Exercise 4: Task 5

1) $F(u,v) = \mathcal{F}\{f(x,y)\}$

$$\mathcal{F}\{f(x,y)\, e^{j2\pi(u_0 x/M + v_0 y/N)}\} =$$

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{j2\pi(u_0 x/M + v_0 y/N)}\, e^{-j2\pi(ux/M + v_0 y/N)} =$$

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi[(u-u_0)x/M + (v-v_0)y/N]} =$$

$$\boxed{F(u-u_0,\ v-v_0)}$$

$$\mathcal{F}^{-1}\{F(u,v)\, e^{-j2\pi(x_0 u/M + y_0 v/N)}\} =$$

$$\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)\, e^{-j2\pi(x_0 u/M + y_0 v/N)}\, e^{j2\pi(ux/M + vy/N)} =$$

$$\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)\, e^{j2\pi[(x-x_0)u/M + (y-y_0)v/N]} =$$

$$\boxed{f(x-x_0,\ y-y_0)}$$

2) $x = r\cos\theta \qquad u = \omega\cos\phi$

$y = r\sin\theta \qquad v = \omega\sin\phi$

→ define rotated coordinates (x', y') in terms of x, y, and θ_0

$\theta' = \theta + \theta_0$

$x = r\cos(\theta' - \theta_0)$

$x = r\cos\theta'\cos\theta_0 + r\sin\theta'\sin\theta_0$

$x = x'\cos\theta_0 + y'\sin\theta_0$

$y = r\sin(\theta' - \theta_0)$

$y = r\sin\theta'\cos\theta_0 - r\cos\theta'\sin\theta_0$

$y = y'\cos\theta_0 - x'\sin\theta_0$

→ find the 2D DFT of $f(x', y')$

$$\mathcal{F}\{f(x', y')\} =$$

$$\sum_{x'}\sum_{y'} f(x', y')\, e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} =$$

$$\sum_{x'}\sum_{y'} f(x', y')\, e^{-j2\pi\left[\frac{u}{M}(x'\cos\theta_0 + y'\sin\theta_0) + \frac{v}{N}(y'\cos\theta_0 - x'\sin\theta_0)\right]} =$$

$$\sum_{x'}\sum_{y'} f(x', y')\, e^{-j2\pi\left[\frac{x'}{M}(u\cos\theta_0 - v\sin\theta_0) + \frac{y'}{N}(v\cos\theta_0 + u\sin\theta_0)\right]} =$$

→ Recognize rotated coordinates (u', v')

$u' = u\cos\theta_0 - v\sin\theta_0 = r\cos(\phi + \theta_0)$

$v' = v\cos\theta_0 + u\sin\theta_0 = r\sin(\phi + \theta_0)$

$$\sum_{x'}\sum_{y'} f(x', y')\, e^{-j2\pi\left(\frac{x'u'}{M} + \frac{y'v'}{N}\right)} =$$

$\boxed{F(u', v'), \text{ s.t. } (u', v') \text{ are the coordinates } (u,v) \text{ rotated by } \theta_0}$