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Creativity is intelligence having fun
- Albert Einstein

206
Discrete Structures II

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Lecture 8 | Counting | Monday October 9th 2023

What we will cover today

Combinatorics

- Recap
  - Counting (Partition, Difference, Product Rule)
- Today
  - Counting
    - Product Rule
    - Bijection Rule
  - Permutations/Combinations (Intro…)
- Next
  - Pigeonhole Principle

Course Outline

- Part I
  - Recap of basics – sets, function, proofs, induction
  - Basic counting techniques
  - Pigeonhole principle
  - Generating functions
- Part II
  - Sample spaces and events
  - Basics of probability
  - Independence, conditional probability
  - Random variables, expectation, variance
  - Moment generating functions
- Part III
  - Graph Theory
  - Machine learning and statistical inference

Counting

Counting

- In the next few lectures
  - Fundamental tools and techniques for counting
  - Sum Rule
  - Product Rule
  - Difference Method
  - Bijection Method
  - Permutations/Combinations
  - Inclusion Exclusion
  - Binomial/Multinomial coefficients

Fundamental Blocks

→ Intermediate

→ Advanced

Lehman, 475-480

Product Rule $A \times B$

- Elements of $A \times B$ are ordered pairs:

$$A \times B = \{(x,y) : x \in A, y \in B\}$$

Product Rule:

$$|A \times B| = |A| \cdot |B|$$

- To create $A \times B$ elements, we choose one element from A and also one element from B.

e.g., If there are 4 types of coffee

{espresso, americano, cappuccino, latte}

and 3 types of sugar

{raw sugar, white sugar, and brown sugar}

then there are $12 = 4 \times 3$ ways to make a coffee.

Product Method: Creating Pairs

- If I roll a white and black die, how many possible outcomes do I see?

$A =$ all outcomes of black die
$B =$ all outcomes of white die

all outcomes $= |A \times B| = |A| \cdot |B| = 36$

Question: Can you make the above question not solvable with the product rule?

Remember: Now we are leaving behind us our ability to count elements and start developing skills that help us count sets without explicitly counting their elements

Last Time...

Product Rule: Creating Sequences

- A restaurant has a menu with 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts.
  - How many ways to choose a complete meal?

A = all possible complete meals

= { (App, Entree, Salad, Dessert)
⋮
}

$|A| = 5 \times 6 \times 3 \times 7$

5 choices for Appetizers
6 " " Entre
3 " " Salad
7 " " Dessert

Product Rule

- A restaurant has a menu with 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts
  - How many ways to choose a meal if I'm allowed to skip some (or all) the courses?

A = { (App, Entree, Salad, Dessert)
(App)
(Entree)
⋮
}

Step 1: Make all elements the same length by including a null option. For ex: (Entree) becomes (null, Entree, null, null)

Step 2: 6 choices for Appetizer, 7 for Entree, 4 for Salad, 8 Dessert

Answer = 6×7×4×8

When to use Partition vs. Product Rule?

- Key: Do you have to choose an arrangement of a set A AND an arrangement of set B, or an arrangement of A OR an arrangement of B?

- If you choose arrangements for both, you use the **product rule**: The set of all possible choices is the cartesian product of the choices for one, and the choices for the other.

- If you choose an arrangement from one OR from the other, you use the **sum rule**: *The set of all possible choices is the sum (disjoint union) of the choices for one and the choices for the other.*

Practice: Counting Passwords...

- You are signing up for an account on FlixBiz.com. The password has the following requirements
  - The password must be 6 to 8 characters long.
  - Each password is an uppercase letter or digit.
  - Each password must contain **at least** one digit.

Partition Method

Q: How many possible passwords?

$A_6 \to$ all passwords with length 6

$A_7 \to$ " 7

$A_8 \to$ " 8

all passwords $= |A_6| + |A_7| + |A_8|$

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Last Time…

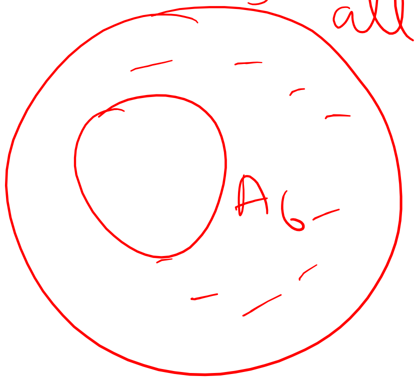
Exercise: Counting Passwords…

$A_6 =$ all valid passwords of length 6

$S =$ all passwords of length 6

$B = S \setminus A_6$

$B =$ all passwords of length 6 with no digits



Difference Method

=>

Find Contrapositive

(see Hint on next slide)

$|A_6| = |S| - |B|$

$|S| = 36^6$

$|B| = 26^6$

$\Rightarrow |A_6| = 36^6 - 26^6$

Partition Method

$|A_7| = 36^7 - 26^7$

$|A_8| = 36^8 - 26^8$

Last Time...

Hint: When to use Difference Method

When you are asked to count something that exists in

"at least" one place, consider counting the opposite

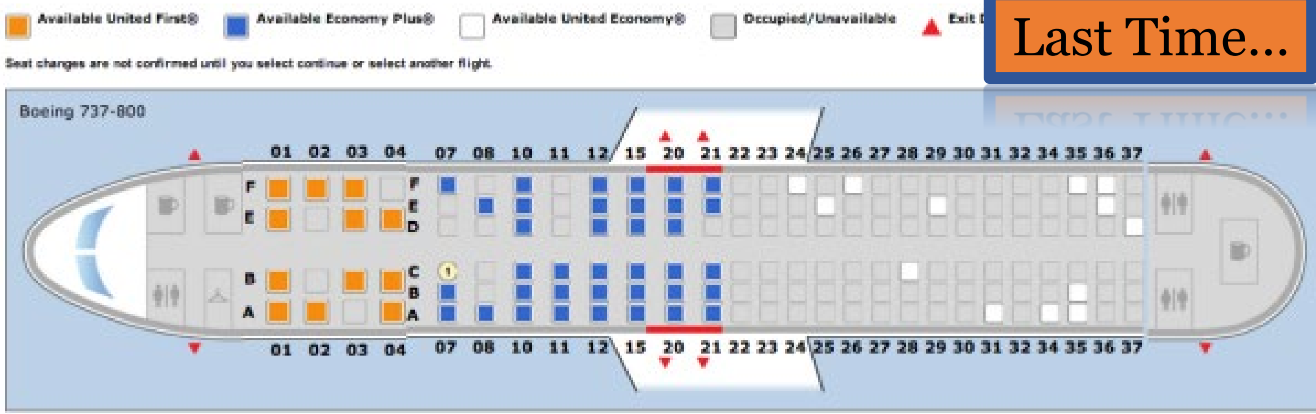
(that is "nowhere")

Which means: You need to be able to find the

"contrapositive argument".

Generalized Product Rule

Last Time...



- How many ways to assign 100 passengers to 100 seats?

Let $P_1, \ldots, P_{100}$ be the passengers.

100 choices for seat of P_1

99 choices for seat of P_2

$\vdots$

1 choices for seat of P_{100}

$\Rightarrow$ answer $= 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$

Generalized Product Rule – Order is important

- Suppose every object of a set S, can be constructed by a sequence of n choices with P_1 possibilities for the first choice, P_2 possibilities for the second choice, and so on
- **IF**
  - Each sequence of choices constructs an object in S.
  - No two different sequences **create the same** object
- THEN
  - $|S| = P_1 \times P_2 \times \cdots P_n$

Generalized Product Rule

- How many subsets of a 100 element set?

(revising the first problem we ever saw…)

2 choices for each element (include it or not)

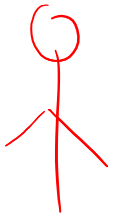
answer = 2^{100}

Counting Pitfalls – and how to avoid them

- You are signing up for an account on FlixBiz.com. The password has the following requirements
  - The password must be 6 to 8 characters long.
  - Each password is an uppercase letter or digit.
  - Each password must contain at least one digit.

$A_6 \to$ all valid pass words of length 6

Can I use the Generalized Product Rule?



- Pick position of first digit $\to$ 6 ways $\longrightarrow$ wrong
- Pick value $\longrightarrow$ 10 ways
- Pick remaining values $\to 36^5$

$|A_6| = 6 \times 10 \times 36^5 \neq 36^6 - 26^6$

We are overcounting…Why?

Counting Pitfalls

Process 1

→ pick position →

→ pick value →

→ pick remaining →

S = all valid passwords

AB4CDE ✓

A64BGZ

⋮

→ every choice sequence in Process 1 maps to a unique element of set

Can I use the Generalized Product Rule?

→ Given element of S must be able uniquely decode how we got to it

→ In process 1 multiple ways to reach A64BGZ

Hint! Counting Pitfalls when…

- …Many choice sequences lead to the same element.
  - This is when the elements can be interchangeable, e.g., when we deal with two white dices, which are the same when it comes to counting their value

- …Imposing an ordering when the problems does not have any.

Product Rule – Counting Pitfalls cont'd

- How many binary strings of length 8 with exactly two 0's?

Pick location of 1^{st} 0 → 8 ways

Pick location of 2^{nd} 0 → 7 ways

answer = $8 \cdot 7 = 56$

Pick location of 1^{st} 0

Pick location of 2^{nd} 0

Choice sequence

Two ways to reach this

00111111

01011111

⋮

Actual set

Product Rule

- How many binary strings of length 8 with exactly two 0's?

$A_1 =$ all strings with 2 0's such that 1st zero from left appears at position 1.

$A_2 =$ 1st zero is at position 2

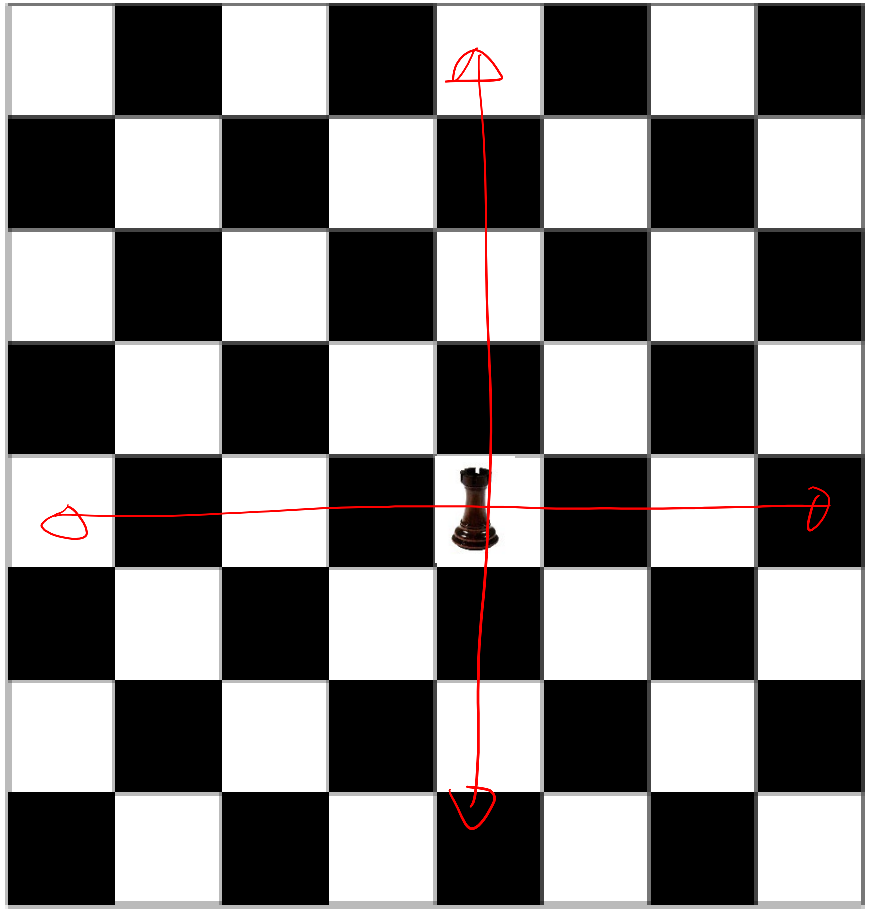
$A_3 =$ 1st zero is at position 3

$\vdots$

$A_8 =$ 1st zero is at position 8

$$\left.\begin{aligned} |A_1| &= 7 \\ |A_2| &= 6 \\ |A_3| &= 5 \\ |A_4| &= 4 \\ |A_5| &= 3 \\ |A_6| &= 2 \\ |A_7| &= 1 \\ |A_8| &= 0 \end{aligned}\right\} 28$$

Generalized Product Rule



- Given two rooks labeled 1 and 2
- How many ways to place them so that they don't threaten each other?

– 64 choices for first rook
– (64-15) choices for second rook

answer (64·49)

Generalized Product Rule

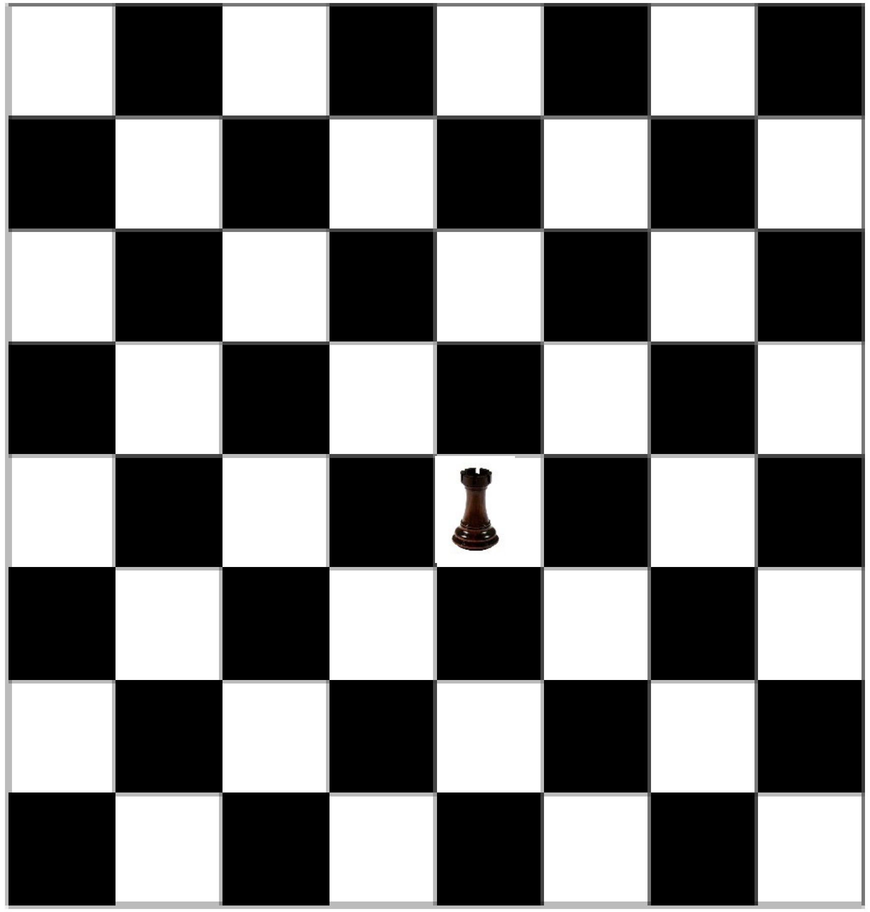
Difference Method

$B = S \setminus A$

$=$ all ways where they threaten

$|S| = 64 \cdot 63$

$|B| = 64 \cdot 14$



How many ways to place two rooks so that they don't threaten each other?

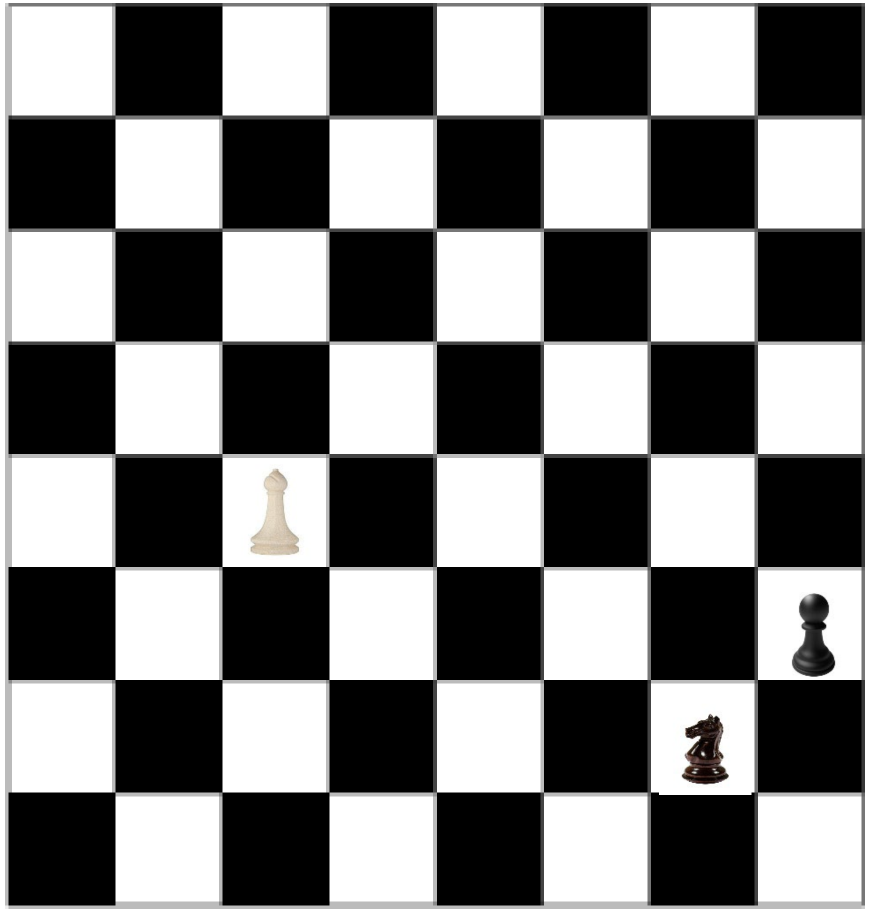
$S =$ all ways

$A =$ all ways such that they don't threaten

$\Rightarrow |A| = |S| - |B| = 64 \cdot 49$

Generalized Product Rule

Potential Pitfall????



YES! If we had two (interchangeable) knights!!!

How many ways to place a knight, bishop, and pawn so that no two share a row or column?

Pick row for bishop $\rightarrow$ 8
Pick colum for bishop $\rightarrow$ 8
Pick row for knight $\rightarrow$ 7 ways
Pick col for knight $\rightarrow$ 7 ways
Pick row for pawn $\rightarrow$ 6 ways
Pick col for pawn $\rightarrow$ 6 ways

ans $= 8^2 7^2 6^2$