Chapter 6: Inner product spaces

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A: Inner products and Norms

Problem: 1

Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbb{R}^3$

Proof. For $((1,1),(1,1)),((-1.-1),(1,1)) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have

$$|1 \cdot 1| + |1 \cdot 1| = 2$$

for both two vectors. But on the same hand, we also have

$$\begin{aligned} ((1,1),(1,1)) + ((-1,-1),(1,1)) &= ((0,0),(1,1)) \\ &= |0 \cdot 1| + |0 \cdot 1| = 0 \end{aligned}$$

□

This could not be an inner product since it violates the additivity property of inner products.

Problem: 2

Show that the function that takes $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1 y_1 + x_3 y_3$ is not an inner product on $\mathbb{R}^3$

Proof. It take $(0, 1, 0)$ to zero while $(0, 1, 0) \neq 0$ □

Problem: 3

Suppose $\mathbb{F} = \mathbb{R}$ and $V \neq \{0\}$. Replace the positivity condition (which states that $\langle v, v \rangle \geq 0$ for all $v \in V$) in the definition of an inner product with the condition that $\langle v, v \rangle > 0$ for some $v \in V$. Show that this new definition of an inner product does not change the set of functions from $V \times V$ to $\mathbb{R}$ that are inner products on V.

Proof. We show that the two condition are equivalent on the given space. Suppose the positivity condition is satisfied, then the new condition is obviously true for some $v \in V$.

Now we suppose the new condition is satisfied, then for all $v \in V$, we have

$$\langle v, v \rangle =$$

□

Problem: 8