University 8 May 1945 - GUELMA

1^{st} year . Algebra 01.

October 2024.

Series of exercises n° 01

Exercise 01 Let $A, B \subset \mathbb{N}$. Translate the following statement into symbolic form:

$A = \phi$, $A \cap B \neq \phi$, $A \not\subset B$.

Exercise 02. Write the negation of the following statement:

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \mathbb{R}, \ |x - x_0| < \alpha \Rightarrow |f(x) - f(x_0)| < \varepsilon,$$

$x_0 \in \mathbb{R}$

Exercise 03. Consider the following propositions:

P_1: "$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \ x + y > 0$".

P_2: "$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \ x + y > 0$."

P_3: "$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \ x + y > 0$."

P_4: "$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \ y^2 > x$."

1) Are these propositions true or false?

2) Give the negation of each proposition.

Exercise 04. Let $n \in \mathbb{N}$, prove that:

$n(n+1)$ is even. (by cases)

Exercise 05 Prove that:

$\forall n \in \mathbb{N}, \ \dfrac{8n(n+1)}{2} + 1$ is a square number.

(Direct reasoning)

Exercise 06. Prove that:

If n^2 is odd then n is odd.

(by contraposition)

Exercise 07. By contradiction, show that:

$\forall n \in \mathbb{N}$, n^2 is even $\Rightarrow$ n is even

Exercise 08. Show that, the following statements are false:

(a) : " $\forall x \in \mathbb{R},\ x^2+1=0$ ".

(b) : " $\forall x \in]0,1[,\ \dfrac{2x}{x^2(1-x^2)} < 1$ ".

(by counter-example)

Exercise 09. Using the principle of induction (recurrence) show that:

$$\forall n \in \mathbb{N}^*:\ 1^3+2^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$$

Exercise 10 Let $a \in \mathbb{R}^*$, prove by induction that:

$$\forall n \in \mathbb{N}:\ (1+a)^n \geq 1+na$$

Exercise 11 By contradiction, prove that:

$\sqrt{2}$ is irrational

Exercise 12 Let $n \in \mathbb{Z}$, prove that:

If $5n-7$ is even, then n is odd.

(by contraposition)