Chapter 5: Eigenvalues, Eigenvectors, and invariant subspaces

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A: Invariant Subspaces

Problem 6

Prove or give a counterexample: if V is a finite-dimensional vector space and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Proof. Because every operator on V leaves $\{0\}$ invariant, the question turns to prove the existence of an operator under which only $\{0\}$ and V is invariant.

The case where $\dim V \leq 1$ is trivial. Suppose $\dim V \geq 2$. We could always construct such an operator U. Suppose there exists an invariant subspace U_1 of V under an operator T that is neither $\{0\}$ nor V for which $\dim U_1 < \dim V$. Define T which rotates U_1 to W, where $W \oplus U_1 = V$. $\forall u \in U_1, T(u)$ have some component in W, which is not in U_1. Thus, U is not invariant under U, which is a contradiction. Therefore, $U = \{0\}$ or $U = V$. □