Chapter 4 and 5 : Real Functions of One Real Variable

Introduction

This chapter is devoted to the functions of a real variable which are often modeled for the study of curves and mechanical calculations. In this regard, we present the foundations of the functions of a real variable, where the objective is to know and interpret the notion of the limit, continuity and derivability of a function, and to present some of their properties.

Contents

Real function

The concept of a function is the fundamental concept of calculus and analysis. Real function f of one real variable is a mapping from the set $D \subseteq \mathbb{R}$, a subset in real numbers $\mathbb{R}$, to the set of all real numbers $\mathbb{R}$.

$$f : D \to \mathbb{R}, \qquad x \longmapsto f(x)$$

• D is the domain of the function f, where $D = \left\{x \in \mathbb{R},\ f(x) \text{ makes sense}\right\}$

1 Limits

Limits are used to analyze the local behavior of functions near points of interest. A function f is said to have a limit ℓ at x_0 if it is possible to make the function arbitrarily close to ℓ by choosing values closer and closer to x_0. Note that the actual value at x_0 is irrelevant to the value of the limit.

The notation is as follows:

$$\lim_{x \to x_0} f(x) = \ell$$

which is read as "the limit of $f(x)$ as x approaches x_0 is ℓ"

1.1 Limit at a Point

We consider values of a function that approaches a value from either inferior or superior.

• **The left-hand limit** of a function f as it approaches x_0 is the limit

$$\lim_{x \to x_0^-} f(x) = \ell$$

Which indicates that the limit is defined in terms of a number less than the given number x_0.

• **The right-hand limit** of a function f as it approaches x_0 is the limit

$$\lim_{x \to x_0^+} f(x) = \ell$$

Which indicates that the limit is defined in terms of a number greater than the given number x_0.

• $\lim\limits_{x \to x_0} f(x) = \ell$ if and only if both the left- hand and right-hand limits at $x = x_0$ exist and share the same value.

$$\lim_{x \to x_0^-} f(x) = \ell = \lim_{x \to x_0^+} f(x).$$

Example : Compute the limit : $\lim_{x\to 0}|x|$

• The right-hand limit at $x=0$: $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$

• The left-hand limit at $x=0$: $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-} -x = 0$

So the right-hand and left-hand limits are equal. Then $\lim_{x\to 0}|x| = 0$

Infinite Limits

• If a function is defined on either side of x_0, but the limit as x approaches x_0 is infinity or negative infinity, then the function has an infinite limit, we write

$$\lim_{x\to x_0} f(x) = \infty$$

•The graph of the function will have a vertical asymptote at x_0.

Limits at Infinity

• Limits at infinity are used to describe the behavior of functions as the independent variable increases or decreases without bound. we write

$$\lim_{x\to \pm\infty} f(x) = \ell$$

• The graph of the function will have a horizontal asymptote at $y=\ell$.

1.2 Operations on Limits

○ Assume that $\lim_{x\to x_0} f(x) = \ell \in \mathbb{R}$, $\lim_{x\to x_0} g(x) = m \in \mathbb{R}$ and $c \in \mathbb{R}$. Therefore :

$\lim_{x\to x_0} f(x)$	$\lim_{x\to x_0} g(x)$	$\lim_{x\to x_0} (f+g)(x)$	$\lim_{x\to x_0} (f\times g)(x)$
ℓ	m	$\ell+m$	$\ell\times m$
$+\infty$	m	$+\infty$	$\begin{cases} +\infty & \text{Si } m>0 \\ -\infty & \text{Si } m<0 \\ \text{Indeterminate} & \text{Si } m=0 \end{cases}$
$-\infty$	m	$-\infty$	$\begin{cases} -\infty & \text{Si } m>0 \\ +\infty & \text{Si } m<0 \\ \text{Indeterminate} & \text{Si } m=0 \end{cases}$
$+\infty$	$+\infty$	$+\infty$	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$+\infty$
$-\infty$	$+\infty$	Indeterminate	$-\infty$

○ $\lim_{x\to x_0} cf(x) = c\lim_{x\to x_0} f(x) = c\ell$

○ $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \dfrac{\ell}{m}$ if $m \neq 0$

○ **Limit of Composition :** Suppose that $\lim_{x\to x_0} g(x) = \ell$ and $\lim_{x\to \ell} f(x) = \ell'$, then

$$\lim_{x\to x_0} f(g(x)) = \ell'$$

Comparative Growth

Suppose that f and g are two functions such that $\lim_{x\to+\infty} f(x) = +\infty$, and $\lim_{x\to+\infty} g(x) = +\infty$. We say that f grows faster than g as $x \to +\infty$ if the following holds:

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = +\infty \qquad \text{or equivalently,} \qquad \lim_{x\to+\infty} \frac{g(x)}{f(x)} = 0$$

Results:

• Exponential functions grow faster than every polynomial functions and polynomial functions grow faster than logarithmic functions. Let n be positive number:

1. $\lim_{x\to\infty} \dfrac{e^x}{x^n} = \infty$ and, $\lim_{x\to\infty} \dfrac{x^n}{e^x} = 0$
2. $\lim_{x\to\infty} \dfrac{x^n}{\ln(x)} = \infty$ and, $\lim_{x\to\infty} \dfrac{\ln(x)}{x^n} = 0$

Indeterminate Form

An indeterminate form is an expression involving two functions whose limit cannot be determined solely from the limits of the individual functions.

$$+\infty - \infty, \quad 0.\infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}$$

1.3 Evaluating Limits in Indeterminate Form

We present some methods that allows us to transform an indeterminate form into one that allows for direct evaluation.

● **Polynomial function as** $x \to \pm\infty$ with indeterminate form $+\infty - \infty$

Factor out the highest power of x in the polynomial function.

Example:

Find $\lim_{x\to+\infty} -2x^3 + 4x - 1$,

We write, $\lim_{x\to+\infty} -2x^3(1 - \dfrac{2}{x^2} + \dfrac{1}{2x^3})$. Thus, $\lim_{x\to\infty} -2x^3 = -\infty$ and $\lim_{x\to\infty} (1 - \dfrac{2}{x^2} + \dfrac{1}{2x^3}) = 1$

Therefore, $\lim_{x\to+\infty} -2x^3 + 4x - 1 = \lim_{x\to+\infty} -2x^3 = -\infty$.

● **Rational function as** $x \to \pm\infty$ with indeterminate form $\dfrac{\infty}{\infty}$

Divide out the highest power of x in both the numerator and denominator.

Example: $\lim\limits_{x\to+\infty} \dfrac{x^2-1}{x+3}$. Both numerator and denominator approach $+\infty$ as $x \to +\infty$. Thus

$$\lim_{x\to+\infty} \frac{x^2-1}{x+3} = \lim_{x\to+\infty} \frac{x^2(1-\frac{1}{x^2})}{x(1+\frac{3}{x})} = +\infty$$

● **Factoring Method** $\left(\dfrac{0}{0}\right.$ **form** $\left.\right)$

Factoring method is a technique to finding limits that works by canceling out common factors.

Example:

Find $\lim\limits_{x\to3} \dfrac{x^2-9}{x-3}$

Using the substitution rule gives $\lim\limits_{x\to3} \dfrac{x^2-9}{x-3} = \dfrac{0}{0}$

find the common divisor which is $(x-3)$ and divide both the numerator and denominator by it,

$$\begin{aligned}\lim_{x\to3} \frac{x^2-9}{x-3} &= \lim_{x\to3} \frac{(x-3)(x+3)}{x-3}\\ &= \lim_{x\to3}(x+3)\\ &= 6\end{aligned}$$

● **L'Hospital's Rule** $\left(\dfrac{0}{0}\right.$ **or** $\dfrac{\infty}{\infty}$ **form** $\left.\right)$

Suppose f and g are differentiable and $g'(x) \neq 0$ near x_0 (except possibly at x_0). Suppose that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}, \text{ or } \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Then,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Example:

Find $\lim\limits_{x\to-2} \dfrac{x+2}{x^2+3x+2}$

Using the substitution rule gives $\lim\limits_{x\to-2} \dfrac{x+2}{x^2+3x+2} = \dfrac{0}{0}$. Apply L'Hospital's Rule

$$\begin{aligned}\lim_{x\to-2} \frac{x+2}{x^2+3x+2} &= \lim_{x\to-2} \frac{(x+2)'}{(x^2+3x+2)'}\\ &= \lim_{x\to-2} \frac{1}{2x+3}\\ &= -1.\end{aligned}$$

● **Conjugate multiplication**

This method useful for fraction functions that contain square roots. It rationalizes the numerator or denominator of a fraction, which means getting rid of square roots.

Example :

Evaluate $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$

By substitution, we find : $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}=\dfrac{0}{0}$

Multiply the numerator and denominator by the conjugate of $\sqrt{x}-2$ which is $\sqrt{x}+2$, we obtain

$$\begin{aligned}
\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}\\
&= \lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x}+2)}\\
&= \lim_{x\to 4}\frac{1}{\sqrt{x}+2} \quad (\text{Cancel the } (x-4))\\
&= \frac{1}{4}
\end{aligned}$$

Alternative methods to evaluate limits

● **Squeeze Theorem**

Suppose that $g(x)\leqslant f(x)\leqslant h(x)$ for all x close to x_0 but not equal to x_0. If $\lim\limits_{x\to x_0} g(x)=\ell=\lim\limits_{x\to x_0} h(x)$, then

$$\lim_{x\to x_0} f(x)=\ell$$

The quantity x_0 and ℓ may be a finite number or $\pm\infty$.

Results: we represent two important limits :

$$\lim_{x\to+\infty}\frac{\sin(x)}{x}=0,\qquad \lim_{x\to+\infty}\frac{1-\cos(x)}{x}=0$$

● **Monotone Limits**

Suppose that the limits of f and g both exist as $x\to x_0$. if $f(x)\leqslant g(x)$ when x is near x_0, then

$$\lim_{x\to x_0} f(x)\leqslant \lim_{x\to x_0} g(x)$$

Some Special Limits

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1,\qquad \lim_{x\to 0}\frac{\tan(x)}{x}=1,\qquad \lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{\ln(x+1)}{x}=1,\qquad \lim_{x\to 0}\frac{\exp(x)-1}{x}=1$$

2 Continuity

Continuous functions are functions that take nearby values at nearby points.

2.1 Continuity at Point

Definition 2.1

- *Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. we say that f is continuous at a point $x_0 \in I$ if,*

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Otherwise, f is said to be discontinous at x_0.

- *We say that f is continuous on I if f is continuous at every point of I.*

Checking Continuity at a Point

A function f is continuous at $x = x_0$ if the following three conditions hold:

1. $f(x_0)$ is defined (that is, x_0 belongs to the domain of f)
2. $\lim_{x \to x_0} f(x)$ exists (that is, left-hand limit = right-hand limit)
3. $\lim_{x \to x_0} f(x) = f(x_0)$

One-sided continuity :

- f is **left continuous at a point** x_0 if, $\lim_{x \to x_0^-} f(x) = f(x_0)$
- f is **right continuous at a point** x_0 if, $\lim_{x \to x_0^+} f(x) = f(x_0)$
- f is continuous at x_0 if and only if these two limits exist and are equal.

$$\lim_{x \to x_0^-} f(x) = f(x_0) = \lim_{x \to x_0^+} f(x)$$

Remark 1

○ *Every polynomial function is continuous on $\mathbb{R}$.*

○ *Every rational function is continuous on its domain.*

○ sin *and* cos *are continuous everywhere on $\mathbb{R}$*

○ *The square root is continuous on $\mathbb{R}^+$*

2.2 Operations on Continuity

The basic properties of continuous functions follow from those of limits:
If $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ are continuous at x_0 of I, and λ is a constant, then :

1. $f+g$ is continuous at x_0
2. λf is continuous at x_0
3. fg is continuous at x_0
4. If $f(x_0) \neq 0$, then $\dfrac{1}{f}$ is continuous at x_0.

Theorem 1 *Let $f: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$ two functions such that $f(I) \subseteq J$. If f is continuous at x_0 of I and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Example :

Determine whether $h(x) = \cos(x^2 - 5x + 2)$ is continuous.
Note that, $h(x) = f(g(x))$, where $f(x) = \cos(x)$ and $g(x) = x^2 - 5x + 2$
Since both f and g are continuous for all x, then h is continuous for all x.

Continuous extension : When we can remove a discontinuity by redefining the function at that point, we call the discontinuity removable. (Not all discontinuities are removable, however.)

If $\lim\limits_{x \to x_0} f(x) = \ell$, but $f(x_0)$ is not defined, we define a new function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \neq x_0 \\ \ell & \text{for } x = x_0 \end{cases}$$

which is continuous at x_0. It is called the continuous extension of $f(x)$ to x_0.

Example :

Show that the following function have continuous extension, and find the extension :

$$f(x) = \frac{x^2-1}{x^3+1}, \qquad \text{for } x \neq -1$$

Here $f(-1)$ has not been defined.

$$\begin{aligned} \lim_{x \to -1} \frac{x^2-1}{x^3+1} &= \lim_{x \to -1} \frac{(x+1)(x-1)}{(x+1)(x^2-x+1)} \\ &= \lim_{x \to -1} \frac{x-1}{x^2-x+1} \\ &= \frac{-2}{3} \end{aligned}$$

Thus, $\lim_{x\to -1} f(x)$ exists, therefore f has a removable discontinuity at $x_0 = -1$.
Hence, The continuous extension is

$$\tilde{f}(x) = \begin{cases} \dfrac{x^2-1}{x^3+1} & \text{for } x \neq -1 \\ -\dfrac{2}{3} & \text{for } x = -1 \end{cases}$$

○ As one consequence of previous results, the image of interval under a continuous function is an interval :

Theorem 2 *Let $f : I \to \mathbb{R}$ be a continuous function on an interval I, then $f(I)$ is an anterval.*

I	$f(I)$	
	f is strictly increasing	f is strictly decreasing
$[a,b]$	$[f(a), f(b)]$	$[f(b), f(a)]$
$[a,b[$	$[f(a), \lim_{x\to b^-} f(x)[$	$]\lim_{x\to b^-} f(x), f(a)]$
$]a,b]$	$]\lim_{x\to a^+} f(x), f(b)]$	$[f(b), \lim_{x\to a^+} f(x)[$
$]a,b[$	$]\lim_{x\to a^+} f(x), \lim_{x\to a^+} f(x)[$	$]\lim_{x\to a^+} f(x), \lim_{x\to a^+} f(x)[$

Theorem 3 *Let $f : I \to \mathbb{R}$ is the function defined on $I \subseteq \mathbb{R}$. Assume that f is continuous and strictly monotonic on the closed interval I, then*

1. *f establishes a bijection of the interval I into the image interval $f(I)$.*

2. *$f^{-1} : f(I) \to I$ is continuous and strictly monotonic on $f(I)$*

2.3 Intermediate Value Theorem (IVT)

The intermediate value theorem describes a key property of continuous functions. It states that a continuous function on an interval takes on all values between any two of its values.

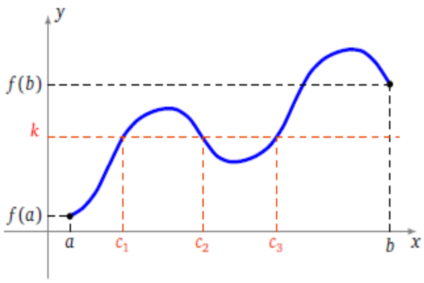
Theorem 4 *Let $f : [a,b] \longrightarrow \mathbb{R}$ such that*

- *f is continuous on the closed interval $[a,b]$*

- *k be any number between $f(a)$ and $f(b)$.*

Then, there exists at least $c \in]a,b[$ such that $f(c) = k$.

Geometric interpretation



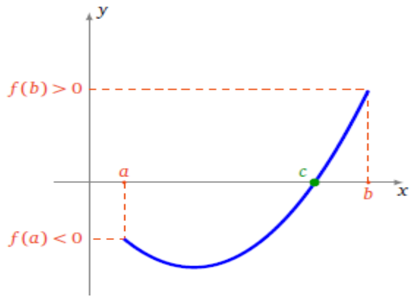
The most used version of the intermediate value theorem given as :

Theorem 5 *Let $f : [a, b] \longrightarrow \mathbb{R}$ such that*

- *f is continuous on the closed interval $[a, b]$,*
- *$f(a).f(b) < 0$*

Then, there exists at least $c \in]a, b[$ such that $f(c) = 0$.

Geometric interpretation



Example :

Show that the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has a solution in the interval $[1, 2]$.
Consider the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ over the closed interval $[1, 2]$
The function f is a polynomial, therefore it is continuous over $[1, 2]$.
We have $f(1) = -1$ and $f(2) = 12$, hence $f(1)f(2) < 0$
by the Mean-Value-Theorem there exists a value c in the interval $]1, 2[$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$ in the interval $]1, 2[$.

3 Derivability

3.1 Derivability at a Point

Below, we note I a non-empty interval of $\mathbb{R}$.

Definition 3.1

Let $f : I \to \mathbb{R}$ be a function, and let $x_0 \in I$. we say that f is differentiable at x_0 if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists, and finite. This limit is called the derivative of f at x_0, we note $f'(x_0)$.

Remark 2

Alternative formula for the derivative :

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Geometric interpretation of the derivative :

If f is differentiable at x_0, then the curve representing the function f have a tangent to the point $(x_0, f(x_0))$, with the slope $f'(x_0)$.

One-sided derivatives :

In analogy to one-sided limits, we define one-sided derivatives

• **The left-hand derivative** of a function f at x_0

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

• **The right- hand derivative** of a function f at x_0

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

f is differentiable at x_0 if and only if these two limits exist and are equal.

Example :

Show that $f(x) = |x-1|$ is not differentiable at $x = 0$

• The right-hand derivative at $x = 0$:

$$\lim_{x \to 1^+} \frac{|x-1| - 0}{x-1} \lim_{x \to 1^+} \frac{x-1}{x-1} = 1$$

• The left-hand derivative at $x = 0$:

$$\lim_{x \to 1^-} \frac{|x-1| - 0}{x-1} \lim_{x \to 1^+} \frac{-(x-1)}{x-1} = -1$$

So the right-hand and left-hand derivatives differ.

Remark 3

We say that a function f is differentiable on an interval I when f is differentiable in any point of I .

Theorem 6 *If f has a derivative at $x = a$, then f is continuous at $x = a$.*

3.2 Operations on derivative

Let $f, g : I \to \mathbb{R}$ two functions. We assume that f and g are differentiable of x. Therefore,

1) $f + g$ is differentiable , and

$$(f+g)'(x) = f'(x) + g'(x)$$

2) fg is differentiable , and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

3) If $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable , and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Theorem 7 (Derivatives of composite functions) *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ two functions such that $f(I) \subseteq J$. If f is differentiable of x, and g is differentiable of $f(x)$, then $g \circ f$ is differentiable of x and*

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

Common Derivatives

$f(x)$	$f'(x)$
$c,\ c \in \mathbb{R}$	0
$cx,\ c \in \mathbb{R}$	c
$x^n,\ n \geqslant 1$	nx^{n-1}
$\frac{1}{x}$	$\frac{-1}{x^2}$
$\frac{1}{x^n},\ n \geqslant 1$	$\frac{-n}{x^{n+1}}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\ln(x),\ x > 0$	$\frac{1}{x}$
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\sin(cx),\ c \in \mathbb{R}$	$c\cos(cx)$

Applications of Derivatives

Derivatives have various applications in Mathematics, We'll learn about these two applications of derivatives :

1. **Monotonicity of functions**

Derivatives can be used to determine whether a function is increasing, decreasing or constant on an interval.

Theorem 8 *Let f be a differentiable function on an intervalle I :*

1. *f is increasing on $I \Longleftrightarrow \forall x \in I, \quad f'(x) \geqslant 0$*
2. *f is decreasing on $I \Longleftrightarrow \forall x \in I, \quad f'(x) \leqslant 0$*
3. *f is constant on $I \Longleftrightarrow \forall x \in I, \quad f'(x) = 0$*

2. **Extremum of Functions**

An extremum of a function is the point where we get the maximum or minimum value of the function in some interval.

• Let $f : I \to \mathbb{R}$ be a function, and let $c \in I$. We say that c is a **critical point** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

Let $f : I \to \mathbb{R}$ is differentiable, and $c \in I$ be a critical point of f. Then

1. If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the maximum value of f.
2. If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the minimum value of f.

Example :

Find the extremum of $f(x) = 3x^2 - 18x + 5$ on $[0, 7]$.

First, we find all possible critical points :

$$\begin{aligned} &f'(x) = 0 \\ &6x - 18 = 0 \\ &x = 3 \end{aligned}$$

for $x \in [0, 3[$, we have $f'(x) < 0$ and for $x \in]3, 7]$, we have $f'(x) > 0$ Then $f(3) = -22$ is the muximun value of f on $[0, 7]$.

3.3 Rolle's Theorem

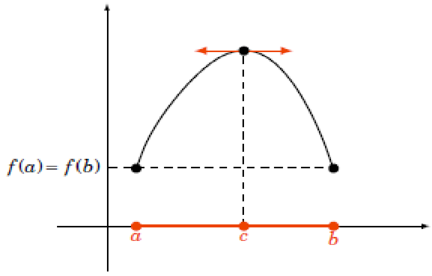
In analysis, special case of the mean-value theorem of differential calculus is Rolle's theorem.

Theorem 9 *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *such that*

- f *is continuous on the closed interval* $[a,b]$,
- f *is differentiable on the open interval* $]a,b[$,
- $f(a) = f(b)$.

Then, there exists $c \in]a,b[$ *such that* $f'(c) = 0$.

Geometric interpretation



There exists at least one point of graph of f where the tangent is horizontal.

Example : Let $g(x) = (1-x)f(x)$

with f is a continuous function on $[0,1]$, differentiable on $]0,1[$ and verify $f(0) = 0$
Show that

$$\exists c \in]0,1[, \quad f'(c) = \frac{f(c)}{1-c}$$

Apply Rolle's theorem :

1) g is continuous $[0,1]$ because it is the product of two continuous functions on $[0,1]$ (f is a continuous function on $[0,1]$ and $x \longmapsto 1-x$ continuous polynômial on $\mathbb{R}$ hence on $[0,1]$).

2) g is differentiable on $]0,1[$ since it is the product of two differentiable functions on $]0,1[$.

3) $g(0) = f(0) = 0, \qquad g(1) = 0 \times f(1) = 0$. Hence $g(0) = g(1)$

According to Rolle's theorem: $\exists c \in]0,1[, \quad g'(c) = 0$.

Where

$$g'(c) = -f(c) + (1-c)f'(c)$$

It follows,

$$f'(c) = \frac{f(c)}{1-c}.$$