1st year, Algebra 1 November 2024

Series of exercises n° 04

Exercise 01. Let $G = \mathbb{R}^* \times \mathbb{R}$ and $*$ is a composition law in G defined by:

$$\forall (x,y),(x',y') \in G: \quad (x,y) * (x',y') = (xx', xy' + y)$$

1) Prove that $*$ is an internal composition law.
2) Show that $*$ is associative, has a neutral element and inverse element.
3) What can we deduce?
4) Prove that $*$ is not commutative.

Exercise 02. In $\mathbb{R}_+^*$ we define an internal composition law by:

$$\forall x, y \in \mathbb{R}_+^*, \quad x * y = \sqrt{x^2 + y^2}$$

a) Prove that $*$ is commutative, associative and 0 is the neutral element
b) Show that all elements of $\mathbb{R}_+^*$ have no inverse for $*$

Exercise 03. Let $*$ an internal composition law defined on a set $A =]-1, 1[$ by:

$$\forall x, y \in A, \quad x * y = \frac{x+y}{1+xy}$$

1) Prove that $(A,*)$ is an abelian group
2) Let $B =]0,1[$, the set $(B,*)$ is it a subgroup of of $(A,*)$.

Exercise 04. Let $A =]1, +\infty[$ and Δ is a composition law in A defined by:

$$\forall x, y \in A, \quad x \Delta y = (x-1)(y-1)+1$$

1) Show that Δ is an internal composition law.
2) Prove that (A, Δ) is a commutative group.
3) The set $\left(\left\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\right\}, \Delta\right)$ is it a subgroup of a group (A, Δ)

Exercise 05. Let $G = \mathbb{R} \times \mathbb{R}$ be a set, we define two internal composition laws in G by:

$\forall (x,y), (x',y') \in G,$

$(x,y) + (x',y') = (x+x', y+y')$

$(x,y) * (x',y') = (xx', xy'+x'y)$

1) Prove that $(G, +)$ is an abelian group.

2)
- ⓐ Show that the law $*$ is commutative
- ⓑ Show that the law $*$ is associative
- ⓒ Give the neutral element of $*$
- ⓓ Prove that $(G, +, *)$ a commutative ring. $(G, +, *)$ is it a ring with 1, (with unity)?

Exercise 06

① Prove that $(\mathbb{Z}^2, +, \cdot)$ is a ring.

② Prove that $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$ is a commutative group

EXERCISE 07. Let (E, Δ) be an abelian group, $E = \mathbb{R} \setminus \{3\}$ and, $\forall a, b \in E$: $a \Delta b = ab + 3(a+b) + 6$

Let, $f : (E, \Delta) \longrightarrow (\mathbb{R}^*, \cdot)$, defined by: $f(a) = \lambda a + 3$

- Determine λ so that f is a group homomorphism

Exercise 08. Let, $g : (\mathbb{R}^*, \cdot) \to (\mathbb{R}, +)$ be a bijective function, such that: $g(x) = \ln x$

- Prove that g is an isomorphism.

Permutations.

Exercise 09. List all the elements of S_3

Exercise 10. Consider S_4, let:

$$\varphi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 1 \end{pmatrix}$$

1. Is φ_3 an element of S_4? Explain?
2. Determine the following: φ_1^2, $\varphi_1\varphi_2$, $\varphi_2\varphi_1$.
3. Find φ_1^{-1}, φ_2^{-1}, $\varphi_2^{-1}\varphi_1^{-1}$, $(\varphi_1\varphi_2)^{-1}$.

Exercise 11. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 4 & 8 & 7 & 6 & 2 & 1 \end{pmatrix}$$

1) Compute σ^{-1}.
2) Write σ as a product of cycles, with disjoint supports
3) Give sign σ.
4) Write σ^2 in table notation and in cycles notation
5) Give sign σ^2.