combra.cs.rutgers.edu

206
Discrete Structures II

Konstantinos P. Michmizos

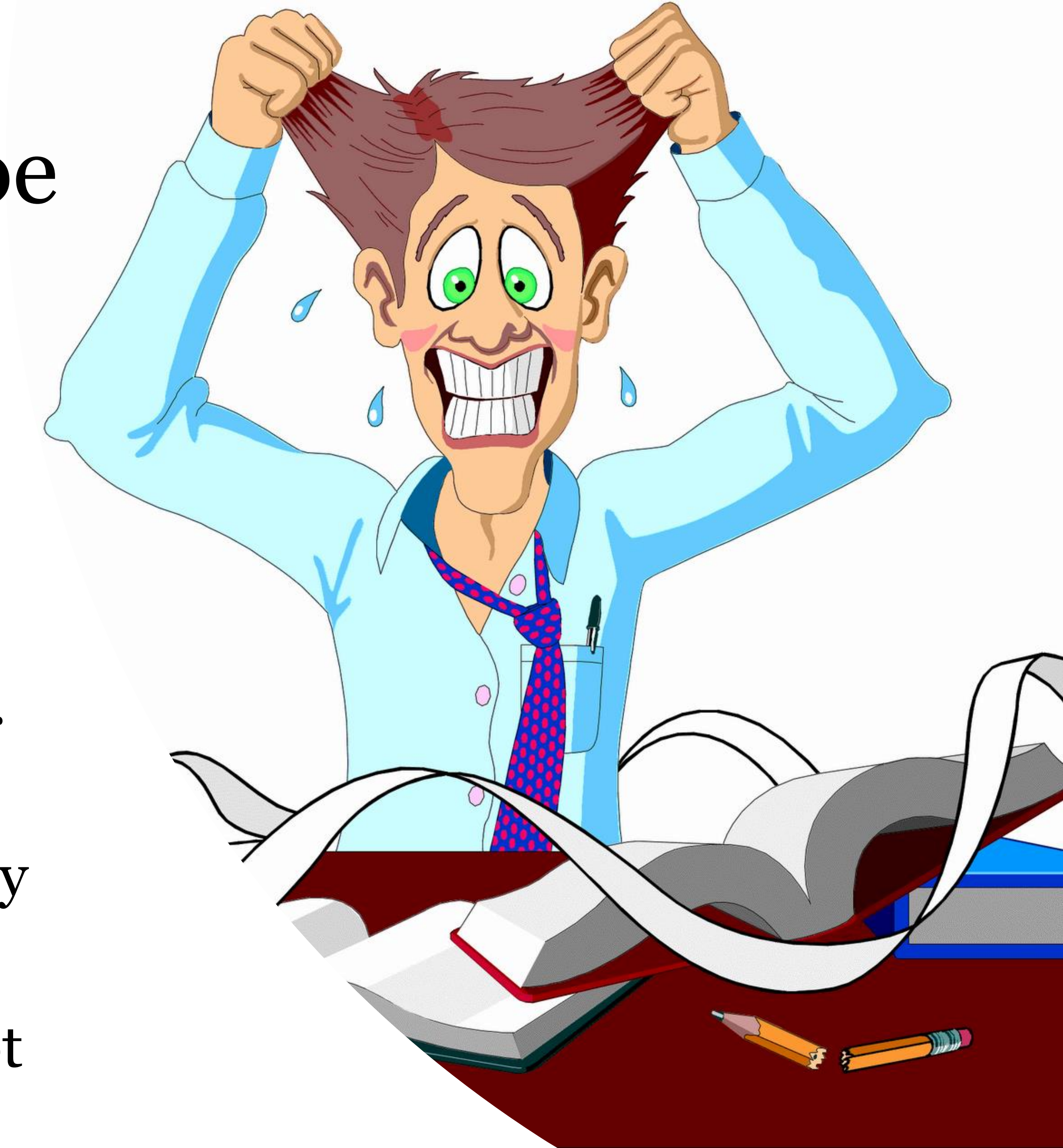
Computational Brain Lab

Computer Science | Rutgers University | NJ, USA

Lecture 18 | Combinatorial Proofs & Binomial Coefficients| Monday November 13th 2023

Midterm – It won't be that bad, promise!

- 130 points
- 60 points are easy
- 70 points are easier
- Focus on what you know better
- Typically, more points per problem means higher difficulty
- Do not panic
- The person next to you does not know more than you do!



Midterm – Material to cover

- **Proofs/Induction**
- Sum Rule
- **Difference Method**
- Bijection Method
- **Product Rule**
- Generalized product rule
- **Permutation/Combinations**
- Inclusion-Exclusion / **Pigeonhole Principle**
- **Combinatorial Proofs** and **Binomial Coefficients**

Good luck!

You have tried hard to eliminate the factor of "luck"

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Please see EXTRA PROBLEMS posted on canvas

Combinatorial Proofs

In general, to give a combinatorial proof for a binomial identity, say A = B you do the following:

1. **Find a counting problem** you will be able to answer in two ways.
2. Explain why one answer to the counting problem is A.
3. Explain why the other answer to the counting problem is B.

Since both A and B are the answers to the same question, we must have A=B.

The tricky thing is coming up with the question. This is not always obvious, but it gets easier the more counting problems you solve.

Combinatorial Proofs – Hints!

- Define a set S.
- Show that $|S| = n$ by counting one way.
- Show that $|S| = m$ by counting ***another way***.
- Conclude that $n = m$.

Combinatorial Proofs

- Proving algebraic identities via counting

IDENTITY

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + +3ab^2 \pm b^3$

$(a \pm b)^4 = a^4 \pm 4a^3b + +6a^2b^2 \pm 4ab^3 + +b^4$

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$

$(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 6abc$$
$$+3(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2)$$

$$(a_1 + a_2 + \cdots a_n)^2 =$$
$$= {a_1}^2 + {a_2}^2 + \cdots {a_n}^2 + 2(a_1a_2 + a_1a_3 + \cdots a_{n-1}a_n)$$

$a^2 - b^2 = (a - b)(a + b)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2$$
$$= (a^2 + \sqrt{2}ab + b^2)(a^2 - \sqrt{2}ab + b^2)$$

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$$
$$= (a + b)(a - b)(a^2 + b^2)$$

$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

Combinatorial Proofs

- Prove that $\binom{n}{k} = \binom{n}{n-k}$

$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$

$LHS = \frac{n!}{(n-k)!\,k!} \longrightarrow$ use formula

$RHS = \frac{n!}{k!\,(n-k)!} \longrightarrow$ use formula

$= LHS$

Combinatorial Proofs

- Prove that $\binom{n}{k} = \binom{n}{n-k}$

Alternate Proof

— Define a counting problem

— In this case counting problem = # ways to select k out of n people

— One way to count = $\binom{n}{k}$ = LHS

— Another way to count = choose n-k people to not select

= $\binom{n}{n-k}$

— Both ways solving the same problem, Hence $\binom{n}{k} = \binom{n}{n-k}$

Example #1

- How many 10-letter words use exactly four A’s, three B’s, two C’s and one D?

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1}$$

or…

$$\binom{10}{1}\binom{9}{2}\binom{7}{3}\binom{4}{4}$$

Example #2

$$1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1 = \binom{n+2}{3}$$

<u>**Define our Question**</u>: We need to count the number of ways to select 3 things from a group of n + 2 things.

Let's name those things 1, 2, 3, . . . , n + 2.

In other words, we want to find 3-element subsets of those numbers (since order does not matter, subsets are exactly the right thing to think about).

We now have to explain why the left-hand-side also gives the number of these subsets.

Consider the question "How many 3-element subsets are there of the set {1, 2, 3, . . . , n + 2}?□

Example #2 – cont'ed

[key thought] Break this problem up into cases by <u>what the middle number in the subset is</u>.

Say each subset is {a, b, c} <u>written in increasing order</u>.

We count the number of subsets for each distinct value of b. The smallest possible value of b is 2, and the largest is $n + 1$.

When $b = 2$, there are $1 \cdot n$ subsets: 1 choice for a and n choices (3 through $n + 2$) for c.

When $b = 3$, there are $2 \cdot (n - 1)$ subsets: 2 choices for a and $n - 1$ choices for c.

When $b = 4$, there are $3 \cdot (n - 2)$ subsets: 3 choices for a and $n - 2$ choices for c.

…When $b = n + 1$, there are n choices for a and only 1 choice for c, so $n \cdot 1$ subsets.

Example #2 – Done!

- Therefore the total number of subsets is

$$1n + 2(n - 1) + 3(n - 2) + \cdots + (n - 1)2 + n1.$$

$$1n + 2(n - 1) + 3(n - 2) + \cdots + (n - 1)2 + n1 = \binom{n+2}{3}$$

Example #3

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Define the Question: How many pizzas can you make using n toppings when there are 2n toppings to choose from?

Example #3

Left Side: Divide the toppings into two groups of n toppings (perhaps n meats and n veggies).

Any choice of n toppings must include some number from the first group and some number from the second group.

Consider each possible number of meat toppings separately:

0 meats: $\binom{n}{0}\binom{n}{n}$, since you need to choose 0 of the n meats and n of the n veggies.

1 meat: $\binom{n}{1}\binom{n}{n-1}$, since you need 1 of n meats so $n-1$ of n veggies.

2 meats: $\binom{n}{2}\binom{n}{n-2}$. Choose 2 meats and the remaining $n-2$ toppings from the n veggies.

And so on. The last case is n meats, which can be done in $\binom{n}{n}\binom{n}{0}$ ways.

Example #3 – Done!

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$$

Combinatorial Proofs – Example #4

- Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Let $S = \{1, 2, \cdots n\}$

Counting problem: How many subsets of S ?

RHS = 2 choices for each element

$\longrightarrow$ Hence, # subsets $= 2^n$

LHS = use partition rule

$\longrightarrow$ Count all subsets of size $0 \rightarrow \binom{n}{0}$

$\longrightarrow$ " " size $1 \rightarrow \binom{n}{1}$ $= \sum_{k=0}^{n} \binom{n}{k}$

$\longrightarrow$ " size $n \rightarrow \binom{n}{n}$

Combinatorial Proofs – Example #5

- Prove that $\sum_{k=0}^{\cancel{n}\ n/2} \binom{n}{2k} = 2^{n-1}$

Problem: # even sized subsets of n elements

RHS = 2^{n-1}

LHS = Use partition method

— subsets of size 0 → $\binom{n}{0}$

— subsets of size 2 → $\binom{n}{2}$ → $\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}$

— subsets of size 4 → $\binom{n}{4}$

⋮

Combinatorial Proofs

- Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Certainly a valid proof, but also entirely useless
Even if you understand the proof perfectly, it does not tell you why the identity is true.

Proof. By the definition of $\binom{n}{k}$, we have

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = \frac{(n-1)!}{(n-k)!(k-1)!}$$

and

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!}.$$

Thus, starting with the right-hand side of the equation:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!} \\
&= \frac{(n-1)!k}{(n-k)!\,k!} + \frac{(n-1)!(n-k)}{(n-k)!\,k!} \\
&= \frac{(n-1)!(k+n-k)}{(n-k)!\,k!} \\
&= \frac{n!}{(n-k)!\,k!} \\
&= \binom{n}{k}.
\end{aligned}$$

The second line (where the common denominator is found) works because $k(k-1)! = k!$ and $(n-k)(n-k-1)! = (n-k)!$. QED

Combinatorial Proofs – Example #6

- Prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Counting Problem: # ways to choose k out of n people

— LHS = $\binom{n}{k}$

RHS: Use partition method

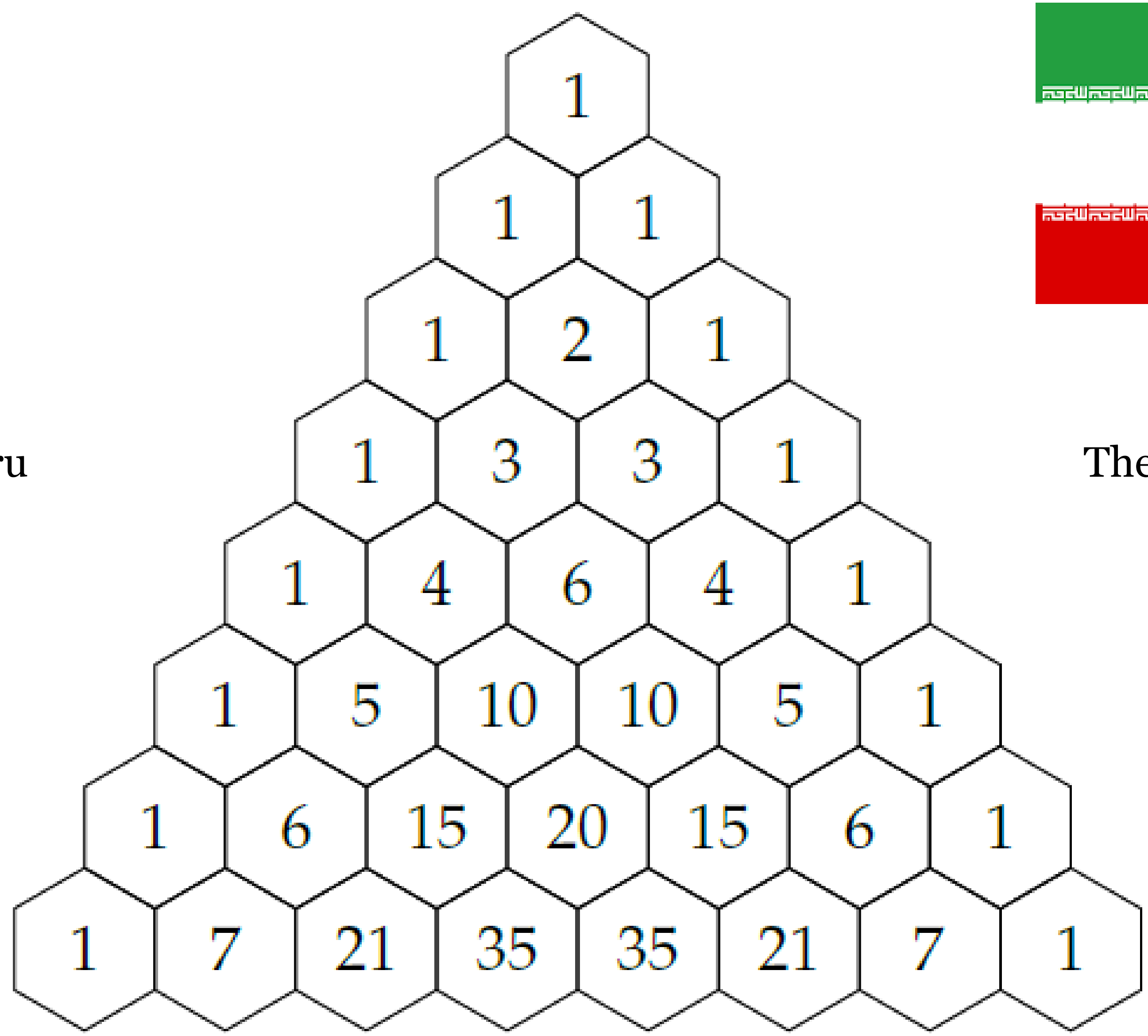
Case1: # ways to chose k out of n such that element 1 is chosen

Case2: # ways to chose k out of n such that element 1 is not chosen

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

The Staircase of Mount Meru

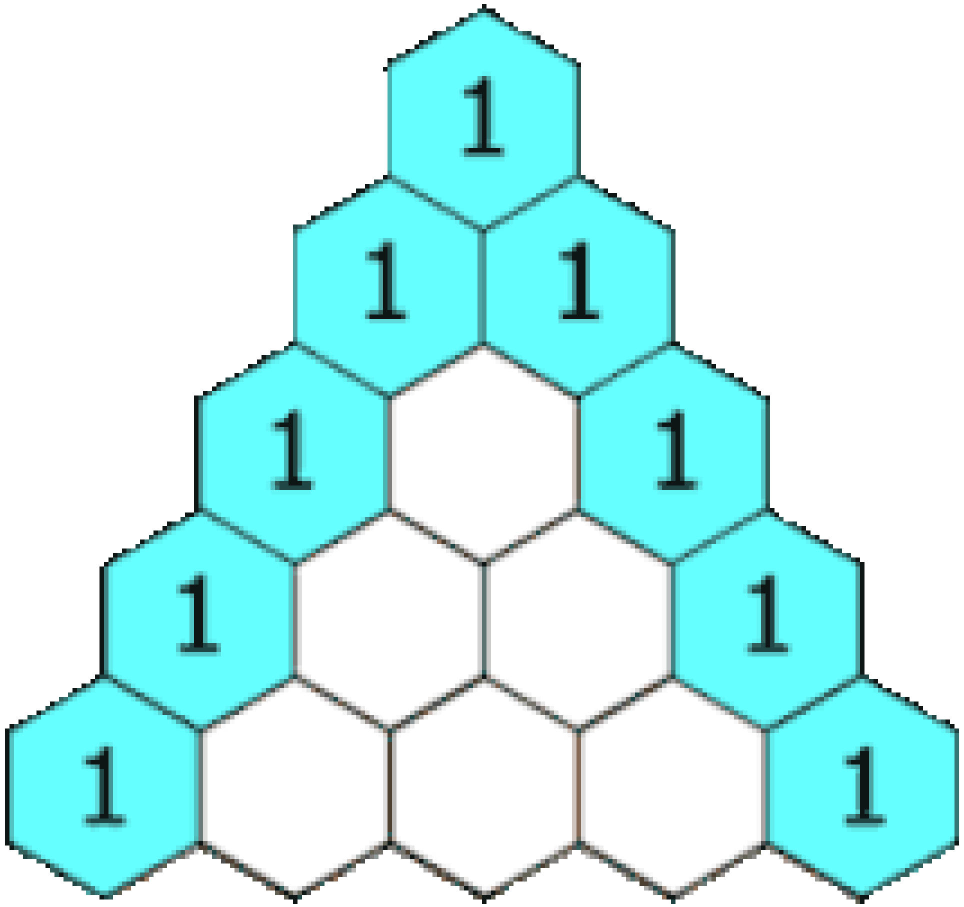
The Yang Hui's Triangle



The Khayyam Triangle

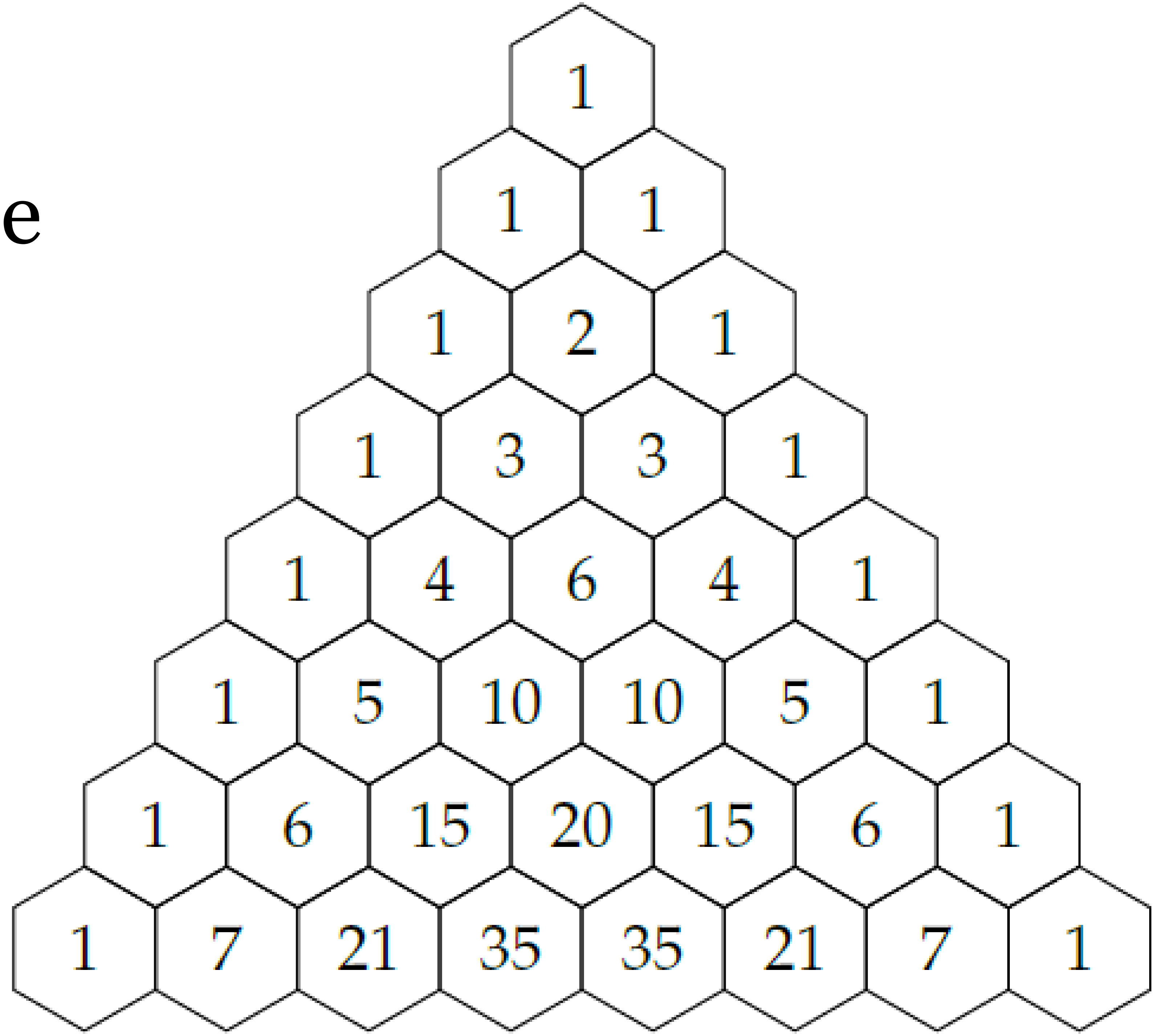
The Pascal's Triangle

How to construct a Pascal’s triangle

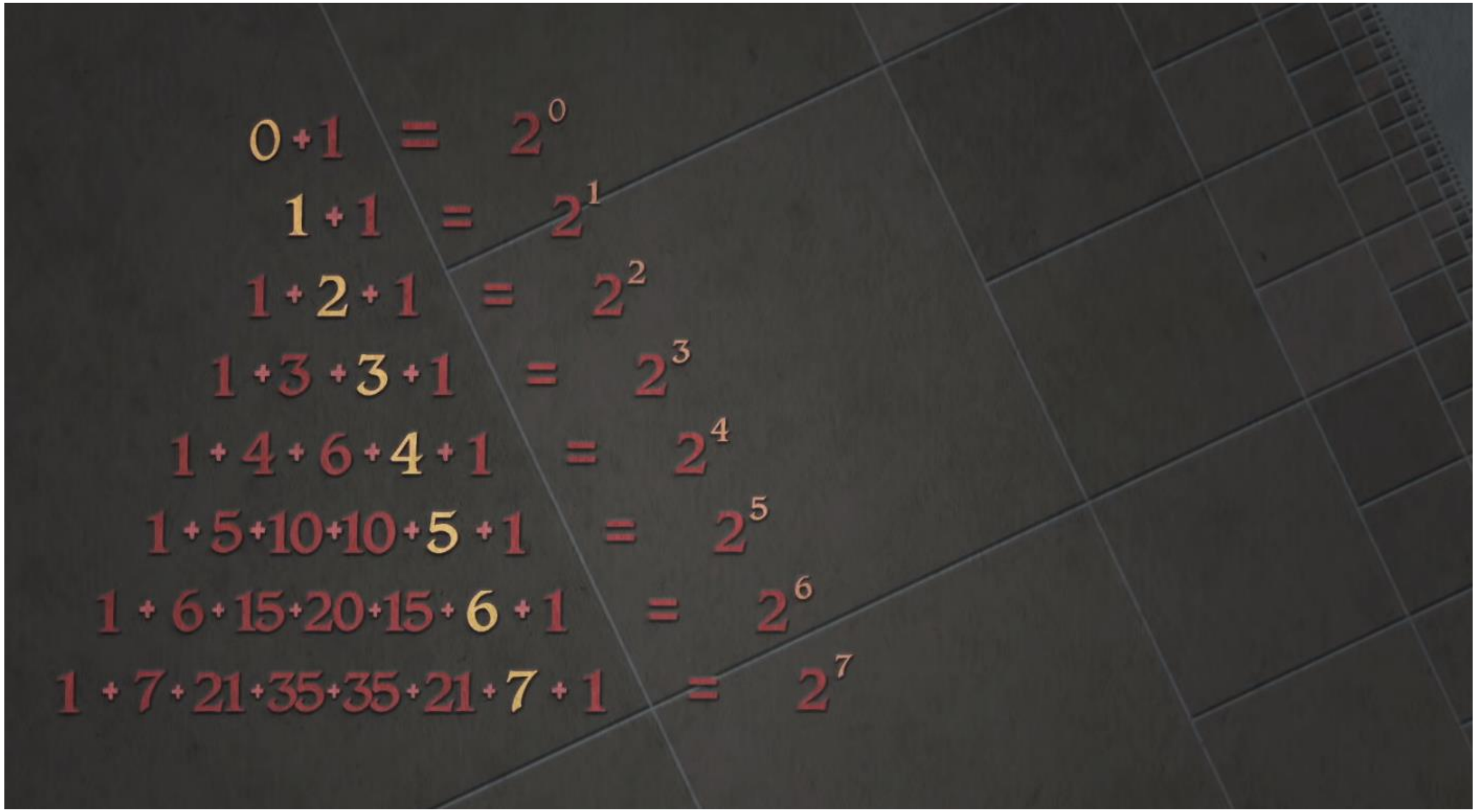


Pascal’s Triangle

- 1. The entries on the border of the triangle are all 1.
- 2. Any entry not on the border is the sum of the two entries above it.
- 3. The triangle is symmetric. In any row, entries on the left side are mirrored on the right side.
- 4. The sum of all entries on a given row is a power of 2. (Check this!)



Pascal's Triangle



Look at Sierpinski Triangle…

Pascal's Triangle

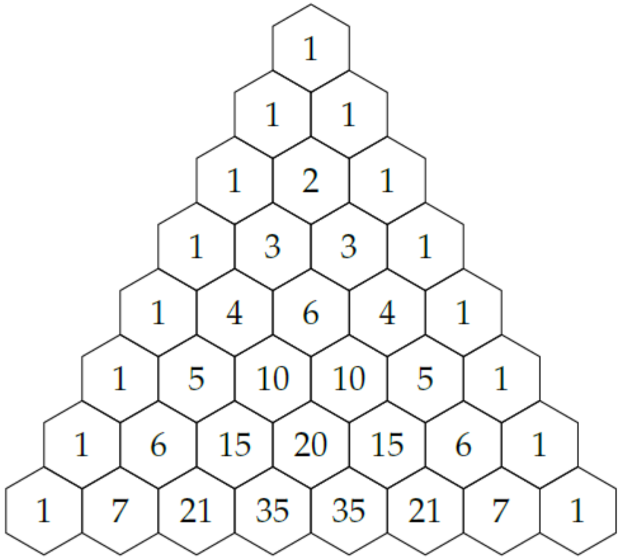
Line 4:

1, 4, 6, 4, 1

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

- Each entry in Pascal's triangle is in fact a binomial coefficient.

- We will be using Pascal's triangle (and other counting methods we have learned) to prove binomial identities, i.e., equations that involve binomial coefficients



Binomial Coefficients

- $\binom{n}{k}$, known as the ***Binomial Coefficient***.
  - Number of ways to pick k out of n distinct objects.
  - Intimately connected to algebraic polynomials.

Examples:

$$\left.\begin{array}{l} 1+x \\ 1+x+3x^2 \\ 5x^3-2x^2+7x-8 \end{array}\right\} \rightarrow \text{univariate polynomials}$$

General: $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_nx^n \rightarrow$ degree n polynomial

Basic Problem: Given a polynomial, infer the co-efficients

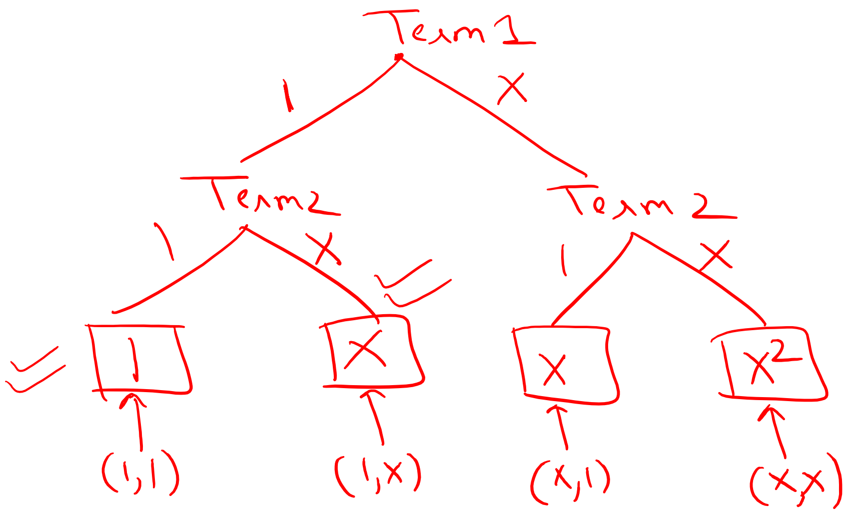
Binomial Coefficients

- $(1+x)^2 = 1 + 2x + x^2$

Given: $(1+x)^2 \longrightarrow (1+x)\cdot(1+x) = 1 + x + x + x^2$

$= 1 + \underline{\underline{2x}} + x^2$

$(1+x)^2 \longrightarrow \underset{\text{Term 1}}{(1+x)}\,\underset{\text{Term 2}}{(1+x)}$



Co-efficient of x = # ways to reach x from root node

$= 2$

ways to reach x

= # ways to choose x once out of 2 terms $= \binom{2}{1}$

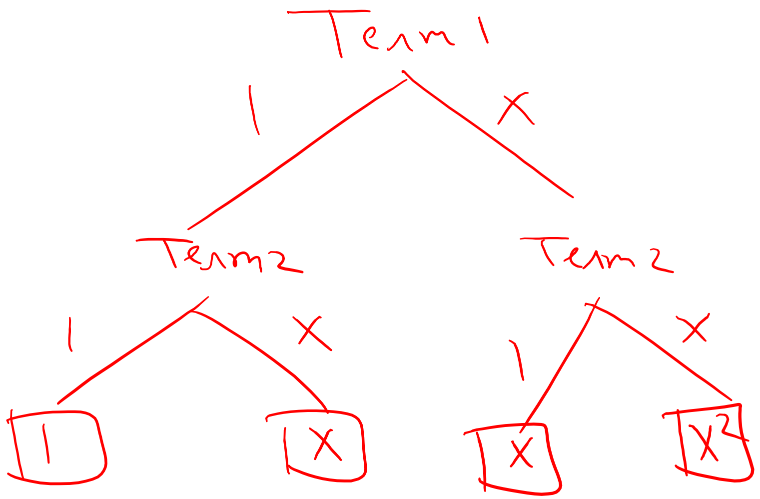
Binomial Coefficients

- $(1 + x)^2 = 1 + 2x + x^2$

ways to reach $x = \binom{2}{1} = 2$

ways to reach $x^2 = \binom{2}{2} = 1$

ways to reach $1 = \binom{2}{2} = 1$



Binomial Coefficients

- $(1+x)^3 = 1 + 3x + 3x^2 + x^3$

$(1+x)\,(1+x)\,(1+x)$

Term1 Term2 Term3

Co-efficient of x

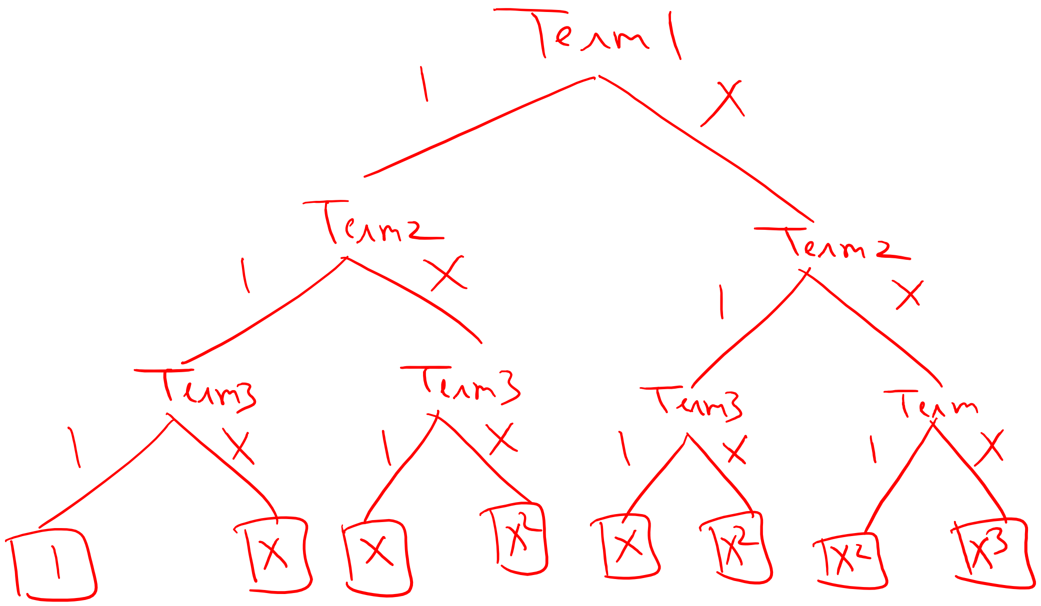
$= \#$ ways to reach x

$= \binom{3}{1} = 3$

Co-efficient of x^2

$= \#$ ways to reach x^2

$= \binom{3}{2} = 3$



Binomial Coefficients

- $(1 + x)^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$

$c_n = \#$ ways to reach $x^n = \binom{n}{n}$

$c_{n-1} = \#$ ways to reach $x^{n-1} = \binom{n}{n-1}$

$\vdots$

$c_k = \#$ ways to reach $x^k = \binom{n}{k}$ — Binomial Coefficients

$\vdots$

$c_0 = \#$ ways to reach $x^0 = \binom{n}{0} = 1$

Binomial Coefficients

- $(1+x)^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$

- What is c_k?
  - Number of paths in the choice tree with exactly k x's.
  - $= \binom{n}{k}$

The Binomial Formula – Univariate Case

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

The Binomial Formula – Example 1

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

- $x = 1$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n}$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$\Rightarrow$ Proof that the size of the powerset is 2^n

The Binomial Formula – Example 2

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

- $x = -1$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} \cdots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

$\to$ # subsets of odd size = # subsets of even size

The Binomial Formula - Example 3

- Prove that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

$$\frac{\partial}{\partial x}\left[(1+x)^n\right] = \frac{\partial}{\partial x}\left[\binom{n}{0} + x\binom{n}{1} + \underline{x^2\binom{n}{2}} + \cdots x^k\binom{n}{k} + \cdots x^n\binom{n}{n}\right]$$

$$= n(1+x)^{n-1} = \binom{n}{1} + 2x\binom{n}{2} + \cdots kx^{k-1}\binom{n}{k} + \cdots nx^{n-1}\binom{n}{n}$$

Set $x=1$

$$n\cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots k\binom{n}{k} + \cdots n\binom{n}{n}$$

$$\Rightarrow \quad n\cdot 2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

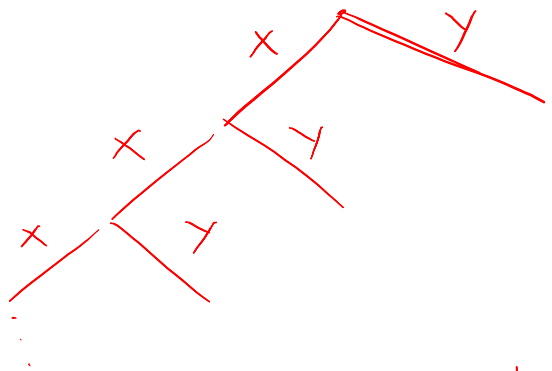
The Binomial Formula – Multivariate

$$(x + y)^n$$

multivariate polynomial

$$(x+y)^n = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots a_n y^n$$

$$\underbrace{(x+y)}_{T1}\underbrace{(x+y)}_{T2} \cdots \underbrace{(x+y)}_{Tn}$$



$a_0 = \#$ ways to reach x^n

$= \binom{n}{n} = \binom{n}{0}$

$a_1 = \#$ ways to reach $x^{n-1}y$

$= \binom{n}{n-1} = \binom{n}{1}$

$a_k = \#$ ways to reach $x^{n-k}y^k$

$= \binom{n}{n-k} = \binom{n}{k}$

The Binomial Formula - Multivariate

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

The Binomial Formula – Example 4

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Find coefficient of $x^{10}y^5$ in $(x+y)^{15}$

- 15 terms
- Want to get $x^{10}y^5$
- $\#$ ways $= \binom{15}{10} = \binom{15}{5}$

The Binomial Formula – Example

Find coefficient of $x^{10}y^5$ in $(19x+4y)^{15}$

$(19x+4y)(19x+4y) \cdots (19x+4y)$

— Want to get $x^{10}y^5$

— # ways $= \binom{15}{10} = \binom{15}{5}$

$\to$ Co-efficient $= \binom{15}{10}(19)^{10}(4)^5$

Coefficient of $x^{10}y^5$ in $(19x-4y)^{15}$

Co-efficient $= \binom{15}{10}(19)^{10}(-4)^5$

The Multinomial Formula – 3 variables

$$(x+y+z)^n$$

$(x+y+z)^n = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-1} z - - - -$

$+ a_k x^a y^b z^c + - - -$

$x^a y^b z^c \rightarrow$ choice sequence must have a xs, b ys and c zs.

$\rightarrow$ any arrangement of a xs, b ys, c zs gives a valid way to get $x^a y^b z^c$

Coefficient of $x^a y^b z^c$

= # arrangements

$= \dfrac{n!}{a!\,b!\,c!}$

The Multinomial Formula

$$(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \frac{n!}{k_1!\,k_2!\,k_3!} x^{k_1} y^{k_2} z^{k_3}$$

Midterm – Material to cover

- **Proofs/Induction**
- Sum Rule
- **Difference Method**
- Bijection Method
- **Product Rule**
- Generalized product rule
- **Permutation/Combinations**
- Inclusion-Exclusion / **Pigeonhole Principle**
- **Combinatorial Proofs** and **Binomial Coefficients**