Chapter 2: Finite dimensional vector space

Linear Algebra Done Right, by Sheldon Axler

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A: Span and Linear independence

B: Bases

Problem 1

Find all vector spaces that have exactly one basis.

Proof. Consider all lines passing through origin. □

Problem 3

(a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Proof. (a) We can verify that $(3,1,0,0,0),\ (0,0,7,1,0),\ (0,0,0,0,1)$ is a basis of U, and it spans U.
Idea: We start from the basis of $\mathbb{R}^5$ and try to fit it into the given condition.

(b) $(3,1,0,0,0),\ (0,0,7,1,0),\ (0,0,0,0,1),\ (0,1,0,0,0),\ (0,0,1,0,0)$
Idea: Downgrade all vector to one dimensional (to obtain the basis of $\mathbb{R}^5$).

(c) $\operatorname{span}\{(0,0,0,0,1),\ (0,1,0,0,0)\}$
Idea: Refer to the definition of direct sum.

□

Problem 5

Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has a degree 2.

Proof. Consider

$$\begin{aligned} p_0 &= a_1 \\ p_1 &= a_2 x \\ p_2 &= a_3 x^3 \\ p_3 &= a_4 x^2 + a_5 x^3 \end{aligned}$$

It is easy to verify that they are linearly independent, and span $\mathcal{P}_3(\mathbb{F})$ □