Properties.

For any three propositions p, q and r, we have:

1) $p \wedge q \Leftrightarrow q \wedge p$ (commutativity of $\wedge$)
2) $p \vee q \Leftrightarrow q \vee p$ (" " $\vee$)
3) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ ($\wedge$ associative)
4) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ ($\vee$ associative)
5) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
6) $\bar{q} \Rightarrow \bar{p}$ is the contraposition of $p \Rightarrow q$

Exercice. For any three propositions p, q and r:

1) Determine whether each statement is true or false
   a) 8 is odd and 6 is even
   b) 8 is odd or 6 is even
   c) If 8 is odd, then 6 is even
   d) 8 is odd iff 6 is even
2) Create a truth table for $\bar{p} \Rightarrow \bar{q}$
3) Create a truth table for the statement $p \vee \bar{q}$
4) Create a truth table for the statement $p \wedge \overline{(q \vee r)}$

3) Predicate

Def. A predicate or

- Methods of proof.

3

1) Direct methods

To prove $p \Rightarrow q$ start the proof by assuming your hypothesis and use definitions to reach the desired conclusion.

Exple. The sum of any two odd integers is even

Proof. Assume m and n are two integers

By definition of odd integers, $m = 2k+1$, $n = 2\ell+1$, $k, \ell \in \mathbb{Z}$

$$m+n = 2k+1+2\ell+1$$
$$= 2(k+\ell+1) = 2h, \text{ where } h = k+\ell+1 \in \mathbb{Z}$$

Since $2h$ is the definition of an even integer

$\Longrightarrow$ $m+n$ is an even integers

consequently, we can conclude that the sum of two integers is even

2) Proof by cases. ✓Exple1 For any integer k, prove that: $3k^2+k$ is even

*Exple2 The product of two consecutive integers is even

(this can be reworded). $\forall n \in \mathbb{Z}$, $n(n+1)$ is even

Proof. Suppose n is an integer, then n is either even or odd

- Case1. Suppose n is even, then: $n = 2k$, $k \in \mathbb{Z}$

this means that: $n(n+1) = 2k(2k+1) = 2\ell$, $\ell \in \mathbb{Z}$

$\Rightarrow$ $n(n+1)$ is even

- Case2. Suppose n is odd, then: $n = 2k+1$, $k \in \mathbb{Z}$.

$\Rightarrow n(n+1) = (2k+1)(2k+2) = 2(2k+1)(k+1) = 2\ell$, $\ell = (2k+1)(k+1) \in \mathbb{Z}$

Thus, $n(n+1)$ is even

Therefore, in either case, whether k is odd or even $3k^2+k$ is even. It follows that for any integer k, $3k^2+k$ is even

3) Contraposition

Recall that $p \Rightarrow q$ has the same truth as its contrapositive $\overline{q} \Rightarrow \overline{p}$

Exple. $\forall n \in \mathbb{Z}$, n^2 is even then n is even

Proof. Suppose, n is not even, that is: n is odd

then, $n = 2k+1$, $k \in \mathbb{Z}$, we have:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\ell + 1, \quad \ell = 2k^2 + 2k \in \mathbb{Z}$$

this means that: n^2 is odd

thus, if n^2 is even, then n is even

4) Contradiction

In a proof by contradiction: to proof that P is true suppose ~~instead~~ that not(P) is true to arrive at a conclusion you know to be false, then you may conclude that not(P) must be false and thus P must be true

Exple. If $\frac{a}{1+b} = \frac{b}{1+a}$ then $a = b$, $a, b \geq 0$

Proof. We have $a, b \geq 0$, suppose that $\frac{a}{1+b} = \frac{b}{1+b}$ and $a \neq b$

so, $\frac{a}{1+b} = \frac{b}{1+a} \Rightarrow a(1+a) = b(b+1) \Rightarrow a^2 - b^2 = b - a$

$$\Rightarrow (a-b)(a+b) = b - a = -(a-b)$$

since $a \neq b \Rightarrow a - b \neq 0 \Rightarrow \frac{(a-b)(a+b)}{a-b} = \frac{-(a-b)}{a-b}$

$\Rightarrow a + b = -1$, this contradicts our assumption that $a, b \geq 0$

5) The principle of induction (by recurrence)

To proof the statement $P(n)$ is true, $\forall n \in \mathbb{N}$ it is enough to show:

(1) $P(0)$ is true

(2) $\forall n \in \mathbb{N}$, if $P(n)$ is true, then $P(n+1)$ is true, i.e,

$$P(n) \Rightarrow P(n+1), \quad \forall n \in \mathbb{N}.$$

Exple. Prove that, $P(n)$: « $\forall n \in \mathbb{N}^*$, $1+2+\dots+n = \frac{n(n+1)}{2}$

Proof. By induction (récurrence)

· for, $n = 1$

$P(1)$: $1 = \frac{1(1+1)}{2} \Rightarrow 1 = 1 \Rightarrow P(1)$ is true

Now suppose that $P(n)$ is true for $n \in \mathbb{N}$, and we must prove that $P(n+1)$: $1+2+\dots+(n+1) = \frac{(n+1)(n+2)}{2}$, $\forall n \in \mathbb{N}$ is true

We have, $1+2+\dots+n = \frac{n(n+1)}{2}$

adding $(n+1)$ to both sides, we obtain:

$$1+2+\dots+n+n+1 = \frac{n(n+1)}{2} + (n+1)$$

Then, $1+2+\dots+n = \frac{n(n+1)+2(n+1)}{2}$

therefore, we have: $1+2+\dots+n+1 = \frac{(n+1)(n+2)}{2}$

which is $P(n+1)$

Consequently $P(n)$ is true $\forall n \in \mathbb{N}$.

6) Counter-example

To show that the statement $P(n)$ is false it is enough to find an element n such that $P(n)$ is false, i.e, ~~to~~ find a counter-example

Exple. Prove that: « $\forall n \in \mathbb{N}$, n is even $\Rightarrow n^2+1$ is even » is false

This statement is false, because:

for, $n = 2$: $n^2+1 = 4+1 = 5$ is not even

(It's a counter-example).