Chapter n° 3 Binary relations on a set

A binary relation $\mathcal{R}$ on a set A is a relation from A to A and is a subset of $A \times A$

expl: $A = \{0,1,2\}$ and $\mathcal{R} = \{(0,1),(0,2),(1,2)\}$

Def Let $\mathcal{R}$ a relation on set A

1) $\mathcal{R}$ is reflexive, iff $\forall x \in A : x \mathcal{R} x$
2) $\mathcal{R}$ is symmetric iff $\forall x, y \in A : x \mathcal{R} y \Rightarrow y \mathcal{R} x$
3) $\mathcal{R}$ is antisymmetric iff $\forall x, y \in A : x \mathcal{R} y \wedge y \mathcal{R} x \Rightarrow x = y$
4) $\mathcal{R}$ is transitive iff $\forall x, y, z \in A : x \mathcal{R} y \wedge y \mathcal{R} z \Rightarrow x \mathcal{R} z$

Exple We define on $\mathbb{R}$ the relation $\mathcal{R}$ by :

$$\forall x, y \in \mathbb{R} : x \mathcal{R} y \Leftrightarrow x = y$$

Show that 1, 2, 3 and 4 are cheked ?

1) Equivalence relations

Def ① A relation $\mathcal{R}$ on a set A is called an equivalence relation if it is : reflexive, symetric and transitive.

② Let $\mathcal{R}$ be an equivalence relation on a set A, then for all $a \in A$ the set : $[a] = \{x \in A / x \mathcal{R} a\}$ is called the equivalence class.

③ The set of all equivalence classes is denoted by

$$A/_{\mathcal{R}} = \{[a] / a \in A\}$$

Exple $\forall x, y \in \mathbb{R}$, $x \mathcal{R} y \Leftrightarrow x^2 - x = y^2 - y$

$\mathcal{R}$ is an equivalence relation, and we have :

$$[0] = \{x \in \mathbb{R} / x \mathcal{R} 0\} = \{x \in \mathbb{R} / x^2 - x = 0 - 0\} = \{x \in \mathbb{R} / x(x-1) =$$
$$= \{x \in \mathbb{R} / x = 0 \vee x = 1\} = \{0, 1\}$$

$$[1] = \{x \in \mathbb{R} / x \mathcal{R} 1\} = \{x \in \mathbb{R} / x^2 - x = 1 - 1\} = \{0, 1\}$$

2) Partial order relation

A binary relation $\mathcal{R}$ is partial order (or, partial ordering) iff the relation is reflexive, antisymmetric and transitive

Definition: The elements x and y of a set A are comparable if either $x\mathcal{R}y$ or $y\mathcal{R}x$

* If any two elements are comparable, the partial ordering is called total order relation (total ordering)

Exple

$\forall x, y \in \mathbb{R}, \; x\mathcal{R}y \Leftrightarrow x \leq y$

$\mathcal{R}$ is a Partial order relation and is a total ordering, because; $\forall x, y \in \mathbb{R}$: $x \leq y$ or $y \leq x$.

Exple. $\forall (x, y), (x', y') \in \mathbb{R}^2, \; (x,y)\,\mathcal{R}\,(x',y') \Leftrightarrow x \leq x' \wedge y \leq y'$

- $\mathcal{R}$ is a Partial order relation.
- $\mathcal{R}$ is not a total ordering, because:

$\exists (1,2)$ and $(3,0) \in \mathbb{R}^2$ such that: $(1,2) \not\mathcal{R} (3,0)$ and $(3,0) \not\mathcal{R} (1,2)$.

Exercise. Let $f: E \longrightarrow F$, we define the relation $\mathcal{R}$ by

$\forall x, y \in E, \; x\mathcal{R}y \Leftrightarrow f(x) = f(y)$

- Prove that $\mathcal{R}$ is a equivalence relation
- Give the equivalence class of $x \in E$

Remark:

Definition (binary relation)

A binary relation from a set A to a set B is a set of ordred pairs (x, y) where $x \in A$ and $y \in B$ ($A \times B$)

Whene an ordred pair is in a relation, we write: $x\mathcal{R}y$ or $(x,y) \in \mathcal{R}$, it means that element x is related to element y in relation $\mathcal{R}$