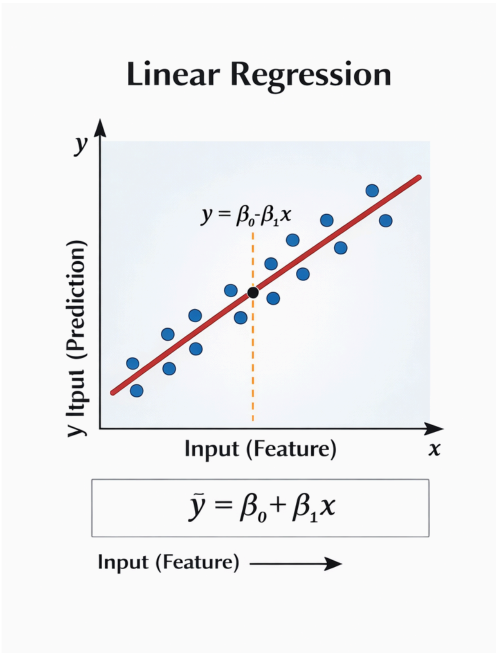
# Documentation-Linear Regression




## 1 Problem Formulation

### 1.1 Input Space

The input space consists of feature vectors $\mathbf{X} \in \mathbb{R}^{\mathbf{d}}$,

where each data point contains **d** numerical features.

### 1.2 Output Space

The output space is continuous and real-valued: $\mathbf{Y} \in \mathbb{R}$

### 1.3 Data Distribution

Assume data samples $(\mathbf{x_i}, \mathbf{y_i})$ are drawn independently and identically distributed (i.i.d.) from an unknown joint distribution $\mathbf{P(X, Y)}$.

### 1.4 Learning Objective (Expected Risk Minimization)

The goal is to learn a function $\mathbf{f(x)}$ that minimizes the expected prediction error:

# 2 Model Specification

### 2.1 Hypothesis Function

Linear Regression assumes a linear relationship between input features and output:

$$\mathbf{f(x) = w^T x + b}$$

where:

- $\mathbf{w} \in \mathbb{R}^d$ are model coefficients
- $\mathbf{b} \in \mathbb{R}$ is the bias term

### 2.2 Parameter Space

$$\mathbf{\Theta = \{ (w, b) \mid w \in \mathbb{R}^d, b \in \mathbb{R} \}}$$

### 2.3 Structural Assumptions

- Linearity between features and target
- Additive noise model
- Independence of observations

# 3 Loss Function

### 3.1 Explicit Mathematical Form

The most common loss function is Mean Squared Error (MSE): $\mathbf{L(y, \hat{y}) = (y - \hat{y})^2}$

### 3.2 Why This Loss? (Statistical Reasoning)

- Penalizes larger errors more heavily
- Differentiable and mathematically tractable
- Corresponds to Maximum Likelihood Estimation under Gaussian noise assumption

### 3.3 Convexity Properties

The squared error loss is convex with respect to model parameters, ensuring a unique global minimum.

## 4 Objective Function

### 4.1 Empirical Risk Expression

$$J(w, b) = (1/n) \sum_{i=1}^{n} (y_i - (w^T x_i + b))^2$$

### 4.2 Regularized Formulation (if used)

- Ridge (L2) Regularization: $J(w) = (1/n) \sum (y_i - w^T x_i)^2 + \lambda\|w\|^2$
- Lasso (L1) Regularization: $J(w) = (1/n) \sum (y_i - w^T x_i)^2 + \lambda\|w\|_1$

## 5 Optimization Method

### 5.1 Closed-form or iterative?

Closed-form solution via Normal Equation: $w = (X^T X)^{-1} X^T y$

### 5.2 Gradient expression

Iterative solution via Gradient Descent

$$\nabla J(w) = -(2/n) X^T (y - Xw)$$

### 5.3 Convergence guarantees

Since the objective is convex, gradient descent converges to a global minimum with a proper learning rate.

### 5.4 Computational complexity

- Closed-form: **$O(d^3)$** due to matrix inversion
- Gradient Descent: **$O(nd)$** per iteration

## 6 Statistical Interpretation

### 6.1 MLE / MAP connection

Linear regression is equivalent to Maximum Likelihood Estimation (MLE) assuming:

$$Y = w^T X + \epsilon$$

Where **$\epsilon \sim N(0, \sigma^2)$**

### 6.2 Noise model assumption

Errors are assumed to be normally distributed with constant variance.

### 6.3 Probabilistic meaning of outputs

The model predicts the conditional mean:

**$E[Y \mid X]$**

## 7️⃣ Regularization & Generalization

### 7.1 Bias–variance tradeoff

- High bias → Underfitting
- High variance → Overfitting
- Regularization balances both

### 7.2 Overfitting behavior

Occurs when model complexity is high or when multicollinearity exists.

### 7.3 Capacity control mechanism

Regularization limits coefficient magnitude to prevent overfitting.

## 8️⃣ Theoretical Properties

### 8.1 Convexity / Global optimality

The optimization problem is convex → guarantees global optimum.

### 8.2 Consistency (if applicable)

Under standard assumptions, the estimator is statistically consistent as sample size increases.

### 8.3 Stability considerations

Sensitive to outliers due to squared loss.

# 9️⃣ Computational Complexity

### 9.1 Training time complexity

- Normal Equation: **$O(d^3)$**
- Gradient Descent: **$O(nd \times \text{iterations})$**

### 9.2 Inference time complexity

**$O(d)$**

### 9.3 Memory complexity

**$O(nd)$**

# 🔟 Limitations

### 10.1 When it fails

- Non-linear relationships
- High multicollinearity
- Heteroscedastic data

### 10.2 Assumption violations

- Non-normal residuals
- Correlated features
- Dependent observations

### 10.3 Sensitivity issues (outliers, scaling, etc.)

- Sensitive to outliers
- Requires feature scaling for numerical stability