
# Linear Algebra Done Right Solution

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# Chapter 2 Finite-dimensional vector space

## A Span and Linear independence

## B Bases

✍ **Exercise 2.B.1** Find all vector spaces that have exactly one basis.

**Proof**

Consider all lines passing through origin.

✍ **Exercise 2.B.2**

(a) Let $U$ be the subspace of $\mathbb{R}^5$ defined by

$$U = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4$$

Find a basis of $U$.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace $W$ of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

**Proof**

(a) We can verify that $(3, 1, 0, 0, 0),\ (0, 0, 7, 1, 0),\ (0, 0, 0, 0, 1)$ is a basis of $U$, and it spans U.

Idea: We start from the basis of $\mathbb{R}^5$ and try to fit it into the given condition.

(b) $(3, 1, 0, 0, 0),\ (0, 0, 7, 1, 0),\ (0, 0, 0, 0, 1),\ (0, 1, 0, 0, 0),\ (0, 0, 1, 0, 0)$

Idea: Downgrade all vector to one dimensional (to obtain the basis of $\mathbb{R}^5$).

(c) $\operatorname{span}\{(0, 0, 0, 0, 1),\ (0, 1, 0, 0, 0)\}$

Idea: Refer to the definition of direct sum.

✍ **Exercise 2.B.3** Prove or disprove: there exists a basis $p_0, p_1, p_2, p_3$ of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials $p_0, p_1, p_2, p_3$ has a degree 2.

**Proof** Consider

$$\begin{aligned} p_0 &= a_1 \\ p_1 &= a_2 x \\ p_2 &= a_3 x^3 \\ p_3 &= a_4 x^2 + a_5 x^3 \end{aligned}$$

It is easy to verify that they are linearly independent, and span $\mathcal{P}_3(\mathbb{F})$

✍ **Exercise 2.B.4** Suppose $v_1, v_2, v_3, v_4$ is a basis of $V$. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of $V$.

**Proof** We could obtain $v_1, v_2, v_3$ with the following operation

$$v_3 + v_4 - v_4 = v_3 \tag{2.1}$$

$$v_2 + v_3 - (v_3 + v_4 - v_4) = v_2 \tag{2.2}$$

$$v_1 + v_2 - (v_2 + v_3 - (v_3 + v_4 - v_4)) = v_1 \tag{2.3}$$

Which suggests that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ also spans $V$.

Next, we prove that they are linearly independent.

$$
\begin{aligned}
&a(v_1 + v_2 + b(v_2 + v_3) + c(v_3 + v_4) + dv_4 && (2.4)\\
=&av_1 + (a+b)v_2 + (b+c)v_3 + (c+d)v_4 && (2.5)
\end{aligned}
$$

Since $v_1, v_2, v_3, v_4$ is a basis, we have $a = b = c = d = 0$. This implies that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also linearly independent.

✍ **Exercise 2.B.5** Suppose $U$ and $W$ are subspaces of $V$ such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of $U$ and $w_1, \dots, w_n$ is a basis of $W$. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of $V$.

**Proof** The linear independency of $u_1, \dots, u_m, w_1, \dots, w_n$ is directly come from the definition of direct sum since $U \cap W = 0$. Since $u_1, \dots, u_m$ is a basis of U. We can write any $u \in U$ as a linear combination of $u_1, \dots, u_m$. Similarly, we can write any $w \in W$ as a linear combination of $w_1, \dots, w_n$. Therefore, we can write any $v \in V$ as a linear combination of $u_1, \dots, u_m, w_1, \dots, w_n$.

## C Dimension

✍ **Exercise 2.C.1** Suppose $V$ is finite-dimensional and $U$ is a subspace of $V$ such that $\dim U = \dim V$. Prove that $U = V$.

**Proof** Let $u_1, \dots, u_n$ be a basis of $U$. Since $U$ is a subspace of $V$, we can extend the basis of $U$ to a basis of $V$. Since $\dim U = \dim V$, every independent list of right length is also a base suggests that $u_1, \dots, u_n$ also spans $V$. Therefore, $U = V$.

✍ **Exercise 2.C.2** Show that the subspaces of $\mathbb{R}^2$ are precisely the zero subspace, all lines in $\mathbb{R}^2$ through the origin, and $\mathbb{R}^2$ itself.

**Proof** $\dim \mathbb{R}^2 = 2 \implies \dim \text{subspaces of } \mathbb{R}^2 \leq 2$

$$
\begin{aligned}
\text{When} \quad &\dim U = 0, & U = \{0\}. && (2.6)\\
&\dim U = 1, & U \text{ is a line passing through origin.} && (2.7)\\
&\dim U = 2, & U = \mathbb{R}^2. && (2.8)
\end{aligned}
$$

# Chapter 3 Linear maps

## A The vector space of linear maps

✍ **Exercise 3.A.1** Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that $T$ is a linear map if and only if $b = 0$ and $c = 0$.

**Proof** Consider $T(1,1,1) = T(1,0,0) + T(0,1,1)$

$$\begin{aligned} T(1,1,1) &= (1+b, 6+c) \\ T(1,0,0) &= (2+b, 6) \\ T(0,1,1) &= (-1+b, 0) \end{aligned}$$

Therefore, $b = 0, c = 0$

## B Null space and Range

✍ **Exercise 3.B.1** Give an example of a linear map $T$ such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

**Proof** Consider $T : \mathcal{P}(4) \mapsto \mathcal{P}(1)$, $T(az^4 + bz^3 + cz^2 + dz + e) = (dz + e)$. Then $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

✍ **Exercise 3.B.2** Suppose $v_1, \ldots, v_m$ is a list of vectors in $V$. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of $T$ corresponds to $v_1, \ldots, v_m$ spanning $V$.

(b) What property of $T$ corresponds to $v_1, \ldots, v_m$ being linearly independent.

(a) **Proof** $\forall v \in V$, we have $T(z_1, \ldots, z_m) = v$, which means $v_1, \ldots, v_m$ span $V$. This suggests that $\operatorname{range} T$ is equal to $V$. Hence, $T$ is surjective.

(b) **Proof** If $v_1, \ldots, v_m$ are linearly independent, then $T(z_1, \ldots, z_m) = 0$ implies $z_1 = \cdots = z_m = 0$. This suggests that $\operatorname{null} T = \{0\}$, hence $T$ is injective.

✍ **Exercise 3.B.3** Show that

$$T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$

**Proof** Suppose $\dim \operatorname{null} T > 2$, by F.T. of linear maps,

$$\begin{aligned} \dim T = 5 &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &> 2 + \dim \operatorname{range} T \\ \dim \operatorname{range} T &< 3 \end{aligned}$$

Hence, $T \notin \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$, and thus not a subspace of $\mathcal{L}(R^5, R^4)$.

✍ **Exercise 3.B.4** Give an example of a linear map $T : \mathbb{R}^4 \mapsto \mathbb{R}^4$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

**Proof** Consider $T : R^4 \mapsto R^4$, $T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0)$. Then $\operatorname{range} T = \operatorname{null} T$.

✍ **Exercise 3.B.5** Prove that there does not exist a linear map $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\operatorname{range} T = \operatorname{null} T$.

**Proof** Suppose there exists a linear map $T:\mathbb{R}^5 \mapsto \mathbb{R}^5$ such that range $T = \operatorname{null} T$. Then by F.T. of linear maps, we have $\dim \operatorname{range} T = \dim \operatorname{null} T$, which implies $\dim \operatorname{range} T = 5 - \dim \operatorname{range} T$, or $\dim \operatorname{range} T = 2.5$, which is not an integer. Hence, such a linear map does not exist.

**Exercise 3.B.6** Suppose $V$ and $W$ are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V,W) :$ $T$ is not injective $\}$ is not a subspace of $\mathcal{L}(V,W)$.

**Proof** $T$ is not injective suggests that $\dim \operatorname{null} V > 0$. Then By the F.T. of linear maps, we have

$$\begin{aligned}\dim V &= \dim \operatorname{null} V + \dim W \\ &\geq 1 + \dim W \geq 1 + \dim V\end{aligned}$$

Contradicts! Hence, $T \notin \mathcal{L}(V,W) \implies$ not a subspace.

**Exercise 3.B.7** Suppose $T \in \mathcal{L}(V,W)$ is injective and $v_1,\ldots,v_n$ is linearly independent in $V$. Prove that $Tv_1,\ldots,Tv_n$ is linearly independent in $W$.

**Proof** $T$ is injective $\implies \operatorname{null} T = \{0\}$. Therefore, we have

$$T(\lambda_1 v_1 + \cdots + \lambda_n v_n) \iff \lambda_j = 0$$

where $j \in \{1,\ldots,n\}$. Since $v_1,\ldots,v_n$ are linearly independent. It follows that $Tv_1,\ldots,Tv_n$ are linearly independent.

**Exercise 3.B.8** Suppose $v_1,\ldots,v_n$ spans $V$ and $T \in \mathcal{L}(V,W)$. Prove that the list $Tv_1,\ldots,Tv_n$ spans range $T$.

**Proof** Suppose $v_1,\ldots,v_n$ spans $V$. Then $\forall v \in V$, we have $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then

$$T(v) = T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 Tv_1 + \cdots + \lambda_n Tv_n.$$

This suggests that $\operatorname{range}(T) \subset \operatorname{span}(Tv_1,\ldots,Tv_n)$. Also, $\operatorname{span}(Tv_1,\ldots,Tv_n)$ is the smallest containing subspace of $W$ implying that it is a subset of range $W$, hence range $T = \operatorname{span}(Tv_1,\ldots,Tv_n)$.

**Exercise 3.B.9** Suppose $S_1,\ldots,S_n$ are injective linear maps such that $S_1S_2\ldots S_n$ makes sense. Prove that $S_1S_2\ldots S_n$ is injective.

**Proof** By F.T. of linear maps, we have

$$\begin{aligned}\dim S_1 &= \dim \operatorname{null} S_1 + \dim \operatorname{range} S_1 \\ &= 0 + \dim \operatorname{range} S_1 = \dim \operatorname{range} S_1\end{aligned}$$

It follows that

$$\dim S_1 = \dim \operatorname{range} S_1 = \dim S_2 = \cdots = \dim S_n = \dim \operatorname{range} S_n$$

Hence, $\dim \operatorname{null} S_1S_2\ldots S_n = 0 \iff S_1S_2\ldots S_n$ is injective.

**Exercise 3.B.10** Suppose that $V$ is finite-dimensional and that $T \in \mathcal{L}(V,W)$. Prove that there exists a subspace $U$ of $V$ such that $\operatorname{null} T = U$ and
$U \cap \operatorname{range} T = \{0\}$.

**Proof** Let $u_1,\ldots,u_n$ be a basis for $\operatorname{null} T$. Then $\operatorname{span}(u_1,\ldots,u_n)$ is a subspace of $V$. Since it is linear independent, it can be extended to a basis of V, say $u_1,\ldots,u_n,v_1,\ldots,v_m$. Then $V$ is the direct sum of the spanning of $u_1,\ldots,u_n$ and $v_1,\ldots,v_m$. Take $U = \operatorname{span}(v_1,\ldots,v_m)$

**Exercise 3.B.11** Suppose $T$ is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\operatorname{null} T = \{(x_1,x_2,x_3,x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that $T$ is surjective.

**Proof** A basis of $\operatorname{null} T$ is

$$\{(5,1,0,0),(0,0,7,1)\}$$

Then by F.T. of linear maps, we have

$$\dim \operatorname{range} T = 4 - \dim \operatorname{null} T = 4 - 2 = 2$$

Therefore, $\operatorname{range} T = \mathbb{R}^2 \implies T$ is surjective.

✍ **Exercise 3.B.12** Suppose $U$ is a 3-dimensional subspace of $\mathbb{R}^8$ and that $T$ is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

**Proof** By F.T. of linear maps, we have

$$\dim \operatorname{range} T = 8 - \dim \operatorname{null} T = 8 - 3 = 5$$

This suggests that $\operatorname{range} T = \mathbb{R}^5 \implies T$ is surjective.

✍ **Exercise 3.B.13** Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

**Proof** Suppose there exists such a linear map, then, by F.T. of linear maps, we have

$$\dim \operatorname{range} T = 5 - \dim \operatorname{null} T = 5 - 2 = 2$$

Contradicts!

✍ **Exercise 3.B.14** Suppose there exists a linear map on $V$ whose null space and range are both finite-dimensional. Prove that $V$ is finite dimensional.

**Proof** WLOG, let $\dim \operatorname{null} T = m$ and $\dim \operatorname{range} T = n$. Then by F.T. of linear maps, we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = m + n < \infty$$

✍ **Exercise 3.B.15** Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists an injective linear map from $V$ to $W$ if and only if $\dim V \leq \dim W$.

**Proof** ($\implies$) Suppose there is an injective linear map $T : V \mapsto W$. By F.T. of linear maps, we have

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= 0 + \dim \operatorname{range} T = \dim \operatorname{range} T \end{aligned}$$

($\impliedby$) Suppose $\dim V \leq \dim W$. Then there exists a basis of $V$ that can be extended to a basis of $W$. Let $v_1, \ldots, v_n$ be a basis of $V$ and $w_1, \ldots, w_n$ be a basis of $W$. Define $T : V \mapsto W$ by $T(v_i) = w_i$. Then $T$ is injective.

✍ **Exercise 3.B.16** Suppose $V$ and $W$ are finite-dimensional and that $U$ is a subspace of $V$. Prove that there exists

$$T \in \mathcal{L}(V, W)$$

such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

**Proof** ($\implies$) Suppose there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$. Since $\operatorname{range} T$ is a subspace of $W$, we have $\dim Range \leq \dim W$. The rest of the proof follows by the F.T. of linear maps.

($\impliedby$) Suppose $\dim U \geq \dim V - \dim W$, we have

$$\dim U + \dim W \geq \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Of course we could find such a $T$.

✍ **Exercise 3.B.17** Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T$ is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity map on $V$.

**Proof** ($\implies$) T is injective $\iff \dim \operatorname{null} T = 0$. Then there exists a basis of $V$ that can be extended to a basis of $W$. Let $v_1, \ldots, v_n$ be a basis of $V$ and $w_1, \ldots, w_n$ be a basis of $W$. Define $S : W \mapsto V$ by $S(w_i) = v_i$ and $T : V \mapsto W$ by $T(v_k) = w_k$ Then

$$\begin{aligned} ST(v) &= ST(a_1v_1 + \cdots + a_nv_n) \\ &= S(a_1w_1 + \cdots + a_nw_n) \\ &= a_1v_1 + \cdots + a_nv_n \\ &= v \end{aligned}$$

( $\Longleftarrow$ ) Since $v_k$ is a basis, $\operatorname{null} T = \{0\}$. This suggests that $T$ is injective.

## C Matrix

**Exercise 3.C.1** Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of $V$ and $W$, the matrix of $T$ has at least $\dim \operatorname{range} T$ nonzero entries.

**Proof** Follows from the F.T. of linear maps, we have $\dim \operatorname{range} T = \dim W - \dim \operatorname{null} T$. Since $\operatorname{null} T$ becomes all the zero entries, therefore the matrix of $T$ has at least $\dim \operatorname{range} T$ nonzero entries.

**Exercise 3.C.2** Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p\prime$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of $D$ with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

**Proof** Easy to verify that $\frac{1}{3}x^3, \frac{1}{2}x^2, x, 1$ is a list of basis of $\mathcal{P}_3(\mathbb{R})$, and its derivative is $x^2, x, 1$ which is a basis of $\mathcal{P}_2(\mathbb{R})$.

**Exercise 3.C.3** Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $\mathcal{M}(T)$ are zero except for the entries in row $j$, column $j$, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

**Proof** Let $v_1, \ldots, v_n$ be a basis of $V$ such that $\forall i \in 1, \ldots, k, Tv_i = 1$, where $k = \dim \operatorname{range} T$. Of course, it is a basis of $\operatorname{range} T$. Expressing this as a matrix gives the desired result.

**Exercise 3.C.4** Suppose $v_1, \ldots, v_m$ is a basis of $V$ and $W$ is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of $W$ such that all entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first column.

**Proof** Let $v_1, \ldots, v_m$ be the trivial basis of $V$. Then $Tv_1, \ldots, Tv_m$ spans $\operatorname{range} T$. After finite steps of procedure, we can obtain a list, $Tv_1, \ldots, Tv_m$ which is a basis of $W$, say $w_1, \ldots, w_m$. Then the first column of $\mathcal{M}(T)$ is the desired result.

**Exercise 3.C.5** Suppose $w_1, \ldots, w_n$ is a basis of $W$ and $V$ is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of $V$ such that all entries in the first row of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row.

**Proof** We could always find a basis of $V$ such that $\exists i \in \{1, \ldots, n\}, v_i = (1, \ldots, 0)$ For list $v_1, \ldots, v_m$, if $m \leq n$, we obtain the desired result. Otherwise, we could let the $v_{m+1}, \ldots, v_n$ be the basis of $\operatorname{null} T$.

**Exercise 3.C.6** Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exists a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

**Proof** ( $\Longrightarrow$ ) Suppose $\dim \operatorname{range} T = 1$, then for $v_1, \ldots, v_n$, a basis of $V$, $Tv_1, \ldots, Tv_n \in \operatorname{range} T$ suggests that they are linearly dependent to each other. Hence, we could obtain the desire by letting $Tv_1, \ldots, Tv_n = (1, \ldots, 1)$

( $\Longleftarrow$ ) Suppose there exists a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1. Then, $\dim \operatorname{range} T = 1$.

**Exercise 3.C.7** Suppose $A$ is an m-by-n matrix and $C$ is an n by p matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for $1 \leq j \leq m$. In other words, show that row $j$ of $AC$ equals (row $j$ of $A$) times $C$.

**Proof**

$$(AC)_{j,i} = \sum_{k=1}^{n} A_{j,k}C_{k,i} = (A_{j,\cdot}C)_i$$

## D Invertibility and Isomorphic Vector spaces

✍ **Exercise 3.D.1** Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

**Proof** The proof for the invertibility is trivial, since $U, V, W$ are bijective to each other. Then we only have to prove the equation. It follows from the fact that $STT^{-1}S^{-1} = I = T^{-1}S^{-1}ST$

✍ **Exercise 3.D.2** Suppose $V$ is finite-dimensional and $\dim V > 1$. Prove that the set of non-invertible operators on $V$ is not a subspace of $\mathcal{L}(V)$.

**Proof** Suppose $T, S$ are non-invertible operators on $V$. Then $\operatorname{null} T \neq \{0\}$ and $\operatorname{null} S \neq \{0\}$. Then $\operatorname{null} T + \operatorname{null} S \neq \{0\}$, which suggests that $T + S$ is not invertible.

✍ **Exercise 3.D.3** Suppose $V$ is finite-dimensional, $U$ is a subspace of $V$, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for ever $u \in U$ if and only if $S$ is injective.

**Proof** ( $\Longrightarrow$ ) This is obvious since T is bijective.

( $\Longleftarrow$ ) Suppose $S$ is injective. Then we could extend $u_1, \ldots, u_n$ to a basis of $V$. Then we could define $T$ by $Tu_i = Su_i$ for $i = 1, \ldots, n$.

✍ **Exercise 3.D.4** Suppose $W$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$

**Proof** ( $\Longrightarrow$ ) Suppose $\operatorname{null} T_1 = \operatorname{null} T_2$, this implies that $\operatorname{range} T_1 = \operatorname{range} T_2$. Let $w_1, \ldots w_m$ be a basis of $\operatorname{range} T_1$ and $\operatorname{range} T_2$. Then we could define $S$ by $Sw_i = T_1 v_i$ for $i = 1, \ldots, m$.

( $\Longleftarrow$ ) Suppose there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$. Since $S$ is invertible, this suggests that it has to be injective such that $\operatorname{null} S = \{0\}$. Then $\operatorname{null} T_1 = \operatorname{null} T_2$.

✍ **Exercise 3.D.5** Suppose $V$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$

**Proof** ( $\Longrightarrow$ ) Suppose $\operatorname{range} T_1 = \operatorname{range} T_2$. This suggests that there exists a basis $w_1, \ldots, w_n$of $\operatorname{range} T_1$ and $\operatorname{range} T_2$. For such a basis, we could always find a corresponding list $v_1, \ldots, v_m$ with which $T_2$ maps onto $w_1, \ldots, w_n$we could define $S$ by $Su_i = v_i$ for $i = 1, \ldots, m$.

( $\Longleftarrow$ ) Suppose there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$. Since $S$ is bijective, this implies that $\operatorname{range} T_1 = \operatorname{range} T_2$.

✍ **Exercise 3.D.6** Suppose $V$ and $W$ are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that there exist invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = S T_2 R$ if and only if $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

**Proof** The same with question 4.

✍ **Exercise 3.D.7** Suppose $V$ and $W$ are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that $E$ is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E$

(a) **Proof** $0(v) \in E$ therefore $E$ is not empty, and $T_1 v = T_2 v = 0$ implies $(T_1 + T_2)v = 0$. $Tv = 0$ implies $\lambda Tv = 0$ for all $\lambda \in \mathbb{F}$. Hence, $E$ is a subspace of $\mathcal{L}(V, W)$.

(b) **Proof** Suppose $v \neq 0$. Then there is a basis $v_1, \ldots, v_n$ of $V$ extended from $v$, and we can choose a basis $w_1, \ldots, w_m$ of $W$. Since $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{m,n}$ and $Tv = 0$ implies that the first column of $\mathcal{M}(T)$ is zero. Hence, $\dim E = m(n-1)$.

✍ **Exercise 3.D.8** Suppose $V$ is finite-dimensional and $T : V \to W$ is a surjective linear map of $V$ onto $W$. Prove that there is a subspace $U$ of $V$ such that $T|_U$ is an isomorphism of $U$ onto $W$.

**Proof** Since $T$ is surjective, any $w \in W$ we could find a $v \in V$ such that $Tv = w$. Define $x_1 \sim x_2 :\iff T(x_1) = T(x_2)$. We could define $U = \{[v] : \forall v \in \text{V}\}$. As such $T|_U$ is bijective, henceforth an isomorphism.

✍ **Exercise 3.D.9** Suppose $V$ is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST$ is invertible if and only if both S and T are invertible.

**Proof** ( $\Longrightarrow$ ) Suppose ST is invertible. Then $(ST)^{-1} = T^{-1}S^{-1}$, this suggests that $S$ and $T$ are invertible.

( $\Longleftarrow$ ) Suppose $S$ and $T$ are invertible. Then $S^{-1}T^{-1} = (ST)^{-1}$, this suggests that $ST$ is invertible.

✍ **Exercise 3.D.10** Suppose $V$ is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$

**Proof** Directly from the definition of inverse.

✍ **Exercise 3.D.11** Suppose $V$ is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that $T$ is invertible and that $T^{-1} = US$.

**Proof** The associativity implies that both S and U are invertible. Then $T = S^{-1}IU^{-1} = S^{-1}U^{-1}$. Hence, $T^{-1} = US$.

✍ **Exercise 3.D.12** Show that the result in the previous exercise can fail without the hypothesis that $V$ is finite-dimensional.

**Proof** TODO: 3.D.12 Pending.

✍ **Exercise 3.D.13** Suppose $V$ is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that $RST$ is surjective. Prove that $S$ is injective.

**Proof** Since $RST$ is surjective, this suggests that $R$ is surjective. Then $R$ is injective, hence $S$ is injective.

✍ **Exercise 3.D.14** Suppose $v_1, \ldots, v_n$ is a basis of $V$. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(V)$$

is an isomorphism of $V$ onto $\mathbb{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_n$.

**Proof** T is injective: $\forall v_i \in V, Tv = 0 \iff v = 0 \iff v = (0, \ldots, 0)$

T is surjective: $\forall (w_1, \ldots, w_n) \in \mathbb{F}^{n,1}, T(a_1v_1 + \ldots + a_nv_n) = (a_1, \ldots, a_n)$ since $v_1, \ldots, v_n$ is a basis of $V$. Henceforth, T is an isomorphism.

✍ **Exercise 3.D.15** Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an m-by-n matrix $A$ such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$

**Proof** Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{F}^{n,1}$ and $w_1, \ldots, w_m$ be the standard basis of $\mathbb{F}^{m,1}$. Then $T(e_i) = a_{1i}w_1 + \ldots + a_{mi}w_m$. Then $T(x) = Ax$.

✍ **Exercise 3.D.16** Suppose $V$ is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}, TS \in \mathcal{L}(V)$for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$

**Proof**

✍ **Exercise 3.D.17** Suppose $V$ is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$

**Proof** Suppose $\mathcal{E} \neq \mathcal{L}(V)$. The supposition suggests that there exists some linear mapping in $\mathcal{L}(V)$ which is not in $\mathcal{E}$. We denote it by $\varphi$. Now, note that $\varphi \in \mathcal{L}(V)$, we have $\varphi T \in \mathcal{E}$

Suppose $\mathcal{E} \neq \{0\}$, we will show that $\mathcal{E} = \mathcal{L}(V)$. Suppose $\mathcal{E}$ contains the identity map, then the prove is completed, so we assume identity map is not in $\mathcal{E}$.

✍ **Exercise 3.D.18** Show that $V$ and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

**Proof** $\dim \mathcal{L}(\mathbb{F}, V) = \dim \mathbb{F} * \dim V = 1 * \dim V = \dim V$

✍ **Exercise 3.D.19** Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that $T$ is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

(a) Prove that $T$ is surjective.

(b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$

(a) **Proof** $\deg Tp = \deg p$ implies that $\dim T = \dim \mathcal{P}(\mathbb{R})$. Hence, $T$ is surjective.

(b) **Proof** We prove it by induction: $\deg Tp = \deg p$ for $\dim p = 1$ (T is injective). Suppose it holds for $\dim p = n$, then, due to $T$ is surjective, every $p \in \mathcal{P}(\mathbb{R})$ can be expressed as $T(p)$ uniquely. For p = n + 1, suppose that $\deg Tp < \deg p$, then this implies that $T$ is not injective, which is a contradiction. Hence, $\deg Tp = \deg p$ for all $n \in \mathbb{N}$.

## E Products and Quotients of Vector Spaces

✍ **Exercise 3.E.1** Suppose $T$ is a function from $V$ to $W$. The **graph** of $T$ is the subset of $V \times W$ defined by

$$\operatorname{graph} T = \{(v, Tv) : v \in V\}.$$

Prove that $T$ is a linear map if and only if the graph of $T$ is a subspace of $V \times W$.

**Proof** Since $T$ is a well-defined function, the graph of $T$ is not empty. Then, due to the properties of linear maps and vector space, T is also a subspace of $V \times W$. Hence, the graph of $T$ is a subspace of $V \times W$.

✍ **Exercise 3.E.2** Suppose $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that each $V_i$ is finite-dimensional.

**Proof** Suppose $\exists \dim V_i = \infty (i \in \{1, \ldots, m\})$. Then it follows that $\dim(V_1 \times \cdots \times V_m) = \sum \dim V_i = \infty$. This is a contradiction. Hence, each $V_i$ is finite-dimensional.

✍ **Exercise 3.E.3** Give an example of a vector space $V$ and subspaces $U_1, U_2$ of $V$ such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

**Proof** TODO: 3.E.3 Give an example Pending.

✍ **Exercise 3.E.4** Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic.

**Proof** Clearly, for all $T(v_1, \ldots, v_m) = w (w \in W)$, $T \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$. Then we could define a map $S : \mathcal{L}(V_1 \times \cdots \times V_m, W) \to \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ by $S(T) = (T_1, \ldots, T_m)$ where $T_i(v_i) = T(0, \ldots, v_i, \ldots, 0)$. We will verify that $S$ is an isomorphism.

First we show that $S$ is injective: Suppose $S(T) = 0$, then $T_i = 0$ for all $i \in \{1, \ldots, m\}$. Then $T = 0$.

Then we show that $S$ is surjective: This is obvious since for every $(T_1, \ldots, T_m) \in \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$, we could definGe $T(v_1, \ldots, v_m) = (T_1 v_1, \ldots, T_m v_m)$, where $S(T) = (T_1, \ldots, T_m)$.

Hence, $S$ is an isomorphism and $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ is isomorphic to $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$.

✍ **Exercise 3.E.5** Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ is isomorphic to $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$.

**Proof** Define $S : \mathcal{L}(V, W_1 \times \cdots \times W_m) \mapsto \mathcal{L}(V, W_m)$ by $S(T) = (T_1, \ldots, T_m)$ where $T_i(v) = (0, \ldots, T(v), \ldots, 0)$.

We will verify that $S$ is an isomorphism by constructing an inverse function of $S$: define $S^{-1}(T_1(v), \ldots, T_m(v)) = T_1(v) + \cdots + T_m(v)$. $S \circ S^{-1} = \operatorname{id}(T)$

✍ **Exercise 3.E.6** For $n$ a positive integer, define $V_n$ by

$$V^n = \underbrace{V \times \cdots \times V}_{n}$$

Prove that $V^n$ and $\mathcal{L}(\mathbb{F}^n, V)$ are isomorphic vector spaces.

**Proof** Define $S(v_1, \ldots, v_n) = T(\lambda_1, \ldots, \lambda_n)$ where $T \in \mathcal{L}(\mathbb{F}^n, V)$, we will show that $S$ is an isomorphism by constructing an inverse function of $S$. Let $S^{-1}(T(\lambda_1, \ldots, \lambda_n)) = (T_1(\lambda), \ldots, T_n(\lambda))$ where $T_i(\lambda) = (0, \ldots, T(\lambda), \ldots, 0)$. $S \circ S^{-1} = \mathrm{id}$. Hence, They are isomorphic.

✍ **Exercise 3.E.7** Suppose $v, x$ are vectors in $V$ and $U, W$ are subspaces of $V$ such that $v + U = x + W$. Prove that $U = W$

**Proof** Rearrange the equation,

$$v + U = x + W$$
$$v - x + U = W$$

This implies that $U = W$.

✍ **Exercise 3.E.8** Prove that a nonempty subset $A$ of $V$ is an affine subset of $V$ if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$

**Proof** ( $\Longrightarrow$ ) For all affine subset in the form $t + S$, every element can be expressed as $t + s$. Then for all $t + s_1, t + s_2 \in t + S$, $\lambda(t + s_1) + (1 - \lambda)(t + s_2) = t + \lambda s_1 + (1 - \lambda)s_2 \in t + S$ for all $t, s_1, s_2 \in t + S$ and all $\lambda \in \mathbb{F}$ since $S$ is a subspace.

( $\Longleftarrow$ ) If $\lambda v + (1 - \lambda)w \in A$ holds for all $v, w \in A$ and all $\lambda \in \mathbb{F}$. Rearrange the formula we get $\lambda v + (1 - \lambda)w = v + (1 - \lambda)(v + w)$. This implies that $A$ is an affine subset.

## F Duality

✍ **Exercise 3.F.1** Explain why every linear functional is either surjective or the zero map.

**Proof** Suppose $f$ is not surjective, the $\dim \operatorname{range} f < \dim \mathbb{F}$ which could only be $0$. Therefore, $f$ is the zero map.

Suppose $f$ is not the zero map, then $\dim \operatorname{range} f = \dim \mathbb{F}$, which implies that $f$ is surjective.

✍ **Exercise 3.F.2** Give three distinct examples of linear functional on $\mathbb{R}^{[0,1]}$.

**Proof** TODO: 3.F.2 Give three distinct examples Pending.

✍ **Exercise 3.F.3** Suppose $V$ is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

**Proof** Since $v \neq 0$, we could extend $v$ to a basis $v_1, \ldots, v_n$ of $V$. Define $\varphi(v) = 1$ and $\varphi(v_i) = 0$ for $i \neq 1$. Then $\varphi$ is a linear functional.

✍ **Exercise 3.F.4** Suppose $V$ is finite-dimensional and $U$ is a subspace of $V$ such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

**Proof** Let $u_1, \ldots, u_m$ be a basis of $U$ and extend to $u_1, \ldots, u_m, v_1, \ldots, v_n$ a basis of $V$. Define $\varphi(u_i) = 0$ for $i = 1, \ldots, m$ and $\varphi(v_i) = 1$ for $i = 1, \ldots, n$. Then $\varphi$ is a linear functional.

✍ **Exercise 3.F.5** Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

**Proof** Define $S(f) = (f_1, \ldots, f_m)$ where $f_i(v_1, \ldots, v_m) = (0, \ldots, f_i, \ldots, 0)$. And let $S^{-1}(f_1, \ldots, f_m) = f_1 + \cdots + f_m$. Then $S$ is an isomorphism.

✍ **Exercise 3.F.6** Suppose $V$ is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma : V' \mapsto \mathbb{F}^m$ by

$$\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)).$$

(a) Prove that $v_1, \ldots, v_m$ spans $V$ if and only if $\Gamma$ is injective.

(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if $\Gamma$ is surjective.

(a) **Proof** ($\Longrightarrow$) Suppose $v_1, \ldots, v_m$ spans $V$, then for all $\varphi \in V'$, $\Gamma(\varphi) = 0$ implies that $\varphi(v_1) = \cdots = \varphi(v_m) = 0$, that is, $\varphi = 0$. Hence, $\Gamma$ is injective.

($\Longleftarrow$) Suppose $\Gamma$ is injective, then $\varphi(v_1) = \cdots = \varphi(v_m) = 0$ implies that $\varphi = 0$. Hence, $v_1, \ldots, v_m$ is surjective, that is, spanning $V$.

(b) **Proof** ($\Longrightarrow$) Suppose $v_1, \ldots, v_m$ is linearly independent. Then for all $(\lambda_1, \ldots, \lambda_m) \in \mathbb{F}^m$, $\lambda_1\varphi(v_1) + \cdots + \lambda_m\varphi(v_m) = 0$ implies that $\lambda_1 = \cdots = \lambda_m = 0$ or $\forall v_i = 0$. Therefore, for each $\varphi(v_i)$, $\varphi$ spans $\mathbb{F}$. Hence, $\Gamma$ is surjective.

($\Longleftarrow$) Suppose $\Gamma$ is surjective, then for all $v_i$, $\varphi \neq 0$. That is, $v_1, \ldots, v_m$ is linearly independent. Otherwise, suppose $v_1, \ldots, v_m$ is linearly dependent. Let $v_k$ be the one can be expressed by all other $v$'s. Then $\dim \Gamma(\varphi) < m = \dim \mathbb{F}^m$, suggesting that $\Gamma$ is not surjective, contradiction! Therefore, $v_1, \ldots, v_m$ is linearly independent.

✍ **Exercise 3.F.7** Suppose $m$ is a positive integer. Show that the dual basis of the basis $1, x, \ldots, x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the $j^{\text{th}}$ derivative of $p$, with the understanding that the $0^{th}$ derivative of $p$ is $p$.

**Proof** Since $1, x, \ldots, x^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$, let $p = a_1 + a_2x + \cdots + a_mx_m$. Note that $\frac{p^j(0)}{j!} = a_j$, then $\varphi_j(p) = a_j$ if and only if $i = j$. Hence, $\varphi_0, \varphi_1, \ldots, \varphi_m$ is the dual basis of $1, x, \ldots, x^m$.

✍ **Exercise 3.F.8** Suppose $m$ is a positive integer.

(a) Show that $1, x-5, \ldots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$.

(b) What is the dual basis of the basis in part(a)?

(a) **Proof** This is obvious since we can generate an upper-triangle matrix with $1, x-5, \ldots, (x-5)^m$

(b) **Proof** The dual basis is $\varphi_0, \varphi_1, \ldots, \varphi_m$ where $\varphi_j(p) = \frac{p^{(j)}(5)}{j!}$

✍ **Exercise 3.F.9** Suppose $v_1, \ldots, v_n$ is a basis of $V$ and $\varphi_1, \ldots, \varphi_n$ is the corresponding dual basis of $V'$. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$$

**Proof** Notice that

$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_1) = \psi(v_1)$$
$$\vdots$$
$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_n) = \psi(v_n)$$

Therefore, $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$ as they coincide at a basis of $V$.

✍ **Exercise 3.F.10** Suppose $A$ is an m-by-n matrix with $A \neq 0$. Prove that the rank of $A$ is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbb{F}^m$ and $(d_1, \ldots d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$

**Proof** ( $\Longrightarrow$ ) Suppose the rank of $A$ is 1. Then there exists a

✍ **Exercise 3.F.11** Show that the dual map of the identity map on $V$ is the identity map on $V'$.

**Proof** Let $\mathrm{id}_V : V \mapsto V$ be the identity map on $V$. Then the dual map of $\mathrm{id}_V$ is the linear map $\mathrm{id}_V'(\varphi) := \varphi \circ \mathrm{id}_V = \varphi$ where $\varphi \in \mathcal{L}(V, \mathbb{F})$. Hence, the dual map of the identity map on $V$ is the identity map on $V'$.

✍ **Exercise 3.F.12** Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

**Proof** ( $\Longrightarrow$ ) Suppose $T' = 0$. By definition, it suggests that $T'(\varphi) = \varphi(T(v)) = 0$ for all $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $v \in V$. Therefore, it implies that $T = 0$.

( $\Longleftarrow$ ) Suppose $T = 0$. By definition and with the fact that all linear maps map 0 to 0, it suggests that $T' = 0$.

✍ **Exercise 3.F.13** Suppose $V$ and $W$ are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism.

**Proof** Since $\dim \mathcal{L}(V, W) = \dim(W', V')$, the identity map is bijective. Hence, the map is an isomorphism.

✍ **Exercise 3.F.14** Suppose $U \subset V$. Explain why $U^0 = \{\varphi \in V' : U \subset \operatorname{null} \varphi\}$

**Proof** By definition, of course that the zero map 0 is in null space.

✍ **Exercise 3.F.15** Suppose $V$ is finite-dimensional and $U \subset V$. Show that $U = \{0\}$ if and only if $U^0 = V'$.

**Proof** ( $\Longrightarrow$ ) Suppose $U = \{0\}$. Then it suggests that $\forall \varphi \in \mathcal{L}(U, \mathbb{F}), \varphi(u) = 0$. That is, $U^0 = V'$.

( $\Longleftarrow$ ) Directly came from the definition.

✍ **Exercise 3.F.16** Suppose $V$ is finite-dimensional and $U$ is a subspace of $V$. Show that $U = V$ if and only if $U^0 = \{0\}$

**Proof** ( $\Longrightarrow$ ) Suppose $U = V$. Then $U^0 = V^0 := \mathcal{L}(V, \mathbb{F})$. For a vector space, this could only happen when $V = \{0\}$

( $\Longleftarrow$ ) $U^0 = \{0\}$ suggests that $\dim U = \dim V$. That is, $U = V$

✍ **Exercise 3.F.17** Suppose $U$ and $W$ are subsets of $V$ with $U \subset W$. Prove that $W^0 \subset U^0$.

**Proof** Since $U \subset W$, this suggests that $\forall \varphi \in W^0, \varphi(u) = 0$ for all $u \in U$. Hence, $W^0 \subset U^0$.

✍ **Exercise 3.F.18** Suppose $V$ is finite-dimensional and $U$ and $W$ are subspaces of $V$ with $W^0 \subset U^0$. Prove that $U \subset W$.

**Proof** Following from $W^0 \subset U^0$, it suggests that $\dim U > \dim W$. Henceforth, $U \subset W$.

✍ **Exercise 3.F.19** Suppose $U, W$ are subspaces of $V$. Show that $(U + W)^0 = U^0 \cap W^0$

**Proof** Let $u_1, \ldots, u_m$ be a basis of $U$ and $u_1, \ldots, u_m, w_1, \ldots, w_n$ be a basis of $W$. Then

# Chapter 5 Eigenvalues, eigenvectors, and invariant subspaces

## A Invariant Subspaces

**Exercise 5.A.1** Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$.

(a) Prove that if $U \subset \operatorname{null} T$, then $U$ is invariant under $T$.

(b) Prove that if $\operatorname{range} T \subset U$, then $U$ is invariant under $T$.

**Proof**

(a) Let $u \in U$. Then $T(u) \in \operatorname{null} T$ because $u \in \operatorname{null} T$. Therefore, $T(u) \in U$.

(b) Let $u \in U$. Then $T(u) \in U$ because $T(u) \in \operatorname{range} T \subset U$.

**Exercise 5.A.2** Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null} S$ is invariant under $T$.

**Proof** We have $ST(\operatorname{null} S) = T(S(\operatorname{null} S)) = T(0) = 0$. This implies that $T(\operatorname{null} S) \subset \operatorname{null} S$.

**Exercise 5.A.3** Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{range} S$ is invariant under $T$.

**Proof** TODO: 5.A.12 Pending.

**Exercise 5.A.4** Suppose that $T \in \mathcal{L}(V)$ and $U_1, \dots U_m$ are subspaces of $V$ invariant under $T$. Prove that $U_1 + \cdots + U_m$ is invariant under $T$.

**Proof**

$$\begin{aligned} T(U_1 + \cdots + U_m) =& T(U_1) + \cdots + T(U_m) \\ T(U_1) \subset U_1 \subset& U_1 + \cdots + U_m \\ &\cdots \\ T(U_m) \subset U_m \subset& U_1 + \cdots + U_m \end{aligned}$$

Since $U_1 + \cdots + U_m$ is still a subspace, $T(U_1 + \cdots + U_m) \subset U_1 + \cdots + U_m$.

**Exercise 5.A.5** Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of $V$ that are invariant under $T$ is invariant under $T$.

**Proof** Every invariant subspace contains $\{0\}$, and also which is the smallest one. Hence, the intersection is $\{0\}$ and is trivially invariant under $T$.

**Exercise 5.A.6** Prove or give a counterexample: if $V$ is a finite-dimensional vector space and $U$ is a subspace of $V$ that is invariant under every operator on $V$, then $U = \{0\}$ or $U = V$.

**Proof** Because every operator on $V$ leaves $\{0\}$ invariant, the question turns to prove the existence of an operator under which only $\{0\}$ and $V$ is invariant.

The case where $\dim V \leq 1$ is trivial. Suppose $\dim V \geq 2$. We could always construct such an operator $U$. Suppose there exists an invariant subspace $U_1$ of $V$ under an operator $T$ that is neither $\{0\}$ nor $V$ for which $\dim U_1 < \dim V$. Define $T$ which rotates $U_1$ to $W$, where $W \oplus U_1 = V$. $\forall u \in U_1, T(u)$ have some component in $W$, which is not in $U_1$. Thus, $U$ is not invariant under $U$, which is a contradiction. Therefore, $U = \{0\}$ or $U = V$.

**Exercise 5.A.7** Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$$

(a) Find all Eigenvalues and eigenvectors of $T$.
(b) Find all invariant subspaces of $T$.

**Proof**

(a) **Proof** the eigenvalues and the corresponding eigenvectors are i and $(0, \ldots, x_i, \ldots, 0)$
(b) **Proof** The invariant subspaces are $\{0\}$, $\mathbb{F}^n$, and the subspaces spanned by the eigenvectors.

✍ **Exercise 5.A.8** Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of $T$.

**Proof**

$$\begin{aligned} \lambda p(x) &= Tp(x) \\ \lambda a_4x^4 + \lambda a_3x^3 + \lambda a_2x^2 + \lambda a_1x + \lambda a_0 &= x(4a_4x^3 + 3a_3x^2 + 2a_2x + a_1) \\ &= 4a_4x^4 + 3a_3x^3 + 2a_2x^2 + a_1x \end{aligned}$$

The eigenvalues and eigenvectors are $i$ and $ix^i$ respectively.

✍ **Exercise 5.A.9** Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Prove that there exists $\alpha \in \mathbb{F}$ such that $|\alpha - \lambda| < \frac{1}{1000}$ and $T - \alpha I$ is invertible.

**Proof** We only need to make $\alpha$ not be an eigenvalue of $T$. We could achieve this by the following procedure: suppose $\lambda$ is an eigenvalue of $T$, then $T - \lambda I$ is not invertible. We could then choose $\alpha = \lambda + \frac{1}{1000+i}$, where $i \in \{1, \ldots, \dim V + 1\}$, which is not an eigenvalue of $T$.

✍ **Exercise 5.A.10** Suppose $V = U \oplus W$, where $U$ and $W$ are nonzero subspaces of $V$. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of $P$.

**Proof** The eigenvalues are 1 and 0, and the eigenvectors are $u$ and $w$ respectively.

✍ **Exercise 5.A.11** Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that $T$ and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of $T$ and those of $S^{-1}TS$?

(a) **Proof** Let $\lambda$ be a eigenvalue of $T$ and $v$ be the corresponding eigenvector. Then $T(v) = \lambda v$. Now we will verify that $\lambda$ is also an eigenvalue of $S^{-1}TS$.
$S^{-1}(T)S(S^{-1}(v)) = S^{-1}(\lambda v) = \lambda S^{-1}(v)$ since $S$ is invertible. Therefore, eigenvalue for $T$ is also an eigenvalue of $S^{-1}TS$. Similarly, let $\lambda$ be an eigenvalue of $S^{-1}TS$ and $v$ be the corresponding eigenvector such that $S^{-1}TS(v) = \lambda v$. Notice that $S(S^{-1}TS)S^{-1} = T$. Hence, an eigenvalue for $S^{-1}TS$ is also an eigen value for $T$. Therefore, the eigenvalues are the same.

(b) **Proof** The eigenvectors of $S^{-1}TS$ are $S^{-1}v$.

✍ **Exercise 5.A.12** Suppose $V$ is a complex vector space, $T \in \mathcal{L}(V)$, and the matrix of $T$ with respect to some basis of $V$ contains only real entries. Show that if $\lambda$ is an eigenvalue of $T$, then so is $\bar{\lambda}$.

**Proof** Suppose $\lambda$ is an eigenvalue of $T$,

✍ **Exercise 5.A.13** Show that the operator $T \in \mathcal{L}(\mathbb{C}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

**Proof** Suppose $\lambda$ is an eigenvalue of $T$, and $(0, z_1, z_2, \ldots)$ be the corresponding eigenvector. Then $T(z_1, z_2, \ldots) = \lambda(0, z_1, z_2, \ldots)$. This implies that $z_1 = 0$, and $z_2 = \lambda z_1 = 0$, and so on. Therefore, the eigenvector is $(0, 0, 0, \ldots)$, which is not an eigenvector.

✍ **Exercise 5.A.14** Suppose $n$ is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$$

in other words, $T$ is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of $T$.

**Proof** The eigenvalues are $n$ and 0, and the eigenvectors are $(1, \ldots, 1)$, and $\{(x_1, \ldots, x_n) \in \mathbb{F}^n / \{0\} : x_1 + \cdots + x_n = 0\}$ respectively.

TODO: Sec.A 20 and beyond

## B Eigenvectors and Upper-Triangular Matrices

✍ **Exercise 5.B.1** Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer $n$ such that $T^n = 0$

(a) Prove that $I - T$ is invertible and that

$$(I-T)^{-1} = I + T + \cdots + T^{n-1}.$$

(b) Explain how you would guess the formula above.

(a) **Proof** Notice that

$$(I-T)(I+T+\cdots+T^{n-1}) = (I+T+\cdots+T^{n-1})(I-T) = I - T^n = I$$

(b) **Proof**

✍ **Exercise 5.B.2** Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$. Suppose $\lambda$ is an eigenvalue of $T$. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$

**Proof** Suppose $(T-2I)(T-3I)(T-4I) = 0$, this implies that $T$ is upper triangular. Therefore, the eigenvalues are either 2, 3, or 4.

✍ **Exercise 5.B.3** Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of $T$. Prove that $T = I$.

**Proof** Suppose $T^2 = I$. Following from $T^2 * T = T$, we have $T^{2n} = I$. Now if $T \neq I$, then a matrix of $T$ with respect to some basis which is upper triangular has eigenvalues 1 and -1. This is a contradiction.

✍ **Exercise 5.B.4** Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{Range} P$.

**Proof** $P^2 = P$ implies that $P$ is invariant under $V$. Therefore, $V = \operatorname{null} P \oplus \operatorname{Range} P$.

✍ **Exercise 5.B.5** Suppose $S, T \in \mathcal{L}(V)$ and $S$ is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that

$$p(STS^{-1}) = Sp(T)S^{-1}$$

**Proof** From the properties of polynomials, we have

$$\begin{aligned}
p(STS^{-1}) &= a_0 I + a_1 STS^{-1} + \cdots + a_n (STS^{-1})^n \\
&= a_0 I + a_1 STS^{-1} + \cdots + a_n STS^{-1}STS^{-1}\cdots STS^{-1} \\
&= a_0 I + a_1 STS^{-1} + \cdots + a_n ST^nS^{-1} \\
&= Sp(T)S^{-1}
\end{aligned}$$

✍ **Exercise 5.B.6** Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$ invariant under $T$. Prove that $U$ is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$

**Proof** From the properties of polynomials, we have

$$\begin{aligned}
p(T)(U) &= a_0 I + a_1 T + \cdots + a_n T^n(U) \\
&= a_0 I(U) + a_1 T(U) + \cdots + a_n T^n(U) \\
&= U
\end{aligned}$$

✍ **Exercise 5.B.7** Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of $T^2$ if and only if 3 or -3 is an eigenvalue of $T$.

**Proof** The upper triangle matrix of $T^2$ with respect to some basis has 9 on the diagonal, namely eigenvalue, therefore, the eigenvalues of $T$ are $\sqrt{9} = \pm 3$

✍ **Exercise 5.B.8** Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -1$.

**Proof**

✍ **Exercise 5.B.9** Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let $p$ be a nonzero polynomial of smallest degree such that $p(T)v = 0$. Prove that every zero of $p$ is an eigenvalue of $T$.

**Proof** Suppose $\lambda \in \mathbb{F}$ is a zero of $p$. Then by the fundamental theorem of algebra, we have $p(x) = (x-\lambda)q(x)$, where $q(x)$ is a polynomial of degree $n-1$. Therefore, $p(T)v = (T-\lambda I)q(T)v = 0$. Since $p$ is of smallest degree, $q(T)v \neq 0$. Hence, $\lambda$ is an eigenvalue of $T$.

✍ **Exercise 5.B.10** Suppose $T \in \mathcal{L}(V)$ and $v$ is an eigenvetor of $T$ with eigenvalue $\lambda$. Suppose $p \in \mathcal{P}(\mathbb{F})$. Prove that $p(T)v = p(\lambda)v$

**Proof**

$$\begin{aligned} p(T)v &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= a_0 v + a_1 \lambda v + \cdots + a_n \lambda^n v \\ &= p(\lambda)v \end{aligned}$$

✍ **Exercise 5.B.11** Suppose $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{C})$ is a polynomial, and $\alpha \in \mathbb{C}$. Prove that $\alpha$ is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue $\lambda$ of $T$.

**Proof** Suppose $\alpha$ is an eigenvalue of $p(T)$, we have $p(T) = (T-\lambda I)q(T)$.

## C Eigenspaces and Diagonal Matrices

✍ **Exercise 5.C.1** Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{Range} T$.

**Proof** Since $T$ is diagonalizable, $V$ has a basis of eigenvectors of $T$. Therefore, $V = \operatorname{null} T \oplus \operatorname{Range} T$.

✍ **Exercise 5.C.2** Prove the converse of the statement in the exercise above or give a counterexample to the converse.

**Proof**

✍ **Exercise 5.C.3** Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$

(b) $V = \operatorname{null} T + \operatorname{range} T$

(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$

**Proof** (a) $\iff$ (b): This is trivial.

(b) $\to$ (c): Suppose $V = \operatorname{null} T + \operatorname{range} T$. By Theorem, we have

$$\begin{aligned} \dim V &= \dim(\operatorname{null} T + \operatorname{range} T) \\ &= \dim \operatorname{null} T + \dim \operatorname{range} T + \dim(\operatorname{null} T \cap \operatorname{range} T) \end{aligned}$$

✍ **Exercise 5.C.4** Give an example to show that the exercise above is false without the hypothesis that $V$ is finite-dimensional.

**Proof**

✍ **Exercise 5.C.5** Suppose $V$ is a 1finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that $T$ is diagonalizable if and only if

$$V = \operatorname{null}(T-\lambda I) \oplus \operatorname{range}(T-\lambda I)$$

for every $\lambda \in \mathbb{C}$

**Proof** Suppose $T$ is diagonalizable, the eigenvectors of $T$ form a basis of $V$. This impiles that $T-\lambda I$ has same dimension to $V$. That is, $V = \operatorname{null}(T-\lambda I) \oplus \operatorname{range}(T-\lambda I)$.

✍ **Exercise 5.C.6** Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as $T$ (not necessarily with the same eigenvalues). Prove that $ST = TS$.

**Proof** Since $S$ and $T$ have the same eigenvectors. Then $S$ and $T$ are diagonalizable, and this implies that $ST = TS$.

✍ **Exercise 5.C.7** Suppose $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

**Proof** Suppose $\dim E(8, T) = 4$, then we have 4 independent eigenvectors of $T$ with eigenvalue 8. And the diagonal matrix will have 4 8's on the diagonal. Therefore, Suppose $T - 2I$ or $T - 6I$ is not invertible, that is, 2 or 6 is an eigenvalue of $T$. the upper-triangular matrix of $T$ will have 2 or 6 on the diagonal since $\dim T \leq 5$, which is a contradiction.

✍ **Exercise 5.C.8** Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $E(\lambda T) = E(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

# Chapter 6 Inner product spaces

## A Inner products and Norms

✍ **Exercise 6.A.1** Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $|x_1y_1| + |x_2y_2|$ is not an inner product on $\mathbb{R}^3$

**Proof** For $((1,1),(1,1)), ((-1.-1),(1,1)) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have

$$|1 \cdot 1| + |1 \cdot 1| = 2$$

for both two vectors. But on the same hand, we also have

$$\begin{aligned}((1,1),(1,1)) + ((-1,-1),(1,1)) &= ((0,0),(1,1)) \\ &= |0 \cdot 1| + |0 \cdot 1| = 0\end{aligned}$$

This could not be an inner product since it violates the additivity property of inner products.

✍ **Exercise 6.A.2** Show that the function that takes $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1y_1 + x_3y_3$ is not an inner product on $\mathbb{R}^3$

**Proof** It take $(0,1,0)$ to zero while $(0,1,0) \neq 0$

✍ **Exercise 6.A.3** Suppose $\mathbb{F} = \mathbb{R}$ and $V \neq \{0\}$. Replace the positivity condition (which states that $\langle v, v\rangle \geq 0$ for all $v \in V$) in the definition of an inner product with the condition that $\langle v, v\rangle > 0$ for some $v \in V$. Show that this new definition of an inner product does not change the set of functions from $V \times V$ to $\mathbb{R}$ that are inner products on $V$.

**Proof** We show that the two condition are equivalent on the given space. Suppose the positivity condition is satisfied, then the new condition is obviously true for some $v \in V$.

Now we suppose the new condition is satisfied, then for all $v \in V$, we have

$$\langle v, v\rangle =$$

✍ **Exercise 6.A.4** aa

# Chapter 7 Operators on inner product spaces

## A Self-adjoint and Normal Operators

✍ **Exercise 7.A.1** Suppose $n$ is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).$$

Find a formula for $T^*(z_1, \ldots, z_n)$

**Proof** By definition, we have

$$\begin{aligned}
&\langle T(z_1, \ldots, z_n), (x_1, \ldots, x_n)\rangle \\
=&\langle (0, z_1, \ldots, z_{n-1}), (z_1, \ldots, z_n)\rangle \\
=&0 \cdot x_1 + z_1 \cdot x_2 + \cdots + z_{n-1} \cdot x_n \\
=&z_1 \cdot 0 + z_1 \cdot x_2 + \cdots + z_{n-1} \cdot x_n + z_n \cdot 0 \\
=&\langle (z_1, \ldots, z_n), (x_2, x_3, \ldots, 0)\rangle
\end{aligned}$$

Therefore, $T^*(z_1, \ldots, z_n) = (z_2, \ldots, z_n, 0)$.

✍ **Exercise 7.A.2** Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that $\lambda$ is an eigenvalue of $T$ if and only if $\overline{\lambda}$ is an eigenvalue of $T^*$.

**Proof** Suppose $\lambda$ is an eigenvalue of $T$, and choose $v \in V$ be the eigenvector. Then, we have

$$\begin{aligned}
\langle T(v), v\rangle &= \langle \lambda v, v\rangle \\
&= \lambda\langle v, v\rangle \\
&= \langle v, \overline{\lambda} v\rangle \\
&= \langle v, T^*(v)\rangle
\end{aligned}$$

Therefore, $\lambda$ is an eigenvalue of $T$ if and only if $\overline{\lambda}$ is an eigenvalue of $T^*$

✍ **Exercise 7.A.3** Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$. Prove that $U$ is invariant under $T$ if and only if $U^{\perp}$ is invariant under $T^*$.

**Proof** Choose $v \in U$ and $w \in U^{\perp}$, suppose $U$ is invariant under $T$. Then, since $T(v) \in U$, we have

$$0 = \langle T(v), w\rangle = \langle v, T^*(w)\rangle$$

. This implies that $T^*(w) \in U^{\perp}$. Therefore, $T^*$ is invariant under $U^{\perp}$.

✍ **Exercise 7.A.4** Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) $T$ is injective if and only if $T^*$ is surjective;

(a) $T$ is surjective if and only if $T^*$ is injective.

(a) **Proof** Suppose $T$ is injective.

## B The spectral theorem

✍ **Exercise 7.B.1** True or false (and give a proof of your answer): There exists $T \in \mathcal{L}(R_b^3)$ such that $T$ is not self-adjoint (with respect to the usual inner product) and such that there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of $T$

**Proof** The statement is false: suppose there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of $T$. Then we could find a diagonal matrix with respect the the basis consisting of eigenvectors of $T$. Clearly a diagonal matrix equals its transpose. Hence, $T = T^*$, that is, $T$ is self-adjoint.

**Exercise 7.B.2** Suppose that $T$ is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of $T$. Prove that $T^2 - 5T + 6I = 0$

**Proof** Suppose $T$ is self-adjoint, then we could find an eigenvector or $T$ with $\|u\| = 0$. Let $U = \operatorname{span} u$, then $U$ and $U^\perp$ is invariant under $T$, $T|_U \in L(U)$ and $T|_U \in L(U)$ are self-adjoint. We could choose an orthonormal basis in both and the new list created by combining them is also an orthonormal basis consisting of eigenvectors. For any $v \in V$, we could decompose v onto the orthonormal basis. Then, following from $(T^2-5T+6I)v = (T-2I)(T-3I)v$. Since $\lambda \in \{2,3\}$, one of them must be zero. Therefore, $T^2-5T+6I = 0$ as desired.

**Exercise 7.B.3** Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^3)$ such that 2 and 3 are the only eigenvalues of $T$ and $T^2 - 5T + 6I \neq 0$.

**Proof** Consider $T$ such that there doesn't exist an orthonormal basis consisting of eigenvectors of $T$. Then we could find a $v \in V$ such that v cannot expressed as $a_1\lambda_1 + \cdots + a_n\lambda_n$ where $\lambda_n$ are eigenvectors of $T$. Then $T(v)$ does not equals to 0.

## C Positive Operators and Isometries

**Exercise 7.C.1** Prove or give a counterexaples: If $T \in \mathcal{L}(V)$ is self-adjoint and there exists an orthonormal basis $e_1, \ldots, e_n$ of $V$ such that $\langle Te_j, e_j\rangle \geq 0$ for each $j$, then $T$ is a positive operator.

**Proof** TODO: Pending

**Exercise 7.C.2** Suppose $T$ is a positive operator on $V$. Suppose $v, w \in V$ are such that

$$Tv = w \text{and} Tw = v$$

Prove that $v = w$.

**Proof** Suppose $T$ is a positive operator. Then we have

$$\langle T(v-w), v-w\rangle \geq 0$$

On the other hand, we have

$$\begin{aligned}\langle T(v-w), v-w\rangle &= \langle Tv - Tw, v-w\rangle \\ &= -\langle v-w, v-w\rangle \leq 0\end{aligned}$$

Therefore, $\langle v-w, v-w\rangle = 0$ implies $v = w$.

**Exercise 7.C.3** Suppose $T$ is a positive operator on $V$ and $U$ is a subspace of $V$ invariant under $T$. Prove that $T|_U$ is a positive operator on $U$.

**Proof** Suppose $T$ is a positive operator on $V$ and $U$ is invariant under $T$. Then, by definition, for any $u \in U$, $T|_U$ defines a operator in $\mathcal{L}(U)$ and for any $u \in U \subseteq V$, we have

$$\langle Tu, u\rangle \geq 0$$

. Therefore, $T|_U$ is a positive operator on $U$.

## D Polar Decomposition and Singular Value Decomposition

**Exercise 7.D.1** Fix $u, x \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u\rangle\, x$$

for every $v \in V$. Prove that

$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$$

for every $v \in V$.

**Proof** First we need to find the $T^*$. By definition, we have

$$\begin{aligned}\langle v, T^*w\rangle &= \langle Tv, w\rangle \\ &= \langle \langle v, u\rangle x, w\rangle \\ &= \langle v, u\rangle\langle x, w\rangle \\ &= \langle v, \langle w, x\rangle u\rangle \\ T^*v &= \langle v, x\rangle u\end{aligned}$$

Then

$$\begin{aligned}T^*Tv &= T^*\langle v, u\rangle x \\ &= \langle v, u\rangle T^*x \\ &= \langle v, u\rangle\langle x, x\rangle u \\ &= \langle v, u\rangle \|x\|^2 u\end{aligned}$$

Notice that the first two terms are scalars, the eigenvector could only be $u$ and its eigenvalue is $\langle u, u\rangle \|x\|^2 = \|u\|^2 \|x\|^2$.

Thus, $\sqrt{T^*T}v = \frac{T^*Tv}{\sqrt{\lambda}} = \frac{\langle v,u\rangle \|x\|^2 u}{\|u\|\|x\|} = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$

✍ **Exercise 7.D.2** Find a singular values of the differentiation operator $D \in \mathcal{P}(\mathbb{R}^2)$ defined by $Dp = p'$, where the inner product on $\mathcal{P}(\mathbb{R}^2)$ is $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\mathrm{d}x$