Chapter n°2

- Sets and Functions

1) Sets

~~In mathematics, the term set is used anywell~~

($x \in A$) or (x in A) means that x is an element of a set A

Exple.

- The set of even integers can be written $= \{2n, n \in \mathbb{Z}\}$
- $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geqslant 0\}$: the set of positive real numbers
- A set with no element in it is called: the empty set and is denoted ϕ (phi)
- The set of all natural numbers less than 0 is: $\{n \in \mathbb{N} / n < 0\} = \phi$

- Subset. $A = \{1, 3, 4\}$, $B = \{1, 4\}$

- B is a subset of A, and we write: $B \subseteq A$

$S = \{1, 3, 5\}$ is not a subset of A: $S \nsubseteq A$

- For any set A, $\phi \subseteq A$.

- Union and intersection

If $A, B \subseteq E$, we can define:

1) The union of A and by: $A \cup B = \{x \in E / x \in A \vee x \in B\}$

2) The intersection of A and B by $A \cap B = \{x \in E / x \in A / x \in B\}$

- When the intersection of two sets is ϕ, we say that the two sets are disjoint

Exple. $A =]-5, 3]$ and $B =]3, 6]$ are disjoint, because $A \cap B = \phi$

- Difference and Complement

If $A, B \subseteq E$, the set

- $A \setminus B = \{x \in E / x \in A \wedge x \notin B\}$ is called the difference $A \setminus B$

- The set: $\{x \in E / x \notin A\}$ is called is called the complement of A in E and is denoted by: $E \setminus A$ or C_E^A or $\bar{A}$ or A^c

and is the elements outside A.

$C_B^A = \{x / x \in B \wedge x \notin A\}$

$C_E^{\phi} = E$, $C_E^{E} = \phi$, $A \cap A^c = \phi$

$A \cup A^c = E$

- The symmetrical difference of two sets A and B is denoted:

$A \Delta B = \{x \,/\, (x \in A \setminus B) \text{ ou } (x \in B \setminus A)\}$

$= \{x / x \in (A \setminus B) \cup (B \setminus A)\}$

$= (A \setminus B) \cup (B \setminus A)$

Remark.

$A \setminus \phi = A$,

$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Properties. $A, B, C \subseteq E$, $E \neq \phi$

1) $\left.\begin{matrix}(A \cap B) \cap C = A \cap (B \cap C) \\ (A \cup B) \cup C = A \cup (B \cup C)\end{matrix}\right\}$ associative law

2) $\left.\begin{matrix}A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\end{matrix}\right\}$

distributive law

3) $\left.\begin{matrix}A \cup B = B \cup A \\ A \cap B = B \cap A\end{matrix}\right\}$ commutative law

4) $A \subset B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$

5) $\left.\begin{matrix}(A \cup B)^c = A^c \cap B^c \\ (A \cap B)^c = A^c \cup B^c\end{matrix}\right\}$ Demorgan's laws

Proof. $(A \cup B)^c = A^c \cap B^c$?

$(A \cup B)^c = \{x \in E / x \in (A \cup B)^c\}$

$= \{x \in E / x \notin A \cup B\}$

$= \{x \in E / x \notin A \wedge x \notin B\}$

$= \{x \in E / x \in A^c \wedge x \in B^c\}$

$= \{x \in E / x \in A^c \cap B^c\}$

$= A^c \cap B^c$

and:

$(A \cap B)^c = \{x \in E / x \in (A \cap B)^c\}$

$= \dots\dots\dots\dots$

- Cartesian product

The cartesian product of two sets X and Y is

$X \times Y = \{(a, b) / a \in X, b \in Y\}$

is the set of all ordered pairs (a, b).

Exple. $A = \{1, 2, 3\}$, $B = \{4, 6\}$

$A \times B = \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6)\}$

$B \times A = \{(4,1), (4,2), (4,3), (6,1), (6,2), (6,3)\}$

- Finite sets and Cardinality

- Finite sets are sets having a finite number of elements.
- Let A be a finite set, the number of elements in set A is called the cardinality of A denoted: $n(A)$

Exple. Let

$E = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 2, 3, 4, 5, 6\}$ and

$B = \{2, 4, 6, 8\}$

We have:

1) $A \subset E$ and 2) $B \subset E$

3) $A \not\subset B$, because: $1 \in A$, $1 \notin B$

4) $B \not\subset A$, 5) $A \cap B = \{2, 4, 6\}$

6) $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$

7) $A \setminus B = A - B = C_A^B = \{1, 3, 5\}$

8) $B \setminus A = B - A = C_B^A = \{8\}$

The cardinal number: (cardinality)

$n(A \cap B) = 3$, $n(A \cup B) = 7$

$n(A) = 6$, $n(B) = 4$

We have:

$7 = 6 + 4 - 3$, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Exple. Let: ~~$A = \{1, 2\}$~~

$A = \{1, 2\}$ and $B = \{1, 2, 3\}$

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

$B \times A = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

$n(A) = 2$, $n(B) = 3$

$n(A \times B) = 6$, $n(B \times A) = 6$

Then,

$$n(A \times B) = n(A) \times n(B)$$

$$6 = 2 \times 3$$

- Power set

The power set (powerset) of a set E is the set of all subsets of E including ϕ and E itself}, denoted: $\mathcal{P}(E)$

Exple.

$A = \{0, 1, 2\}$, the powerset of A is

$\mathcal{P}(A) = \{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

$n(A) = 3$, $n(\mathcal{P}(A)) = 8 = 2^{n(A)} = 2^3$

Then, $\boxed{n(\mathcal{P}(A)) = 2^{n(A)}}$

2) Functions.

A function f is a relation or mapping between two sets (E and F) that associated each element (input) from the first set E to exactly one element (output) of F, we write,

$$f: E \longrightarrow F$$
$$x \longmapsto y = f(x)$$

E: domain, F: codomain

Def. Two functions f and g are equal iff:

1) they have the same domain (E) and the same codomain (F)
2) $f(x) = g(x), \ \forall x \in E$

- The identity function

the identity function for any set X is:

$$I_d: X \longrightarrow X$$
$$x \longmapsto Id(x) = x$$

(associated each point $x \in X$ with itself)

• Function composition

Suppose you have two function $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$, then you can make a new function, $g \circ f: X \to Z$

This makes sens because the codomain of f = the domain of g = Y (the same)

Exple. Let f and g two functions defined in $\mathbb{R}$ by

$f = \sqrt{x-1}$, $g(x) = \dfrac{2x}{x^2+1}$

1) Calculate $g \circ f$ and $f \circ g$.

2) Deduce that, the composition of two funcions is not commutative.

Properties. Let:

$f: E \to F$, $g F \to G$, $h: G \to H$

1) $f \circ I_d = I_d \circ f = f$
2) $h \circ (g \circ f) = (h \circ g) \circ f$

Image of function

Let, $f: X \to Y$, then the image of f is:

$Im f = \{ f(x) \mid x \in X \} \subseteq Y$

Injection, surjection, bijection

Let, $f : X \longrightarrow Y$

1) **f is injective** (or : one-to-one function)

iff :

$\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\Leftrightarrow$

$\forall x_1, x_2 \in X : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

2) **f is surjective** (or onto function)

iff : $\forall y \in Y, \exists x \in X : y = f(x)$

3) f is bijective iff it is injective and surjective, exactly iff

$\forall y \in Y, \exists! x \in E : y = f(x)$

Exple. $f : \mathbb{R} \longrightarrow \mathbb{R}$

$x \mapsto f(x) = 5x + 3$

1) **f injective?** $\forall x_1, x_2 \in \mathbb{R}$:

$f(x_1) = f(x_2) \Rightarrow 5x_1 + 3 = 5x_2 + 3$

$\Rightarrow x_1 = x_2 \Rightarrow f$ inject

2) **f surjective?** (We try to find x)

$\forall y \in \mathbb{R}, \exists x \in E$? such that

$y = f(x) \Rightarrow y = 5x + 3$

$\Rightarrow x = \frac{y-3}{5} \in \mathbb{R}$

$\Rightarrow \exists x \in \mathbb{R} \Rightarrow f$ surjective

So, f inject + surject $\Rightarrow$ f bijective

Exple $f : \mathbb{N} \longrightarrow \mathbb{N}$

$x \mapsto f(x) = 4x + 2$

We have, $\forall y \in \mathbb{N}$:

$y = f(x) \Rightarrow y = 4x + 2$

$\Rightarrow x = \frac{y-2}{4} \notin \mathbb{N}$

$\Rightarrow \forall y \in \mathbb{N}, \nexists x \in \mathbb{N} : y = f(x)$

$\Rightarrow$ f is not surject

$\Rightarrow$ is not bijective

Inverse function

Def. If, $f : E \to F$ is bijective $\Rightarrow$ f is invertible and its inverse is f^{-1} defined by : $f^{-1} : F \to E$

$y \mapsto f^{-1}(y)$

We have :

$y = f(x) \Leftrightarrow x = f^{-1}(y)$

For example $f : \mathbb{R} \to \mathbb{R}$

$x \mapsto 5x + 3$

f is bijective $\Rightarrow$ f is invertible and its inverse is

$f^{-1}(x) = ?$

We have, $y = f(x) \Rightarrow$

$y = 5x + 3 \Rightarrow x = \frac{y-3}{5}$

Then, $\boxed{f^{-1}(x) = \frac{x-3}{5}}$, $\forall x \in \mathbb{R}$

Remark.

(1) If f is bijective $\Rightarrow f^{-1}$ exists and is bijective

(2) $f: E \to F$, if $A \subseteq E$ and $B \subseteq F$

$f(A) = \{ f(x) \mid x \in A \} \subset F$

$f^{-1}(B) = \{ f^{-1}(y) \mid y \in B \}$

$= \{ x \mid f(x) \in B \} \subset E$

Exple. Let $A, B \subset \mathbb{R}$

$f(x) = x^3$, $A = [0,2]$, $B = [-1,1]$

We have:

- $f(A) = \{ f(x) \mid x \in A \} = \{ f(x) \mid 0 \le x \le 2 \}$

$= \{ x^3 \mid 0 \le x^3 \le 8 \} = [0, 8]$

- $f^{-1}(B) = \{ x \mid f(x) \in B \}$

$= \{ x \mid -1 \le x^3 \le 1 \}$

$= \{ x \mid -1 \le x \le 1 \} = [-1, 1]$

Proposition. E, F and G three finite sets

$f: E \to F$, $g: F \to G$

1) $g \circ f$ injective $\Rightarrow f$ injective
2) $g \circ f$ surject $\Rightarrow g$ surjet
3) f and g are inject $\Rightarrow g \circ f$ inject
4) g and f surject $\Rightarrow g \circ f$ surject
5) g and f biject $\Rightarrow g \circ f$ biject
6) $(f^{-1})^{-1} = f$, 7) $f^{-1} \circ f = I_{dE}$
8) ~~f~~ $f \circ f^{-1} = I_{dF}$

Proof.
$$E \xrightarrow{f} F \xrightarrow{g} G$$
$$x \mapsto f(x) = y \mapsto g(y) = g(f(x)) = z$$

1) Let $x_1, x_2 \in E$:

$f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow g \circ f(x_1) = g \circ f(x_2)$

$\Rightarrow x_1 = x_2$ (because $g \circ f$ injective)

2) If $g \circ f$ surject, then $\forall z \in G$, $\exists x \in E$: $g \circ f(x) = z$

$\Rightarrow z = g(f(x)) = g(y)$, (we can choose $y = f(x)$)

$\Rightarrow \forall z \in G$, $\exists y \in F$: $z = g(y)$

$\Rightarrow g$ surjective

7) $\forall x \in E$, $y = f(x) \in F$

$f^{-1} \circ f: E \xrightarrow{f} F \xrightarrow{f^{-1}} E$

$x \longmapsto f^{-1} \circ f(x)$

then,

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$

$\Rightarrow \boxed{f^{-1} \circ f = I_{dE}}$

8) $f \circ f^{-1}$: $F \xrightarrow{f^{-1}} E \xrightarrow{f} F$

$y \longmapsto f \circ f^{-1}(y)$

then,

$f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y$

$\Rightarrow f \circ f^{-1} = I_{dF}$

~~Remark If g and~~

Remark. Let $f: X \to Y$, $g: Y \to Z$

If, f and g are invertible:

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proposition. Let, $f: E \longrightarrow F$

and, $A_1, A_2 \subset E$ and, $B_1, B_2 \subset F$

Then, we have:

1) $A_1 \subset A_2 \Longrightarrow f(A_1) \subset f(A_2)$
2) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$
3) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
4) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$
5) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
6) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
7) $A \subset f^{-1}(f(A))$ $/ A \subset E$
8) $f(f^{-1}(B)) \subset B$ $/ B \subset F$.

Proof.

$\boxed{1}$ Let $y \in f(A_1)$, $\exists x \in A_1 : y = f(x)$

We have:

$A_1 \subset A_2 \Rightarrow x \in A_2$ and

$y = f(x) \in f(A_2) \Rightarrow f(A_1) \subset f(A_2)$

$\boxed{3}$ Let $y \in f(A_1 \cup A_2) \Longrightarrow$

$\exists x \in A_1 \cup A_2 / y = f(x) \Longrightarrow$

$\exists x \in A_1 \vee x \in A_2$ and

$y = f(x) \in f(A_1) \vee y = f(x) \in f(A_2)$

$\Rightarrow y \in f(A_1) \cup f(A_2)$

7) Let $x \in A \Rightarrow f(x) \in f(A)$

$\Rightarrow x \in f^{-1}(f(A))$

$\Rightarrow A \subset f^{-1}(f(A))$

4) Suppose that: $B_1 \subset B_2$

and, $\forall x \in E$, $x \in f^{-1}(B_1)$

$\Rightarrow f(x) \in B_1 \subset B_2$

$\Rightarrow f(x) \in B_2 \Rightarrow x \in f^{-1}(B_2)$

$\Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$

Exple. $f(x) = x^2$, ~~$A = [-1, 0$~~

$A = [-1, 0]$, $B = [0, 1]$

$A \cap B = \{0\}$

$f(A \cap B) = f(\{0\}) = \{0\}$

$f(A) = f([-1, 0]) = [0, 1]$

$f(B) = f([0, 1]) = [0, 1]$

$f(A) \cap f(B) = [0, 1]$

$\Rightarrow f(A \cap B) \subset f(A) \cap f(B)$

Restriction and Extention

Def. Let $f: X \longrightarrow Y$

and let $A \subset X$,

1) The function g:

$g : A \longrightarrow Y$ is the restriction of the function f to A

denoted by :

$f_{/A} : A \longrightarrow Y$

2). f is an extention of g to X

<u>Exple.</u> $f : \mathbb{R} \longrightarrow \mathbb{R}$, $\boxed{f(x) = x^2}$

we known that $\mathbb{R}_+ \subset \mathbb{R}$, then we can define the restriction of f to $\mathbb{R}_+$ by :

$$f_{/\mathbb{R}_+} : \mathbb{R}_+ \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2$$