# Chapter 5: Eigenvalues, Eigenvectors, and invariant subspaces

Zelong Kuang

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## Contents

## A: Invariant Subspaces

**Problem: 1**

Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$.

(a) Prove that if $U \subset \operatorname{null} T$, then $U$ is invariant under $T$.

(b) Prove that if $\operatorname{range} T \subset U$, then $U$ is invariant under $T$.

*Proof.* (a) Let $u \in U$. Then $T(u) \in \operatorname{null} T$ because $u \in \operatorname{null} T$. Therefore, $T(u) \in U$.

(b) Let $u \in U$. Then $T(u) \in U$ because $T(u) \in \operatorname{range} T \subset U$.

□

**Problem: 2**

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null} S$ is invariant under $T$.

*Proof.* We have $ST(\operatorname{null} S) = T(S(\operatorname{null} S)) = T(0) = 0$. This implies that $T(\operatorname{null} S) \subset \operatorname{null} S$. □

**Problem: 3**

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{range} S$ is invariant under $T$.

*Proof.* TODO: 5.A.12 Pending. □

**Problem: 4**

Suppose that $T \in \mathcal{L}(V)$ and $U_1, \dots U_m$ are subspaces of $V$ invariant under $T$. Prove that $U_1 + \cdots + U_m$ is invariant under $T$.

*Proof.*

$$\begin{aligned} T(U_1 + \cdots + U_m) =& T(U_1) + \cdots + T(U_m) \\ T(U_1) \subset U_1 \subset& U_1 + \cdots + U_m \\ \cdots& \\ T(U_m) \subset U_m \subset& U_1 + \cdots + U_m \end{aligned}$$

Since $U_1 + \cdots + U_m$ is still a subspace, $T(U_1 + \cdots + U_m) \subset U_1 + \cdots + U_m$. □

**Problem: 5**

Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of $V$ that are invariant under $T$ is invariant under $T$.

*Proof.* Every invariant subspace contains $\{0\}$, and also which is the smallest one. Hence, the intersection is $\{0\}$ and is trivially invariant under $T$. $\square$

**Problem: 6**

Prove or give a counterexample: if $V$ is a finite-dimensional vector space and $U$ is a subspace of $V$ that is invariant under every operator on $V$, then $U = \{0\}$ or $U = V$.

*Proof.* Because every operator on $V$ leaves $\{0\}$ invariant, the question turns to prove the existence of an operator under which only $\{0\}$ and $V$ is invariant.

The case where $\dim V \leq 1$ is trivial. Suppose $\dim V \geq 2$. We could always construct such an operator $U$. Suppose there exists an invariant subspace $U_1$ of $V$ under an operator $T$ that is neither $\{0\}$ nor $V$ for which $\dim U_1 < \dim V$. Define $T$ which rotates $U_1$ to $W$, where $W \oplus U_1 = V$. $\forall u \in U_1, T(u)$ have some component in $W$, which is not in $U_1$. Thus, $U$ is not invariant under $U$, which is a contradiction. Therefore, $U = \{0\}$ or $U = V$. $\square$

**Problem: 10**

Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$$

(a) Find all Eigenvalues and eigenvectors of $T$.

(b) Find all invariant subspaces of $T$.

*Proof.*

(a) *Proof.* the eigenvalues and the corresponding eigenvectors are i and $(0, \ldots, x_i, \ldots, 0)$ $\square$

(b) *Proof.* The invariant subspaces are $\{0\}$, $\mathbb{F}^n$, and the subspaces spanned by the eigenvectors. $\square$

$\square$

**Problem: 12**

Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of $T$.

*Proof.*

$$\begin{aligned} \lambda p(x) &= Tp(x) \\ \lambda a_4 x^4 + \lambda a_3 x^3 + \lambda a_2 x^2 + \lambda a_1 x + \lambda a_0 &= x(4a_4 x^3 + 3a_3 x^2 + 2a_2 x + a_1) \\ &= 4a_4 x^4 + 3a_3 x^3 + 2a_2 x^2 + a_1 x \end{aligned}$$

The eigenvalues and eigenvectors are $i$ and $ix^i$ respectively. ☐

**Problem: 13**

Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Prove that there exists $\alpha \in \mathbb{F}$ such that $|\alpha - \lambda| < \frac{1}{1000}$ and $T - \alpha I$ is invertible.

*Proof.* We only need to make $\alpha$ not be an eigenvalue of $T$. We could achieve this by the following procedure: suppose $\lambda$ is an eigenvalue of $T$, then $T - \lambda I$ is not invertible. We could then choose $\alpha = \lambda + \frac{1}{1000+i}$, where $i \in \{1, \ldots, \dim V + 1\}$, which is not an eigenvalue of $T$. ☐

**Problem: 14**

Suppose $V = U \oplus W$, where $U$ and $W$ are nonzero subspaces of $V$. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of $P$.

*Proof.* The eigenvalues are 1 and 0, and the eigenvectors are $u$ and $w$ respectively. ☐

**Problem: 15**

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that $T$ and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of $T$ and those of $S^{-1}TS$?

(a) *Proof.* Let $\lambda$ be a eigenvalue of $T$ and $v$ be the corresponding eigenvector. Then $T(v) = \lambda v$. Now we will verify that $\lambda$ is also an eigenvalue of $S^{-1}TS$.

$S^{-1}(T)S(S^{-1}(v)) = S^{-1}(\lambda v) = \lambda S^{-1}(v)$ since $S$ is invertible. Therefore, eigenvalue for $T$ is also an eigenvalue of $S^{-1}TS$. Similarly, let $\lambda$ be an eigenvalue of $S^{-1}TS$ and $v$ be the corresponding eigenvector such that $S^{-1}TS(v) = \lambda v$. Notice that $S(S^{-1}TS)S^{-1} = T$. Hence, an eigenvalue for $S^{-1}TS$ is also an eigen value for $T$. Therefore, the eigenvalues are the same. □

(b) *Proof.* The eigenvectors of $S^{-1}TS$ are $S^{-1}v$. □

**Problem: 16**

Suppose $V$ is a complex vector space, $T \in \mathcal{L}(V)$, and the matrix of $T$ with respect to some basis of $V$ contains only real entries. Show that if $\lambda$ is an eigenvalue of $T$, then so is $\bar{\lambda}$.

*Proof.* Suppose $\lambda$ is an eigenvalue of $T$, □

**Problem: 18**

Show that the operator $T \in \mathcal{L}(\mathbb{C}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

*Proof.* Suppose $\lambda$ is an eigenvalue of $T$, and $(0, z_1, z_2, \ldots)$ be the corresponding eigenvector. Then $T(z_1, z_2, \ldots) = \lambda(0, z_1, z_2, \ldots)$. This implies that $z_1 = 0$, and $z_2 = \lambda z_1 = 0$, and so on. Therefore, the eigenvector is $(0, 0, 0, \ldots)$, which is not an eigenvector. □

**Problem: 19**

Suppose $n$ is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$$

in other words, $T$ is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of $T$.

*Proof.* The eigenvalues are $n$ and 0, and the eigenvectors are $(1, \ldots, 1)$, and $\{(x_1, \ldots, x_n) \in \mathbb{F}^n / \{0\} : x_1 + \cdots + x_n = 0\}$ respectively. □

TODO: Sec.A 20 and beyond

## B: Eigenvectors and Upper-Triangular Matrices

**Problem: 1**

Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer $n$ such that $T^n = 0$

(a) Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + \cdots + T^{n-1}.$$

(b) Explain how you would guess the formula above.

(a) *Proof.* Notice that

$$(I - T)(I + T + \cdots + T^{n-1}) = (I + T + \cdots + T^{n-1})(I - T) = I - T^n = I$$

□

(b) *Proof.* □

**Problem: 2**

Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose $\lambda$ is an eigenvalue of $T$. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$

*Proof.* Suppose $(T - 2I)(T - 3I)(T - 4I) = 0$, this implies that $T$ is upper triangular. Therefore, the eigenvalues are either 2, 3, or 4. □

**Problem: 3**

Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of $T$. Prove that $T = I$.

*Proof.* Suppose $T^2 = I$. Following from $T^2 * T = T$, we have $T^{2n} = I$. Now if $T \neq I$, then a matrix of $T$ with respect to some basis which is upper triangular has eigenvalues 1 and -1. This is a contradiction. □

**Problem: 4**

Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{Range} P$.

*Proof.* $P^2 = P$ implies that $P$ is invariant under $V$. Therefore, $V = \operatorname{null} P \oplus \operatorname{Range} P$. □

**Problem: 5**

Suppose $S, T \in \mathcal{L}(V)$ and $S$ is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that

$$p(STS^{-1}) = Sp(T)S^{-1}$$

*Proof.* From the properties of polynomials, we have

$$\begin{aligned}
p(STS^{-1}) &= a_0 I + a_1 STS^{-1} + \cdots + a_n (STS^{-1})^n \\
&= a_0 I + a_1 STS^{-1} + \cdots + a_n STS^{-1}STS^{-1} \cdots STS^{-1} \\
&= a_0 I + a_1 STS^{-1} + \cdots + a_n ST^n S^{-1} \\
&= Sp(T)S^{-1}
\end{aligned}$$

□

**Problem: 6**

Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$ invariant under $T$. Prove that $U$ is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$

*Proof.* From the properties of polynomials, we have

$$\begin{aligned}
p(T)(U) &= a_0 I + a_1 T + \cdots + a_n T^n(U) \\
&= a_0 I(U) + a_1 T(U) + \cdots + a_n T^n(U) \\
&= U
\end{aligned}$$

□

**Problem: 7**

Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of $T^2$ if and only if 3 or -3 is an eigenvalue of $T$.

*Proof.* The upper triangle matrix of $T^2$ with respect to some basis has 9 on the diagonal, namely eigenvalue, therefore, the eigenvalues of $T$ are $\sqrt{9} = \pm 3$ □

**Problem: 8**

Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -1$.

*Proof.* □

**Problem: 9**

Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let $p$ be a nonzero polynomial of smallest degree such that $p(T)v = 0$. Prove that every zero of $p$ is an eigenvalue of $T$.

*Proof.* Suppose $\lambda \in \mathbb{F}$ is a zero of $p$. Then by the fundamental theorem of algebra, we have $p(x) = (x - \lambda)q(x)$, where $q(x)$ is a polynomial of degree $n - 1$. Therefore, $p(T)v = (T - \lambda I)q(T)v = 0$. Since $p$ is of smallest degree, $q(T)v \neq 0$. Hence, $\lambda$ is an eigenvalue of $T$. □

**Problem: 10**

Suppose $T \in \mathcal{L}(V)$ and $v$ is an eigenvetor of $T$ with eigenvalue $\lambda$. Suppose $p \in \mathcal{P}(\mathbb{F})$. Prove that $p(T)v = p(\lambda)v$

*Proof.*

$$\begin{aligned} p(T)v &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= a_0 v + a_1 \lambda v + \cdots + a_n \lambda^n v \\ &= p(\lambda) v \end{aligned}$$

□

**Problem: 11**

Suppose $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{C})$ is a polynomial, and $\alpha \in \mathbb{C}$. Prove that $\alpha$ is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue $\lambda$ of $T$.

*Proof.* Suppose $\alpha$ is an eigenvalue of $p(T)$, we have $p(T) = (T - \lambda I)q(T)$. □

## C: Eigenspaces and Diagonal Matrices

**Problem: 1**

Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{Range} T$.

*Proof.* Since $T$ is diagonalizable, $V$ has a basis of eigenvectors of $T$. Therefore, $V = \operatorname{null} T \oplus \operatorname{Range} T$. □

**Problem: 2**

Prove the converse of the statement in the exercise above or give a counterexample to the converse.

*Proof.* □

**Problem: 3**

Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$

(b) $V = \operatorname{null} T + \operatorname{range} T$

(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$

*Proof.* (a) $\iff$ (b): This is trivial.
(b) $\to$ (c): Suppose $V = \operatorname{null} T + \operatorname{range} T$. By Theorem, we have

$$\begin{aligned} \dim V &= \dim(\operatorname{null} T + \operatorname{range} T) \\ &= \dim \operatorname{null} T + \dim \operatorname{range} T + \dim(\operatorname{null} T \cap \operatorname{range} T) \end{aligned}$$

□

**Problem: 4**

Give an example to show that the exercise above is false without the hypothesis that $V$ is finite-dimensional.