1. $A = \begin{pmatrix} 16 & 1 \\ 1 & 16 \end{pmatrix}$

$$|\lambda E - A| = \begin{vmatrix} \lambda - 16 & -1 \\ -1 & \lambda - 16 \end{vmatrix} = 0$$

$$(\lambda - 16)^2 - 1 = 0$$

$$\lambda_1 = 17 \quad \lambda_2 = 15$$

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i| = 17$$

$$\mathrm{Cond}_2(A) = \frac{17}{15}$$

2.

(1) 迭代函数 $\varphi(x) = 4 + \frac{2}{3}\cos x$

对 $\forall x_0$ 有 $3 \le \varphi(x) \le 5$

有 $\varphi(x) \in (-\infty, +\infty)$

$\varphi'(x) = -\frac{2}{3}\sin x$,

$|\varphi'(x)| \le \frac{2}{3} = L < 1$

∴ 迭代法收敛

方程根 $\alpha \in (3.4)$, $g'(\alpha) \ne 0$

∴ 收敛阶为1

(2) $x_0 = 4$

$x_1 = 4 + \frac{2}{3}\cos 4$

$L = \frac{2}{3}$

$$\therefore k > \frac{\ln \frac{10^{-3} \times \frac{1}{3}}{\frac{2}{3}\cos 4}}{\ln \frac{2}{3}} > 6$$

∴ 迭代 7 步

3 G-S 迭代法的计算公式为：

$$\begin{cases} x_1^{(k+1)} = 1 - 2x_2^{(k)} + 2x_3^{(k)} \\ x_2^{(k+1)} = 2 - x_1^{(k+1)} - x_3^{(k)} \\ x_3^{(k+1)} = 3 - 2x_1^{(k+1)} - 2x_2^{(k+1)} \end{cases}$$

② $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$

G-S 迭代法的迭代矩阵为：

$$G = -(D-L)^{-1}U = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - G| = \lambda(\lambda-2)^3 = 0$$

$\rho(G) = 2 > 1$ 所以 G-S 迭代法不收敛

4. 是，非负性和齐次性易证

三角不等式可由柯西不等式得出

$$\|x\| = \sqrt{x_1^2+2x_2^2+x_3^2} = (x_1, x_2, x_3)\begin{bmatrix}1&&\\&\sqrt{2}&\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&\sqrt{2}&\\&&1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

令 $y = \begin{bmatrix}1&&\\&\sqrt{2}&\\&&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = (x_1, \sqrt{2}x_2, x_3)^T$

则 $\|x\| = \sqrt{y^Ty} = \|y\|_2$

$\therefore$ $\|x\|$ 是 R^3 上的范数

5. 迭代矩阵的谱半径

$$\rho(I-\alpha A)=\max_{1\le i\le n}|1-\alpha\lambda_i|$$，λ_i 为 A 的特征值

$0<\alpha<1$

$0<|\lambda_i|<2$　　代入得

$\therefore\ 0<\max_{1\le i\le n}|1-\alpha\lambda_i|<1$

$\therefore$ 谱半径小于 1

$\therefore$ 收敛