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A: Polar Decomposition and Singular Value Decomposition **3**

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Problem: 1

Fix $u, x \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u \rangle \, x$$

for every $v \in V$. Prove that

$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v, u \rangle u$$

for every $v \in V$.

Proof. First we need to find the T^*. By definition, we have

$$\begin{aligned} \langle v, T^*w \rangle &= \langle Tv, w \rangle \\ &= \langle \langle v, u \rangle x, w \rangle \\ &= \langle v, u \rangle \langle x, w \rangle \\ &= \langle v, \langle w, x \rangle u \rangle \\ &= \langle v, Tw \rangle \\ T^*v = \langle v, x \rangle u & \end{aligned}$$

Then

$$\begin{aligned} T^*Tv &= T^*\langle v, u \rangle x \\ &= \langle v, u \rangle T^*x \\ &= \langle v, u \rangle \langle x, x \rangle u \\ &= \langle v, u \rangle \left\|x\right\|^2 u \end{aligned}$$

Notice that the first two terms are scalars, the eigenvector could only be u and its eigenvalue is $\langle u, u \rangle \left\|x\right\|^2 = \left\|u\right\|^2 \left\|x\right\|^2$.
Thus, $\sqrt{T^*T}v = \frac{T^*Tv}{\sqrt{\lambda}} = \frac{\langle v,u \rangle \|x\|^2 u}{\|u\|\|x\|} = \frac{\|x\|}{\|u\|}\langle v, u \rangle u$ $\square$