Mathematics
Analysis 1

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FOR STUDENTS IN FIRST YEAR engineer

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Chapter 1

Real numbers

1.1 Number Sets

In mathematics very often we study sets whose elements are the real numbers. Some special number sets which are frequently encountered are defined as follow.

- **N** is the set of Natural numbers : $N = \{0, 1, 2, 3, ...\}$

- **Z** is the set of Integers : $Z = \{....., -3, -2, -1, 0, 1, 2, 3, ...\}$

- **D** is the set of Decimal numbers : $D = \{\frac{p}{10^n}, p \in \mathbb{Z}, n \in \mathbb{N}^*\}$

  Example : $1.234 = \frac{1234}{10^3}$ is a decimal number.

- **Q** is the set of Rational numbers : $\mathbb{Q} = \{\frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}^*\}$.
  Rational numbers are numbers that can be expressed as the quotient of two integers (ie a fraction) with a denominator that is not zero. Note that all terminating decimals or repeating decimals (or periodic decimal expansion) are a rational numbers.

  Example :

$$\frac{3}{5} = 0.6 \quad \text{(terminating decimals).}$$

$$\frac{1}{3} = 0.33333...and \quad 1.179\mathbf{325325325}... \quad \text{(repeating decimals)}$$

2

– $\mathbb{R}$ is the set of Real numbers, numbers that can be represented by any decimal expansion, limited or not.

  Examples : 123.10100010000100001.........

– $\mathbb{R}\backslash\mathbb{Q} =$ is the set of Irrational numbers which are not rational.

  Examples : $-\sqrt{3},\ \sqrt{2},\ \pi$.

Lemma 1.1 : A number is rational if and only if it admits a periodic or finite decimal writing.

– **C** is the set of Complex numbers $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$ Recall that a complex number is formed by adding a real number to a real multiple of i, where $i = \sqrt{-1}$.

– We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

– The set of **Even numbers** contains the elements $0, \pm 2, \pm 4, \pm 6, ...$ which are those of the form $2n$ for some integer n.

– The set of **Odd numbers** is the set of integers which are not even. Hence odd numbers are $\pm 1, \pm 3, \pm 5, ...$ which can be written as $2n + 1$ for some integer n.

1.2 Operation of real numbers

3

- For every a, $b \in \mathbb{R}$ there is a unique real number $a+b$, called their *sum*.
- For every a, $b \in \mathbb{R}$ there is a unique real number $a \cdot b$, called their *product*.
- For $a \in \mathbb{R}$ there is a unique real number $-a$ called its *negative* or its *additive inverse*.
- For $a \in \mathbb{R}$ with $a \neq 0$ there is a unique real number $\frac{1}{a}$ called its *reciprocal* or its *multiplicative inverse*.
- There is a special element $0 \in \mathbb{R}$ called *zero* or the *additive identity*.
- There is a special element $1 \in \mathbb{R}$ called *one* or the *multiplicative identity*.

For all a, b, $c \in \mathbb{R}$, we have

- $a + b = b + a$ (+ is commutative)
- $a + (b + c) = (a + b) + c$ (+ is associative)
- $a + 0 = a$ (additive identity)
- $a + (-a) = 0$ (additive inverses)
- $a \cdot b = b \cdot a$ ($\cdot$ is commutative)
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ ($\cdot$ is associative)
- $a \cdot 1 = a$ (multiplicative identity)
- if $a \neq 0$ then $a \cdot \frac{1}{a} = 1$ (multiplicative inverses)
- $a \cdot (b + c) = a \cdot b + a \cdot c$ ($\cdot$ distributes over +)

- $0 \neq 1$ (to avoid total collapse)

This is a long list of properties, called *axioms*.

I have given a name to each axiom, so that in proofs we can say which property we are using at each stage. Some people prefer to number the axioms, but there is no standard way to do this, and I find it hard to remember the numbers and easier to remember the names.

I think of the axioms as coming in bundles, to help me to remember them. The first four tell us that 'addition behaves nicely'. The next four tell us that 'multiplication behaves nicely'. Then there is an axiom that tells us that 'addition and multiplication interact nicely', and a final technical detail to clarify that $\mathbb{R} \neq \{0\}$.

As you study abstract algebra (such as linear algebra and group theory) this year, you'll find yourself coming across lists of axioms such as this again.

Definition. Let $\mathbb{F}$ be a set with operations $+$ and $\cdot$ that satisfy the axioms above. Then we say that $\mathbb{F}$ is a *field*.

Example. We've just said that $\mathbb{R}$ is a field. The rational numbers $\mathbb{Q}$ form a field. The complex numbers $\mathbb{C}$ form a field. You'll meet other fields too, in other courses. The integers $\mathbb{Z}$ do not form a field.

In the next section, we'll use these axioms to deduce the basic properties of arithmetic in $\mathbb{R}$. This reasoning would apply equally to any field. We'll need to make further assumptions about $\mathbb{R}$, but we'll postpone this till we've studied basic arithmetic.

3 Properties of arithmetic in $\mathbb{R}$

In the last section, we saw the arithmetic axioms for $\mathbb{R}$. Now we'll deduce some properties of arithmetic in $\mathbb{R}$.

Proposition 1. *Let a, b, c, x, y be real numbers.*

(i) *If $a + x = a$ for all a then $x = 0$ (uniqueness of 0).*

(ii) *If $a + x = a + y$ then $x = y$ (cancellation for $+$).*

(iii) $-0 = 0$.

(iv) $-(-a) = a$.

(v) $-(a + b) = (-a) + (-b)$.

(vi) *If $a \cdot x = a$ for all $a \neq 0$ then $x = 1$ (uniqueness of 1).*

(vii) *If $a \neq 0$ and $a \cdot x = a \cdot y$ then $x = y$ (cancellation for $\cdot$).*

(viii) *If $a \neq 0$ then $\frac{1}{\frac{1}{a}} = a$.*

(ix) $(a + b) \cdot c = a \cdot c + b \cdot c$.

(x) $a \cdot 0 = 0$.

(xi) $a \cdot (-b) = -(a \cdot b)$. *In particular,* $(-1) \cdot a = -a$.

(xii) $(-1) \cdot (-1) = 1$.

(xiii) *If $a \cdot b = 0$ then $a = 0$ or $b = 0$. If $a \neq 0$ and $b \neq 0$ then $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$.*

Remark.

- (ii) shows the uniqueness of $-a$, the additive inverse of a.
- (vii) shows the uniqueness of $\frac{1}{a}$, the multiplicative inverse of a (if $a \neq 0$).
- As we'll see shortly, (i)–(v) can be proved using only the four axioms about $+$.
- Similarly, (vi)–(viii) can be proved using only the four axioms about $\cdot$.

- (ix)–(xiii) between them use all the axioms.
- It's worth proving results like this in a sensible order! Once we've proved a property, we can add it to the list of properties we can assume in subsequent parts. You'll see that we prove some later parts using earlier parts.

Proof. (i) Suppose that $a + x = a$ for all a. Then

$$\begin{aligned} x &= x + 0 \quad \text{(additive identity)} \\ &= 0 + x \quad \text{(+ is commutative)} \\ &= 0 \quad \text{(by hypothesis, with } a = 0\text{).} \end{aligned}$$

(ii) Suppose that $a + x = a + y$. Then

$$\begin{aligned} y &= y + 0 \quad \text{(additive identity)} \\ &= y + (a + (-a)) \quad \text{(additive inverses)} \\ &= (y + a) + (-a) \quad \text{(+ is associative)} \\ &= (a + y) + (-a) \quad \text{(+ is commutative)} \\ &= (a + x) + (-a) \quad \text{(hypothesis)} \\ &= (x + a) + (-a) \quad \text{(+ is commutative)} \\ &= x + (a + (-a)) \quad \text{(+ is associative)} \\ &= x + 0 \quad \text{(additive inverses)} \\ &= x \quad \text{(additive identity).} \end{aligned}$$

(iii) We have $0 + 0 = 0$ (additive identity)

and $0 + (-0) = 0$ (additive inverses)

so $0 + 0 = 0 + (-0)$, so $0 = -0$ (cancellation for + (ii)).

(iv) We have

$$
\begin{aligned}
(-a) + a &= a + (-a) \quad (+ \text{ is commutative}) \\
&= 0 \quad (\text{additive inverses})
\end{aligned}
$$

and $(-a) + (-(-a)) = 0$ (additive inverses),

so $(-a) + a = (-a) + (-(-a))$,

so $a = -(-a)$ (cancellation for + (ii)).

(v) Exercise (see Sheet 1).

(vi)–(viii) Exercise — similar to (i), (ii), (iv).

(ix) (This is another form of distributivity, similar to the axiom but different!)

We have

$$
\begin{aligned}
(a + b) \cdot c &= c \cdot (a + b) \quad (+ \text{ is commutative}) \\
&= c \cdot a + c \cdot b \quad (\cdot \text{ distributes over } +) \\
&= a \cdot c + b \cdot c \quad (\cdot \text{ is commutative – twice}).
\end{aligned}
$$

(x) We have $a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$ ($\cdot$ distributes over +),

and also

$$
\begin{aligned}
a \cdot (0 + 0) &= a \cdot 0 \quad (\text{additive identity}) \\
&= a \cdot 0 + 0 \quad (\text{additive identity})
\end{aligned}
$$

so $a \cdot 0 = 0$ (cancellation for + (ii))

(xi) We have

$$
\begin{aligned}
a \cdot b + a \cdot (-b) &= a \cdot (b + (-b)) \quad (\cdot \text{ distributes over } +) \\
&= a \cdot 0 \quad (\text{additive inverses})
\end{aligned}
$$

and $a \cdot b + (-(a \cdot b)) = 0$ (additive inverses),

so $a \cdot (-b) = -(a \cdot b)$ (cancellation for $+$ (ii)).

(xii) We have

$$
\begin{aligned}
(-1) \cdot (-1) &= -((-1) \cdot 1) \quad ((\text{xi}) \text{ with } a = -1,\, b = 1) \\
&= -(-1) \quad (\text{multiplicative identity}) \\
&= 1 \quad ((\text{iv})).
\end{aligned}
$$

(xiii) Suppose, for a contradiction, that $a \neq 0$, $b \neq 0$ but $a \cdot b = 0$. Then

$$
\begin{aligned}
0 &= \left(\frac{1}{a} \cdot \frac{1}{b}\right) \cdot 0 \quad ((\text{x})) \\
&= 0 \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\cdot \text{ is commutative}) \\
&= (a \cdot b) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\text{hypothesis}) \\
&= (b \cdot a) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\cdot \text{ is commutative}) \\
&= \left((b \cdot a) \cdot \frac{1}{a}\right) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\
&= (b \cdot \left(a \cdot \frac{1}{a}\right)) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\
&= (b \cdot 1) \cdot \frac{1}{b} \quad (\text{multiplicative inverses}) \\
&= b \cdot \frac{1}{b} \quad (\text{multiplicative identity}) \\
&= 1 \quad (\text{multiplicative inverses})
\end{aligned}
$$

and this is a contradiction ($0 \neq 1$).

So if $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Note that on the way we showed that if $a \neq 0$ and $b \neq 0$ then $a \cdot b \neq 0$ and $(a \cdot b) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) = 1$ so $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$ (cancellation for $\cdot$ (vii)).

$\square$

From now on, we can use all of these properties. We shan't give such detailed, one-axiom-at-a-time, derivations in the remainder of the course — but we could, if we needed to!

Remark. (This remark is not part of the course!) You might be concerned that if we don't go back to the axioms every time then we might overlook an unproved step, or might make a mistake. But going back to the axioms every time is not practical: a research paper might take tens of pages to give a proof, referring back to other results, which themselves build on results, which build on results, ... This is where *proof verification* comes in: computers can take care of this detailed checking, leaving humans to focus on things that humans are good at (like having creative ideas for proofs).

One interesting project in this area (there are others, not just this one) is Xena `https://xenaproject.wordpress.com/what-is-the-xena-project/` — this is aimed at undergraduates, which is why I'm mentioning it. Some of you might find it interesting. There's an article about proof verification and Xena in the London Mathematical Society Newsletter, pages 32–36 of `https://www.lms.ac.uk/sites/lms.ac.uk/files/files/NLMS_484-forweb2.pdf`.

Now back to the course ....

6 The modulus of a real number

Definition. Let $a \in \mathbb{R}$. The *modulus* $|a|$ of a is defined to be

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0. \end{cases}$$

(It is also sometimes called the *absolute value* of a.)

Remark. The modulus is well defined (that is, this is a legitimate definition) thanks to the 'positive, negative or 0' property (essentially trichotomy).

Here are some basic properties of the modulus.

Proposition 6. *Take a, b, $c \in \mathbb{R}$. Then*

(i) $|-a| = |a|$*;*

(ii) $|a| \geqslant 0$*;*

(iii) $|a|^2 = a^2$;

(iv) $|ab| = |a||b|$;

(v) $-|a| \leqslant a \leqslant |a|$;

(vi) *if* $c \geqslant 0$*, then* $|a| \leqslant c$ *if and only if* $-c \leqslant a \leqslant c$*; and similarly with weak inequalities (*$\leqslant$*,* $\geqslant$*) replaced by strict (*$<$*,* $>$*).*

Proof. (i), (ii) Immediate from the definition, since $a > 0$ if and only if $-a < 0$.

(iii) Check using the definition and trichotomy – go through the cases and also use $(-a)(-a) = a^2$.

(iv) Check the cases using the definition and trichotomy.

(v) If $a \geqslant 0$, then $-|a| \leqslant 0 \leqslant a = |a|$.

If $a < 0$, then $-|a| = a < 0 \leqslant |a|$.

(vi) Assume that $c \geqslant 0$.

($\Rightarrow$) Suppose that $|a| \leqslant c$. Then, by (v), $-c \leqslant -|a| \leqslant a \leqslant |a| \leqslant c$, and we're done by transitivity (Proposition 3).

($\Leftarrow$) Suppose that $-c \leqslant a \leqslant c$. Then $-a \leqslant c$ and $a \leqslant c$. But $|a|$ is a or $-a$, so $|a| \leqslant c$.

Similarly for the version with strict inequalities.

$\square$

Theorem 7 (Triangle Inequality)**.** *Take* $a, b \in \mathbb{R}$*. Then*

(i) $|a + b| \leqslant |a| + |b|$;

(ii) $|a + b| \geqslant ||a| - |b||$.

Remark. (ii) is called the *Reverse Triangle Inequality*.

Proof. (i) We have $-|a| \leqslant a \leqslant |a|$ and $-|b| \leqslant b \leqslant |b|$, by Proposition 6.

We can add these (see Sheet 1 Q2); using properties of addition, we get $-(|a|+|b|) \leqslant a+b \leqslant |a|+|b|$.

By Proposition 6 (vi) (with $c = |a|+|b| \geqslant 0$), this gives $|a+b| \leqslant |a|+|b|$.

(ii) By (i), we have $|a| = |a+b+(-b)| \leqslant |a+b| + |-b| = |a+b| + |b|$,

so $|a+b| \geqslant |a| - |b|$.

Similarly (swap a and b), $|a+b| \geqslant |b| - |a|$.

Now $||a|-|b||$ is $|a|-|b|$ or $|b|-|a|$, so $|a+b| \geqslant ||a|-|b||$.

$\square$

1.3 The greatest integer function

Definition 1.2 For real numbers x, the greatest integer function denoted by $[x]$ is the largest integer value less or equal than x. We write this as

$$\begin{aligned} f: \ & \mathbb{R} \longrightarrow \mathbb{Z} \\ & x \longrightarrow [x]. \end{aligned}$$

Example 1.8 $[1.65] = 1, \quad [0.016] = 0, \quad [-3.14] = -4, \quad [-1.96] = -2.$

Properties. $\forall x \in \mathbb{R}$, the following inequalities are satisfied

- $\forall x \in \mathbb{R}, \ [x] \in \mathbb{Z}$
- $[x] \leq x < [x] + 1, \quad$ and $\ x - 1 < [x] \leq x.$
- $\forall x \in \mathbb{R}$, and $m \in \mathbb{Z}, \ [x + m] = [x] + m$
- $\forall x, y \in \mathbb{R}, \ x \leq y \Longrightarrow [x] \leq [y].$

Proof.
For any $x \in \mathbb{R}$, we have $[x] \leq x \leq [x] + 1$
which implies

$$[x] + m \leq x + m \leq [x] + m + 1$$

for all $m \in \mathbb{Z}$. On the other hand

$$[x + m] \leq x + m \leq [x + m] + 1$$

since $[x + m]$ is the largest integer less then $x + m$, we have

$$[x] + m \leq [x + m]$$

Similarly, $[x + m] + 1$ is the smallest integer greater than or equal to $x + m$, so

$$[x + m] + 1 \leq [x] + m + 1$$

Combining these, we get

$$[x + m] \leq [x] + m$$

From $[x]+m \leq [x+m]$ and $[x+m] \leq [x]+m$, we conclude $[x+m] = [x]+m$.

Note : The integer part function is increasing function but not strictly increasing.

8 Upper and lower bounds

In the next few sections of the course, we going to explore the difference between $\mathbb{Q}$ and $\mathbb{R}$, as discussed in the last paragraph of the previous section.

One helpful example to have in mind is the square root of two. We know that $\sqrt{2}$ is not in $\mathbb{Q}$ (you have probably seen a proof of this elsewhere). But there *is* a positive real number that squares to give 2. One goal of these sections of the course is to prove the existence of this positive real number (which we call $\sqrt{2}$).

The key property, which $\mathbb{R}$ has and $\mathbb{Q}$ does not, is called *completeness*. But before we get there, we need a few preliminary definitions.

Definition. Let $S \subseteq \mathbb{R}$. Take $b \in \mathbb{R}$. We say that

- b is an *upper bound* of S if $s \leqslant b$ for all $s \in S$;
- b is a *lower bound* of S if $s \geqslant b$ for all $s \in S$;
- S is *bounded above* if S has an upper bound;
- S is *bounded below* if S has a lower bound;
- S is *bounded* if S is bounded above and below.

10 Supremum, infimum and completeness

Some upper bounds are more interesting than others. The set $[0, 1]$ has upper bounds including 15, 1, 1.7 and infinitely many more. Of these, 1 feels special. This is the focus of our next definition.

Definition. Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the *supremum* of S, written $\sup S$, if

(i) $s \leqslant \alpha$ for all $s \in S$; (α is an upper bound of S)

(ii) if $s \leqslant b$ for all $s \in S$ then $\alpha \leqslant b$ (α is the least upper bound of S).

Remark. If S has a supremum, then $\sup S$ is unique. (Check you can show this!)

Now that we have defined the supremum, we can state our final key property of $\mathbb{R}$ (in addition to the properties that make it an ordered field).

Completeness axiom for the real numbers Let S be a non-empty subset of $\mathbb{R}$ that is bounded above. Then S has a supremum.

Remark. There are two conditions on S here: non-empty, and bounded above. They are both crucial!

It is easy to forget the non-empty condition, but it has to be there: the empty set does not have a supremum, because every real number is an upper bound for the empty set — there is no *least* upper bound.

The condition that S is bounded above is also necessary: a set with no upper bound certainly has no supremum.

Example.

- Let $S = [1, 2)$. Then 2 is an upper bound, and is the least upper bound: if $b < 2$ then b is not an upper bound because $\max(1, 1 + \frac{b}{2}) \in S$ and $\max(1, 1 + \frac{b}{2}) > b$. Note that in this case $\sup S \notin S$.
- Let $S = (1, 2]$. Then we again have $\sup S = 2$, and this time $\sup S \in S$.

The supremum is the least upper bound of a set. There's an analogous definition for lower bounds.

Definition. Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the *infimum* of S, written $\inf S$, if

- $s \geqslant \alpha$ for all $s \in S$; (α is a lower bound of S)
- if $s \geqslant b$ for all $s \in S$ then $\alpha \geqslant b$ (α is the greatest lower bound of S).

Let's explore some useful properties of sup and inf.

Proposition 8. *(i) Let S, T be non-empty subsets of $\mathbb{R}$, with $S \subseteq T$ and with T bounded above. Then S is bounded above, and $\sup S \leqslant \sup T$.*

(ii) Let $T \subseteq \mathbb{R}$ be non-empty and bounded below. Let $S = \{-t : t \in T\}$. Then S is non-empty and bounded above. Furthermore, $\inf T$ exists, and $\inf T = -\sup S$.

Remark. (ii) and a similar result with sup and inf swapped essentially tell us that we can pass between sups and infs. Any result we prove about sup will have an analogue for inf. Also, we could have phrased the Completeness Axiom in terms of inf instead of sup. Proposition 8(ii) tells us that we don't need separate axioms for sup and inf.

Proof. (i) Since T is bounded above, it has an upper bound, say b.

Then $t \leqslant b$ for all $t \in T$, so certainly $t \leqslant b$ for all $t \in S$, so b is an upper bound for S.

Now S, T are non-empty and bounded above, so by completeness each has a supremum.

Note that $\sup T$ is an upper bound for T and hence also for S, so $\sup T \geqslant \sup S$ (since $\sup S$ is the *least* upper bound for S).

(ii) Since T is non-empty, so is S.

Let b be a lower bound for T, so $t \geqslant b$ for all $t \in T$.

Then $-t \leqslant -b$ for all $t \in T$, so $s \leqslant -b$ for all $s \in S$, so $-b$ is an upper bound for S.

Now S is non-empty and bounded above, so by completeness it has a supremum.

Then $s \leqslant \sup S$ for all $s \in S$, so $t \geqslant -\sup S$ for all $t \in T$, so $-\sup S$ is a lower bound for T.

Also, we saw before that if b is a lower bound for T then $-b$ is an upper bound for S.

Then $-b \geqslant \sup S$ (since $\sup S$ is the *least* upper bound),

so $b \leqslant -\sup S$.

So $-\sup S$ is the greatest lower bound.

So $\inf T$ exists and $\inf T = -\sup S$.

$\square$

You might be wondering how all this relates to familiar notions of maximum and minimum so let's explore that.

Definition. Let $S \subseteq \mathbb{R}$ be non-empty. Take $M \in \mathbb{R}$. We say that M is the *maximum* of S if

(i) $M \in S$; (M is an element of S)

(ii) $s \leqslant M$ for all $s \in S$ (M is an upper bound for S).

Remark.
- If S is empty or S is not bounded above then S does not have a maximum. (Check this!)
- Let $S \subseteq \mathbb{R}$ be non-empty and bounded above, so (by completeness) $\sup S$ exists.

  Then S has a maximum if and only if $\sup S \in S$.

  Also, if S has a maximum then $\max S = \sup S$.

  (Check this!)

Definition. Let $S \subseteq \mathbb{R}$ be non-empty. Take $m \in \mathbb{R}$. We say that m is the *minimum* of S if

(i) $m \in S$; (m is an element of S)

(ii) $s \geqslant m$ for all $s \in S$ (m is a lower bound for S).

Here is a key result about the supremum, which we'll use *a lot*. It is a quick consequence of the definition, but it will be useful to have formulated it in this way.

1.4 Characterization of sup and inf

Proposition 5.1. Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded from above, then $\alpha = \sup A$, if and only if the following conditioner satisfied :

i) $x \leq \alpha \quad \forall \ x \in A$

ii) for any $\varepsilon > 0$, there exists $a \in A$ such that $\alpha - \varepsilon < a$.

Remark.

- If A is nonempty and bounded above, then exists $\alpha = \sup A \in \mathbb{R}$
- If A is nonempty and not bounded above, then $\sup A = \infty$
- If $A = \emptyset$, then $\sup A = -\infty$, Any real number is an upper bound of $\emptyset$.

Examples.

a) Let $A =]-\infty, 2[$, then $\sup A = 2$

b) Let $A =]-\infty, 2]$, then $\sup A = 2$

c) Let $A = \{x^2, -2 < x < 1\}$, then $A =]0, 4[$, $\sup A = 4$

Proposition 5.2. Let $A \subseteq \mathbb{R}$ be a nonempty set and bounded below then $\beta = \inf A$, if and only if the following conditioner satisfied :

i) $\beta \leq x \quad \forall \ x \in A$

ii) for any $\varepsilon > 0$, there exists $a \in A$ such that $a < \beta + \varepsilon$.

1.5 Archimedeum property

The completeness axiom implies the Archimedean property, which asserts that each real number is strictly less than some natural number.

Theorem 6.1. (Archimedean Property for $\mathbb{R}$). For each $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $x < n$.

Proof. Let $x \in \mathbb{R}$. We prove that there exists a natural number $n \in \mathbb{N}$ such that $x < n$. Suppose, for a contradiction, that $n < x$ for all $n \in \mathbb{N}$. Thus,

$\mathbb{N} \subset \mathbb{R}$ is bounded above. Hence, by the completeness axiom, $\sup(\mathbb{N}) = \alpha$ exists. Now because $\alpha - 1 < \alpha$ there is an $m \in \mathbb{N}$ such that $\alpha - 1 < m$. Therefore,$\alpha < m + 1 = n \in \mathbb{N}$; contradicting the fact that α is an upper bound for $\mathbb{N}$. This contradiction completes the proof.

In our next theorem, we show that the Archimedean property implies two useful results.

Theorem 6.2. Each of the following statements hold :

a) For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, if $x > 0$, then there is an $n \in \mathbb{N}$ such that $y < nx$.

b) For all $x \in \mathbb{R}$, if $x > 0$, then there is an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

Proof. We first prove (a). Let $x, y \in \mathbb{R}$ where $x > 0$. Consider the real number $\frac{y}{x}$. By Theorem 6.1, there is an $n \in \mathbb{N}$ such that $\frac{y}{x} < n$. We conclude that $y < nx$ and the proof of (a) is complete.
Now to prove (b), let $x > 0$. From (a) (where we take $y = 1$), we conclude that there is an $n \in \mathbb{N}$ such that $0 < 1 < nx$. Thus, $0 < \frac{1}{n} < x$.

1.6 The Density of the Rational Numbers

Definition. Let $A \subset \mathbb{R}$. We say that A is dense in $\mathbb{R}$ if for all $x, y \in \mathbb{R}$ if $x < y$, then there exists a $a \in A$ such that $x < a < y$.

Theorem 7.1. (Density of $\mathbb{Q}$ in $\mathbb{R}$). For all $x, y \in \mathbb{R}$ if $x < y$, then there exists a $q \in \mathbb{Q}$ such that $x < q < y$.

Proof. Let $x, y \in \mathbb{R}$ be such that $x < y$. Therefore, $(y - x) > 0$. Theorem 6.2(a) implies that there is an $n \in \mathbb{N}$ such that $1 < n(y - x)$. Thus, $1 < ny - nx$.
states that there is an $m \in \mathbb{Z}$ such that $nx < m < ny$. Thus, $x < \frac{m}{n} < y$ and $q = \frac{m}{n} \in \mathbb{Q}$ is as required.