Chapter 3: Linear maps

Linear Algebra Done Right, by Sheldon Axler

Contents

A: The vector space of linear maps

Problem: 1

Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is a linear map if and only if $b = 0$ and $c = 0$.

Proof. Consider $T(1, 1, 1) = T(1, 0, 0) + T(0, 1, 1)$

$$\begin{aligned} T(1, 1, 1) &= (1 + b, 6 + c) \\ T(1, 0, 0) &= (2 + b, 6) \\ T(0, 1, 1) &= (-1 + b, 0) \end{aligned}$$

Therefore, $b = 0, c = 0$ □

B: Null space and Range

Problem: 1

Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Proof. Consider $T : \mathcal{P}(4) \mapsto \mathcal{P}(1)$, $T(az^4 + bz^3 + cz^2 + dz + e) = (dz + e)$. Then $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$. □

Problem: 3

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V.

(b) What property of T corresponds to $v_1, \ldots, v_m$ being linearly independent.

(a) *Proof.* $\forall v \in V$, we have $T(z_1, \ldots, z_m) = v$, which means $v_1, \ldots, v_m$ span V. This suggests that $\operatorname{range} T$ is equal to V. Hence, T is surjective. □

(b) *Proof.* If $v_1, \ldots, v_m$ are linearly independent, then $T(z_1, \ldots, z_m) = 0$ implies $z_1 = \cdots = z_m = 0$. This suggests that $\operatorname{null} T = \{0\}$, hence T is injective. □

Problem: 4

Show that
$$T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2$$
is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$

Proof. Suppose $\dim \operatorname{null} T > 2$, by F.T. of linear maps,

$$\begin{aligned}\dim T = 5 &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &> 2 + \dim \operatorname{range} T \\ \dim \operatorname{range} T &< 3\end{aligned}$$

Hence, $T \notin \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$, and thus not a subspace of $\mathcal{L}(R^5, R^4)$. □

Problem: 5

Give an example of a linear map $T : \mathbb{R}^4 \mapsto \mathbb{R}^4$ such that
$$\operatorname{range} T = \operatorname{null} T.$$

Proof. Consider $T : R^4 \mapsto R^4$, $T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0)$. Then $\operatorname{range} T = \operatorname{null} T$. □

Problem: 6

Prove that there does not exist a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\operatorname{range} T = \operatorname{null} T$.

Proof. Suppose there exists a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\operatorname{range} T = \operatorname{null} T$. Then by F.T. of linear maps, we have $\dim \operatorname{range} T = \dim \operatorname{null} T$, which implies $\dim \operatorname{range} T = 5 - \dim \operatorname{range} T$, or $\dim \operatorname{range} T = 2.5$, which is not an integer. Hence, such a linear map does not exist. □

Problem: 7

Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective } \}$ is not a subspace of $\mathcal{L}(V, W)$.

Proof. T is not injective suggests that $\dim \operatorname{null} V > 0$. Then By the F.T. of linear maps, we have

$$\begin{aligned}\dim V &= \dim \operatorname{null} V + \dim W \\ &\geq 1 + \dim W \geq 1 + \dim V\end{aligned}$$

Contradicts! Hence, $T \notin \mathcal{L}(V, W) \implies$ not a subspace. □

Problem: 9

Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Proof. T is injective $\implies$ null $T = \{0\}$. Therefore, we have

$$T(\lambda_1 v_1 + \cdots + \lambda_n v_n) \iff \lambda_j = 0$$

where $j \in \{1, \ldots, n\}$. Since $v_1, \ldots, v_n$ are linearly independent. It follows that $Tv_1, \ldots, Tv_n$ are linearly independent. $\square$

Problem: 10

Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \ldots, Tv_n$ spans range T.

Proof. Suppose $v_1, \ldots, v_n$ spans V. Then $\forall v \in V$, we have $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then

$$T(v) = T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 Tv_1 + \cdots + \lambda_n Tv_n.$$

This suggests that $\operatorname{range}(T) \subset \operatorname{span}(Tv_1, \ldots, Tv_n)$. Also, $\operatorname{span}(Tv_1, \ldots, Tv_n)$ is the smallest containing subspace of W implying that it is a subset of range W, hence range $T = \operatorname{span}(Tv_1, \ldots, Tv_n)$. $\square$

Problem: 11

Suppose $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \ldots S_n$ makes sense. Prove that $S_1 S_2 \ldots S_n$ is injective.

Proof. By F.T. of linear maps, we have

$$\begin{aligned} \dim S_1 &= \dim \operatorname{null} S_1 + \dim \operatorname{range} S_1 \\ &= 0 + \dim \operatorname{range} S_1 = \dim \operatorname{range} S_1 \end{aligned}$$

It follows that

$$\dim S_1 = \dim \operatorname{range} S_1 = \dim S_2 = \cdots = \dim S_n = \dim \operatorname{range} S_n$$

Hence, $\dim \operatorname{null} S_1 S_2 \ldots S_n = 0 \iff S_1 S_2 \ldots S_n$ is injective. $\square$

Problem: 12

Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that null $T = U$ and $U \cap \operatorname{range} T = \{0\}$.

Proof. Let $u_1, \ldots, u_n$ be a basis for $\operatorname{null} T$. Then $\operatorname{span}(u_1, \ldots, u_n)$ is a subspace of V. Since it is linear independent, it can be extended to a basis of V, say $u_1, \ldots, u_n, v_1, \ldots, v_m$. Then V is the direct sum of the spanning of $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$. Take $U = \operatorname{span}(v_1, \ldots, v_m)$ $\square$

Problem: 13

Suppose T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is surjective.

Proof. A basis of $\operatorname{null} T$ is

$$\{(5, 1, 0, 0), (0, 0, 7, 1)\}$$

Then by F.T. of linear maps, we have

$$\dim \operatorname{range} T = 4 - \dim \operatorname{null} T = 4 - 2 = 2$$

Therefore, $\operatorname{range} T = \mathbb{R}^2 \implies T$ is surjective. $\square$

Problem: 14

Suppose U is a 3-dimensional subspace of $\mathbb{R}^8$ and that T is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

Proof. By F.T. of linear maps, we have

$$\dim \operatorname{range} T = 8 - \dim \operatorname{null} T = 8 - 3 = 5$$

This suggests that $\operatorname{range} T = \mathbb{R}^5 \implies T$ is surjective. $\square$

Problem: 15

Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Proof. Suppose there exists such a linear map, then, by F.T. of linear maps, we have

$$\dim \operatorname{range} T = 5 - \dim \operatorname{null} T = 5 - 2 = 2$$

Contradicts! $\square$

Problem: 16

Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite dimensional.

Proof. WLOG, let $\dim \operatorname{null} T = m$ and $\dim \operatorname{range} T = n$. Then by F.T. of linear maps, we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = m + n < \infty$$

$\square$

Problem: 17

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Proof. ($\Longrightarrow$) Suppose there is an injective linear map $T : V \mapsto W$. By F.T. of linear maps, we have

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= 0 + \dim \operatorname{range} T = \dim \operatorname{range} T \end{aligned}$$

($\Longleftarrow$) Suppose $\dim V \leq \dim W$. Then there exists a basis of V that can be extended to a basis of W. Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. Define $T : V \mapsto W$ by $T(v_i) = w_i$. Then T is injective. $\square$

Problem: 19

Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists

$$T \in \mathcal{L}(V, W)$$

such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. ($\Longrightarrow$) Suppose there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$. Since $\operatorname{range} T$ is a subspace of W, we have $\dim Range \leq \dim W$. The rest of the proof follows by the F.T. of linear maps.

($\Longleftarrow$) Suppose $\dim U \geq \dim V - \dim W$, we have

$$\dim U + \dim W \geq \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Of course we could find such a T. $\square$

Problem: 20

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Proof. ($\Longrightarrow$) T is injective $\iff$ $\dim \operatorname{null} T = 0$. Then there exists a basis of V that can be extended to a basis of W. Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. Define $S : W \mapsto V$ by $S(w_i) = v_i$ and $T : V \mapsto W$ by $T(v_k) = w_k$ Then

$$\begin{aligned} ST(v) &= ST(a_1 v_1 + \cdots + a_n v_n) \\ &= S(a_1 w_1 + \cdots + a_n w_n) \\ &= a_1 v_1 + \cdots + a_n v_n \\ &= v \end{aligned}$$

($\Longleftarrow$) Since v_k is a basis, $\operatorname{null} T = \{0\}$. This suggests that T is injective. □

C: Matrix

Problem: 1

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Proof. Follows from the F.T. of linear maps, we have $\dim \operatorname{range} T = \dim W - \dim \operatorname{null} T$. Since $\operatorname{null} T$ becomes all the zero entries, therefore the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries. □

Problem: 2

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p\prime$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Proof. Easy to verify that $\frac{1}{3}x^3, \frac{1}{2}x^2, x, 1$ is a list of basis of $\mathcal{P}_3(\mathbb{R})$, and its derivative is $x^2, x, 1$ which is a basis of $\mathcal{P}_2(\mathbb{R})$. □

Problem: 3

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are zero except for the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

Proof. Let $v_1, \dots, v_n$ be a basis of V such that $\forall i \in 1, \dots, k, Tv_i = 1$, where $k = \dim \operatorname{range} T$. Of course, it is a basis of $\operatorname{range} T$. Expressing this as a matrix gives the desired result. $\square$

Problem: 4

Suppose $v_1, \dots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \dots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$) are 0 except for possibly a 1 in the first column.

Proof. Let $v_1, \dots, v_m$ be the trivial basis of V. Then $Tv_1, \dots, Tv_m$ spans $\operatorname{range} T$. After finite steps of procedure, we can obtain a list, $Tv_1, \dots, Tv_m$ which is a basis of W, say $w_1, \dots, w_m$. Then the first column of $\mathcal{M}(T)$ is the desired result. $\square$

Problem: 5

Suppose $w_1, \dots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \dots, v_m$ of V such that all entries in the first row of $\mathcal{M}(T)$ (with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$) are 0 except for possibly a 1 in the first row.

Proof. We could always find a basis of V such that $\exists i \in \{1, \dots, n\}, v_i = (1, \dots, 0)$ For list $v_1, \dots, v_m$, if $m \leq n$, we obtain the desired result. Otherwise, we could let the $v_{m+1}, \dots, v_n$ be the basis of $\operatorname{null} T$. $\square$

Problem: 6

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

Proof. ($\implies$) Suppose $\dim \operatorname{range} T = 1$, then for $v_1, \dots, v_n$, a basis of V, $Tv_1, \dots, Tv_n \in \operatorname{range} T$ suggests that they are linearly dependent to each other. Hence, we could obtain the desire by letting $Tv_1, \dots, Tv_n = (1, \dots, 1)$

($\impliedby$) Suppose there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1. Then, $\dim \operatorname{range} T = 1$. $\square$

Problem: 10

Suppose A is an m-by-n matrix and C is an n by p matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C.

Proof.

$$(AC)_{j,i} = \sum_{k=1}^{n} A_{j,k}C_{k,i} = (A_{j,\cdot}C)_i$$

□

D: Invertibility and Isomorphic Vector spaces

Problem: 1

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that ST is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. The proof for the invertibility is trivial, since U, V, W are bijective to each other. Then we only have to prove the equation. It follows from the fact that $STT^{-1}S^{-1} = I = T^{-1}S^{-1}ST$ $\square$

Problem: 2

Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of non-invertible operators on V is not a subspace of $\mathcal{L}(V)$.

Proof. Suppose T, S are non-invertible operators on V. Then $\operatorname{null} T \neq \{0\}$ and $\operatorname{null} S \neq \{0\}$. Then $\operatorname{null} T + \operatorname{null} S \neq \{0\}$, which suggests that $T + S$ is not invertible. $\square$

Problem: 3

Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for ever $u \in U$ if and only if S is injective.

Proof. ($\Longrightarrow$) This is obvious since T is bijective.

($\Longleftarrow$) Suppose S is injective. Then we could extend $u_1, \ldots, u_n$ to a basis of V. Then we could define T by $Tu_i = Su_i$ for $i = 1, \ldots, n$. $\square$

Problem: 4

Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$

Proof. ($\Longrightarrow$) Suppose $\operatorname{null} T_1 = \operatorname{null} T_2$, this implies that $\operatorname{range} T_1 = \operatorname{range} T_2$. Let $w_1, \ldots w_m$ be a basis of $\operatorname{range} T_1$ and $\operatorname{range} T_2$. Then we could define S by $Sw_i = T_1 v_i$ for $i = 1, \ldots, m$.

($\Longleftarrow$) Suppose there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$. Since S is invertible, this suggests that it has to be injective such that $\operatorname{null} S = \{0\}$. Then $\operatorname{null} T_1 = \operatorname{null} T_2$. $\square$

Problem: 5

Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$

Proof. ($\implies$) Suppose $\operatorname{range} T_1 = \operatorname{range} T_2$. This suggests that there exists a basis $w_1, \ldots, w_n$of $\operatorname{range} T_1$ and $\operatorname{range} T_2$. For such a basis, we could always find a corresponding list $v_1, \ldots, v_m$ with which T_2 maps onto $w_1, \ldots, w_n$we could define S by $Su_i = v_i$ for $i = 1, \ldots, m$.

($\impliedby$) Suppose there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$. Since S is bijective, this implies that $\operatorname{range} T_1 = \operatorname{range} T_2$. □

Problem: 6

Suppose V and W are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that there exist invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = ST_2R$ if and only if $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

Proof. The same with question 4. □

Problem: 7

Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E$

(a) *Proof.* $0(v) \in E$ therefore E is not empty, and $T_1 v = T_2 v = 0$ implies $(T_1 + T_2)v = 0$. $Tv = 0$ implies $\lambda Tv = 0$ for all $\lambda \in \mathbb{F}$. Hence, E is a subspace of $\mathcal{L}(V, W)$. □

(b) *Proof.* Suppose $v \neq 0$. Then there is a basis $v_1, \ldots, v_n$ of V extended from v, and we can choose a basis $w_1, \ldots, w_m$ of W. Since $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{m,n}$ and $Tv = 0$ implies that the first column of $\mathcal{M}(T)$ is zero. Hence, $\dim E = m(n-1)$. □

Problem: 8

Suppose V is finite-dimensional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Proof. Since T is surjective, any $w \in W$ we could find a $v \in V$ such that $Tv = w$. Define $x_1 \sim x_2 :\iff T(x_1) = T(x_2)$. We could define $U = \{[v] : \forall v \in \text{V}\}$. As such $T|_U$ is bijective, henceforth an isomorphism. □

Problem: 9

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof. ($\implies$) Suppose ST is invertible. Then $(ST)^{-1} = T^{-1}S^{-1}$, this suggests that S and T are invertible.

($\impliedby$) Suppose S and T are invertible. Then $S^{-1}T^{-1} = (ST)^{-1}$, this suggests that ST is invertible. □

Problem: 10

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$

Proof. Directly from the definition of inverse. □

Problem: 11

Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Proof. The associativity implies that both S and U are invertible. Then $T = S^{-1}IU^{-1} = S^{-1}U^{-1}$. Hence, $T^{-1} = US$. □

Problem: 12

Show that the result in the previous exercise can fail without the hypothesis that V is finite-dimensional.

Proof. TODO: 3.D.12 Pending. □

Problem: 13

Suppose V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove that S is injective.

Proof. Since RST is surjective, this suggests that R is surjective. Then R is injective, hence S is injective. $\square$

Problem: 14

Suppose $v_1, \ldots, v_n$ is a basis of V. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(V)$$

is an isomorphism of V onto $\mathbb{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_n$.

Proof. T is injective: $\forall v_i \in V$, $Tv = 0 \iff v = 0 \iff v = (0, \ldots, 0)$
T is surjective: $\forall (w_1, \ldots, w_n) \in \mathbb{F}^{n,1}$, $T(a_1v_1 + \ldots + a_nv_n) = (a_1, \ldots, a_n)$ since $v_1, \ldots, v_n$ is a basis of V.
Henceforth, T is an isomorphism. $\square$

Problem: 15

Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an m-by-n matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$

Proof. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{F}^{n,1}$ and $w_1, \ldots, w_m$ be the standard basis of $\mathbb{F}^{m,1}$. Then $T(e_i) = a_{1i}w_1 + \ldots + a_{mi}w_m$. Then $T(x) = Ax$. $\square$

Problem: 17

Suppose V is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$, $TS \in \mathcal{L}(V)$for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$

Proof. $\square$

Problem: 17 variant

Suppose V is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$

Proof. Suppose $\mathcal{E} \neq \mathcal{L}(V)$. The supposition suggests that there exists some linear mapping in $\mathcal{L}(V)$ which is not in $\mathcal{E}$. We denote it by φ. Now, note that $\varphi \in \mathcal{L}(V)$, we have $\varphi T \in \mathcal{E}$

Suppose $\mathcal{E} \neq \{0\}$, we will show that $\mathcal{E} = \mathcal{L}(V)$. Suppose $\mathcal{E}$ contains the identity map, then the prove is completed, so we assume identity map is not in $\mathcal{E}$. $\square$

Problem: 18

Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

Proof. $\dim \mathcal{L}(\mathbb{F}, V) = \dim \mathbb{F} * \dim V = 1 * \dim V = \dim V$ $\square$

Problem: 19

Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

(a) Prove that T is surjective.

(b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$

(a) *Proof.* $\deg Tp = \deg p$ implies that $\dim T = \dim \mathcal{P}(\mathbb{R})$. Hence, T is surjective. □

(b) *Proof.* We prove it by induction: $\deg Tp = \deg p$ for $\dim p = 1$ (T is injective). Suppose it holds for $\dim p = n$, then, due to T is surjective, every $p \in \mathcal{P}(\mathbb{R})$ can be expressed as $T(p)$ uniquely. For p = n + 1, suppose that $\deg Tp < \deg p$, then this implies that T is not injective, which is a contradiction. Hence, $\deg Tp = \deg p$ for all $n \in \mathbb{N}$. □

E: Products and Quotients of Vector Spaces

Problem: 1

Suppose T is a function from V to W. The **graph** of T is the subset of $V \times W$ defined by

$$\operatorname{graph} T = \{(v, Tv) : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Proof. Since T is a well-defined function, the graph of T is not empty. Then, due to the properties of linear maps and vector space, T is also a subspace of $V \times W$. Hence, the graph of T is a subspace of $V \times W$. □

Problem: 2

Suppose $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that each V_i is finite-dimensional.

Proof. Suppose $\exists \dim V_i = \infty (i \in \{1, \ldots, m\})$. Then it follows that $\dim(V_1 \times \cdots \times V_m) = \sum \dim V_i = \infty$. This is a contradiction. Hence, each V_i is finite-dimensional. □

Problem: 3

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Proof. TODO: 3.E.3 Give an example Pending. $\square$

Problem: 4

Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic.

Proof. Clearly, for all $T(v_1, \ldots, v_m) = w(w \in W)$, $T \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$. Then we could define a map $S : \mathcal{L}(V_1 \times \cdots \times V_m, W) \to \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ by $S(T) = (T_1, \ldots, T_m)$ where $T_i(v_i) = T(0, \ldots, v_i, \ldots, 0)$. We will verify that S is an isomorphism.

First we show that S is injective: Suppose $S(T) = 0$, then $T_i = 0$ for all $i \in \{1, \ldots, m\}$. Then $T = 0$.

Then we show that S is surjective: This is obvious since for every $(T_1, \ldots, T_m) \in \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$, we could definGe $T(v_1, \ldots, v_m) = (T_1 v_1, \ldots, T_m v_m)$, where $S(T) = (T_1, \ldots, T_m)$.

Hence, S is an isomorphism and $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ is isomorphic to $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$. $\square$

Problem: 5

Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ is isomorphic to $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$.

Proof. Define $S : \mathcal{L}(V, W_1 \times \cdots \times W_m) \mapsto \mathcal{L}(V, W_m)$ by $S(T) = (T_1, \ldots, T_m)$ where $T_i(v) = (0, \ldots, T(v), \ldots, 0)$. We will verify that S is an isomorphism by constructing an inverse function of S: define $S^{-1}(T_1(v), \ldots, T_m(v)) = T_1(v) + \cdots + T_m(v)$. $S \circ S^{-1} = \mathrm{id}(T)$ $\square$

Problem: 6

For n a positive integer, define V_n by

$$V^n = \underbrace{V \times \cdots \times V}_{n}$$

Prove that V^n and $\mathcal{L}(\mathbb{F}^n, V)$ are isomorphic vector spaces.

Proof. Define $S(v_1, \ldots, v_n) = T(\lambda_1, \ldots, \lambda_n)$ where $T \in \mathcal{L}(\mathbb{F}^n, V)$, we will show that S is an isomorphism by constructing an inverse function of S. Let $S^{-1}(T(\lambda_1, \ldots, \lambda_n)) = (T_1(\lambda), \ldots, T_n(\lambda))$ where $T_i(\lambda) = (0, \ldots, T(\lambda), \ldots, 0)$. $S \circ S^{-1} = \mathrm{id}$. Hence, They are isomorphic. $\square$

Problem: 7

Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$

Proof. Rearrange the equation,

$$v + U = x + W$$
$$v - x + U = W$$

This implies that $U = W$. $\square$

Problem: 8

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$

Proof. ($\implies$) For all affine subset in the form $t + S$, every element can be expressed as $t + s$. Then for all $t + s_1, t + s_2 \in t + S$, $\lambda(t + s_1) + (1 - \lambda)(t + s_2) = t + \lambda s_1 + (1 - \lambda)s_2 \in t + S$ for all $t, s_1, s_2 \in t + S$ and all $\lambda \in \mathbb{F}$ since S is a subspace.

($\impliedby$) If $\lambda v + (1 - \lambda)w \in A$ holds for all $v, w \in A$ and all $\lambda \in \mathbb{F}$. Rearrange the formula we get $\lambda v + (1 - \lambda)w = v + (1 - \lambda)(v + w)$. This implies that A is an affine subset. $\square$

F: Duality

Problem: 1

Explain why every linear functional is either surjective or the zero map.

Proof. Suppose f is not surjective, the $\dim \operatorname{range} f < \dim \mathbb{F}$ which could only be 0. Therefore, f is the zero map.

Suppose f is not the zero map, then $\dim \operatorname{range} f = \dim \mathbb{F}$, which implies that f is surjective. $\square$

Problem: 2

Give three distinct examples of linear functional on $\mathbb{R}^{[0,1]}$.

Proof. TODO: 3.F.2 Give three distinct examples Pending. $\square$

Problem: 3

Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Proof. Since $v \neq 0$, we could extend v to a basis $v_1, \ldots, v_n$ of V. Define $\varphi(v) = 1$ and $\varphi(v_i) = 0$ for $i \neq 1$. Then φ is a linear functional. $\square$

Problem: 4

Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Proof. Let $u_1, \ldots, u_m$ be a basis of U and extend to $u_1, \ldots, u_m, v_1, \ldots, v_n$ a basis of V. Define $\varphi(u_i) = 0$ for $i = 1, \ldots, m$ and $\varphi(v_i) = 1$ for $i = 1, \ldots, n$. Then φ is a linear functional. $\square$

Problem: 5

Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

Proof. Define $S(f) = (f_1, \ldots, f_m)$ where $f_i(v_1, \ldots, v_m) = (0, \ldots, f_i, \ldots, 0)$. And let $S^{-1}(f_1, \ldots, f_m) = f_1 + \cdots + f_m$. Then S is an isomorphism. $\square$

Problem: 6

Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma : V' \mapsto \mathbb{F}^m$ by

$$\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)).$$

(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.

(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

(a) *Proof.* ($\Longrightarrow$) Suppose $v_1, \ldots, v_m$ spans V, then for all $\varphi \in V'$, $\Gamma(\varphi) = 0$ implies that $\varphi(v_1) = \cdots = \varphi(v_m) = 0$, that is, $\varphi = 0$. Hence, Γ is injective.

($\Longleftarrow$) Suppose Γ is injective, then $\varphi(v_1) = \cdots = \varphi(v_m) = 0$ implies that $\varphi = 0$. Hence, $v_1, \ldots, v_m$ is surjective, that is, spanning V. $\square$

(b) *Proof.* ($\Longrightarrow$) Suppose $v_1, \ldots, v_m$ is linearly independent. Then for all $(\lambda_1, \ldots, \lambda_m) \in \mathbb{F}^m$, $\lambda_1\varphi(v_1) + \cdots + \lambda_m\varphi(v_m) = 0$ implies that $\lambda_1 = \cdots = \lambda_m = 0$ or $\forall v_i = 0$. Therefore, for each $\varphi(v_i)$, φ spans $\mathbb{F}$. Hence, Γ is surjective.

($\Longleftarrow$) Suppose Γ is surjective, then for all v_i, $\varphi \neq 0$. That is, $v_1, \ldots, v_m$ is linearly independent. Otherwise, suppose $v_1, \ldots, v_m$ is linearly dependent. Let v_k be the one can be expressed by all other v's. Then $\dim \Gamma(\varphi) < m = \dim \mathbb{F}^m$, suggesting that Γ is not surjective, contradiction! Therefore, $v_1, \ldots, v_m$ is linearly independent. $\square$

Problem: 7

Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \ldots, x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

Proof. Since $1, x, \ldots, x^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$, let $p = a_1 + a_2 x + \cdots + a_m x_m$. Note that $\frac{p^j(0)}{j!} = a_j$, then $\varphi_j(p) = a_j$ if and only if $i = j$. Hence, $\varphi_0, \varphi_1, \ldots, \varphi_m$ is the dual basis of $1, x, \ldots, x^m$. $\square$

Problem: 8

Suppose m is a positive integer.

(a) Show that $1, x-5, \ldots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$.

(b) What is the dual basis of the basis in part(a)?

(a) *Proof.* This is obvious since we can generate an upper-triangle matrix with $1, x-5, \ldots, (x-5)^m$ $\square$

(b) *Proof.* The dual basis is $\varphi_0, \varphi_1, \ldots, \varphi_m$ where $\varphi_j(p) = \frac{p^{(j)}(5)}{j!}$ $\square$

Problem: 9

Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$$

Proof. Notice that

$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_1) = \psi(v_1)$$
$$\vdots$$
$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_n) = \psi(v_n)$$

Therefore, $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$ as they coincide at a basis of V. $\square$

Problem: 11

Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbb{F}^m$ and $(d_1, \dots d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$

Proof. ($\implies$) Suppose the rank of A is 1. Then there exists a $\square$

Problem: 12

Show that the dual map of the identity map on V is the identity map on V'.

Proof. Let $\mathrm{id}_V : V \mapsto V$ be the identity map on V. Then the dual map of id_V is the linear map $\mathrm{id}'_V(\varphi) := \varphi \circ \mathrm{id}_V = \varphi$ where $\varphi \in \mathcal{L}(V, \mathbb{F})$. Hence, the dual map of the identity map on V is the identity map on V'. $\square$

Problem: 15

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Proof. ($\implies$) Suppose $T' = 0$. By definition, it suggests that $T'(\varphi) = \varphi(T(v)) = 0$ for all $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $v \in V$. Therefore, it implies that $T = 0$.

($\impliedby$) Suppose $T = 0$. By definition and with the fact that all linear maps map 0 to 0, it suggests that $T' = 0$. $\square$

Problem: 16

Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism.

Proof. Since $\dim \mathcal{L}(V, W) = \dim(W', V')$, the identity map is bijective. Hence, the map is an isomorphism. $\square$

Problem: 17

Suppose $U \subset V$. Explain why $U^0 = \left\{\varphi \in V' : U \subset \operatorname{null} \varphi\right\}$

Proof. By definition, of course that the zero map 0 is in null space. $\square$

Problem: 18

Suppose V is finite-dimensional and $U \subset V$. Show that $U = \{0\}$ if and only if $U^0 = V'$.

Proof. ($\implies$) Suppose $U = \{0\}$. Then it suggests that $\forall \varphi \in \mathcal{L}(U, \mathbb{F}), \varphi(u) = 0$. That is, $U^0 = V'$.

($\impliedby$) Directly came from the definition. ☐

Problem: 19

Suppose V is finite-dimensional and U is a subspace of V. Show that $U = V$ if and only if $U^0 = \{0\}$

Proof. ($\implies$) Suppose $U = V$. Then $U^0 = V^0 := \mathcal{L}(V, \mathbb{F})$. For a vector space, this could only happen when $V = \{0\}$

($\impliedby$) $U^0 = \{0\}$ suggests that $\dim U = \dim V$. That is, $U = V$ ☐

Problem: 20

Suppose U and W are subsets of V with $U \subset W$. Prove that $W^0 \subset U^0$.

Proof. Since $U \subset W$, this suggests that $\forall \varphi \in W^0, \varphi(u) = 0$ for all $u \in U$. Hence, $W^0 \subset U^0$. ☐

Problem: 21

Suppose V is finite-dimensional and U and W are subspaces of V with $W^0 \subset U^0$. Prove that $U \subset W$.

Proof. Following from $W^0 \subset U^0$, it suggests that $\dim U > \dim W$. Henceforth, $U \subset W$. ☐

Problem: 22

Suppose U, W are subspaces of V. Show that $(U + W)^0 = U^0 \cap W^0$

Proof. Let $u_1, \ldots, u_m$ be a basis of U and $u_1, \ldots, u_m, w_1, \ldots, w_n$ be a basis of W. Then ☐

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