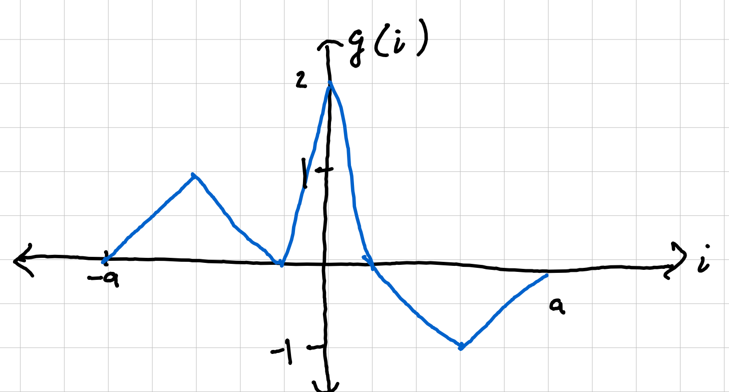
Exercise 3 : Task 4



Define : $h(i) = \begin{cases} i & 0 \le i \le \frac{a}{5} \\ 0 & \text{else} \end{cases}$

$$g(i) = \tfrac{1}{2}h\left(\tfrac{1}{2}(i+a)\right) + \tfrac{1}{2}h\left(-\tfrac{1}{2}(i+\tfrac{1}{5}a)\right) + h\left(i+\tfrac{a}{5}\right) + h\left(-(i-\tfrac{1}{5}a)\right) - \tfrac{1}{2}h\left(\tfrac{1}{2}(i-\tfrac{1}{5}a)\right) - \tfrac{1}{2}h\left(-\tfrac{1}{2}(i-a)\right)$$

a) - Find $(f*g)(i)$ such that:

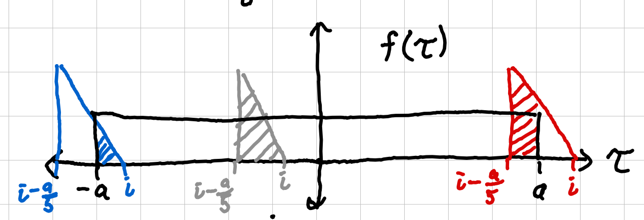
$$f(i) = \begin{cases} 1 & -a \le i \le a \\ 0 & \text{else} \end{cases}$$

- First compute $(f*h)(i) = p(i)$

$$p(i) = (f*h)(i) = \int_{-\infty}^{\infty} f(\tau)\, h(i-\tau)\, d\tau$$

$$p(i) = \int_{-a}^{a} (1)\, h(i-\tau)\, d\tau$$

- This integral is nonzero in 3 cases:



$-a \le i < -a + \frac{a}{5}$

$-a + \frac{a}{5} \le i < a$

$a \le i < a + \frac{a}{5}$

$$p(i) = \begin{cases} \int_{-a}^{i} h(i-\tau)\, d\tau & -a \le i < -a + \frac{a}{5} \\ \int_{i-a/5}^{i} h(i-\tau)\, d\tau & -a + \frac{a}{5} \le i < a \\ \int_{i-\frac{a}{5}}^{a} h(i-\tau)\, d\tau & a \le i \le a + \frac{a}{5} \end{cases}$$

$$p(i) = \begin{cases} \int_{-a}^{i} (i-\tau)\, d\tau & -a \le i < -a + \frac{a}{5} \\ \int_{i-a/5}^{i} (i-\tau)\, d\tau & -a + \frac{a}{5} \le i < a \\ \int_{i-\frac{a}{5}}^{a} (i-\tau)\, d\tau & a \le i \le a + \frac{a}{5} \end{cases}$$

$$p(i) = \begin{cases} \left[i\tau - \frac{1}{2}\tau^2\right]_{-a}^{i} & -a \le i < -a + \frac{a}{5} \\ \left[i\tau - \frac{1}{2}\tau^2\right]_{i-\frac{a}{5}}^{i} & -a + \frac{a}{5} \le i < a \\ \left[i\tau - \frac{1}{2}\tau^2\right]_{i-\frac{a}{5}}^{a} & a \le i \le a + \frac{a}{5} \end{cases}$$

$$p(i) = \begin{cases} (i^2 - \frac{1}{2}i^2) - (-ai - \frac{a^2}{2}) & -a \le i < -a + \frac{a}{5} \\ (i^2 - \frac{1}{2}i^2) - \left(i(i-\frac{a}{5}) - \frac{1}{2}(i-\frac{a}{5})^2\right) & -a + \frac{a}{5} \le i < a \\ (ia - \frac{1}{2}a^2) - \left(i(i-\frac{a}{5}) - \frac{1}{2}(i-\frac{a}{5})^2\right) & a \le i \le a + \frac{a}{5} \end{cases}$$

$$p(i) = \begin{cases} \frac{1}{2}(i+a)^2 & -a \le i < -a + \frac{a}{5} \\ \frac{1}{50}a^2 & -a + \frac{a}{5} \le i < a \\ -\frac{1}{2}i^2 + ai - \frac{12}{25}a^2 & a \le i \le a + \frac{a}{5} \end{cases}$$

- Using $p(i)$ and linearity properties of convolutions, we can find $(f*g)(i)$:

$$(f*g)(i) = \tfrac{1}{2}p\left(\tfrac{1}{2}(i+a)\right) + \tfrac{1}{2}p\left(-\tfrac{1}{2}(i+\tfrac{1}{5}a)\right) + p\left(i+\tfrac{1}{5}a\right) + p\left(-(i-\tfrac{1}{5}a)\right) - \tfrac{1}{2}p\left(-\tfrac{1}{2}(i-\tfrac{1}{5}a)\right) - \tfrac{1}{2}p\left(-\tfrac{1}{2}(i-a)\right)$$

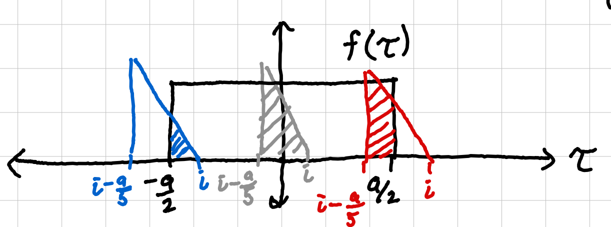
b) - We can use the same method as part a) but with:

$$f(i) = \begin{cases} 2 & -\frac{a}{2} \le i \le \frac{a}{2} \\ 0 & \text{else} \end{cases}$$

$$p(i) = (f*h)(i) = \int_{-\infty}^{\infty} f(\tau)\, h(i-\tau)\, d\tau$$

$$p(i) = \int_{-a/2}^{a/2} (2)\, h(i-\tau)\, d\tau$$

- Here, our boundaries change



$\frac{-a}{2} \le i < \frac{-a}{2} + \frac{a}{5}$

$\frac{-a}{2} + \frac{a}{5} \le i < \frac{a}{2}$

$\frac{a}{2} \le i < \frac{a}{2} + \frac{a}{5}$

$$p(i) = \begin{cases} 2\int_{-a/2}^{i} h(i-\tau)\, d\tau & -\frac{a}{2} \le i < \frac{-a}{2} + \frac{a}{5} \\ 2\int_{i-a/5}^{i} h(i-\tau)\, d\tau & -\frac{a}{2} + \frac{a}{5} \le i < \frac{a}{2} \\ 2\int_{i-a/5}^{a/2} h(i-\tau)\, d\tau & \frac{a}{2} \le i < \frac{a}{2} + \frac{a}{5} \end{cases}$$

$$p(i) = \begin{cases} 2\left[i\tau - \frac{1}{2}\tau^2\right]_{-a/2}^{i} & -\frac{a}{2} \le i < \frac{-a}{2} + \frac{a}{5} \\ 2\left[i\tau - \frac{1}{2}\tau^2\right]_{i-a/5}^{i} & -\frac{a}{2} + \frac{a}{5} \le i < \frac{a}{2} \\ 2\left[i\tau - \frac{1}{2}\tau^2\right]_{i-a/5}^{a/2} & \frac{a}{2} \le i < \frac{a}{2} + \frac{a}{5} \end{cases}$$

$$p(i) = \begin{cases} 2\left[(i^2 - \frac{1}{2}i^2) - (-i\frac{a}{2} - \frac{1}{2}(\frac{-a}{2})^2)\right] & -\frac{a}{2} \le i < \frac{-a}{2} + \frac{a}{5} \\ 2\left[(i^2 - \frac{1}{2}i^2) - (i(i-\frac{a}{5}) - \frac{1}{2}(i-\frac{a}{5})^2)\right] & -\frac{a}{2} + \frac{a}{5} \le i < \frac{a}{2} \\ 2\left[(i\frac{a}{2} - \frac{1}{2}\frac{a^2}{4}) - (i(i-\frac{a}{5}) - \frac{1}{2}(i-\frac{a}{5})^2)\right] & \frac{a}{2} \le i < \frac{a}{2} + \frac{a}{5} \end{cases}$$

$$p(i) = \begin{cases} i^2 + ai + \frac{a^2}{4} & -\frac{a}{2} \le i < \frac{-a}{2} + \frac{a}{5} \\ \frac{a^2}{25} & -\frac{a}{2} + \frac{a}{5} \le i < \frac{a}{2} \\ -i^2 + 2ai - \frac{23}{50}a^2 & \frac{a}{2} \le i < \frac{a}{2} + \frac{a}{5} \end{cases}$$

- We can then compute $(f*g)(i)$ the same way as before

$$(f*g)(i) = \tfrac{1}{2}p\left(\tfrac{1}{2}(i+a)\right) + \tfrac{1}{2}p\left(-\tfrac{1}{2}(i+\tfrac{1}{5}a)\right) + p\left(i+\tfrac{1}{5}a\right) + p\left(-(i-\tfrac{1}{5}a)\right) - \tfrac{1}{2}p\left(-\tfrac{1}{2}(i-\tfrac{1}{5}a)\right) - \tfrac{1}{2}p\left(-\tfrac{1}{2}(i-a)\right)$$