Chapter 5: Eigenvalues, Eigenvectors, and invariant subspaces

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Contents

A: Invariant Subspaces

Problem 1

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subset \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subset U$, then U is invariant under T.

Proof. (a) Let $u \in U$. Then $T(u) \in \operatorname{null} T$ because $u \in \operatorname{null} T$. Therefore, $T(u) \in U$.

(b) Let $u \in U$. Then $T(u) \in U$ because $T(u) \in \operatorname{range} T \subset U$.

□

Problem 2

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null} S$ is invariant under T.

Proof. We have $ST(\operatorname{null} S) = T(S(\operatorname{null} S)) = T(0) = 0$. This implies that $T(\operatorname{null} S) \subset \operatorname{null} S$. □

Problem 3

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{range} S$ is invariant under T.

Proof. Pending. □

Problem 4

Suppose that $T \in \mathcal{L}(V)$ and $U_1, \dots U_m$ are subspaces of V invariant under T. Prove that $U_1 + \cdots + U_m$ is invariant under T.

Proof.

$$\begin{aligned} T(U_1 + \cdots + U_m) &= T(U_1) + \cdots + T(U_m) \\ T(U_1) \subset U_1 &\subset U_1 + \cdots + U_m \\ &\cdots \\ T(U_m) \subset U_m &\subset U_1 + \cdots + U_m \end{aligned}$$

Since $U_1 + \cdots + U_m$ is still a subspace, $T(U_1 + \cdots + U_m) \subset U_1 + \cdots + U_m$. □

Problem 5

Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V that are invariant under T is invariant under T.

Proof. Every invariant subspace contains $\{0\}$, and also which is the smallest one. Hence, the intersection is $\{0\}$ and is trivially invariant under T. $\square$

Problem 6

Prove or give a counterexample: if V is a finite-dimensional vector space and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Proof. Because every operator on V leaves $\{0\}$ invariant, the question turns to prove the existence of an operator under which only $\{0\}$ and V is invariant.

The case where $\dim V \leq 1$ is trivial. Suppose $\dim V \geq 2$. We could always construct such an operator U. Suppose there exists an invariant subspace U_1 of V under an operator T that is neither $\{0\}$ nor V for which $\dim U_1 < \dim V$. Define T which rotates U_1 to W, where $W \oplus U_1 = V$. $\forall u \in U_1, T(u)$ have some component in W, which is not in U_1. Thus, U is not invariant under U, which is a contradiction. Therefore, $U = \{0\}$ or $U = V$. $\square$

Problem 8

Define $T in \mathcal{L}(\mathbb{F}^2)$ by

$$T(w, z) = (z, w).$$

Find all eigenvalues and eigenvectors of T.