Solution of Series n°01.

EX01.

$A = \phi$: $\forall x \in \mathbb{N}$, $x \notin A$

$A \cap B \neq \phi$: $\exists x \in \mathbb{N}$, $x \in A \wedge x \in B$

$A \subset B$: $\forall x \in \mathbb{N}$, $x \in A \Rightarrow x \in B$

$A \not\subset B$: $\forall x \in \mathbb{N}$, $\exists x \in A \wedge x \notin B$

EX02

We Know that $\overline{(P \Rightarrow Q)} \Leftrightarrow \overline{\overline{P} \vee Q} \Leftrightarrow P \wedge \overline{Q}$

- the negation: $\exists \varepsilon > 0, \forall \alpha > 0, \exists x \in \mathbb{R}, |x - x_0| < \alpha \wedge |f(x) - f(x_0)| \geq \varepsilon$.

EX03. P_1: is false, because its negation which is

$\overline{P_1}$: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \leq 0$ is true, for example: we can take, $y = -(x+1)$, and we have: $x + y = x - x - 1 = -1 \leq 0$

P_2: is true, for a given x, we can take (for example) $y = -x + 1$, and we have: $x + y = 1 > 0$

$\overline{P_2}$: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y \leq 0$

P_3: is false, (for example), $x = -1$, $y = 0$: $x + y = -1 \neq 0$

$\overline{P_3}$: $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \leq 0$

P_4: is true, we can take, $x = -1$, $y^2 > -1$

$\overline{P_4}$: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^2 \leq x$.

EX04. Suppose that n is an integer, then n is either even or odd

Case1. n is even, then: $n = 2k$, $k \in \mathbb{Z}$. This means that,

$n(n+1) = 2k(2k+1) = 2k'$, $k' \in \mathbb{Z} \Rightarrow n(n+1)$ is even

Case2: n is odd, then: $n = 2k + 1$, $k \in \mathbb{Z} \Rightarrow n(n+1) =$

$(2k+1)(2k+2) = 2(2k+1)(k+1) = 2k'$, $k' \in \mathbb{Z} \Rightarrow n(n+1)$ is even

$\therefore$ In both cases $n(n+1)$ is even.

EX05. $\forall n \in \mathbb{N}$, $8\frac{n(n+1)}{2} + 1 = 4n^2 + 4n + 1 = (2n+1)^2 \Rightarrow$ It's a square number

EX06. The contraposition of: n^2 odd $\Rightarrow n$ is odd ; is: $(\overline{q} \Rightarrow \overline{p})$

n even $\Rightarrow n^2$ even

n is even: $n = 2k \Rightarrow n^2 = 4k^2 = 2(2k^2) = 2k'$, $k' \in \mathbb{N} \Rightarrow n^2$ even

EX07. By contradiction, $\forall n \in \mathbb{N}$: we suppose that, n^2 is even and n is an odd number.

We have: n is odd $\Rightarrow n = 2k+1$, $k \in \mathbb{N}$, then: $n^2 = (2k+1)^2 \Rightarrow$

$n^2 = 4k^2 + 4k + 1 = 2k(k+2) + 1 = 2(k^2 + 2k) + 1 = 2k' + 1$, $k' \in \mathbb{N}$

$\Rightarrow n^2$ is an odd number, this contradicts our assumption that n^2 is even. ~~Consequently then~~

So the assumption that n is odd is false.

Consequently: n^2 is even $\Rightarrow n$ is even

EX08. (By countre-example)

(a): $\forall x \in \mathbb{R}$, we can choose; $x = 1$: $1^2 + 1 = 2 \neq 0$

(b): $\forall x \in]0,1[$, we can choose; $x = \frac{1}{2}$: $\frac{2 \times \frac{1}{2}}{(\frac{1}{2})^2 (1 - (\frac{1}{2})^2)} = \frac{4}{\frac{3}{4}} = \frac{16}{3} > 1$

EX09. By induction (recurrence).

$P(n)$: « $\forall n \in \mathbb{N}^*$, $1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$ »

for, $n = 1$; $P(1)$: $1 = \frac{1^2(1+1)^2}{4}$,

$\Rightarrow 1 = 1 \Rightarrow P(1)$ is true

Now suppose that $P(n)$ is true for $n \in \mathbb{N}^*$, and we must prove that $P(n+1)$: « $\forall n \in \mathbb{N}^*$: $1^3 + \dots + (n+1)^3 = \frac{(n+1)^2 (n+2)^2}{4}$ » is true

We have: $1^3 + \dots + (n+1)^3 = 1^3 + \dots + n^3 + (1+n)^3$

$= \frac{n^2(n+1)^2}{4} + \frac{4(1+n)^3}{4} = \frac{(n+1)^2(n^2+4n+4)}{4}$

$= \frac{(n+1)^2 + (n+2)^2}{4} \Rightarrow P(n+1)$ is true

Consequently: $P(n)$ is true, $\forall n \in \mathbb{N}^*$

EX10. $P(n+1)$: « $\forall n \in \mathbb{N}$; $(1+a)^{n+1} \geq 1 + (n+1)a$. »

We have, $(1+a)^{n+1} = (1+a)(1+a)^n \geq (1+a)(1+n.a) = 1 + (n+1)a + na^2$

$\forall n \in \mathbb{N}$: $na^2 \geq 0 \Rightarrow 1 + (n+1)a + na^2 \geq 1 + (n+1)a \Rightarrow P(n+1)$ is true $\forall n \in \mathbb{N}$.

Ex11. Suppose that, the proposition is false, so $\sqrt{2}$ is rational. Then there exist integers a, b so that $\sqrt{2} = \frac{a}{b}$; $b \neq 0$, (this fraction is in simplest form i.e, a and b are coprime.)

By squaring both sides, we have:

$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$, so $2 = \frac{a^2}{b^2}$, then $a^2 = 2b^2$, therefore a^2 is even and hence a must also be even.

Thus, $\exists\, k \in \mathbb{Z}$ so that $a = 2k \Rightarrow a^2 = 4k^2$ and $a^2 = 2b^2 = 4k^2$, dividing by 2 yields: $b^2 = 2k^2$.

Therefore, b^2 is even and b must also be even.

Since a and b are both even, they are both divisible by 2, but by assumption, a and b are coprime, so this is impossible.

Therefore, it cannot be the case that the proposition is false

Thus $\sqrt{2}$ is irrational.

Ex12. Let $n \in \mathbb{Z}$,

the contraposition of: $5n-7$ is even $\Rightarrow n$ is odd, is

n is even $\Rightarrow 5n-7$ is odd

Suppose that: n is even, then: $n = 2k$, $k \in \mathbb{Z}$.

Then, $5n-7 = 5(2k)-7 = 10k-7 = 2(5k-4)+1$

$= 2k'+1$, $k' \in \mathbb{Z}$.

Then, $5n-7$ is odd.