

Noto Sans CJK SC

Chapter 2: Finite dimensional vector space

Linear Algebra Done Right, by Sheldon Axler

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A: Span and Linear independence

B: Bases

Problem: 1

Find all vector spaces that have exactly one basis.

Proof. Consider all lines passing through origin. □

Problem: 3

(a) Let U be the subspace of \mathbb{R}^5 defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}$$

Find a basis of U .

(b) Extend the basis in part (a) to a basis of \mathbb{R}^5 .

(c) Find a subspace W of \mathbb{R}^5 such that $\mathbb{R}^5 = U \oplus W$.

Proof. (a) We can verify that $(3, 1, 0, 0, 0)$, $(0, 0, 7, 1, 0)$, $(0, 0, 0, 0, 1)$ is a basis of U , and it spans U .

Idea: We start from the basis of \mathbb{R}^5 and try to fit it into the given condition.

(b) $(3, 1, 0, 0, 0)$, $(0, 0, 7, 1, 0)$, $(0, 0, 0, 0, 1)$, $(0, 1, 0, 0, 0)$, $(0, 0, 1, 0, 0)$

Idea: Downgrade all vector to one dimensional (to obtain the basis of \mathbb{R}^5).

(c) $\text{span}\{(0, 0, 0, 0, 1), (0, 1, 0, 0, 0)\}$

Idea: Refer to the definition of direct sum. □

Problem: 5

Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has a degree 2.

Proof. Consider

$$p_0 = a_1$$

$$p_1 = a_2x$$

$$p_2 = a_3x^3$$

$$p_3 = a_4x^2 + a_5x^3$$

It is easy to verify that they are linearly independent, and $\text{span } \mathcal{P}_3(\mathbb{F})$ □

Problem: 6

Suppose v_1, v_2, v_3, v_4 is a basis of V . Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V .

Proof. We could obtain v_1, v_2, v_3 with the following operation

$$v_3 + v_4 - v_4 = v_3 \quad (1)$$

$$v_2 + v_3 - (v_3 + v_4 - v_4) = v_2 \quad (2)$$

$$v_1 + v_2 - (v_2 + v_3 - (v_3 + v_4 - v_4)) = v_1 \quad (3)$$

Which suggests that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ also spans V .

Next, we prove that they are linearly independent.

$$a(v_1 + v_2) + b(v_2 + v_3) + c(v_3 + v_4) + dv_4 \quad (4)$$

$$= av_1 + (a + b)v_2 + (b + c)v_3 + (c + d)v_4 \quad (5)$$

Since v_1, v_2, v_3, v_4 is a basis, we have $a = b = c = d = 0$. This implies that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also linearly independent. \square

Problem: 8

Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that u_1, \dots, u_m is a basis of U and w_1, \dots, w_n is a basis of W . Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V .

Proof. The linear independency of $u_1, \dots, u_m, w_1, \dots, w_n$ is directly come from the definition of direct sum since $U \cap W = 0$. Since u_1, \dots, u_m is a basis of U . We can write any $u \in U$ as a linear combination of u_1, \dots, u_m . Similarly, we can write any $w \in W$ as a linear combination of w_1, \dots, w_n . Therefore, we can write any $v \in V$ as a linear combination of $u_1, \dots, u_m, w_1, \dots, w_n$. \square

C: Dimension**Problem: 1**

Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Let u_1, \dots, u_n be a basis of U . Since U is a subspace of V , we can extend the basis of U to a basis of V . Since $\dim U = \dim V$, every independent list of right length is also a base suggests that u_1, \dots, u_n also spans V . Therefore, $U = V$. \square

Problem: 2

Show that the subspaces of \mathbb{R}^2 are precisely the zero subspace, all lines in \mathbb{R}^2 through the origin, and \mathbb{R}^2 itself.

Proof. $\dim \mathbb{R}^2 = 2 \implies \dim \text{subspaces of } \mathbb{R}^2 \leq 2$

$$\text{When } \dim U = 0, \quad U = \{0\}. \quad (6)$$

$$\dim U = 1, \quad U \text{ is a line passing through origin.} \quad (7)$$

$$\dim U = 2, \quad U = \mathbb{R}^2. \quad (8)$$

\square