Chapter 2: Finite dimensional vector space

Linear Algebra Done Right, by Sheldon Axler

Contents

A: Span and Linear independence

B: Bases

Problem 1

Find all vector spaces that have exactly one basis.

Proof. Consider all lines passing through origin. □

Problem 3

(a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Proof. (a) We can verify that $(3, 1, 0, 0, 0),\ (0, 0, 7, 1, 0),\ (0, 0, 0, 0, 1)$ is a basis of U, and it spans U.
Idea: We start from the basis of $\mathbb{R}^5$ and try to fit it into the given condition.

(b) $(3, 1, 0, 0, 0),\ (0, 0, 7, 1, 0),\ (0, 0, 0, 0, 1),\ (0, 1, 0, 0, 0),\ (0, 0, 1, 0, 0)$
Idea: Downgrade all vector to one dimensional (to obtain the basis of $\mathbb{R}^5$).

(c) $\operatorname{span}\{(0, 0, 0, 0, 1),\ (0, 1, 0, 0, 0)\}$
Idea: Refer to the definition of direct sum.

□

Problem 5

Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has a degree 2.

Proof. Consider

$$\begin{aligned} p_0 &= a_1 \\ p_1 &= a_2 x \\ p_2 &= a_3 x^3 \\ p_3 &= a_4 x^2 + a_5 x^3 \end{aligned}$$

It is easy to verify that they are linearly independent, and span $\mathcal{P}_3(\mathbb{F})$ □

Problem 6

Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Proof. We could obtain v_1, v_2, v_3 with the following operation

$$v_3 + v_4 - v_4 = v_3 \tag{1}$$
$$v_2 + v_3 - (v_3 + v_4 - v_4) = v_2 \tag{2}$$
$$v_1 + v_2 - (v_2 + v_3 - (v_3 + v_4 - v_4)) = v_1 \tag{3}$$

Which suggests that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ also spans V.
Next, we prove that they are linearly independent.

$$a(v_1 + v_2 + b(v_2 + v_3) + c(v_3 + v_4) + dv_4 \tag{4}$$
$$=av_1 + (a + b)v_2 + (b + c)v_3 + (c + d)v_4 \tag{5}$$

Since v_1, v_2, v_3, v_4 is a basis, we have $a = b = c = d = 0$. This implies that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also linearly independent. $\square$

Problem 8

Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof. The linear independency of $u_1, \ldots, u_m, w_1, \ldots, w_n$ is directly come from the definition of direct sum since $U \cap W = 0$. Since $u_1, \ldots, u_m$ is a basis of U. We can write any $u \in U$ as a linear combination of $u_1, \ldots, u_m$. Similarly, we can write any $w \in W$ as a linear combination of $w_1, \ldots, w_n$. Therefore, we can write any $v \in V$ as a linear combination of $u_1, \ldots, u_m, w_1, \ldots, w_n$. $\square$

C: Dimension

Problem 1

Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Let $u_1, \ldots, u_n$ be a basis of U. Since U is a subspace of V, we can extend the basis of U to a basis of V. Since $\dim U = \dim V$, every independent list of right length is also a base suggests that $u_1, \ldots, u_n$ also spans V. Therefore, $U = V$. $\square$

Problem 2

Show that the subspaces of $\mathbb{R}^2$ are precisely the zero subspace, all lines in R^2 through the origin, and $\mathbb{R}^2$ itself.

Proof. $\dim R^2 = 2 \implies \dim \text{subspaces of } R^2 \leq 2$

$$\text{When} \quad \dim U = 0, \qquad U = \{0\}. \tag{6}$$

$$\dim U = 1, \qquad U \text{ is a line passing through origin.} \tag{7}$$

$$\dim U = 2, \qquad U = R^2. \tag{8}$$

$\square$