1 Scalar product in a discrete space

Reminder : the partial sum of a geometric series is $\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}$ for $q \neq 1$. For the calculations, you may need the Riemann Zêta function:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

The series converges for all $s > 1$ (with $\zeta(1) = +\infty$) ; some particular values $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(3) \approx 1.202$ (Apéry constant).

We work here in the discrete domain and the scalar product between 2 functions f_i and f_j is defined as $\langle f_i, f_j \rangle = \sum_{n \in \mathbb{Z}} f_i[n] f_j[n]$. Let $u[n]$ denote the step function:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

We introduce the following functions:

$$\begin{aligned} f_1[n] &= \delta[n] - \frac{1}{2}\delta[n-1], \\ f_2[n] &= \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & n \leq 0 \end{cases}, \\ f_3[n] &= \begin{cases} \frac{1}{|n|} & n \neq 0 \\ 1 & n = 0 \end{cases}, \\ f_4[n] &= 1, \\ f_5[n] &= (\frac{1}{2})^n u[n], \\ f_6[n] &= (\frac{3}{2})^n u[n]. \end{aligned}$$

Compute the following scalar products: $\langle f_1, f_2 \rangle$, $\langle f_1, f_3 \rangle$, $\langle f_2, f_3 \rangle$, $\langle f_4, f_5 \rangle$, $\langle f_5, f_6 \rangle$.

2 Solution

1. $\langle f_1, f_2 \rangle = \sum_{n=1}^{+\infty} (\delta[n] - \frac{1}{2}\delta[n-1]) \frac{1}{n^2} = -\frac{1}{2}$.
2. $\langle f_1, f_3 \rangle = f_1[0] + \sum_{n \in \mathbb{Z} \setminus \{0\}} (\delta[n] - \frac{1}{2}\delta[n-1]) \frac{1}{|n|} = 1 - \frac{1}{2} = \frac{1}{2}$.
3. $\langle f_2, f_3 \rangle = \sum_{n=1}^{+\infty} \frac{1}{n^2} \frac{1}{|n|} = \sum_{n=1}^{+\infty} \frac{1}{n^3} = \zeta(3) \approx 1.202$.
4. $\langle f_4, f_5 \rangle = \sum_{n=0}^{+\infty} (\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2$.
5. $\langle f_5, f_6 \rangle = \sum_{n=0}^{+\infty} (\frac{1}{2})^n (\frac{3}{2})^n = \sum_{n=0}^{+\infty} (\frac{3}{4})^n = \frac{1}{1-\frac{3}{4}} = 4$.