Chapter n°1.

Logic theory.

② We know that there are four types of sentences ~~and~~

- Declarative (most interested), Imperative, Exclamative, and interogative

① We are going to look at propositions, what they are ?

Definition (Logical proposition)

Logical proposition (statement, formula) is a declarative sentence that can be either true (denoted T or 1) or false (denoted F or 0), but not both.

- The thruth or falsity of a logical proposition is called its truth value

Exples.

1) « 3 is a prime number. »
2) « X is a vowel. »
3) « $2+3=6$. »

All of the above sentences are propositions (are a declarative sentences)

4) « Is it hot outside ? »
5) « Look out ! »

Sentences 4 and 5 are not a propositions (are not a declarative sentences)

6) « $2x = 2+x$. » is ~~and~~ a declarative sentence but if, $x = 2$ we have $2\times 2 = 2+2$ is true, and if, $x = 1$ we have $2\times 1 = 2+1$ is false

The sentence (6) is neither true nor false, hence is not a proposition (depends on the value of x)

Remark

We will use letters such as: $p, q, r, s, \dots$, or $A, B, C, \dots$ to present propositions.

The letters are called logical variables.

Logical operators (or logical connectives)

There are several ways to combine simple propositions to form compound propositions using connectives.

The connectives we introduce are: $\wedge$: and, $\vee$: or, $\Rightarrow$: implies, $\Leftrightarrow$: iff (if and only if), $\bar{p}$: not p

1) Negation

Def. If p is an arbitrary proposition then, the negation of p is written $\bar{p}$, and will be true if p is false.

The truth table of $\bar{p}$ is

p	$\bar{p}$
1	0
0	1

Exple. p: "It is raining today"

$\bar{p}$: "It is not raining today"

Q: "$1 \times 2 = 2$", $\bar{Q}$: "$1 \times 2 \neq 2$"

Remark

The number of rows in truth table is 2^n, where n is the number of propositions.

2) Conjunction.

Def. If p and q are arbitrary propositions, the conjection of p and q is written "$p \wedge q$" which means p and q and will be true iff p and q are true

The truth table of $p \wedge q$ is

Number of rows is $2^n = 2^2 = 4$

Exple

1) (4 is an even number) $\wedge$ (3 is a prime number)

This proposition is true

2) $(3 \leq 2) \wedge (4 \geq 1)$ is false

3) Disjunction

Def. If p and q are arbitrary proposit. The disjunction of p and q is written "$p \vee q$" which means "p or q" and will be true iff either p is thrue or q is true, or both p and q are true.

The truth table of $p \vee q$

Exple. $(2 \geq 3) \vee (4 \geq 1)$

This propo. is true

4) Implication

Def. If p and q are arbitrary proposition, the statement "If p then q" denoted "$p \Rightarrow q$" is called an implication (conditional statement), and will be false when p is true and q is false.

Truth table

Exple. p: "3 = 7 - 4" is true
q: 7 = 4 - 3 is false
1) $p \Rightarrow q$ is false
2) $q \Rightarrow p$ is true.

~~$\overline{p \wedge q} \equiv \overline{p} \vee \overline{q}$~~

5) Equivalence (biconditional)

Def. For any two proposition p and q the statement "p if and only if q" denoted "$p \Leftrightarrow q$" is called a biconditional and will be true iff either
1) p and q are both true, or
2) p and q are both false.

Exple. p: "3 = 7 - 4", q: "7 = 4 - 3"
$p \Leftrightarrow q$ is false.

~~Proposition~~ Remark

$\overline{p \wedge q} = \overline{p} \vee \overline{q}$
$\overline{p \vee q} = \overline{p} \wedge \overline{q}$
Demorgan's laws

$(p \Rightarrow q) \Leftrightarrow \overline{p} \vee q$

$[p \Leftrightarrow q] \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$

~~Properties~~

Predicate

Def. A predicate or propositional function (دالة قضوية) is a statement containing variables and that may be true or false depending on the values of these variables.

Exple.
$P(x)$: "$x^2 < 36$."
$P(x,y)$: "$x - y = 3$."
$P(x,y,z)$: "$x + y - z = 0$."

If we replace x with 2 in the predicate $P(x)$ we obtain $2^2 < 36$, which is true.
but if we replace it with 7, we get: $7^2 < 36$, which is false

Quantifiers

There are two main quantifiers:
the existential quantifier:
and the universal quantifier

Def. The phrase: "for all", "for any", for every", "for each" is called universal quantifier, and is denoted by "$\forall$"

The phrase: "for some x", "There exists an x" is called an existential quantifier and is denoted "$\exists x$"

Exple. "$\forall x, f(x)$" ~~read~~:

① A sentence "$\forall$" read "For all values x, the predicate $f(x)$ is true"

② "For all real x, y: $x>y$ if $x^2>y^2$"
$\Leftrightarrow$ "$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 > y^2 \Rightarrow x > y$"

③ For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $|x - a| < \delta$."

$\Leftrightarrow$

"$\forall \varepsilon > 0, \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

④ "There is an integer for which its square is four"
$\Leftrightarrow$ "$\exists x \in \mathbb{Z}, \ x^2 = 4$".

Remark. You can commute similar quantifiers. However dissimilar quantifiers do not commute in general and the sentence must be read left to right

- The order of quantifiers is very important.

Exple:

1) "$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y > 0$" is true

2) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : x + y > 0$" is false.

Negation of rules of quantifiers

Expls.

1) $\overline{\forall x \in E, P(x)} \Leftrightarrow \exists x \in E, \overline{P(x)}$

2) $\overline{\forall x \in [1, +\infty[: x^2 \geq 1}$ $\Leftrightarrow$

3) $\overline{\exists x \in E, P(x)}$ $\Leftrightarrow$

4) $\overline{\forall y \in \mathbb{Z}, \exists x \in \mathbb{R}, x < y}$ $\Leftrightarrow$

5) $\overline{\forall x \in \mathbb{Z} : x = x^3}$ $\Leftrightarrow$

Ex. Are the following sentences true or false. Give its negation

1) "$\exists x \in \mathbb{N}, \forall y \in \mathbb{R}, x \leq y$." $\to$ F

2) "$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}. (xy)^2 = 0$" $\to$ F

3) $\exists x \in \mathbb{N}, \forall y \in \mathbb{Z}$