Chapter 7: Operators on Inner Product Spaces

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Contents

A: The spectral theorem

Problem: 1

True or false (and give a proof of your answer): There exists $T \in \mathcal{L}(R_b^3)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T

Proof. The statement is false: suppose there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T. Then we could find a diagonal matrix with respect the the basis consisting of eigenvectors of T. Clearly a diagonal matrix equals its transpose. Hence, $T = T^*$, that is, T is self-adjoint. □

Problem: 2

Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that $T^2 - 5T + 6I = 0$

Proof. Suppose T is self-adjoint, then we could find an eigenvector or T with $\|u\| = 0$. Let $U = u$, then U and $U^\perp$ is invariant under T, $T|_U \in L(U)$ and $T|_U \in L(U)$ are self-adjoint. We could choose an orthonormal basis in both and the new list created by combining them is also an orthonormal basis consisting of eigenvectors. For any $v \in V$, we could decompose v onto the orthonormal basis. Then, following from $(T^2 - 5T + 6I)v = (T - 2I)(T - 3I)v$. Since $\lambda \in \{2, 3\}$, one of them must be zero. Therefore, $T^2 - 5T + 6I = 0$ as desired. □

Problem: 3

Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^3)$ such that 2 and 3 are the only eigenvalues of T and $T^2 - 5T + 6I \neq 0$.

Proof. Consider T such that there doesn't exist an orthonormal basis consisting of eigenvectors of T. Then we could find a $v \in V$ such that v cannot expressed as $a_1\lambda_1 + \cdots + a_n\lambda_n$ where λ_n are eigenvectors of T. Then $T(v)$ does not equals to 0. □

/sectionPositive Operators and Isometries

Problem: 1

Prove or give a counterexaples: If $T \in \mathcal{L}(V)$ is self-adjoint and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\langle Te_j, e_j \rangle \geq 0$ for each j, then T is a positive operator.

Proof. TODO: Pending □

Problem: 2

Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w \text{and} Tw = v$$

Prove that $v = w$.

Proof. Suppose T is a positive operator. Then we have

$$\langle T(v-w), v-w\rangle \geq 0$$

On the other hand, we have

$$\begin{aligned}\langle T(v-w), v-w\rangle &= \langle Tv - Tw, v-w\rangle \\ &= -\langle v-w, v-w\rangle \leq 0\end{aligned}$$

Therefore, $\langle v-w, v-w\rangle = 0$ implies $v = w$. □

B: Polar Decomposition and Singular Value Decomposition

Problem: 1

Fix $u, x \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u\rangle\, x$$

for every $v \in V$. Prove that

$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$$

for every $v \in V$.

Proof. First we need to find the T^*. By definition, we have

$$\begin{aligned}\langle v, T^*w\rangle &= \langle Tv, w\rangle \\ &= \langle\langle v, u\rangle x, w\rangle \\ &= \langle v, u\rangle\langle x, w\rangle \\ &= \langle v, \langle w, x\rangle u\rangle \\ T^*v &= \langle v, x\rangle u\end{aligned}$$

Then

$$
\begin{aligned}
T^*Tv &= T^*\langle v, u\rangle x \\
&= \langle v, u\rangle T^* x \\
&= \langle v, u\rangle \langle x, x\rangle u \\
&= \langle v, u\rangle \left\|x\right\|^2 u
\end{aligned}
$$

Notice that the first two terms are scalars, the eigenvector could only be u and its eigenvalue is $\langle u, u\rangle \left\|x\right\|^2 = \left\|u\right\|^2 \left\|x\right\|^2$.
Thus, $\sqrt{T^*T}v = \frac{T^*Tv}{\sqrt{\lambda}} = \frac{\langle v,u\rangle \|x\|^2 u}{\|u\|\|x\|} = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$ $\square$

Problem: 6

Find a singular values of the differentiation operator $D \in \mathcal{P}(\mathbb{R}^2)$ defined by $Dp = p'$, where the inner product on $\mathcal{P}(\mathbb{R}^2)$ is $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$