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206
Discrete Structures II

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Lecture 17 | Combinatorial Proofs & Binomial Coefficients | Wednesday November 8th 2023

So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- ~~Permutation/Combinations~~
- ~~Inclusion-Exclusion / Pigeonhole Principle~~
- Combinatorial Proofs and Binomial Coefficients

Pigeonhole Principle

If m pigeons are in n holes and $m > n$, then at least $\left\lceil \frac{m}{n} \right\rceil$ pigeons are in the same hole.

$\left\lceil \frac{m}{n} \right\rceil$ = nearest integer higher than $\frac{m}{n}$

Ceiling of m over n rounds the ratio to the larger integer

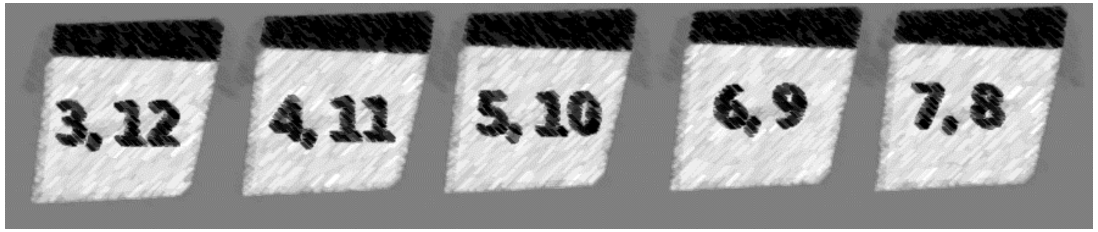
$m = 20$

$n = 9$

$\left\lceil \frac{20}{9} \right\rceil = 3$

PHP - Example

- Prove that if 6 integers are selected from {3,4,5,6,7,8,9,10,11,12}, there must be 2 integers whose sum is 15.
- Solution: Label 5 boxes



- We select 6 integers and place them in one of the boxes above, based on its label
- By PHP: One box must have at least 2 integers

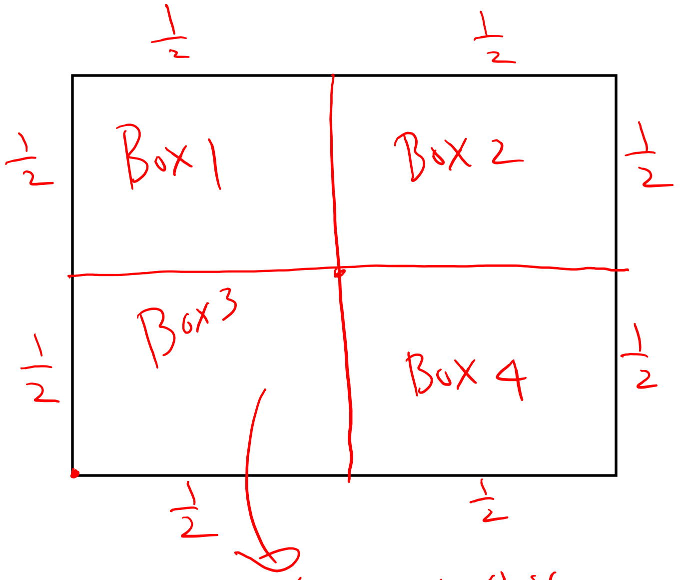
Pigeonhole Principle

- Consider any 5 points in the interior of a square of unit length. Show that one can find two points that are at a distance of at most $\frac{\sqrt{2}}{2}$.

— If we pick 5 points
Then by PHP some
box must have ≥ 2 points

— Maximum distance between
any two points in a box
is when they are diagonally
opposite. The distance is

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$



Let's say this
box has 2 points

Pigeonhole Principle

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 People be $P_1, P_2, P_3, P_4, P_5, P_6$

Define 2 boxes

friends to P1

Strangers to P1

— Every remaining person goes to one of these boxes depending on whether she/he knows P1 or not.

— By pigeonhole principle one of the two boxes must have at least $\lceil \frac{5}{2} \rceil = 3$ People

Pigeonhole Principle

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 people be $P_1, P_2, P_3, P_4, P_5, P_6$

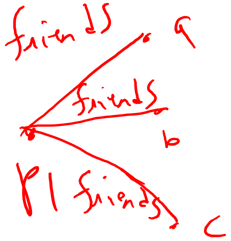
Define 2 boxes

friends of P1

Strangers to P1

— Case 1: Friends of P1 box has ≥ 3 people.

Let a, b, c be any 3 people in the box



— If any of a, b, c know each other then together with P1 they form a group of 3 mutual friends

— otherwise a, b, c is a group of 3 mutual strangers

Pigeonhole Principle

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 People be $P_1, P_2, P_3, P_4, P_5, P_6$

Define 2 boxes

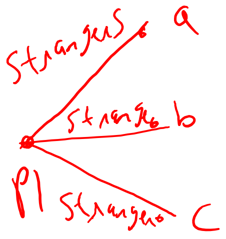
friends of P1

Strangers to P1

— Case: Strangers to P1 box has ≥ 3 people.

Let a, b, c be any 3 people in the box

— If any two of a, b, c are strangers then together with P1 they form a group of 3 mutual strangers.



— Otherwise a, b, c form a group of 3 mutual friends.

Pigeonhole Principle

(PHP)

- There are n people in a room. Show that there must exist two people with the same number of acquaintances.

Make n boxes numbered 0 to $n-1$

Box 0 Box 1 Box 2 - - - - Box $n-1$

— Box i has people who have i acquaintances

— We have n boxes and n people so can't directly apply PHP and infer something useful.

— However, notice that if Box 0 is non-empty then Box $n-1$ must be empty and vice-versa.

Pigeonhole Principle

(PHP)

- There are n people in a room. Show that there must exist two people with the same number of acquaintances.

Make n boxes numbered 0 to $n-1$

$\boxed{\text{Box } 0}$ $\boxed{\text{Box } 1}$ $\boxed{\text{Box } 2}$ - - - - $\boxed{\text{Box } n-1}$

Case 1: Box 0 is empty

Then n people go to remaining $n-1$ boxes and by PHP some box must have ≥ 2 people

Case 2: Box $n-1$ is empty

Then again n people go to $n-1$ boxes and by PHP some box must have ≥ 2 people

Pigeonhole Principle

- There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

Apples are Pigeons
and
Baskets are Boxes

One way to model the process

Dosen't Work

1. We don't know how many apples in total
2. Even if we know, apples can't independently go to any box. There is a constraint that each box has ≤ 24 apples

Pigeonhole Principle

- There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

Baskets are pigeons ← Another way

and

Apples are Boxes

Works

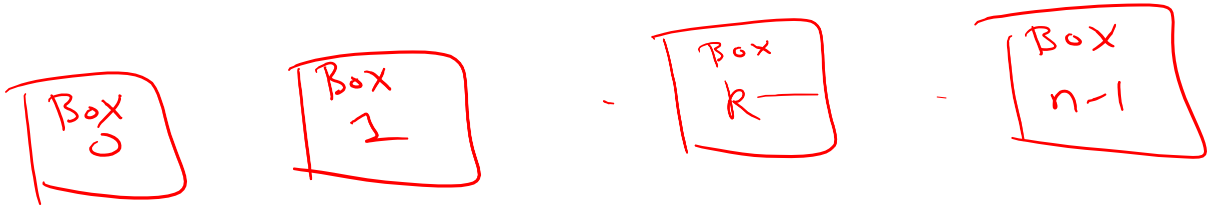
- Create 24 boxes
- Each pigeon (basket) goes to the box corresponding to number of apples that the basket has
- By PHP, must exist a box with at least $\lceil \frac{50}{24} \rceil = 3$ pigeons

Pigeonhole Principle

- Suppose S is a set of $n + 1$ distinct integers. Show that there must exist $a, b \in S$ such that $a - b$ is divisible by n.

— $n+1$ integers as pigeons

— Lets create n boxes



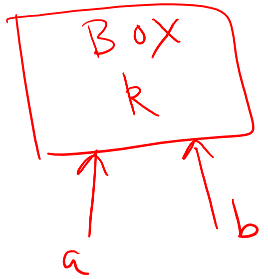
Any $a \in S$ goes to box i if a divided by n leaves i as remainder

By PHP there exists Box k that has ≥ 2 integers

Let k be one such box that contains a, b

Pigeonhole Principle

- Suppose S is a set of $n+1$ distinct integers. Show that there must exist $a, b \in S$ such that $a - b$ is divisible by n.



Then, we must have that $a = r_1 n + k$ and $b = r_2 n + k$

for integers r_1 and r_2.

But then $a - b = (r_1 - r_2)n$ is divisible by n.

Pigeonhole Principle

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub groups of people (with no common person) the sum of whose ages is the same.

Option 1: View people as boxes
and age as pigeons

Then we have 10 boxes and 60 pigeons

– Multiple pigeons going to a box means a person having multiple ages

– Doesn't make sense and violates the constraint of the problem

Pigeonhole Principle

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub groups of people (with no common person) the sum of whose ages is the same.

Option 2: View ages as boxes and people as pigeons

Then we have 60 boxes and 10 pigeons

— # pigeons < # boxes, hence PHP is not very useful

Pigeonhole Principle

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub groups of people (with no common person) the sum of whose ages is the same.

— We really care about subgroups of people and their total age.

— Let's call pigeons as subgroups of people and put them in a box corresponding to total age

Pigeonhole Principle

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub groups of people (with no common person) the sum of whose ages is the same.

How many subgroups of 10 people $= 2^{10} - 1 = 1023$

$\boxed{\text{Box } 1}$ $\boxed{\text{Box } 2}$ - $\boxed{\text{Box } i}$ — — $\boxed{\text{Box } 600}$

— Subgroup S goes to box i if Sum of ages in $S = i$

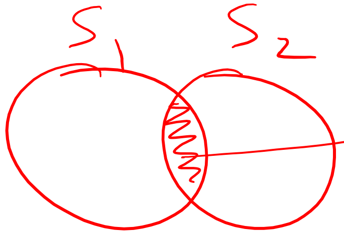
— By PHP there must exist a box that has at least $\left\lceil \frac{1023}{600} \right\rceil = 2$ pigeons

— Let S_1 and S_2 be two subgroups that go to Box i

Pigeonhole Principle

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub groups of people (with no common person) the sum of whose ages is the same.

Sum of ages in S_1 = Sum of ages in S_2



Let $A = S_1 \cap S_2$

— Then sum of ages in $S_1 \setminus A$ = sum of ages in $S_2 \setminus A$

— $S_1 \setminus A$ and $S_2 \setminus A$ are disjoint subgroups