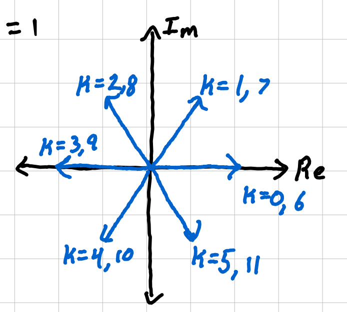
Exercise 4: Task 3

$W_N = e^{j\frac{2\pi}{N}}$

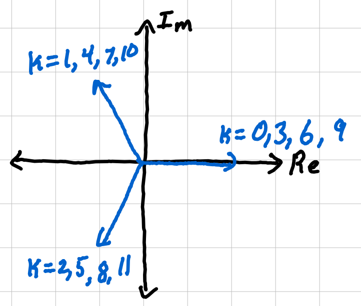
1) W_6^{nk} for $k = 0, 1 \dots 11$

n=1



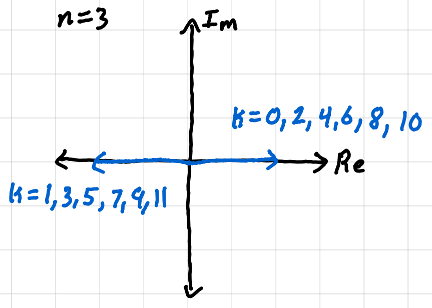
$k=0 : e^{j\frac{2\pi}{6}(1)(0)} = 1$
$k=1 : e^{j\frac{2\pi}{6}(1)(1)} = e^{j\frac{\pi}{3}}$
$k=2 : e^{j\frac{2\pi}{3}}$
$k=3 : e^{j\pi}$
$k=4 : e^{j\frac{4\pi}{3}}$
$k=5 : e^{j\frac{5\pi}{3}}$
$k=6 : e^{j2\pi} = 1$
(repeats for $k = 6, 7, \dots, 11$)

n=2



$k=0 : e^{j\frac{\pi}{3}(2)(0)} = 1$
$k=1 : e^{j\frac{\pi}{3}(2)(1)} = e^{j\frac{2\pi}{3}}$
$k=2 : e^{j\frac{\pi}{3}(2)(2)} = e^{j\frac{4\pi}{3}}$
$k=3 : e^{j\frac{\pi}{3}(2)(3)} = e^{j2\pi} = 1$
(repeats)

n=3



$k=0 : e^{j\frac{\pi}{3}(3)(0)} = 1$
$k=1 : e^{j\pi} = -1$
$k=2 : e^{j2\pi} = 1$
(repeats like this)

2) $W_N^{Nk} = \left(e^{j\frac{2\pi}{N}}\right)^{Nk} = e^{j\frac{2\pi}{N}Nk} = e^{j2\pi k} = 1$

3) For n is divisible by N, $\frac{n}{N}$ is always an integer. This means $2\pi\frac{n}{N}k$ will always be an integer multiple of 2π, making $W_N^{nk} = 1$.

$$\sum_{k=0}^{N-1} W_N^{nk} = \sum_{k=0}^{N-1} e^{j2\pi\frac{n}{N}k} = \sum_{k=0}^{N-1} 1 = N$$

For n is not divisible by 1, the values of W_N^{nk} will rotate around the complex plane until they reach the $W_N^{nk}\big|_{k=N} = W_N^{nk}\big|_{k=0} = 1$. In this case, the imaginary and real components will sum to 0.

$$\sum_{k=0}^{N-1} W_N^{nk} = \sum_{k=0}^{N-1} Re\{W_N^{nk}\} + j\sum_{k=0}^{N-1} Im\{W_N^{nk}\} = 0$$

This can also be proven with the sum of a geometric series:

$$\sum_{k=0}^{N-1} e^{j2\pi\frac{nk}{N}} = \frac{1 - e^{j2\pi n}}{1 - e^{j2\pi n/N}} = \frac{1-1}{1 - e^{j2\pi n/N}} = 0$$

4) The Fourier modes in a DFT in $\mathbb{R}$ are:

$$\psi_k(n) = e^{j2\pi\frac{nk}{N}} = W_N^{nk}$$

In a DFT, $n \in \{0, 1, \dots, N-1\}$ so n will never be divisible by N.

$$\langle \psi_k(n), \psi_h^*(n) \rangle = \langle W_N^{nk}, W_N^{nh^*} \rangle$$
$$= \sum_{n=1}^{N-1} e^{j2\pi\frac{n}{N}k} e^{-j2\pi\frac{n}{N}h}$$
$$= \sum_{n=1}^{N-1} e^{j2\pi\frac{n}{N}(k-h)}$$
$$= 0 \quad , \quad \text{s.t. } k \neq h$$

The Fourier modes are orthogonal because $\langle \psi_h(n), \psi_h^*(n) \rangle = 0$