 2 → G
Door 3 → Car





The New York Times

Quandary Created by Gain In Detecting Breast Cancer

SAUDIS WOULD LIFT BOYCOTT IF ISRAEL HALTS SETTLEMENT

States Seeking Aid on Budgets From Pensions

After 16 Years of War, Beirut Tentatively Reclaims Peace

U.N. Says Kurds Take Most Parts Of an Iraqi City

Behind Monty Hall's Doors: Puzzle, Debate and Answer?

Communities and Their Residents Age Gracefully

# Monty Hall Problem

Door 1 → Goat
Door 2 → Goat
Door 3 → Car

- Announcer hides prize behind one of 3 doors. You select some door at random. Announcer opens one of others with no prize. You can decide to keep or switch.
- What to do?

A → I win If I switch

B → door with no prize is revealed

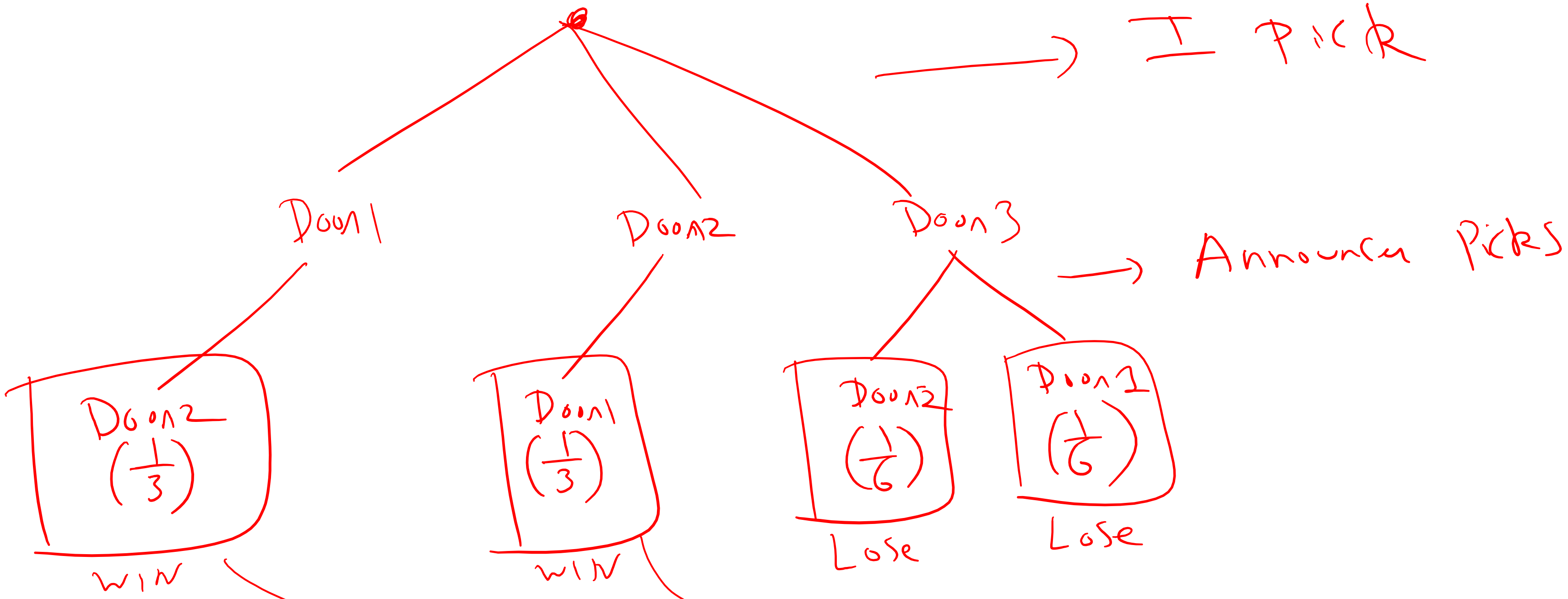
$P(A|B) > \frac{1}{2}$ or $< \frac{1}{2}$ ??

Will show $P(A|B) = \frac{2}{3}$

In this case $P(B)=1$ [Because of rules of the game]

# Monty Hall Problem

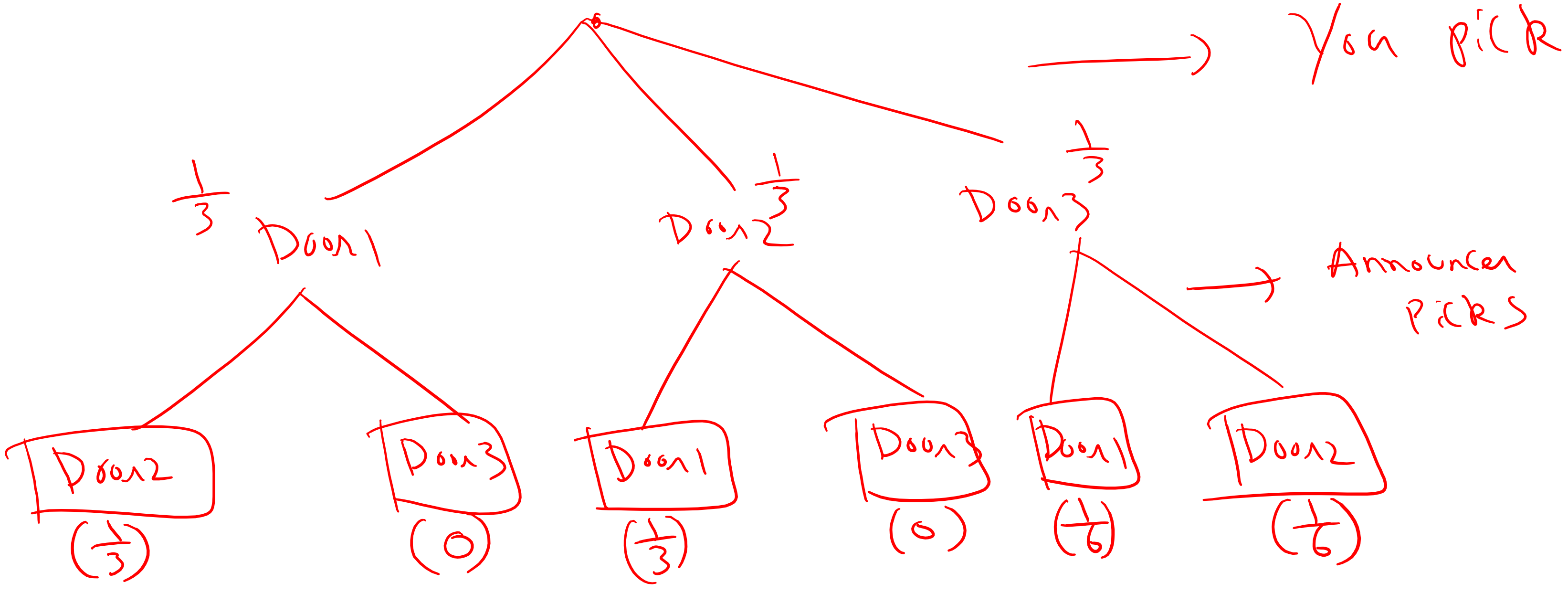
Door 1 → G
Door 2 → G
Door 3 → Car




$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

# Monty Hall Problem

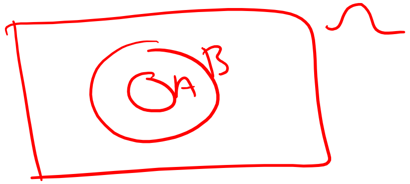
Door 1 $\to$ G
Door 2 $\to$ G
Door 3 $\to$ Car




$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

# 3. Direct Use of Formula

- Let P(N) stands for the probability of a new-born to reach the age of N years. We are given that
- P(50) = .913,
- P(55) = .881.
- What is the probability that a 50 year-old man will reach the age of 55?

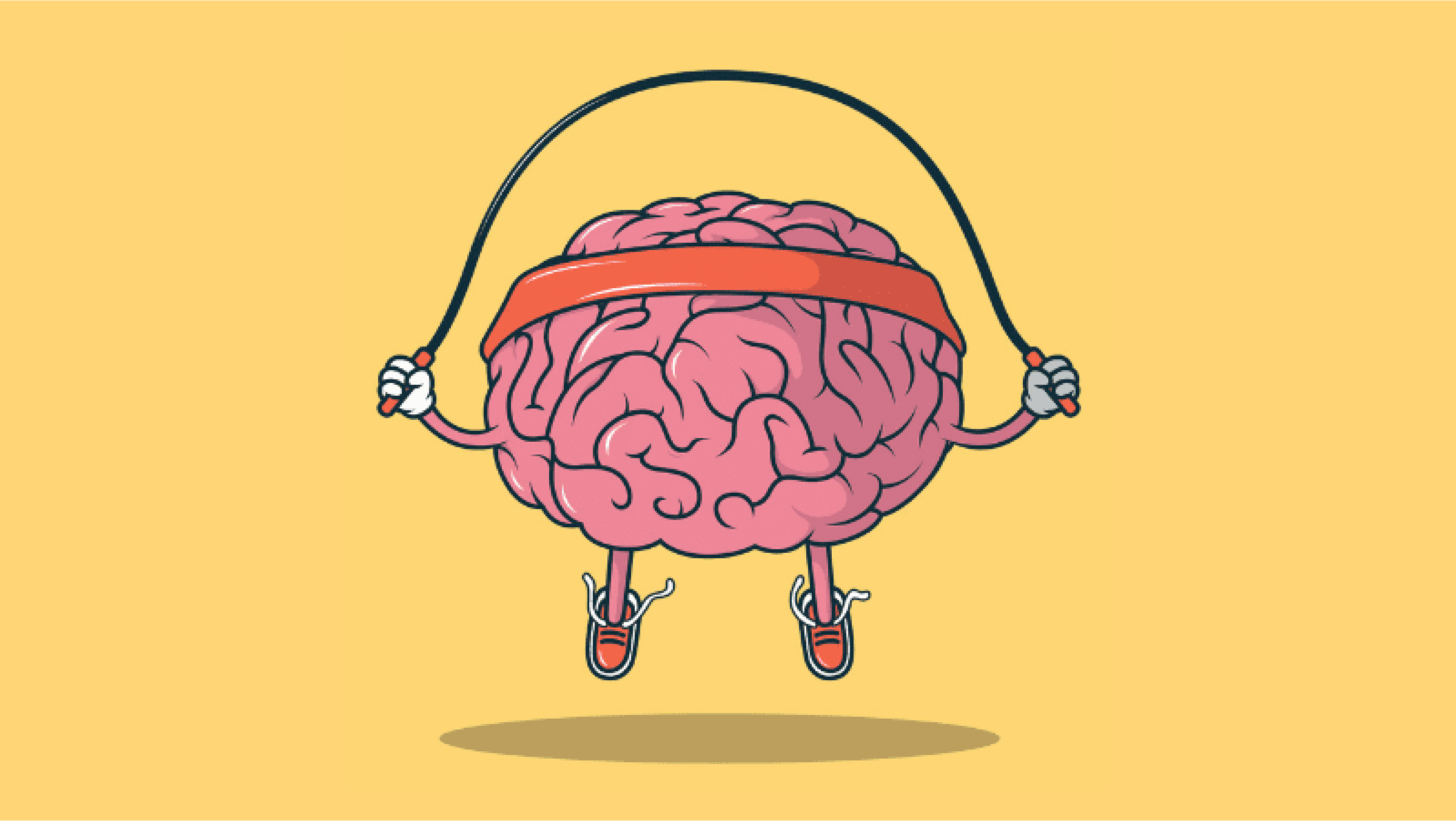



A → person lives till 55

B → person lives till 50

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(55 \cap 50)}{P(50)} = \frac{P(55)}{P(50)} = \frac{.881}{.913}$$

# Brain Break – 1 min

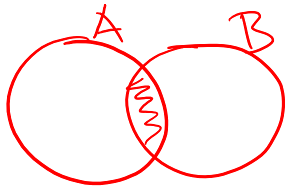


# Conditional Probabilities

- Suppose we roll a white and a black die. What is the probability that the white die is 1 given that the sum is 7?
- $A$ = white die is 1
- $B$ = sum if 7
- We want $P(A|B)$

$\rightarrow$ Conditional Probability

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$




We know B has happened

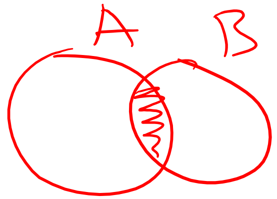
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ $\rightarrow$ Formula for conditional probability

$$= \frac{|A \cap B|}{|B|}$$ $\rightarrow$ If equally likely outcomes

# Conditional Probabilities

- Suppose we roll a white and a black die. What is the probability that the white die is 1 given that the sum is 7?
- $A$ = white die is 1
- $B$ = sum if 7
- We want $P(A|B)$

$$= \frac{|A \cap B|}{|B|}$$




$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{6}$$

# Conditional Probabilities

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

If equally likely outcomes $\frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|}$

$$P(A|B,C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$