

Chapter 3: Linear maps

Linear Algebra Done Right, by Sheldon Axler

Contents

A: The vector space of linear maps	3
B: Null space and Range	3
C: Matrix	8
D: Invertibility and Isomorphic Vector spaces	11
E: Products and Quotients of Vector Spaces	16
F: Duality	18

A: The vector space of linear maps

Problem: 1

Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is a linear map if and only if $b = 0$ and $c = 0$.

Proof. Consider $T(1, 1, 1) = T(1, 0, 0) + T(0, 1, 1)$

$$T(1, 1, 1) = (1 + b, 6 + c)$$

$$T(1, 0, 0) = (2 + b, 6)$$

$$T(0, 1, 1) = (-1 + b, 0)$$

Therefore, $b = 0, c = 0$

□

B: Null space and Range

Problem: 1

Give an example of a linear map T such that $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$.

Proof. Consider $T : \mathcal{P}(4) \mapsto \mathcal{P}(1)$, $T(az^4 + bz^3 + cz^2 + dz + e) = (dz + e)$. Then $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$. □

Problem: 3

Suppose v_1, \dots, v_m is a list of vectors in V . Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m.$$

- (a) What property of T corresponds to v_1, \dots, v_m spanning V .
- (b) What property of T corresponds to v_1, \dots, v_m being linearly independent.

(a) *Proof.* $\forall v \in V$, we have $T(z_1, \dots, z_m) = v$, which means v_1, \dots, v_m span V . This suggests that $\text{range } T$ is equal to V . Hence, T is surjective. □

(b) *Proof.* If v_1, \dots, v_m are linearly independent, then $T(z_1, \dots, z_m) = 0$ implies $z_1 = \dots = z_m = 0$. This suggests that $\text{null } T = \{0\}$, hence T is injective. □

Problem: 4

Show that

$$T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null } T > 2$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$

Proof. Suppose $\dim \text{null } T > 2$, by F.T. of linear maps,

$$\begin{aligned} \dim T = 5 &= \dim \text{null } T + \dim \text{range } T \\ &> 2 + \dim \text{range } T \\ \dim \text{range } T &< 3 \end{aligned}$$

Hence, $T \notin \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$, and thus not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. \square

Problem: 5

Give an example of a linear map $T : \mathbb{R}^4 \mapsto \mathbb{R}^4$ such that

$$\text{range } T = \text{null } T.$$

Proof. Consider $T : \mathbb{R}^4 \mapsto \mathbb{R}^4$, $T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0)$. Then $\text{range } T = \text{null } T$. \square

Problem: 6

Prove that there does not exist a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\text{range } T = \text{null } T$.

Proof. Suppose there exists a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\text{range } T = \text{null } T$. Then by F.T. of linear maps, we have $\dim \text{range } T = \dim \text{null } T$, which implies $\dim \text{range } T = 5 - \dim \text{range } T$, or $\dim \text{range } T = 2.5$, which is not an integer. Hence, such a linear map does not exist. \square

Problem: 7

Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Proof. T is not injective suggests that $\dim \text{null } V > 0$. Then By the F.T. of linear maps, we have

$$\begin{aligned} \dim V &= \dim \text{null } V + \dim \text{range } T \\ &\geq 1 + \dim W \geq 1 + \dim V \end{aligned}$$

Contradicts! Hence, $T \notin \mathcal{L}(V, W) \implies$ not a subspace. \square

Problem: 9

Suppose $T \in \mathcal{L}(V, W)$ is injective and v_1, \dots, v_n is linearly independent in V . Prove that Tv_1, \dots, Tv_n is linearly independent in W .

Proof. T is injective $\implies \text{null } T = \{0\}$. Therefore, we have

$$T(\lambda_1 v_1 + \dots + \lambda_n v_n) \iff \lambda_j = 0$$

where $j \in \{1, \dots, n\}$. Since v_1, \dots, v_n are linearly independent. It follows that Tv_1, \dots, Tv_n are linearly independent. \square

Problem: 10

Suppose v_1, \dots, v_n spans V and $T \in \mathcal{L}(V, W)$. Prove that the list Tv_1, \dots, Tv_n spans $\text{range } T$.

Proof. Suppose v_1, \dots, v_n spans V . Then $\forall v \in V$, we have $v = \lambda_1 v_1 + \dots + \lambda_n v_n$. Then

$$T(v) = T(\lambda_1 v_1 + \dots + \lambda_n v_n) = \lambda_1 T v_1 + \dots + \lambda_n T v_n.$$

This suggests that $\text{range}(T) \subset \text{span}(Tv_1, \dots, Tv_n)$. Also, $\text{span}(Tv_1, \dots, Tv_n)$ is the smallest containing subspace of W implying that it is a subset of $\text{range } T$, hence $\text{range } T = \text{span}(Tv_1, \dots, Tv_n)$. \square

Problem: 11

Suppose S_1, \dots, S_n are injective linear maps such that $S_1 S_2 \dots S_n$ makes sense. Prove that $S_1 S_2 \dots S_n$ is injective.

Proof. By F.T. of linear maps, we have

$$\begin{aligned} \dim S_1 &= \dim \text{null } S_1 + \dim \text{range } S_1 \\ &= 0 + \dim \text{range } S_1 = \dim \text{range } S_1 \end{aligned}$$

It follows that

$$\dim S_1 = \dim \text{range } S_1 = \dim S_2 = \dots = \dim S_n = \dim \text{range } S_n$$

Hence, $\dim \text{null } S_1 S_2 \dots S_n = 0 \iff S_1 S_2 \dots S_n$ is injective. \square

Problem: 12

Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $\text{null } T = U$ and $U \cap \text{range } T = \{0\}$.

Proof. Let u_1, \dots, u_n be a basis for $\text{null } T$. Then $\text{span}(u_1, \dots, u_n)$ is a subspace of V . Since it is linear independent, it can be extended to a basis of V , say $u_1, \dots, u_n, v_1, \dots, v_m$. Then V is the direct sum of the spanning of u_1, \dots, u_n and v_1, \dots, v_m . Take $U = \text{span}(v_1, \dots, v_m)$ \square

Problem: 13

Suppose T is a linear map from \mathbb{F}^4 to \mathbb{F}^2 such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is surjective.

Proof. A basis of $\text{null } T$ is

$$\{(5, 1, 0, 0), (0, 0, 7, 1)\}$$

Then by F.T. of linear maps, we have

$$\dim \text{range } T = 4 - \dim \text{null } T = 4 - 2 = 2$$

Therefore, $\text{range } T = \mathbb{R}^2 \implies T$ is surjective. \square

Problem: 14

Suppose U is a 3-dimensional subspace of \mathbb{R}^8 and that T is a linear map from \mathbb{R}^8 to \mathbb{R}^5 such that $\text{null } T = U$. Prove that T is surjective.

Proof. By F.T. of linear maps, we have

$$\dim \text{range } T = 8 - \dim \text{null } T = 8 - 3 = 5$$

This suggests that $\text{range } T = \mathbb{R}^5 \implies T$ is surjective. \square

Problem: 15

Prove that there does not exist a linear map from \mathbb{F}^5 to \mathbb{F}^2 whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Proof. Suppose there exists such a linear map, then, by F.T. of linear maps, we have

$$\dim \text{range } T = 5 - \dim \text{null } T = 5 - 2 = 2$$

Contradicts! \square

Problem: 16

Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite dimensional.

Proof. WLOG, let $\dim \text{null } T = m$ and $\dim \text{range } T = n$. Then by F.T. of linear maps, we have

$$\dim V = \dim \text{null } T + \dim \text{range } T = m + n < \infty$$

□

Problem: 17

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Proof. (\implies) Suppose there is an injective linear map $T : V \mapsto W$. By F.T. of linear maps, we have

$$\begin{aligned} \dim V &= \dim \text{null } T + \dim \text{range } T \\ &= 0 + \dim \text{range } T = \dim \text{range } T \end{aligned}$$

(\impliedby) Suppose $\dim V \leq \dim W$. Then there exists a basis of V that can be extended to a basis of W . Let v_1, \dots, v_n be a basis of V and w_1, \dots, w_n be a basis of W . Define $T : V \mapsto W$ by $T(v_i) = w_i$. Then T is injective. □

Problem: 19

Suppose V and W are finite-dimensional and that U is a subspace of V . Prove that there exists

$$T \in \mathcal{L}(V, W)$$

such that $\text{null } T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. (\implies) Suppose there exists $T \in \mathcal{L}(V, W)$ such that $\text{null } T = U$. Since $\text{range } T$ is a subspace of W , we have $\dim \text{Range } T \leq \dim W$. The rest of the proof follows by the F.T. of linear maps.

(\impliedby) Suppose $\dim U \geq \dim V - \dim W$, we have

$$\dim U + \dim W \geq \dim V = \dim \text{null } T + \dim \text{range } T$$

Of course we could find such a T . □

Problem: 20

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V .

Proof. (\implies) T is injective $\iff \dim \text{null } T = 0$. Then there exists a basis of V that can be extended to a basis of W . Let v_1, \dots, v_n be a basis of V and w_1, \dots, w_n be a basis of W . Define $S : W \mapsto V$ by $S(w_i) = v_i$ and $T : V \mapsto W$ by $T(v_k) = w_k$. Then

$$\begin{aligned} ST(v) &= ST(a_1v_1 + \dots + a_nv_n) \\ &= S(a_1w_1 + \dots + a_nw_n) \\ &= a_1v_1 + \dots + a_nv_n \\ &= v \end{aligned}$$

(\impliedby) Since v_k is a basis, $\text{null } T = \{0\}$. This suggests that T is injective. \square

C: Matrix

Problem: 1

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W , the matrix of T has at least $\dim \text{range } T$ nonzero entries.

Proof. Follows from the F.T. of linear maps, we have $\dim \text{range } T = \dim W - \dim \text{null } T$. Since $\text{null } T$ becomes all the zero entries, therefore the matrix of T has at least $\dim \text{range } T$ nonzero entries. \square

Problem: 2

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Proof. Easy to verify that $\frac{1}{3}x^3, \frac{1}{2}x^2, x, 1$ is a list of basis of $\mathcal{P}_3(\mathbb{R})$, and its derivative is $x^2, x, 1$ which is a basis of $\mathcal{P}_2(\mathbb{R})$. \square

Problem: 3

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are zero except for the entries in row j , column j , equal 1 for $1 \leq j \leq \dim \text{range } T$.

Proof. Let v_1, \dots, v_n be a basis of V such that $\forall i \in 1, \dots, k, Tv_i = 1$, where $k = \dim \text{range } T$. Of course, it is a basis of $\text{range } T$. Expressing this as a matrix gives the desired result. \square

Problem: 4

Suppose v_1, \dots, v_m is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis w_1, \dots, w_n of W such that all entries in the first column of $\mathcal{M}(T)$ (with respect to the bases v_1, \dots, v_m and w_1, \dots, w_n) are 0 except for possibly a 1 in the first column.

Proof. Let v_1, \dots, v_m be the trivial basis of V . Then Tv_1, \dots, Tv_m spans $\text{range } T$. After finite steps of procedure, we can obtain a list, Tv_1, \dots, Tv_m which is a basis of W , say w_1, \dots, w_m . Then the first column of $\mathcal{M}(T)$ is the desired result. \square

Problem: 5

Suppose w_1, \dots, w_n is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis v_1, \dots, v_m of V such that all entries in the first row of $\mathcal{M}(T)$ (with respect to the bases v_1, \dots, v_m and w_1, \dots, w_n) are 0 except for possibly a 1 in the first row.

Proof. We could always find a basis of V such that $\exists i \in \{1, \dots, n\}, v_i = (1, \dots, 0)$. For list v_1, \dots, v_m , if $m \leq n$, we obtain the desired result. Otherwise, we could let the v_{m+1}, \dots, v_n be the basis of $\text{null } T$. \square

Problem: 6

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \text{range } T = 1$ if and only if there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

Proof. (\implies) Suppose $\dim \text{range } T = 1$, then for v_1, \dots, v_n , a basis of V , $Tv_1, \dots, Tv_n \in \text{range } T$ suggests that they are linearly dependent to each other. Hence, we could obtain the desire by letting $Tv_1, \dots, Tv_n = (1, \dots, 1)$

(\impliedby) Suppose there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1. Then, $\dim \text{range } T = 1$. \square

Problem: 10

Suppose A is an m -by- n matrix and C is an n by p matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C .

Proof.

$$(AC)_{j,i} = \sum_{k=1}^n A_{j,k}C_{k,i} = (A_{j,\cdot}C)_i$$

□

D: Invertibility and Isomorphic Vector spaces

Problem: 1

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that ST is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. The proof for the invertibility is trivial, since U, V, W are bijective to each other. Then we only have to prove the equation. It follows from the fact that $STT^{-1}S^{-1} = I = T^{-1}S^{-1}ST$ \square

Problem: 2

Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of non-invertible operators on V is not a subspace of $\mathcal{L}(V)$.

Proof. Suppose T, S are non-invertible operators on V . Then $\text{null } T \neq \{0\}$ and $\text{null } S \neq \{0\}$. Then $\text{null } T + \text{null } S \neq \{0\}$, which suggests that $T + S$ is not invertible. \square

Problem: 3

Suppose V is finite-dimensional, U is a subspace of V , and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Proof. (\implies) This is obvious since T is bijective.

(\impliedby) Suppose S is injective. Then we could extend u_1, \dots, u_n to a basis of V . Then we could define T by $Tu_i = Su_i$ for $i = 1, \dots, n$. \square

Problem: 4

Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{null } T_1 = \text{null } T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

Proof. (\implies) Suppose $\text{null } T_1 = \text{null } T_2$, this implies that $\text{range } T_1 = \text{range } T_2$. Let w_1, \dots, w_m be a basis of $\text{range } T_1$ and $\text{range } T_2$. Then we could define S by $Sw_i = T_1v_i$ for $i = 1, \dots, m$.

(\impliedby) Suppose there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$. Since S is invertible, this suggests that it has to be injective such that $\text{null } S = \{0\}$. Then $\text{null } T_1 = \text{null } T_2$. \square

Problem: 5

Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{range } T_1 = \text{range } T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$.

Proof. (\implies) Suppose $\text{range } T_1 = \text{range } T_2$. This suggests that there exists a basis w_1, \dots, w_n of $\text{range } T_1$ and $\text{range } T_2$. For such a basis, we could always find a corresponding list v_1, \dots, v_m with which T_2 maps onto w_1, \dots, w_n we could define S by $Su_i = v_i$ for $i = 1, \dots, m$.

(\impliedby) Suppose there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$. Since S is bijective, this implies that $\text{range } T_1 = \text{range } T_2$. \square

Problem: 6

Suppose V and W are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that there exist invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = ST_2R$ if and only if $\dim \text{null } T_1 = \dim \text{null } T_2$.

Proof. The same with question 4. \square

Problem: 7

Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

- (a) Show that E is a subspace of $\mathcal{L}(V, W)$.
- (b) Suppose $v \neq 0$. What is $\dim E$?

(a) *Proof.* $0(v) \in E$ therefore E is not empty, and $T_1 v = T_2 v = 0$ implies $(T_1 + T_2)v = 0$. $Tv = 0$ implies $\lambda Tv = 0$ for all $\lambda \in \mathbb{F}$. Hence, E is a subspace of $\mathcal{L}(V, W)$. \square

(b) *Proof.* Suppose $v \neq 0$. Then there is a basis v_1, \dots, v_n of V extended from v , and we can choose a basis w_1, \dots, w_m of W . Since $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{m,n}$ and $Tv = 0$ implies that the first column of $\mathcal{M}(T)$ is zero. Hence, $\dim E = m(n - 1)$. \square

Problem: 8

Suppose V is finite-dimensional and $T : V \rightarrow W$ is a surjective linear map of V onto W . Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W .

Proof. Since T is surjective, any $w \in W$ we could find a $v \in V$ such that $Tv = w$. Define $x_1 \sim x_2 : \iff T(x_1) = T(x_2)$. We could define $U = \{[v] : \forall v \in V\}$. As such $T|_U$ is bijective, henceforth an isomorphism. \square

Problem: 9

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof. (\implies) Suppose ST is invertible. Then $(ST)^{-1} = T^{-1}S^{-1}$, this suggests that S and T are invertible.

(\impliedby) Suppose S and T are invertible. Then $S^{-1}T^{-1} = (ST)^{-1}$, this suggests that ST is invertible. \square

Problem: 10

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$

Proof. Directly from the definition of inverse. \square

Problem: 11

Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Proof. The associativity implies that both S and U are invertible. Then $T = S^{-1}IU^{-1} = S^{-1}U^{-1}$. Hence, $T^{-1} = US$. \square

Problem: 12

Show that the result in the previous exercise can fail without the hypothesis that V is finite-dimensional.

Proof. TODO: 3.D.12 Pending. \square

Problem: 13

Suppose V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove that S is injective.

Proof. Since RST is surjective, this suggests that R is surjective. Then R is injective, hence S is injective. \square

Problem: 14

Suppose v_1, \dots, v_n is a basis of V . Prove that the map $T : V \rightarrow \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(V)$$

is an isomorphism of V onto $\mathbb{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis v_1, \dots, v_n .

Proof. T is injective: $\forall v_i \in V, Tv = 0 \iff v = 0 \iff v = (0, \dots, 0)$
 T is surjective: $\forall (w_1, \dots, w_n) \in \mathbb{F}^{n,1}, T(a_1v_1 + \dots + a_nv_n) = (a_1, \dots, a_n)$ since v_1, \dots, v_n is a basis of V .
Henceforth, T is an isomorphism. \square

Problem: 15

Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an m -by- n matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$

Proof. Let e_1, \dots, e_n be the standard basis of $\mathbb{F}^{n,1}$ and w_1, \dots, w_m be the standard basis of $\mathbb{F}^{m,1}$. Then $T(e_i) = a_{1i}w_1 + \dots + a_{mi}w_m$. Then $T(x) = Ax$. \square

Problem: 17

Suppose V is finite-dimensional and \mathcal{E} is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}, TS \in \mathcal{L}(V)$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$

Proof. \square

Problem: 17 variant

Suppose V is finite-dimensional and \mathcal{E} is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$

Proof. Suppose $\mathcal{E} \neq \mathcal{L}(V)$. The supposition suggests that there exists some linear mapping in $\mathcal{L}(V)$ which is not in \mathcal{E} . We denote it by φ . Now, note that $\varphi \in \mathcal{L}(V)$, we have $\varphi T \in \mathcal{E}$

Suppose $\mathcal{E} \neq \{0\}$, we will show that $\mathcal{E} = \mathcal{L}(V)$. Suppose \mathcal{E} contains the identity map, then the prove is completed, so we assume identity map is not in \mathcal{E} . \square

Problem: 18

Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

Proof. $\dim \mathcal{L}(\mathbb{F}, V) = \dim \mathbb{F} * \dim V = 1 * \dim V = \dim V$ □

Problem: 19

Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

- (a) Prove that T is surjective.
- (b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$

- (a) *Proof.* $\deg Tp = \deg p$ implies that $\dim T = \dim \mathcal{P}(\mathbb{R})$. Hence, T is surjective. \square
- (b) *Proof.* We prove it by induction: $\deg Tp = \deg p$ for $\dim p = 1$ (T is injective). Suppose it holds for $\dim p = n$, then, due to T is surjective, every $p \in \mathcal{P}(\mathbb{R})$ can be expressed as $T(p)$ uniquely. For $p = n + 1$, suppose that $\deg Tp < \deg p$, then this implies that T is not injective, which is a contradiction. Hence, $\deg Tp = \deg p$ for all $n \in \mathbb{N}$. \square

E: Products and Quotients of Vector Spaces**Problem: 1**

Suppose T is a function from V to W . The **graph** of T is the subset of $V \times W$ defined by

$$\text{graph } T = \{(v, Tv) : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Proof. Since T is a well-defined function, the graph of T is not empty. Then, due to the properties of linear maps and vector space, T is also a subspace of $V \times W$. Hence, the graph of T is a subspace of $V \times W$. \square

Problem: 2

Suppose V_1, \dots, V_m are vector spaces such that $V_1 \times \dots \times V_m$ is finite-dimensional. Prove that each V_i is finite-dimensional.

Proof. Suppose $\exists \dim V_i = \infty (i \in \{1, \dots, m\})$. Then it follows that $\dim(V_1 \times \dots \times V_m) = \sum \dim V_i = \infty$. This is a contradiction. Hence, each V_i is finite-dimensional. \square

Problem: 3

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Proof. TODO: 3.E.3 Give an example Pending. \square

Problem: 4

Suppose V_1, \dots, V_m are vector spaces. Prove that $\mathcal{L}(V_1 \times \dots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$ are isomorphic.

Proof. Clearly, for all $T(v_1, \dots, v_m) = w (w \in W), T \in \mathcal{L}(V_1 \times \dots \times V_m, W)$. Then we could define a map $S : \mathcal{L}(V_1 \times \dots \times V_m, W) \rightarrow \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$ by $S(T) = (T_1, \dots, T_m)$ where $T_i(v_i) = T(0, \dots, v_i, \dots, 0)$. We will verify that S is an isomorphism.

First we show that S is injective: Suppose $S(T) = 0$, then $T_i = 0$ for all $i \in \{1, \dots, m\}$. Then $T = 0$.

Then we show that S is surjective: This is obvious since for every $(T_1, \dots, T_m) \in \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$, we could define $T(v_1, \dots, v_m) = (T_1 v_1, \dots, T_m v_m)$, where $S(T) = (T_1, \dots, T_m)$.

Hence, S is an isomorphism and $\mathcal{L}(V_1 \times \dots \times V_m, W)$ is isomorphic to $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$. \square

Problem: 5

Suppose W_1, \dots, W_m are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \dots \times W_m)$ is isomorphic to $\mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m)$.

Proof. Define $S : \mathcal{L}(V, W_1 \times \dots \times W_m) \rightarrow \mathcal{L}(V, W_m)$ by $S(T) = (T_1, \dots, T_m)$ where $T_i(v) = (0, \dots, T(v), \dots, 0)$. We will verify that S is an isomorphism by constructing an inverse function of S : define $S^{-1}(T_1(v), \dots, T_m(v)) = T_1(v) + \dots + T_m(v)$. $S \circ S^{-1} = \text{id}(T)$ \square

Problem: 6

For n a positive integer, define V^n by

$$V^n = \underbrace{V \times \dots \times V}_n$$

Prove that V^n and $\mathcal{L}(\mathbb{F}^n, V)$ are isomorphic vector spaces.

Proof. Define $S(v_1, \dots, v_n) = T(\lambda_1, \dots, \lambda_n)$ where $T \in \mathcal{L}(\mathbb{F}^n, V)$, we will show that S is an isomorphism by constructing an inverse function of S . Let $S^{-1}(T(\lambda_1, \dots, \lambda_n)) = (T_1(\lambda), \dots, T_n(\lambda))$ where $T_i(\lambda) = (0, \dots, T(\lambda), \dots, 0)$. $S \circ S^{-1} = \text{id}$. Hence, They are isomorphic. \square

Problem: 7

Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$

Proof. Rearrange the equation,

$$\begin{aligned} v + U &= x + W \\ v - x + U &= W \end{aligned}$$

This implies that $U = W$. □

Problem: 8

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$

Proof. (\implies) For all affine subset in the form $t + S$, every element can be expressed as $t + s$. Then for all $t + s_1, t + s_2 \in t + S$, $\lambda(t + s_1) + (1 - \lambda)(t + s_2) = t + \lambda s_1 + (1 - \lambda)s_2 \in t + S$ for all $t, s_1, s_2 \in t + S$ and all $\lambda \in \mathbb{F}$ since S is a subspace.

(\impliedby) If $\lambda v + (1 - \lambda)w \in A$ holds for all $v, w \in A$ and all $\lambda \in \mathbb{F}$. Rearrange the formula we get $\lambda v + (1 - \lambda)w = v + (1 - \lambda)(v + w)$. This implies that A is an affine subset. □

F: Duality**Problem: 1**

Explain why every linear functional is either surjective or the zero map.

Proof. Suppose f is not surjective, the $\dim \text{range } f < \dim \mathbb{F}$ which could only be 0. Therefore, f is the zero map.

Suppose f is not the zero map, then $\dim \text{range } f = \dim \mathbb{F}$, which implies that f is surjective. □

Problem: 2

Give three distinct examples of linear functional on $\mathbb{R}^{[0,1]}$.

Proof. TODO: 3.F.2 Give three distinct examples Pending. □

Problem: 3

Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Proof. Since $v \neq 0$, we could extend v to a basis v_1, \dots, v_n of V . Define $\varphi(v) = 1$ and $\varphi(v_i) = 0$ for $i \neq 1$. Then φ is a linear functional. \square

Problem: 4

Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Proof. Let u_1, \dots, u_m be a basis of U and extend to $u_1, \dots, u_m, v_1, \dots, v_n$ a basis of V . Define $\varphi(u_i) = 0$ for $i = 1, \dots, m$ and $\varphi(v_i) = 1$ for $i = 1, \dots, n$. Then φ is a linear functional. \square

Problem: 5

Suppose V_1, \dots, V_m are vector spaces. Prove that $(V_1 \times \dots \times V_m)'$ and $V_1' \times \dots \times V_m'$ are isomorphic vector spaces.

Proof. Define $S(f) = (f_1, \dots, f_m)$ where $f_i(v_1, \dots, v_m) = (0, \dots, f_i, \dots, 0)$. And let $S^{-1}(f_1, \dots, f_m) = f_1 + \dots + f_m$. Then S is an isomorphism. \square

Problem: 6

Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Define a linear map $\Gamma : V' \mapsto \mathbb{F}^m$ by

$$\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m)).$$

- (a) Prove that v_1, \dots, v_m spans V if and only if Γ is injective.
- (b) Prove that v_1, \dots, v_m is linearly independent if and only if Γ is surjective.

- (a) *Proof.* (\implies) Suppose v_1, \dots, v_m spans V , then for all $\varphi \in V'$, $\Gamma(\varphi) = 0$ implies that $\varphi(v_1) = \dots = \varphi(v_m) = 0$, that is, $\varphi = 0$. Hence, Γ is injective.
(\impliedby) Suppose Γ is injective, then $\varphi(v_1) = \dots = \varphi(v_m) = 0$ implies that $\varphi = 0$. Hence, v_1, \dots, v_m is surjective, that is, spanning V . \square
- (b) *Proof.* (\implies) Suppose v_1, \dots, v_m is linearly independent. Then for all $(\lambda_1, \dots, \lambda_m) \in \mathbb{F}^m$, $\lambda_1\varphi(v_1) + \dots + \lambda_m\varphi(v_m) = 0$ implies that $\lambda_1 = \dots = \lambda_m = 0$ or $\forall v_i = 0$. Therefore, for each $\varphi(v_i)$, φ spans \mathbb{F} . Hence, Γ is surjective.

(\Leftarrow) Suppose Γ is surjective, then for all $v_i, \varphi \neq 0$. That is, v_1, \dots, v_m is linearly independent. Otherwise, suppose v_1, \dots, v_m is linearly dependent. Let v_k be the one can be expressed by all other v 's. Then $\dim \Gamma(\varphi) < m = \dim \mathbb{F}^m$, suggesting that Γ is not surjective, contradiction! Therefore, v_1, \dots, v_m is linearly independent. \square

Problem: 7

Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \dots, x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p , with the understanding that the 0^{th} derivative of p is p .

Proof. Since $1, x, \dots, x^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$, let $p = a_1 + a_2x + \dots + a_mx^m$. Note that $\frac{p^{(j)}(0)}{j!} = a_j$, then $\varphi_j(p) = a_j$ if and only if $i = j$. Hence, $\varphi_0, \varphi_1, \dots, \varphi_m$ is the dual basis of $1, x, \dots, x^m$. \square

Problem: 8

Suppose m is a positive integer.

- (a) Show that $1, x - 5, \dots, (x - 5)^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$.
- (b) What is the dual basis of the basis in part(a)?

(a) *Proof.* This is obvious since we can generate an upper-triangle matrix with $1, x - 5, \dots, (x - 5)^m$ \square

(b) *Proof.* The dual basis is $\varphi_0, \varphi_1, \dots, \varphi_m$ where $\varphi_j(p) = \frac{p^{(j)}(5)}{j!}$ \square

Problem: 9

Suppose v_1, \dots, v_n is a basis of V and $\varphi_1, \dots, \varphi_n$ is the corresponding dual basis of V' . Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$$

Proof. Notice that

$$\begin{aligned} (\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n)(v_1) &= \psi(v_1) \\ &\vdots \\ (\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n)(v_n) &= \psi(v_n) \end{aligned}$$

Therefore, $\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$ as they coincide at a basis of V . \square

Problem: 11

Suppose A is an m -by- n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbb{F}^m$ and $(d_1, \dots, d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$

Proof. (\implies) Suppose the rank of A is 1. Then there exists a \square

Problem: 12

Show that the dual map of the identity map on V is the identity map on V' .

Proof. Let $\text{id}_V : V \mapsto V$ be the identity map on V . Then the dual map of id_V is the linear map $\text{id}'_V(\varphi) := \varphi \circ \text{id}_V = \varphi$ where $\varphi \in \mathcal{L}(V, \mathbb{F})$. Hence, the dual map of the identity map on V is the identity map on V' . \square

Problem: 15

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Proof. (\implies) Suppose $T' = 0$. By definition, it suggests that $T'(\varphi) = \varphi(T(v)) = 0$ for all $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $v \in V$. Therefore, it implies that $T = 0$.

(\impliedby) Suppose $T = 0$. By definition and with the fact that all linear maps map 0 to 0, it suggests that $T' = 0$. \square

Problem: 16

Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism.

Proof. Since $\dim \mathcal{L}(V, W) = \dim(W', V')$, the identity map is bijective. Hence, the map is an isomorphism. \square

Problem: 17

Suppose $U \subset V$. Explain why $U^0 = \{\varphi \in V' : U \subset \text{null } \varphi\}$

Proof. By definition, of course that the zero map 0 is in null space. \square

Problem: 18

Suppose V is finite-dimensional and $U \subset V$. Show that $U = \{0\}$ if and only if $U^0 = V'$.

Proof. (\implies) Suppose $U = \{0\}$. Then it suggests that $\forall \varphi \in \mathcal{L}(U, \mathbb{F}), \varphi(u) = 0$. That is, $U^0 = V'$.

(\impliedby) Directly came from the definition. \square

Problem: 19

Suppose V is finite-dimensional and U is a subspace of V . Show that $U = V$ if and only if $U^0 = \{0\}$

Proof. (\implies) Suppose $U = V$. Then $U^0 = V^0 := \mathcal{L}(V, \mathbb{F})$. For a vector space, this could only happen when $V = \{0\}$

(\impliedby) $U^0 = \{0\}$ suggests that $\dim U = \dim V$. That is, $U = V$ \square

Problem: 20

Suppose U and W are subsets of V with $U \subset W$. Prove that $W^0 \subset U^0$.

Proof. Since $U \subset W$, this suggests that $\forall \varphi \in W^0, \varphi(u) = 0$ for all $u \in U$. Hence, $W^0 \subset U^0$. \square

Problem: 21

Suppose V is finite-dimensional and U and W are subspaces of V with $W^0 \subset U^0$. Prove that $U \subset W$.

Proof. Following from $W^0 \subset U^0$, it suggests that $\dim U > \dim W$. Henceforth, $U \subset W$. \square

Problem: 22

Suppose U, W are subspaces of V . Show that $(U + W)^0 = U^0 \cap W^0$

Proof. Let u_1, \dots, u_m be a basis of U and $u_1, \dots, u_m, w_1, \dots, w_n$ be a basis of W . Then \square