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Wear a jacket – it's getting cold outside

206
Discrete Structures II

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Lecture 5 | Proofs – Proofs - Proofs | Wednesday September 27th 2023

What we will cover today

Combinatorics

- Recap
  - Proofs (Direct, Contrapositive, Case Analysis, Contradiction, Induction)
- Today
  - Counting
    - Partition Method
    - Difference Method
    - Product Rule

Next Time

- Bijection Rule

Course Outline

- Part I
  - Recap of basics – sets, function, proofs, induction
  - Basic counting techniques
  - Pigeonhole principle
  - Generating functions
- Part II
  - Sample spaces and events
  - Basics of probability
  - Independence, conditional probability
  - Random variables, expectation, variance
  - Moment generating functions
- Part III
  - Graph Theory
  - Machine learning and statistical inference

Proving an Implication via Direct Proof

- To prove: $P \Rightarrow Q$
  - Assume that P is true.
  - Show that Q logically follows

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So far...

Direct Proof

- To prove: $P \Rightarrow Q$
  - Assume that P is true.
  - Show that Q logically follows

The sum of two even numbers is even.

Proof $x = 2m,\ y = 2n$

$$x+y = 2m+2n$$
$$= 2(m+n)$$

The product of two odd numbers is odd.

Proof $x = 2m+1,\ y = 2n+1$

$$xy = (2m+1)(2n+1)$$
$$= 4mn + 2m + 2n + 1$$
$$= 2(2mn+m+n) + 1$$

Example of Proving an Implication

P Q

- Theorem: $1 \leq x \leq 2 \Rightarrow x^2 - 3x + 2 \leq 0$

Assume $1 \leq x \leq 2$

Step 1: $x^2 - 3x + 2 = (x-1)(x-2)$

Step 2: $1 \leq x \Rightarrow (x-1) \geq 0$

Step 3: $x \leq 2 \Rightarrow (x-2) \leq 0$

Step 4: $(x-1) \geq 0,\ (x-2) \leq 0 \Rightarrow (x-1)(x-2) \leq 0$

Intuition: When x grows, 3x grows faster than x^2 in that range.

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proving "if P, then Q" is equivalent to proving "if not Q, then not P"

Proof by Contrapositive

- To prove: $P \Rightarrow Q$
  - Prove that $\neg Q \Rightarrow \neg P$.

  - Assume ¬Q is true and show that $\neg P$ follows logically.

*i.e., we will **assume the opposite of our desired conclusion** and show that this fancy opposite conclusion could never be true in the first place.*

proving “if P, then Q” is equivalent to proving “if not Q, then not P”

Example of Proof by Contrapositive

- Theorem: *If r is irrational, then $\sqrt{r}$ is irrational.*

Rational Number

R is rational ⇔ there are integers a and b such that

numerator

$$r = \frac{a}{b}$$

denominator

and b ≠ 0.

Remember:

1. A number is rational if it is equal to a ratio of integers

*2. The **sum** of two rational numbers is always a rational number*

*3. The **difference** of two rational numbers is always a rational number*

*4. The **product** of two rational numbers is always a rational number*

*5. The **quotient** of two rational numbers is always a rational number* proving “if P, then Q” is equivalent to proving “if not Q, then not P”

Example of Proof by Contrapositive

- Theorem: *If r is irrational, then $\sqrt{r}$ is irrational.*

Proof:

We shall prove the contrapositive –

“if $\sqrt{r}$ is rational, then r is rational.”

Since $\sqrt{r}$ is rational, $\sqrt{r} = a/b$ for some integers a,b.

So $r = a^2/b^2$. Since a,b are integers, a^2,b^2 are integers.

Therefore, r is rational. Q.E.D.

(Q.E.D.) "which was to be demonstrated", or “quite easily done”. ☺

quod erat demonstrandum

Intuition: Square roots and absolute values are our worst enemies in proofs

So far...

Example of Proof by Case Analysis

- Theorem: *For all* $x \in \mathbb{R}, -5 \leq |x+2| - |x-3| \leq 5$

*We **hate** absolute values so we want to avoid them as fast as possible*

Two possible ways:-)

One of them is our goal here: To identify all possible cases

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So far...

Example of Proof by Case Analysis

- Theorem: *For all* $x \in \mathbb{R}, -5 \leq |x+2| - |x-3| \leq 5$

$|x+2| = x+2$, if $x+2 \geq 0$ or $x \geq -2$

$|x+2| = -(x+2)$, if $x+2 < 0$ or $x < -2$

$|x-3| = x-3$ if $x \geq 3$

$|x-3| = -(x-3)$ if $x < 3$

$x < -2$ $\quad$ $-2 \leq x < 3$ $\quad$ $x \geq 3 \rightarrow$ Case I

Values of x $\quad$ Case III $\quad$ -2 $\quad$ Case II $\quad$ 3

Example of Proof by Case Analysis

- Theorem: *For all* $x \in \mathbb{R}, -5 \leq |x+2| - |x-3| \leq 5$

Case I: $x \geq 3$, Want $-5 \leq (x+2) - (x-3) \leq 5$

Want $-5 \leq 5 \leq 5$ $\square$

Case II: $-2 \leq x < 3$, Want $-5 \leq (x+2) - -(x-3) \leq 5$

Want $-5 \leq x+2+x-3 \leq 5$

Want $-5 \leq 2x-1 \leq 5$ $\square$

Case III $x < -2$, Want $-5 \leq -(x+2) - -(x-3) \leq 5$

Want $-5 \leq -(x+2)+x-3 \leq 5$

Want $-5 \leq -5 \leq 5$ $\square$

Proof by Contradiction

$$\frac{\bar{P} \to \mathrm{F}}{P}$$

To prove P, you prove that **not P** would lead to a ridiculous result,

and ***so P must be true***.

I am working 20 hours per day.

If I had won the lottery, then I would not be working 20 hours per day.

$\therefore$ I have not won the lottery.

Proof by Contradiction – Work Chart

- To prove P
  - Assume P is false.
  - Logically deduce something that is known to be false.

Proof by Contradiction – Work Chart

To prove a proposition P by contradiction:

1. Write, “We use proof by contradiction.”
2. Write, “Suppose P is false.”
3. Deduce something known to be false (a logical contradiction).
4. Write, “This is a contradiction. Therefore, P must be true.”

Example of Proof by Contradiction

- Theorem: *There are infinitely many primes*

A **prime number** (or a prime) is a natural number greater than 1 that cannot be formed by multiplying two smaller natural numbers.

This is one of the most famous, most often quoted, and most beautiful proofs in all of mathematics. Its origins date back more than 2000 years to Euclid

Example of Proof by Contradiction

- Theorem: *There are infinitely many primes*

Assume: **There are finitely many primes** – And let p_1, p_2, …, p_N be all the primes.

Now we construct a new number, $p = p_1 p_2 \ldots p_N + 1$

Clearly, *p* is larger than any of the primes, so it does not equal one of them.
Therefore it cannot be prime and must be **composite**, i.e., divisible by at least one of the primes.

But our assumption was that *p* is not prime and therefore divisible by any prime number.

On the other hand, we know that any number must be divisible by *some* prime (*fundamental theorem of arithmetic or the unique factorization theorem or the unique-prime-factorization theorem*)

This leads to a **contradiction**, and therefore the assumption must be false.

So there must be infinitely many primes.

Lehman, 115-129

Induction

- Let $P(m)$ be a predicate of non-negative integers
- You want to prove that $P(m)$ is true for all non-negative integers.
- Step 1: Prove that $P(0)$ is true
- Step 2: Prove that $P(n) \Rightarrow \mathrm{P(n + 1)}$ for all non-negative integers.

New...

Example of Induction

- Theorem: *For all* $n \in \mathbb{N}$,
  - $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

New...

Example of Induction

- Theorem: *For all* $n \in \mathbb{N}$,
  - $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$P(n)$

Base Case: $P(0)$ is true, for $n = 0$
LHS = 0
RHS = 0

Inductive Step: $P(n) \Rightarrow P(n+1)$

Assume $P(n)$ is true. $1+2+\cdots+n = \frac{n(n+1)}{2}$

Want: $1+2+\cdots+n+n+1 = \frac{(n+1)(n+2)}{2}$

$1+2+\cdots+n+n+1 = \frac{n(n+1)}{2} + n+1 = \frac{(n+1)(n+2)}{2} = \text{RHS}$ $\square$

Intuition: During induction, my goal is to construct what I have assumed as true

New Stuff

Counting

Counting

- Basic Question: What is the size of a given set?

- Easy when the set is explicitly defined.
  - $X = \{1,2,3,4\}$, what is $|X|$?

- Tricky when set is implicit or a defined via set operations.
  - How many ways to get flush in the game of poker?
  - How many ways to assign time slots to courses at Rutgers?
  - How many operations before my algorithm terminates?

Counting

- In the next few lectures
  - Fundamental tools and techniques for counting
  - Sum Rule
  - Product Rule
  - Difference Method
  - Bijection Method
  - Permutations/Combinations
  - Inclusion Exclusion
  - Binomial/Multinomial coefficients

Fundamental Blocks

→ Intermediate

→ Advanced

Sum Rule

- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- How many students are there in total in both sections?

Sum Rule:
If A and B are **disjoint** sets, then $|A \cup B| = |A| + |B|$

$A =$ all students in section 5
$B =$ all students in section 6

$|A \cup B| = |A| + |B| = 60 + 71 = 131$

Example: Sum Rule

- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- There are 80 students in section 1 of 206.
- There are 80 students in section 2 of 206.
- How many students are there in total?

$60 + 71 + 80 + 80 = 291$

Example: Sum Rule

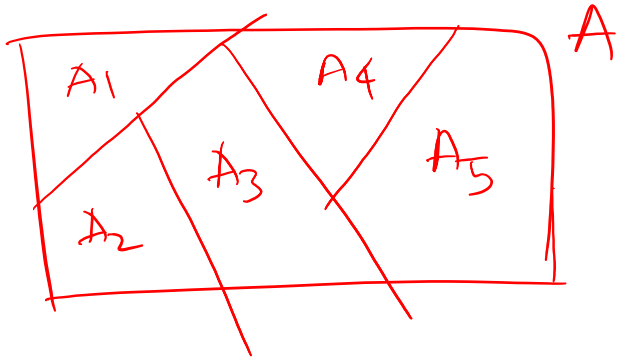
- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- There are 80 students in section 1 of 206.
- There are 80 students in section 2 of 206.
- How many students are there in total?

Sum Rule:
If $A_1, A_2, \ldots A_n$ are **disjoint** sets, then
$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$

Partition Method – How to

- To find the size of a set A,
  - Partition it into a union of disjoint sets $A_1, A_2, \ldots, A_n$
  - Use sum rule

- Example: How many students are there in total in 206?



Partition Method

- To find the size of a set A,
  - Partition it into a union of disjoint sets $A_1, A_2, \dots, A_n$
  - Use sum rule

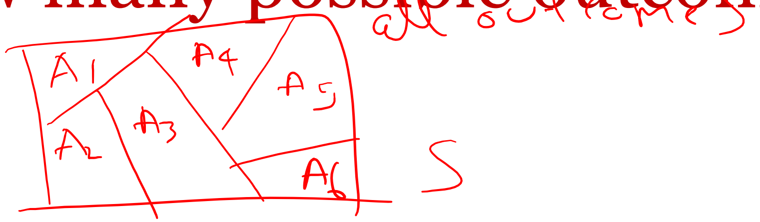
- If I roll a white and black die, how many possible outcomes do I see?

$$S = \left\{ \begin{array}{l} (1,1), (1,2), \dots, (1,6) \\ (2,1), (2,2), \dots, (2,6) \\ \vdots \\ (6,1), (6,2), \dots, (6,6) \end{array} \right\}$$

$$|S| = 36$$

Partition Method

- If I roll a white and black die, how many possible outcomes do I see?



A_1 = all outcomes with black die = 1

A_2 = all outcomes with black die = 2

⋮

A_6 = all outcomes with black die = 6

$$|S| = |A_1| + |A_2| + \dots + |A_6|$$
$$= 6 \cdot 6 = 36$$

Partition Method

- Possible outcomes where white and black die have different values?

$S =$ all possible outcomes

$A_1 =$ all outcomes with black die $= 1$

$A_2 =$ black die $= 2$

$\vdots$

$A_6 =$ black die $= 6$

$|A_1| = 5, \ |A_2| = 5,$

$|S| = 5 + 5 + 5 + \cdots 5 = 30$

Partition Method

- Possible outcomes where white die has a larger value then the black die?

$A_1 =$ all outcomes with black die $= 1$

$S =$ all possible outcomes

$\vdots$

$A_6 =$ black die $= 6$

$|A_1| = 5, \ |A_2| = 4, \ |A_3| = 3$
$|A_4| = 2, \ |A_5| = 1, \ |A_6| = 6$

$$|S| = 5+4+3+2+1 = \frac{5(5+1)}{2}$$
$$= 15$$