1 Convolution

Let h be the discrete signal $h[n] = \frac{1}{2}\delta[n] + \frac{1}{3}\delta[n-1]$, with δ being the Kronecker delta function.

1. A function f which satisfies $f[n] = 0$ for all $n < 0$ is called a causal function. We want to find a **causal** solution $g[n]$ of the equation $(h*g)[n] = \delta[n]$.
   (a) Write explicitly the equation $(h * g)[n] = \delta[n]$.
   (b) What is the value of $g[0]$?
   (c) Give the relationship between $g[n]$ and $g[n-1]$ for all $n > 0$.
   (d) Deduce from the previous results the expression of $g[n]$.
2. Does the solution belong to the space of absolutely convergent sequences $\ell_1(\mathbb{Z})$?

2 Solution

1. (a) Using the expression of $h[n]$, we obtain $\frac{1}{2}g[n] + \frac{1}{3}g[n-1] = \delta[n]$.
   (b) At $n = 0$, we have $\frac{1}{2}g[0] + \frac{1}{3}g[-1] = 1 \Leftrightarrow g[0] = 2$. Since $g[n]$ must be causal, we must have $g[-1] = 0$.
   (c) For $n > 0$, we have $\frac{1}{2}g[n] + \frac{1}{3}g[n-1] = 0 \Leftrightarrow g[n] = -\frac{2}{3}g[n-1]$.
   (d) From the previous questions, we have for g a geometric sequence with common ratio $-\frac{2}{3}$, which gives $g[n] = (-\frac{2}{3})^n g[0] = 2(-\frac{2}{3})^n$ for $n \geq 0$. Since $g[n]$ must be causal, we hence have $g[n] = 2(-\frac{2}{3})^n$ for all $n \geq 0$, which verify the equation $(h * g)[n] = \delta[n]$ for all $n \in \mathbb{Z}$.
2. We have $\|g\|_{\ell_1} = 2\sum_{n=0}^{+\infty} \left|(-\frac{2}{3})^n\right| = \frac{2}{1-\frac{2}{3}} = 3 < +\infty$, therefore $g \in \ell_1(\mathbb{Z})$.