Chapter 3: Linear maps

Linear Algebra Done Right, by Sheldon Axler

Contents

A: The vector space of linear maps

Problem S

uppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is a linear map if and only if $b = 0$ and $c = 0$.

Proof. Consider $T(1, 1, 1) = T(1, 0, 0) + T(0, 1, 1)$

$$T(1, 1, 1) = (1 + b, 6 + c) \tag{1}$$
$$T(1, 0, 0) = (2 + b, 6) \tag{2}$$
$$T(0, 1, 1) = (-1 + b, 0) \tag{3}$$

Therefore, $b = 0, c = 0$ □

B: Null space and Range

Problem 1

Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Proof. Consider $T : \mathcal{P}(4) \mapsto \mathcal{P}(1)$, $T(az^4 + bz^3 + cz^2 + dz + e) = (dz + e)$. Then $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$. □

Problem 3

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V.

(b) What property of T corresponds to $v_1, \ldots, v_m$ being linearly independent.

Proof. (a) $\forall v \in V$, we have $T(z_1, \ldots, z_m) = v$, which means $v_1, \ldots, v_m$ span V. This suggests that $\operatorname{range} T$ is equal to V. Hence, T is surjective.

(b) If $v_1, \ldots, v_m$ are linearly independent, then $T(z_1, \ldots, z_m) = 0$ implies $z_1 = \cdots = z_m = 0$. This suggests that $\operatorname{null} T = \{0\}$, hence T is injective. □

Problem 4

Show that
$$T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2$$
is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$

Proof. Suppose $\dim \operatorname{null} T > 2$, by F.T. of linear maps,

$$\begin{aligned} \dim T = 5 &= \dim \operatorname{null} T + \dim \operatorname{range} T & (4) \\ &> 2 + \dim \operatorname{range} T & (5) \\ \dim \operatorname{range} T &< 3 & (6) \end{aligned}$$

Hence, $T \notin \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$, and thus not a subspace of $\mathcal{L}(R^5, R^4)$. □

Problem 5

Give an example of a linear map $T : \mathbb{R}^4 \mapsto \mathbb{R}^4$ such that
$$\operatorname{range} T = \operatorname{null} T.$$

Proof. Consider $T : R^4 \mapsto R^4$, $T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0)$. Then $\operatorname{range} T = \operatorname{null} T$. □

Problem 6

Prove that there does not exist a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\operatorname{range} T = \operatorname{null} T$.

Proof. Suppose there exists a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\operatorname{range} T = \operatorname{null} T$. Then by F.T. of linear maps, we have $\dim \operatorname{range} T = \dim \operatorname{null} T$, which implies $\dim \operatorname{range} T = 5 - \dim \operatorname{range} T$, or $\dim \operatorname{range} T = 2.5$, which is not an integer. Hence, such a linear map does not exist. □

Problem 7

Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective } \}$ is not a subspace of $\mathcal{L}(V, W)$.

Proof. T is not injective suggests that $\dim \operatorname{null} V > 0$. Then By the F.T. of linear maps, we have

$$\begin{aligned} \dim V &= \dim \operatorname{null} V + \dim W & (7) \\ &\geq 1 + \dim W \geq 1 + \dim V & (8) \end{aligned}$$

Contradicts! Hence, $T \notin \mathcal{L}(V, W) \implies$ not a subspace. □

Problem 9

Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Proof. T is injective $\implies$ null $T = \{0\}$. Therefore, we have

$$T(\lambda_1 v_1 + \cdots + \lambda_n v_n) \iff \lambda_j = 0$$

where $j \in \{1, \ldots, n\}$. Since $v_1, \ldots, v_n$ are linearly independent. It follows that $Tv_1, \ldots, Tv_n$ are linearly independent. □

Problem 10

Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \ldots, Tv_n$ spans range T.

Proof. Suppose $v_1, \ldots, v_n$ spans V. Then $\forall v \in V$, we have $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then

$$T(v) = T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 Tv_1 + \cdots + \lambda_n Tv_n.$$

This suggests that range$(T) \subset$ span$(Tv_1, \ldots, Tv_n)$. Also, span$(Tv_1, \ldots, Tv_n)$ is the smallest containing subspace of W implying that it is a subset of range W, hence range $T =$ span$(Tv_1, \ldots, Tv_n)$. □

Problem 11

Suppose $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \ldots S_n$ makes sense. Prove that $S_1 S_2 \ldots S_n$ is injective.

Proof. By F.T. of linear maps, we have

$$\dim S_1 = \dim \operatorname{null} S_1 + \dim \operatorname{range} S_1 \tag{9}$$
$$= 0 + \dim \operatorname{range} S_1 = \dim \operatorname{range} S_1 \tag{10}$$

It follows that

$$\dim S_1 = \dim \operatorname{range} S_1 = \dim S_2 = \cdots = \dim S_n = \dim \operatorname{range} S_n$$

Hence, $\dim \operatorname{null} S_1 S_2 \ldots S_n = 0 \iff S_1 S_2 \ldots S_n$ is injective. □

Problem 12

Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that null $T = U$ and $U \cap \operatorname{range} T = \{0\}$.

Proof. Let $u_1, \ldots, u_n$ be a basis for null T. Then span$(u_1, \ldots, u_n)$ is a subspace of V. Since it is linear independent, it can be extended to a basis of V, say $u_1, \ldots, u_n, v_1, \ldots, v_m$. Then V is the direct sum of the spanning of $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$. Take $U =$ span$(v_1, \ldots, v_m)$ □

Problem 13

Suppose T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is surjective.

Proof. A basis of $\operatorname{null} T$ is

$$\{(5, 1, 0, 0), (0, 0, 7, 1)\}$$

Then by F.T. of linear maps, we have

$$\dim \operatorname{range} T = 4 - \dim \operatorname{null} T = 4 - 2 = 2$$

Therefore, $\operatorname{range} T = \mathbb{R}^2 \implies T$ is surjective. $\square$

Problem 14

Suppose U is a 3-dimensional subspace of $\mathbb{R}^8$ and that T is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

Proof. By F.T. of linear maps, we have

$$\dim \operatorname{range} T = 8 - \dim \operatorname{null} T = 8 - 3 = 5$$

This suggests that $\operatorname{range} T = \mathbb{R}^5 \implies T$ is surjective. $\square$

Problem 15

Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Proof. Suppose there exists such a linear map, then, by F.T. of linear maps, we have

$$\dim \operatorname{range} T = 5 - \dim \operatorname{null} T = 5 - 2 = 2$$

Contradicts! $\square$

Problem 16

Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite dimensional.

Proof. WLOG, let $\dim \operatorname{null} T = m$ and $\dim \operatorname{range} T = n$. Then by F.T. of linear maps, we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = m + n < \infty$$

$\square$

Problem 17

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Proof. ($\Longrightarrow$) Suppose there is an injective linear map $T : V \mapsto W$. By F.T. of linear maps, we have

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T && (11) \\ &= 0 + \dim \operatorname{range} T = \dim \operatorname{range} T && (12) \end{aligned}$$

($\Longleftarrow$) Suppose $\dim V \leq \dim W$. Then there exists a basis of V that can be extended to a basis of W. Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. Define $T : V \mapsto W$ by $T(v_i) = w_i$. Then T is injective. ☐

Problem 19

Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists

$$T \in \mathcal{L}(V, W)$$

such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. ($\Longrightarrow$) Suppose there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$. Since $\operatorname{range} T$ is a subspace of W, we have $\dim \mathit{Range} \leq \dim W$. The rest of the proof follows by the F.T. of linear maps.

($\Longleftarrow$) Suppose $\dim U \geq \dim V - \dim W$, we have

$$\dim U + \dim W \geq \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \qquad (13)$$

Of course we could find such a T. ☐

Problem 20

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Proof. ($\Longrightarrow$) T is injective $\iff$ $\dim \operatorname{null} T = 0$. Then there exists a basis of V that can be extended to a basis of W. Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. Define $S : W \mapsto V$ by $S(w_i) = v_i$ and $T : V \mapsto W$ by $T(v_k) = w_k$ Then

$$\begin{aligned} ST(v) &= ST(a_1 v_1 + \cdots + a_n v_n) && (14) \\ &= S(a_1 w_1 + \cdots + a_n w_n) && (15) \\ &= a_1 v_1 + \cdots + a_n v_n && (16) \\ &= v && (17) \end{aligned}$$

($\Longleftarrow$) Since v_k is a basis, $\operatorname{null} T = \{0\}$. This suggests that T is injective. ☐

C: Matrix

Problem 1

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Proof. Follows from the F.T. of linear maps, we have $\dim \operatorname{range} T = \dim W - \dim \operatorname{null} T$. Since $\operatorname{null} T$ becomes all the zero entries, therefore the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries. □

Problem 2

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p\prime$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Proof. Easy to verify that $\frac{1}{3}x^3, \frac{1}{2}x^2, x, 1$ is a list of basis of $\mathcal{P}_3(\mathbb{R})$, and its derivative is $x^2, x, 1$ which is a basis of $\mathcal{P}_2(\mathbb{R})$. □

Problem 3

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are zero except for the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

Proof. Let $v_1, \ldots, v_n$ be a basis of V such that $\forall i \in 1, \ldots, k, Tv_i = 1$, where $k = \dim \operatorname{range} T$. Of course, it is a basis of $\operatorname{range} T$. Expressing this as a matrix gives the desired result. □

Problem 4

Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first column.

Proof. □