Series n°2 (Solution)

EXO1

Since $A \subset B$ $\quad A \cap B = A$ and $A \cup B = B$

. $A \times B = \{(1,0), (1,1), (1,2), (1,3), (3,0), (3,1), (3,2), (3,3), (2,0), (2,1), (2,2), (2,3)\}$

. $n(A \times B) = n(A) \times n(B)$ $\quad$ {the Cardinality, or (cardinal number) = = number of elements}

$12 = 2 \times 3$

EXO2

$\complement_{\mathbb{R}}^{A} = [1,2]$, $\complement_{\mathbb{R}}^{B} \cap \complement_{\mathbb{R}}^{C} = [1,2]$, $\complement_{\mathbb{R}}^{(B \cup C)} = [1,2]$

EXO3

① . Let $x \in (A \cap B) \cup B^c \iff x \in (A \cap B)$ ou $x \in B^c$

$\iff (x \in A \wedge x \in B) \vee (x \in B^c)$

$\iff (x \in A \vee x \in B^c) \wedge (x \in B \vee x \in B^c)$

$\iff x \in (A \cup B^c) \wedge x \in (B \cup B^c)$

$\iff x \in (A \cup B^c) \cap (B \cup B^c)$

$\iff x \in (A \cup B^c) \cap E$

$\iff x \in (A \cup B^c)$

. Let $x \in (A \setminus B) \setminus C \iff x \in (A \setminus B) \wedge x \notin C$

$\iff (x \in A \wedge x \notin B) \wedge (x \notin C)$

$\iff x \in A \wedge (x \notin B \wedge x \notin C)$

$\iff x \in A \wedge (x \notin B \cup C)$

$\iff x \in A \setminus (B \cup C)$

② Simplify $\overline{A \cup B} \cap \overline{(C \cup \bar{A})} = \bar{A} \cap \bar{B} \cap \bar{C} \cap A = A \cap \bar{A} \cap \bar{B} \cap \bar{C}$

$= \emptyset \cap \bar{B} \cap \bar{C} = \emptyset$

$\overline{(A \cap B)} \cup \overline{(C \cap \bar{A})} = \bar{A} \cup \bar{B} \cup \bar{C} \cup A = \bar{A} \cup A \cup \bar{B} \cup \bar{C}$

$= E \cup \bar{B} \cup \bar{C} = E$

EXO4

$\complement_E^A \setminus \complement_E^B = \{x \in E / x \in \complement_E^A \wedge x \notin \complement_E^B\} = \{x \in E / x \notin A \wedge x \in B\} = \{x \in E / x \in B \setminus A\}$

$= B \setminus A$

$A \setminus (B \cap C) = \{x \in E / (x \in A) \wedge (x \notin B \cap C)\} = \{x \in E / (x \in A) \wedge (x \notin B \vee x \notin C)\}$

$= \{x \in E / (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)\} = \{x \in E / x \in (A \setminus B) \cup (A \setminus C)\}$

$= (A \setminus B) \cup (A \setminus C)$

EX05. $f : [0,1] \longrightarrow [0,2]$, $f(x) = 2 - x$

We know that: $f(A) = \{f(x) \mid x \in A\}$, $f^{-1}(B) = \{x \in E / f(x) \in B\}$

We have: $f(\{\frac{1}{2}\}) = \{f(x) / x = \frac{1}{2}\} = \{f(\frac{1}{2}) = 2 - \frac{1}{2} = \frac{3}{2} / x = \frac{1}{2}\} = \{\frac{3}{2}\}$

$f^{-1}(\{0\}) = \{x \in [0,1] / f(x) = 0\} = \{x \in [0,1] / 2 - x = 0\}$

$= \{x \in [0,1] / x = 2 \notin [0,1]\} = \phi$

Since $f^{-1}(\{0\}) = \phi$ this means that the element $0 \in [0,2]$ hasn't an input in $[0,1]$, i.e, $\nexists x \in [0,1] : f(x) = 0$

then, f is not surjective $\Rightarrow f$ isn't bijective.

EX06

① $A \Delta B = (A \setminus B) \cup (B \setminus A) = \{(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\} =$

$= \{(x \in A \vee x \in B) \wedge (x \in A \vee x \notin A) \wedge (x \notin B \vee x \in B) \wedge (x \notin B \vee x \notin A)\}$

$= \{(x \in A \cup B) \wedge (x \in A \cup A^c) \wedge (x \in B^c \cup B) \wedge (x \notin A \cap B)\}$

$= \{x \in (A \cup B) \setminus (A \cap B)\} = (A \cup B) \setminus (A \cap B)$

② $A, B \subset \mathcal{P}(E)$, $A \Delta B = (A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = B \Delta A$

③ $\forall A \in \mathcal{P}(E)$, $A \Delta \phi = (A \setminus \phi) \cup (\phi \setminus A) = A \cup \phi = A$

$A \Delta A = (A \setminus A) \cup (A \setminus A) = \phi \cup \phi = \phi$

EX07

1) $g : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$, $g(x) = \frac{1}{x - 1}$

$\forall x_1, x_2 \in \mathbb{R} \setminus \{1\}$, $g(x_1) = g(x_2) = \frac{1}{x_1 - 1} = \frac{1}{x_2 - 1} \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2 \Rightarrow g$ is injective

2) $\forall y \in \mathbb{R}$, is there $x \in \mathbb{R} \setminus \{1\}$ such that $y = g(x)$?

We have: $y = g(x) \Rightarrow y = \frac{1}{x - 1} \Rightarrow x = \frac{1}{y} - 1$, for $y = 0$

there isn't an input $x \in \mathbb{R} \setminus \{0\}$ such that $y = g(x)$.

$\Rightarrow g$ isn't surject $\Rightarrow g$ non bijective.

EXO8.

① f is not injective (one-to-one), because:
$f(2) = \frac{4}{5} = f(\frac{1}{2})$ and $2 \neq \frac{1}{2}$

② f isn't surjective (onto), because: $\nexists x \in \mathbb{R}$ such that
$f(x) = 2$ and this equation $\Leftrightarrow 2x = 2(1+x^2)$
$\Leftrightarrow x^2 - x + 1 = 0$
which has no solution in $\mathbb{R}$.

- Consequently, f is not bijective
then has no inverse.

EXO9. $g\circ f : \mathbb{R}_*^+ \xrightarrow{f} \mathbb{R}_*^+ \xrightarrow{g} \mathbb{R}$

① $g\circ f : \mathbb{R}_+^* \longrightarrow \mathbb{R}$
$x \longmapsto g\circ f(x)$

$g\circ f(x) = g[f(x)] = -g(x)$

② f bijective? $\forall y \in \mathbb{R}_*^+, \exists! x \in \mathbb{R}_*^+ : y = f(x)$?

EX10.

f bijective $\Rightarrow f$ invertible and its inverse is:
$f^{-1} : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R} \setminus \{2\}$

$y = f(x) \Rightarrow y = \frac{x+1}{x+2}$
$\Rightarrow 2y + xy = x+1$
$\Rightarrow x = \frac{1-2y}{y-1}$

then,
$f^{-1}(x) = \frac{1-2x}{x-1}$, $\forall x \in \mathbb{R} \setminus \{1\}$.