

Chapter 3: Linear maps

Linear Algebra Done Right, by Sheldon Axler

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A: The vector space of linear maps

Problem S

Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is a linear map if and only if $b = 0$ and $c = 0$.

Proof. Consider $T(1, 1, 1) = T(1, 0, 0) + T(0, 1, 1)$

$$T(1, 1, 1) = (1 + b, 6 + c) \tag{1}$$

$$T(1, 0, 0) = (2 + b, 6) \tag{2}$$

$$T(0, 1, 1) = (-1 + b, 0) \tag{3}$$

Therefore, $b = 0, c = 0$ □

B: Null space and Range

Problem 1

Give an example of a linear map T such that $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$.

Proof. Consider $T : \mathcal{P}(4) \mapsto \mathcal{P}(1)$, $T(az^4 + bz^3 + cz^2 + dz + e) = (dz + e)$. Then $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$. □

Problem 3

Suppose v_1, \dots, v_m is a list of vectors in V . Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m.$$

- (a) What property of T corresponds to v_1, \dots, v_m spanning V .
- (b) What property of T corresponds to v_1, \dots, v_m being linearly independent.

Proof. (a) $\forall v \in V$, we have $T(z_1, \dots, z_m) = v$, which means v_1, \dots, v_m span V . This suggests that $\text{range } T$ is equal to V . Hence, T is surjective.

- (b) If v_1, \dots, v_m are linearly independent, then $T(z_1, \dots, z_m) = 0$ implies $z_1 = \dots = z_m = 0$. This suggests that $\text{null } T = \{0\}$, hence T is injective. □

Problem 4

Show that

$$T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null } T > 2$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$

Proof. Suppose $\dim \text{null } T > 2$, by F.T. of linear maps,

$$\dim T = 5 = \dim \text{null } T + \dim \text{range } T \quad (4)$$

$$> 2 + \dim \text{range } T \quad (5)$$

$$\dim \text{range } T < 3 \quad (6)$$

Hence, $T \notin \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$, and thus not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. \square

Problem 5

Give an example of a linear map $T : \mathbb{R}^4 \mapsto \mathbb{R}^4$ such that

$$\text{range } T = \text{null } T.$$

Proof. Consider $T : \mathbb{R}^4 \mapsto \mathbb{R}^4$, $T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0)$. Then $\text{range } T = \text{null } T$. \square

Problem 6

Prove that there does not exist a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\text{range } T = \text{null } T$.

Proof. Suppose there exists a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\text{range } T = \text{null } T$. Then by F.T. of linear maps, we have $\dim \text{range } T = \dim \text{null } T$, which implies $\dim \text{range } T = 5 - \dim \text{range } T$, or $\dim \text{range } T = 2.5$, which is not an integer. Hence, such a linear map does not exist. \square

Problem 7

Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Proof. T is not injective suggests that $\dim \text{null } V > 0$. Then By the F.T. of linear maps, we have

$$\dim V = \dim \text{null } V + \dim \text{range } T \quad (7)$$

$$\geq 1 + \dim \text{range } T \geq 1 + \dim V \quad (8)$$

Contradicts! Hence, $T \notin \mathcal{L}(V, W) \implies$ not a subspace. \square

Problem 9

Suppose $T \in \mathcal{L}(V, W)$ is injective and v_1, \dots, v_n is linearly independent in V . Prove that Tv_1, \dots, Tv_n is linearly independent in W .

Proof. T is injective $\implies \text{null } T = \{0\}$. Therefore, we have

$$T(\lambda_1 v_1 + \dots + \lambda_n v_n) \iff \lambda_j = 0$$

where $j \in \{1, \dots, n\}$. Since v_1, \dots, v_n are linearly independent. It follows that Tv_1, \dots, Tv_n are linearly independent. \square

Problem 10

Suppose v_1, \dots, v_n spans V and $T \in \mathcal{L}(V, W)$. Prove that the list Tv_1, \dots, Tv_n spans $\text{range } T$.

Proof. Suppose v_1, \dots, v_n spans V . Then $\forall v \in V$, we have $v = \lambda_1 v_1 + \dots + \lambda_n v_n$. Then

$$T(v) = T(\lambda_1 v_1 + \dots + \lambda_n v_n) = \lambda_1 T v_1 + \dots + \lambda_n T v_n.$$

This suggests that $\text{range}(T) \subset \text{span}(Tv_1, \dots, Tv_n)$. Also, $\text{span}(Tv_1, \dots, Tv_n)$ is the smallest containing subspace of W implying that it is a subset of $\text{range } T$, hence $\text{range } T = \text{span}(Tv_1, \dots, Tv_n)$. \square

Problem 11

Suppose S_1, \dots, S_n are injective linear maps such that $S_1 S_2 \dots S_n$ makes sense. Prove that $S_1 S_2 \dots S_n$ is injective.

Proof. By F.T. of linear maps, we have

$$\dim S_1 = \dim \text{null } S_1 + \dim \text{range } S_1 \quad (9)$$

$$= 0 + \dim \text{range } S_1 = \dim \text{range } S_1 \quad (10)$$

It follows that

$$\dim S_1 = \dim \text{range } S_1 = \dim S_2 = \dots = \dim S_n = \dim \text{range } S_n$$

Hence, $\dim \text{null } S_1 S_2 \dots S_n = 0 \iff S_1 S_2 \dots S_n$ is injective. \square

Problem 12

Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $\text{null } T = U$ and $U \cap \text{range } T = \{0\}$.

Proof. Let u_1, \dots, u_n be a basis for $\text{null } T$. Then $\text{span}(u_1, \dots, u_n)$ is a subspace of V . Since it is linearly independent, it can be extended to a basis of V , say $u_1, \dots, u_n, v_1, \dots, v_m$. Then V is the direct sum of the spanning of u_1, \dots, u_n and v_1, \dots, v_m . Take $U = \text{span}(v_1, \dots, v_m)$ \square

Problem 13

Suppose T is a linear map from \mathbb{F}^4 to \mathbb{F}^2 such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is surjective.

Proof. A basis of $\text{null } T$ is

$$\{(5, 1, 0, 0), (0, 0, 7, 1)\}$$

Then by F.T. of linear maps, we have

$$\dim \text{range } T = 4 - \dim \text{null } T = 4 - 2 = 2$$

Therefore, $\text{range } T = \mathbb{R}^2 \implies T$ is surjective. \square

Problem 14

Suppose U is a 3-dimensional subspace of \mathbb{R}^8 and that T is a linear map from \mathbb{R}^8 to \mathbb{R}^5 such that $\text{null } T = U$. Prove that T is surjective.

Proof. By F.T. of linear maps, we have

$$\dim \text{range } T = 8 - \dim \text{null } T = 8 - 3 = 5$$

This suggests that $\text{range } T = \mathbb{R}^5 \implies T$ is surjective. \square

Problem 15

Prove that there does not exist a linear map from \mathbb{F}^5 to \mathbb{F}^2 whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Proof. Suppose there exists such a linear map, then, by F.T. of linear maps, we have

$$\dim \text{range } T = 5 - \dim \text{null } T = 5 - 2 = 2$$

Contradicts! \square

Problem 16

Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite dimensional.

Proof. WLOG, let $\dim \text{null } T = m$ and $\dim \text{range } T = n$. Then by F.T. of linear maps, we have

$$\dim V = \dim \text{null } T + \dim \text{range } T = m + n < \infty$$

\square

Problem 17

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Proof. (\implies) Suppose there is an injective linear map $T : V \mapsto W$. By F.T. of linear maps, we have

$$\dim V = \dim \text{null } T + \dim \text{range } T \quad (11)$$

$$= 0 + \dim \text{range } T = \dim \text{range } T \quad (12)$$

(\impliedby) Suppose $\dim V \leq \dim W$. Then there exists a basis of V that can be extended to a basis of W . Let v_1, \dots, v_n be a basis of V and w_1, \dots, w_n be a basis of W . Define $T : V \mapsto W$ by $T(v_i) = w_i$. Then T is injective. \square

Problem 19

Suppose V and W are finite-dimensional and that U is a subspace of V . Prove that there exists

$$T \in \mathcal{L}(V, W)$$

such that $\text{null } T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. (\implies) Suppose there exists $T \in \mathcal{L}(V, W)$ such that $\text{null } T = U$. Since $\text{range } T$ is a subspace of W , we have $\dim \text{Range } T \leq \dim W$. The rest of the proof follows by the F.T. of linear maps.

(\impliedby) Suppose $\dim U \geq \dim V - \dim W$, we have

$$\dim U + \dim W \geq \dim V = \dim \text{null } T + \dim \text{range } T \quad (13)$$

Of course we could find such a T . \square

Problem 20

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V .

Proof. (\implies) T is injective $\iff \dim \text{null } T = 0$. Then there exists a basis of V that can be extended to a basis of W . Let v_1, \dots, v_n be a basis of V and w_1, \dots, w_n be a basis of W . Define $S : W \mapsto V$ by $S(w_i) = v_i$ and $T : V \mapsto W$ by $T(v_k) = w_k$. Then

$$ST(v) = ST(a_1v_1 + \dots + a_nv_n) \quad (14)$$

$$= S(a_1w_1 + \dots + a_nw_n) \quad (15)$$

$$= a_1v_1 + \dots + a_nv_n \quad (16)$$

$$= v \quad (17)$$

(\impliedby) Since v_k is a basis, $\text{null } T = \{0\}$. This suggests that T is injective. \square

C: Matrix

Problem 1

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W , the matrix of T has at least $\dim \text{range } T$ nonzero entries.

Proof. Follows from the F.T. of linear maps, we have $\dim \text{range } T = \dim W - \dim \text{null } T$. Since $\text{null } T$ becomes all the zero entries, therefore the matrix of T has at least $\dim \text{range } T$ nonzero entries. \square

Problem 2

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Proof. Easy to verify that $\frac{1}{3}x^3, \frac{1}{2}x^2, x, 1$ is a list of basis of $\mathcal{P}_3(\mathbb{R})$, and its derivative is $x^2, x, 1$ which is a basis of $\mathcal{P}_2(\mathbb{R})$. \square

Problem 3

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are zero except for the entries in row j , column j , equal 1 for $1 \leq j \leq \dim \text{range } T$.

Proof. Let v_1, \dots, v_n be a basis of V such that $\forall i \in 1, \dots, k, Tv_i = v_i$, where $k = \dim \text{range } T$. Of course, it is a basis of $\text{range } T$. Expressing this as a matrix gives the desired result. \square

Problem 4

Suppose v_1, \dots, v_m is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis w_1, \dots, w_n of W such that all entries in the first column of $\mathcal{M}(T)$ (with respect to the bases v_1, \dots, v_m and w_1, \dots, w_n) are 0 except for possibly a 1 in the first column.

Proof. \square